An Introduction to Differential Geometry

A Senior Comprehensive Project
by
Mitchell Lachat
Allegheny College
Meadville, PA

May 6, 2020

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science.

Project Advisor: Dr. Brent Carswell
Second Reader: Dr. Craig Dodge

I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

Mitchell Lachat

Acknowledgements

I want to thank the following:

Dr. Lobello: For his effort in making me love mathematics. He is the reason that I became a mathematics major and I hope that he can impact the lives of many more students in the same way that he impacted mine.

Dr. Carswell: For the growing pains in my mathematics career. His classes were difficult but I learned so many things from your courses. Hard work has a big payoff and to really understand material, you need to struggle with it.

Dr. Dodge: For my new found love of geometry. I loved every minute of being in your geometry class. It was my first time doing a flipped classroom but it allowed me to explore the material in a way that I would not have otherwise. I will be interested in this topic for the rest of my life.

Dr. Weir: For the compassion that you show. I will never forget when I was deciding whether to take complex analysis and you came to me with how complex analysis has application is astrophysics. You love to show people the beauty of analysis.

Dr. Leech: For giving me the opportunities to teach others. While being a tutor for calculus I, II, and III, I was able to learn the skills needed to assist in learning for people new to the subject. I gained a lot of knowledge that I will be using throughout the rest of my life.

Dr. Werner: For always being a friendly face in the department. You were always there to say hello when I was doing work in the student lounge. I also gained a large number of mathematical skills in your junior seminar

that I will never forget.

All Mathematics Majors: For the memories. We had a lot of great moments doing work together, playing SET, our just hanging out. We will always have this bond.

Betsy and Everyone in Ballroom: For your friendship. Betsy is more of a friend than a professor to me. Ballroom was a way for me to de-stress, and it was some of the most fun that I have had my whole time at Allegheny College.

Dr. Niblock, Carol, Vicky, Ward, and the Choirs: For the music that we made together. I will never forget the amount of memories that we had singing in the choir room or the stairwell. Even in opera scenes, without a performance, I had some of my favorite experiences.

Abstract

Differential geometry can be applied to a variety of subjects including general relativity. Differential geometry is typically a graduate concept but I believe that this paper makes it digestible for final year undergraduate mathematics majors. We begin with the idea of a regular surface followed by change of parameters. We then transition to talk about tangent planes, the first fundamental form, and area. We conclude with Gauss maps including the main properties of these maps.

Contents

1 Introduction

A common topic in astrophysics is general relativity. General relativity leads to the study of stellar dynamics, black holes, and gravitational waves. General relativity is a graduate level concept that is daunting to some undergraduates even as they enter graduate school. Therefore, the purpose of this project is to present some preliminary mathematics that can serve as a bridge to general relativity. The focus is on differential geometry where the material presented can be applied to future study of general relativity.

The connection between differential geometry and general relativity is shown beautiful in a paper written by one of the most well know theorists to work on the subject, John Archibald Wheeler (1911-2008). The paper is titled *Problems on the Frontiers Between General Relativity and Differential Geometry* and was published in the Review of Modern Physics in 1962 [1]. Wheeler explains that there is much unknown in both differential geometry and general relativity. In reality, these problems are the same and any progress in one immediately boosts the other.

Thus it is worthwhile to look at some of the fundamental differential geometry that can be used in the context of differential geometry. To do this, we will use *Differential Geometry of Curves & Surfaces* by Manfredo P. do Carmo as a guide [2]. We will touch on subject matter taught in the common 300 level mathematics courses such as MATH*340 Introduction to Real Analysis but will not do proofs that would appear in these classes. It is expected that the reader has a grasp on the basic analysis concepts provided by the Allegheny College mathematics curriculum.

2 Regular Surfaces

The first chapter of do Carmo is mostly covered in calculus and real analysis so please refer to the text if more background is required. Therefore we will begin working with regular surfaces by defining a regular surface in $\mathbb{R}^3$. We will then talk about differentiability and the connection that $\mathbb{R}^3$ has with $\mathbb{R}^2$ in the context of differentials. We will wrap up with an introduction of the first fundamental form that will be used to determine quantities such as the length of curves and the area of regions.

2.1 Defining and Showing a Surface is Regular

The main idea of a regular surface is that portions of a plane are deformed and rearranged in such a way that we have a surface with no edges, points, or self-intersections and in such a way that the existence of tangent plane in meaningful. We can state this idea more formally for a surface S in the following way:

Definition 2.1. The subset $S \subset \mathbb{R}^3$ is a normal surface if, for each $p \in S$, there exists a neighborhood V of p in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable,
2. $\mathbf{x}$ is a homeomorphism,
3. For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

Condition 3 is called the regularity condition.

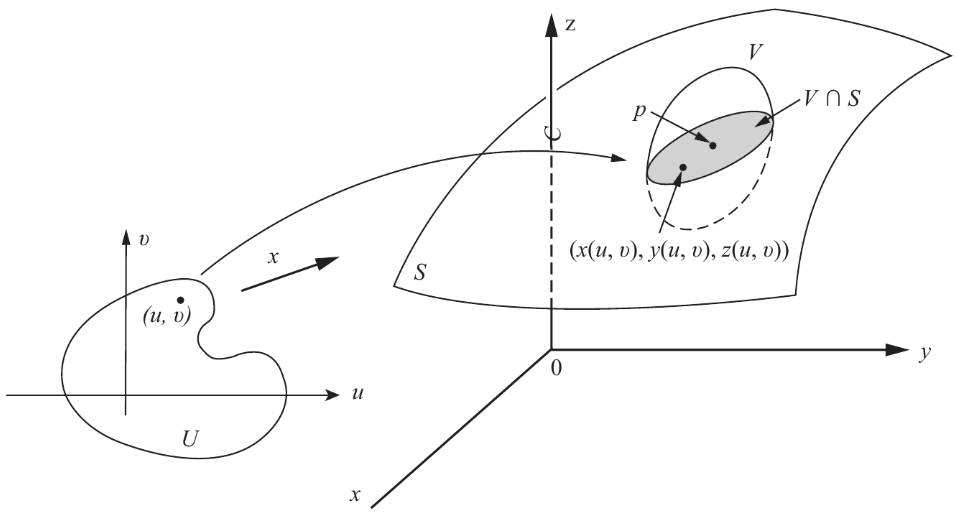

Figure 1: Shows the mapping of $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ (found on page 54 [2]).

