||Jai Sri Gurudev||
Sri Adichunchanagiri Shikshana Trust (R.)
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P.B. No.20, B.B Road, Chikkaballapur - 562 101, Karnataka
www.sjcit.ac.in

Department of Mathematics

Hand written notes

Of

MATHEMATICS FOR COMPUTER SCIENCE

Semester -3

SUB CODE: BCS301

Module 1: PROBABILITY DISTRIBUTIONS
Review of basic probability theory. Random variables (discrete and continuous), probability mass and density functions. Mathematical expectation, mean and variance. Binomial, Poisson and normal distributions- problems (derivations for mean and standard deviation for Binomial and Poisson distributions only)-Illustrative examples. Exponential distribution.

Written by

CHOWDAPPA MANNE R
ASSISTANT PROFESSOR
E-mail id: chowdappamr.maths@sjcit.ac.in
Mobil: 9972021124

Module-1 : Probability distributions

Probability : When an event is performed, we will get 'n' possible cases and 'm' favourable cases.

Therefore $\dfrac{\text{No. of favourable cases}}{\text{No of possible cases}}$ gives the probability of an event, denoted by $P(E)$

$$\Rightarrow \boxed{P(E) = \frac{m}{n}}$$

Probability of non happening of an event is given by $P(\bar{E}) = \dfrac{n-m}{n} = q$ (say)

$$\Rightarrow \quad P(\bar{E}) = 1 - \frac{m}{n}$$

$$\Rightarrow \quad q = 1 - p \quad \Rightarrow \boxed{p + q = 1}$$

$\Rightarrow$ Sum of probability of success and probability of failure is always equal to unity

Random variables :

In a random experiment, if a real variable is associated with every outcome then it is called a random variable or stochastic variable.

Random variables are usually denoted by $X, Y, Z \dots$

The set of all real numbers of a random variable X is called the range of X.

Ex: suppose a coin is tossed twice, we will get the sample space (No of possible outcomes) i.e $S = \{HH, HT, TH, TT\}$

If a random variable X represents the number of heads turning up, then

$X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$, $X(TT) = 0$

$\Rightarrow$ Range of $X = \{0, 1, 2\}$

If a random variable Y represents the number of tails turning up, then

$Y(HH) = 0$, $Y(HT) = 1$, $Y(TH) = 1$, $Y(T,T) = 2$

$\Rightarrow$ Range of $Y = \{0, 1, 2\}$

Discrete and Continuous Random Variables

*) If a random variable takes finite or countablly infinite number of values then it is called a discrete random variable.

Ex *) Tossing a coin and observing the outcome

*) Throwing a 'die' and observing the numbers on the face.

*) If a random variable takes non countable, infinite numbers of values then it is called non discrete or Continuous random variable

Ex *) Conducting a survey on the life of Electric bulbs.

Probability distributions (P.D)

- Discrete P.D
  - Binomial Distribution
  - poission Distribution
- Continuous P.D.
  - Exponential Distribution
  - Normal Distribution

Discrete probability distribution [Probability Function]

If we assigne a real numbers $p(x_i)$ for each value of x_i of a discrete random variable X such that

*) i) $P(x_i) = P_i = P(X = x_i) \geq 0 \Rightarrow P(x_i) \geq 0$

ii) $\sum_i P(x_i) = 1$, then the Function $p(x)$ is Called Probability Function

*) The set of values $(x_i, P(x_i))$ is called a discrete (finite) probability distribution

$X = x_i$	x_1	x_2	x_3	...
$P(X) = P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	--

*) The function $P(x)$ is called probability density function (p.d.f)

*) The function $\boxed{f(x) = \sum_{i=1}^{n} P(x_i)}$ is called Cumulative distribution function (c.d.f)

Mean and variance of Discrete P.D

i) Mean $(\mu) = \sum_i x_i \cdot P(x_i)$

ii) Variance $(V) = \sum_i (x_i - \mu)^2 \cdot P(x_i)$

iii) standard deviation $(\sigma) = \sqrt{V}$

Note: Variance can also put in the form

$$\boxed{V = \sum_i x_i^2 P(x_i) - \mu^2}$$

Examples

(1) A coin is tossed twice, and a random variable X represents the number of heads turning up. Find the discrete probability distribution for X. Also find its mean and variance.

Solution: when a coin is tossed twice, the associated sample space is

$$S = \{HH, HT, TH, TT\}$$

$X \to$ number of head turns up

$\Rightarrow$ Random variable X takes the values 2, 1, 1, 0

$\Rightarrow$ $P(HH) = 1/4$, $P(HT) = 1/4$, $P(TH) = 1/4$, $P(TT) = 1/4$

$P(X=0$, i.e no head$) = P(TT) = 1/4$

$P(X=1$, i.e one head$) = P(HT) + P(TH) = \frac{2}{4} = 1/2$

$P(X=2$ i.e two heads$) = P(HH) = 1/4$

$\therefore$ The associated probability distribution for X is

x_i	0	1	2
$P(x_i)$	1/4	1/2	1/4

$\Rightarrow P(x_i) > 0$ & $\sum P(x) = 1$

$\Rightarrow$ Distribution is discrete

Mean $= \mu = \sum x_i . P(x_i)$

$= x_1 P(x_1) + x_2 P(x_2) + x_3 . P(x_3)$

$= 0(1/4) + 1(1/2) + 2(1/4)$

$= 0 + \frac{1}{2} + \frac{1}{2}$

$\boxed{\mu = 1}$

Variance $= V = \sum (x_i - \mu)^2 . P(x_i)$

$V = (x_1 - 1)^2 P(x_1) + (x_2 - 1)^2 P(x_2) + (x_3 - 1)^2 P(x_3)$

$V = (0-1)^2 (1/4) + (1-1)^2 (1/2) + (2-1)^2 (1/4)$

$\boxed{V = 1/2}$

② ** The probability distribution of a finite random variable X is given by the following table

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	$2K$	0.3	K

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Find the value of K, mean and variance.

$P(X \leq 2)$

Solution: For a probability distribution we must have $p(x_i) \geq 0$ and $\Sigma p(x) = 1$

Here $p(x_i) \geq 0$ if $k \geq 0$

To find k use $\Sigma p(x_i) = 1$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k + 0.6 = 1$$

$$\Rightarrow \boxed{k = 0.1} \text{ which is true } (k \geq 0)$$

Thus the probability distribution is

x	-2	-1	0	1	2	3
$p(x)$	0.1	0.1	0.2	0.2	0.3	0.1

Mean $(\mu) = \Sigma x_i \cdot p(x_i)$

$$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4) + x_5 p(x_5) + x_6 p(x_6)$$

$$\mu = (-2)(0.1) + (-1)(0.1) + (0)(0.2) + 1(0.2) + 2(0.3) + 3(0.1)$$

$$\Rightarrow \boxed{\mu = 0.8}$$

Variance $(V) = \Sigma x_i^2 \cdot p(x_i) - \mu^2$

$$V = [0.4 + 0.1 + 0.2 + 1.2 + 0.9] - (0.8)^2$$

$$\boxed{V = 2.16}$$

(3) A random variable x has the following probability function for various values of x

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

i) Find K ii) Evaluate $p(x<6)$, $p(x\geq 6)$ and $p(3<x\leq 6)$ Q.P DEC/JAN 2024 -6-

iii) Find Cumulative distribution function.

Solution: For a probability function we must have $p(x)\geq 0$ & $\Sigma p(x)=1$

$p(x)\geq 0$ if $K\geq 0$

i) To find K we $\Sigma p(x)=1$

$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K=1$$

$$10K^2+9K=1$$

$$\Rightarrow 10K^2+9K-1=0$$

$$\Rightarrow K=0.1 \text{ and } K=-1$$

Since $K\geq 0$ $K\neq -1$

$\therefore \boxed{K=0.1}$ is valid.

