UNIT-1
INTEGRATION

Define an integral

A function $F(x)$ is called an anti derivative or integral of a function $f(x)$ on an interval I if

$F'(x) = f(x)$, for every value of x in I

(i.e) If the derivative of a function $F(x)$ w.r.to $f(x)$,then we say that the integral of $f(x)$ w.r.to x is $F(x)$.

(i.e) $\int f(x)dx = F(x)$

- **Evaluate** $\int \frac{x^2-5x+1}{x} dx$.

Solution:

$$\int \frac{x^2-5x+1}{x} dx = \int \left(\frac{x^2}{x} - \frac{5x}{x} + \frac{1}{x}\right) dx$$
$$= \int \left(x - 5 + \frac{1}{x}\right) dx$$
$$= \frac{x^2}{2} - 5x + logx + \mathrm{C}$$

- **Evaluate:** $\int \frac{x^2+2x-1}{\sqrt{x}} dx$.

Solution:

$$\int \frac{x^2+2x-1}{\sqrt{x}} dx = \int (x^2+2x-1)x^{-\frac{1}{2}} dx$$
$$= \int (x^{2-\frac{1}{2}} + 2x^{1-\frac{1}{2}} - x^{-\frac{1}{2}}) dx$$
$$= \int (x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}) dx$$
$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + \mathrm{C}$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \mathrm{C}$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \mathrm{C}$$

$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + \mathrm{C}$$

- **Evaluate:** $\int \boldsymbol{cos5x \cos 3x\, dx}$

  Solution:

  We know that,

$$cosC \cos D = \frac{1}{2}[\cos(C - D) + \cos(C + D)]$$

$$\therefore \int cos5x \cos 3x\, dx = \int \frac{1}{2}[\cos(5x - 3x) + \cos(5x + 3x)]dx$$

$$= \frac{1}{2}\int [\cos 2x + \cos 8x]dx$$

$$= \frac{1}{2}\left[\int cos2x\, dx + \int cos8x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{sin2x}{2} + \frac{sin8x}{8}\right] + \mathrm{C}$$

- **Evaluate:** $\int \sqrt{\boldsymbol{1 - cos2x}}\, \boldsymbol{dx}$.

  Solution:

$$\int \sqrt{1 - cos2x}\, dx = \int \sqrt{2sin^2x} \qquad [\text{since } 2sin^2\theta = 1 - cos2\theta\ ]$$

$$= \sqrt{2}\int sinx\ \ dx$$

$$= \sqrt{2}(-cosx) + \mathrm{C} = -\sqrt{2}cosx + \mathrm{C}$$

- **Evaluate:** $\int \sqrt{\boldsymbol{1 + sin2x}}\, \boldsymbol{dx}$.

  Solution:

  We know that,

$$sin^2x + cos^2x = 1\ and\ \sin 2x = 2sinxcosx$$

$$\therefore \int \sqrt{1 + sin2x}\, dx = \int \sqrt{(sin^2x + cos^2x) + 2sinxcosx}\ \ dx$$

$$= \int \sqrt{(sinx + cosx)^2)}\ \ dx$$

$$= \int (sinx + cosx)\ \ dx$$

$$= (-cosx + sinx) + C = sinx - cosx + \mathrm{C}$$

- **Integrate:** $\int \boldsymbol{\frac{sinx}{cos^2x} dx}$.

  Solution:

$$\int \frac{sinx}{cos^2x}dx = \int \frac{sinx}{(cosx)(cosx)}dx$$
$$= \int \left(\frac{sinx}{cosx}\right)\left(\frac{1}{cosx}\right)dx$$
$$= \int tanx\ secx\ dx = secx + C$$

- **Integrate:** $\int \boldsymbol{cos^3x\ dx}$.

  Solution:

  We know that,

$$cos\ 3x = 4cos^3x - 3\ cos\ x$$
$$\Rightarrow\ 4cos^3x = cos\ 3x + 3\ cos\ x$$
$$\Rightarrow\quad cos^3x = \frac{1}{4}(cos\ 3x + 3\ cos\ x)$$
$$\therefore\quad \int cos^3x\ dx = \frac{1}{4}\int (cos\ 3x + 3\ cos\ x)\ dx$$
$$= \frac{1}{4}\left[\frac{sin\ 3x}{3} + 3\ sinx\right] + \mathrm{C}$$

- **Integrate:** $\int \boldsymbol{sin3x\cos 2x\ dx}$.

  Solution:

  We know that,

$$sinC\sin D = \frac{1}{2}[\sin(C+D) + \sin(C-D)]$$
$$\therefore \int sin3x\cos 2x\ dx = \int \frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)]dx$$
$$= \int \frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)]dx$$
$$= \frac{1}{2}\int [\sin 5x + \sin x]dx$$
$$= \frac{1}{2}\left[\int \sin 5x\ \ dx + \int \sin x\ \ dx\right]$$
$$= \frac{1}{2}\left[\frac{-cos5x}{5} - cosx\right] + \mathrm{C}$$
$$= \frac{-1}{2}\left[\frac{cos5x}{5} + cosx\right] + \mathrm{C}$$

- **Integrate:** $\int \frac{\boldsymbol{dx}}{\boldsymbol{x^2+2x+5}}$.

  Solution:

$$x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 2^2$$

$$\therefore \int \frac{dx}{x^2+2x+5} = \int \frac{dx}{2^2+(x+1)^2}$$

We know that,

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\therefore \int \frac{dx}{x^2+2x+5} = \int \frac{dx}{2^2+(x+1)^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

- **Integrate:** $\int \frac{dx}{x^2+2x+10}$.

  <u>Solution:</u>

$$x^2+2x+10 = x^2+2x+1+9 = (x+1)^2+3^2$$

$$\therefore \int \frac{dx}{x^2+2x+10} = \int \frac{dx}{3^2+(x+1)^2}$$

We know that,

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\therefore \int \frac{dx}{x^2+2x+10} = \int \frac{dx}{3^2+(x+1)^2} = \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + \mathrm{C}$$

- **Evaluate** $\int \frac{dx}{\sin^2 x \cos^2 x}$

  <u>Solution:</u>

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{(\sin^2 x+\cos^2 x)dx}{\sin^2 x\cos^2 x}$$

$$= \left(\int \frac{\sin^2 x}{\sin^2 x\cos^2 x} + \frac{\cos^2 x\,dx}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x\,dx$$

$$= tanx - cotx + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{sin7x\ .cos5xdx}$

  <u>Solution:</u>

We know that,

$$sinCsinD = \frac{1}{2}[\sin(C+D) + \sin\,(C-D)]$$

$$\therefore sin7x\,.cos5xdx = \frac{1}{2}[\sin(7x+5x) + \sin\,(7x-5x)]$$

$$= \frac{1}{2}[\sin(12x) + \sin\,(2x)]$$

$$\therefore \int sin7x\ .cos5xdx = \frac{1}{2}\int[\sin(12x) + \sin\,(2x)]dx$$

$$= \frac{1}{2}\left[\frac{-cos12x}{12} - \frac{cos2x}{2}\right] + \mathrm{C}$$
$$= -\frac{1}{4}\left[cos2x + \frac{cos12x}{6}\right] + \mathrm{C}$$

Integration by Method of substitution

- **Integrate:** $\int \frac{2x}{1+x^2} dx$.

  Solution:

$$\text{Let } u = 1 + x^2$$
$$du = 2x\,dx$$
$$\therefore \int \frac{2x}{1+x^2} dx = \int \frac{1}{u} du$$
$$= \log u + C = \log(1 + x^2) + C$$

- **Integrate:** $\int \frac{cosx}{sinx} dx$.

  Solution:

$$\text{Let } u = sinx$$
$$du = cosx\,dx$$
$$\therefore \int \frac{cosx}{sinx} dx = \int \frac{1}{u} du$$
$$= \log u + C = \log(\sin x) + C$$

- **Integrate:** $\int \frac{sec^2x}{tanx} dx$.

  Solution:

$$\text{Let } u = tanx$$
$$du = sec^2x\,dx$$
$$\therefore \int \frac{sec^2x}{tanx} dx = \int \frac{1}{u} du$$
$$= \log u + C = \log(\tan x) + C$$

- Integrate: $\int \frac{dx}{(1+x^2)\tan^{-1}x}$.

  Solution:

$$\text{Let } u = \tan^{-1}x$$
$$du = \frac{1}{1+x^2} dx$$
$$\therefore \int \frac{dx}{(1+x^2)\tan^{-1}x} = \int \frac{du}{u} = \log u + \mathrm{C} = \log(\tan^{-1}x) + \mathrm{C}$$

- **Evaluate:** $\int \frac{cosx-sinx}{cosx+sinx} dx$.

  Solution:

$$\text{Let } u = cosx + sinx$$

$$du = (-sinx + cosx)\, dx$$

$$\therefore \int \frac{cosx - sinx}{cosx + sinx} dx = \int \frac{1}{u} du = log\, u + C \ = log(cosx + sinx) + \mathrm{C}$$

- **Integrate:** $\int \frac{\boldsymbol{4x+3}}{\boldsymbol{2x^2+3x+5}} \boldsymbol{dx}$.

  Solution:

$$\text{Let } u = 2x^2 + 3x + 5$$

$$du = (4x + 3)\, dx$$

$$\therefore \int \frac{4x + 3}{2x^2 + 3x + 5} dx = \int \frac{1}{u} du \ = \log u + C \ = \log(2x^2 + 3x + 5) + C$$

- **Integrate:** $\int \frac{\boldsymbol{1+cosx}}{\boldsymbol{x+sinx}} \boldsymbol{dx}$.

  Solution:

$$Let\ u = x + sinx$$

$$du = (1 + cosx)\, dx$$

$$\therefore \int \frac{1 + cosx}{x + sinx} dx = \int \frac{1}{u} du \ = \log u + \mathrm{C} \ = \log(x + sinx) + \mathrm{C}$$

- **Integrate:** $\int \frac{1}{x(\log x)^n} dx$.

  Solution:

$$\text{Let } u = logx$$

$$du = \frac{1}{x}\ dx$$

$$\therefore \int \frac{1}{x(logx)^n} dx = \int \frac{1}{u^n} du \ = \int u^{-n}\, du$$

$$= \frac{u^{-n+1}}{-n + 1} + C = \frac{(\log x)^{1-n}}{1 - n} + C$$

- **Evaluate** $\int \boldsymbol{5x^4\ .\, e^{x^5} dx}$.

  Solution:

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$$\text{Let } u = x^5$$

$$du = 5x^4\ dx$$

$$\therefore \int 5x^4 e^{x^5}\, dx = \int e^u\, du \ = e^u + C = e^{x^5} + C$$

- **Evaluate:** $\int \frac{\boldsymbol{sec^2(logx)}}{\boldsymbol{x}} \boldsymbol{dx}$.

  Solution:

$$\text{Let } u = logx$$

$$du = \frac{1}{x}\ dx$$

$$\therefore \int \frac{sec^2(logx)}{x}dx = \int sec^2 u\, du = tan\, u + \mathrm{C} = tan(log\, x) + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{e^{2logx} \,.\, e^{x^3} dx}$ **.**

  Solution:

$$\int e^{2logx} \,.\, e^{x^3} dx = \int e^{logx^2} \,.\, e^{x^3} dx$$

$$= \int x^2 \,.\, e^{x^3} dx$$

Put $t = x^3 \quad \Longrightarrow dt = 3x^2 dx$

$$\Longrightarrow x^2 dx = \frac{dt}{3}$$

$$\therefore \int x^2 \,.\, e^{x^3} dx = \int e^t \frac{dt}{3} = \frac{1}{3} e^t + \mathrm{C} = \frac{1}{3} e^{x^3} + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{\frac{e^{tanx}}{cos^2 x} dx}$

  Solution:

Let $t = tanx \quad \Longrightarrow dt = sec^2 x dx$

$$\therefore \int \frac{e^{tanx}}{cos^2 x} dx = \int e^{tanx} . sec^2 x\, dx$$

$$= \int e^t dt = e^t + \mathrm{C} = e^{tanx} + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{\frac{logx}{x}} \mathbf{d}\boldsymbol{x}$

  Solution:

Let $t = logx \quad \Longrightarrow dt = \frac{1}{x} dx$

$$\therefore \int \frac{logx}{x} = \int t\, dt = \frac{t^2}{2} + \mathrm{C} = \frac{(logx)^2}{2} + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{x \,.\, e^{x^2} dx}$

  Solution:

Let $u = x^2 \quad \Longrightarrow du = 2x\, dx$

$$\Longrightarrow \frac{du}{2} = xdx$$

$$\therefore \int x \,.\, e^{x^2} dx = \int e^u . \frac{du}{2} = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + \mathrm{C} = \frac{1}{2} e^{x^2} + \mathrm{C}$$

- **Evaluate** $\int \boldsymbol{\frac{sin(logx)}{x} dx}$**.**

  Solution:

Let $t = logx \quad \Longrightarrow dt = \frac{1}{x} dx$

$$\therefore \int \frac{sin(logx)}{x} dx = \int sint\, dt = -cost + \mathrm{C} \quad = -\cos(logx) + \mathrm{C}$$

- **Evaluate** $\int \frac{\boldsymbol{dx}}{\boldsymbol{x.logx}}$.

Solution:

Let $t = logx \quad \Longrightarrow dt = \frac{1}{x}dx$

$$\therefore \int \frac{dx}{x.\,logx} = \int \frac{1}{t}dt \quad = logt + \mathrm{C}$$

- **Evaluate** $\int \frac{\boldsymbol{cot(x)}}{\boldsymbol{log(sinx)}} \boldsymbol{dx}$.

Solution:

Let $\quad t = log\ (sinx)$ and $u = sinx$

$\therefore t = log\, u \qquad du = cosx\, dx$

$$\Longrightarrow dt = \frac{1}{u}du \quad = \frac{1}{sinx}cosx\, dx = cotx\, dx$$

$$\therefore \quad \int \frac{cotx}{log(sinx)}dx = \int \frac{dt}{t} \quad = \quad logt + \mathrm{C} = log\big(log(sinx)\big) + \mathrm{C}$$

- **Evaluate** $\int \sqrt{\boldsymbol{9 - 4x^2}}\ \boldsymbol{dx}$

Solution:

Here, $\int \sqrt{9 - 4x^2}\ dx = \int \sqrt{3^2 - (2x)^2}\ dx$

Let $u = 2x \quad \Longrightarrow \quad du = 2dx \quad \Longrightarrow dx = \frac{du}{2}$

$$\therefore \int \sqrt{9 - 4x^2}\ dx \ = \frac{1}{2}\int \left(\sqrt{3^2 - (u)^2}\right)\left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}\left\{u\sqrt{3^2 - (u)^2} + 3^2 sin^{-1}\left(\frac{u}{3}\right)\right\}\right]$$

$$= \frac{1}{4}\left[2x\sqrt{9 - 4x^2} + 9sin^{-1}\left(\frac{2x}{3}\right)\right]$$

Integration by parts

Let u & v be two functions of x then

$$\int udv = uv - \int vdu$$

- **Evaluate** $\int \boldsymbol{x^3 e^{x^2} dx}$.

Solution:

Put $t = x^2 \ \Longrightarrow dt = 2xdx$

$$\Longrightarrow \frac{dt}{2} = xdx$$

$$\int x^3 e^{x^2} dx = \int x^2 e^{x^2} x dx$$
$$= \int t\, e^t \frac{dt}{2} = \frac{1}{2}\int t\, e^t dt$$

Use Integration by parts method

Let $u = t,\qquad dv = e^t dt$

$du = dt,\qquad v = e^t$

$$\therefore \quad \frac{1}{2}\int t\, e^t dt = \frac{1}{2}\left[te^t - \int e^t\, dt\right]$$
$$= \frac{1}{2}[te^t - e^t] + C$$
$$= \frac{1}{2}e^t[t-1] + C$$
$$\therefore \int x^3 e^{x^2} dx = \frac{1}{2}e^{x^2}[x^2-1] + C \quad (since\ t = x^2)$$

- **Evaluate** $\int \boldsymbol{x \tan^{-1} x\,.\,dx}$.

<u>**Solution:**</u> Use Integration by parts method

(i.e) $\int udv = uv - \int vdu$

Let $u = \tan^{-1} x,\qquad dv = xdx$

$du = \frac{1}{1+x^2}dx,\qquad v = \frac{x^2}{2}$

$$\therefore \quad \int x \tan^{-1} x\,.\,dx = \frac{x^2}{2}\tan^{-1} x - \int (\frac{x^2}{2})\frac{1}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int (\frac{x^2+1-1}{1+x^2})dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left[\int (\frac{x^2+1}{1+x^2})dx - \int (\frac{1}{1+x^2})dx\right] + C$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left[\int dx - \tan^{-1} x\right] + C$$
$$= \frac{1}{2}[x^2\tan^{-1} x + \tan^{-1} x - x] + C$$

- **Integrate:** $\int \boldsymbol{x^2 e^{-x} dx}$.

<u>**Solution:**</u>

Use Integration by parts method

Let $u = x^2,\qquad dv = e^{-x}dx$

$du = 2xdx,\qquad v = -e^{-x}$

$$\therefore \quad \int x^2 e^{-x} dx = x^2(-e^{-x}) - \int (-e^{-x})2xdx$$
$$= -x^2 e^{-x} + 2\int xe^{-x}dx$$

Now $u = x, \quad dv = e^{-x}dx$

$du = dx, \quad v = -e^{-x}$

$\therefore \int x^2e^{-x}dx = -x^2e^{-x} + 2[-xe^{-x} - \int(-e^{-x})dx]$

$= -x^2e^{-x} + 2[-xe^{-x} + (-e^{-x})] + C = -e^{-x}[x^2 + 2x + 2] + C$

- **Evaluate $\int x^2e^{3x}dx$.**

<u>**Solution:**</u>

Applying integration by parts,

Let $u = x^2\,, \quad dv = e^{3x}dx$

$$du = 2xdx, \quad v = \frac{e^{3x}}{3}$$

$$\therefore \int x^2e^{3x}dx = \frac{x^2e^{3x}}{3} - \int \frac{e^{3x}}{3}2xdx$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx \quad ------------(1)$$

Again applying integration by parts

Let $u = x\,, \quad dv = e^{3x}dx$

$$du = dx, \quad v = \frac{e^{3x}}{3}$$

$\therefore$ Equation (1) $\Rightarrow$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left\{x.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}dx\right\}$$

$$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\int e^{3x}dx$$

$$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\left(\frac{e^{3x}}{3}\right) + c$$

$$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$$

- **Evaluate $\int x^2 \sin^{-1} x\, dx$.**

<u>**Solution:**</u>

Use Integration by parts method

Put $u = \sin^{-1} x, \quad dv = x^2 dx$

$du = \frac{1}{\sqrt{1-x^2}} dx, \quad v = \frac{x^3}{3}$

$$\therefore \int x^2 \sin^{-1} x \, dx = \frac{x^3}{3} \sin^{-1} x - \int \left(\frac{x^3}{3}\right) \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^3}{3} \sin^{-1} x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx \text{ -------(1)}$$

To find $\int \frac{x^3}{\sqrt{1-x^2}} dx$

Put $t = 1 - x^2 \quad \Longrightarrow x^2 = 1 - t$

$dt = -2x dx \quad \Longrightarrow -\frac{dt}{2} = x dx$

$$\therefore \int \frac{x^3}{\sqrt{1-x^2}} dx = \int \frac{x^2}{\sqrt{1-x^2}} x \, dx$$

$$= \int \frac{(1-t)}{\sqrt{t}} \left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{2} \int \left[\frac{1}{\sqrt{t}} - \sqrt{t}\right] dt$$

$$= -\frac{1}{2} \left[2\sqrt{t} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right] = -\left[\sqrt{t} - \frac{1}{3} t^{\frac{3}{2}}\right]$$

$$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}}$$

$\therefore$ Equation (1) gives

$$\int x^2 \sin^{-1} x \, dx = \frac{x^3}{3} \sin^{-1} x - \frac{1}{3}\left[-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]$$

$$= \frac{x^3}{3} \sin^{-1} x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{\frac{3}{2}}$$

- **Evaluate** $\int \boldsymbol{x^2 cos2xdx}$**. (L6)**

<u>**Solution:**</u>

Applying integration by parts,

$$\text{Let } u = x^2, \qquad dv = cos2xdx$$

$$du = 2xdx, \qquad v = \frac{sin2x}{2}$$

$$\therefore \int x^2 cos2xdx = x^2 \frac{sin2x}{2} - \int \frac{sin2x}{2}.2xdx$$

$$= x^2 \frac{sin2x}{2} - \int x.sin2x\,dx$$

Again applying integration by parts

$$\text{Let } u = x, \quad dv = sin2xdx$$

$$du = dx, \qquad v = \frac{-cos2x}{2}$$

$$\therefore \int x^2 cos2xdx = x^2 \frac{sin2x}{2} - \left\{\frac{x(-cos2x)}{2} - \int \frac{(-cos2x)}{2} dx\right\}$$

$$= x^2 \frac{sin2x}{2} + \frac{xcos2x}{2} - \frac{1}{2}\int cos2xdx$$

$$= x^2 \frac{sin2x}{2} + \frac{xcos2x}{2} - \frac{1}{4} sin2x + c$$

DEFINITE INTEGRAL

- **Evaluate:** $\int_0^2 (\boldsymbol{x^2 + 2x + 2})\,\boldsymbol{dx}$.

<u>**Solution:**</u>

$$\int_0^2 (x^2 + 2x + 2)\,dx = \int_0^2 x^2\,dx + \int_0^2 2x\,dx + \int_0^2 2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{2x^2}{2}\right]_0^2 + [2x]_0^2$$

$$= \frac{(2^3 - 0)}{3} + (2^2 - 0) + 2(2 - 0)$$

$$= \frac{8}{3} + 4 + 4 = \frac{8}{3} + 8 = \frac{8 + 24}{3} = \frac{32}{3}$$

- **Evaluate:** $\int_0^{\frac{\pi}{2}} \boldsymbol{sin^2 x\,dx}$.

<u>**Solution:**</u>

We know that,

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left[x - \left(\frac{\sin 2x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - \frac{1}{2}\left(\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right) - \frac{1}{2}(\sin\pi - 0)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \qquad [\text{since} \sin\pi = 0]$$

- **Evaluate:** $\int_0^1 \frac{1}{1+x^2}\,dx$ **.**

Solution:

We know that,

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\therefore \qquad \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

- **Evaluate:** $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx$**.**

Solution:

$$\text{Put } u = 1 + \sin x$$

$$\frac{du}{dx} = \cos x \implies du = \cos x\, dx$$

When $x = 0, u = 1$

When $x = \frac{\pi}{2}, u = 1 + 1 = 2$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{cosx}{1+sinx}\,dx = \int_1^2 \frac{du}{u} = [log\,u]_1^2$$
$$= log2 - log1 = log2 \qquad \qquad [\text{since } log1 = 0]$$

- **Evaluate** $\int_1^2 \log x\,dx$.

Solution:

Using Integration by parts, $\int u\,dv = uv - \int v\,du,$

Let $\quad u = \log\ x, \qquad dv = dx.$

$$\therefore du = \frac{1}{x}dx, \qquad \int dv = \int dx \implies v = x$$

$$\therefore \quad \int_1^2 logx\,dx = [xlogx]_1^2 - \int_1^2 x\frac{1}{x}dx$$
$$= [2log2 - 1log1] - [x]_1^2 \qquad \qquad [\text{since log1} = 0]$$
$$= 2log2 - 0 - (2-1)$$
$$= 2log2 - 1$$
$$= log2^2 - 1 = log4 - 1$$

- **Evaluate** $\int_0^{\frac{\pi}{4}} \tan x\,dx$.

Solution:

We know that,

$$\int tanx\,dx = -lo\,g(cosx) + \mathrm{C}$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} tanx\,dx = [-\,log(cosx)]_0^{\frac{\pi}{4}}$$
$$= -\left[log\left(cos\frac{\pi}{4}\right) - log\ (\cos 0)\right]$$
$$= -\left[log\left(\frac{1}{\sqrt{2}}\right) - log1\right]$$
$$= -\left[log1 - log\sqrt{2} - log1\right]$$
$$= log\sqrt{2}$$

- **Evaluate:** $\int_{-a}^{a} e^x\,dx$.

Solution:

We know that,

$$\int e^x\,dx = e^x + C$$

$$\therefore \quad \int_{-a}^{a} e^x\,dx = [e^x]_{-a}^{a}$$
$$= e^a - e^{-a}$$

- **Evaluate:** $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$.

  <u>Solution:</u>

  We know that,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx$$
$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] = \frac{\pi}{4}$$

- **Evaluate** $\int_0^{\pi/2} \frac{Cosx}{1+Sinx}\,dx$

  <u>Solution:</u>

Put $t = sinx$ when $x = 0; t = sin0 = 0$

$\Longrightarrow \quad dt = cosxdx$ when $x = \frac{\pi}{2}; t = sin\frac{\pi}{2} = 1$

$$\therefore \int_0^1 \frac{dt}{1+t} = [log(1+t)]_0^1$$
$$= log(1+1) - log(1+0)$$
$$= log2 - log1 = log2 \qquad [\text{since } log1 = 0]$$

- **Evaluate** $\int_0^{\frac{\pi}{2}} \sec^2 x\,dx$.

  <u>Solution:</u>

$$\int_0^{\frac{\pi}{2}} \sec^2 x\,dx = [tanx]_0^{\frac{\pi}{2}} = tan\frac{\pi}{2} - tan0 = \infty$$

- **Evaluate** $\int_0^{\frac{\pi}{2}} \frac{sinx}{1+cosx}\,dx$.

