Quasi-periodicity Detection via Repetition Invariance (Supplementary Materials)

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1 Theorems

Theorem 1 (Linear Trend from Invariance). *If ϕ satisfies the linear accumulation rule and it is prefix and suffix-invariant, then $\{\phi(\mathbf{X}_{[0:t]})\}$, the trajectory of $\phi(\mathbf{X})$ over expanding windows follows a linear trend on intervals where $\mathbf{X}$ is quasi-periodic.*

Proof. We recall the definition of prefix and suffix-invariant linear accumulation rule:

$$\phi(\mathbf{P}|\mathbf{U}|\mathbf{U}|...|\mathbf{U}|\mathbf{Q}) = n \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q}) \tag{1}$$

Prefix and suffix-invariance implies Eqn. 1 holds. Assume the expanding window is over a quasi-periodic interval, and let $\mathbf{X}_{[0:t]} = \mathbf{P}|\mathbf{U}|\mathbf{U}|...|\mathbf{U}|\mathbf{Q}$, where $\mathbf{Q}$ is an incomplete repetition of $\mathbf{U}$. Since $\mathbf{Q}$ is a part of $\mathbf{U}$, every unique value of $\mathbf{Q}$ (denoted by $\mathbf{Q}_k$, where k is a phase identifier) corresponds to a phase in the repetition cycle. As the prefix $\mathbf{P}$ is fixed, Eqn. 1 implies that $\gamma(\mathbf{P}, \mathbf{Q}_k)$ is uniquely determined by each k. Therefore starting from the beginning of any repetition cycle n_0, the value of $\phi(\mathbf{X}_{[0:t]})$ evolves from $n_0 \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{0})$ to $(n_0+1) \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{0})$, and its value at every phase k is determined by $n_0 \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q}_k)$. Moreover, the difference between any two $\phi(\mathbf{X}_{[0:t]})$ values with the same phase k is:

$$(n_0 \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q}_k)) - (n_1 \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q}_k)) = (n_0 - n_1) \cdot \phi(\mathbf{U})$$

which is parallel to $\phi(\mathbf{U})$.

Therefore $\{\phi(\mathbf{X}_{[0:t]})\}$ repeats the pattern $\gamma(\mathbf{P}, \mathbf{Q})$ with offset $\phi(\mathbf{U})$ per cycle. Additionally, each set of points with the same phase $\{\mathbf{P}|\mathbf{U}|\mathbf{U}|...|\mathbf{U}|\mathbf{Q}_k\}$ are collinear, and all these lines for different phase k's are parallel. The linear trend is the line $\lambda \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{0})$ where λ is the free parameter, and each ordered subset $\{\phi(\mathbf{X}_{[0:t]}) | \mathbf{X}_{[0:t]}$ is in phase $k\}$ extends along the line $\lambda \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q}_k)$.

Theorem 2 (LAR of Signed Areas). *The signed area terms of $\mathbf{X} = \mathbf{P}|\mathbf{U}|...|\mathbf{U}$, $Sig(\mathbf{X})^{ij}, i \neq j$ satisfy the prefix-invariant linear accumulation rule if $\mathbf{U}|\mathbf{U}|...|\mathbf{U}$ is quasi-periodic.*

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Proof. We show this by expanding the signature terms according to definition. Let a, c be the start and end time index of $\mathbf{X}$ and b be the time index of the beginning of the repetition segment.

$$\begin{aligned} S^{ij}_{[a,c]} &= \int_a^c dX^i_{t_1} \int_a^{t_1} dX^j_{t_2} \\ &= \int_a^b dX^i_{t_1} \int_a^{t_1} dX^j_{t_2} + \int_b^c dX^i_{t_1} \int_a^{t_1} dX^j_{t_2} \\ &= S^{ij}_{[a,b]} + \int_b^c dX^i_{t_1} (\int_a^b dX^j_{t_2} + \int_b^{t_1} dX^j_{t_2}) \end{aligned}$$

The quasi-periodicity of $UU...U$ implies that $X_b = X_c$, therefore:

$$\begin{aligned} S^{ij}_{[a,c]} = S^{ij}_{[a,b]} + (\int_a^b dX^j_{t_2})(X^i_c - X^i_b) + \int_b^c dX^i_{t_1} \int_b^{t_1} dX^j_{t_2} \\ = S^{ij}_{[a,b]} + S^{ij}_{[b,c]} \end{aligned}$$

Now recursively apply this relationship to $S^{ij}_{[b,c]}$ by treating the first U as the new prefix $\mathbf{P}$, we have:

$$S^{ij}_{[a,c]} = S^{ij}_{[a,b]} + n \cdot S^{ij}_{[b,b']} = S^{ij}(\mathbf{P}) + nS^{ij}(\mathbf{U}) \tag{2}$$

Where n is the number of repetitions, and $[b, b']$ is a quasi-periodic cycle.