$\therefore$ The probability distribution is

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

ii) $P(x<6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$P(x<6) = 0 + 0.1 + 0.2 + 0.2 + 0.3 + 0.01$

$\boxed{P(x<6) = 0.81}$

*) $P(x \geq 6) = P(6) + P(7)$

$= (0.02) + (0.17)$

$\boxed{P(x \geq 6) = 0.19}$

*) $P(3 < x \leq 6) \Rightarrow P(x>3)$ & $P(x \leq 6)$

$\Rightarrow P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$= 0.3 + 0.01 + 0.02$

$\boxed{P(3 < x \leq 6) = 0.33}$

iii) Cumulative distribution function of X is

$$f(x) = \sum_{i=1}^{n} P(x_i) = P(X \leq x)$$

$\Rightarrow$

x	0	1	2	3	4	5	6	7
$f(x)$	0	$0+0.1$ $=0.1$	$0.1+0.2$ $=0.3$	$0.3+0.2$ $=0.5$	$0.5+0.3$ $=0.8$	$0.8+0.01$ 0.81	$0.81+0.02$ $=0.83$	$0.83+0.17$ $=1$

(4) Show that the following distribution represents a discrete probability distribution. Find the mean and variance.

x	10	20	30	40
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Solution: For a discrete probability distribution we must have $P(x) \geq 0$ & $\Sigma P(x) = 1$

We observe that $P(x) > 0$

& $\Sigma P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

$\therefore$ The given distribution represents discrete probability distribution.

Mean $(\mu) = \Sigma x \cdot P(x) = \frac{10}{8} + \frac{60}{8} + \frac{90}{8} + \frac{40}{8}$

$$\boxed{\mu = 25}$$

Variance $(V) = \Sigma (x-\mu)^2 \cdot P(x)$

$= 225 \cdot \frac{1}{8} + 25 \cdot \frac{3}{8} + 25 \cdot \frac{3}{8} + 225 \cdot \frac{1}{8}$

$V = \frac{600}{8}$

$$\boxed{V = 75}$$

Binomial distribution :

If p is a probability of success and q is the probability of failure, the probability of x success out of n trials is given by

$$\boxed{P(x) = {}^nC_x\, p^x q^{n-x}}$$

Binomial distribution is given by

x	0	1	2	---	n
$P(x)$	q^n	${}^nC_1\, p q^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	---	p^n

${}^nC_0 = 1$

${}^nC_n = 1$

${}^nC_1 = n$

$P(x)$ is said to be discrete if (i) $P(x) \geq 0$
(ii) $\Sigma P(x) = 1$

$$\Rightarrow \Sigma P(x) = q^n + {}^nC_1 q^{n-1} \cdot p + {}^nC_2\, q^{n-2} \cdot p^2 + \dots + p^n$$

$$= (q+p)^n \quad \rightarrow \text{by Binomial expansion}$$

$$= 1^n$$

$$\boxed{\Sigma P(x) = 1}$$

$\Rightarrow$ Binomial distribution is discrete

Mean & Variance of Binomial Distribution

Mean $= np$

$\Rightarrow \boxed{\mu = np}$

Variance $(V) = npq$

$S.D = \sqrt{npq}$

Poisson Distribution

This is referred as limiting form of Binomial distribution.
i.e when n is very large ($n \to \infty$) and p (the probability of success) is vary small ($p \to 0$)

$\therefore$ np tends to a fixed finite constant say 'm'

$\Rightarrow$ $\boxed{np = m}$

So poisson distribution is given by

$$\boxed{P(x) = \frac{m^x e^{-m}}{x!}}$$

Mean and Variance of poisson Distribution

Mean $(\mu) = m$

Variance $(V) = m$

$S.D = \sqrt{m}$

Mean and variance of poisson distribution is same i.e m

Obtain the mean and Variance of Binomial distribution. [Dec.2023 | Jan 2024] Q.P
-6-M-

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x \cdot p(x)$$

$${}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n(n-1)!}{x(x-1)!\,(n-x)!}\, p\, p^{x-1} \cdot q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!}\, p^{(x-1)} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{n-1}$$

$$\sum_{x=0}^{n} {}^{n}C_{x}\, p^{x} q^{n-x} = (p+q)^{n}$$

$$p+q = 1$$

$\boxed{\mu = np}$ is mean of Binomial distribution.

$$\text{Variance } (V) = \sum_{x=0}^{n} x^{2} p(x) - \mu^{2} \quad -(1)$$

$$\text{Consider } \sum_{x=0}^{n} x^{2} p(x) = \sum_{x=0}^{n} (x(x-1) + x)\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, p(x) + \sum_{x=0}^{n} x\, p(x)$$

$$\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x} q^{n-x} + \mu$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!}\, p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!}\, p^{2}\cdot p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{((n-2)-(x-2))!\,(x-2)!} \cdot p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$\sum_{x=0}^{n} x^2 p(x) = n(n-1)p^2 (p+q)^{n-2} + np$$

$$\sum_{x=0}^{n} x^2 p(x) = n(n-1)p^2 + np \qquad \Big|\; p+q=1$$

$$\Big|\; \mu = np$$

$$(1) \Rightarrow \quad V = n(n-1)p^2 + np - n^2p^2$$

$$V = n^2p^2 - np^2 + np - n^2p^2$$

$$V = np - np^2 \qquad \Big|\; p+q=1$$

$$V = np(1-p) \qquad \Big|\; q = 1-p$$

$$\boxed{\text{Variance} = npq}$$

Obtain the mean and variance of Poisson distribution. [Q.P June/July 2024 - 06-marks]

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x . P(x)$$

$$= \sum_{x=0}^{n} x . \frac{m^x e^{-m}}{x!}$$

$$= \sum_{x=0}^{n} x . \frac{m . m^{x-1} . e^{-m}}{x(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{n} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m} \left[\frac{m^0}{0!} + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right]$$

$$= m e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right] \quad \begin{array}{|l} m^0 = 1 \\ 0! = 1 \end{array}$$

$$= m e^{-m} . [e^m]$$

$\Rightarrow$ $\boxed{\mu = m}$ is mean of poisson distribution

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 P(x) - \mu^2 \quad - (1)$$

Consider $\sum_{x=0}^{n} x^2 P(x) = \sum_{x=0}^{n} (x(x-1) + x) P(x)$

$$= \sum_{x=0}^{n} x(x-1) P(x) + \sum_{x=0}^{n} x P(x)$$

$$\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} x(x-1) \frac{m^x e^{-m}}{x!} + m$$

$$= \sum_{x=0}^{n} x(x-1) \frac{m^x e^{-m}}{x(x-1)(x-2)!} + m$$

$$= \sum_{x=2}^{n} \frac{m^{x-2} \cdot m^2 e^{-m}}{(x-2)!} + m$$

$$= m^2 e^{-m} \sum_{x=2}^{n} \frac{m^{x-2}}{(x-2)!} + m$$

$$\sum x^2 p(x) = m^2 e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right] + m$$

$$= m^2 e^{-m} \cdot e^{m} + m$$

$$\sum x^2 p(x) = m^2 + m \quad - (2)$$

Substituting (2) in (1)

$$V = m^2 + m - m^2$$

$$\boxed{V = m} \qquad \therefore \; S.D = \sqrt{m}$$

Mean and variance of poisson distribution is same i.e 'm'

(5) The probability of pen manufactured by a factory being defective is 1/10, if 12 such pens are being manufactured. what is the probability that

a) Exactly 2 are defective.

b) Atleast 2 are defective.

c) none of them are defective.

Solution: Given the probability of defective pen 1/10 $\Rightarrow P = \frac{1}{10} \Rightarrow \boxed{P = 0.1}$

Let the probability of non-defective pen is q

W.K.T $p + q = 1$

$\Rightarrow q = 1 - p = 1 - 1/10$

$q = 9/10 = 0.9$

$\Rightarrow \boxed{q = 0.9}$

Given $n = 12$.