The mapping can be seen in Figure 1. For $\mathbf{x}$ to be differentiable, if

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \qquad (u, v) \in U,$$

then the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ must have continuous partial derivatives of all orders in U. For $\mathbf{x}$ to be a homeomorphism, we need $\mathbf{x}$ to be continuous and $\mathbf{x}$ to have an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous. By condition 1, we know that $\mathbf{x}$ is continuous because if $\mathbf{x}$ is differentiable at a point, then $\mathbf{x}$ is continuous at that point. So to prove condition 2, we would just have to show that $\mathbf{x}^{-1}$ exists and is continuous.

Condition 3 can be expressed in another more convenient way. We will find the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ coordinates (u, v) and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$

of $\mathbb{R}^3$ with coordinates (x, y, z). Let $q = (u_0, v_0)$. Then the vector e_1 is tangent to the curve $v = v_0$ with $\mathbf{x}(u, v_0) = (x(u, v_0),\, y(u, v_0),\, z(u, v_0))$. So the tangent vector at $\mathbf{x}(q)$ is

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial \mathbf{x}}{\partial u}$$

where the derivatives are computed at q and the component vectors are in the basis $\{f_1, f_2, f_3\}$. Thus by the definition of differential,

$$d\mathbf{x}_q(e_1) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial \mathbf{x}}{\partial u}.$$

Similarly, the vector e_2 is tangent to the curve $u = u_0$ so

$$d\mathbf{x}_q(e_2) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \frac{\partial \mathbf{x}}{\partial v}.$$

Therefore the matrix of the linear map $d\mathbf{x}_q$ is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

in the above basis.

Condition 3 now has another meaning. The condition is met if the column vectors of the above matrix are linearly independent. To more easily determine whether the column vectors are independent, we can also check whether one of the Jacobian determinants,

$$\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(x,z)}{\partial(u,v)},$$

is not zero at point q where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

This also checks for linear independence as we know that for two vectors to be linearly independent, their determinant must be 0. We are doing this for each pairing of x, y, and z so as long as we have a nonzero determinant we have linear independence and have satisfied condition 3 of Definition 2.1.

Example 2.2. Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\}$ is a regular surface and find parametrizations whose coordinate neighborhoods cover it.

Solution. The first step is to find a maps $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ that parametrize the cylinder. First, let's look at

$$\mathbf{x}_1(x, z) = (x, \sqrt{1 - x^2}, z), \qquad x \in U$$

where $U = \{(x, z) \in \mathbb{R}^2; x^2 < 1\}$, which is a parametrization of a part of the cylinder.

Since $x^2 < 1$, the function $\sqrt{1 - x^2}$ has continuous partial derivatives of all orders. So $\mathbf{x_1}$ is differentiable and condition 1 holds.

We know that condition 3 holds because

$$\frac{\partial(x,z)}{\partial(x,z)} \equiv 1.$$

For condition 2, we know that $\mathbf{x}_1$ is one-to-one. Also, notice that $\mathbf{x}_1^{-1}$ is the restriction of the continuous projection $\pi(x,y,z) = (x,z)$ to the set $x_1(U)$. Therefore, $\mathbf{x}_1^{-1}$ is continuous in $\mathbf{x}_1(U)$.

All of the conditions are met for $\mathbf{x}_1$. Notice that $\mathbf{x}_1$ only parametrizes a portion of the cylinder in question. To parametrize the rest of the cylinder, we must include

$$\begin{aligned}
\mathbf{x}_2(x,z) &= (x, -\sqrt{1-x^2}, z),\\
\mathbf{x}_3(y,z) &= (+\sqrt{1-y^2}, y, z),\\
\mathbf{x}_4(y,z) &= (-\sqrt{1-y^2}, y, z).
\end{aligned}$$

It can easily be shown that $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ also satisfy all three conditions of Definition 2.1. We can see that our set of parametrizations covers the entire cylinder because $\mathbf{x}_1$ and $\mathbf{x}_2$ cover all of the cylinder except the lines parametrized by $(1,0,z)$ and $(-1,0,z)$ but $\mathbf{x}_3$ and $\mathbf{x}_4$, respectively, cover these parametrized lines. Therefore, we have proven that the given cylinder is a regular surface.

2.2 Change of Parameters and Differentiable Functions

Now that we have an intuition for regular surfaces, we need to focus on the coordinate neighborhoods around points. Notice that in Example 2.2 a point in any of the quadrants of the xy-plane does not only belong to one coordinate neighborhood. Each of these points belongs to two coordinate neighborhoods. So we need a way to transform from one coordinate neighborhood to another where the transformation is differentiable.

Proposition 2.3. (Change of Parameters) Let p be a point of a regular surface S, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the "change of coordinates" $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.

Proposition 2.3 can be expressed in another way similar to how we represented Definition 2.1. Let $\mathbf{x}$ and $\mathbf{y}$ be given in the following way:

$$\begin{aligned} \mathbf{x}(u,v) &= (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in U, \\ \mathbf{y}(\xi,\eta) &= (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)), \qquad (\xi,\eta) \in V. \end{aligned}$$

Then the change of coordinates h is given by

$$u = u(\xi,\eta), \qquad v = v(\xi,\eta), \qquad (\xi,\eta) \in \mathbf{y}^{-1}(W),$$

where u and v have continuous partial derivatives of all orders. Also, h must be differentiable such that

$$\xi = \xi(u,v), \qquad \eta = \eta(u,v), \qquad (u,v) \in \mathbf{x}^{-1}(W).$$

In all of these cases, both ξ and η also have continuous partial derivatives of all orders. We know that for two matrices A and B, that if $|A||B| = 1$, then $|B| = 1/|A|$ where $|A| \neq 0$. So if

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \frac{\partial(\xi,\eta)}{\partial(u,v)} = 1, \tag{1}$$

the Jacobian determinents of both h and h^{-1} are nonzero everywhere. We know that Equation 1 is true because $\frac{\partial a}{\partial b}\frac{\partial b}{\partial a} = 1$ for some a, b by the chain rule.

The proof for Proposition 2.3 uses a theorem that that is widely used through the text known as the *Inverse Function Theorem.* The *Inverse Function Theorem* states the following:

> If $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable mapping and the differential $dF : \mathbb{R}^n \to \mathbb{R}^n$ at a point $p \in U$ is an isomorphism, then there exists a neighborhood V of $p \in U$ and a neighborhood W of $F(p) \in \mathbb{R}^n$ such that $F : V \to W$ has a differentiable inverse $F^{-1} : W \to V$.

