DISCRETE MATHEMATICS UNIT 1

Discrete Mathematics (Savitribai Phule Pune University)

Set Theory and Logic

· Logic ⇒

<u>Truth table</u> :- A table giving all possible truth values of a statement form, corresponding to the truth values assigned to its variables, is called a truth ~~value~~ table

Negation $\sim p$

p	$\sim p$
T	F
F	T

Conjunction (and) $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or) $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional if.... then $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional if and only if $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
~~F~~T	F	F
F	T	F
F	F	T

Exclusive or $p \bar{\vee} q$ (either p or q is true but not both)

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Multiplication $p \times q$

p	q	$p \times q$
T	T	T
T	F	F
F	T	F
F	F	T

Construct the truth tables for the following statement forms:-

1) $(\sim p \vee q) \rightarrow q$

$\Rightarrow$ Soln.

p	q	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \rightarrow q$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

2) $\sim(p \wedge q) \vee (p \times q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \times q$	$\sim(p \wedge q) \vee (p \times q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	T	T	T

3) $(\sim p \rightarrow r) \wedge (p \times q)$

p	q	r	$\sim p$	$\sim p \rightarrow r$	$p \times q$	$(\sim p \rightarrow r) \wedge (p \times$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Construct the truth table.

1) $(\sim p \vee q) \rightarrow q$

2) $\sim(p \wedge q) \vee (p \times q)$

3) $(\sim p \rightarrow r) \wedge (p \times q)$

1) $(\sim p \vee q) \rightarrow q$

p	q	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \rightarrow q$

Tautology

We have seen how to construct the truth table of various statement forms. The last column in the truth table gives the truth values of the statement form for all possible assignment of truth values to its variables.

Defination ⇒

* A statement form is called Tautology, if it always assumes the truth value 'T' irrespective of the truth values assigned to its variables.

* A statement form is called a Contradiction if its alway assumes the truth values 'F' irrespective of the truth values assigned to its variables.

* A statement form which is neither a tautology nor a contradiction is called a Contingency.

example ⇒

i) $p \rightarrow (q \rightarrow p)$

ii) $(p \wedge q) \wedge \sim(p \vee q)$

iii) $(p \wedge q) \longrightarrow p$

* Equivalence of Statement forms.

examples ⇒ i) $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are logically equivalent

ii) $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent

iii) p and $p \wedge p$ are logically equivalent.

Set

- A set is a collection of definite, distinguishable objects of our intuition to be conceived as a whole.
- The objects are called as elements or members of the set.
- A set is described as

$$A = \{ x \mid P(x) \}$$

to be read as element a is in the set iff & only if the statement $P(a)$ is true.

⇒ Axiomatic set theory:

An alternative or modified version of Cantor's set theory free from contradiction was developed by Zermelo & fraenkel. This version is called as Axiomatic set as follows:

i) ZF1 (Axioms of extension)

If A & B are sets and if for all x, $x \in A$ iff $x \in B$ then $A = B$

ii) ZF2 (Axiom schema of subsets)

For any set A, there exists a set B such that for all x, $x \in B$ iff $P(x)$ where $P(x)$ is any condition on x which contains no free occurance of B.

Sets

- A set is a collection of objects
- An object in the collection is called an element or member of the set
- A set may contain finite no of elements or infinite no of elements.

e.g. 1) The set of all telephone nos in the directory (finite)

2) The set of all points in the plane (infinite)

- A set is generally denoted by capital letters
- Elements of the set are denoted by small letters
- If x is an element of set A, we express

$x \in A$ (x belongs to A)

- If x is not an element of A, we write

$x \notin A$ (x does not belongs to)

Ways of representation of set

1) Listing method

The elements of listed within braces

e.g. $A = \{2, 4, 8, ...\}$

2) Statement form

A statement describing the set, especially where the elements share a common characteristic

e.g. A set of all the students of class

3) Set builder notation

A more concise or compact way of describing the set is to specify the property shared by all the elements of the set. This property is denoted by $P(x)$ & written as

$\{x \mid P(x)\}$

where,

$\{ \}$ denote the set of

& $\mid$ denote such that

e.g.