Since 'n' is very small we can apply Binomial probability function

$$P(x) = n_{C_x} p^x q^{n-x}$$

$\Rightarrow P(x) = {}^{12}C_x (0.1)^x (0.9)^{12-x}$ — ①

(i) probability of exactly 2 are defective

$\Rightarrow P(x=2) = {}^{12}C_2 (0.1)^2 (0.9)^{10}$ ⇐ ①

$$\boxed{P(x=2) = 0.2301}$$

ii) probability of atleast 2 are defective

$\Rightarrow P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(0) + P(1)]$

$= 1 - \left[{}^{12}C_0 (0.1) + {}^{12}C_1 (0.1)(0.9)^{11}\right]$

$= 1 - [0.2824 + 0.3389]$

$= 0.341$

| ${}^nC_0 = 1$
| ${}^nC_1 = n$

iii) probability of none of them are defective

$\Rightarrow P(x=0) = {}^{12}C_0 (0.1)^0 (0.9)^{12}$

$= (0.9)^{12}$

$$\boxed{P(x=0) = 0.2824}$$

| ${}^nC_0 = 1$
| $(0.1)^0 = 1$

(6) If the probability that an individual will suffer a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individual.

(i) Exactly 3

(ii) Mor than 2 will suffer from a bad reaction.

solution: Since n is very large & p is very small, we can apply poisson distribution.

Given $n = 2000$, $P = 0.001$

$m = n \times p = (2000)(0.01)$

$\boxed{m = 2}$

poisson distribution function

$$P(X=x) = \frac{m^x e^{-m}}{x!}$$

$$P(x) = \frac{2^x e^{-2}}{x!} \quad - (1)$$

i) probability of exactly 3 is given by

$$P(x = 3) = \frac{2^3 \cdot e^{-2}}{3!} = 0.180 \quad \Leftarrow (1)$$

ii) probability of more than 2 is given by

$$P(x>2) = 1-[P[x=0] + P[x=1] + P(x=2)]$$

$$= 1-\left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!}\right]$$

$$= 1-5e^{-2}$$

$$\boxed{P(x>2) = 0.323}$$

(7) The probability that a man aged 60 years will live upto 70 is 0.65. what is the probability that out of 10 men aged 60, atleast 7 of them will live up to 70.

Solution:

Let x represents the number of men aged 60 years living upto 70 years.

$p+q=1$

$\Rightarrow$ $p = 0.65$

$q = 0.35$

$n = 10.$

we have Binomial distribution

$$P(x) = {}^nC_x\, p^x q^{n-x}$$

$$P(x) = {}^{10}C_x (0.65)^x (0.35)^{10-x} \quad -(1)$$

Here we need to find the probability of at least 7 $\Rightarrow P(x \geq 7)$

$\Rightarrow P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$

$$P(x \geq 7) = 10_{C_7}(0.65)^7(0.35)^3 + 10_{C_8}(0.65)^8(0.35)^2 + 10_{C_9}(0.65)^9(0.35)^1 + 10_{C_{10}}(0.65)^{10}(0.35)^0$$

$$\boxed{P(x \geq 7) = 0.5138}$$

$\Rightarrow$ Probability of atleast 7 men aged 60 will live up to 70 is 0.5138

(8) ** In 800 families with 5 childern each, how many families would be expected to have (i) 3 boys ii) 5 girls iii) Either 2 (or) 3 boys iv) at most 2 girls, by assuming probabilities for boys and girls to be equal.

[Q.P June/July 2024 of MA114]

Solution: Let us assume

$P =$ probability of having a boy $= \frac{1}{2}$

$q =$ probability of having a girl $= \frac{1}{2}$

Let x reprasents the number of boys in the family

Let the Binomial distribution be

$$P(x) = n_{C_x} p^x q^{n-x}$$

since $n=5$

$$P(x) = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

$$\left(\frac{1}{2}\right)^{5-x} = \frac{1^{5-x}}{2^{5-x}} = \frac{1}{2^5 \cdot 2^{-x}} = \frac{1}{2^5} \cdot 2^x$$

$$P(x) = {}^5C_x \frac{1}{2^x} \cdot \frac{2^x}{2^5}$$

$$P(x) = {}^5C_x \cdot \frac{1}{2^5} \Rightarrow \boxed{P(x) = \frac{1}{32} \cdot {}^5C_x}$$

Since we need to calculate the result for 800 families

$800 \cdot P(x) = f(x)$ (say)

$$\Rightarrow f(x) = 800 \cdot \frac{{}^5C_x}{32}$$

$$\boxed{f(x) = 25 \cdot {}^5C_x}$$

i) Number of families with 3 boys is given by

$f(3) = 25 \cdot {}^5C_3$

$f(3) = 250$

$\Rightarrow$ 250 families Expected to have 3 boys

ii) Number of families with 5 girls (0 Boys)

$\Rightarrow f(0) = 25 \cdot {}^5C_0 = 25$ ${}^nC_0 = {}^nC_n = 1$

$\Rightarrow \boxed{f(0) = 25}$

25 families Expected to have 5 girls.

iii) Families with either 2 or 3 boys is

$f(2) + f(3) = 25 \cdot 5c_2 + 25 \cdot 5c_3$

$f(2) + f(3) = 500$

$\Rightarrow$ 500 families expected to have either 2 or 3 boys.

iv) Number of families with atmost 2 girls means

5 Boys and 0 girls
4 Boys and 1 girl
3 Boys and 2 girl

$\Rightarrow f(5) + f(4) + f(3) = 400$ since x takes the values of boys

$\Rightarrow$ 400 families with atmost 2 girls

9. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1, that a line is busy. If 10 lines are chosen at random, what is the probability that

i) no line is busy
ii) all lines are busy
iii) at least one line is busy
iv) atmost 2 lines are busy.

Solution: Let p be the probability of busy telephone lines $p+q=1$

$\Rightarrow p = 0.1 \quad \Rightarrow q = 0.9$

Given $n = 10$.

By Binomial distribution

$$P(x) = nC_x \, p^x q^{n-x}$$

$$\boxed{P(x) = 10C_x (0.1)^x (0.9)^{10-x}} \quad -(1).$$

(i) probability of no line is busy is given by

$(1) \Rightarrow P(0) = 10C_0 (0.1)^0 (0.9)^{10}$

$$\boxed{P(0) = 0.3487}$$

ii) probability of all lines are busy

$(1) \Rightarrow P(10) = 10C_{10} (0.1)^{10} (0.9)^0$ $\qquad nC_n = 1$

$P(10) = (0.1)^{10}$

iii) probability of atleast one line is busy

$\Rightarrow P(x \geq 1) = P(1) + P(2) + P(3) + \cdots + P(10)$

(or)

$P(x \geq 1) = 1 - P(x < 1)$

$= 1 - [P(0)]$

$= 1 - 0.3487$

$$\boxed{P(x \geq 1) = 0.6513}$$

iv) P(at most 2 lines are busy) $= P(x \leq 2)$

$= P(0) + P(1) + P(2)$

$= 0.3487 + 0.3874 + 0.1937$

$= 0.9298$

(10) If the mean and S.D of number of Correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$.
Find an Estimate, the number of Candidates answering Correctly

(i) 8 or more questions

ii) 2 or less

iii) 5 questions.

Solution:

Given mean $(\mu) = 2.5$

S.D $(\sigma) = \sqrt{1.875}$

mean of Binomial $\mu = np$.

S.D of Binomial $\sigma = \sqrt{npq}$

$\Rightarrow \quad np = 2.5 \qquad npq = 1.875$

$\Rightarrow (2.5)q = 1.875$

$\Rightarrow \boxed{q = 0.75}$

$\therefore \boxed{p = 0.25}$

$np = 2.5 \quad \Rightarrow \boxed{n = 10}$ $\Rightarrow$ Total number of questions

Here n is not 4096

We need to estimate the result for 4096 students $\Rightarrow \boxed{f(x) = 4096 \cdot p(x)}$

$$P(x) = {}^{10}C_x (0.25)^x (0.75)^{10-x}$$

(i) numbers of candidates answering correctly 8 or more questions is given by

$$f(x \geq 8) = f(8) + f(9) + f(10)$$

$$= {}^{10}C_8 (0.25)^8 (0.75)^2 + {}^{10}C_9 (0.25)^9 (0.75)^1 + {}^{10}C_{10} (0.25)^{10} (0.75)^0$$

$$\boxed{f(x \geq 8) = 2}$$

$\Rightarrow$ out of 4096 students 2 students answered correctly 8 or more questions.

ii) 2 or less $\Rightarrow f(0) + f(1) + f(2)$

$$\boxed{f(0) + f(1) + f(2) = 2153}$$

$\Rightarrow$ out of 4096 students 2153 students answered 2 or less questions.

iii) Number of students answering correctly 5 questions

$$\Rightarrow f(x=5) = {}^{10}C_5 (0.25)^5 (0.75)^5$$

$$\boxed{f(x=5) = 239}$$

$\Rightarrow$ 239 students correctly answered 5 questions out of 10

*) In a certain factory turning out razor blades there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing

i) no defective

ii) one defective

iii) two defective blades in a Consignment of 10,000 Packets.