  <u>Solution:</u>

Let $u = 1 + cosx$ when $x = 0, u = 1 + cos0 = 1 + 1 = 2$

$$du = -sinxdx \qquad\qquad x = \frac{\pi}{2}, u = 1 + cos\frac{\pi}{2} = 1 + 0 = 1$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{sinx}{1 + cosx} dx = \int_2^1 \frac{-du}{u} = \int_1^2 \frac{du}{u}$$

$$= [logu]_1^2 = log2 - log1 = log2$$

- **Evaluate $\int_0^{\frac{\pi}{3}} \sin^3 x \ dx$.**

<u>**Solution:**</u>

We know that, $\quad sin3x = 3sinx - 4sin^3x$

$$\Longrightarrow \quad sin^3x = \frac{1}{4}(3sinx - sin3x)$$

$$\therefore \quad \int_0^{\frac{\pi}{3}} \sin^3 x \ dx = \frac{1}{4}\int_0^{\frac{\pi}{3}} (3sinx - sin3x)\, dx$$

$$= \frac{1}{4}\left[-3cosx + \frac{cos3x}{3}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left[-3cos\frac{\pi}{3} + \frac{cos3\left(\frac{\pi}{3}\right)}{3} - \left(-3cos0 + \frac{1}{3}cos0\right)\right]$$

$$= \frac{1}{4}\left[-3\left(\frac{1}{2}\right) + \frac{1}{3}(-1) + 3 - \frac{1}{3}\right] = \frac{1}{4}\left[\frac{-3}{2} - \frac{1}{3} - \frac{1}{3} + 3\right]$$

$$= \frac{1}{4}\left[\frac{-9 - 2 - 2 + 18}{6}\right] = \frac{1}{4}\left(\frac{5}{6}\right) \quad = \frac{5}{24}$$

- **Evaluate $\int_0^{\frac{\pi}{2}} \boldsymbol{sin2x.cos3x\ dx}$.**

<u>**Solution:**</u>

We know that,

$$cosAsinB = \frac{1}{2}[sin(A + B) - \sin(A - B)]$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} sin2x.cos3x\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}[sin(3x + 2x) - \sin(3x - 2x)]dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{2}} [sin(5x) - \sin(x)]dx$$

$$= \frac{1}{2}\left[\frac{-cos5x}{5} + cosx\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{-1}{5}\cos\left(5.\frac{\pi}{2}\right) + cos\frac{\pi}{2} + cos0 - cos0\right]$$

$$= \frac{1}{2}\left[\frac{-1}{5}\cos\frac{\pi}{2} + 0\right] \qquad \left[\text{since } \cos\frac{5\pi}{2} = \cos\left(360 + \frac{\pi}{2}\right) = \cos\frac{\pi}{2}\right]$$

$$= \frac{1}{2}(0) = 0$$

- **Evaluate** $\int_{0}^{\frac{\pi}{2}} \sqrt{sinx}\, cosx\, dx$.

  <u>**Solution:**</u>

  Let $u = sinx$ when $x = 0; u = sin0 = 0$

  $\Longrightarrow du = cosxdx$ when $x = \frac{\pi}{2}; u = sin\frac{\pi}{2} = 1$

$$\therefore \int_{0}^{1} \sqrt{u}du = \int_{0}^{1} u^{\frac{1}{2}}du = \left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{0}^{1} = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0}^{1} = \frac{2}{3}[1-0] = \frac{2}{3}$$

Evaluate $\int_{-2}^{3} (4 - x^2)dx$.

<u>**Solution:**</u>

$$\int_{-2}^{3} (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{3} = \left(4 \times 3 - \frac{3^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= (12 - 9) + \left(8 - \frac{8}{3}\right) = 3 + \left(\frac{24 - 8}{3}\right)$$

$$= 3 + \frac{16}{3} = \frac{25}{3}$$

- **Evaluate** $\int_{0}^{1} \frac{\sin^{-1} xdx}{\sqrt{1-x^2}}$.

<u>**Solution:**</u>

Let $t = \sin^{-1} x \quad \Longrightarrow dt = \frac{1}{\sqrt{(1-x^2)}}dx$

When $x = 0 \Longrightarrow t = \sin^{-1}(0)$

$\Longrightarrow t = 0$

When $x = 1 \Longrightarrow t = \sin^{-1}(1)$

$$\Longrightarrow t = \frac{\pi}{2}$$

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} t\, dt = \left(\frac{t^2}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

- **Evaluate $\int_0^1 xe^x dx$.**

Solution:

Let $u = x, \quad dv = e^x dx$

$du = dx, \quad v = e^x$

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = (1e^1 - 0) - [e^x]_0^1$$

$$= e - (e^1 - e^0)$$

$$= e - e + 1 = 1$$

- **Evaluate $\int_0^1 x5^x dx$.**

Solution:

Let $u = x, \quad dv = 5^x dx$

$du = dx, \quad v = \frac{5^x}{log5}$

$$\therefore \int_0^1 x5^x dx = \left[x\frac{5^x}{log5}\right]_0^1 - \int_0^1 \frac{5^x}{log5} dx$$

$$= \left(\frac{(1)5^1}{log5} - 0\right) - \frac{1}{log5}\left[\frac{5^x}{log5}\right]_0^1 = \frac{5}{log5} - \frac{1}{(log5)^2}[5^1 - 5^0]$$

$$= \frac{5}{log5} - \frac{4}{(log5)^2} \quad = \frac{5log5 - 4}{(log5)^2}$$

- **Evaluate $\int_1^2 e^{\sqrt{x}} dx$.**

Solution:

Let $t = \sqrt{x} = x^{\frac{1}{2}}$ when $x = 1; t = \sqrt{1} = 1$

$\Longrightarrow dt = \frac{1}{2}x^{\frac{1}{2}-1}dx$ when $x = 2; t = \sqrt{2}$

$\Longrightarrow dx = 2\sqrt{x}dt = 2tdt$

$$\therefore \int_1^2 e^{\sqrt{x}}\, dx = \int_1^{\sqrt{2}} e^t . 2t dt$$

Using the method of integration by parts

Let $\quad u = t \ , dv = e^t dt$

$du = dt, v = e^t$

$$\therefore \int_1^2 e^{\sqrt{x}}\ dx = \int_1^{\sqrt{2}} e^t . 2tdt = 2\int_1^{\sqrt{2}} t . e^t dt$$

$$= 2\left[(te^t)_1^{\sqrt{2}} - \int_1^{\sqrt{2}} e^t dt\right]$$

$$= 2\left[\sqrt{2}e^{\sqrt{2}} - 1e^1 - [e^t]_1^{\sqrt{2}}\right]$$

$$= 2\left[\sqrt{2}e^{\sqrt{2}} - e - \left[e^{\sqrt{2}} - e\right]\right]$$

$$= 2e^{\sqrt{2}}\left(\sqrt{2} - 1\right)$$

- **Evaluate** $\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{1+\cos^2 x}$ **.**

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Let $t = \cos x \ \Longrightarrow dt = -\sin x\, dx$

$\Longrightarrow -dt = \sin x\, dx$

When $x = 0 \Longrightarrow t = \cos 0 = 1$

When $x = \dfrac{\pi}{2} \Longrightarrow t = \cos\dfrac{\pi}{2} = 0$

$$\therefore \quad \mathrm{I} = \int_1^0 \frac{-dt}{1+t^2} = -[\tan^{-1} t]_1^0$$

$$= -[\tan^{-1} 0 - \tan^{-1} 1]$$

$$= -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

- **Evaluate** $\int_1^2 \frac{dx}{x^2+5x+6}$.

Solution:

$$\text{Let } \frac{1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

$$\Rightarrow 1 = A(x+3) + B(x+2)$$

Put $x = -2$; $\quad 1 = A(-2+3) + 0$

$$\Rightarrow A = 1$$

Put $x = -3$; $\quad 1 = 0 + B(-3+2)$

$$\Rightarrow B = -1$$

$$\int \frac{dx}{(x+2)(x+3)} = \int \left(\frac{1}{(x+2)} + \frac{(-1)}{(x+3)}\right) dx$$

$$= log(x+2) - log(x+3) + C$$

$$= log\left(\frac{x+2}{x+3}\right) + C$$

- **Evaluate** $\int_0^{\frac{\pi}{2}} \boldsymbol{x^2 sinxdx}$.

<u>**Solution:**</u>

Use Integration by parts method

Let $u = x^2$, $\quad dv = sinxdx$

$du = 2xdx$, $\quad v = -cosx$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 sinxdx = [x^2(-cosx)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-cosx)(2x)dx$$

$$= -\left(\left(\frac{\pi}{2}\right)^2 cos\frac{\pi}{2} - 0\right) + 2\int_0^{\frac{\pi}{2}} (xcosx)dx$$

$$= 0 + 2\int_0^{\frac{\pi}{2}} (xcosx)dx$$

Again Using Integration by parts method,

Let $u = x$, $\quad dv = cosxdx$

$du = dx$, $\quad v = sinx$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 sinxdx = 2\left[(xsinx)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} sinxdx\right]$$

$$= 2\left[\left(\frac{\pi}{2} sin\frac{\pi}{2} - 0\right) - (-cosx)_0^{\frac{\pi}{2}}\right]$$

$$= \pi + \left(cos\frac{\pi}{2} - cos0\right) = \pi - 1$$

- **Evaluate** $\int_0^{1} \boldsymbol{x^2 e^x\, dx}$.

<u>**Solution:**</u>

Use Integration by parts method

Let $u = x^2, \qquad dv = e^x dx$

$du = 2xdx, \qquad v = e^x$

$$\therefore \int_0^1 x^2 e^x \, dx = [x^2 e^x]_0^1 - \int_0^1 e^x \, 2xdx$$

$$= (e - 0) - 2\int_0^1 e^x \, xdx$$

$$= e - 2\int_0^1 e^x \, xdx$$

Again using Integration by parts method

Let $u = x, \qquad dv = e^x dx$

$du = dx, \qquad v = e^x$

$$\therefore \int_0^1 x^2 e^x \, dx = e - 2\left[(xe^x)_0^1 - \int_0^1 e^x \, dx\right]$$

$$= e - 2[(e - 0) - [e^x]_0^1]$$

$$= e - 2[e - (e^1 - e^0)]$$

$$= e - 2[e - e^1 + e^0] = e - 2$$

- **Evaluate $\int_0^\infty xe^{-x^2} \, dx$.**

Solution:

Let $t = x^2$

$$\Longrightarrow \qquad dt = 2xdx \Longrightarrow xdx = \frac{dt}{2}$$

when $x = 0; \ t = 0$

when $x = \infty; t = \infty$

$$\int_0^\infty e^{-t} \frac{dt}{2} = \frac{1}{2}\int_0^\infty e^{-t} \, dt = \frac{1}{2}[-e^{-t}]_0^\infty$$

$$= -\frac{1}{2}[e^{-\infty} - e^0] = -\frac{1}{2}[0 - 1] = \frac{1}{2}$$

- **Evaluate: $\int_0^{\frac{\pi}{2}} \frac{cosx}{(1+sinx)+(2+sinx)} dx$.**

Solution:

Put $t = sinx$

$\Longrightarrow dt = cosxdx$

when $\quad x = 0; t = sin0 = 0$

when $\quad x = \frac{\pi}{2}; t = sin\frac{\pi}{2} = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{cosx}{(1 + sinx) + (2 + sinx)} dx = \int_0^1 \frac{dt}{3 + 2t} = \left[\frac{1}{2} log(3 + 2t)\right]_0^1$$

$$= \frac{1}{2}[log(3 + 2) - log(3 + 0)]$$

$$= \frac{1}{2}[log(5) - log(3)] = \frac{1}{2} log\left(\frac{5}{3}\right)$$

- **Evaluate:** $\int_0^1 \boldsymbol{xe^{-2\,x}\,dx}$**.**

<u>**Solution:**</u>

Let $\quad u = x \qquad , \ dv = e^{-2\,x}dx$

$\Rightarrow du = dx \qquad , v = \frac{e^{-2\,x}}{-2}$

$$\therefore \int_0^1 xe^{-2\,x}\,dx = \left[x\left(\frac{e^{-2\,x}}{-2}\right)\right]_0^1 - \int_0^1 \frac{e^{-2\,x}}{-2} dx$$

$$= \frac{-1}{2}[1. e^{-2} - 0] + \frac{1}{2}\int_0^1 e^{-2\,x}dx$$

$$= \frac{-1}{2}[e^{-2}] + \frac{1}{2}\left[\frac{e^{-2\,x}}{-2}\right]_0^1$$

$$= \frac{-1}{2}[e^{-2}] - \frac{1}{4}(e^{-2} - 1) = e^{-2}\left(\frac{-1}{2} - \frac{1}{4}\right) + \frac{1}{4}$$

$$= e^{-2}\left(-\frac{3}{4}\right) + \frac{1}{4} \quad = \frac{1}{4}(1 - 3e^{-2})$$

- **Evaluate:** $\int_0^{\frac{\pi}{2}} \boldsymbol{sin2x\,cosx\,dx}$**.**

<u>**Solution:**</u>

We know that $sinA\,cosB = \frac{1}{2}[sin(A + B) + sin(A - B)]$

$$\therefore \ sin2x \ \ cosx = \tfrac{1}{2}[sin(2x+x)+sin(2x-x)]$$

$$= \tfrac{1}{2}[sin(3x)+sin(x)]$$

$$\therefore \int_0^{\frac{\pi}{2}} sin2x\ cosx\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} [sin(3x)+sin(x)]\, dx$$

$$= \frac{1}{2}\left[\frac{-cos(3x)}{3} - cosx\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}\left[\frac{1}{3}\left(cos\left(3\frac{\pi}{2}\right)+cos0\right) - \left(cos\frac{\pi}{2}+cos0\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{3}\left(\frac{-1}{\sqrt{2}}+1\right) - (0+1)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{3}\left(\frac{-1}{\sqrt{2}}\right)\right] = \frac{1}{6\sqrt{2}}$$

<u>INTEGRATION BY PARTIAL PRACTION</u>

- **Evaluate** $\int \frac{\mathbf{9}}{(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\mathbf{2})^2}\, \boldsymbol{dx}$.

<u>Solution:</u>

$$\frac{9}{(x+1)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

$$\Longrightarrow \quad 9 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Put $x=-1$; $9 = A(-1+2)^2$

$\Longrightarrow A = 9$

Put $x=-2$; $9 = C(-2+1)^2$

$\Longrightarrow C = 9$

Put $x=0$; $9 = A(0+2)^2 + B(0+1)(0+2) + C(0+1)$

$\Longrightarrow 9 = A(4) + B(2) + C$

$$\Rightarrow 9 = 9(4) + B(2) + 9$$
$$\Rightarrow 2B = -36$$
$$\Rightarrow B = -18$$
$$\therefore \int \frac{9}{(x+1)(x+2)^2}\,dx = \int \left[\frac{9}{(x+1)} + \frac{(-18)}{(x+2)} + \frac{9}{(x+2)^2}\right] dx$$

$$= 9\,log(x+1) - 18\,log(x+2) + 9\int (x+2)^{-2}dx$$
$$= 9\,[log(x+1) - 2\,log(x+2)] + 9\left(\frac{-1}{x+2}\right)$$
$$= 9\left[log(x+1) - 2\,log(x+2) - \frac{1}{x+2}\right]$$

- **Evaluate** $\int \frac{\mathbf{3-2x}}{(\boldsymbol{x^2+x+1})}\,\boldsymbol{dx}$.

<u>Solution:</u>

Let $\quad 3 - 2x = A\frac{d}{dx}(x^2+x+1) + B$
$$\Rightarrow 3 - 2x = A(2x+1) + B$$
Put $x = \frac{-1}{2}$; $\quad 3 - 2\left(\frac{-1}{2}\right) = A\left(2\left(\frac{-1}{2}\right) + 1\right) + B$
$$\Rightarrow 3 + 1 = B$$
$$\Rightarrow B = 4$$
Put $x = 0$; $\quad 3 = A(0+1) + B$
$$\Rightarrow 3 = A + 4$$
$$\Rightarrow A = -1$$
$$\therefore \int \frac{3-2x}{(x^2+x+1)}dx = \int \frac{-1(2x+1)+4}{(x^2+x+1)}dx$$
$$= -\int \frac{(2x+1)}{(x^2+x+1)}dx + 4\int \frac{1}{(x^2+x+1)}dx$$
$$= -log(x^2+x+1) + 4\int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}dx$$
$$= -log(x^2+x+1) + 4\int \frac{1}{(x+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$
$$= -log(x^2+x+1) + 4\left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right] + C$$

$$= -log(x^2 + x + 1) \; + \frac{8}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

- **Evaluate $\int_{-1}^{1} \boldsymbol{log} \, \frac{3-x}{3+x} \boldsymbol{dx}$. (L6)**

<u>**Solution:**</u>

Let $f(x) = \; log \; \left[\frac{3-x}{3+x}\right]$

Then $f(-x) = log \; \left[\frac{3+x}{3-x}\right] = log \left[\frac{3-x}{3+x}\right]^{-1}$

$$= -log \; \left[\frac{3-x}{3+x}\right] = -f(x)$$

$\therefore f(x)$ is an odd function of x.

But we know that,

$\int_{-a}^{a} f(x)dx = 0, \; when \; f(x)$ is an odd function of x

$$\therefore \quad \int_{-1}^{1} log \; \left[\frac{3-x}{3+x}\right] dx \; = 0$$

- **Integrate: $\int \frac{7x-6}{x^2-3x+2} \boldsymbol{dx}$.**

<u>**Solution:**</u>

Here $\frac{7x-6}{x^2-3x+2} = \frac{7x-6}{(x-2)(x-1)}$

Let $\frac{7x-6}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$

$$= \frac{A(x-1) + B(x-2)}{(x-2)(x-1)}$$

$\therefore \qquad 7x - 6 = A(x-1) + B(x-2)$

Put $x = 1$; $\;\; 7(1) - 6 = 0 + B(1-2)$

$\Longrightarrow B = -1$

Put $x = 2$; $\;\; 7(2) - 6 = A(2-1) + 0$

$\Longrightarrow A = 8$

$$\therefore \int \frac{7x-6}{x^2-3x+2} dx = \int \left[\frac{8}{(x-2)} + \frac{(-1)}{(x-1)}\right] dx$$

$$= 8\log(x-2) - \log(x-1) + C$$

- **Integrate: $\int \frac{3x+1}{2x^2+x+1} \boldsymbol{dx}$.**

<u>**Solution:**</u>

Let $3x+1 = A\dfrac{d}{dx}(2x^2+x+1)+B$

Then, $3x+1 = A(4x+1)+B$

when $x = \dfrac{-1}{4}$; $\quad 3\left(\dfrac{-1}{4}\right)+1 = 0+B$

$$\Rightarrow B = \frac{1}{4}$$

when $x = 0$; $\quad A+B = 1$

$$\Rightarrow A = 1-\frac{1}{4} = \frac{3}{4}$$

$$\therefore \int \frac{3x+1}{2x^2+x+1}\,dx = \int \frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+1}\,dx$$

$$= \frac{3}{4}\int \frac{3x+1}{2x^2+x+1}\,dx + \frac{1}{4}\int \frac{1}{2x^2+x+1}\,dx$$

$$= \frac{3}{4}\log(2x^2+x+1) + \frac{1}{4}\int \frac{1}{2\left[\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2\right]}\,dx$$

$$= \frac{3}{4}\log(2x^2+x+1) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\frac{\sqrt{7}}{4}}\right)\tan^{-1}\left(\frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right)+C$$

$$= \frac{3}{4}\log(2x^2+x+1) + \left(\frac{1}{2\sqrt{7}}\right)\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+C$$

- **Integrate:** $\int \frac{\boldsymbol{2x-1}}{\sqrt{\boldsymbol{x^2-5x-6}}}\,\boldsymbol{dx}$.

Solution:

Let $2x-1 = A\dfrac{d}{dx}(x^2-5x-6)+B$

$\Rightarrow \qquad 2x-1 = A(2x-5)+B$

when $x = \frac{5}{2}$; $\quad 2\left(\frac{5}{2}\right)-1 = 0+B \qquad \Rightarrow B = 4$

$$\text{when } x=0; \quad -1 = 5A + B \quad \Longrightarrow -1 = 5A + 4 \quad \Longrightarrow A = 1$$

$$\therefore \int \frac{2x-1}{\sqrt{x^2-5x-6}}dx = \int \frac{1(2x-5)+4}{\sqrt{x^2-5x-6}}dx$$

$$= \int \frac{2x-5}{\sqrt{x^2-5x-6}}dx + 4\int \frac{1}{\sqrt{x^2-5x-6}}dx$$

$$= 2\sqrt{x^2-5x-6} + 4\int \frac{1}{\sqrt{\left(x-\frac{5}{2}\right)^2 - \left(\frac{7}{2}\right)^2}}dx$$

$$= 2\sqrt{x^2-5x-6} + 4log\left[\left(x-\frac{5}{2}\right) + \sqrt{\left(x-\frac{5}{2}\right)^2 - \left(\frac{7}{2}\right)^2}\right] + C$$

- **Integrate:** $\int \frac{\boldsymbol{x+1}}{\sqrt{\boldsymbol{2x^2+x-3}}}\boldsymbol{dx}$.

<u>Solution:</u>

Let $x+1 = A\frac{d}{dx}(2x^2+x-3) + B$

Then, $x+1 = A(4x+1) + B$

when $x = -\frac{1}{4}$; $\quad -\frac{1}{4} + 1 = 0 + B \Longrightarrow B = \frac{3}{4}$

$$\text{when } x=0; \quad 1 = A + B \quad \Longrightarrow A = \frac{1}{4}$$

$$\therefore \int \frac{x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{\frac{1}{4}(4x+1)+\frac{3}{4}}{\sqrt{2x^2+x-3}}dx$$

$$= \frac{1}{4}\int \frac{(4x+1)}{\sqrt{2x^2+x-3}}dx + \frac{3}{4}\int \frac{1}{\sqrt{2x^2+x-3}}dx$$

$$= \frac{1}{4}(2)\sqrt{2x^2+x-3} + \frac{3}{4}\int \frac{1}{\sqrt{(x+\frac{1}{4})^2 - \left(\frac{5}{4}\right)^2}}dx$$

$$= \frac{1}{4}(2)\sqrt{2x^2 + x - 3} + \frac{3}{4}\log\left[\left(x + \frac{1}{4}\right) + \sqrt{\left(x + \frac{1}{4}\right)^2 - \left(\frac{5}{4}\right)^2}\right] + C$$

- **Evaluate: $\int_1^2 \frac{dx}{x^2+5x+6}$.**

<u>Solution:</u>

Here $x^2 + 5x + 6 = (x + 3)(x + 2)$

Consider

$$\frac{1}{x^2 + 5x + 6} = \frac{A}{x + 3} + \frac{B}{x + 2}$$

$$= \frac{A(x + 2) + B(x + 3)}{(x + 3)(x + 2)}$$

$$\Longrightarrow \quad 1 = A(x + 2) + B(x + 3)$$

when $x = -2$; $1 = 0 + B(x + 3) \quad \Rightarrow B = 1$

when $x = -3$; $1 = A(-3 + 2) + 0 \quad \Rightarrow A = -1$

$$\therefore \int_1^2 \frac{dx}{x^2 + 5x + 6} = \int_1^2 \left(\frac{-1}{x + 3} + \frac{1}{x + 2}\right) dx$$

$$= [-log(x + 3) + log(x + 2)]_1^2$$
$$= [-log(5) + log(4)]—(-log(4) + log(3)]$$
$$= -log(5) + log(4) + log(4) - log(3)$$
$$= -log(5) + 2log(4) - log(3)$$
$$= -log(5) + log4^2 - log3$$
$$= -log(5) + log16 - log3 \quad = log\left(\frac{16}{15}\right)$$

Properties of definite integrals

1. $\int_a^b f(x)dx = \int_b^a f(y)dy$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_0^a f(x)dx = \int_0^a f(a - x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
5. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)(dx)$ if $f(x)$ is an even function

   $= 0$ if $f(x)$ is an odd function

- **Write any two properties of definite integrals.**

<u>Solution:</u>

$$(i) \quad \int_a^b f(x)dx = -\int_b^a f(x)dx$$
$$(ii) \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

- **If $\boldsymbol{f(x)}$ is an even function then prove that $\boldsymbol{\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)(dx)}$**

<u>**Solution:**</u>

Consider

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \qquad ------ \text{(i)}$$

Let $x = -t$ in the first integral of the R.H.S.

$x = -t \Longrightarrow dx = -dt$

when $x = -a; t = a$

when $x = 0; t = 0$

$\therefore$Equation (i) $\Rightarrow$

$$\int_{-a}^{a} f(x)dx = \int_a^0 f(-t)(-dt) + \int_0^a f(x)dx$$
$$= -\int_a^0 f(-t)(dt) + \int_0^a f(x)dx$$
$$= \int_0^a f(-t)dt + \int_0^a f(x)dx$$
$$= \int_0^a f(-x)dx + \int_0^a f(x)dx \qquad -------- \text{(ii)}$$

$\left[\text{Since } \int_a^b f(x)dx = \int_a^b f(y)dy\right]$

Given function is an even function.