We will take the *Inverse Function Theorem* as fact for it is not the focus of this work. If you would like a proof of the theorem, please look at Buck's *Advanced Calculus* or Spivak's *Calculus on Manifolds* [3, 4].

Proof. The first thing that must be shown is $h = \mathbf{x}^{-1} \circ \mathbf{y}$ is a homeo-

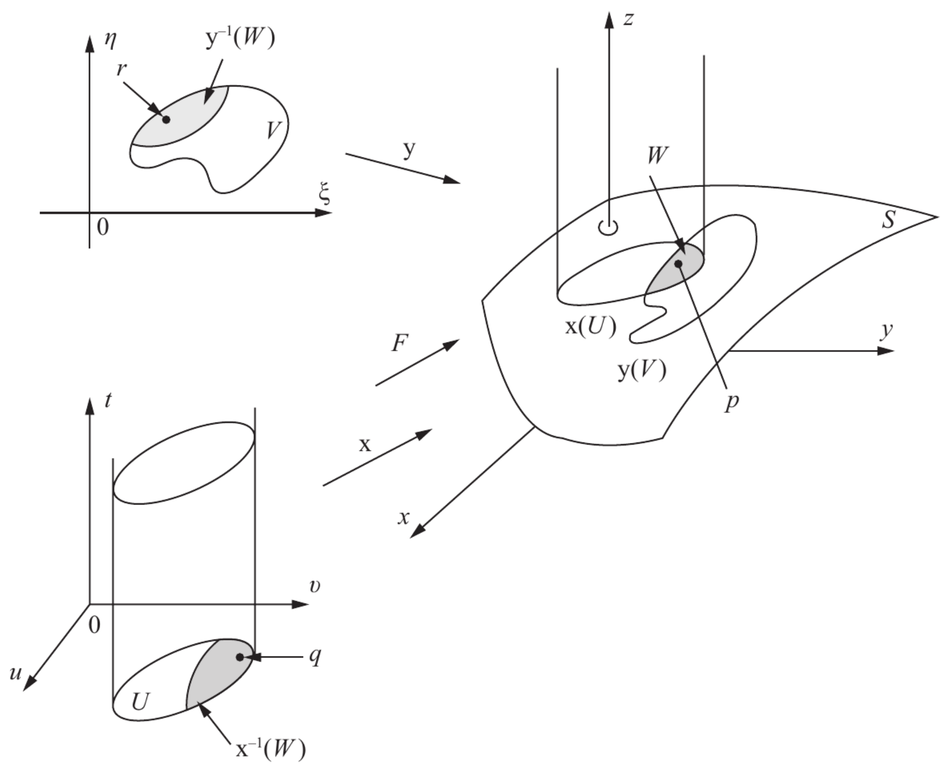

Figure 2: Shows the relationship between two sets of parameters can be both be used to describe the surface (found on page 73 [2]).

morphism. We already know that $\mathbf{x}^{-1}$ is a homeomorphism and $\mathbf{y}$ is a homeomorphism. Since h is composed of homeomorphisms, h must be a homeomorphism.

Now we will show that h is differentiable. Let $r \in \mathbf{y}^{-1}(W)$ and $q = h(r)$. Then since $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parametrization, we know that one of the Jacobian determinants at point q must not be equal to zero.

We want to use the inverse function theorem to show that we have a neighborhood around q where we have a differentiable inverse. To do this, let us extend $\mathbf{x}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ where

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t), \qquad u, v \in U, t \in \mathbb{R}.$$

Then we have a vertical cylinder C over U that gets mapped to another vertical cylinder over $\mathbf{x}(U)$. Introducing t into $\mathbf{x}$ does not change differentiability. So F is differentiable and the restriction $F|_{U\times\{0\}} = \mathbf{x}$. By calculating the determinant of the differential dF at point q we find that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

Therefore, dF is also an isomorphism. We know that F is a parametrization $\mathbf{x}$ is a parametrization. With these conditions, we can invoke the inverse function theorem to state that there exists a neighborhood M of $\mathbf{x}(q)$ in $\mathbb{R}^3$ such that F^{-1} exists and is differentiable in M.

Since $\mathbf{y}$ is continuous, by definition there exists a neighborhood N of r in V such that $\mathbf{y}(N) \subset M$. Notice that $h|_N = F^{-1} \circ y|_N$ is a composition of differential maps. Using the chain rule for maps[1], we can conclude that h is differentiable in r. Because r is arbitrary, we know that h is differentiable on $\mathbf{y}^{-1}(W)$. We can now use a similar argument to show that h^{-1} is differentiable. Thus h is a diffeomorphism. $\square$

[1]The proof for this is fairly straight forward. Consider a third function $\alpha : (-\epsilon.\epsilon) \to U$ where the point in question is an element of U and $\alpha(0) = p$. Then setting α' equal to a given curve w_1 and defining $dF(w_1) = w_2$ at p, we can show that $d(G \circ F)(w_1) = dG \circ dF(w_1)$ via $(d/dt)(G \circ F \circ \alpha)_{t=0}$ since $dG(w_2) = (d/dt)(G \circ F \circ \alpha)_{t=0}$.

We will now give a rigorous definition for what is meant by a function to be differentiable on a regular surface.

Definition 2.4. Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$. Moreover, f is differentiable in V if it is differentiable at all points of V.

There is one last useful proposition which states that for a point in $\mathbb{R}^2$, some neighborhood around that point can be mapped to a regular surface.

Proposition 2.5. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface where $\mathbf{x}$ is a collection of parametrization functions and let $q \in U$. Then there exists a neighborhood V of q in $\mathbb{R}^2$ such that $\mathbf{x}(V) \subset \mathbb{R}^3$ is a regular surface.*

Proof. We write $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Since the parametrization of $\mathbf{x}$ is regular, WLOG, we know that $(\partial(x, y)/\partial(u, v))(q) \neq 0$. Let $F : U \times \mathbb{R} \to \mathbb{R}^3$ be defined by

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t), \qquad u, v \in U, t \in \mathbb{R}.$$

Then

$$\det(dF_q) = \frac{\partial(x, y)}{\partial(u, v))}(q) \neq 0.$$

By the inverse value function, there exist a neighborhoods W_1 of q and W_2 of $F(q)$ such that $F : W_1 \to W_2$ is a diffeomorphism. Then we can set

$V = W_1 \cap U$ and see that $F|_V = \mathbf{x}|_V$. Therefore $\mathbf{x}(V)$ is a diffeomorphism to V. Because $\mathbf{x}(V)$ is a diffeomorphism, we know that condition 1 and 2 of Definition 2.1 are satisfied. Since we already showed that condition 3 was met, we know that $\mathbf{x}(V)$ is a regular surface. $\square$

Example 2.6. Show that the parabaloid $z = x^2 + y^2$ is a diffeomorphism to a plane.

