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CSCE 222

MIDTERM II CORRECTIONS

Fall 2015

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Mitesh Patel

[10] **PROBLEM 1**

Give a recursive algorithm for exponentiation (computing a^n) that has complexity $O(\log n)$. **Prove** that your algorithm is correct.

Algorithm: power(a,n)
if *n==0* **then**
  return *1*
end
else
  if *n is even* **then**
    return (power(a,$\frac{n}{2}$))^2
  end
  else
    return a(power(a,$\frac{n-1}{2}$))^2
  end
end

Proof using strong induction:
Prove: power(a,n) = a^n

Basis: Let n = 0:
power(a,0) = 1
$a^0 = 1$ so our basis step checks out.

Inductive:
Assume power(a,i) = a^i $0 \leq i \leq k$ and $k \geq 0$
Show power(a,k+1) = a^{k+1}
We have two consider 2 cases: even and odd

1) k+1 even so k+1 = 2j
power(a,k+1) = power(a,2j) = power(a,$\frac{2j}{2}$)^2 = $(a^j)^2 = a^{2j} = a^{k+1}$

2) k+1 odd so k+1 = 2j+1
power(a,k+1) = power(a,2j+1) = a power(a,$\frac{2j+1-1}{2}$)^2 = a $(a^j)^2$ = a (a^{2j}) = $a^{2j+1} = a^{k+1}$

[10] **PROBLEM 2**

1. How many different strings can be made from the word PEPPERCORN when all the letters are used?

   PEPPERCORN has 10 letters: n = 10!
   using formula C(n,r) = $\frac{n!}{r!(n-r)!}$
   We can observe there are 3 Ps , 2 Es, 2 Rs, 1 C, 1 O, 1 N.
   P = 3!, E = 2!, R = 2!, C = 1!, O = 1!, N =1!
   therefore: $\frac{10!}{PERCON}$
   plugging in: $\frac{10!}{3!2!2!1!1!1!}$
   $= \frac{10!}{3!2!2!}$

2. How many of these strings (from part 1) start and end with the letter P?

   P _ _ _ _ _ _ _ _ P
   n = 10 - 2 = 8!
   P = 3 - 2 = 1!
   E, R, N, C, O still same from part a
   therefore: $\frac{8!}{PERCON}$
   plugging in: $\frac{8!}{1!2!2!1!1!1!}$
   $= \frac{8!}{2!2!}$

3. In how many of these strings (from part 1) are the three letter Ps consecutive?

   PPP _ _ _ _ _ _ _
   PPP treated as one block of string so P = 1!
   therefore: $\frac{8!}{PERCON}$
   plugging in: $\frac{8!}{1!2!2!1!1!1!}$
   $= \frac{8!}{2!2!}$

[10] **PROBLEM 3**

A model for the number of points the Aggies score each game is that the number of points scored is the average of the number of points scored in the previous two games.

1. Find a recurrence relation for $\{S_n\}$, where S_n is the Aggies' score in the n-th game.
   $S_n = \frac{S_{n-1}+S_{n-2}}{2}$

2. Find a closed form solution for S_n if 38 points were scored in the first game ($n = 0$) and 56 points were scored in the second game.
   $S_n = \frac{S_{n-1}+S_{n-2}}{2}$
   $S_0 = 38$
   $S_1 = 56$
   Find characteristic equation:
   $r^n = \frac{r^{n-1}+r^{n-2}}{2}$
   $2r^n = r^{n-1} + r^{n-2}$
   Find roots:
   $2r^n = r^{n-1} + r^{n-2}$
   Dividing each side by r^{n-2} we get:
   $2r^2 = r + 1$
   $2r^2 - r - 1 = 0$
   Using magic factoring method:
   $r^2 - r - 2 = 0$
   $(r - \frac{2}{2})(r + \frac{1}{2})$
   $r_1 = 1$, $r_2 = -\frac{1}{2}$
   Must be true that $S(n) = \alpha(r_1)^n + \beta(r_2)^n$
   $S(n) = \alpha(1)^n + \beta(-\frac{1}{2})^n = \alpha + \beta(-\frac{1}{2})^n$
   $S(0) = 38 = \alpha + \beta(-\frac{1}{2})^0 = \alpha + \beta$
   $S(1) = 56 = \alpha + \beta(-\frac{1}{2})^1 = \alpha - \beta/2$
   Solve for α and β using system of equations:
   $-38 = -\alpha - \beta$
   $+\ 56 = \alpha - \beta/2$
   ____
   $18 = -3/2\ \beta$
   $\beta = -12$, $\alpha = 50$ (by plugging in β into one of the equations)
   SOLUTION: $S(n) = 50 - 12(-\frac{1}{2})^n$