$\therefore f(-x) = f(x)$

Equations (ii) $\Rightarrow$

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)(dx) + \int_0^a f(x)(dx) = 2\int_0^a f(x)(dx)$$

- **Evaluate:** $\boldsymbol{\int_0^{\frac{\pi}{2}} \frac{(sinx)^{\frac{3}{2}}}{(sinx)^{\frac{3}{2}}+(cosx)^{\frac{3}{2}}} dx.}$

Solution:

Let $f(x) = \dfrac{(sinx)^{\frac{3}{2}}}{(sinx)^{\frac{3}{2}} + (cosx)^{\frac{3}{2}}}$ ------------- (I)

$\therefore \int_0^{\frac{\pi}{2}} f(x)dx = I$ (Let) ------------(1)

Now $f\left(\frac{\pi}{2} - x\right) = \dfrac{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}$

$= \dfrac{(cosx)^{\frac{3}{2}}}{(cosx)^{\frac{3}{2}} + (sinx)^{\frac{3}{2}}}$ ----------(II)

since $sin\left(\frac{\pi}{2} - x\right) = cosx$

$cos\left(\frac{\pi}{2} - x\right) = sinx$

By the property of definite integral

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx \quad \text{------------}(2)$$

From (1) and (2) we get

$$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx = I \quad \text{------------}(3)$$

$$(1) + (3) \Rightarrow 2\mathrm{I} = \int_0^{\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{(sinx)^{\frac{3}{2}}}{(sinx)^{\frac{3}{2}} + (cosx)^{\frac{3}{2}}}\right)dx + \int_0^{\frac{\pi}{2}} \left(\frac{(cosx)^{\frac{3}{2}}}{(cosx)^{\frac{3}{2}} + (sinx)^{\frac{3}{2}}}\right)dx$$

[From I & II]

$$= \int_0^{\frac{\pi}{2}} \frac{(sinx)^{\frac{3}{2}} + (cosx)^{\frac{3}{2}}}{(sinx)^{\frac{3}{2}} + (cosx)^{\frac{3}{2}}}dx$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} (1)dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}$$

- **Evaluate:** $\boldsymbol{\int_0^{\frac{\pi}{2}} \frac{sin^4x}{sin^4x + cos^4x}dx}$**.**

Solution:

Let $f(x) = \dfrac{sin^4x}{sin^4x + cos^4x}$ ------------- (I)

$\therefore \int_0^{\frac{\pi}{2}} f(x)dx = I$ (Let) ---------- (1)

Now

$$f\left(\frac{\pi}{2} - x\right) = \frac{sin^4\left(\frac{\pi}{2} - x\right)}{sin^4\left(\frac{\pi}{2} - x\right) + cos^4\left(\frac{\pi}{2} - x\right)}$$

$$= \frac{(cosx)^4}{(cosx)^4 + (sinx)^4} \quad \text{------------- (II)}$$

$since \quad sin\left(\frac{\pi}{2} - x\right) = cosx$

$cos\left(\frac{\pi}{2} - x\right) = sinx$

By the property of definite integral

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx \quad \text{------------(2)}$$

From (1) and (2) we get

$$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx = \mathrm{I} \quad \text{------------(3)}$$

$$(1) + (3) \Rightarrow 2I = \int_0^{\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{sin^4x}{sin^4x + cos^4x}dx + \int_0^{\frac{\pi}{2}} \frac{(cosx)^4}{(cosx)^4 + (sinx)^4}dx$$

[From I & II]

$$= \int_0^{\frac{\pi}{2}} \frac{(sinx)^4 + (cosx)^4}{(sinx)^4 + (cosx)^4}dx$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} (1)dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \qquad \therefore I = \frac{\pi}{4}$$

- **Evaluate: $\int_0^{\frac{\pi}{2}} \boldsymbol{log(tanx)dx}$.**

Solution:

Let $\quad I = \int_0^{\frac{\pi}{2}} log(tanx)dx \quad ------------ (1)$

$$= \int_0^{\frac{\pi}{2}} log\left(tan\left(\frac{\pi}{2} - x\right)\right)dx$$

By the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\frac{\pi}{2}} log(cotx)dx \quad ------------ (2)$$

(1)+(2) gives

$$2I = \int_0^{\frac{\pi}{2}} [log(tanx) + log(cotx)]dx$$

$$= \int_0^{\frac{\pi}{2}} [log(tanx)(cotx)]dx = \int_0^{\frac{\pi}{2}} [log(1)]dx \quad [since \ \ log1 = 0] \ \therefore I = 0$$

Area & Volume

Area $= \int_a^b f(x)dx$

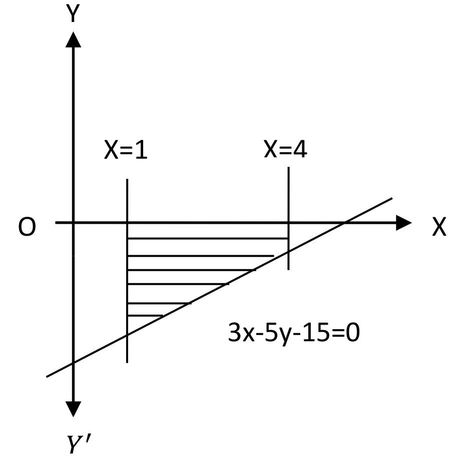
Volume $= \int_a^b \pi[f(x)]^2 dx$

- **Using integration find the area of the region bounded by the lines $\boldsymbol{3x - 5y - 15 = 0, x = 1, x = 4}$ and the $\boldsymbol{x}$-axis.**

Solution:

The line $3x - 5y - 15 = 0$ lies below the x-axis in the interval $x = 1$ and $x = 4$

$\therefore$ Required Area $= \int_1^4 (-y)dx$



$$= \int_1^4 \left(-\frac{1}{5}\right)(3x - 15)dx$$

$$= \frac{3}{5}\int_1^4 (5 - x)dx$$

$$= \frac{3}{5}\left[5x - \frac{x^2}{2}\right]_1^4$$

$$= \frac{3}{5}\left[5(4-1) - \frac{1}{2}(16-1)\right]$$

$$= \frac{3}{5}\left[15 - \frac{15}{2}\right] = \frac{9}{2}$$

- **Using integration find the volume of the solid that results when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a>b>0) is revolved in the minor axis.**

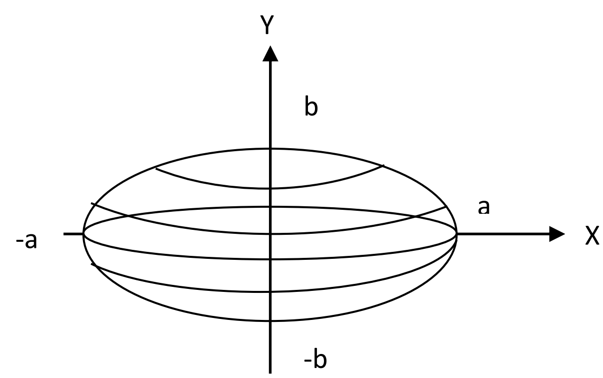
Solution:

Volume of the solid is obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,about the minor axis.

(i.e) the revolution is about Y-axis.

Limit for y varies from $-b$ to b

From the given equation,



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{we get,}$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\Rightarrow x^2 = a^2\left(1 - \frac{y^2}{b^2}\right) = \frac{a^2}{b^2}(b^2 - y^2)$$

$$\therefore \quad \text{Volume} = \int \pi x^2 dy$$

$$= \int_{-b}^{b} \pi \frac{a^2}{b^2}(b^2 - y^2)dy = 2\pi\frac{a^2}{b^2}\int_0^b (b^2 - y^2)dy$$

$$= 2\pi\frac{a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_0^b = 2\pi\frac{a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right]$$

$$= 2\pi\frac{a^2}{b^2}\left[\frac{2b^3}{3}\right] = \frac{4\pi}{3}a^2 b \text{ cubic units.}$$

- **Using integration find the volume of the solid that results when the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a>b>0) is revolved in the major axis.**

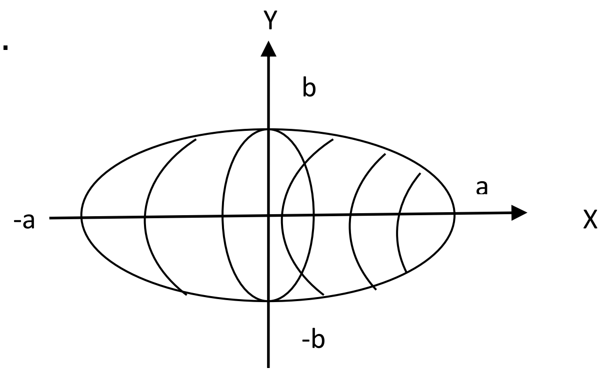
Solution:

Volume of the solid is obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ about the major axis.

(i.e) the revolution is about X-axis.

Limit for x varies from $-a$ to a

From the given equation,



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{we get,}$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore \text{Volume} = \int \pi y^2 dx = \int_{-a}^{a} \pi\frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= 2\pi\frac{b^2}{a^2}\int_0^a (a^2 - x^2)dx$$

$$= 2\pi\frac{b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\frac{b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\pi\frac{b^2}{a^2}\left[\frac{2a^3}{3}\right]$$

$$= \frac{4\pi}{3}b^2 a \text{ cubic units.}$$

- **Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$**

Solution:

The parabola $y^2 = 16x$ is symmetrical about the x axis.

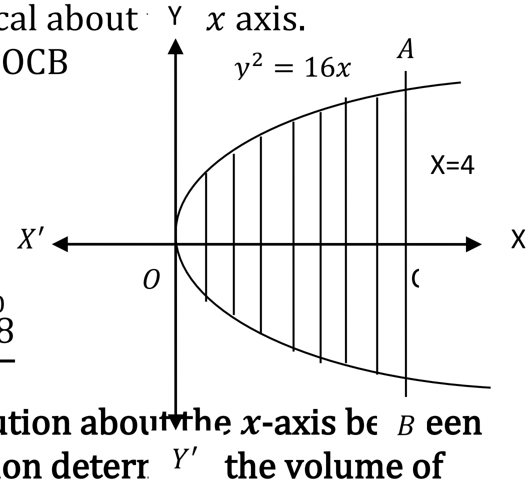
$\therefore$ Required area =area AOCA+ area BOCB

=2(area AOCA)

$$= 2\int_0^4 y dx \;= 2\int_0^4 \sqrt{16x} dx$$

$$= 8\int_0^4 \sqrt{x} dx \quad = 8 \times \frac{2}{3} \times \left[x^{\frac{3}{2}}\right]_0^4$$

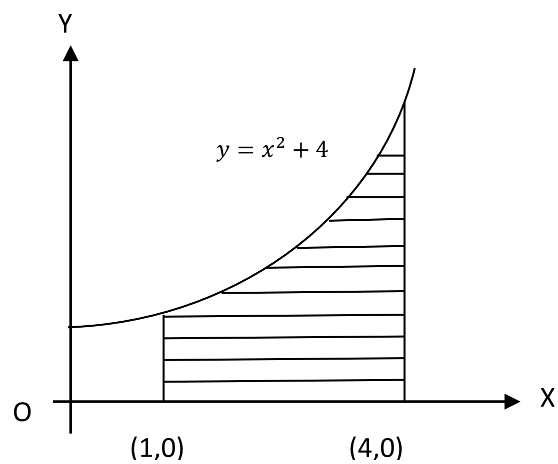
$$= \frac{16}{3} \times (4)^{\frac{3}{2}} \;= \frac{16}{3} \times 8 \;= \frac{128}{3}$$



- **The curve $y = x^2 + 4$ *is* rotated one revolution about the x-axis between the limits $x = 1$ *and* $x = 4$. Using integration determine the volume of the solid of revolution produced. (L3)**

Solution:

Volume $= \int_1^4 \pi y^2 dx$



$$= \int_1^4 \pi(x^2 + 4)^2 dx$$

$$= \int_1^4 \pi(x^4 + 8x^2 + 16) dx$$

$$= \pi\left[\frac{x^5}{5} + 8\frac{x^3}{3} + 16x\right]_1^4$$

$$= \pi\left[\frac{4^5}{5} + 8\frac{4^3}{3} + 16(4) - (\frac{1}{5} + \frac{8}{3} + 16)\right]$$

$$= \pi[439.47 - 18.87]$$

$= 420.6\,\pi$ cubic units.

UNIT-II
MULTIPLE INTEGRALS

Double Integrals

Integration of a function of two variables f(x, y) with respect to x and y in the given region is called Double Integration It is denoted by the symbol

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} \mathrm{f(x, y)dxdy}.$$

In the case of constant limits the order of integration is not taken into account and the integration is performed done first wrt x and then wrt y.

PROBLEMS UNDER CONSTANT LIMITS

1. **Evaluate:** $\int_{1}^{b}\int_{1}^{a}\frac{dx\,dy}{xy}$

Solution:

$$\int_{1}^{b}\int_{1}^{a}\frac{dxdy}{xy} = \int_{1}^{b}\left(\int_{1}^{a}\frac{dx}{x}\right)\frac{dy}{y}$$

$$= \int_{1}^{b}[logx]_1^a\frac{dy}{y}$$

$$= (loga - log1)\int_{1}^{b}\frac{dy}{y}$$

$$= (loga - 0)[logy]_1^b = loga(logb - log1) = logalogb$$

2. **Evaluate:** $\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\sin(\theta + \emptyset)\, d\theta\, d\emptyset$

Solution:

$$=\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}(\sin\theta cos\emptyset + cos\theta sin\emptyset)\, d\theta\, d\emptyset$$

$$=\int_{0}^{\frac{\pi}{2}}[-cos\theta cos\emptyset + sin\theta sin\emptyset]_0^{\frac{\pi}{2}}\, d\emptyset$$

$$=\int_{0}^{\frac{\pi}{2}}[0 + sin\emptyset + cos\emptyset + 0]\, d\emptyset = [-cos\emptyset + sin\emptyset]_0^{\frac{\pi}{2}} \quad = 0 + 1 + 1 + 0 = 2$$

3. **Evaluate:** $\int_{0}^{\pi}\int_{0}^{sin\theta} r\, dr\, d\theta.$

Solution:

$$\int_{0}^{\pi}\int_{0}^{sin\theta} rdrd\theta = \int_{0}^{\pi}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta \; = \frac{1}{2}\int_{0}^{\pi} sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(\frac{1-cos2\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{sin2\theta}{2}\right]_0^{\pi} = \frac{1}{4}\left[\pi - \frac{sin2\pi}{2}\right] \quad = \frac{\pi}{4}$$

4. Evaluate: $\boldsymbol{\int_0^{\pi}\int_0^{a(1+cos\theta)} r\, dr\, d\theta}$ **.**

Solution:

$$\int_0^{\pi}\int_0^{a(1+cos\theta)} r\, dr\, d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1+cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1+2cos\theta+cos^2\theta)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(1+2cos\theta+\frac{1+cos2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi}(2+4cos\theta+1+cos2\theta)d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi}(3+4cos\theta+cos2\theta)d\theta$$

$$= \frac{a^2}{4}\left[3\theta+4sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[3\pi+4sin\pi+\frac{\sin 2\pi}{2}\right] \quad [Since\ sin\pi = sin2\pi = 0]$$

$$= \frac{3\pi a^2}{4}$$

5. Evaluate: $\boldsymbol{\int_0^{3}\int_0^{2} e^{x+y}\, dx\, dy}$**.**

Solution:

$$\int_0^3\int_0^2 e^{x+y}dxdy = \int_0^3\int_0^2 e^x e^y dxdy$$

$$= \int_0^3 [e^x]_0^2 e^y dy \quad = \int_0^3 (e^2 - e^0)\, e^y dy$$

$$= \int_0^3 (e^2-1)\, e^y dy = (e^2-1)[e^y]_0^3$$

$$= (e^2-1)(e^3-1)$$

DOUBLE INTEGRATION UNDER VARIABLE LIMITS

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} \mathrm{f(x,y)dxdy}$,we first integrate f(x,) with respect to x partially treating y as a constant between x_0 and x_1.The resulting function got after the inner integration and substitution of limits will be a function of y between the limits $y_0\ and y_1$.The order in which the integrations are performed in the double integral is as follows.

$$\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} \mathrm{f(x,y)dx}\right] dy$$

PROBLEMS UNDER VARIABLE LIMITS

1. **Evaluate:$\int_0^1\int_0^x dx\, dy$. (L6)**

Solution:

$$\int_0^1\int_0^x dxdy = \int_0^1\int_0^x dydx$$

$$= \int_0^1 [y]_0^x dx$$

$$= \int_0^1 (x-0)dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

2. **Evaluate:$\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\, dx\, dy$. (L6)**

Solution:

$$\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dx\,dy = \int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx$$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\left(\frac{1}{x}\right)}\right]_0^{x^2} dx \ = \int_0^1 \left(xe^{\frac{y}{x}}\right)_0^{x^2} dx$$

$$= \int_0^1 (xe^x - xe^0)dx$$

$$= \int_0^1 (xe^x - x)dx$$

$$= \int_0^1 xe^x dx - \int_0^1 xdx$$

$$= [xe^x]_0^1 - \int_0^1 e^x dx - \int_0^1 xdx$$

$$= (e-0) - [e^x]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= (e-0) - (e-1) - \frac{1}{2}$$

$$= \frac{1}{2}$$

3. **Evaluate:** $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$ **. (L6)**

Solution:

$$\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^2 x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^3 - \frac{x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - 4\frac{x^3}{3} \right) dx$$

$$= \left[\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{1}{3}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) - \frac{4}{3}\frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{1}{3}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) - \frac{x^4}{3} \right]_0^1$$

$$= \left[\frac{1}{\left(\frac{7}{2}\right)} + \frac{1}{3}\frac{1}{\left(\frac{5}{2}\right)} - \frac{1}{3} \right] = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{3}{35}$$

4. **Evaluate:** $\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dx\, dy$. **(L6)**

Solution:

$$\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dx\, dy = \int_0^1 \int_0^{x^2} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{x^2} dx$$

$$= \int_0^1 \left[x^4 + \frac{x^6}{3} \right] dx$$

$$= \left[\frac{x^5}{5} + \left(\frac{1}{3}\right) \frac{x^7}{7} \right]_0^1 = \frac{1}{5} + \frac{1}{21} = \frac{21 + 5}{105} = \frac{26}{105}$$

1. **Evaluate:** $\int_0^1 \int_x^1 \frac{y\, dx\, dy}{x^2 + y^2}$ **.(L6)** **[8marks]**

Solution:

$$\text{Let I} = \int_0^1 \int_x^1 \frac{y\, dx\, dy}{x^2 + y^2}$$

$$= \int_0^1 \left(\int_x^1 \frac{y}{x^2 + y^2}\, dy \right) dx = \int_0^1 \left(\frac{1}{2} \int_x^1 \frac{2y}{x^2 + y^2}\, dy \right) dx$$

$$= \frac{1}{2} \int_0^1 [\log(x^2 + y^2)]_x^1 dx = \frac{1}{2} \int_0^1 (\log(x^2 + 1) - \log(2x^2)) dx$$

$$(i.e)\ \text{I} = \frac{1}{2} \int_0^1 log\left(\frac{x^2 + 1}{2x^2} \right) dx$$

Using integration by parts

Let $u = log\left(\frac{x^2 + 1}{2x^2} \right)$, $\quad dv = dx\ , \Longrightarrow\ v = x$

$$\Longrightarrow\ du = \frac{1}{\left(\frac{x^2+1}{2x^2}\right)} \times \left(\frac{2x^2(2x) - (x^2 + 1)4x}{(2x^2)^2} \right)$$

$$= \frac{2x^2}{x^2 + 1} \times \frac{4x^3 - (x^2 + 1)4x}{(2x^2)^2}$$

$$= \frac{4x(x^2 - x^2 - 1)}{2x^2(x^2 + 1)} = \frac{-2}{x(x^2 + 1)}$$

$$\therefore \mathrm{I} = \frac{1}{2}\left\{\left[x.\,log\left(\frac{x^2+1}{2x^2}\right)\right]_0^1 - \int_0^1 x.\left(\frac{-2}{x(x^2+1)}\right)dx\right\}$$

$$= \frac{1}{2}\left\{\left[x.\,log\left(\frac{x^2+1}{2x^2}\right)\right]_0^1 + (2tan^{-1}x\,)_0^1\right\}$$

$$= \frac{1}{2}[log1 + 2tan^{-1}1] = \frac{1}{2}\left[2\frac{\pi}{4}\right] = \frac{\pi}{4}$$

2. **Evaluate:** $\int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{y\,dx\,dy}{x^2+y^2}$. **(L6)** **[8marks]**

Solution:

$$\text{Let I} = \int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{y\,dx\,dy}{x^2+y^2}$$

$$= \int_0^4 \left(\int_{\frac{y^2}{4}}^{y} \frac{dx}{x^2+y^2}\right) ydy$$

$$= \int_0^4 \left[\frac{1}{y}tan^{-1}\left(\frac{x}{y}\right)\right]_{\frac{y^2}{4}}^{y} ydy = \int_0^4 \left[tan^{-1}\left(\frac{x}{y}\right)\right]_{\frac{y^2}{4}}^{y} dy$$

$$= \int_0^4 \left[tan^{-1}(1) - tan^{-1}\left(\frac{\frac{y^2}{4}}{y}\right)\right]dy$$

$$= \int_0^4 \frac{\pi}{4}dy - \int_0^4 tan^{-1}\left(\frac{y}{4}\right)dy$$

$$(i.e)\;\; I = \frac{\pi}{4}[y]_0^4 - \int_0^4 tan^{-1}\left(\frac{y}{4}\right)dy = \pi - \int_0^4 tan^{-1}\left(\frac{y}{4}\right)dy$$

Consider $\int_0^4 tan^{-1}\left(\frac{y}{4}\right)dy$

Using integration by parts

Let $u = tan^{-1}\left(\frac{y}{4}\right)$, $\quad dv = dy \implies v = y$

$$du = \frac{1}{\left(\frac{y}{4}\right)^2 + 1}.\left(\frac{1}{4}\right) \;\;,$$

$$\therefore \int_0^4 tan^{-1}\left(\frac{y}{4}\right) dy = \left[\left(tan^{-1}\frac{y}{4}\right) y\right]_0^4 - \int_0^4 y.\frac{1}{\left(\frac{y}{4}\right)^2+1}.\frac{1}{4}dy$$

$$= [4tan^{-1}(1) - 0] - \int_0^4 \frac{4^2}{4}.\left(\frac{y}{y^2+4^2}\right) dy$$

$$= 4\frac{\pi}{4} - 2\int_0^4 \frac{2y}{y^2+4^2} dy = \pi - 2[log(y^2+4^2)]_0^4$$

$$= \pi - 2[log32 - log16]$$

$$= \pi - 2log\frac{32}{16} = \pi - 2log2$$

$$\therefore \; I = \pi - \pi + 2log2 = 2log2 = log2^2 = log4$$

3. **Evaluate**: $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dx\, dy$ **. (L6)** **[8marks]**

Solution:

$$\text{Let I} = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dx\, dy$$

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dy\, dx = \int_0^a \left[\frac{y^2}{2}\right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a [(a^2-x^2)-(a-x)^2]dx$$

$$= \frac{1}{2}\int_0^a [a^2-x^2-a^2+2ax-x^2]dx$$

$$= \frac{1}{2}\int_0^a [2ax-2x^2]dx$$

$$= a\int_0^a xdx - \int_0^a x^2dx = a\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^3}{3}\right]_0^a$$

$$= a\left(\frac{a^2}{2}\right) - \frac{a^3}{3} \quad = \frac{a^3}{2} - \frac{a^3}{3} \quad = \frac{a^3}{6}$$

4. **Evaluate:** $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$**. (L6)** **[8marks]**

Solution:

$$\text{Let I} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, d\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2}\, dy\, dx$$

$$= \int_0^a \left(0 + \frac{a^2-x^2}{2} sin^{-1}(1) - (0+0)\right) dx$$

$$= \int_0^a \frac{a^2-x^2}{2}\frac{\pi}{2} dx = \frac{\pi}{4}\int_0^a (a^2-x^2)dx$$

$$= \frac{\pi}{4}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{\pi}{4}.\frac{2a^3}{3} = \frac{\pi a^3}{6}$$

TRIPLE INTEGRALS

Extending the concept of double integrals one step further ,we get the triple integral $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz$.

PROBLEMS UNDER CONSTANT LIMITS

1. **Evaluate:** $\int_0^1\int_0^2\int_0^3 \boldsymbol{xyz\,dx\,dy\,dz}$**. (L6)**

Solution:

$$\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz = \int_0^1\int_0^2 \left[\frac{x^2}{2}\right]_0^3 yz\,dydz = \int_0^1\int_0^2 \left(\frac{9}{2}\right) yz\,dydz$$

$$= \frac{9}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 z\,dz = \frac{9}{2}\int_0^1 \left(\frac{4}{2}\right) z\,dz$$

$$= \left(\frac{9}{2}\right)\left(\frac{4}{2}\right)\int_0^1 z\,dz = \left(\frac{9}{2}\right)\left(\frac{4}{2}\right)\left[\frac{z^2}{2}\right]_0^1$$

$$= \left(\frac{9}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right) = \frac{9}{2}$$

2. **Evaluate:** $\int_0^a \int_0^b \int_0^c xyz\, dx\, dy\, dz$. **(L6)**

<u>**Solution:**</u>

$$\int_0^a \int_0^b \int_0^c xyz\, dx\, dy\, dz = \left(\int_0^a x dx\right)\left(\int_0^b y dy\right)\left(\int_0^c z dz\right)$$

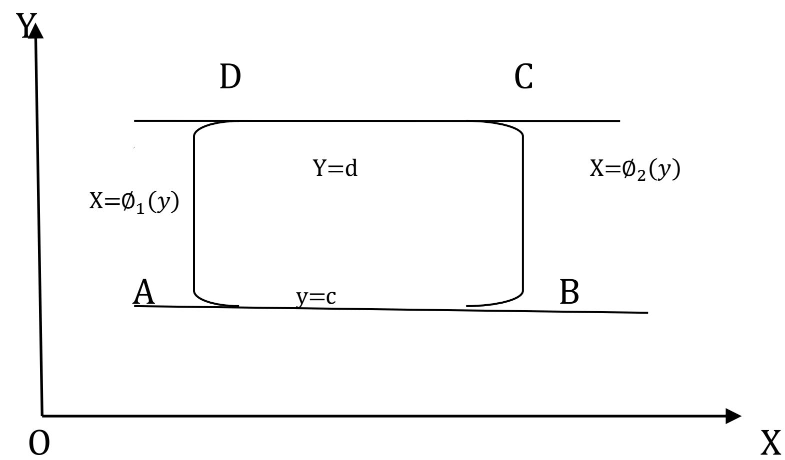
$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{a^2}{2} \times \frac{b^2}{2} \times \frac{c^2}{2} = \frac{(abc)^2}{8}$$

<u>SKETCHING THE REGION OF INTEGRATION</u>

Consider the double integral $\int_c^d \int_{\emptyset_1}^{\emptyset_2} f(x,y)dxdy$.As stated above x varies from $\emptyset_1(y) to\ \emptyset_2(y)$ and y varies from c to d.