Corollary 1. *Linear accumulation rule for signed areas is prefix and suffix-invariant.*

Proof. For $\mathbf{X} = \mathbf{P}|\mathbf{U}|\mathbf{U}...\mathbf{U}|\mathbf{Q}$ with a, b, c, d as the time indices for the start of $\mathbf{P}$, end of $\mathbf{P}$, start of $\mathbf{Q}$, end of $\mathbf{Q}$, respectively, we have:

$$\begin{aligned} S^{ij}_{[a,d]} &= S^{ij}_{[a,c]} + S^{i}_{[c,d]} S^{j}_{[a,c]} + S^{ij}_{[c,d]} \\ &= S^{ij}_{[a,b]} + S^{ij}_{[b,c]} + S^{i}_{[c,d]} S^{j}_{[a,b]} + S^{ij}_{[c,d]} \\ &= S^{ij}(\mathbf{P}) + nS^{ij}(\mathbf{U}) + S^{i}(\mathbf{Q}) S^{j}(\mathbf{P}) + S^{ij}(\mathbf{Q}) \\ &= nS^{ij}(\mathbf{U}) + S^{ij}(\mathbf{P}|\mathbf{Q}) \end{aligned}$$

Therefore the linear accumulation rule for signed areas is prefix and suffix-invariant.

Corollary 2. *Truncated log signatures of order* 3 *and above do not follow prefix and suffix-invariant linear accumulation rule.*

Proof. Let p, q, u be the log signatures of $\mathbf{P}, \mathbf{Q}, \mathbf{U}$, and let $r = log(e^p \otimes e^{\lambda u})$. We should interpret the expression e^p as the exponential map of p, which is the signature of $\mathbf{P}$. We would like to find $L := \Pi_3 LogSig(e^r \otimes e^q)$.

For any log signature, the order 0 term is 0, and the depth 1 log signature terms are identical to the order 1 signatures, namely the coordinate displacements for each dimension. If $\mathbf{X} = \mathbf{P}|\mathbf{U}|\mathbf{U}|...|\mathbf{U}$ is quasi-periodic for its $\mathbf{U}$ repetitions, then $U = Sig(\mathbf{U})$ has 0 in its order 1 terms, and by extension $u = LogSig(\mathbf{U})$ also has 0 in its order 1 terms. Therefore u has 2 as its lowest possible order with non-zero terms. On the other hand, the lowest order term for p is at least order 1. This implies that for any bracketed expression of u and p, its lowest order term is at least order $2 \cdot Count(u) + Count(p)$. Also because log signatures do not have a non-zero order 0 term, $\otimes$ or $[.,.]$ operations on truncated log signatures move the highest order terms of the operands outside of the truncation limit, for example:

$$\Pi_n(p \otimes q) = \Pi_{n-1}p \otimes \Pi_{n-1}q$$

$$\Pi_n[p,q] = [\Pi_{n-1}p, \Pi_{n-1}q]$$

Which allows us to do the following simplification:

$$\begin{aligned}
L &= \Pi_3(q + r + \frac{1}{2}[q,r] + \frac{1}{12}([q,[q,r]] - [r,[q,r]])) \\
&= q + \Pi_3 r + \frac{1}{2}[q, \Pi_2 r] \\
&\quad + \frac{1}{12}([q,[q,\Pi_1 r]] - [\Pi_1 r,[q,\Pi_1 r]]) \\
&= q + p + \lambda(u + \frac{1}{2}[u,p]) \\
&\quad + \frac{1}{2}[q, p + \lambda u] + \frac{1}{12}([q,[q,p]] - [p,[q,p]]) \\
&= \lambda(u + \frac{1}{2}[u,p] + \frac{1}{2}[q,u]) + \Pi_3 log(e^p \otimes e^q)
\end{aligned}$$

Therefore log signatures truncated at depth 3 have prefix and suffix-dependent terms in its linear coefficient, thus cannot be written in the form $\lambda \cdot \phi(\mathbf{U}) + \gamma(\mathbf{P}, \mathbf{Q})$. The same result holds for depth 4 and above, because these terms will still exist in their expansions.

Notice, however, in the above expression, if $\mathbf{Q}$ is a subsequence of $\mathbf{U}$ (i.e. an “incomplete” repetition), then for every $\mathbf{Q}$ that represents the same repetition phase, given fixed prefix $\mathbf{P}$, a linear trend still exists between equal-phase (i.e. same q) points, but the trend is different for each phase of the repetition, which is harder to exploit for repetition detection, but can be of interest to quantify the degree of repetition compliance. This weaker form of linear trend breaks at log signature order 5, at which terms quadratic in λ start to show up in the simplified BCH expansion.

