Single-Variable Calculus Cheat Sheet

Comprehensive Definitions, Formulas, and Key Results

Timofei Gerasimov

February 26, 2025

1 Limits and Continuity

Limit (Informal)

$\lim_{x \to a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.

Key Limit Laws

- Sum/Difference: $\lim_{x \to a} [f(x) \pm g(x)] = \lim f(x) \pm \lim g(x)$
- Product: $\lim_{x \to a} [f(x)g(x)] = (\lim f(x))(\lim g(x))$
- Quotient: $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$ (if denominator $\neq 0$)
- Power: $\lim_{x \to a} [f(x)]^n = (\lim f(x))^n$

Continuity

Continuity at a Point

A function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Intermediate Value Theorem

If f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = N$.

2 Derivatives

Derivative

The derivative of f at $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

if this limit exists.

Basic Rules

- Power Rule: $\dfrac{d}{dx} x^n = nx^{n-1}$
- Constant Multiple: $\dfrac{d}{dx} [c \cdot f(x)] = c\, f'(x)$
- Sum/Difference: $\dfrac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$
- Product Rule: $(fg)' = f'g + fg'$
- Quotient Rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- Chain Rule: $\dfrac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

Derivatives of Common Functions

- $\dfrac{d}{dx}[e^x] = e^x$
- $\dfrac{d}{dx}[a^x] = a^x \ln(a)$
- $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$
- $\dfrac{d}{dx}[\sin x] = \cos x$
- $\dfrac{d}{dx}[\cos x] = -\sin x$
- $\dfrac{d}{dx}[\tan x] = \sec^2 x$
- $\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx}[\arctan x] = \dfrac{1}{1 + x^2}$

L'Hôpital's Rule

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists.

Mean Value Theorem

Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

First and Second Derivative Tests

- **First Derivative Test:** If $f'(x)$ changes sign from positive to negative at $x = c$, then f has a local maximum at c. If $f'(x)$ changes from negative to positive, then f has a local minimum.
- **Second Derivative Test:** If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

3 Applications of Derivatives

Critical Points and Extrema

A critical point is where $f'(x) = 0$ or $f'(x)$ does not exist. Local maxima or minima are typically found at these points.

Concavity and Inflection Points

- $f''(x) > 0$: f is concave up.
- $f''(x) < 0$: f is concave down.
- An inflection point occurs where $f''(x)$ changes sign.

4 Integration

Riemann Integral (Informal)

The definite integral of f over $[a,b]$ is defined as the limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Fundamental Theorem of Calculus Part 1

If f is continuous on $[a,b]$ and

$$F(x) = \int_a^x f(t)\,dt,$$

then F is differentiable on (a,b) and $F'(x) = f(x)$.

Fundamental Theorem of Calculus Part 2

If F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Common Antiderivatives

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
- $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
- $\displaystyle\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$
- $\displaystyle\int \csc x\,dx = -\ln\left|\csc x + \cot x\right| + C$

Average Value of a Function

The average value of f on $[a,b]$ is given by:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

5 Techniques of Integration

Substitution

If $u = g(x)$, then $du = g'(x)\,dx$, so:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

Partial Fraction Decomposition

For a rational function $\frac{P(x)}{Q(x)}$, decompose it into simpler fractions before integrating.

6 Applications of Integration

Areas and Volumes

- **Area under a curve:** $\int_a^b f(x)\,dx$
- **Volume by revolution (disk method):** $V = \pi \int_a^b [f(x)]^2\,dx$

Arc Length

The length of a curve is:

$$\text{Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

7 Sequences and Series

Sequences

A sequence $\{a_n\}$ converges to L if for every $\epsilon > 0$, there exists N such that for all $n > N$,

$$|a_n - L| < \epsilon.$$

Infinite Series

A series $\sum_{n=1}^{\infty} a_n$ converges if its partial sums converge to a limit.

Common Tests for Convergence

- **Comparison Test**
- **Ratio Test:** $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. If $L < 1$, the series converges absolutely.
- **Root Test:** $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. If $L < 1$, the series converges absolutely.
- **Integral Test:** If f is positive, continuous, and decreasing for $x \geq 1$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

8 Power Series and Taylor Series

Power Series

A power series is of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

It converges for $|x - a| < R$, where R is the radius of convergence.

Taylor/Maclaurin Series

For a function f infinitely differentiable at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$, this is called a Maclaurin series.

Key Expansions (Around 0)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Taylor's Theorem Remainder (Lagrange Form)

For f with $n+1$ continuous derivatives on an interval containing a and x,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some ξ between a and x.

9 Fundamental Constants and Functions

Definition of e

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Natural Logarithm

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

10 Short Proof Outlines

FTC Part 1 (Idea)

Define $F(x) = \int_a^x f(t)\,dt$. Show that by the definition of derivative and properties of the integral, $F'(x) = f(x)$.

Derivative of $\sin x$

Show $\sin'(0) = \lim_{h\to 0} \frac{\sin(h)-0}{h} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$. Then use the sine addition formula to generalize for any x.