$\emptyset_1(y) \leq \emptyset_2(y)$ and c$\leq y \leq d$.These inequalities determine a region in the xy-plane whose boundaries are the curves x= $\emptyset_1(y)$, $x = \emptyset_2(y)$ and the lines y=c, y=d and which is shown on the following figure.This region ABCD is known as the region of integration of the above double integral.



Similarly for the double integral $\int_c^d \int_{\emptyset_1}^{\emptyset_2} f(x,y)dydx$,we interchange x and y accordingly in the above diagram.

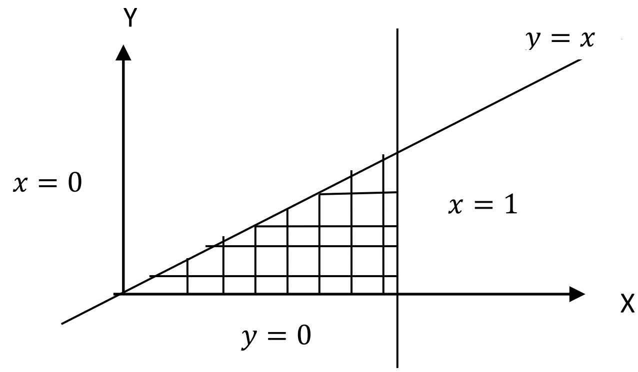
PROBLEMS

1. **Draw the region of integration** $\int_0^1 \int_0^x f(x,y)\,dy\,dx$. **(L3)**

Solution:

The limits are given by

$$y = 0, y = x$$
$$x = 0, x = 1$$

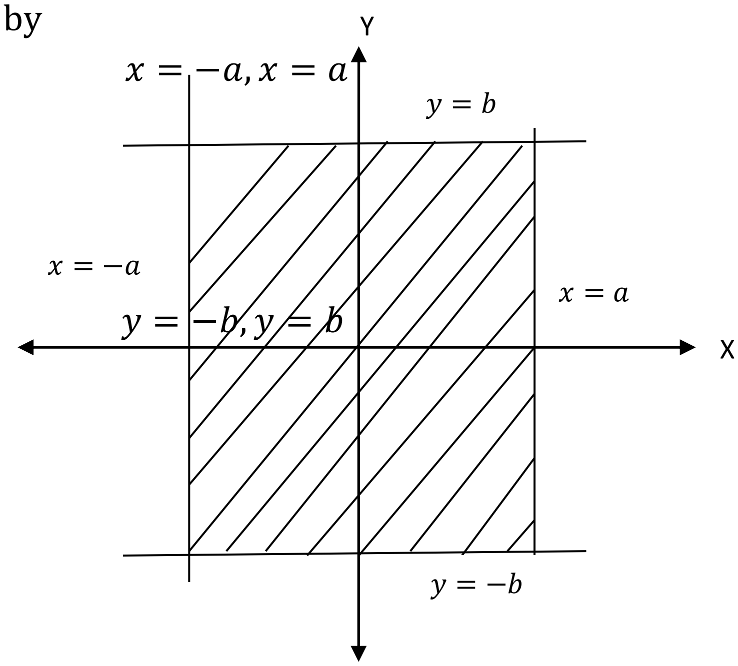


2. **Draw the region of integration** $\int_{-b}^{b} \int_{-a}^{a} f(x,y)\,dx\,dy$. **(L3)**

Solution:

The limits are given by

$$x = -a, x = a$$
$$y = -b, y = b$$



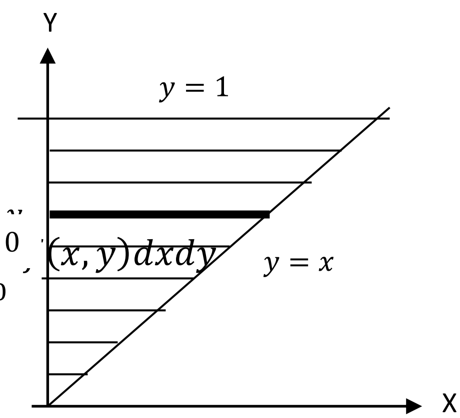
LIMITS OF INTEGRATION

1. **Determine the limits of integration in $\iint_R f(x,y)dxdy$ where R is the region in the first quadrant and bounded by $x = 0, x = y, y = 1$.(L6)**

Solution:

Given

$x = 0, x = y, y = 1.$



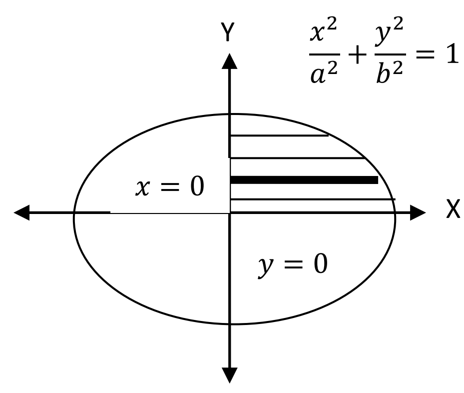
$$\therefore \iint_R f(x,y)dxdy = \int_0^1 \int_0^y f(x,y)dxdy$$

1. **Determine the limits of integration in $\iint_R f(x,y)dxdy$ where R is the region in the first quadrant and bounded by $x = 0, y = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.(L6)**

Solution:

Given $x = 0, y = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore I = \int_0^a \int_0^{\sqrt{a^2\left(1-\frac{y^2}{b^2}\right)}} f(x,y)dxdy$$



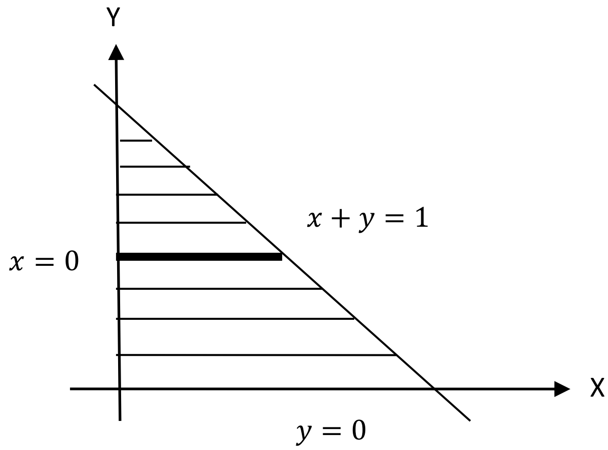
2. **Determine the limits of integration in $\iint_R f(x,y)dxdy$ where R is the region in the first quadrant and bounded by $x = 0, y = 0$, $x + y = 1$. (L6)**

<u>Solution:</u>

Given $x = 0, y = 0, x + y = 1.$

The limits of the integration is given by

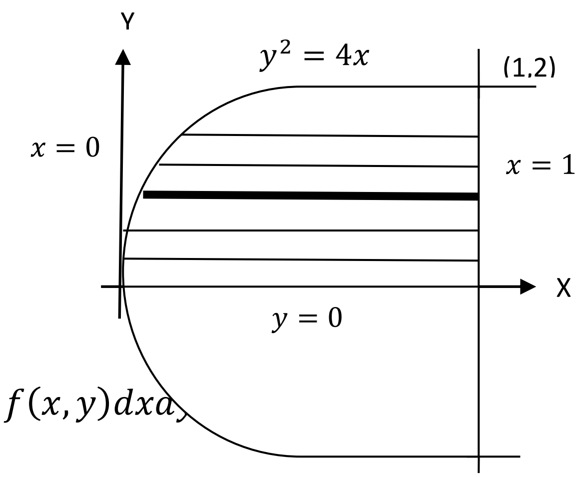
$$I = \int_0^1 \int_0^{1-y} f(x,y)dxdy$$



3. **Determine the limits of integration in $\iint_R f(x,y)dxdy$ where R is the region in the first quadrant and bounded by $x = 1, y = 0$, $y^2 = 4x$.(L6)**

<u>Solution:</u>

Given $x = 1, y = 0, y^2 = 4x.$
The limits of the integration
is given by



$$I = \int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)dxdy$$

<u>CHANGE OF ORDER OF INTEGRATION</u>

When the limits for inner integration are functions of a variable ,the change in the order of integration will result inchange of the limits of integration.

i.e.the double integral $\int_c^d \int_{\emptyset_1(y)}^{\emptyset_2(y)} f(x,y)dxdy$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$when the order of integration is changed.This process of converting aa given double integral into its equivalent double integral by changing the order of integration is called "*change of order of integration*".

PROBLEMS

1. **Change the order of integration in $\int_0^a \int_0^x f(x,y)\,dx\,dy$. (L3)**

Solution:

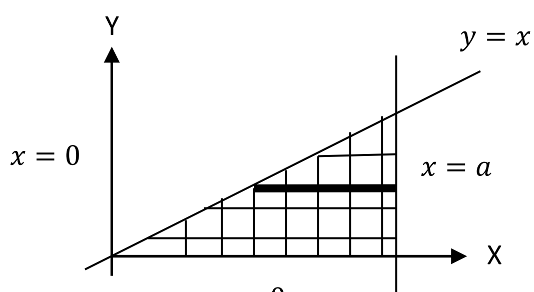
Let I= $\int_0^a \int_0^x f(x,y)\,dx\,dy = \int_0^a \int_0^x f(x,y)\,dy\,dx$

The region of integration is given by

$$y = 0, y = x$$
$$x = 0, x = a$$

After changing the order of integration,the integral I becomes

$$I = \int_0^a \int_y^a f(x,y)\,dxdy$$



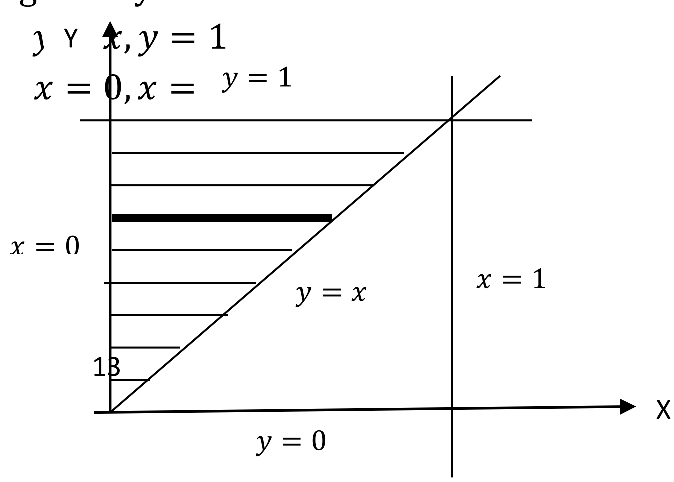
2. **Change the order of integration in $\int_0^1 \int_x^1 f(x,y)\,dx\,dy$. (L3)**

Solution:

Let I= $\int_0^1 \int_x^1 f(x,y)\,dx\,dy = \int_0^1 \int_x^1 f(x,y)\,dy\,dx$

The region of integration is given by

$$y = x, y = 1$$
$$x = 0, x = 1$$



After changing the order of integration,the integral I becomes

$$I = \int_0^1 \int_0^y f(x, y)\, dx\, dy$$

3. **Change the order of integration in $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$. (L3)**

<u>**Solution:**</u>

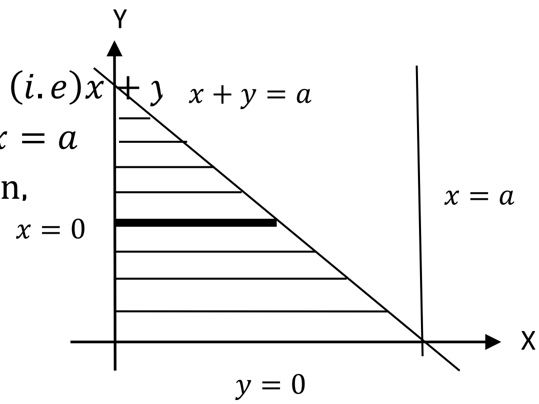
Let I= $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$

$= \int_0^a \int_0^{a-x} f(x, y)\, dy\, dx$

The region of integration is given by

$y = 0, y = a - x \quad (i.e) x + y$

$x = 0, x = a$



After changing the order of integration,
the integral I becomes

$$I = \int_0^a \int_0^{a-y} f(x, y)\, dx\, dy$$

4. **Change the order of integration $\int_0^1 \int_0^1 \frac{x}{x^2+y^2}\, dydx$. (L3)**

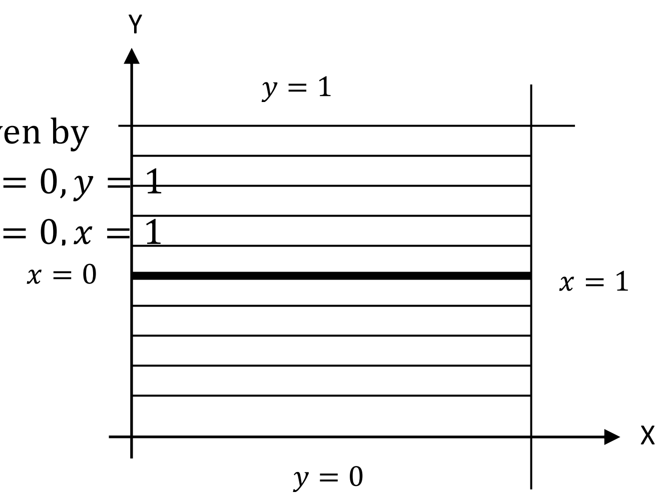
<u>**Solution:**</u>

Let I= $\int_0^1 \int_0^1 \frac{x}{x^2+y^2}\, dydx$

The region of integration is given by

$y = 0, y = 1$

$x = 0, x = 1$



After changing the order of integration,the integral I becomes

$$I = \int_0^1 \int_0^1 \frac{x}{x^2 + y^2}\, dxdy$$

5. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)\, dy\, dx$.(L3)

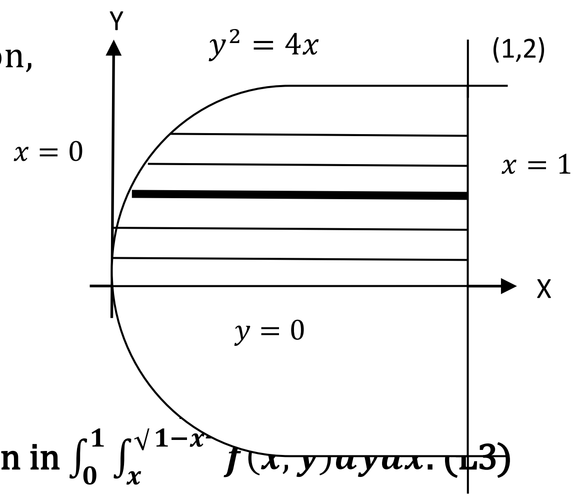
<u>Solution:</u>

Let $I = \int_0^1 \int_0^{2\sqrt{x}} f(x,y) dydx$

The region of integration is given by

$$y = 0, y = 2\sqrt{x} (i.e) y^2 = 4x$$

$x = 0, x = 1$



After changing the order of integration,

the integral I becomes

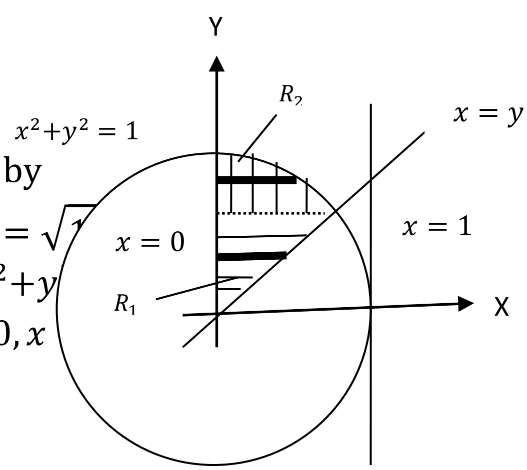
$$I = \int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y) dx\, dy$$

6. Change the order of integration in $\int_0^1 \int_x^{\sqrt{1-x^2}} f(x,y)dydx$. (L3)

<u>Solution:</u>

Let $I = \int_0^1 \int_x^{\sqrt{1-x^2}} f(x,y) dydx$

The region of integration is given by



$y = x, y = \sqrt{1}$

$(i.e) x^2+y$

$x = 0, x$

After changing the order of integration
the integral I becomes

I= $\int\int f dxdy + \int\int f dxdy$

$R_1 R_2$

$$(i.e)\ \text{I} = \int_0^{\frac{1}{\sqrt{2}}} \int_0^{y} f(x,y)dx\,dy + \int_{\frac{1}{\sqrt{2}}}^{1} \int_0^{\sqrt{1-y^2}} f(x,y)dx\,dy$$

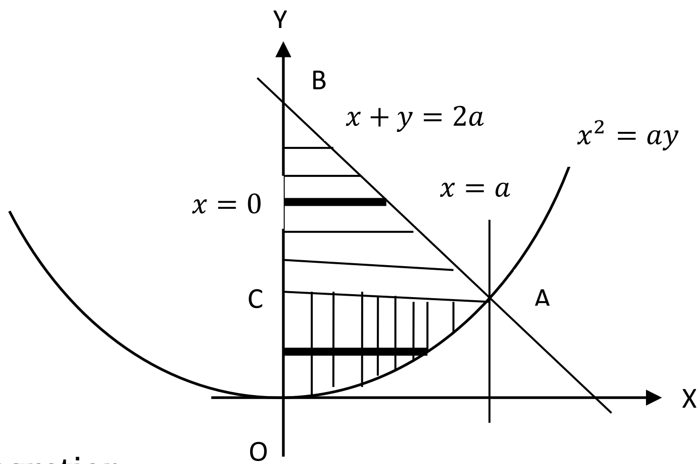
7. Change the order of integration $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy$

and hence evaluate it. (L6) **[16marks]**

Solution:

$$\text{Let } I = \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$$

$$= \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$$



The region of integration is

$$y = \frac{x^2}{a}, \qquad y = 2a - x$$

$(i.e) x^2 = ay, y = 2a - x$

$\& x = 0, x = a$

After changing the order of integration,
the integral I becomes

I= $\int\int xydxdy$

R

$= \int\int xydxdy + \int\int xydxdy$

OAC CAB

$$= \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy + \int_a^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{1}{2}\left[\int_0^a ay^2dy + \int_a^{2a} y(2a-y)^2dy\right]$$

$$= \frac{1}{2}\left[\int_0^a ay^2 dy + \int_a^{2a} y(4a^2 - 4ay + y^2)dy\right]$$

$$= \frac{1}{2}\left[a\int_0^a y^2 dy + \int_a^{2a} (4a^2y - 4ay^2 + y^3)dy\right]$$

$$= \frac{1}{2}\left[a\left[\frac{y^3}{3}\right]_0^a + 4a^2\left[\frac{y^2}{2}\right]_a^{2a} - 4a\left[\frac{y^3}{3}\right]_a^{2a} + \left[\frac{y^4}{4}\right]_a^{2a}\right]$$

$$= \frac{1}{2}\left\{a.\frac{a^3}{3} + 4a^2\left(\frac{4a^2}{2} - \frac{a^2}{2}\right) - 4a\left(\frac{8a^3}{3} - \frac{a^3}{3}\right) + \left(\frac{16a^4}{4} - \frac{a^4}{4}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{a^4}{3} + 4a^2\left(\frac{3a^2}{2}\right) - 4a\left(\frac{7a^3}{3}\right) + \frac{15a^4}{4}\right\}$$

$$= \frac{1}{2}\left\{\frac{a^4}{3} + \frac{12a^4}{2} - \frac{28a^4}{3} + \frac{15a^4}{4}\right\}$$

$$= \frac{1}{2}\left\{\frac{9a^4}{12}\right\} \quad = \frac{3}{8}a^4$$

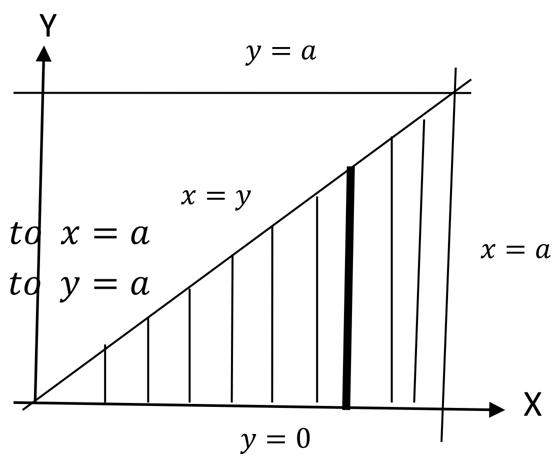
8. **Change the order of integration and hence evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}$.(L6)**

[16marks]

Solution:

Let $I = \int_0^a \int_y^a \frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}$

The region of integration is



$x = y$ to $x = a$

$y = 0$ to $y = a$

After changing the order of the integration,the integral I becomes

$$I = \int\int \frac{x\,dy\,dx}{\sqrt{(x^2+y^2)}}$$

$$= \int_0^a \int_0^x \frac{x}{\sqrt{(x^2+y^2)}}dydx$$

$$= \int_0^a x\left\{\log\left(y+\sqrt{y^2+x^2}\right)\right\}_{y=0}^{y=x} dx$$

$$= \int_0^a x\{\log(x + x\sqrt{2}) - logx\}dx$$

$$= \int_0^a x.\log\left(\frac{x(1+\sqrt{2})}{x}\right) dx$$

$$= \log(1+\sqrt{2})\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2}{2}\log(1+\sqrt{2})$$

9. **Change the order of integration and hence evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}\,dx\,dy}{y}$. (L6)**

[16marks]

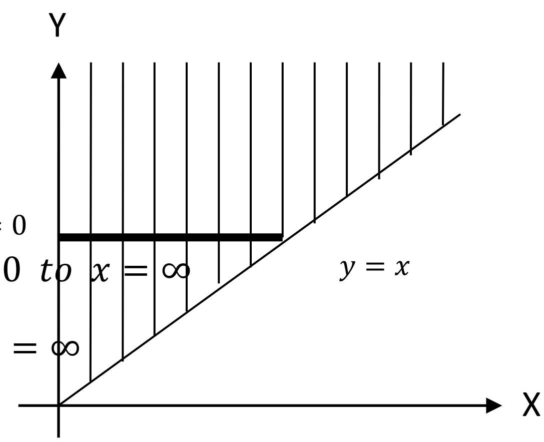
<u>Solution:</u>

$$\text{Let}I = \int_0^\infty \int_x^\infty \frac{e^{-y}\,dy\,dx}{y}$$

The region of integration is

$x = 0\ to\ x = \infty$

$y = x\ to\ y = \infty$



After changing the order of integration, the integral I becomes

$$I = \int\int \frac{e^{-y}\,dx\,dy}{y} = \int_0^\infty \int_0^y \frac{e^{-y}\,dx\,dy}{y}$$

$$= \int_0^\infty \frac{e^{-y}}{y}[x]_0^y dy = \int_0^\infty \frac{e^{-y}}{y}.ydy$$

$$= \int_0^\infty e^{-y}dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

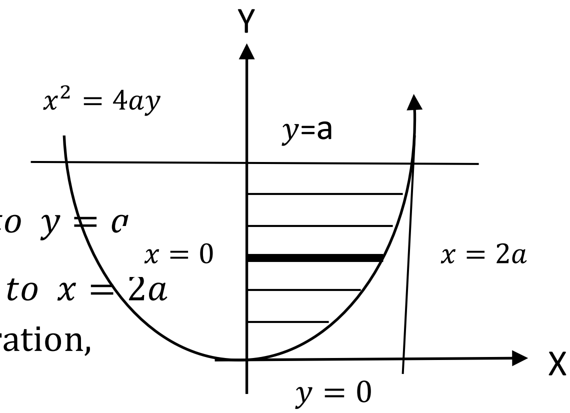
$$= -1[e^{-\infty} - e^0] = -1[0-1] = 1$$

10. **Change the order of integration $\int_0^{2a}\int_{\frac{x^2}{4a}}^{a}(x+y)\,dx\,dy$ and hence evaluate. (L6)** **[16marks]**

<u>**Solution:**</u>

$$\text{Let}I = \int_0^{2a}\int_{\frac{x^2}{4a}}^{a}(x+y)\,dy\,dx$$

The region of integration is



$$y = \frac{x^2}{4a} \quad to \quad y = a$$

$$x = 0 \quad to \quad x = 2a$$

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \int_0^{a}\int_0^{\sqrt{4ay}}(x+y)\,dx\,dy$$

$$= \int_0^{a}\left[\frac{x^2}{2} + yx\right]_0^{\sqrt{4ay}} dy$$

$$= \int_0^{a}\left[\frac{4ay}{2} + y\sqrt{4ay}\right] dy$$

$$= 2a\left[\frac{y^2}{2}\right]_0^{a} + 2\sqrt{a}\int_0^{a} y^{\frac{3}{2}}dy$$

$$= a(a^2) + 2\sqrt{a}\left[\frac{y^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right]_0^{a} = a^3 + \frac{4}{5}a^{\left(\frac{1}{2}+\frac{5}{2}\right)}$$

$$= a^3 + \frac{4}{5}a^3 = \frac{9a^3}{5}$$

11. **Change the order of integration $\int_0^{4}\int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$ and hence evaluate it. (L6)** **[16marks]**

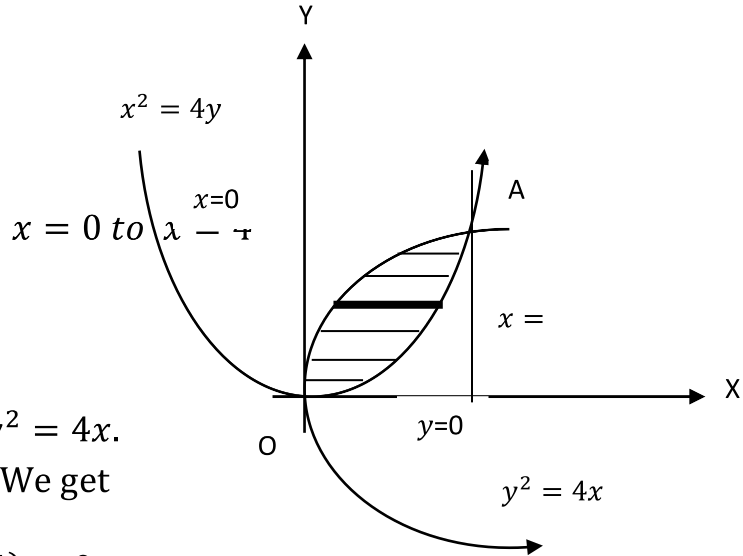
<u>**Solution:**</u>

Let $\mathrm{I}=\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx$

The region of integration is

$$y=\frac{x^2}{4}\quad to\quad y=2\sqrt{x}$$

$(i.e) x^2=4y, y^2=4x$



The points of intersection of the two parabolas are obtained by solving the equations $x^2=4y$ & $y^2=4x$.