[Q.P Dec. 2023 | Jan 2024 - 07 - Mark]

Solution: Given, the probability of defective blade $= \frac{1}{500}$

i.e $p = \frac{1}{500} = 0.002$

$n = 10$

$m = n \times p = 10 \times 0.02 \Rightarrow m = 0.02$

Poisson distribution, $P(x) = \frac{m^x e^{-m}}{x!}$

$$P(x) = \frac{(0.02)^x e^{-0.02}}{x!}$$

Since we need to estimate the result for 10,000 packets

$e^{-0.02} = 0.9802$

$\Rightarrow f(x) = 10{,}000 \times P(x)$

$f(x) = 10{,}000 \times \frac{(0.02)^x}{x!} \times 0.9802$

$$\Rightarrow f(x) = \frac{9802\,(0.02)^x}{x!} \quad -(1)$$

i) number of packets Containing no defective blades

(1) $\Rightarrow$ $f(0) = \dfrac{9802\,(0.02)^0}{0!}$

$\Rightarrow \boxed{f(0) = 9802}$

ii) number of packets Containing one defective blade

(1) $\Rightarrow$ $f(1) = \dfrac{9802\,(0.02)}{1!} \simeq 196.04$

$\boxed{f(1) \simeq 196}$

iii) number of packets Containing two defective blades

(1) $\Rightarrow$ $f(2) = \dfrac{9802(0.02)^2}{2!} = 1.9604$

$f(2) \simeq 2$

4) A Communication Channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of

i) no error during a micro second

ii) one error per micro second

iii) atleast one error per micro second

iv) two errors. v) atmost two errors.

Solution: Given

The probability of transmission error $= 0.001$

$\Rightarrow p = 0.001$

$n = 12$

$\Rightarrow m = n \times p = 12 \times 0.001$

$\Rightarrow m = 0.012$

The poisson distribution

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$\Rightarrow P(x) = \frac{(0.012)^x e^{-0.012}}{x!} \quad -(1)$$

i) P(no error during a micro second)

$= P(x=0)$

$$(1) \Rightarrow P(0) = \frac{(0.012)^0 e^{-0.012}}{0!}$$

$$\Rightarrow \boxed{P(0) = 0.988072}$$

ii) P(one error per micro second)

$\Rightarrow P(x=1)$

$$(1) \Rightarrow P(1) = \frac{(0.012) e^{-0.012}}{1!}$$

$$\boxed{P(1) = 0.01186}$$

iii) P(Atleast one error) = 1 - P(0)

$= 1 - 0.988072$

$= 0.01193$

iv) P(two errors) = P(2)

$= \frac{(0.012)^2 \cdot e^{-0.012}}{2}$

$\boxed{P(2) = 0.000071}$

v) probability of atmost two errors

$= P(0) + P(1) + P(2)$

$= 0.988072 + 0.01186 + 0.000071$

$= 0.99999714$

$\boxed{P(\text{Atmost two}) \simeq 1}$

(11) The number of accident in a year to taxi drivers in a city follows a poisson distribution with mean 3. out of 1000 taxi drivers find approximately the number of drivers with

(i) no accident in a year.

(ii) more than 3 accident in a year.

(iii) at most 2 accidents in a year.

Q.P [June/July 2024] 07 marks

Solution: Given mean $(\mu) = 3$

$\Rightarrow m = 3$ in poission

poission distribution is given by

$$P(x) = \frac{m^x e^{-m}}{x!}$$

$$\Rightarrow P(x) = \frac{3^x e^{-3}}{x!}$$

Since we need to estimate the result for 1000 taxi drivers

$$f(x) = 1000 \cdot P(x)$$

$$f(x) = 1000 \cdot \frac{3^x e^{-3}}{x!}$$

$e^{-3} = 0.05$

$$\boxed{f(x) = \frac{50 \cdot 3^x}{x!}}$$

(i) No accident in a year is given by

$$f(0) = \frac{50 \cdot 3^0}{0!} = 50$$

$3^0 = 1$

$0! = 1$

$\Rightarrow$ out of 1000 drivers 50 drivers not make any accident in a year.

ii) probability of more than 3 accidents in a year is given by

$$p(x>3) = 1-p(x\leq 3)$$

$$= 1-[p(0)+p(1)+p(2)+p(3)]$$

$$= 1-\left[e^{-3}\left(\frac{3^0}{0!}+\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}\right)\right]$$

$$= 1-[0.05(1+3+4.5+4.5]$$

$$p(x>3) = 0.35$$

∴ number of drivers out of 1000 with more than 3 accidents in a year $= 1000 \times 0.35$
$= 350$

iii) $P(\text{at most } 3 \text{ accidents}) = f(0)+f(1)+f(2) = 425$

(12) 2% of the fuses manufactured by a factory are found to be defective. Find the probability that a box containing 200 fuses contains

(i) no defective fuses

(ii) 3 or more defective fuses.

Continuous probability distribution

If for every x belonging to the range of a Continuous random variable X, then the real number $f(x)$ satisfying the Conditions

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

Then $f(x)$ is called Continuous probability function (or) probability density function (pdf)

*) $P(a \leq x \leq b) = \int_{a}^{b} f(x) \cdot dx$

Mean and variance of Continuous distribution

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$\text{Variance}(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

Also $V = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (\mu)^2$

(13) A random Variable x has the following density function $p(x) = \begin{cases} Kx^2, & -3 \le x \le 3 \\ 0 & \text{Elsewhere} \end{cases}$

Evaluate K and find

(i) $p(1 \le x \le 2)$, ii) $p(x \le 2)$ iii) $p(x > 1)$

[Q.P. Dec.2023/ JAN.2024 -6-]

solution: A probability density function must have $p(x) \geq 0$ if $K \geq 0$ &

$$\int_{-\infty}^{\infty} f(x).dx = 1$$

i.e $\int_{-3}^{3} Kx^2 \, dx = 1 \Rightarrow K\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$

$$\Rightarrow \frac{K}{3}\left[3^3 + 3^3\right] = 1$$

$18K = 1 \Rightarrow \boxed{K = 1/18} \Rightarrow f(x) = \frac{x^2}{18}, -3 \le x \le$

i) $P(1 \le x \le 2) = \int_{1}^{2} \frac{x^2}{18} dx = \frac{1}{18}\left[\frac{x^3}{3}\right]_1^2$

$$= \frac{1}{54}\left[8-1\right]$$

$$\boxed{P(1 \le x \le 2) = \frac{7}{54}}$$

ii) $P(x \le 2) = P(-3 \le x \le 2)$

$$= \int_{-3}^{2} f(x) \cdot dx$$

$$= \int_{-3}^{3} \frac{x^2}{18} \cdot dx = \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{3}$$

$$= \frac{1}{54}\left[8 + 27\right]$$

$$\boxed{P(x \le 2) = \frac{35}{54}}$$

iii) $P(x > 1) = \int_{1}^{3} f(x)\, dx$

$$= \int_{1}^{3} \frac{x^2}{18}\, dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{1}^{3} = \frac{1}{54}\left[27 - 1\right]$$

$$= \frac{26}{54}$$

$$\boxed{P(x > 1) = \frac{13}{27}}$$

(14) Find the constant K such that

$$f(x) = \begin{cases} Kx^2, & 0 < x < 3 \\ 0 & \text{Otherwise} \end{cases} \quad \text{is a p.d.f}$$

Also compute (i) $P(1 < x < 2)$

(ii) $P(x \le 1)$

(iii) $P(x > 1)$

(iv) Mean, v) Variance.