[10] **PROBLEM 4**

Set up a divide and conquer recurrence relation for the number of comparisons used in the following algorithm. Then, **use the master theorem** to give a big-O estimate for the number of comparisons used on a list of length n.

Algorithm: maxProfit(list)

Input: List of prices

Output: The maximum single-sale profit possible by buying at one price and selling at a later price

if *length(list)* ≤ 1 **then**
  return *0*
end
$left$ = left half of list
$right$ = right half of list
return $\max(maxProfit(left), maxProfit(right), \max(right) - \min(left))$

Hint: it is perfectly safe to write $O(f(n))$ terms that appear in the recurrence as a constant multiple of $f(n)$.

T(n) = T(n/2) + T(n/2) + n/2 + n/2 + 1
T(n) = 2T(n/2) + n + 1
T(n) = 2T(n/2) + O(n)

Using master theorem:
three cases

1) O(n^d) when a < b^d
2) O($n^d \log_b n$) when a = b^d
3) O($n^{log_b a}$) when a > b^d
a = 2, b = 2, d = 1, b^d = 2
we see a = b^d so we use second case therefore:
O($n^d \log_b n$) = O($n^1 \log_2 n$) = O(nlogn)

[10] **PROBLEM 5**

Let $R = \{((a,b),(c,d)) \mid ad = bc\}$ be a relation on pairs of positive integers. Is R

Note: you ***must*** *show your work.*

- Reflexive?
  $\forall_{(a,b)}$ (a,b) R (a,b)
  ab = ba so yes it is reflexive

- Symmetric?
  $\forall_{(a,b)(c,d)}$ (a,b) R (c,d) $\implies$ (c,d) R (a,b)
  ad = bc $\implies$ cb = da so yes it is symmetric

- Antisymmetric?
  $\forall_{(a,b)(c,d)}$ ((a,b)R(c,d) $\wedge$ (c,d)R(a,b)) $\implies$ (a,b) = (c,d)
  (ad=bc $\wedge$ cb=da) $\implies$ (a=c $\wedge$ b=d)
  We can prove this is not true by counter example:
  ((5,2) R (2,1)) $\wedge$ ((2,1) R (5,2)) but (5,2) $\neq$ (2,1)

- Transitive?
  $\forall_{(a,b)(c,d)(e,f)}$ ((a,b)R(c,d) $\wedge$ (c,d)R(e,f)) $\implies$ (a,b)R(e,f)
  We can prove generally: (ad=bc $\wedge$ cf = de) $\implies$ (af=be)
  a = $\frac{bc}{d}$
  f = $\frac{de}{c}$
  af = $\frac{bc}{d}\ \frac{de}{c}$ = be so yes it is transitive.

Is R an equivalence relation, a partial ordering, or both?
Yes it is equivalence relation because it is reflexive, symmetric, and transitive.
It is not partial ordering.
If R is an equivalence relation, what is the equivalence class of $(12,8)$?
.[(12,8)] = {(12,8),(24,16),(36,24),(48,32)}
= {(a,b) | $\frac{a}{b} = \frac{12}{8}$}
If R is a partial ordering, is it also a total ordering? If so, order $(12,8),(6,4),(7,5),(12,9)$.
It is NOT partial ordering

[5] **BONUS**

A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

There are two cases to be considered in this problem. 1) When no computers are connected to each other and 2) some computer is connected to all the other computers. Each computer can be directly connected to 0,1,2,3,4,5. But there are only 5 choices that can be considered because of case 1. So we have six computers and 5 choices, therefore the pigeonhole principle says that at least two computers have the same number of direct connections.