Proof. Let U be a subset of $\mathbb{R}^2$ and $\mathbf{x} : U \to \mathbb{R}^3$ be defined by $\mathbf{x}(x, y) = (x, y, x^2 + y^2)$. We will show that $\mathbf{x}$ is a diffeomorphism. We know that $\mathbf{x}$ is differentiable as x, y, and $x^2 + y^2$ are all differentiable. Let us extend $\mathbf{x}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(x, y, t) = (x, y, x^2 + y^2 + t).$$

Notice that $F|_{U\times\{0\}} = \mathbf{x}$. The determinant of the differential dF at a point $q \in U$ is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x & 2y & 1 \end{vmatrix} = 1 \neq 0.$$

Therefore, we know that dF is a parametrization and an isomorphism. Since F is a parametrization, we can apply the inverse function theorem. So there exists a neighborhood M of $\mathbf{x}(q)$ in $\mathbb{R}^3$ such that F^{-1} exists and is differentiable in M. Since q is arbitrary in the plane, we know that F is a diffeomorphism. So by the restriction, we know that $\mathbf{x}$ is a diffeomorphism. $\square$

2.3 Tangent Planes

We will show that the third condition of Definition 2.1 guarantees that for every point on a regular surface the set of tangent vectors to the parametrized curves of the surface that pass through the point are a plane. We define a *tangent vector* to a regular surface S at a point $p \in S$ as the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

Proposition 2.7. *Let* $\mathbf{x} : U \subset R^2 \to S$ *be a parametrization of a regular surface* S *and let* $q \in U$. *The vector subspace of dimension 2,*

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S *at* $\mathbf{x}(q)$.

Proof. Let w be a tangent vector at $\mathbf{x}(q)$. In other words, let $w = \alpha'(0)$ where $\alpha : (-\epsilon, \epsilon) \to \mathbf{x}(U) \subset S$ is differentiable and $\alpha(0) = \mathbf{x}(0)$. Notice that $\beta = \mathbf{x}^{-1} \circ \alpha : (-\epsilon, \epsilon) \to U$ is differentiable because if $q \in \mathbf{x}(W)$ and α is a parametrization containing p then $\mathbf{x}^{-1} \circ \alpha : \alpha^{-1}(W) \to \mathbf{x}^{-1}(W)$ where $W = \mathbf{x}(U) \cap \alpha(U)$ which is differentiable. So U and $\mathbf{x}(U)$ are diffeomorphic. By the definition of differential, we know that $d\mathbf{x}_q(\beta'(0)) = w$. Thus $w \in d\mathbf{x}_q(\mathbb{R}^2)$.

Now let $w = d\mathbf{x}_q(v)$ where $v \in \mathbb{R}^2$. Then v is the velocity vector of the curve $\gamma : (-\epsilon, \epsilon) \to U$ given by

$$\gamma(t) = tv + q.$$

By the definition of the differential, $w = \alpha'(0)$, where $\alpha = \mathbf{x} \circ \gamma$. Combining these results, we know that w is a tangent vector. Notice that w is of dimension 2 as dx_q has $\mathbb{R}^2$ as both the domain and codomain. $\square$

There are many ideas from calculus that we are able to extend for use in differential geometry. One of these is the *Inverse Function Theorem* that we mentioned earlier. We can apply the *Inverse Function Theorem* to prove a version for our applications.

Proposition 2.8. *If S_1 and S_2 are regular surfaces and $\varphi : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\varphi_p$ of φ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.*

Proof. Let S_1 and S_2 be regular surfaces and $\varphi : U \subset S_1 \to S_2$ be a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\varphi_p$ of φ at $p \in U$ is an isomorphism. It follows directly from the Inverse Function Theorem that there exists a neighborhood W of $\varphi(p) \in S_2$ such that $\varphi : C \to W$ has a differentiable inverse $\varphi^{-1} : W \to V$. Since φ and its inverse are differentiable at p, we can conclude that φ is a local diffeomorphism at p. $\square$

Example 2.9. Show that the equation of the tangent plane of a surface which is a graph of a differentiable function $z = f(x, y)$, at a point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Solution. Using our definition of differential, we know that

$$df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix}.$$

To find all linear combinations of our two vectors, we must find the span of the vectors which is the following:

$$\text{span}\left[\begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}\right] = a\begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}.$$

We also know that the span of our vectors has to be the same as Δf where

$$\Delta f(x_0, y_0) = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - f(x_0, y_0) \end{pmatrix}.$$

Therefore,

$$a\begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}((x_0, y_0)) + b\begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}((x_0, y_0)) = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - f(x_0, y_0) \end{pmatrix}$$

for some $a, b \in \mathbb{R}^2$. Thus, $a = x - x_0$ and $b = y - y_0$ and we have

$$(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) = f(x, y) - f(x_0, y_0)$$

which can be rewritten as

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

as desired.

2.4 First Fundamental Form and Area

Now that we have looked at surfaces from the view of differentiability, we will look at some of the geometric properties of regular surfaces. We will begin with what is known as the first fundamental form.

The first fundamental form involves the inner product which will be denoted by $\langle \, , \, \rangle$. Let $T_p(S)$ be the tangent plane at point p of the regular surface S where $T_p(S)$ is a 2-dimensional subspace of $\mathbb{R}^3$ consistent with Proposition 2.7. So if $w_1, w_2 \in T_p(S) \subset \mathbb{R}^3$, then $\langle w_1, w_2 \rangle_p$ is equal to the ordinary dot product of w_1 and w_2 as vectors in $\mathbb{R}^3$. The inner product has a symmetric bilinear form which means that $\langle w_1, w_2 \rangle = \langle w_2, w_1 \rangle$ and $\langle w_1, w_2 \rangle$ is linear in both w_1 and w_2. For each inner product at point p, there exists a corresponding quadratic form $I_p : T_p(S) \to R$.