Solution : For a p.d.f we must have

$f(x) \geq 0$ if $K \geq 0$ and

$$\int_{-\infty}^{\infty} f(x).dx = 1$$

$$\Rightarrow \int_0^3 Kx^2.dx = 1$$

$$K\left[\frac{x^3}{3}\right]_0^3 = 1 \Rightarrow \frac{K}{3}\left[3^3\right] = 1$$

$$9K = 1$$

$$\boxed{K = 1/9} \Rightarrow f(x) = \frac{x^2}{9}, \ 0<x<3$$

(i) $$P(1 < x < 2) = \int_1^2 f(x).dx$$

$$= \int_1^2 \frac{x^2}{9}.dx = \left[\frac{x^3}{27}\right]_1^2 = \frac{7}{27}$$

ii) $P(x \le 1) = P(0 \le x \le 1)$

$$= \int_0^1 f(x).dx = \int_0^1 \frac{x^2}{9}.dx$$

$$P(x \le 1) = \left[\frac{x^3}{27}\right]_0^1 = \frac{1}{27}$$ //

iii) $P(x > 1) = P(1 < x < 3)$

$$P(x > 1) = \int_1^3 \frac{x^2}{9}.dx = \left(\frac{x^3}{27}\right)_1^3 = \frac{26}{27}$$ //

iv) $$\text{Mean} = \mu = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^3 x.\frac{x^2}{9}\,dx$$

$$= \int_0^3 \frac{x^3}{9}.dx = \left[\frac{x^4}{36}\right]_0^3 = \frac{81}{36}$$

$$\boxed{\mu = 9/4}$$

v) $$\text{Variance} = V = \int_{-\infty}^{\infty} x^2 f(x).dx - \mu^2$$

$$= \int_0^3 x^2.\frac{x^2}{9}.dx - \left(\frac{9}{4}\right)^2$$

$$= \left[\frac{x^5}{45}\right]_0^3 - \frac{81}{16} = \frac{81}{15} - \frac{81}{16} = \frac{81}{240}$$

$$\boxed{V = \frac{27}{80}}$$

* Find c such that $f(x) = \begin{cases} \frac{x}{6}+c & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ is pdf

also find $p(1 \le x \le 2)$

Solution: For pdf we must have $f(x) \ge 0$, which holds good only when $c \ge 0$

Consider $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^3 \left(\frac{x}{6}+c\right) dx = 1$$

$$\Rightarrow \frac{1}{6}\int_0^3 x\,dx + \int_0^3 c.dx = 1$$

$$\Rightarrow \frac{1}{6}\left(\frac{x^2}{2}\right)_0^3 + c\,[x]_0^3 = 1$$

$$\frac{1}{12}[9-0] + c[3-0] = 1$$

$$\frac{3}{4} + 3c = 1 \Rightarrow \frac{3+12c}{4} = 1$$

$$\Rightarrow 3+12c = 4 \Rightarrow 12c = 1$$

$$\therefore \boxed{c = 1/12} \quad \text{i.e } c \ge 0$$

$$P(1 \le x \le 2) = \int_1^2 f(x).dx$$

$$P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + c\right) \cdot dx$$

$$= \int_1^2 \left[\frac{x}{6} + \frac{1}{12}\right] dx$$

$$= \frac{1}{6}\left(\frac{x^2}{2}\right)_1^2 + \frac{1}{12}\left(x\right)_1^2$$

$$= \frac{1}{12}\left[4-1\right] + \frac{1}{12}\left(2-1\right)$$

$$= \frac{3}{12} + \frac{1}{12}$$

$$= \frac{4}{12}$$

$$\therefore \boxed{P(1 \le x \le 2) = \frac{1}{3}}$$

* Find K such that $f(x) = \begin{cases} Kxe^{-x} & , 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

is a pdf. Find mean also.

Solution: For a pdf we must have $f(x) \geq 0$ if $K \geq 0$

To find K consider $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$\Rightarrow \int_0^1 Kxe^{-x} \cdot dx = 1$$

$$\Rightarrow \quad K\int_0^1 x\, e^{-x}\, dx = 1$$

using Bernoulli's rule of integration

$$\int uv\,dx = uv_{(1)} - u'v_{(2)} + u''v_{(3)} - u'''v_{(4)} + \cdots$$

$$v_{(1)} = \int v\,dx \qquad u' = \frac{du}{dx}$$

$$v_{(2)} = \int v_{(1)}\,dx \qquad u'' = \frac{d^2u}{dx^2}$$

$$\vdots \qquad \vdots$$

$$\therefore \quad K\left[x\left(\frac{e^{-x}}{-1}\right) - \left(\frac{e^{-x}}{1}\right) + 0\right]_0^1 = 1$$

$$K\left[-(x\,e^{-x})_0^1 - (e^{-x})_0^1\right] = 1$$

$$K\left[-(e^{-1} - 0) - (e^{-1} - e^{0})\right] = 1$$

$$K\left[-e^{-1} - e^{-1} + 1\right] = 1$$

$$K\left[1 - 2e^{-1}\right] = 1$$

$$K\left[1 - \frac{2}{e}\right] = 1 \Rightarrow K\left[\frac{e-2}{e}\right] = 1$$

$$\Rightarrow \boxed{K = \frac{e}{e-2}} \quad \text{i.e } K > 0$$

which is true.

$$\text{Mean }(\mu) = \int_{-\infty}^{\infty} x . f(x)\, dx$$

$$= \int_0^1 x .\ Kx\bar{e}^{x}\, dx$$

$$= K\int_0^1 x^2 \bar{e}^{x}\, dx$$, Again applying Bernoulli's rule

$$= K\left[x^2\left(\frac{\bar{e}^{x}}{-1}\right) - (2x)\left(\frac{\bar{e}^{x}}{1}\right) + 2\left(\frac{\bar{e}^{x}}{-1}\right)\right]_0^1$$

$$= K\left[-(x^2\bar{e}^{x})_0^1 - 2(x\bar{e}^{x})_0^1 - 2(\bar{e}^{x})_0^1\right]$$

$$= K\left[-(\bar{e}^{1} - 0) - 2(\bar{e}^{1} - 0) - 2(\bar{e}^{1} - e^{0})\right]$$

$$= K\left[-\bar{e}^{1} - 2\bar{e}^{1} - 2\bar{e}^{1} + 2\right]$$

$$= K\left[2 - 5\bar{e}^{1}\right] = K\left[2 - \frac{5}{e}\right]$$

$$= K\left[\frac{2e-5}{e}\right] = \frac{e}{e-2}\left[\frac{2e-5}{e}\right]$$

$$\mu = \frac{2e-5}{e-2}$$ 14

$$\boxed{\mu = 0.6077}$$

Exponential Distribution :

The Continuous probability distribution having the probability function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise}, \text{ where } \alpha > 0 \end{cases}$$

is known as the exponential distribution.