$A = \{x \mid x > 10\}$

is read as "A is set of all x such that x is greater than 10"

Let $A=\{a,b,\{a,b\},\{\{a,b\}\}\}$

Identify true or false

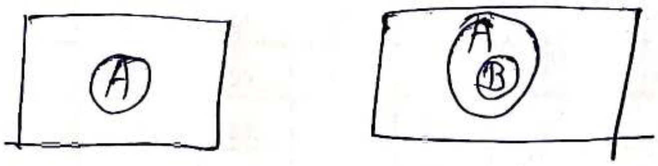
1) $a \in A$
2) $\{a\} \in A$
3) $\{a,b\} \in A$
4) $\{\{a,b\}\} \in A$
5) $\{a,b\} \subseteq A$
6) $\{a,\{b\}\} \subseteq A$

Determine true or false

1) If $A \in B$ and $B \subseteq C$ then $A \in C$
2) If $A \in B$ and $B \subseteq C$ then $A \subseteq C$
3) If $A \subseteq B$ and $B \in C$ then $A \in C$
4) If $A \subseteq B$ and $B \in C$ then $A \subseteq C$

Venn diagram

- It is pictorial description of a set
- A rectangle represent the universal set
- interior of the rectangle represent elements in the set.
- A circle drawn within rectangle depicts an arbitary set



$B \subseteq A$

Set operations

1) Complement of a set

Let A be a set, then complement of A denoted by, $\overline{A}$ is defined as

$$\overline{A} = \{x \mid x \notin A\}$$

e.g.

If $U = N = \{1, 2, 3, \ldots\}$

& $E = \{2, 4, 6, \ldots\}$

then $\overline{E} = \{1, 3, 5, \ldots\}$

Note that,

$\overline{\phi} = U$ and $\overline{U} = \phi$ where

$U \rightarrow$ Universal set

2) Union of sets

- The union of 2 sets A & B is the set consisting of all elements which are in A or in B or in both sets A & B.

It is denoted by $A \cup B$

i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

e.g. If $A = \{2, 4, 6, 8, 10\}$

$B = \{1, 2, 6, 8, 12, 15\}$

then $A \cup B = \{1, 2, 4, 6, 8, 10, 12, 15\}$

Note: for any set

$$\boxed{\begin{array}{l} A \cup \phi = A \\ A \cup U = U \\ A \cup \overline{A} = U \end{array}}$$

3) Intersection of sets

- The intersection of 2 sets A & B denoted by $A \cap B$ is the set consisting of elements which are in A as well as B

Thus, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A \cap B = \phi$

then, sets are said to be disjoint

e.g. If $A = \{a, b, c, g\}$ & $B = \{d, e, f, g\}$

then $A \cap B = \{g\}$

Note

$A \cap \phi = \phi$	$A \cap U = A$	$A \cap \overline{A} = \phi$

4) Difference of sets (Relative complement)

- Let A & B be any 2 sets

The difference is A - B

i.e. $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ is the (relative) complement of B

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$ is the complement of A in B

e.g. If $A = \{1, 2, 3, ---, 10\}$

$B = \{1, 3, 5, ---9\}$

then $A - B = \{2, 4, 6, 8, 10\}$

$B - A = \phi$

Please Note

1) $\overline{A} = U - A$

2) $A - A = \phi$

3) $A - \overline{A} = A$ & $\overline{A} - A = \overline{A}$

4) $A - \phi = \phi$

5) $A - B = A \cap \overline{B}$

6) $A - B = B - A$ iff $A = B$

7) $A - B = A$ iff $A \cap B = \phi$.

8) $A - B = \phi$ iff $A \subseteq B$

5) Symmetric difference

The symmetric difference of 2 sets A & B denoted by $A \oplus B$ is defined as

$$A \oplus B = \{x \mid x \in A - B \text{ or } x \in B - A\}$$

in other words,

$$A \oplus B = (A - B) \cup (B - A)$$

e.g. If $A = \{a, b, e, g\}$, $B = \{d, e, f, g\}$

then $A \oplus B = \{a, b, d, f\}$

please note,

1) $A \oplus A = \phi$
2) $A \oplus \phi = A$
3) $A \oplus U = \overline{A}$
4) $A \oplus \overline{A} = U$
5) $A \oplus B = (A \cup B) - (A \cap B)$