Definition 2.10. The quadratic form I_p on $T_p(S)$, defined by $I_p(w) = \langle w, w \rangle = |w|^2 \geq 0$, is called the *first fundamental form* of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

It is useful to express the the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to the parametrization $\mathbf{x}(u, v)$ at p. The tangent vector $w \in T_p(S)$ is tangent to a parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t)), t \in (-\epsilon, \epsilon)$, with

$p = \alpha(0) = \mathbf{x}(u_0, v_0)$. Then we have the following:

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle_p \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v')^2, \end{aligned}$$

with the values of the functions are evaluated at $t = 0$. We can define function in the following way:

$$\begin{aligned} E(u_0, v_0) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \\ F(u_0, v_0) &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \\ G(u_0, v_0) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p, \end{aligned}$$

where the funtions are in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S)$. Our first fundamental form then becomes

$$I_p(\alpha'(0)) = E(u')^2 + 2Fu'v' + G(v')^2.$$

Notice that while p is in the coordinate neighborhood of $\mathbf{x}(u, v)$, $E(u, v)$, $F(u, v)$, and $G(u, v)$ are differentiable in that neighborhood.

We can use the first fundamental form to compute the area of a surface. First, we need a definition for the area of a surface. A regular domain of a surface S is an open and connected subset of S such that its boundary is the image in S of a circle by a differentiable homeomorphism with a nonzero differential except at a finite number of points. A region of S is the union of a domain with its boundary. A region of $S \subset \mathbb{R}^3$ is bounded if it is contained in some ball of $\mathbb{R}^3$. The function $|\mathbf{x}_u \wedge \mathbf{x}_v|$ is the area of the parallelogram formed by the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$. We use the notation "$\wedge$" instead of "$\times$" for the cross product to allow for easier readability.

Definition 2.11. Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. The positive number

$$\iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv = A(R), \qquad Q = \mathbf{x}^{-1}(R),$$

is called the area of R.

Notice that

$$|\mathbf{x}_u \wedge \mathbf{x}_v|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle^2 \langle \mathbf{x}_v, \mathbf{x}_v \rangle^2.$$

Therefore, we find that

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{EG - F^2}.$$

Now we can write our equation for area as

$$\iint_Q \sqrt{EG - F^2} \, du \, dv = A(R), \qquad Q = \mathbf{x}^{-1}(R)$$

which is much easier to compute.

Notice that $Q = \mathbf{x}^{-1}(R)$ means that Q is the region in $\mathbb{R}^2$ that maps to our surface S. In the case of a torus, we can see this in action in Figure 3.

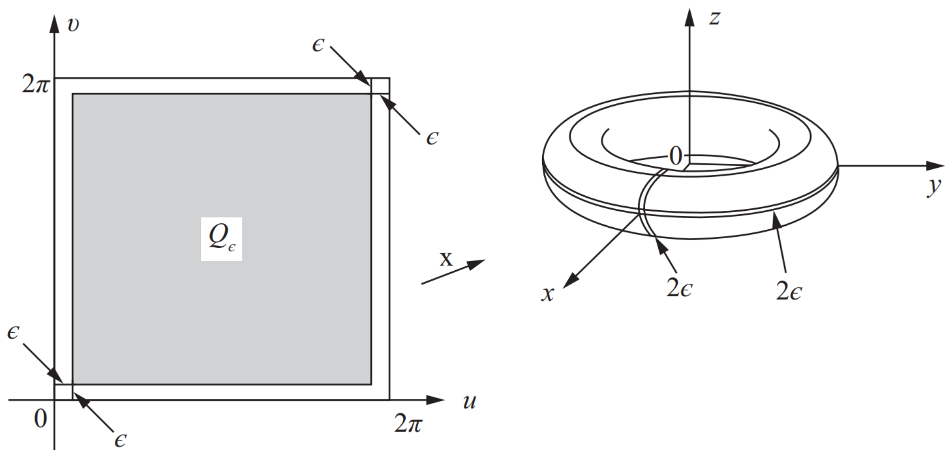

Figure 3: Shows the area of a regular surface S. This specifically illustrates the way in which Q would be mapped to the surface of a torus (found on page 101 [2]).

Example 2.12. Show that the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + {f_x}^2 + {f_y}^2}\, dx\, dy$$

where Q is the normal projection of R onto the xy plane.

Solution. So we need to find the integrand of $A(R)$ from Definition 2.11. We can do this by finding $\sqrt{EG - F^2}$. For this, we need to define $\mathbf{x}$. Since Q is the normal projection, we know that $\mathbf{x}(x, y) = (x, y, f(x, y))$. So $\mathbf{x}_x = (1, 0, f_x)$ and $\mathbf{x}_y = (0, 1, f_y)$. Then we have the following for E, F, and G:

$$E = 1 + {f_x}^2$$

$$F = f_x f_y$$

$$G = 1 + {f_y}^2.$$

Therefore, our integrand is

$$\begin{aligned}\sqrt{EG - F^2} &= \sqrt{(1 + {f_x}^2)(1 + {f_y}^2) - {f_x}^2{f_y}^2} \\ &= \sqrt{1 + {f_x}^2 + {f_y}^2 + {f_x}^2{f_y}^2 - {f_x}^2{f_y}^2} \\ &= \sqrt{1 + {f_x}^2 + {f_y}^2}.\end{aligned}$$

Thus, the integral for the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + {f_x}^2 + {f_y}^2}\, dx\, dy$$

where Q is the normal projection of R onto the xy plane as desired.

3 The Gauss Map

In this section, we will be looking at curvature. Specifically, we want to have a way to measure how quickly a surface moves away from from a tangent plane. We will define the Gauss map and outline its fundamental properties.

3.1 Defining the Gauss Map

Before we get into our definition of the Gauss map, we need to have a general understanding of orientation. We do this by first defining a unit normal vector at each point on a surface. With a parametrization $x : U \subset \mathbb{R}^2 \to S$ of a regular surface S at a point $p \in S$, we can define a unit normal vector at each point of $\mathbf{x}(U)$ with

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), \qquad q \in \mathbf{x}(U). \tag{2}$$

We now have a differentiable map $N : \mathbf{x}(U) \to \mathbb{R}^3$ where at each $q \in \mathbf{x}(U)$ there is a unit normal vector $N(q)$. When $V \subset S$ is an open set in S and $N : V \to \mathbb{R}^3$ is a differentiable map, we say that N is a differentiable field of unit normal vectors on V.

A surface is orientable if there can be a differentiable field of unit normal vectors defined on the whole surface. Naively, it would seem that all surfaces are orientable. However, this is not the case and a prime example is the Möbius strip. If you look at one point p which has value N and follow the strip around a full loop until you reach p again, you find that you have a value of $-N$. Therefore, N is not continuous and we have no defined "side" of the surface. If we choose a differentiable field of unit vectors that is orientable, we have an orientation.

Now that we have a general understanding of orientation, we can get our first look at the Gauss map.