By solving these equations ,We get

$$\left(\frac{x^2}{4}\right)^2=4x \quad \Longrightarrow \quad x(x^3-64)=0$$

$$\Longrightarrow x=0\ , x=4$$

When $x=0\ , y=0$

$$x=4\ , y=4$$

$(i.e)$The points of intersections are $O(0,0)$ &$A(4,4)$.

After changing the order of integration the integral I becomes,

$$\mathrm{I}=\int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\, dy=\int_0^4 [x]_{\frac{y^2}{4}}^{2\sqrt{y}} dy$$

$$=\int_0^4\left(2\sqrt{y}-\frac{y^2}{4}\right)dy=\left[2\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1}{4}\frac{y^3}{3}\right]_0^4$$

$$=\left[\frac{4}{3}y^{\frac{3}{2}}-\frac{y^3}{12}\right]_0^4 \quad =\frac{4}{3}(4)^{\frac{3}{2}}-\frac{4^3}{12}$$

$$=\frac{4}{3}\times 8-\frac{16}{3}\quad =\frac{32}{3}-\frac{16}{3}\quad =\frac{16}{3}$$

12. Change the order of integration $\int_0^1 \int_y^{2-y} xy\, dx\, dy$ and hence evaluate it. (L6)

[16marks]

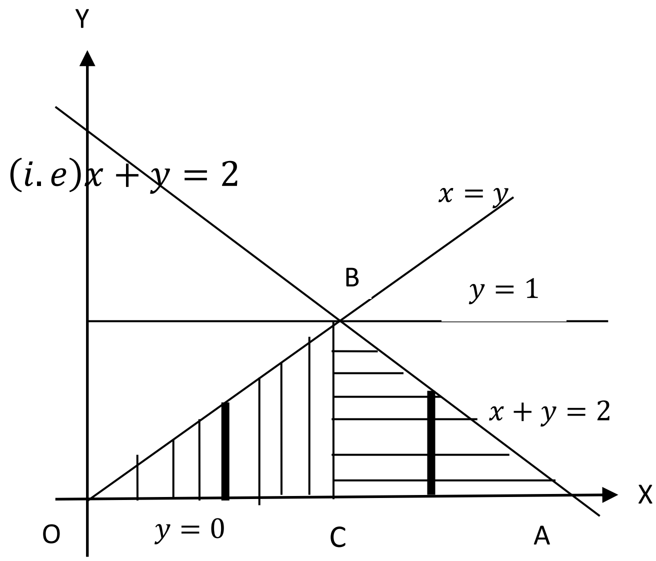
Solution:

Let $I = \int_0^1 \int_y^{2-y} xy\, dx\, dy$

The region of integration is bounded by the lines

$x = y, x = 2 - y$

& $y = 0, y = 1$



After changing the order of integration, the integral I becomes

I= $\int \int xydydx$

$$= \int \int xydydx + \int \int xydydx$$

OCB BCA

$$= \int_0^1 \int_0^x xydydx + \int_1^2 \int_0^{2-x} xydydx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^x dx + \int_1^2 x \left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \int_1^2 \frac{x}{2}(2 - x)^2 dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \int_1^2 \frac{x}{2}(4 - 4x + x^2) dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \frac{1}{2}\int_1^2 (4x - 4x^2 + x^3) dx$$
$$= \left[\frac{x^4}{8}\right]_0^1 + \frac{1}{2}\left[4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$$
$$= \frac{1}{8} + \frac{1}{2}\left[2 \times 4 - \frac{4}{3} \times 8 + \frac{16}{4} - 2 + \frac{4}{3} - \frac{1}{4}\right]$$
$$= \frac{1}{8} + \frac{1}{2}\left[\frac{5}{12}\right] \qquad = \frac{8}{24} \qquad = \frac{1}{3}$$

VOLUME INTEGRAL

Let V be a region of space ,bounded by a closed surface. Let $f(x, y, z)$ be a continuous function defined at all points of V. Then

$\int\int\int f(x, y, z) dxdydz$ is called the volume integral of $f(x, y, z)$ over the region V.

1. Give the volume bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$ in triple integration. (L2)

Solution:

Given $x^2 + y^2 + z^2 \leq 1$

The limits of $z, y \& x$ are

$$z = 0 \quad to \quad z = \sqrt{1 - x^2 - y^2}$$
$$y = 0 \quad to \quad y = \sqrt{1 - x^2}$$
$$x = 0 \quad to \quad x = 1$$

$$\text{Volume} = \int\int\int dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

2. Give the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in triple integration. (L2)

Solution:

Given $x^2 + y^2 + z^2 = a^2$

The limits of $z, y \& x$ are

$$z = 0 \quad to \quad z = \sqrt{a^2 - x^2 - y^2}$$
$$y = 0 \quad to \quad y = \sqrt{a^2 - x^2}$$
$$x = 0 \quad to \quad x = a$$

$$\text{Volume} = \int\int\int dzdydx = 8 \times \text{Volume in the first octant region}$$
$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

1. **Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ without transformation. (L1)**

[16marks]

<u>**Solution:**</u>

To find the limits in the first octant:

$$z = 0\ , \qquad x^2 + y^2 + z^2 = a^2$$
$$(i.e)z^2 = a^2 - x^2 - y^2$$
$$(i.e)z = \sqrt{a^2 - x^2 - y^2}$$
$$y = 0\ , \qquad x^2 + y^2 = a^2$$
$$(i.e)y^2 = a^2 - x^2$$
$$(i.e)y = \sqrt{a^2 - x^2}$$
$$x = 0\ , \qquad x^2 = a^2\ (i.e)x = a$$

Volume of the sphere= 8 ×Volume of one octant region

$$= 8\int\int\int dzdydx$$
$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dydx$$
$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$
$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2 - x^2) - y^2}\,dydx$$
$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2 - x^2}\right)^2 - y^2}\ dydx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{(a^2 - x^2) - y^2} + \frac{(a^2 - x^2)}{2} sin^{-1}\frac{y}{\sqrt{(a^2 - x^2)}}\right]_0^{\sqrt{a^2 - x^2}} dx$$

$$\left[\text{Since} \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\therefore \text{ Volume of the sphere } = 8\int_0^a \frac{(a^2 - x^2)}{2}\frac{\pi}{2} dx$$

$$= 8\int_o^a \frac{(a^2 - x^2)}{2}\frac{\pi}{2}\ dx$$

$$= \frac{8\pi}{4}\int_o^a (a^2 - x^2)dx \quad = 2\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 2\left[a^3 - \frac{a^3}{3}\right] = 2\left[\frac{2a^3}{3}\right] = \frac{4a^3}{3}$$

2. **Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which lies in the first octant using triple integration. (L1) 16marks]**

<u>**Solution:**</u>

To find the limits in the first octant:

$$z = 0\ , \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$(i.e)\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$(i.e)z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$y = 0\ , \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(i.e)\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$(i.e)y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$x = 0\ , \qquad \frac{x^2}{a^2} = 1 \qquad (i.e)x = a$$

$$\therefore \quad \text{Volume} = \int\int\int dz dy dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy dx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dy dx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\, dy dx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\right)^2-y^2}\, dy dx$$

$$\left[\text{Since} \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\therefore \text{ Volume} = \frac{c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+\frac{\left(\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\right)^2}{2} sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{c}{b}\int_0^a \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2} sin^{-1}(1) dx \quad = \frac{c}{b}\int_0^a \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}.\frac{\pi}{2} dx$$

$$= \frac{c}{b}\cdot\frac{\pi}{4}\cdot\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx \qquad = \frac{\pi bc}{4a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{\pi bc}{4a^2}\left[a^3 - \frac{a^3}{3}\right] \qquad = \frac{\pi bc}{4a^2}\times\frac{2a^3}{3} = \frac{\pi abc}{6}$$

POLAR COORDINATES

Put $x = rcos\theta, y = r\,sin\theta,$

$dxdy = rdrd\theta$, we have the polar co-ordinates.

4. **Change into polar coordinates:** $\int_0^a \int_y^a f(x,y)\,dx\,dy$ **. (L3)**

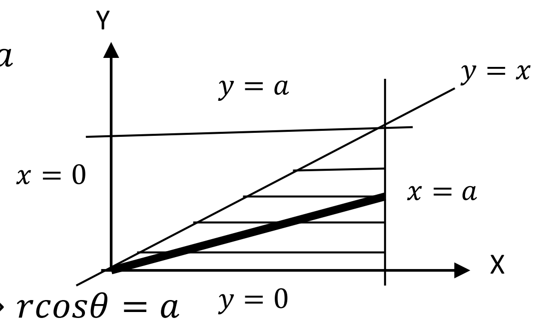
Solution:

Let I= $\int_0^a \int_y^a f(x,y)\,dx\,dy$

The region of integration is given by

$x = y, x = a$

$y = 0 y = a$



Put $x = rcos\theta, y = r\,sin\theta,$

$dxdy = rdrd\theta$

The limits of r& θ are given by

$$r = 0\ , \qquad x = a \Rightarrow rcos\theta = a$$

$$\Rightarrow r = \frac{a}{cos\theta}(i.e)r = asec\theta$$

$$\theta = 0\ ,\ \ \theta = \frac{\pi}{4}$$

After changing into polar coordinates

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \int_0^{asec\theta} f(r,\theta).r\,dr\,d\theta$$

5. **Change into polar co-ordinates** $\int_0^\infty \int_0^y dydx$ **.(L3)**

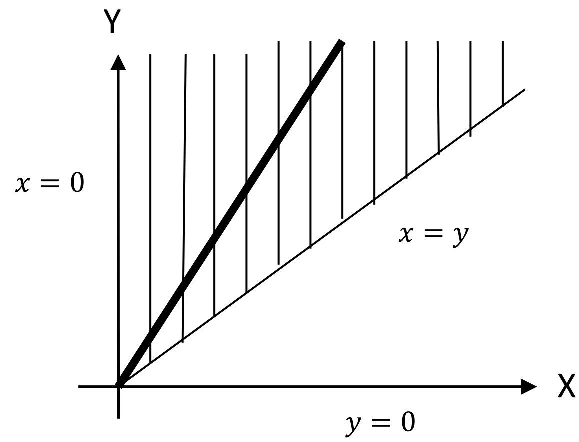
Solution:

Let $\mathrm{I} = \int_0^\infty \int_0^y dydx = \int_0^\infty \int_0^y dxdy$

The region of integration is given by

$x = 0 \quad , \quad x = y$

$y = 0 \quad , \quad y = \infty$



To change into polar

co-ordinates

Put $x = rcos\theta, y = rsin\theta,$

$dxdy = rdrd\theta$

$r = 0\,, r = \infty$ and $\theta = \frac{\pi}{4}\,, \theta = \frac{\pi}{2}$

$$\therefore I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^\infty f(r,\theta)\ rdrd\theta$$

3. **Change into polar coordinates and then evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.(L6)** **[16marks]**

<u>Solution:</u>

Let $\mathrm{I} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

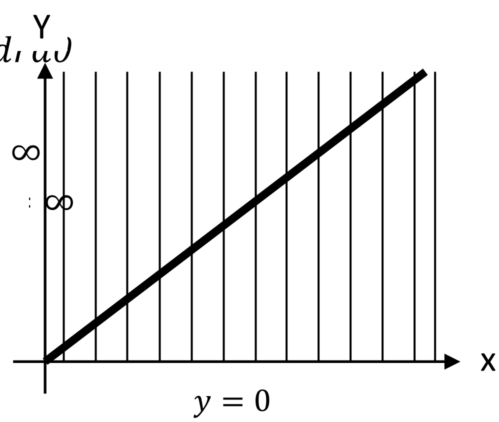
Put $x = rcos\theta, y = r\,sin\theta,$

$dxdy = rdrd\theta$

The region of integration is

$x = 0\ \ to\ \ x = \infty$

$y = 0\ \ to\ \ y = \infty$



After changing into polar co-ordinates the integral I becomes

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\ dr\, d\theta$$

Put $r^2 = t$

$\Longrightarrow 2rdr = dt$

$$\therefore\ rdr = \frac{dt}{2}$$

When $r = 0, t = 0$

$$r = \infty, t = \infty$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \left(e^{-t}\frac{dt}{2}\right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} [-e^{-t}]_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [-e^{-\infty} + e^0] d\theta \quad = \frac{1}{2}\int_0^{\frac{\pi}{2}} [0+1] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \quad = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} \quad = \frac{1}{2}.\frac{\pi}{2} \quad = \frac{\pi}{4}$$

4. Change into polar coordinates and then evaluate

$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{(x^2+y^2)}} dx\, dy$.**(L6)** **[16marks]**

Solution:

Let $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{(x^2+y^2)}} dy\, dx$

Put $x = r\cos\theta, y = r\sin\theta,$

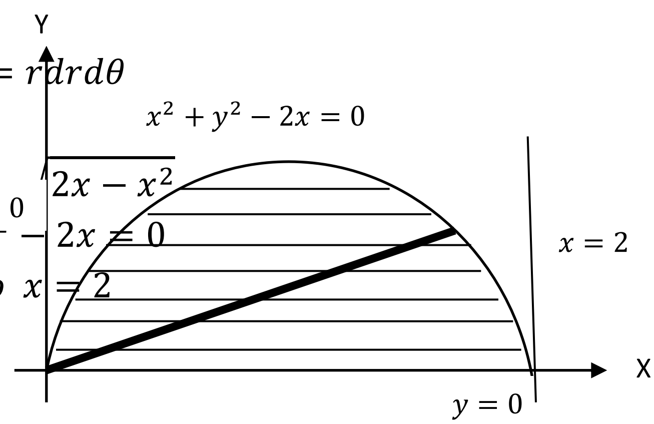
$$dxdy = rdrd\theta$$

The region of integration is

$y = 0$ to $\sqrt{2x-x^2}$

$(i.e) x^2 + y^2 - 2x = 0$

$x = 0$ to $x = 2$



Limits of r & θ are

$$r = 0, x^2 + y^2 - 2x = 0$$

$$(i.e) r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$$

$$\Rightarrow r^2 - 2r\cos\theta = 0 \qquad \Rightarrow r(r - 2\cos\theta) = 0$$

$$\Rightarrow r = 0, (r - 2\cos\theta) = 0 \quad \Rightarrow r = 2\cos\theta$$

$(i.e)\ r = 0, r = 2\cos\theta$ and $\theta = 0, \theta = \frac{\pi}{2}$

After changing into polar co-ordinates the integral I becomes,

$$I = \int_0^{\frac{\pi}{2}} \int_0^{2cos\theta} \frac{(rcos\theta)}{r} r \; dr \, d\theta \quad = \int_0^{\frac{\pi}{2}} \int_0^{2cos\theta} rcos\theta \; dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} cos\theta \left[\frac{r^2}{2}\right]_0^{2cos\theta} d\theta \quad = \frac{4}{2}\int_0^{\frac{\pi}{2}} cos^3\theta d\theta \;= \frac{4}{2} . \frac{(3-1)}{3} = \frac{4}{3}$$

5. **Change into polar coordinates and then evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$. (L6)** [16marks]

<u>**Solution:**</u>

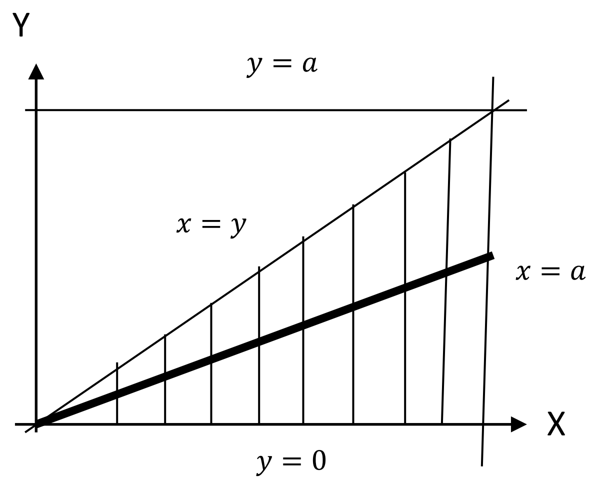
$$\text{Let } I = \int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$$

Put $x = rcos\theta, y = r\, sin\theta, dxdy = rdrd\theta$

The region of integration is

$x = y \;\; to \;\; x = a$

$y = 0 \; to \;\; y = a$



Limits of r & θ are

$r = 0, r = asec\theta$

$\theta = 0, \theta = \frac{\pi}{4}$

After changing in to polar co-ordinates the integral I becomes,

$$I = \int_0^{\frac{\pi}{4}} \int_0^{asec\theta} \frac{(rcos\theta)}{r^2} r \; dr \, d\theta = \int_0^{\frac{\pi}{4}} \int_0^{asec\theta} cos\theta \; dr \, d\theta$$

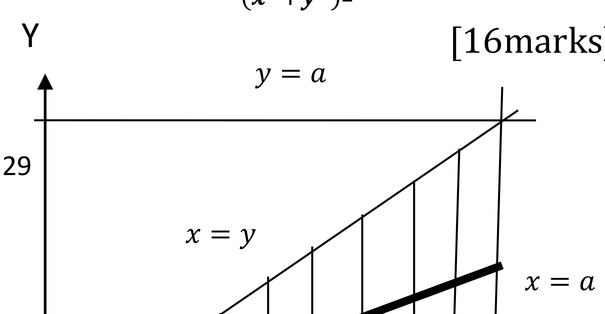
$$= \int_0^{\frac{\pi}{4}} cos\theta [r]_0^{asec\theta} d\theta = \int_0^{\frac{\pi}{4}} cos\theta . asec\theta d\theta$$

$$= a\int_0^{\frac{\pi}{4}} cos\theta . \frac{1}{cos\theta} d\theta \; = a\int_0^{\frac{\pi}{4}} d\theta \; = a[\theta]_0^{\frac{\pi}{4}} \; = a\left(\frac{\pi}{4}\right) = \frac{\pi a}{4}$$

6. **Change into polar coordinates and evaluate $\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} dx\, dy$. (L6)**

[16marks]

<u>**Solution:**</u>



Let I $= \int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} dx\, dy$

Put $x = r cos\theta, y = r\, sin\theta, \ \ dxdy = rdrd\theta$

The region of integration is

$$x = y \ \ to \ \ x = a$$
$$y = 0 \ \ to \ \ y = a$$

The limits of r & θ

$$r = 0, r = asec\theta$$
$$\theta = 0, \theta = \frac{\pi}{4}$$

After changing in to polar co-ordinates the integral I becomes

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \int_0^{asec\theta} \frac{(rcos\theta)^2}{(r^2)^{\frac{3}{2}}} r\ dr\, d\theta = \int_0^{\frac{\pi}{4}} \int_0^{asec\theta} cos^2\theta\ dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} cos^2\theta [r]_0^{asec\theta} d\theta = \int_0^{\frac{\pi}{4}} cos^2\theta . asec\theta d\theta$$

$$= a \int_0^{\frac{\pi}{4}} cos^2\theta . \frac{1}{cos\theta} d\theta \ = a \int_0^{\frac{\pi}{4}} cos\theta d\theta$$

$$= a[sin\theta]_0^{\frac{\pi}{4}} = a\left[sin\frac{\pi}{4} - sin 0\right]$$

$$= a\left[\frac{1}{\sqrt{2}} - 0\right] = \frac{a}{\sqrt{2}}$$

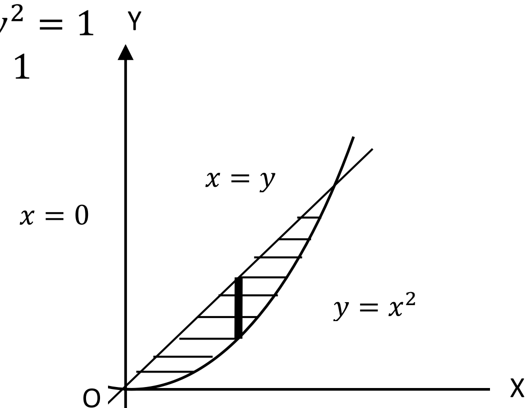
7. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical Polar coordinates. (L6) [16marks]

<u>Solution:</u>

The region of integration is bounded by

$$z = 0 \ , \qquad x^2 + y^2 + z^2 = 1$$
$$y = 0 \ , \qquad x^2 + y^2 = 1$$
$$x = 0 \ , \qquad x = 1$$

To change into spherical polar co-ordinates



Put $x = rsin\theta cos\emptyset, y = rsin\theta sin\emptyset, z = rcos\theta$

So that $x^2 + y^2 + z^2 = r^2$
$\&dzdydx = dxdydz = r^2 sin\theta drd\theta d\emptyset$

The limits are

$r = 0 \quad to\, r = 1,$

$\theta = 0 \quad to\;\; \theta = \frac{\pi}{2},$

$\emptyset = 0 \quad to\;\; \emptyset = \frac{\pi}{2}.$

∴The given integral

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^2 sin\theta}{\sqrt{1-r^2}} d\emptyset d\theta dr \quad = \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r^2 sin\theta}{\sqrt{1-r^2}} [\emptyset]_0^{\frac{\pi}{2}} d\theta dr$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r^2 sin\theta}{\sqrt{1-r^2}} d\theta dr$$

$$= \frac{\pi}{2} \int_0^1 \frac{r^2}{\sqrt{1-r^2}} [-cos\theta]_0^{\frac{\pi}{2}} dr$$

$$= \frac{\pi}{2} \int_0^1 \frac{r^2}{\sqrt{1-r^2}} [0+1] dr$$

$$= \frac{\pi}{2} \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr$$

$$= \frac{\pi}{2} \int_0^1 \frac{-(-r^2)}{\sqrt{1-r^2}} dr$$

$$= -\frac{\pi}{2} \int_0^1 \frac{1-r^2-1}{\sqrt{1-r^2}} dr$$

$$= -\frac{\pi}{2} \int_0^1 \frac{1-r^2}{\sqrt{1-r^2}} dr + \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-r^2}} dr$$

$$= -\frac{\pi}{2} \int_0^1 \sqrt{1-r^2} dr + \frac{\pi}{2} [sin^{-1} r]_0^1$$

$$= \frac{\pi}{2} \left[\frac{r}{2} \sqrt{1-r^2} + \frac{1}{2} sin^{-1} r \right]_0^1 + \frac{\pi}{2} [sin^{-1} r]_0^1$$

$$= -\frac{\pi}{2} \left[\frac{1}{2} . \frac{\pi}{2} \right] + \frac{\pi}{2} \left[\frac{\pi}{2} - 0 \right]$$

$$= -\frac{\pi^2}{8} + \frac{\pi^2}{4} = \frac{\pi^2}{8}$$

8. Evaluate $\int\int\int(\sqrt{1 - x^2 - y^2 - z^2})\,dx\,dy\,dz$ taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$ by transforming to spherical polar coordinates . (L6) [16marks]

<u>Solution:</u>

To change into spherical polar co-ordinates

Put $x = r sin\theta cos\emptyset, y = r sin\theta sin\emptyset, z = r cos\theta$

So that $x^2 + y^2 + z^2 = r^2 \& dzdydx = dxdydz = r^2 sin\theta dr d\theta d\emptyset$

The limits are

$$r = 0 \quad to\ r = 1,$$
$$\theta = 0 \quad to\ \theta = \frac{\pi}{2},$$
$$\emptyset = 0 \quad to\ \emptyset = \frac{\pi}{2}.$$

The given integral= I = 8 ×the first octant region

$$= 8\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{1 - r^2}\,.\,r^2 sin\theta d\emptyset d\theta dr$$

$$= 8\int_0^1\int_0^{\frac{\pi}{2}} r^2\sqrt{1 - r^2}\ sin\theta\ [\emptyset]_0^{\frac{\pi}{2}} d\theta dr$$

$$= 8\ \frac{\pi}{2}\int_0^1 r^2\sqrt{1 - r^2}[-cos\theta]_0^{\frac{\pi}{2}} dr$$

$$= 4\pi\int_0^1 r^2\sqrt{1 - r^2}[0 + 1]dr$$

$$= 4\pi\int_0^1 r^2\sqrt{1 - r^2}\ dr$$

Put $r = sin\theta$ when $r = 0, \theta = 0$

$dr = cos\theta\ d\theta$ when $r = 1, \theta = \frac{\pi}{2}$

$$\therefore I = 4\pi\int_0^{\frac{\pi}{2}} sin^2\theta . cos\theta . cos\theta\ d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} sin^2\theta . cos^2\theta \, d\theta$$

$$= 4\pi . \frac{(2-1)}{(2+2)(4-2)} . \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$ [By using reduction formula]

[**Reduction formula is**, If m and n are even,

$$\int_0^{\frac{\pi}{2}} sin^m\theta . cos^n\theta d\theta = \frac{(m-1)(m-3)(m-5) \ldots .3.1 \times (n-1)(n-3) \ldots ..3.1}{(m+n)(m+n-2)(m+n-4) \ldots ..4.2} \times \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} sin^2\theta . cos^2\theta d\theta = \frac{(2-1) \times (2-1)}{(2+2)(2+2-2)} \times \frac{\pi}{2} = \frac{1}{8} \times \frac{\pi}{2} = \frac{\pi}{16}]$$]

Define line integral

Let C be the segment of a continuous curve joining A(a, b) &B(c, d) and$f(x, y)$ be a single valued and continuous function of x & y defined at all points of C. Then $\int f(x, y) ds$is defined as the line integral along the curve C.