Algebra of set Operations :-

i) Commutative ⇒ • $A \cup B = B \cup A$

• $A \cap B = B \cap A$

ii) Associativity ⇒

• $A \cup (B \cup C) = (A \cup B) \cup C$

• $A \cap (B \cap C) = (A \cap B) \cap C$

iii) Distributivity ⇒

• $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

• $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

iv) ~~Indem~~ Idempotent laws ⇒

• $A \cup A = A$

• $A \cap A = A$

v) Absorption law ⇒

• $A \cup (A \cap B) = A$

• $A \cap (A \cup B) = A$

vi) De Morgan's law ⇒

• $\overline{A \cup B} = \bar{A} \cap \bar{B}$

• $\overline{A \cap B} = \bar{A} \cup \bar{B}$

v) Double Complement ⇒

• $\bar{\bar{A}} = A$

Examples :-

i) Determine true or false.

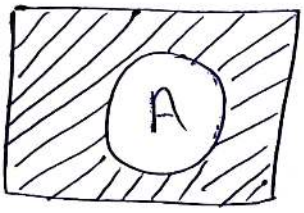
1) $\{a, \phi\} \in \{a, \{a, \phi\}\}$

2) $\{a, b\} \subseteq \{a, b, \{a, b\}\}$

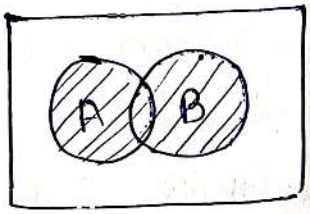
3) $\{a, b\} \in \{a, b, \{a, b\}\}$ =

4) $\{a, c\} \in \{a, b, c, \{a, b, c\}\}$ =

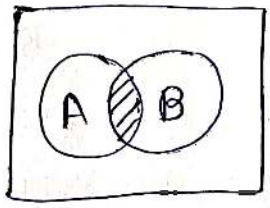
Representation of set Operations on Venn Diagrams.



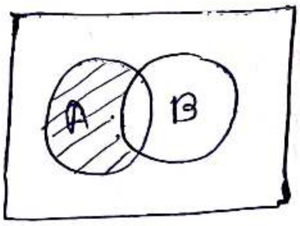
$\bar{A} =$



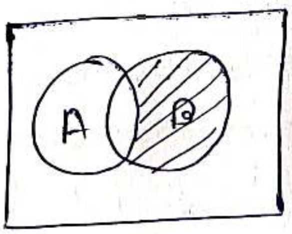
$A \cup B =$



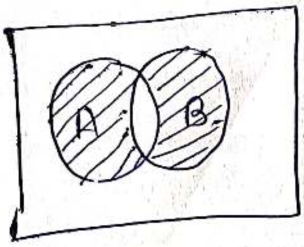
$A \cap B =$



$A - B =$



$B - A =$

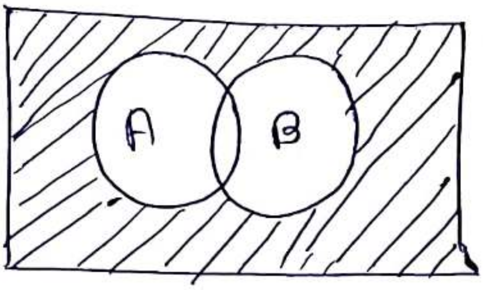


$A \oplus B =$

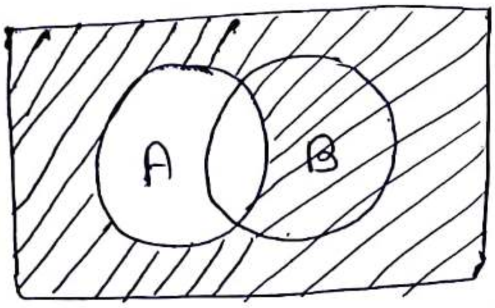
Problem solving using Venn Diagram

1) De Morgan's laws :-

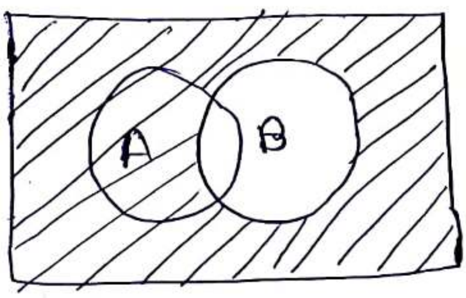
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