Definition 3.1. Let $S \subset \mathbb{R}^3$ be a surface with an orientation N, a differentiable field of unit vectors on V. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}.$$

The map $N : S \to S^2$, thus defined, is called the Gauss map of S.

An illustration of the Gauss map can be seen in Figure 4. The Gauss map takes all of the unit normal vectors at each point of an orientable regular surface and maps them to the unit sphere where the vectors emanate from the center of the unit sphere.

The Gauss map is differentiable because the differential dN_p of N at $p \in S$ is linear from $T_p(S)$ to $T_{N(p)}(S^2)$. Notice that $T_p(S)$ and $T_{N(p)}(S^2)$ are the same vector space due to the unit normal vectors always being normal to the vector space. This is true in both $T_p(S)$ and $T_{N(p)}(S^2)$. Since dN_p of N at $p \in S$ is linear, we know that $dN(p + h) = dN(p) + dN(h)$ for some $h \in S$. Therefore, dN_p is differentiable. We should look at the linear map $dN_p : T_p(S) \to T_{N(p)}(S^2)$ more carefully as it gives a large amount of insight on how surfaces curve.

As we have done before, let $\alpha(t)$ be a parametrized curve in S where $\alpha(0) = p$. Then we should consider $N \circ \alpha(t) = N(t)$ in the sphere S^2. This

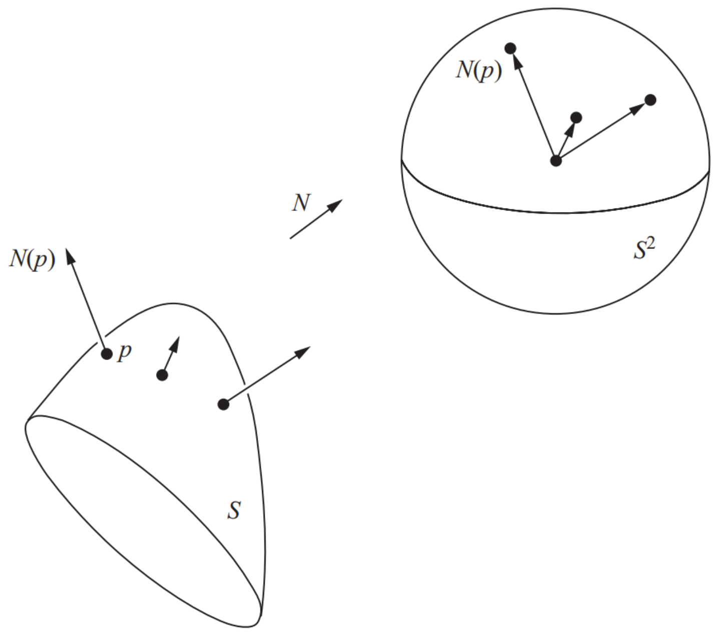

Figure 4: An illustration of a Gauss map which maps the normal unit vectors at each point of an orientable regular surface to the unit sphere (found on page 139 [2]).

restricts the normal vector N to the curve $\alpha(t)$. Notice that the tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$ which measures the rate of change of the normal vector N. So dN_p measures how quickly N moves away from $N(p)$ in a neighborhood of p. We can see how α' affects N on a surface in Figure 5. When dealing with curves, this measure is given by a number, the curvature but for surfaces, it is characterized by a linear map.

We can represent dN using matrices in the following way:

$$dN = \begin{pmatrix} \frac{\partial N_1}{\partial u} & \frac{\partial N_2}{\partial u} \\ \frac{\partial N_1}{\partial v} & \frac{\partial N_2}{\partial v} \end{pmatrix},$$

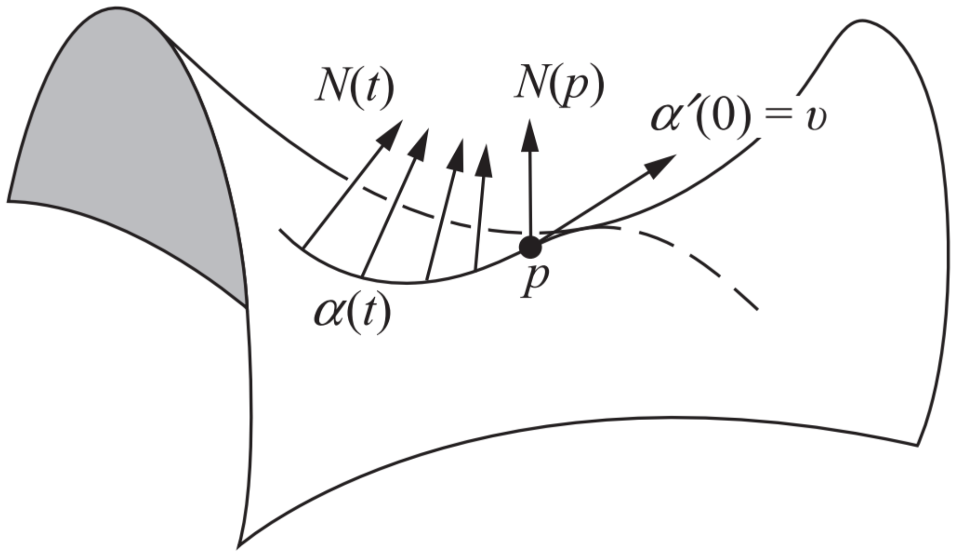

Figure 5: An illustration of how α' is related to a change in N (found on page 139 [2]).

where N_1 is the u-component of N and N_2 is the v-component of N. Notice that we do not use a third component of N as we are only concerned with the change in N with respect to our tangent plane which is 2-dimensional.

Example 3.2. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

1. Paraboloid of revolution $z = x^2 + y^2$

2. Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$

Solution.

1. We can use the parametrization $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$. We find that

$$\mathbf{x}_u = (1, 0, 2u), \qquad \mathbf{x}_v = (0, 1, 2v).$$

Therefore we find that the normal vectors using Equation 2 have the form

$$N = \left(-\frac{u}{\sqrt{u^2+v^2+\frac{1}{4}}}, -\frac{v}{\sqrt{u^2+v^2+\frac{1}{4}}}, \frac{1}{2\sqrt{u^2+v^2+\frac{1}{4}}} \right).$$

Now that we have defined our set of normal unit vectors, we can parametrize in terms of spherical polar coordinates that will allow us to see the behavior of these unit vectors. So parametrizing in terms of spherical polar coordinates, we find that our normal unit vectors are

$$\begin{aligned} r(u,v) &= 1 \\ \tan(\theta(u,v)) &= \tfrac{v}{u} \\ \tan(\phi(u,v)) &= 2\sqrt{u^2+v^2}. \end{aligned}$$

Since $\tan(\phi(u,v)) = 2\sqrt{u^2+v^2}$ can never be negative, we cover only the hemisphere with $z > 0$ of the unit circle.