Define surface integral

Solution:

Let S be a portion of a regular two sided surface .Let $f(x, y, z)$ be a function defined and continuous at all points on S.Then $\int\int f(x, y, z) ds$ is defined as the surface integral of $f(x, y, z)$ over the surface S.

- **Evaluate:**$\int (x^2 dy + y^2 dx)$ **where C is the path** $y = x$ **from (0, 0) to (1, 1).(L6)**

Solution:

Given the path $y = x$ from (0, 0) to (1, 1).

$$\Longrightarrow dy = dx$$

$$\therefore \int (x^2dy + y^2dx) = \int_0^1 (x^2dx + x^2dx) = \int_0^1 2x^2dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

PROBLEMS ON TRIPLE INTEGRAL(VARIABLE LIMITS)

1. **Evaluate:** $\int_0^{log2} \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$**. (L6)** **[8marks]**

   Solution:

$$\text{Let I} = \int_0^{log2} \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^{log2} \int_0^x e^{x+y}[e^z]_0^{x+y}\, dy\, dx$$

$$= \int_0^{log2} \int_0^x e^{x+y}[e^{x+y} - 1]\, dy\, dx$$

$$= \int_0^{log2} \int_0^x e^{2x+2y}\, dy\, dx - \int_0^{log2} \int_0^x e^{x+y}\, dy\, dx$$

$$= \int_0^{log2} \int_0^x e^{2x}e^{2y}\, dy\, dx - \int_0^{log2} \int_0^x e^x e^y\, dy\, dx$$

$$= \int_0^{log2} e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x dx - \int_0^{log2} e^x[e^y]_0^x\, dx$$

$$= \int_0^{log2} e^{2x}\left(\frac{e^{2x}}{2} - \frac{e^0}{2}\right) dx - \int_0^{log2} e^x(e^x - 1)\, dx$$

$$= \int_0^{log2} \left(\frac{e^{4x}}{2} - \frac{e^{2x}}{2}\right) dx - \int_0^{log2} (e^{2x} - e^x)\, dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^x\right]_0^{log2}$$

$$= \left[\frac{e^{4log2}}{8} - \frac{e^{2log2}}{4} - \frac{e^{2log2}}{2} + e^{log2} - \frac{1}{8} + \frac{1}{4} + \frac{1}{2} - 1\right]$$

$$=\frac{16}{8}-\frac{4}{4}-\frac{4}{2}+2-\frac{3}{8} \quad =\frac{5}{8}$$

13. Evaluate: $\int_0^{log2}\int_0^{x}\int_0^{x+logy} e^{x+y+z}\, dx\, dy\, dz$ **. (L6)[16marks]**

Solution:

$$\text{Let I} = \int_0^{log2}\int_0^{x}\int_0^{x+logy} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^{log2}\int_0^{x} e^{x+y}[e^{z}]_0^{x+logy}\, dy\, dx$$

$$= \int_0^{log2}\int_0^{x} e^{x+y}[e^{x+logy}-1]\, dy\, dx$$

$$= \int_0^{log2}\int_0^{x} e^{x+y}e^{x+logy}\, dy\, dx - \int_0^{log2}\int_0^{x} e^{x+y}\, dy\, dx$$

$$= \int_0^{log2}\int_0^{x} e^{2x}e^{y}.\, y\, dy\, dx - \int_0^{log2}\int_0^{x} e^{x}e^{y}\, dy\, dx$$

$$= \int_0^{log2} e^{2x}[ye^{y}-e^{y}]_0^{x}\, dx - \int_0^{log2} e^{x}[e^{y}]_0^{x}\, dx$$

[Using integration by parts]

$$= \int_0^{log2} e^{2x}[xe^{x}-e^{x}+1]\, dx - \int_0^{log2} e^{x}(e^{x}-1)\, dx$$

$$= \int_0^{log2} (xe^{3x}-e^{3x}+e^{2x})\, dx - \int_0^{log2} e^{2x}\, dx + \int_0^{log2} e^{x}\, dx$$

$$= \int_0^{log2} xe^{3x}\, dx - \int_0^{log2} e^{3x}\, dx + \int_0^{log2} e^{2x}\, dx - \int_0^{log2} e^{2x}\, dx + \int_0^{log2} e^{x}\, dx$$

$$= \int_0^{log2} xe^{3x}\, dx - \int_0^{log2} e^{3x}\, dx + \int_0^{log2} e^{x}\, dx$$

$$= \left[x\frac{e^{3x}}{3}\right]_0^{log2} - \int_0^{log2}\frac{e^{3x}}{3}\, dx - \left[\frac{e^{3x}}{3}\right]_0^{log2} + [x]_0^{log2}$$

[Using integration by parts]

$$= \left[log2\frac{e^{3log2}}{3} - 0\right] - \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_0^{log2} - \left(\frac{e^{3log2}}{3} - \frac{e^0}{3}\right) + (e^{log2} - e^0)$$

$$= \ log2\frac{e^{3log2}}{3} - \frac{1}{3}\left(\frac{e^{3log2}}{3} - \frac{e^0}{3}\right) - \left(\frac{e^{log2^3}}{3} - \frac{1}{3}\right) + (2-1)$$

$$= \ log2\left(\frac{8}{3}\right) - \left(\frac{8}{9} - \frac{1}{9}\right) - \left(\frac{8}{3} - \frac{1}{3}\right) + 1$$

$$= \frac{8}{3}\ log2 - \frac{7}{9} - \frac{7}{3} + 1$$

$$= \frac{8}{3}\ log2 - \left(\frac{7+12}{9}\right)$$

$$= \frac{8}{3}\ log2 - \left(\frac{19}{9}\right)$$

14. Evaluate: $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \boldsymbol{xyz\,dx\,dy\,dz}$.**(L6)**

[16marks]

<u>Solution:</u>

$$\text{Let } I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyzdz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} xy[(1-x) - y]^2\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} xy[(1-x)^2 - 2(1-x)y + y^2]\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} [x(1-x)^2y - 2\,(1-x)xy^2 + xy^3]dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[x(1-x)^2\frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + x\frac{y^4}{4}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(1-x)^4}{2} - 2x\frac{(1-x)^4}{3} + x\frac{(1-x)^4}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^4\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^4 \left[\frac{1}{12}\right] dx$$

$$= \frac{1}{24}\int_0^1 (1-(1-x))(1-x)^4) dx$$

$$= \frac{1}{24}\left\{\int_0^1 (1-x)^4 dx - \int_0^1 (1-x)^5 dx\right\}$$

$$= \frac{1}{24}\left[\frac{(1-x)^5}{5}(-1) - \frac{(1-x)^6}{6}(-1)\right]_0^1$$

$$= \frac{1}{24}\left[-\frac{(1-x)^5}{5} + \frac{(1-x)^6}{6}\right]_0^1 = \frac{1}{24}\left[\frac{1}{5} - \frac{1}{6}\right]$$

$$= \frac{1}{24} \times \frac{1}{30} = \frac{1}{720}$$

15. Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z)^3}$, where v is the region of space bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$.(L6) [16marks]

Solution:

Let I $= \iiint \frac{dx\,dy\,dz}{(x+y+z)^3}$

The limits are

$$x = 0 \rightarrow x = 1 - y - z$$
$$y = 0 \rightarrow y = 1 - z$$
$$z = 0 \rightarrow z = 1$$

$$\mathrm{I} = \iiint \frac{dx\,dy\,dz}{(x+y+z)^3} = \int_{z=0}^{z=1}\int_{y=0}^{y=1-z}\int_{x=0}^{x=1-y-z} \frac{dx\,dy\,dz}{(x+y+z)^3}$$

$$= \int_{z=0}^{z=1}\int_{y=0}^{y=1-z} \left(-\frac{1}{2}\right)\left[\frac{1}{(x+y+z+1)^2}\right]_0^{1-y-z} dydz$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-z}\left[\frac{1}{(1-y-z+y+z+1)^2} - \frac{1}{(y+z+1)^2}\right] dy\,dz$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-z}\left[\frac{1}{4} - \frac{1}{(y+z+1)^2}\right] dy\,dz$$

$$= -\frac{1}{2}\int_0^1 \left\{\frac{1}{4}[y]_0^{1-z} + \left[\frac{1}{y+z+1}\right]_0^{1-z}\right\} dz$$
$$= -\frac{1}{2}\int_0^1 \left\{\frac{1}{4}(1-z) + \left[\frac{1}{1-z+z+1} - \frac{1}{z+1}\right]\right\} dz$$
$$= -\frac{1}{2}\int_0^1 \left\{\frac{1}{4}(1-z) + \frac{1}{2} - \frac{1}{z+1}\right\} dz$$
$$= -\frac{1}{2}\left[\frac{z}{4} - \frac{z^2}{8} + \frac{z}{2} - \log(z+1)\right]_0^1$$
$$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{8} + \frac{1}{2} - \log 2\right] = -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$
$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

16. Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$, where v is the volume of the rectangular parallelepiped bounded by $x = 0, x = a, y = 0 y = b$, $z = 0$ and $z = c$. (L6) [16marks]

Solution:

Let I = Volume $= \iiint (x+y+z) dz\, dy\, dx$

$$= \int_0^a \int_0^b \int_0^c (x+y+z) dz\, dy\, dx$$
$$= \int_0^a \int_0^b \left[(x+y)z + \frac{z^2}{2}\right]_0^c dydx$$
$$= \int_0^a \int_0^b \left[c(x+y) + \frac{c^2}{2}\right] dydx$$
$$= \int_0^a \int_0^b \left\{\left(cx + \frac{c^2}{2}\right) + cy\right\} dydx$$
$$= \int_0^a \left[\left(c + \frac{c^2}{2}\right)y + c\frac{y^2}{2}\right]_0^b dx$$
$$= \int_0^a \left(bcx + b\frac{c^2}{2} + \frac{b^2 c}{2}\right) dx$$

$$= \left[bc\frac{x^2}{2} + \frac{bc}{2}(b+c)x\right]_0^a$$

$$= bc\frac{a^2}{2} + \frac{bc}{2}(b+c)a$$

$$= \frac{bca^2}{2} + \frac{b^2ca}{2} + \frac{bc^2a}{2} = \frac{abc}{2}(a+b+c)$$

17. Evaluate $\iiint xyz\,dx\,dy\,dz$, where v is the region of space bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant. (L6) [16marks]

<u>Solution:</u>

Let I $= \iiint xyz\,dx\,dy\,dz$

The limits are

$$z = 0 \rightarrow z = \sqrt{1 - x^2 - y^2}$$

$$y = 0 \rightarrow y = \sqrt{1 - x^2}$$

$$x = 0 \rightarrow x = 1$$

$$\therefore I = \iiint xyz\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy(1 - x^2 - y^2)dydx$$

$$= \frac{1}{2}\left\{\int_0^1 \int_0^{\sqrt{1-x^2}} xydydx - \int_0^1 \int_0^{\sqrt{1-x^2}} x^3ydydx - \int_0^1 \int_0^{\sqrt{1-x^2}} xy^3dydx\right\}$$

$$= \frac{1}{2}\left\{\int_0^1 x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx - \int_0^1 x^3\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx - \int_0^1 x\left(\frac{y^4}{4}\right)_0^{\sqrt{1-x^2}} dx\right\}$$

$$= \frac{1}{2}\left\{\int_0^1 x\frac{(1-x^2)}{2}dx - \frac{1}{2}\int_0^1 x^3(1-x^2)dx - \frac{1}{4}\int_0^1 x(1-x^2)^2dx\right\}$$

$$= \frac{1}{2}\left\{\int_0^1 x(1-x^2)\left[\frac{1}{2}-\frac{x^2}{2}-\frac{(1-x^2)}{4}\right]dx\right\}$$
$$= \frac{1}{2}\int_0^1 x(1-x^2)\left[\frac{2-2x^2-1+x^2}{4}\right]dx$$
$$= \frac{1}{8}\int_0^1 x(1-x^2)^2\,dx \quad = \frac{1}{8}\int_0^1 x(1-2x^2+x^4)\,dx$$
$$= \frac{1}{8}\int_0^1 (x-2x^3+x^5)\,dx \quad = \frac{1}{8}\left[\frac{x^2}{2}-2.\frac{x^4}{4}+\frac{x^6}{6}\right]_0^1$$
$$= \frac{1}{8}\left[\frac{1}{2}-\frac{2}{4}+\frac{1}{6}\right] = \frac{1}{8}\left[\frac{1}{6}\right] = \frac{1}{48}$$

18. Evaluate $\iiint (xy+yz+zx)\,dx\,dy\,dz$, where v is the region bounded
by $x=0, y=0, z=0, x=1, y=2$ and $z=3$.(L6) [8marks]

Solution:

Let I $= \iiint (xy+yz+zx)\,dz\,dy\,dx$

$$= \int_0^1\int_0^2\int_0^3 (xy+yz+zx)\,dz\,dy\,dx$$
$$= \int_0^1\int_0^2 \left[xyz+\frac{yz^2}{2}+\frac{z^2x}{2}\right]_0^3 dydx$$
$$= \int_0^1\int_0^2 \left[xy(3)+y\left(\frac{9}{2}\right)+\frac{9}{2}x\right]dydx$$
$$= \int_0^1 \left[3x\frac{y^2}{2}+\left(\frac{9}{2}\right)\frac{y^2}{2}+\frac{9}{2}xy\right]_0^2 dx$$
$$= \int_0^1 \left[3x.\frac{4}{2}+\left(\frac{9}{2}\right)\frac{4}{2}+\frac{9}{2}x(2)\right]dx$$
$$= \left[6\frac{x^2}{2}+9x+9\frac{x^2}{2}\right]_0^1 = \frac{6}{2}+9+\frac{9}{2}$$
$$= \frac{33}{2}$$

AREA UNDER DOUBLE INTEGRATION

1. **Using double integral, find the area bounded by $y=x$ and $y=x^2$. (L1)**

[8marks]

Solution:

$$\text{Area } = \int\int dydx$$

Limits:

$$y = x^2 \ \ to \ y = x$$

$$x = 0 \quad to \ x = 1$$

$$\therefore \text{ Area } = \int\int dydx$$

$$= \int_0^1 \int_{x^2}^{x} dydx \ = \int_0^1 [y]_{x^2}^{x} dx$$

$$= \int_0^1 [x - x^2] dx \ = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} \ = \frac{1}{6}$$

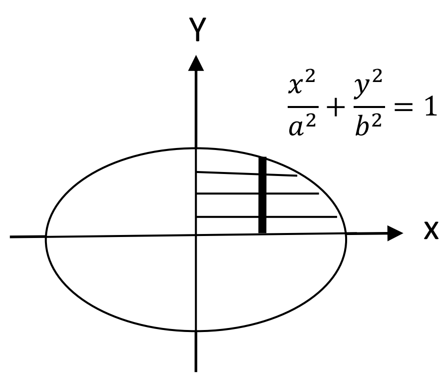
2. **Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (L1)** **[8marks]**

Solution:

Area of the ellipse
$= 4\times$ area of one quadrant

$$= 4 \times \int\int dydx$$

Limits:



$$x = 0 \ , \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(i.e)\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$(i.e)x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

$$(i.e)x = \frac{a}{b}\sqrt{b^2 - y^2}$$

$$y = 0\,, \quad y = b$$

$$\therefore \qquad \text{Area of the ellipse} = 4 \times \int\int dydx$$

$$= 4 \times \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dxdy$$

$$= 4 \times \int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4 \times \int_0^b \frac{a}{b}\sqrt{b^2-y^2}dy$$

$$= 4\ \frac{a}{b} \times \int_0^b \sqrt{b^2-y^2}dy$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2} + \frac{b^2}{2}sin^{-1}\left(\frac{y}{b}\right)\right]_0^b$$

$$\left[\text{Since} \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right]$$

$\therefore$ Area of the ellipse $= \dfrac{4a}{b}\left[0 + \dfrac{b^2}{2}sin^{-1}(1) - 0\right]$

$$= \frac{4a}{b}\left[\frac{b^2}{2}.\frac{\pi}{2}\right] \quad = \pi ab$$

UNIT-III

ORDINARY DIFFERENTIAL EQUATIONS

A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.

Ordinary Differential Equation

An ordinary differential equation (ODE) is a differential equation in which the unknown function (also known as the dependent variable) is a function of a single independent variable.

Partial differential equations

Partial differential equation is a differential equation,in which partial derivative exists.

Both ordinary and partial differential equations are broadly classified as **linear and nonlinear**.

Linear and Non linear

A differential equation is linear if the unknown function and its derivative appear to power 1 (products are not allowed) and **nonlinear** otherwise.

Linear equations

A differential equation, in which the dependent variable and all its derivatives occur in the first degree only, has already been defined as linear. The equation

$$P_0 \frac{d^n y}{dt^n} + P_1 \frac{d^{n-1} y}{dt^{n-1}} + P_2 \frac{d^{n-2} y}{dt^{n-2}} + \ldots + P_{n-1} \frac{dy}{dt} + P_n \mathrm{y} = \mathrm{X}$$

Where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and X are fuctions of X is the general linear equation of order n.

Notations

D$=\frac{d}{dx}$, $D^2=\frac{d^2}{dx^2}$

$\frac{1}{D}=\int dx$, $\frac{1}{D^2}=\iint dxdx$

Complete solution

Y= Complementary function + Particular integral

Auxillary equation

$a_0m^n+a_1m^{n-1}+a_2m^{n-2}+.....+a_n=0$ is known as auxiliary equation.It is obtained by repacing D by m. On solving we get $m_1, m_{2,}$.....,m_n those which are the roots of the auxiliary equation.

Complementary function (CF)

Based on the roots of the auxiliary equation, we have three types of CF.

Case1: m_i's are real and $m_1 = m_2 = m_3 \dots = m_n$

CF=(Ax+B)e^{mx}

Case 2: m_i's are real and $m_1 \neq m_2 \neq m_3 \dots \neq m_n$

CF=Ae^{m_1x}+ Ae^{m_2x}+ A$e^{m_3x} + \cdots$

Case 3: m_i's are imaginary and in the form $a \pm$ ib

CF= e^{ax}(Acosbx+Bsinbx)

Problems

- **Solve $(D^2+3D+2)y = 0$.(L3)**

  Solution:

  Auxiliary equation is $m^2+3m+2=0$

$$(m+1)(m+2)=0$$

$m=-1,-2$

The complimentary function $= Ae^{-x}+Be^{-2x}$

- **Solve$(\mathbf{D^2-6D+9})\mathbf{y}\ =\mathbf{0}$. (L3)**

<u>Solution:</u>

Auxiliary equation is $m^2-6m+9=0$

$$(m-3)(m-3)=0$$

$$m=3,3$$

The complimentary function $=(Ax+B)e^{3x}$

- **Solve $(\boldsymbol{D^2+5D+9})\boldsymbol{y}\ =\mathbf{0}$. (L3)**

<u>Solution:</u>

Auxiliary equation is $m^2+5m+9=0$

$$m=\frac{-5\pm\sqrt{25-36}}{2}\qquad\left[\text{since } m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$m=\frac{-5\pm\sqrt{-11}}{2}=\frac{-5\pm i\sqrt{11}}{2}=\alpha\pm i\beta$$

The complimentary function $=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

$$\alpha=\frac{-5}{2}\ and\ \beta=\frac{\sqrt{11}}{2}$$

$$C.F=e^{\frac{-5}{2}x}\left(A\cos\frac{\sqrt{11}}{2}x+B\sin\frac{\sqrt{11}}{2}x\right)$$

- **Solve$(\boldsymbol{D^2\ +2D-2})\ \boldsymbol{y}\ \ =\ \mathbf{0}$. (L3)**

<u>Solution:</u>

Auxiliary equation is $m^2+2m-2=0$

$$m=\frac{-2\pm\sqrt{4+8}}{2}\qquad\left[m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$m = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$$

$$m = \ -1 - \sqrt{3}\ , \quad -1 + \sqrt{3}$$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$$C.F = Ae^{(-1-\sqrt{3}\,)x} + Be^{(-1+\sqrt{3}\,)x}$$

- **Solve $(\boldsymbol{D^3 + D^2 + 4D + 4})\boldsymbol{y = 0}$. (L3)**

<u>Solution:</u>

Auxiliary equation is $m^3 + m^2 + 4m + 4 = 0$

$$m^2(m+1) + 4(m+1) = 0$$

$$(m^2 + 4)(m+1) = 0$$

$$\Longrightarrow m = -1, m = \pm 2i == \alpha \pm i\beta$$

The complimentary function $== Ae^{m_1 x} + e^{\alpha x}(Bcos\beta x + Csin\beta x)$

$m_1 = -1\ ,\ \alpha = 0\ and\ \beta = 2$

$$C.F = Ae^{-x} + Bcos2x + Csin2x$$

- **Find the complementary function of $(\boldsymbol{D^3 - 7D - 6})\boldsymbol{y} \ = \ \boldsymbol{coshx}$. (L3)**

<u>Solution:</u>

Auxiliary equation $is\ m^3 - 7m - 6 = 0$

$$(-1)^3 - 7(-1) - 6 = 0$$

$m = -1$ satisfies the equation

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$\therefore m = -1, -2, 3$$

The complimentary function $= Ae^{-x} + Be^{-2x} + Ce^{3x}$

- **Find the complementary function of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$. (L1)**

Solution:

If $\frac{d}{dx} = D$, then the given equation becomes

$(D^2 + 4D + 4)y = e^{-2x}$

Auxiliary equation is $m^2 + 4m + 4 = 0$

$$(m+2)^2 = 0$$

$$m = -2, -2$$

The complimentary function $= (Ax + B)e^{mx}$

$C.F = (Ax + B)e^{-2x}$

- **Solve $(D^2 + 1)y = 0$ given $y\,(0) = 0, y'\,(0) = 1$. (L3)**

Solution:

Auxiliary equation *is* $m^2 + 1 = 0$

$$m^2 = -1$$

$m = 0 \pm i = \alpha \pm i\beta$

The complimentary function $= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$$\alpha = 0 \; and \; \beta = 1$$

$C.F \;\; = e^{0x}(A\cos x + B\sin x) \;= A\cos x + B\sin x$

$Differentiating\ wrt\ x, \qquad y'(x) = -A\sin x + B\cos x$

Given that $y(0) = 0 \;\; \Rightarrow A\cos 0 + B\sin 0 = 0 \qquad \Rightarrow A = 0$

and $y'(0) = 1 \;\; \Rightarrow -A\sin 0 + B\cos 0 = 1 \qquad \Rightarrow B = 1$

Substituting the values of A and B in C.F, $y(x) = \sin x$

Particular Integral (PI)

Type 1: X=e^{ax}

Algorithm to find PI

1. Replace D by α.
2. If f(α)=0 then multiply variable in the denominator with respect to the derivative of the corresponding variable and repeat step 1, till we get non zero value in the denominator.