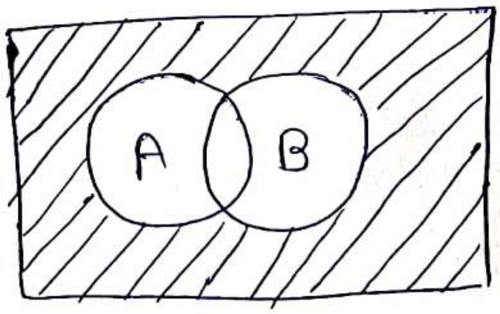
$\overline{A \cup B}$



$\bar{A}$



$\bar{B}$



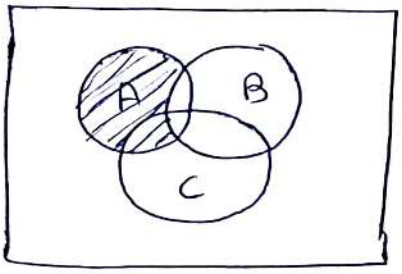
$\bar{A} \cap \bar{B}$

2) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

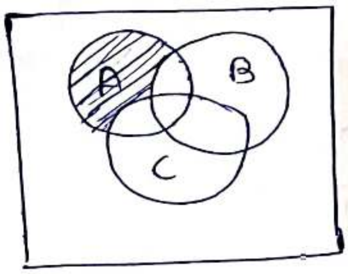
Examples :-

3) Show that $(A - B) - C = A - (B \cup C)$ Using Venn Diagram.

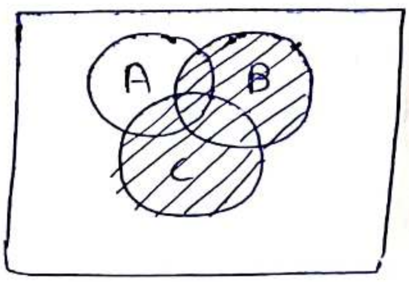
$\Rightarrow$



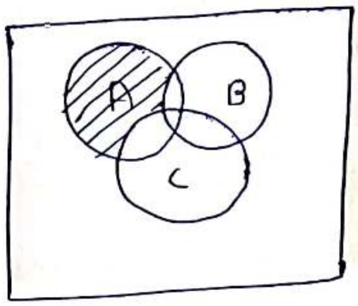
$A - B$



$(A - B) - C$



$B \cup C$



$A - (B \cup C)$

4) Drawing Venn diagram, prove that

$A - (B - C) = (A - B) \cup (A \cap B \cap C)$

Cardinality of finite set $\Rightarrow$

- Let A be a finite set the Cordinality of A, denoted by $|A|$ is the number of elements i the set.
- If $A = \phi$ then $|A| = 0$
- $|A \cup B| = |A| + |B|$ for disjoint set
- $|A - B| = |A| - |A \cap B|$
- Principal of Inclusion & exclusion

Let A and B be finite sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example :-

1) In a survey 2000 people were asked wheather they read India today or business times. It was found that 1200 read India today 900 reads business times and 400 read both. Find how many read at least one magazine and how many read none?

$\Longrightarrow$

$|A| = 1200$ $\quad$ $|B| = 900$ & $|A \cap B| = 400$

- to find out "read at least one magzine" we need to take union.

hence $|A \cup B| = |A| + |B| - |A \cap B|$

$= 1200 + 900 - 400$

$= 1700$

- To find out "how many read none" we need substract above from total strength.

is, $|U - (A \cup B)| = |U| - |A \cup B|$

$= 2000 - 1700$

$= 300$

Q) Among the integers 1 to 300, find how many are not divisible by 3 nor by 5. Find also how many are divisible by 3, but not by 7.