Mean of exponential Distribution $(\mu) = \frac{1}{\alpha}$

$$\Rightarrow \boxed{\mu = \frac{1}{\alpha}}$$

Variance $(\sigma^2) = \frac{1}{\alpha^2}$

Standard deviation $(S.D) = \frac{1}{\alpha}$

$$\Rightarrow \boxed{S.D = \frac{1}{\alpha}}$$

15 If x is an exponential variate with mean 4

Evaluate (i) $P(0 < x < 1)$

(ii) $P(-\infty < x < 8)$

(iii) $P(x \leq 0 \text{ or } x \geq 1)$

Solution: We have exponential function

$$f(x) = \alpha e^{-\alpha x}, x>0. \quad -(1)$$

Given μ(mean) $= 4$

$$\Rightarrow \frac{1}{\alpha} = 4 \Rightarrow \boxed{\alpha = \frac{1}{4}}$$

$$\therefore \boxed{f(x) = \frac{1}{4} e^{-1/4 x}}, \quad x>0$$

(i) $P(0<x<1) = \int_0^1 f(x)\,dx$

$$= \int_0^1 \frac{1}{4} e^{-1/4 x}\,dx$$

$$= \frac{1}{4}\left[\frac{e^{-1/4 x}}{-1/4}\right]_0^1$$

$$= -\left[e^{-1/4} - e^{0}\right]$$

$$\boxed{P(0<x<1) = 0.2212}$$

ii) $P(-\infty<x<8) = \int_{-\infty}^{0} f(x)\,dx + \int_0^8 f(x)\,dx$

$$= 0 + \frac{1}{4}\left[\frac{e^{-1/4 x}}{-1/4}\right]_0^8 = 1 - \frac{1}{e^2}$$

$$\boxed{P(-\infty<x<\infty) = 0.8646}$$

iii) $P(x \le 0 \text{ or } x \ge 1) = P(x \le 0) + P(x \ge 1)$

$= 0 + P(1 \le x \le \infty)$

$= \int_1^\infty f(x) \cdot dx$

$= \int_1^\infty \frac{1}{4} e^{-1/4 x} \cdot dx$

$= \frac{1}{4} \left[\frac{e^{-1/4 x}}{-1/4} \right]_1^\infty$

$\boxed{P(x \le 0 \text{ or } x \ge 1) = 0.7788}$

☆☆☆

16 The length of telephone Conversation on a Cell phone has been an exponential distribution and found an avarage to be 3 minuts. Find the probability that random call made from this phone (i) Ends less than 3 minutes.

(ii) between 3 and 5 minutes.

[Q.P June/July 2024 - 07 marks] (change in mean & conditions)

Solution: Given $\mu = 3$

$\Rightarrow \frac{1}{\alpha} = 3 \Rightarrow \boxed{\alpha = 1/3}$

$\Rightarrow f(x) = \alpha e^{-1/\alpha \cdot x}, x > 0$

$\Rightarrow \boxed{f(x) = \frac{1}{3} e^{-1/3 x}}$

i) probability of random call ends less than 3 minutes is $P(x<3)$

$\Rightarrow$ $P(x<3) = P(0<x<3)$

$$= \int_0^3 f(x) \cdot dx$$

$$= \int_0^3 \frac{1}{3} e^{-1/3 \cdot x} \cdot dx$$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_0^3 = -\left[e^{-1} - e^{0}\right]$$

$$\boxed{P(x<3) = 0.632}$$

ii) probability of random call between 3 and 5 minuts is given by

$$P(3<x<5) = \int_3^5 f(x) \cdot dx$$

$$= \int_3^5 \frac{1}{3} e^{-1/3 x} \cdot dx$$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_3^5$$

$$= -\left[e^{-5/3} - e^{-1}\right]$$

$$\boxed{P(3<x<5) = 0.179}$$

17) ☆☆☆ In a certain town dureation of rainfall is exponential with mean 3 minutes. what is the probability that a rain will last for

i) 10 minutes or more.

ii) less than 10 minutes.

iii) between 10 to 12 minutes.

Q.P [Dec.2023] Jan 2024] 7 marks

Solution: Given $\mu = 3$

$$\frac{1}{\alpha} = 3 \Rightarrow \alpha = 1/3$$

$$\Rightarrow f(x) = \frac{1}{3} e^{-1/3 x} \quad \text{for } x > 0$$

(i) probability of rain last for 10 minutes or more is given by

$$P(x \geq 10) = P(10 \leq x \leq \infty)$$

$$= \int_{10}^{\infty} f(x).dx$$

$e^{-\infty} = 0$

$$= \int_{10}^{\infty} \frac{1}{3} e^{-\frac{1}{3}x}.dx$$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_{10}^{\infty} = -\left[e^{-\infty} - e^{-10/3}\right]$$

$$\boxed{P(x \geq 10) = 0.03567}$$

ii) probability of rain last for less than 10 minutes is

$$P(x < 10) = P(0 < x < 10)$$

$$= \int_0^{10} \frac{1}{3} e^{-\frac{1}{3}x} \cdot dx$$

$$= \frac{1}{3}\left[\frac{e^{-1/3 x}}{-1/3}\right]_0^{10} = -\left[e^{-10/3} - e^{0}\right]$$

$$\boxed{P(x < 10) = 0.9643}$$

iii) $P(10 < x < 12) = \frac{1}{3}\int_{10}^{12} e^{-x/3} \cdot dx$

$$= -\left.e^{-x/3}\right]_{10}^{12}$$

$$= -\left[e^{-12/3} - e^{-10/3}\right]$$

$$\boxed{P(10 < x < 12) = 0.01735}$$

Normal Distribution

A Continuous probability distribution having the probability density function

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}}$$

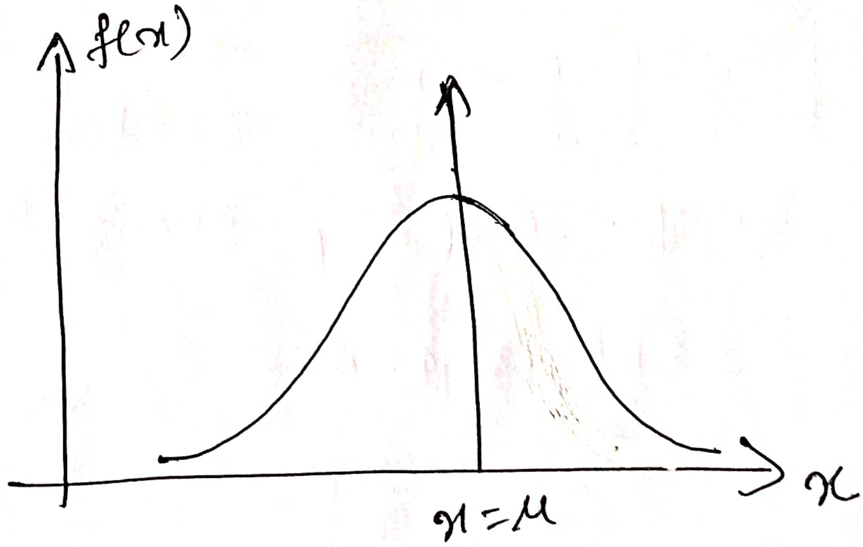
where $-\infty < x < \infty$ & $\sigma > 0$ is known
$-\infty < \mu < \infty$

as Normal distribution.

Mean $= \mu$

Variance $= \sigma^2 \Rightarrow S.D = \sigma$

Graphically $f(x)$



Line $x=\mu$ divides the total area under the curve which is equal to one in two equal parts.

i.e the area to the right as well as to the left of the line $x = \mu$ is $\underline{0.5}$

Standard Normal Distribution

We have $P(a \le x \le b) = \int_a^b f(x).dx$

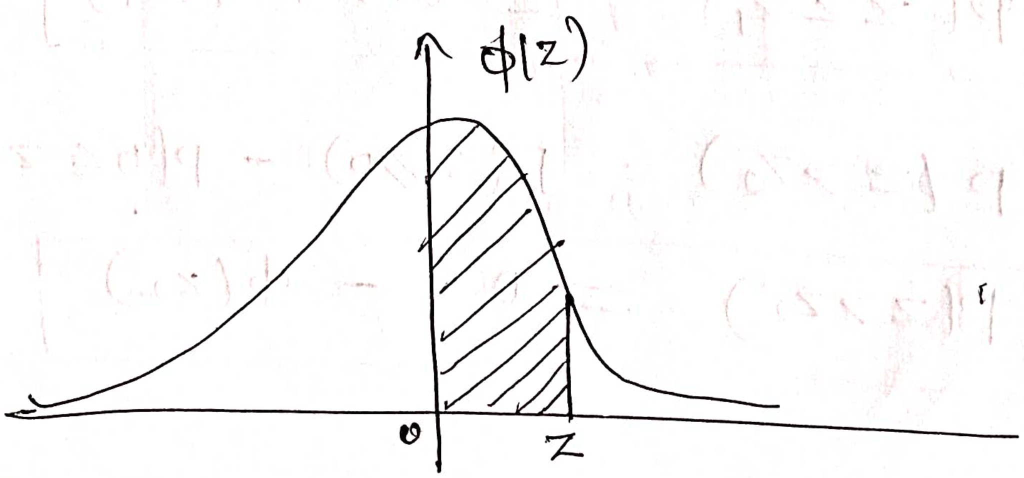
$$\Rightarrow P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/2\sigma^2} . dx$$

putting standard Normal Variate (SNV)

$$Z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma Z$$

$$\Rightarrow \boxed{\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} . dz}$$