- **Solve $(D^4 - 2D^3 + D^2)y = e^{2x}$. (L3)**

  Solution:

  Auxiliary equation *is* $m^4 - 2m^3 + m^2 = 0$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)(m-1) = 0$$

$$m = 0,0,1,1$$

The complimentary function = $(Ax + B)e^{0x} + (Cx + D)e^{1x}$

$$C.F = (Ax + B) + (Cx + D)e^x$$

The particular integral $= \dfrac{e^{2x}}{D^4 - 2D^3 + D^2}$

$$= \frac{e^{2x}}{(2)^4 - 2(2)^3 + (2)^2} \quad (Put\ D = 2)$$

$$= \frac{e^{2x}}{16 - 16 + 4} = \frac{e^{2x}}{4}$$

General solution $= C.F + P.I = (Ax + B) + (Cx + D)e^x + \dfrac{e^{2x}}{4}$

- **Solve$(D-2)^3 y = e^{2x}$. (L3)**

  Solution:

  Auxiliary equation is $(m-2)^3 = 0$

$$(m-2)(m-2)(m-2)=0$$

$$m=2,2,2$$

The complimentary function $=(Ax^2+Bx+C)e^{2x}$

The particular integral $=\dfrac{e^{2x}}{(D-2)^3}=\dfrac{e^{2x}}{(2-2)^3}\quad (D=2)$

$=\dfrac{x^3}{3!}e^{2x}\quad \left\{Since, \dfrac{e^{ax}}{(D-a)^n}=\dfrac{x^n}{n!}e^{nx}\right\}$

General solution $=C.F+P.I$

$=(Ax^2+Bx+C)e^{2x}+\dfrac{x^3}{6}e^{2x}$

- **Find the particular integral of$(D-1)^3y=2sinhx.$ (L1)**

Solution:

The particular integral $=\dfrac{2\,sinhx}{(D-1)^3}=\dfrac{2\left(\dfrac{e^x-e^{-x}}{2}\right)}{(D-1)^3}=\dfrac{e^x-e^{-x}}{(D-1)^3}$

$=\dfrac{e^x}{(D-1)^3}-\dfrac{e^{-x}}{(D-1)^3}$

$=\dfrac{e^x}{(1-1)^3}-\dfrac{e^{-x}}{(-1-1)^3}$

$=\dfrac{x^3}{3!}e^x-\dfrac{e^{-x}}{(-2)^3}\ \left\{Since, \dfrac{e^{ax}}{(D-a)^n}=\dfrac{x^n}{n!}e^{ax}\right\}$

$=\dfrac{x^3}{3!}e^x+\dfrac{e^{-x}}{8}$

- **Solve $(D^3+1)y=5e^{2x}.$ (L3)**

Solution:

Auxiliary equation is $m^3+1=0$

$$(m+1)(m^2-m+1)=0$$

$$m+1=0 \quad or \quad m^2-m+1=0$$

$$m=-1, m=\frac{1\pm\sqrt{1-4(1)(1)}}{2(1)}$$

$$m = -1, m = \frac{1 \pm i\sqrt{3}}{2} = \alpha \pm i\beta$$

The complimentary function $= Ae^{m_1 x} + e^{\alpha x}(Bcos\beta x + Csin\beta x)$

$C.F = Ae^{-x} + e^{\frac{1}{2}x}\left(Bcos\frac{\sqrt{3}}{2}x + Csin\frac{\sqrt{3}}{2}x\right)$

The particular integral $= \frac{5e^{2x}}{D^3 + 1} = \frac{5e^{2x}}{2^3 + 1}$ $\qquad (D = 2)$

$= \frac{5e^{2x}}{8 + 1} = \frac{5e^{2x}}{9}$

General solution $= C.F + P.I$

$$= Ae^{-x} + e^{\frac{1}{2}x}\left(Bcos\frac{\sqrt{3}}{2}x + Csin\frac{\sqrt{3}}{2}x\right) + \frac{5e^{2x}}{9}$$

- **Find the particular integral of$(D^2 + 3D + 2)y = Cosh^2 3x$. (L1)**

Solution:

The particular integral $= \frac{Cosh^2 3x}{D^2 + 3D + 2} = \frac{(Cosh3x)^2}{D^2 + 3D + 2}$

$= \frac{\left(\frac{e^{3x} + e^{-3x}}{2}\right)^2}{D^2 + 3D + 2} = \frac{\left(\frac{e^{6x} + 2e^{3x}e^{-3x} + e^{-6x}}{4}\right)}{D^2 + 3D + 2} = \frac{\left(\frac{e^{6x} + 2 + e^{-6x}}{4}\right)}{D^2 + 3D + 2}$

$= \frac{1}{4}\left[\frac{e^{6x}}{D^2 + 3D + 2} + \frac{2e^{0x}}{D^2 + 3D + 2} + \frac{e^{-6x}}{D^2 + 3D + 2}\right]$

$= \frac{1}{4}\left[\frac{e^{6x}}{36 + 18 + 2} + \frac{2e^{0x}}{0 + 0 + 2} + \frac{e^{-6x}}{36 - 18 + 2}\right]$

$= \frac{1}{4}\left[\frac{e^{6x}}{56} + \frac{2e^{0x}}{2} + \frac{e^{-6x}}{20}\right] = \frac{1}{4}\left[\frac{e^{6x}}{56} + 1 + \frac{e^{-6x}}{20}\right]$

- **Solve $(D^2 - 7D - 6)y = sinh^2 3x$. (L3)** **[8 Marks]**

Solution:

Auxiliary equation is $m^2 - 7m - 6 = 0$

$$m = \frac{7 \pm \sqrt{49+24}}{2} = \frac{7 \pm \sqrt{73}}{2}$$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$$C.F = Ae^{\left(\frac{7-\sqrt{73}}{2}\right)x} + Be^{\left(\frac{7+\sqrt{73}}{2}\right)x}$$

The particular integral $= \dfrac{sinh^2 3x}{D^2-7D-6} = \dfrac{\left(\frac{e^{3x}-e^{-3x}}{2}\right)^2}{D^2-7D-6}$

$$= \frac{1}{4}\left[\frac{e^{6x} - 2e^{3x}e^{-3x} + e^{-6x}}{D^2-7D-6}\right] = \frac{1}{4}\left[\frac{e^{6x} - 2 + e^{-6x}}{D^2-7D-6}\right]$$

$$= \frac{1}{4}\left[\frac{e^{6x}}{D^2-7D-6} - \frac{2}{D^2-7D-6} + \frac{e^{-6x}}{D^2-7D-6}\right]$$

$$= \frac{1}{4}\left[\frac{e^{6x}}{6^2-7(6)-6} - \frac{2e^{0x}}{0-0-6} + \frac{e^{-6x}}{(-6)^2-7(-6)-6}\right]$$

$$= \frac{1}{4}\left[\frac{e^{6x}}{-12} + \frac{2e^{0x}}{6} + \frac{e^{-6x}}{72}\right]$$

$$= \frac{1}{12}\left[-\frac{e^{6x}}{4} + 1 + \frac{e^{-6x}}{24}\right]$$

General solution $= C.F + P.I$

$$y(x) = Ae^{\left(\frac{7-\sqrt{73}}{2}\right)x} + Be^{\left(\frac{7+\sqrt{73}}{2}\right)x} + \frac{1}{12}\left[-\frac{e^{6x}}{4} + \frac{e^{-6x}}{24} + 1\right]$$

Type 2: X=sin ax or cos ax

If X=sin ax or cos ax then the particular integral is given $\frac{X}{f(D)}$ becomes,

Step 1: Replace D^2 by $-\alpha^2$

Step 2: If Dr becomes zero multiply Nr with the functional variable and differentiate Dr with respect to the derivative of the functional variable.

Step 3: If linear order obtained in Dr, take the conjugate of denominator and multiply in both Nr and Dr (to make Nr and Dr in even order)

- **Find the particular integral of $(D^2 + a^2)y = cosax.$ (L1)**

  Solution:

  The particular integral $= \dfrac{cosax}{D^2 + a^2} = \dfrac{cosax}{-a^2 + a^2} (D^2 = -a^2)$

Multiplying the numerator by x and differentiating denominator wrt D,

$$P.I = \frac{xcosax}{2D} = \frac{x}{2}\int cosax\, dx \quad (\frac{1}{D} = \int dx)$$

$$= \frac{x}{2}\left(\frac{sinax}{a}\right) = \frac{xsinax}{2a}$$

- **Find the particular integral of $(D^2 - 6D + 9)y = sinx.$ (L1)**

  Solution:

The particular integral $= \dfrac{sinx}{D^2 - 6D + 9} = \dfrac{sinx}{-1 - 6D + 9} (Put\ D^2 = -1)$

$$= \frac{sinx}{8 - 6D} = \frac{1}{2}\left(\frac{sinx}{4 - 3D}\right)$$

$Multiplying\ and\ dividing\ by\ (4 + 3D)$

$$= \frac{1}{2}\left(\frac{(4 + 3D)sinx}{(4 + 3D)(4 - 3D)}\right) = \frac{1}{2}\left(\frac{(4 + 3D)sinx}{16 - 9D^2}\right)$$

$$= \frac{1}{2}\left(\frac{(4 + 3D)sinx}{16 - 9(-1)}\right) = \frac{1}{2}\left(\frac{4sinx + 3Dsinx}{25}\right)$$

$$= \frac{1}{2}\left(\frac{4sinx + 3cosx}{25}\right) = \frac{1}{50}(4sinx + 3cosx)$$

- **Solve $(D^2 + 1)y = sinx.$ (L3)**

  Solution:

  Auxiliary equation is $m^2 + 1 = 0$

$m^2 = -1 m = 0 \pm i = \alpha \pm i\beta$

The complimentary function is $e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = 0 \; and \; \beta = 1$$

$C.F = e^{0x}(Acosx + Bsinx) = Acosx + Bsinx$

The particular integral $= \dfrac{sinx}{D^2+1} = \dfrac{sinx}{-1+1} \quad (D^2 = -1)$

Multiplying the numerator by x and differentiating denominator wrt D,

$$= \frac{xsinx}{2D} = \frac{x}{2}\int sinx\, dx \qquad \left(\frac{1}{D} = \int dx\right)$$

$$= \frac{x}{2}(-cosx) = \frac{-xcosx}{2}$$

General solution $= C.F + P.I = Acosx + Bsinx - \dfrac{xcosx}{2}$

- **Solve$(D^2 + 4)y = cos2x.$ (L3)**

<u>**Solution:**</u>

Auxiliary equation is $m^2 + 4 = 0$

$m^2 = -4, \;\; m = 0 \pm 2i = \alpha \pm i\beta$

The complimentary function is $e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = 0 \; and \; \beta = 2$$

$C.F = e^{0x}(Acosx + Bsinx) = Acos2x + Bsin2x$

The particular integral $= \dfrac{cos2x}{D^2+2^2} = \dfrac{cos2x}{-4+4} \quad (D^2 = -2^2)$

Multiplying the numerator by x and differentiating denominator wrt D

$$P.I = \frac{xcos2x}{2D} = \frac{x}{2}\int cos2x\, dx \qquad (\frac{1}{D} = \int dx) = \frac{x}{2}\left(\frac{sin2x}{2}\right) = \frac{xsin2x}{4}$$

General solution $= C.F + P.I = Acos2x + Bsin2x + \dfrac{xsin2x}{4}$

- **Solve $(D^2 + 4)\, y = Sin2x.$ (L3)**

<u>***Solution:***</u>

Auxiliary equation is $m^2 + 4 = 0$

$$m^2 = -4, \quad m = 0 \pm 2i = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha x}(A cos\beta x + B sin\beta x)$

$C.F = e^{0x}(Acos2x + Bsin2x) = Acos2x + Bsin2x$

The particular integral $= \dfrac{sin2x}{D^2+4} = \dfrac{sin2x}{-4+4} \quad (D^2 = -4)$

Multiplying the numerator by x and differentiating denominator wrt D,

$$P.I = \frac{xsin2x}{2D} = \frac{x}{2}\int sin2x\, dx \qquad \left(\because \frac{1}{D} = \int dx\right)$$

$$= \frac{x}{2}\left(\frac{-cos2x}{2}\right) = \frac{-xcos2x}{4}$$

General solution $= C.F + P.I = Acos2x + Bsin2x - \dfrac{xcos2x}{4}$

- **Solve $(\boldsymbol{D^2 - 4D + 3})\boldsymbol{y} = \boldsymbol{sin3x\ cos2x}$. (L3) [8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^2 - 4D + 3 = 0$

$$(m-3)(m-1) = 0$$

$$m = 3,1$$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^{3x} + Be^{x}$

The particular integral $= \dfrac{sin3xcos2x}{D^2 - 4D + 3}$

$$\left[Formula:\ sinAcosB = \frac{1}{2}[sin(A+B) + sin(A-B)]\right]$$

$$P.I = \frac{1}{2}\left[\frac{sin(3x+2x) + sin(3x-2x)}{D^2 - 4D + 3}\right] = \frac{1}{2}\left[\frac{sin\,5x + sinx}{D^2 - 4D + 3}\right]$$

$$= P.I_{\ 1} + P.I_{\ 2}$$

$$P.I_{\ 1} = \frac{1}{2}\left[\frac{sin\,5x}{D^2 - 4D + 3}\right] = \frac{1}{2}\left[\frac{sin\,5x}{-25 - 4D + 3}\right] (D^2 = -25)$$

$$= \frac{1}{2}\left[\frac{sin\,5x}{-22-4D}\right] = \frac{1}{-4}\left[\frac{sin\,5x}{11+2D}\right]$$

Multiplying and dividing by $(11-2D)$

$$= -\frac{1}{4}\left[\frac{(11-2D)\,sin\,5x}{(11-2D)(11+2D)}\right] = -\frac{1}{4}\left[\frac{(11-2D)\,sin\,5x}{121-4D^2}\right]$$

$$= \frac{1}{4}\left[\frac{(11sin5x - 2D(sin5x)}{121-4(-25)}\right] \qquad (D^2 = -25)$$

$$= -\frac{1}{4}\left[\frac{(11sin5x - 10cos5x)}{121+100}\right]$$

$$= -\frac{1}{4}\left[\frac{(11sin5x - 10cos5x)}{121+100}\right]$$

$$= -\frac{1}{884}[11sin5x - 10cos5x]$$

$$P.I_{\ 2} = \frac{1}{2}\left[\frac{sin\,x}{D^2\ -\ 4D\ +\ 3}\right] = \frac{1}{2}\left[\frac{sin\,x}{-1-4D+3}\right] \qquad (D^2 = -1)$$

$$= \frac{1}{2}\left[\frac{sin\,x}{2-4D}\right] = \frac{1}{4}\left[\frac{sin\,x}{1-2D}\right]$$

Multiplying and dividing by $(1+2D)$

$$= \frac{1}{4}\left[\frac{(1+2D)\,sin\,x}{(1+2D)(1-2D)} = \frac{1}{4}\left[\frac{(1+2D)\,sin\,x}{1-4D^2}\right]\right]$$

$$= \frac{1}{4}\left[\frac{(1+2D)\,sin\,x}{1-4(-1)}\right] \qquad (D^2 = -1)$$

$$= \frac{1}{4}\left[\frac{(sinx + 2cosx)}{1+4}\right]$$

$$= \frac{1}{4}\left[\frac{(sinx + 2cosx)}{5}\right] = \frac{1}{20}[sinx + 2cosx]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = Ae^{3x} + Be^{x} - \frac{1}{884}[11sin5x - 10cos5x] + \frac{1}{20}[sinx + 2cosx]$$

- **Solve $(\boldsymbol{D^2+3D+2})\boldsymbol{y} = \boldsymbol{e^{2x} + sin3xcosx.}$ (L3)** **[8 Marks]**

<u>Solution:</u>

Auxiliary equation is $m^2 + 3m + 2 = 0$

$$(m+2)(m+1) = 0$$

$m = -2, -1$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^{-x} + Be^{-2x}$

The particular integral $= \dfrac{e^{2x} + sin3xcosx}{D^2+3D+2}$

$$\left[Formula\text{: } sinAcosB = \frac{1}{2}[sin(A+B) + sin(A-B)]\right]$$

$$= \frac{e^{2x}}{D^2+3D+2} + \frac{1}{2}\left[\frac{sin(3x+x) + sin(3x-x)}{D^2+3D+2}\right]$$

$$= \frac{e^{2x}}{D^2+3D+2} + \frac{1}{2}\left[\frac{\sin 4x}{D^2+3D+2}\right] + \frac{1}{2}\left[\frac{\sin 2x}{D^2+3D+2}\right]$$

$$P.I_{\ 1} = \frac{e^{2x}}{D^2+3D+2} = \frac{e^{2x}}{4+6+2} \quad (D=2) = \frac{e^{2x}}{12}$$

$$P.I_{\ 2} = \frac{1}{2}\left[\frac{\sin 4x}{D^2+3D+2}\right] = \frac{1}{2}\left[\frac{\sin 4x}{-16+3D+2}\right] \ (D^2 = -16)$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{-14+3D}\right]$$

Multiplying and dividing by $(-14-3D)$

$$= \frac{1}{2}\left[\frac{(-14-3D)\sin 4x}{(-14-3D)(-14+3D)}\right] = -\frac{1}{2}\left[\frac{(14+3D)\sin 4x}{196-9D^2}\right]$$

$$= -\frac{1}{2}\left[\frac{(14+3D)\sin 4x}{196-9(-16)}\right] = -\frac{1}{2}\left[\frac{14\sin 4x + 3D\sin 4x}{340}\right]$$

$$= -\frac{1}{2}\left[\frac{14\sin 4x + 3D\sin 4x}{340}\right] = -\frac{1}{2}\left[\frac{14\sin 4x + 12cos4x}{340}\right]$$

$$P.I_{\ 2} = -\left[\frac{7\sin 4x + 6\cos 4x}{340}\right]$$

$$P.I_{\ 3} = \frac{1}{2}\left[\frac{\sin 2x}{D^2 + 3D + 2}\right] = \frac{1}{2}\left[\frac{\sin 2x}{-4 + 3D + 2}\right] \quad (D^2 = -4)$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{-2 + 3D}\right]$$

Multiplying and dividing by $(-2 - 3D)$

$$= \frac{1}{2}\left[\frac{(-2 - 3D)\sin 2x}{(-2 - 3D)(-2 + 3D)}\right] = -\frac{1}{2}\left[\frac{(2 + 3D)\sin 2x}{4 - 9D^2}\right]$$

$$= -\frac{1}{2}\left[\frac{(2 + 3D)\sin 2x}{4 - 9(-4)}\right] = -\frac{1}{2}\left[\frac{2\sin 2x + 3D\sin 2x}{40}\right]$$

$$= -\frac{1}{2}\left[\frac{2\sin 2x + 6\cos 2x}{40}\right] = -\left[\frac{\sin 2x + 3\cos 2x}{40}\right]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2} + P.I_{\ 3}$

$$y(x) = Ae^{-x} + Be^{-2x} + \frac{e^{2x}}{12} - \left[\frac{7\sin 4x + 6\cos 4x}{340}\right] - \left[\frac{\sin 2x + 3\cos 2x}{40}\right]$$

- **Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$. (L3)** **[8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^2 - 3m + 2 = 0$

$$(m - 2)(m - 1) = 0$$

$m = 2,1$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^x + Be^{2x}$

The particular integral $= \dfrac{2\cos(2x + 3) + 2e^x}{D^2 - 3D + 2}$

$$P.I_{\ 1} = \frac{2\cos(2x + 3)}{D^2 - 3D + 2}$$

$$= \frac{2\cos(2x + 3)}{-4 - 3D + 2} \qquad (D^2 = -4)$$

$$= \frac{2cos(2x+3)}{-2-3D} = \frac{-2os(2x+3)}{2+3D}$$

$$Multiplying\ and\ dividing\ by\ (2-3D)$$

$$= \frac{-2(2-3D)cos(2x+3)}{(2-3D)(2+3D)} = \frac{-2(2-3D)cos(2x+3)}{4-9D^2}$$

$$= \frac{-2(2-3D)cos(2x+3)}{4-9(-4)} \quad (D^2 = -4) = \frac{-2(2-3D)cos(2x+3)}{4-9(-4)}$$

$$= \frac{-2(2\cos(2x+3) - 3\,\mathrm{D}\cos(2x+3))}{40}$$

$$= \frac{-2(2cos(2x+3) + 6sin(2x+3))}{40}$$

$$= \frac{-4(\cos(2x+3) + 3\sin(2x+3))}{40}$$

$$= -\left[\frac{cos(2x+3) + 3sin(2x+3)}{10}\right]$$

$$P.I_{\ 2} = \frac{2e^x}{D^2 - 3D + 2}$$

$$= \frac{2e^x}{1^2 - 3(1) + 2} \qquad (D = 1) = \frac{2e^x}{0}$$

Multiplying the numerator by x and differentiating denominator wrt D , we have

$$= \frac{2xe^x}{2D-3} = \frac{2xe^x}{2(1)-3} \quad (D = 1) = \frac{2xe^x}{-1} = -2xe^x$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = Ae^x + Be^{2x} - \left[\frac{cos(2x+3) + 3sin(2x+3)}{10}\right] - 2xe^x$$

Type 3

If x is polynomial or algebraic function.

Step 1: Take the common factor and constant term outside simultaneously (Make the D.R. into the binomial form)

Step 2: Shift binomial term to Nr

Step 3: Using binomial expansion subject to power of the algebraic function.

- **Find the particular integral of $(\boldsymbol{D^2+5D+4})\boldsymbol{y} = \boldsymbol{x}$. (L1)**

  Solution:

  The particular integral $= \dfrac{x}{D^2+5D+4} = \dfrac{x}{4\left(1+\dfrac{(D^2+5D)}{4}\right)}$

$$= \frac{1}{4}\left(1+\frac{(D^2+5D)}{4}\right)^{-1} x$$

$$\left[Formula: (1+x)^{-1} = 1-x+x^2-x^3+x^4-\cdots\right]$$

$$D(x)=1, \qquad D^2(x) = D^3(x) = D^4(x)\ \ldots\ldots = 0$$

$$= \frac{1}{4}\left\{1-\frac{(D^2+5D)}{4}+\frac{(D^2+5D)^2}{16}-\cdots\right\}x$$

$$= \frac{1}{4}\left(x-\frac{5Dx}{4}\right) = \frac{1}{4}\left(x-\frac{5}{4}\right)$$

- **Find the particular integral of$(\boldsymbol{D^2+2})\boldsymbol{y} = \boldsymbol{x^2}$. (L1)**

  Solution:

  The particular integral $= \dfrac{x^2}{D^2+2} = \dfrac{x^2}{2\left(1+\dfrac{D^2}{2}\right)} = \dfrac{1}{2}\left(1+\dfrac{D^2}{2}\right)^{-1} x^2$

$$\left[Formula: (1+x)^{-1} = 1-x+x^2-x^3+x^4-\cdots\right]$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\frac{D^6}{8}+\cdots\right)x^2$$

$$= \frac{1}{2}(x^2 - \frac{D^2(x^2)}{2} + \frac{D^4(x^2)}{4} - \frac{D^6(x^2)}{8} + \cdots)$$

$D(x^2) = 2x,\quad D^2(x^2) = 2,\quad D^3(x^2) = D^4(x^2) \ldots \ldots = 0$

$$P.I = \frac{1}{2}\left(x^2 - \frac{1}{2}(2) + 0 - \cdots\right) = \frac{1}{2}(x^2 - 1)$$

- **Solve** $(\boldsymbol{D^2 + 4})\boldsymbol{y} = \boldsymbol{x^4 + cos^2 x}$. **(L3)**

  [8Marks]

  Solution:

Auxiliary equation is $m^2 + 4 = 0,\ m^2 = -4, m = 0 \pm 2i = \alpha \pm i\beta$

The complimentary function $= e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = 0\ and\ \beta = 2$$

$C.F = e^{0x}(Acos2x + Bsin2x) = Acos2x + Bsin2x$

$$P.I = \frac{x^4 + cos^2 x}{D^2 + 4}$$

$$P.I_1 = \frac{x^4}{D^2 + 4} = \frac{x^4}{4\left(1 + \frac{D^2}{4}\right)} = \frac{1}{4}(1 + \frac{D^2}{4})^{-1}x^4$$

$[Formula: (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots]$

$$= \frac{1}{4}(1 - \frac{D^2}{4} + \frac{D^4}{16} - \frac{D^6}{64} + \cdots)x^4$$

$$= \frac{1}{4}(x^4 - \frac{D^2(x^4)}{4} + \frac{D^4(x^4)}{16} - \frac{D^6(x^4)}{64} + \cdots)$$

$D(x^4) = 4x^3, D^2(x^4) = 12x^2,\quad D^3(x^4) = 24x, D^4(x^4) = 24,$

$$and\ D^5(x^4) = D^6(x^4) \ldots = 0$$

$$= \frac{1}{4}(x^4 - \frac{1}{4}(12x^2) + \frac{1}{16}(24) - \cdots) == \frac{1}{8}(2x^4 - 6x^2 + 3)$$

$$P.I_2 = \frac{cos^2 x}{D^2 + 4} = \frac{\left(\frac{1 + cos2x}{2}\right)}{D^2 + 4}$$

$$= \frac{\left(\frac{1}{2}\right) e^{0x}}{D^2 \ + \ 4} + \frac{\left(\frac{cos2x}{2}\right)}{D^2 \ + \ 4} \ = \frac{1}{2}\left[\frac{e^{0x}}{0 \ + \ 4} + \frac{cos2x}{-4 \ + \ 4}\right]$$

Multiplying the numerator by x and differentiating denominator wrt

D in the second term, we have,

$$= \frac{1}{2}\left[\frac{e^{0x}}{4} + \frac{xcos2x}{2D}\right]$$

$$= \frac{1}{2}\left[\frac{e^{0x}}{4} + \frac{x \int cos2x \, dx}{2}\right] \quad \left(\frac{1}{D} = \int dx\right)$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{xsin2x}{4}\right] = \frac{1}{8}[1 + xsin2x]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$y(x) \ = Acos2x + Bsin2x + \frac{1}{8}(2x^4 - 6x^2 + 3) + \frac{1}{8}[1 + xsin2x]$