2 Experiment Details

Hyperparameter Optimisation For the repetition detection experiments on PAMAP2, we reused the hyperparameter settings from [1] to reproduce their results with

NASC and SIMPAD. We found that the performance reported for NASC in [1] is underestimated due to not combining point-wise predictions into contiguous intervals, so we used a simple rolling window majority vote with a window size of 1000 to locally aggregate the predictions and produce more interval-like outputs. R-SIMPAD [2] has one more hyperparameter ("omega") than SIMPAD, and we found that setting $omega = 0.675$ yields similar results as reported by the authors in [2]. For RPSAP, we estimate the best embedding dimension with the False Nearest Neighbour procedure [3] and optimise the rest of its hyperparameters with the Leave-One-Out approach.

On the RecoFit dataset, we use the first 25 observations to automatically search for optimal hyperparameters using Optuna [4] for each method (NASC, R-SIMPAD and RPSAP), then evaluate the performance on the remaining 101 observations.

Ambiguity Masking Due to some labels in both PAMAP2 and RecoFit not clearly indicating whether the activity should be repetitive or not, we use an ambiguity masking procedure similar to [1, 2] in our experiments. We inherit the "repeating activities" (labels 4, 5, 6, 7, 12, 13, 24) and "non-repeating activities" (labels 1, 2, 3, 8, 9, 10) grouping from SIMPAD. For RecoFit, we label an interval as repeating activities if it is either one of "Repetitive exercise", "Repetitive non-arm exercise", "Machines" and "Opposite-hand repetitive exercise", or the "repeats" attribute is larger than 1 in the activities table. We then exclude the segments that are labelled as one of "Walk", "Walk and run", "Ambiguous exercise" and "Ambiguous labeling". In particular, we exclude walking and running because their boundaries and internal breaks are not as precisely labelled as regular exercises.

3 Additional Experiment Results

Here we include an expanded performance comparison table on the RecoFit dataset with results both with ambiguity masking and without.

	NASC	RSIMPAD	RPSAP
naive_acc	0.825±0.061	0.872±0.057	0.884±0.062
naive_prec	0.733±0.144	0.773±0.141	0.809±0.153
naive_rec	0.735±0.167	0.861±0.112	0.858±0.102
naive_f1	0.709±0.104	0.800±0.087	0.817±0.091
masked_acc	0.886±0.049	0.939±0.034	0.946±0.032
masked_prec	0.867±0.077	0.902±0.069	0.945±0.054
masked_rec	0.765±0.160	0.901±0.085	0.886±0.097
masked_f1	0.802±0.111	0.899±0.064	0.911±0.064

Table 1: RecoFit Results

We also show the comparison between RPSAP and R-SIMPAD of the distribution of performance scores per observation, in Fig. 1.

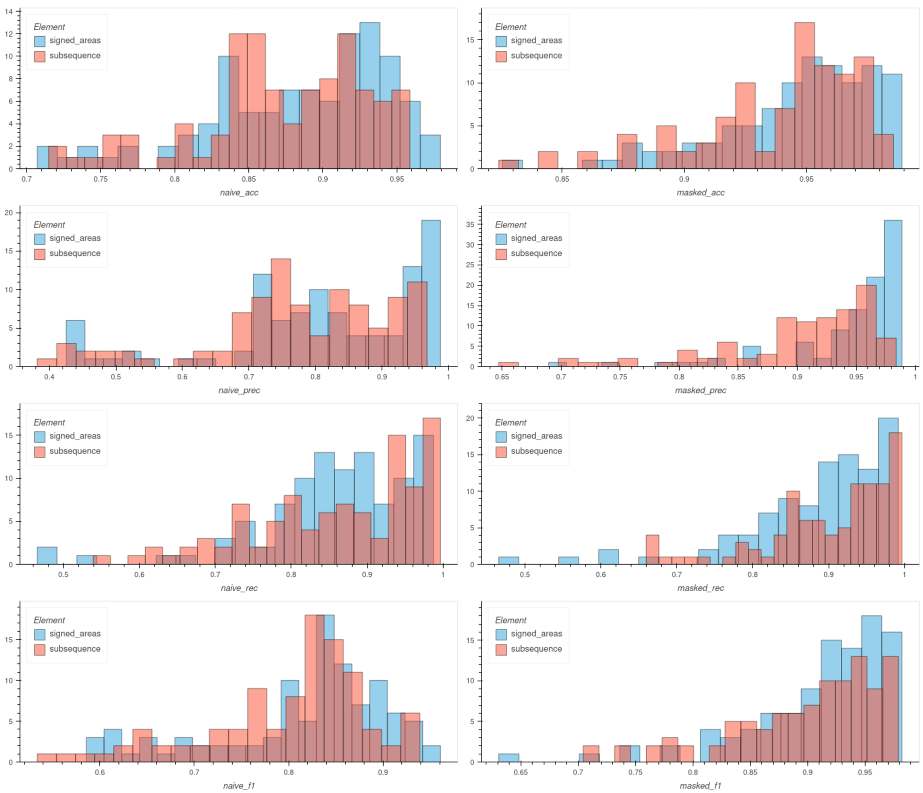

Fig. 1: Performance Distribution Comparison

References

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