This represents the area under the standard normal curve from $z = 0$ to z



Since the area is one, the area on either side of Z is equal to 0 is 0.5 (1/2)

$$\int_{-\infty}^{\infty} \phi(z)\, dz = 1$$

$$\Rightarrow \int_{-\infty}^{0} \phi(z)\, dz + \int_{0}^{\infty} \phi(z) . dz = 1$$

$$\Rightarrow \boxed{0.5 + 0.5 = 1}$$

Standard results

i) $P(-\infty \leq Z \leq \infty) = 1$

ii) $P(-\infty \leq Z \leq 0) = 1/2$

iii) $P(Z \geq 0) = P(0 \leq Z \leq \infty) = 1/2$

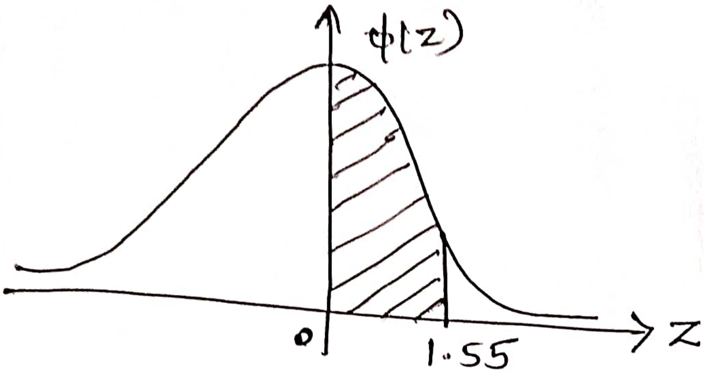
iv) $P(-\infty < Z < z_1) = P(-\infty < Z \leq 0) + P(0 < Z < z_1)$

$$\boxed{P(Z < z_1) = 0.5 + \phi(z_1)}$$

v) $P(Z > z_2) = P(Z \geq 0) - P(0 \leq Z \leq z_2)$

$$\boxed{P(Z > z_2) = 0.5 - \phi(z_2)}$$

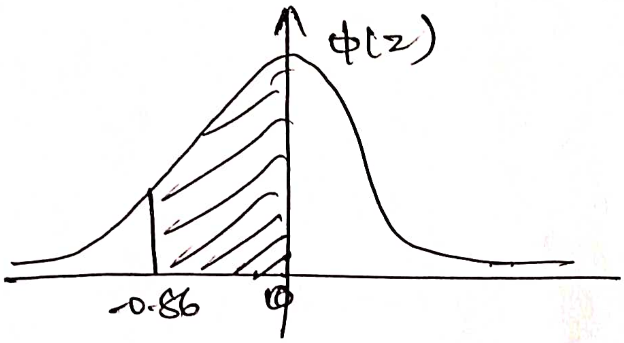
Ex: Area under the standard normal curve b/w $z = 0$ and $z = 1.55$



$$\text{Area} = p(0 \le z \le 1.55)$$
$$= \phi(1.55)$$
$$\boxed{A = 0.4394}$$

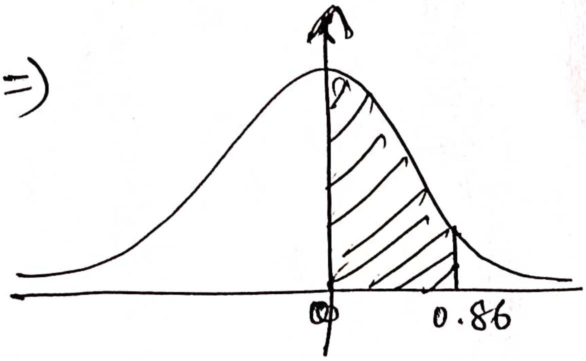
by Normal distribution table

Ex: Area under the standard normal curve b/w $z = -0.86$ & 0



$$\text{Area} = p(-0.86 \le z \le 0)$$
$$= p(0 \le z \le 0.86)$$
$$= \phi(0.86)$$
$$= 0.3051$$

$\Rightarrow$



Both the curves are same since the left & right part of 0 is 0.5 (same)

Areas under the standard normal curve : $\frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2}\, dz$

Normal Probability Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	0.319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4278	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4430	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4648	.4656	.4664	.4671	.4678	.4686	.4693	.4700	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4762	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4874	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.5966	.5967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Note : For values in between those provided, take proportionate increments.

*) Evaluate (i) $P(Z \geq 0.85)$

(ii) $P(-1.64 \leq Z \leq -0.88)$

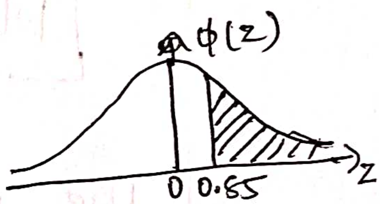
(iii) $P(Z \leq -2.43)$

(iv) $P(|Z| \leq 1.94)$

Solution:

(i) WKT $P(Z > Z_2) = P(Z \geq 0) - P(0 \leq Z \leq Z_2)$

$P(Z > Z_2) = 0.5 - \phi(Z_2)$

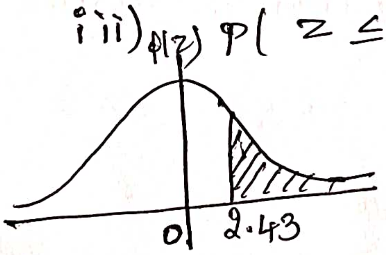


$\Rightarrow P(Z \geq 0.85) = 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

$P(Z \geq 0.85) = 0.1977$

iii) $P(Z \leq -2.43) = P(Z \geq 2.43)$

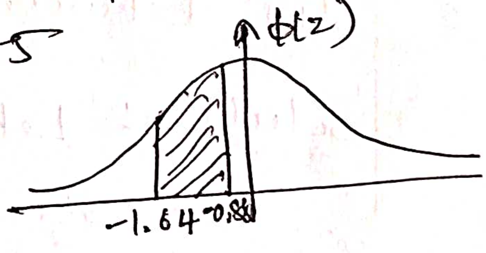


$= P(Z \geq 0) - P(0 \leq Z \leq 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$P(Z \leq -2.43) = 0.0075$



ii) $P(-1.64 \leq Z \leq -0.88)$

$P(-1.64 \leq Z \leq -0.88) = P(0 \leq Z \leq 1.64) - P(0 \leq Z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106$

$= 0.1389$

iv) $p(|z| \leq 1.94) = p(-1.94 \leq z \leq 1.94)$

$= p(-1.94 \leq z \leq 0) + p(0 \leq z \leq 1.94)$

$= p(0 \leq z \leq 1.94) + p(0 \leq z \leq 1.94)$

$= \phi(1.94) + \phi(1.94)$

$= 2\,\phi(1.94)$

$= 2(0.4738)$

$$\boxed{p(|z| \leq 1.94) = 0.9476}$$

8) The marks of 1000 students in an Examination followed a normal distribution with mean 70 and standard deviation 5.