2. Let $U \subset \mathbb{R}^2$ such that $u^2 - v^2 > -1$. Then we can use the following parametrizations:

$$\mathbf{x}_1(u,v) = (\sqrt{v^2-u^2+1}, u, v), \qquad \mathbf{x}_2(u,v) = (-\sqrt{v^2-u^2+1}, u, v).$$

Due to symmetry across the xz-plane, we will only look at $\mathbf{x}_1(u,v)$. We find that

$$\mathbf{x}_{u,1} = (\frac{-u}{\sqrt{v^2-u^2+1}}, 1, 0), \qquad \mathbf{x}_{v,1} = (\frac{v}{\sqrt{v^2-u^2+1}}, 0, 1).$$

Therefore we find that the normal vectors using Equation 2 have the form

$$N = \left(\sqrt{\frac{v^2 - u^2 + 1}{2v^2 + 1}}, \frac{u}{\sqrt{2v^2 + 1}}, \frac{-v}{\sqrt{2v^2 + 1}} \right).$$

Parametrizing in terms of spherical polar coordinates, we find that

$$\begin{aligned} r(u, v) &= 1 \\ \tan(\theta(u, v)) &= \tfrac{u}{\sqrt{v^2 - u^2 + 1}} \\ \tan(\phi(u, v)) &= -\tfrac{\sqrt{2v^2+1}}{v}. \end{aligned}$$

Notice that as $v \to \infty$, $\tan(\phi(u, v)) \to -\sqrt{2}$ and $v \to -\infty$, $\tan(\phi(u, v)) \to \sqrt{2}$. So we cover all θ and all ϕ such that $-\sqrt{2} < \tan(\phi(u, v)) < \sqrt{2}$ of the unit circle.

3.2 Properties of Gauss Maps

Now that we have a definition for the Gauss map along with a general understanding of dN we can define two new concepts: the second fundamental form and normal curvature.

Definition 3.3. The quadratic form II_p, defined in $T_p(S)$ by

$$II_p(v) = -\langle dN_p(v), v \rangle,$$

is called the second fundamental form of S at p.

Definition 3.4. Let C be a regular curve on S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N \rangle$, where n is the normal unit vector to C and N is the normal unit vector to S at p. The number $k_n = k \cos\theta$ is then called the normal curvature of $C \subset S$ at p.

We can think of Definition 3.4 in terms of a projection. The normal curvature is length of the projection of the vector kn to the line normal to the surface at p, where the sign of k_n is given by the orientation N of S at p. We can see this in Figure 6.

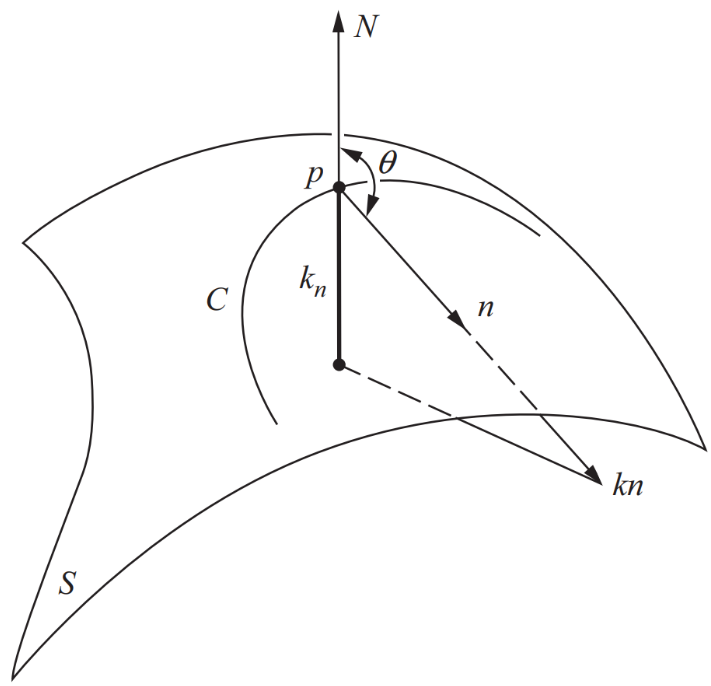

Figure 6: An illustration of normal curvature. We can see that k_n is the projection of kn onto the line normal to the surface (found on page 143 [2]).

We can use the ideas of Definition 3.3 and Definition 3.4 to develop what is known as the Meusnier Theorem.

Proposition 3.5. ***(Meusnier)*** *All curves lying on the surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures.*

Proof. Let $C \subset S$ be a regular curve parametrized by $\alpha(s)$, where s is the arc length of C and $\alpha(0) = p$. Let $N(s)$ be the restriction of the normal vector N to the curve $\alpha(s)$ such that $N(s) = N(\alpha(s))$. Since $N(s)$ is restricted to the curve, which is restricted to the surface, we know that $N(s)$ is always orthogonal to the tangent vector of $\alpha(s)$. Therefore, we know that $\langle N(s), \alpha'(s)\rangle = 0$. Taking the derivative, we know that $\langle N(s), \alpha''(s)\rangle + \langle N'(s), \alpha'(s)\rangle = 0$. Hence,

$$\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle.$$

By definition of dN_p, we know that $dN_p(\alpha'(0)) = N'(0)$. Now we have the following:

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle \\ &= -\langle N(p), kn(p)\rangle = k_n(p). \end{aligned}$$

So the value of the second fundamental form for a unit vector $v \in T_p(S)$ is the same as the normal curvature of a regular curve through p and tangent to v. □

With knowledge of how we can measure curvature in a given direction, we can start laying down a set of labels based on this.

Definition 3.6. The maximum normal curvature k_1 and the minimum normal curvature k_2 over all curves through p are called the principal curvatures at p; the corresponding directions, that is, the directions given by the eigenvectors of dN are e_1 and e_2, called the principle directions at p.

Definition 3.7. If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p, then C is said to be a line of curvature of S.

If we let v be one of these eigenvectors, then for certain values of $|v|$, we can find the normal curvature.

This means that we can create the following definition:

Definition 3.8. Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinent of dN_p is the Gaussian curvature K of S at p. The negative of half the trace of dN_p is called the mean curvature H of S at p.