$\Longrightarrow$

Let A → denote the set of integer 1 - 300, divisible by 3

B → Divisible by 5

C. → Divisible by 7

We need to find (i) $|\bar{A} \cap \bar{B}|$ ii) A - C

Hence $|A| = \left|\frac{300}{3}\right| = 100$ $|B| = \left|\frac{300}{5}\right| = 60$

$|A \cap B| = \frac{300}{5 \times 3} = 20$ ← divisible by 5 & 3

Either divisible by 5 or 3

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 100 + 60 - 20$$
$$= 140$$

Hence nor by 3 or 5 is ⇒ $U - |A \cup B|$

$$= 300 - 140$$
$$= 160$$

(ii) $|A - C| = |A| - |A \cap C|$

$$= |100| - \left|\frac{300}{7 \times 3}\right|$$

$$= 100 - 14$$

$$= 86$$

Formula $|A \cup B| = |A| + B - |A \cap B|$

$|A - B| = |A| - |A \cap B|$

Power set

- A set all subsets of A set A is called the power set of A.
- The power set of A is denoted by $P(A)$
- Hence $P(A) = \{x \mid x \subseteq A\}$
- If A has n element then $P(A)$ has 2^n elements.

example i) $A = \{a, b\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

- If ϕ is an empty set
  find $P(\phi)$, $P(P(\phi))$, $P(P(P(\phi)))$

$\longrightarrow$

$P(\phi) = \{\phi\}$

$P(P(\phi)) = \{\phi, \{\phi\}\}$

$P(P(P(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

ex. $A = \{a, b, c\}$ → find powerset

Mathematical Induction

It is always possible to entend the solution to a group that is one larger than the previous.

<u>Statement</u> :-

Let $p(n)$ be a statement involving a natural number (n)

1. If $p(n)$ is true for $n = n_0$ and
2. Assuming $p(k)$ is true, $(k \geq n_0)$, we prove $p(k+1)$ is also true then $p(n)$ is true for all natural numbers $n \geq n_0$

Step 1 is called the basis of Induction

Step 2 is called as the induction step.

The assumption that $p(n)$ is true for $n = k$ is called as the induction hypothesis.

Example ⇒
prove that $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Sol^n ⟹ Let $p(n)$ be the statement

$$\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

1) for $n=1$ $\quad \frac{1}{(3\times1-2)(3\times1+1)}=\frac{1}{1\cdot4}=1/4$

hence $p(1)$ is true.

2) Assume $p(k)$ is true and prove that $p(k+1)$ is also true.

$$p(k+1)=\underbrace{\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{(3k-2)(3k+1)}}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{1}{3k+1}\left(\frac{k}{1}+\frac{1}{3k+4}\right)$$

$$=\frac{1}{3k+1}\left(\frac{k(3k+4)+1}{3k+4}\right)$$

$$=\frac{1}{3k+1}\times\frac{3k^2+4k+1}{3k+4}$$

$(3k+1)(k+1)$
$=3k^2+3k+k+1$
$=3k^2+4k+1$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{k+1}{3k+4}$$

$$=\frac{k+1}{3k+3+1}$$

$$=\frac{k+1}{[3(k+1)+1]}$$

Hence assuming $p(k)$ is true, $p(k+1)$ is also true,

$\therefore$ $p(n)$ is true for all $n\geq1$.

2) show that $1^3 + 2^3 + \dots + n^3 = (1+2+\dots+n)^2$

Proof
$\Longrightarrow$

1). Basis of Induction

for $n=1$ $\quad 1^3 = 1^2$

2) for $n = k+1$

$$\underbrace{1^3 + 2^3 + \dots + k^3} + (k+1)^3$$

$$= (1+2+\dots+k)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^2 + (k+1)$$

$$= \frac{k^2(k+1)^2 + 4[(k+1)^2 + (k+1)]}{4}$$

$$= \frac{(k+1)^2 (k^2+4k+4)}{4}$$

$$= (k+1)^2 \left(\frac{k^2+4k+4}{4}\right)$$

$$= \frac{(k+1)^2 (k+2)^2}{4} \qquad \Leftarrow (k+2)^2 = k^2+4k+4$$

$$= (k+1)^2 = k^2 + 4k + 4$$

$$= (1+2+\dots+(k+1))^2$$