Find the number of students with

(i) less than 65

(ii) more than 75

(iii) between 65 and 75

Solution: Let x represents the marks of the students

we have standard normal variate (s.n.v)

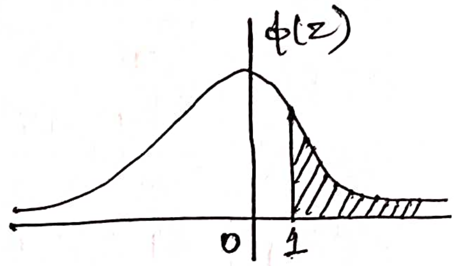
$$z = \frac{x-\mu}{\sigma}$$

Given $\mu = 70$, $\sigma = 5$

$$\Rightarrow z = \frac{x-70}{5}$$

(i) when $x=65$ $Z = \frac{65-70}{5} = -1$

$x=65 \Rightarrow Z=-1$



$P(x \leq 65) = P(Z \leq -1)$
$= P(Z \geq 1)$
$= P(Z \geq 0) - P(0 \leq Z \leq 1)$
$= 0.5 - \phi(1)$
$= 0.5 - 0.3413$

$P(x \leq 65) = 0.1587$

Since we need to estimate for 1000 students

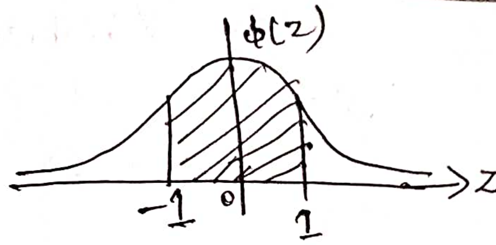
∴ number of students scoring less than 65 marks $= 1000 \times 0.1587$
$= 158.7$
$\simeq 159$

(ii) when $x=75$, $Z = \frac{75-70}{5} = 1$

$\Rightarrow P(x \geq 75) = P(Z \geq 1)$
$= 0.5 - \phi(1)$
$= 0.1587$

∴ number of students scoring more than 75 marks $= 1000 \times 0.1587$
$= 159$

iii) between 65 and 75



$\Rightarrow$ $x=65 \Rightarrow z=-1$

$x=75 \Rightarrow z=1$

$$P(-1 \le z \le 1) = 2P(0 \le z < 1)$$
$$= 2\phi(1)$$
$$= 2(0.3413)$$
$$= 0.6826$$

number of student scoring marks between 65 and 75 $= 1000 \times 0.6826$
$$= 682.6$$
$$= \underline{\underline{683}}$$

*) In a normal distribution, 31% of the items are under 45, 8% of the items are over 64. Find the mean and standard deviation of the distribution.

Solution:- Here we need to find the mean (μ) & S.D (σ) of normal distribution

By data $P(x < 45) = 31\%$ and $P(x > 65) = 8\%$.

$\Rightarrow P(x < 45) = 0.31$ and $P(x > 65) = 0.08$

we have S.n.v $z = \dfrac{x-\mu}{\sigma}$

When $x = 45$

$$z_1 = \frac{45-\mu}{\sigma}$$

$$z_1\sigma = 45-\mu$$

$$\Rightarrow z_1\sigma + \mu = 45 \quad - (1)$$

$$P(x < 45) = 0.31$$

$$\Rightarrow P(z < z_1) = 0.31$$

$$0.5 + \phi(z_1) = 0.31$$

$$\Rightarrow \phi(z_1) = 0.19$$

By normal probability table

$$0.1915 \approx 0.19 = \phi(0.5)$$

$$\phi(z_1) = -\phi(0.5)$$

$$\phi(z_1) = \phi(-0.5)$$

$$\Rightarrow \boxed{z_1 = -0.5}$$

When $x = 64$

$$z_2 = \frac{64-\mu}{\sigma}$$

$$z_2\sigma = 64-\mu$$

$$z_2\sigma + \mu = 64 \quad - (2)$$

$$P(x > 64) = 0.08$$

$$\Rightarrow P(z > z_2) = 0.08$$

$$0.5 - \phi(z_2) = 0.08$$

$$\Rightarrow \phi(z_2) = 0.42$$

By normal probability table

$$0.4192 \approx 0.42 = \phi(1.4)$$

$$\phi(z_2) = \phi(1.4)$$

$$\boxed{z_2 = 1.4}$$

(1) & (2) $\Rightarrow$

$$-0.5\sigma + \mu = 45$$

$$1.4\sigma + \mu = 64$$

Solving $\sigma = 10$, $\mu = 50$

$\therefore$ Mean $= 50$

S.D $= 10$ //

8) In a test on Electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours.

If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for

(i) more than 2100 hours

ii) less than 1950 hours

iii) between 1900 to 2100 hours.

(Given $\phi(1.67) = 0.4525$, $\phi(0.83) = 0.2967$)

[DEC.2023/Jan. 2024]

- 07 marks -

Solution:

Given Mean = 2000 , S.D = 60

i.e $\mu = 2000$, $\sigma = 60$

Let x represent the life of electric bulbs

we have S.n.v $z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

If $x = 2100$, $z = \frac{100}{60} = 1.67$

i) $P(x > 2100) = P(z > 1.67)$

$= P(z \geq 0) - P(0 < z < 1.67)$

$= 0.5 - \phi(1.67)$

$= 0.5 - 0.4525$

$P(x > 2100) = 0.0475$

⇒ Since we need to Estimate the result for 2500 bulbs

⇒ $P(x > 2150) = 2500 \times 0.0475$

$= 118.75$

$\boxed{P(x > 2150) \approx 119}$

⇒ Number of bulbs that are likely to last for more than 2100 hours is 119.

ii) If $x = 1950 \Rightarrow Z = -5/6 = -0.83$

$P(x < 1950) = P(z < -0.83)$

$= P(z > 0.83)$

$= P(z \geq 0) - P(0 < z < 0.83)$

$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967$

$P(x < 1950) = 0.2033$

$= 2500 \times 0.2033$

$= 508.25$

$\boxed{P(x < 1950) \approx 508}$

⇒ 508 bulbs are likely to last for more than 1950 hours.

iii) If $x = 1900$, $Z = -1.67$

& If $x = 2100$, $z = 1.67$

$$P(1900 < x < 2100) = P(-1.67 < z < 1.67)$$
$$= P(-1.67 < z < 0) + P(0 < z < 1.67)$$
$$= P(0 < z < 1.67) + P(0 < z < 1.67)$$
$$= 2P(0 < z < 1.67)$$
$$= 2\,\phi(1.67)$$
$$= 2(0.4525)$$
$$P(1900 < x < 2100) = 0.905$$
$$= 2500 \times 0.905$$
$$= 2262.5$$
$$\boxed{P(1900 < x < 2100) \simeq 2263}$$

$\Rightarrow$ 2263 bulbs are likely to last between 1900 & 2100 hours.

8) In an examination 7% of students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and standard deviation of the marks normally distributed. (Given $P(1.2263) = 0.39$ & $P(1.4757) = 0.43$)

Solution: Here we need to find mean (μ) & standard deviation (σ) for the given data

i.e $P(x < 35) = 0.07$ & $P(x < 60) = 0.89$

Let x represents the marks of students.

we have s.n.v

$$z = \frac{x-\mu}{\sigma}$$

when $x = 35$

$\Rightarrow z_1 = \frac{35-\mu}{\sigma}$ (say)

$z_1\sigma = 35 - \mu$

$z_1\sigma + \mu = 35$ - (1)

$P(x < 35) = 0.07$

$\Rightarrow$

$P(z < z_1) = 0.07$

$0.5 + \phi(z_1) = 0.07$

$\phi(z_1) = -0.43$

using the given data

$\phi(1.4757) = 0.43$

$\Rightarrow \phi(z_1) = -\phi(1.4757)$

$\phi(z_1) = \phi(-1.4757)$

$\Rightarrow z_1 = -1.4757$

when $x = 60$

$z_2 = \frac{60-\mu}{\sigma}$ (say)

$z_2\sigma = 60 - \mu$

$z_2\sigma + \mu = 60$ - (2)

$P(x < 60) = 0.89$

$\Rightarrow P(z < z_2) = 0.89$

$0.5 + \phi(z_2) = 0.89$

$\phi(z_2) = 0.39$

using the given data

$\phi(1.2263) = 0.39$

$\phi(z_2) = \phi(1.2263)$

$\Rightarrow z_2 = 1.2263$

(1) & (2) $\Rightarrow$

$-1.4757\sigma + \mu = 35$ - (3)

$1.2263\sigma + \mu = 60$ - (4)

solving (3) & (4) $\Rightarrow$ $\mu = 48.65$, $\sigma = 9.25$

$\therefore$ $\boxed{\text{Mean} = 48.65,\ S.D = 9.25}$