We can write these values in terms of the principle curvatures with

$$K = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2}.$$

Definition 3.9. A point of a surface S is called

1. Elliptic if $\det(dN_p) > 0$.
2. Hyperbolic if $\det(dN_p) < 0$.
3. Parabolic if $\det(dN_p) = 0$ with $dN_p \neq 0$.
4. Planar if $dN_p = 0$.

We will now look at examples of points that are elliptic, hyperbolic, parabolic, and planer.

For an elliptic surface, consider the point $(0, 0, 1)$ on the unit sphere with the equation $x^2 + y^2 + z^2 = 1$. We can parametrize a portion of the sphere containing the point in question with $\mathbf{x}(u, v) = (u, v, \sqrt{-u^2 - v^2 + 1})$. We find that

$$\mathbf{x}_u = \left(1, 0, -\frac{u}{\sqrt{-u^2 - v^2 + 1}}\right), \qquad \mathbf{x}_v = \left(0, 1, -\frac{v}{\sqrt{-u^2 - v^2 + 1}}\right).$$

So

$$N = \left(u, v, \sqrt{-u^2 - v^2 + 1}\right)$$

and

$$dN = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the determinant is 1 and the unit sphere has a Gaussian curvature of 1 at the point $p = (0, 0, 1)$. So by Definition 3.9 the surface is elliptic at that point. With the correctly chosen parametrization, we would find that any point on a sphere has a Gaussian curvature of 1 and is elliptic at all points on the surface.

For a hyperbolic surface, consider the point $(0, 0, 0)$ on the hyperbolic paraboloid with the equation $z = y - x$. We can parametrize the hyperbolic paraboloid with $\mathbf{x}(u, v) = (u, v, v^2 - u^2)$. We find that

$$\mathbf{x}_u = (1, 0, -2u), \qquad \mathbf{x}_v = (0, 1, 2v).$$

So

$$N = \left(\frac{u}{\sqrt{u^2+v^2+\frac{1}{4}}}, \frac{-v}{\sqrt{u^2+v^2+\frac{1}{4}}}, \frac{1}{2\sqrt{u^2+v^2+\frac{1}{4}}} \right)$$

and

$$dN = \begin{pmatrix} \frac{8v^2+2}{(4u^2+4v^2+1)^{3/2}} & -\frac{uv}{(4u^2+4v^2+1)^{3/2}} \\ \frac{uv}{(4u^2+4v^2+1)^{3/2}} & -\frac{8u^2+2}{(4u^2+4v^2+1)^{3/2}} \end{pmatrix}.$$

Therefore,

$$dN_p = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The determinent is -4 and the hyperbolic paraboloid has Gaussian curvature -4 at the point $p = (0,0,0)$. So by Definition 3.9 the surface is hyperbolic at that point.

For a parabolic surface, consider the the point $p = (0,0,0)$ on the parabolic sheet with equation $y = x^2$. We can parametrize the parabolic sheet with $\mathbf{x}(u,v) = (u, u^2, v)$. We find that

$$\mathbf{x}_u = (1, 2u, 0)\,, \qquad \mathbf{x}_v = (0,0,1)\,.$$

So

$$N = \left(\frac{2u}{\sqrt{4u^2+1}}, -\frac{1}{\sqrt{4u^2+1}}, 0 \right)$$

and

$$dN = \begin{pmatrix} \frac{2}{(4u^2+1)^{3/2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$dN_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The determinent is 0 and the parabolic sheet has Gaussian curvature 0 at the point $p = (0, 0, 0)$. Notice that $dN_p \neq 0$. So by Definition 3.9 the surface is parabolic at that point.

For a planar surface, consider the point $p = (x_0, y_0, z_0)$ on the plane with equation $ax + by + z + d = 0$. We can parametrize the plane with $\mathbf{x}(u, v) = (u, v, -au - bv - d)$. We find that

$$\mathbf{x}_u = (1, 0, -a)\,, \qquad \mathbf{x}_v = (0, 1, -b)\,.$$

So

$$N = (1, 0, 0)$$

and

$$dN = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The determinent is 0 and the plane has Gaussian curvature 0 at the point $p = (0, 0, 0)$. So by Definition 3.9 the surface is planarat that point.

Example 3.10. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$|k| \geq \min(|k_1|, |k_2|),$$

where k_1 and k_2 are the principal curvatures of S at p.

Proof. Let k be the curvature of the regular curve C at point p. Since $K > 0$ and $K = k_1 k_2$, $k_1 \neq 0$ and $k_2 \neq 0$. Let n be the normal vector to C at p and let N be the normal vector to S at p. Then $\langle n, N \rangle = \cos\theta$ where θ is the smallest angle between n and N. Then by Definition 3.4, $k_m = k\cos\theta_m$ for all $m \in \mathbb{N}$. Therefore we have $k_1 = k\cos\theta_1$ and $k_2 = k\cos\theta_2$. Since e_1 and e_2 are the principle directions and they form an orthonormal basis, we know that $\theta_2 = \theta_1 \pm \frac{\pi}{2}$. Thus $k_1 = k\cos\theta_1$ and $k_2 = k\cos(\theta_1 \pm \frac{\pi}{2}) = \mp k\sin\theta_1$. Since $k_1 \neq 0$ and $k_2 \neq 0$, $\cos\theta_1 \neq 0$ and $\sin\theta_1 \neq 0$. So $|k| = |\frac{k_1}{\cos\theta_1}|$ and $|k| = |\frac{k_2}{\mp\sin\theta_1}| = |\frac{k_2}{\sin\theta_1}|$. Since $0 < |\cos\theta_1| \leq 1$ and $0 < |\sin\theta_1| \leq 1$, $|k| \geq \min(|k_1|, |k_2|)$. $\square$

4 Conclusion

This introduction to differential geometry has a covered a few of the most important subject matters to the field. We have seen the idea of regular surfaces. To prevent coordinate dependence, we allowed for a change of parameters. Then we looked at the tangent planes of a regular surface. We were then introduced to the first fundamental form as well as a formalized definition for the area of a regular surface. We finished our study of differential geometry with the Gauss map.

If the reader would like to continue their journey through differential geometry, please look that the second half of our guide text: *Differential Geometry of Curves and Surfaces* [2]. Continue through the text as it covers the intrinsic geometry of curved surfaces and global differential geometry. A follow-up reference which goes more in depth on this topic would be another book by Manfredo P. do Carmo titled *RiemmannGeometry*.

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