$y(x) \ = Acos2x + Bsin2x + \dfrac{1}{8}(2x^4 - 6x^2 + 4 + xsin2x)$

- **Solve $(D^2 + 5D + 4)y = sin2x + x^3 + 3.$ (L3)** **[8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^2 + 5m + 4 \ = 0$

$$(m + 4)(m + 1) = 0$$

$m = -4, -1$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

C.F$= Ae^{-x} + Be^{-4x}$

The particular integral $= \dfrac{sin2x + x^3 + 3}{D^2 + 5D + 4}$

$$P.I_{\ 1} \ = \frac{sin2x}{D^2 + 5D + 4} = \frac{sin2x}{-4 + 5D + 4} \quad (D^2 = -4)$$

$$= \frac{sin2x}{5D} = \frac{1}{5}\int sin2x \, dx = \frac{1}{5}\left(\frac{-cos2x}{2}\right) = \frac{-cos2x}{10}$$

$$P.I_{\ 2} = \frac{x^3+3}{D^2+5D+4} = \frac{x^3+3}{4\left(1+\left(\frac{D^2+5D}{4}\right)\right)}$$

$$= \frac{1}{4}\left(1+\left(\frac{D^2+5D}{4}\right)\right)^{-1}(x^3+3)$$

$$[Formula: (1+x)^{-1} = 1-x+x^2-x^3+x^4-\cdots]$$

$$= \frac{1}{4}(1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2-\left(\frac{D^2+5D}{4}\right)^3+\cdots)(x^3+3)$$

$$= \frac{1}{4}\left[1-\frac{5D}{4}-\frac{D^2}{4}+\frac{1}{16}(D^4+10D^3+25D^2)\right.$$

$$\left.-\frac{1}{64}(D^6+15D^5 \quad +75D^4+125D^3)\right](x^3+3)$$

$$D(x^3+3) = 3x^2, \quad D^2(x^3+3\ ) = 6x,$$

$$D^3(x^3+3\ ) = 6, \quad D^4(x^3+3\ ) = 0 \ldots,$$

$Omitting\ D^4\ onwards,$

$$= \frac{1}{4}\left[1-\frac{5D}{4}-\frac{D^2}{4}+\frac{1}{16}(10D^3+25D^2)-\frac{1}{64}(125D^3)\right](x^3+3)$$

$$= \frac{1}{4}\left[1-\frac{5D}{4}+\left(\frac{25}{16}-\frac{1}{4}\right)D^2+\left(\frac{10}{16}-\frac{125}{64}\right)D^3\right](x^3+3)$$

$$= \frac{1}{4}\left[(x^3+3)-\frac{5D(x^3+3)}{4}+\frac{21}{16}D^2(x^3+3)-\frac{85}{64}D^3(x^3+3)\right]$$

$$= \frac{1}{4}\left[(x^3+3)-\frac{5(3x^2)}{4}+\frac{21}{16}(6x)-\frac{85}{64}(6)\right]$$

$$= \frac{1}{4}\left[(x^3+3)-\frac{15x^2}{4}+\frac{63x}{8}-\frac{255}{32}\right]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = Ae^{-x}+Be^{-4x}-\frac{cos2x}{10}+\frac{1}{4}\left[(x^3+3)-\frac{15x^2}{4}+\frac{63x}{8}+\frac{255}{32}\right]$$

- **Solve** $(\boldsymbol{D^2+5D+9})\boldsymbol{y}=\boldsymbol{3x^2+4x+e^{-3x}}$**. (L3)** **[8 Marks]**

Solution:

Auxiliary equation is $m^2+5m+9 = 0$

$$m=\frac{-5\pm\sqrt{25-4(1)(9)}}{2(1)}=\frac{-5\pm i\sqrt{11}}{2}=\alpha\pm i\beta$$

The complimentary function $=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

$$\alpha=\frac{-5}{2}\ and\ \beta=\frac{\sqrt{11}}{2}$$

$$C.F= e^{\frac{-5}{2}x}\left(A\cos\frac{\sqrt{11}}{2}x+B\sin\frac{\sqrt{11}}{2}x\right)$$

The particular integral $=\dfrac{3x^2+4x+e^{-3x}}{D^2+5D+9}$

$$P.I_{\ 1}=\frac{3x^2+4x}{D^2+5D+9}=\frac{3x^2+4x}{9\left(1+\frac{(D^2+5D)}{9}\right)}$$

$$=\frac{1}{9}\left(1+\frac{(D^2+5D)}{9}\right)^{-1}(3x^2+4x)$$

$$[Formula: (1+x)^{-1}=1-x+x^2-x^3+x^4-\cdots]$$

$$=\frac{1}{9}(1-\frac{(D^2+5D)}{9}+\frac{(D^2+5D)^2}{81}-\cdots)(3x^2+4x)$$

$$=\frac{1}{9}(1-\frac{5D}{9}-\frac{D^2}{9}+\frac{1}{81}(D^4+10D^3+25D^2)-\cdots)(3x^2+4x)$$

$$D(3x^2+4x)=6x+4,\qquad D^2(3x^2+4x\ )=6,$$

$$D^3(3x^2+4x\ )=0,\qquad D^4(3x^2+x)=0\ ...,$$

$$=\frac{1}{9}(1-\frac{5D}{9}-\frac{D^2}{9}+\frac{25}{81}D^2)(3x^2+4x)$$

$$=\frac{1}{9}\left[1-\frac{5D}{9}+\left(\frac{-1}{9}+\frac{25}{81}\right)D^2\right](3x^2+4x)$$

$$= \frac{1}{9}\left[1 - \frac{5D(3x^2+4x)}{9} + \left(\frac{16}{81}\right)D^2(3x^2+4x)\right]$$

$$= \frac{1}{9}\left[3x^2 + 4x - \frac{5(6x+4)}{9} + \left(\frac{16}{81}\right)6\right]$$

$$= \frac{1}{9}\left[3x^2 + 4x - \frac{30x}{9} - \frac{20}{9} + \frac{32}{27}\right] = \frac{1}{9}\left[3x^2 + \frac{2x}{3} - \frac{28}{27}\right]$$

$$P.I_{\ 2} = \frac{e^{-3x}}{D^2+5D+9} = \frac{e^{-3x}}{(-3)^2+5(-3)+9} \quad (D=-3)$$

$$= \frac{e^{-3x}}{9-15+9} = \frac{e^{-3x}}{3}$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = e^{\frac{-5}{2}x}\left(A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right) + \frac{1}{9}\left[3x^2 + \frac{2x}{3} - \frac{28}{27}\right] + \frac{e^{-3x}}{3}$$

Type 4

If x=e^{ax}g(x) [double factor]

Step 1: keep the exponential term ideal, however replace D by (D+a) in f(t)

i.e., P.I=$\frac{e^{ax}X}{f(t)}= e^{ax}\frac{X}{F(D+a)}$

- **Find the particular integral of $(D-1)^2\, y = e^x sinx.$ (L1)**

  Solution:

The particular integral $= \frac{e^x sinx}{(D-1)^2} = \frac{e^x sinx}{(D+1-1)^2} \quad (D \to D+1)$

$$= \frac{e^x sinx}{D^2} = e^x\frac{1}{D}(-cosx) = e^x(-sinx) = -e^x sinx$$

- **Find the particular integral of $(D^2+4D+4)y\ = x\,e^{-2x}$. (L1)**

  Solution:

  The particular integral $= \frac{x\,e^{-2x}.}{D^2\ +\ 4D\ +\ 4}$

$$= e^{-2x} \frac{x}{(D-2)^2 + 4(D-2) + 4} \qquad (D \to D-2)$$

$$= e^{-2x} \frac{x}{D^2 - 4D + 4 + 4D - 8 + 4}$$

$$= e^{-2x} \frac{x}{D^2} = e^{-2x} \frac{1}{D} \int x\, dx = e^{-2x} \frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= e^{-2x} \int \frac{x^2}{2}\, dx = e^{-2x}\left(\frac{x^3}{6}\right) = \frac{x^3}{6} e^{-2x}$$

- **Find the particular integral of $(D^2 + 4D + 5)y = e^{-2x}Cosx$. (L1)**

<u>**Solution:**</u>

The particular integral $= \dfrac{e^{-2x}Cosx.}{D^2 + 4D + 5}$

$$= e^{-2x} \left[\frac{cosx}{(D-2)^2 + 4(D-2) + 5}\right] (D \to D-2)$$

$$= e^{-2x} \frac{cosx}{D^2 - 4D + 4 + 4D - 8 + 5}$$

$$= e^{-2x} \frac{cosx}{D^2+1} = e^{-2x} \frac{cosx}{-1+1} (Put\ D^2 = -1)$$

Multiplying the numerator by x and differentiating denominator wrt D

$$P.I = e^{-2x}\left(\frac{xcosx}{2D}\right) \qquad \left(where\ \frac{1}{D} = \int dx\right)$$

$$y(x) = xe^{-2x}\left(\frac{\int cosdx}{2}\right) = \left(\frac{xsinx}{2}\right)e^{-2x}$$

- **Find the particular integral of $(D^2 + 4D + 4)y = e^{-2x}Cos3x.$ (L1)**

<u>**Solution:**</u>

The particular integral $= \dfrac{e^{-2x}Cos3x}{D^2 + 4D + 4}$

$$= e^{-2x} \frac{cos3x}{(D-2)^2 + 4(D-2) + 4} \qquad (D \to D-2)$$

$$= e^{-2x} \frac{cos3x}{D^2 - 4D + 4 + 4D - 8 + 4} = e^{-2x} \frac{cos3x}{D^2}$$

$$= e^{-2x}\ \frac{cos3x}{-9}\quad (Put\ D^2 = -9) = -\frac{e^{-2x}cos3x}{9}$$

- **Solve $(D^2 - 4D - 5)y = xe^{2x} + 3cos4x + e^{2x}$. (L3)** **[8 Marks]**

<u>Solution:</u>

Auxiliary equation is $m^2 - 4m - 5 = 0$

$$(m-5)(m+1) = 0$$

$$m = 5, -1$$

The complimentary function $= Ae^{m_1x} + Be^{m_2x}$

$C.F = Ae^{-x} + Be^{5x}$

$$P.I = \frac{xe^{2x} + 3cos4x + e^{2x}}{D^2 - 4D - 5}$$

$$P.I_{\ 1} = \frac{xe^{2x}}{D^2 - 4D - 5}$$

$$= e^{2x}\left[\frac{x}{(D+2)^2 - 4(D+2) - 5}\right](D \to D+2)$$

$$= e^{2x}\left[\frac{x}{D^2 + 4D + 4 - 4D - 8 - 5}\right] = e^{2x}\left[\frac{x}{D^2 - 9}\right]$$

$$= \frac{e^{2x}}{-9}\left[\frac{x}{1 - \frac{D^2}{9}}\right] = \frac{e^{2x}}{-9}\left[\left(1 - \frac{D^2}{9}\right)^{-1} x\right]$$

$$[Formula: (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 - \cdots]$$

$$= -\frac{e^{2x}}{9}\left[\left(1 + \frac{D^2}{9} + \frac{D^4}{81} + \cdots\right)x\right] = -\frac{e^{2x}}{9}\left(x + \frac{D^2x}{9} + \frac{D^4x}{81} + \cdots\right)$$

$$D(x) = 1, D^2(x) = D^3(x) = \cdots = 0,$$

$$= -\frac{e^{2x}}{9}(x + 0 + 0 + \cdots) = -\frac{xe^{2x}}{9}$$

$$P.I_{\ 2} = \frac{3cos4x}{D^2 - 4D - 5} = \frac{3cos4x}{-16 - 4D - 5}\qquad (D^2 = -16)$$

$$= \frac{3cos4x}{-21-4D} = \frac{-3cos4x}{21+4D}$$

$$Multiplying\ and\ dividing\ by\ (21-4D)$$

$$= \frac{-3(21-4D)cos4x}{(21-4D)(21+4D)} = \frac{-3(21-4D)cos4x}{(21)^2 - 16\,D^2}$$

$$= \frac{-3(21-4D)cos4x}{441-16(-16)} \qquad (D^2 = -16)$$

$$= -\frac{3(21cos4x - 4Dcos4x)}{697} = -\frac{3(21cos4x + 16sin4x)}{697}$$

$$P.I_{\ 3} = \frac{e^{2x}}{D^2-4D-5}$$

$$= \frac{e^{2x}}{(2)^2-4(2)-5}(D=2) = \frac{e^{2x}}{4-8-5} = -\frac{e^{2x}}{9}$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2} + P.I_{\ 3}$

$$y(x) = Ae^{-x} + Be^{5x} - \frac{xe^{2x}}{9} - \frac{3(21cos4x + 16sin4x)}{697} - \frac{e^{2x}}{9}$$

- **Solve $(D^2 + 3D + 5\ )y = e^x sin^2 x.$ (L3)** **[8 Marks]**

Solution:

Auxiliary equation is $m^2 + 3m + 5\ = 0$

$$m = \frac{-3 \pm \sqrt{9-4(1)(5)}}{2(1)}$$

$$m = \frac{-3 \pm i\sqrt{11}}{2} = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = \frac{-3}{2}\ and\ \beta = \frac{\sqrt{11}}{2}$$

$$C.F = e^{\frac{-3}{2}x}\left(Acos\frac{\sqrt{11}}{2}x + Bsin\frac{\sqrt{11}}{2}x\right)$$

$$P.I = \frac{e^x sin^2x}{D^2 + \ 3D + 5}$$

$$= e^x \left[\frac{sin^2x}{(D+1)^2 + \ 3(D+1) + 5}\right] (D \to D+1)$$

$$= e^x \left[\frac{\left(\frac{1 - cos2x}{2}\right)}{D^2 + 2D + 1 + \ 3D + 3 + 5}\right]$$

$$= \frac{e^x}{2}\left[\frac{1 - cos2x}{D^2 + 5D + 9}\right] = \frac{e^x}{2}\left[\frac{1}{D^2 + 5D + 9}\right] - \frac{e^x}{2}\left[\frac{cos2x}{D^2 + 5D + 9}\right]$$

$$P.I\ _1 = \frac{e^x}{2}\left[\frac{1}{D^2 + 5D + 9}\right] = \frac{e^x}{2}\left[\frac{e^{0x}}{D^2 + 5D + 9}\right]$$

$$= \frac{e^x}{2}\left[\frac{e^{0x}}{0 + 0 + 9}\right] = \frac{e^x}{2}\left[\frac{1}{9}\right] = \frac{e^x}{18}$$

$$P.I\ _2 = -\frac{e^x}{2}\left[\frac{cos2x}{D^2 + 5D + 9}\right]$$

$$= -\frac{e^x}{2}\left[\frac{cos2x}{(-4) + 5D + 9}\right] \qquad (D^2 = -4)$$

$$= -\frac{e^x}{2}\left[\frac{cos2x}{5D + 5}\right] = -\frac{e^x}{10}\left[\frac{cos2x}{D + 1}\right]$$

$$Multiplying\ and\ dividing\ by\ (D-1)$$

$$= -\frac{e^x}{10}\left[\frac{(D-1)cos2x}{(D-1)(D+1)}\right]$$

$$= -\frac{e^x}{10}\left[\frac{(D-1)cos2x}{(D^2 - 1)}\right]$$

$$= -\frac{e^x}{10}\left[\frac{(Dcos2x - cos2x)}{(-4 - 1)}\right] \qquad (D^2 = -4)$$

$$= -\frac{e^x}{10}\left[\frac{(-2sin2x - cos2x)}{-5}\right]$$

$$= \frac{e^x}{50}[-2sin2x - cos2x] = -\frac{e^x}{50}[2sin2x + cos2x]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) \ = e^{\frac{-3}{2}x}\left(Acos\frac{\sqrt{11}}{2}x + Bsin\frac{\sqrt{11}}{2}x\right) + \frac{e^x}{18} - \frac{e^x}{50}[2sin2x + cos2x]$$

- **Solve $(D^2 + 4D + 3)\, y = e^{-x}sinx + xe^{3x}$. (L3)** **[8 Marks]**

Solution:

Auxiliary equation is $m^2 + 4m + 3 \ = 0$

$$(m + 3)(m + 1) = 0$$

$$m = -3, -1$$

The complimentary function $= Ae^{m_1x} + Be^{m_2x}$

$C.F = Ae^{-3x} + Be^{-x}$

The particular integral $= \frac{e^{-x}sinx + xe^{3x}}{D^2 + \ 4D \ + \ 3}$

$$P.I_{\ 1} = \frac{e^{-x}sinx}{D^2 + \ 4D \ + \ 3}$$

$$= e^{-x}\left[\frac{sinx}{(D-1)^2 + \ 4(D-1) + \ 3}\right] (D \to D - 1)$$

$$= e^{-x}\left[\frac{sinx}{D^2 - 2D + 1 + \ 4D - 4 + \ 3}\right]$$

$$= e^{-x}\left[\frac{sinx}{D^2 + \ 2D}\right]$$

$$= e^{-x}\left[\frac{sinx}{-1 + \ 2D}\right] \qquad (D^2 = -1)$$

Multiplying and dividing by $(-1 - 2D)$

$$= e^{-x}\left[\frac{(-1 - 2D)sinx}{(-1 - 2D)(-1 + \ 2D)}\right]$$

$$= e^{-x}\left[\frac{(-1 - 2D)sinx}{(-1)^2 - (2D)^2}\right] = e^{-x}\left[\frac{(-1 - 2D)sinx}{1 - 4D^2}\right]$$

$$= e^{-x}\left[\frac{(-1-2D)sinx}{1-4(-1)}\right] \qquad (D^2 = -1)$$

$$= e^{-x}\left[\frac{(-sinx - 2Dsinx)}{5}\right] = e^{-x}\left[\frac{(-sinx - 2cosx)}{5}\right]$$

$$= -\frac{e^{-x}}{5}[sinx + 2cosx]$$

$$P.I_{\ 2} = \frac{xe^{3x}}{D^2 + 4D + 3} = e^{3x}\left[\frac{x}{(D+3)^2 + 4(D+3) + 3}\right] (D \to D+3)$$

$$= e^{3x}\left[\frac{x}{D^2 + 6D + 9 + 4D + 12 + 3}\right] = e^{3x}\left[\frac{x}{D^2 + 10D + 24}\right]$$

$$= e^{3x}\left[\frac{x}{24\left(1 + \frac{D^2 + 10D}{24}\right)}\right] = \frac{e^{3x}}{24}\left[\left(1 + \frac{D^2 + 10D}{24}\right)^{-1} x\right]$$

$[Formula: (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots]$

$$= \frac{e^{3x}}{24}\left[1 - \left(\frac{D^2 + 10D}{24}\right) + \left(\frac{D^2 + 10D}{24}\right)^2 - + \cdots\right]x$$

$$= \frac{e^{3x}}{24}\left[1 - \frac{D^2}{24} - \frac{10D}{24} + \cdots\right]x = \frac{e^{3x}}{24}\left[x - \frac{D^2x}{24} - \frac{10Dx}{24} + \cdots\right]$$

$$D(x) = 1, D^2(x) = D^3(x) = \cdots = 0,$$

$$= \frac{e^{3x}}{24}\left[x - 0 - \frac{10(1)}{24} + 0 + \cdots\right] = \frac{e^{3x}}{24}\left[x - \frac{10}{24}\right] = \frac{e^{3x}}{24}\left[x - \frac{5}{12}\right]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = Ae^{-3x} + Be^{-x} - \frac{e^{-x}}{5}[sinx + 2cosx] + \frac{e^{3x}}{24}\left[x - \frac{5}{12}\right]$$

- **Solve $(\boldsymbol{D^2 - 2D + 2})\boldsymbol{y = e^{-x}x^2 + 5 + e^{-2x}}$. (L3)** **[8 Marks]**

<u>Solution:</u>

Auxiliary equation is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$m = \frac{2 \pm 2i}{2} = 1 \pm i = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha x}(A cos\beta x + B sin\beta x)$

$$\alpha = 1 and\ \beta = 1$$

C.F$= e^{x}(A cosx + B sinx)$

$$P.I = \frac{e^{-x}x^2 + 5 + e^{-2x}}{D^2 - 2D + 2}$$

$$P.I_1 = e^{-x}\left[\frac{x^2}{(D-1)^2 - 2(D-1) + 2}\right](D \to D - 1)$$

$$= e^{-x}\left[\frac{x^2}{D^2 - 2D + 1 - 2D + 2 + 2}\right]$$

$$= e^{-x}\left[\frac{x^2}{D^2 - 4D + 5}\right]$$

$$= e^{-x}\left[\frac{x^2}{\left(1 + \frac{D^2 - 4D}{5}\right)}\right]$$

$$= \frac{e^{-x}}{5}\left[\left(1 + \frac{D^2 - 4D}{5}\right)^{-1} x^2\right]$$

$$[Formula: (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots]$$

$$= \frac{e^{-x}}{5}\left[1 - \left(\frac{D^2 - 4D}{5}\right) + \left(\frac{D^2 - 4D}{5}\right)^2 - + \cdots\right] x^2$$

$$= \frac{e^{-x}}{5}\left[1 - \frac{D^2}{5} + \frac{4D}{5} + \frac{D^4}{25} - \frac{8D^3}{25} + \frac{16D^2}{25} \ldots\right] x^2$$

$$= \frac{e^{-x}}{5}\left[1 + \frac{4D}{5} + \left(\frac{-1}{5} + \frac{16}{25}\right) D^2 + \frac{D^4}{25} - \frac{8D^3}{25} \ldots\right] x^2$$

$$D(x^2) = 2x,\ D^2(x^2) = 2,\ D^3(x^2) = D^4(x^2) \dots\dots = 0$$

$$= \frac{e^{-x}}{5}\left[x^2 + \frac{4Dx^2}{5} + \left(\frac{11}{25}\right)D^2x^2\right] = \frac{e^{-x}}{5}\left[x^2 + \frac{8x}{5} + \left(\frac{11}{25}\right)2\right]$$

$$= \frac{e^{-x}}{5}\left[x^2 + \frac{8x}{5} + \frac{22}{25}\right] = \frac{e^{-x}}{125}[25x^2 + 40x + 22]$$

$$P.I_2 = \frac{5 + e^{-2x}}{D^2 - 2D + 2} = \frac{5e^{0x} + e^{-2x}}{D^2 - 2D + 2}$$

$$= \frac{5e^{0x}}{D^2 - 2D + 2} + \frac{e^{-2x}}{D^2 - 2D + 2}$$

$$= \frac{5e^{0x}}{0 - 0 + 2}(D = 0) + \frac{e^{-2x}}{4 + 4 + 2}(D = -2) = \frac{5e^{0x}}{2} + \frac{e^{-2x}}{10}$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(x) = e^x(Acosx + Bsinx) + \frac{e^{-x}}{125}[25x^2 + 40x + 22] + \frac{5}{2} + \frac{e^{-2x}}{10}$$

- **Solve $(\boldsymbol{D^2 - 4D + 4})\boldsymbol{y = e^x sin3x.}$ (L3)** **[8 Marks]**

<u>Solution:</u>

Auxiliary equation is $m^2 - 4m + 4 = 0$

$$(m - 2)^2 = 0$$

$$m = 2,2$$

The complimentary function $= (Ax + B)e^{mx}$

$$C.F = (Ax + B)e^{2x}$$

The particular integral $= \dfrac{e^x sin3x}{D^2 - 4D + 4}$

$$= e^x\left[\frac{sin3x}{(D + 1)^2 - 4(D + 1) + 4}\right](D \to D + 1)$$

$$= e^x\left[\frac{sin3x}{D^2 + 2D + 1 - 4D - 4 + 4}\right](D \to D + 1)$$

$$= e^x\left[\frac{sin3x}{D^2 - 2D + 1}\right] = e^x\left[\frac{sin3x}{-9 - 2D + 1}\right]\ (D^2 = -9)$$

$$= e^x\left[\frac{sin3x}{-8-2D}\right] = -\frac{e^x}{2}\left[\frac{sin3x}{4+D}\right]$$

$$Multiplying\ and\ dividing\ by\ (4-D)$$

$$= -\frac{e^x}{2}\left[\frac{(4-D)sin3x}{(4-D)(4+D)}\right] = -\frac{e^x}{2}\left[\frac{(4-D)sin3x}{16-\ D^2}\right]$$

$$= -\frac{e^x}{2}\left[\frac{(4-D)sin3x}{16-(-9)}\right] \qquad (D^2 = -9)$$

$$= -\frac{e^x}{2}\left[\frac{4sin3x - D(sin3x)}{25}\right] = -\frac{e^x}{50}[4sin3x - 3cos3x]$$

General solution $= C.F + P.I$

$$y(x) = (Ax+B)e^{2x} - \frac{e^x}{50}[4sin3x - 3cos3x]$$

- **Solve $(D^2 - 8D + 9)y = e^{-x}(x^2+3)$. (L3)** **[8 Marks]**

Solution:

$$m = \frac{8 \pm \sqrt{64-4(1)(9)}}{2(1)} = \frac{8 \pm \sqrt{28}}{2} = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$$

The complimentary function $= Ae^{m_1 x} + Be^{m_2 x}$

$$C.F = Ae^{(4-\sqrt{7})x} + Be^{(4+\sqrt{7})x}$$

The particular integral $= \dfrac{e^{-x}(x^2+3)}{D^2 - 8D + 9}$

$$P.I = e^{-x}\left[\frac{x^2+3}{(D-1)^2 - 8(D-1) + 9}\right] \qquad (D \to D-1)$$

$$= e^{-x}\left[\frac{x^2+3}{D^2 - 2D + 1 - 8D + 8 + 9}\right]$$

$$= e^{-x}\left[\frac{x^2+3}{D^2 - 2D + 1 - 8D + 8 + 9}\right]$$

$$= e^{-x}\left[\frac{x^2+3}{D^2 - 10D + 18}\right]$$

$$= \frac{e^{-x}}{18}\left[\frac{x^2+3}{1+\left(\frac{D^2-10D}{18}\right)}\right]$$

$$= \frac{e^{-x}}{18}\left[1+\left(\frac{D^2-10D}{18}\right)\right]^{-1}(x^2+3)$$

$$[Formula: (1+x)^{-1} = 1-x+x^2-x^3+x^4-\cdots]$$

$$= \frac{e^{-x}}{18}(1-\left(\frac{D^2-10D}{18}\right)+\left(\frac{D^2-10D}{18}\right)^2-\cdots)(x^2+3)$$

$$= \frac{1}{18}(1+\frac{10D}{18}-\frac{D^2}{18}++\frac{1}{324}(D^4-20D^3+100D^2)-\cdots)(x^2+3)$$

$D(x^2+3) = 2x,\ \ D^2(x^2+3\ ) = 2,\ \ D^3(x^2+3\ ) = 0, D^4(x^2+3) = 0 \ldots,$

$$= \frac{1}{18}(1+\frac{10D}{18}-\frac{D^2}{18}+\frac{1}{324}(100D^2)-\cdots)(x^2+3)$$

$$= \frac{1}{18}\left[1+\frac{10D}{18}+\left(\frac{-1}{18}+\frac{100}{324}\right)D^2\right](x^2+3)$$

$$= \frac{1}{18}\left[x^2+3+\frac{5(2x)}{9}+\left(\frac{82}{324}\right)2\right]$$

$$= \frac{1}{18}\left[x^2+3+\frac{10x}{9}+\left(\frac{41}{82}\right)\right]$$

$$= \frac{1}{18}\left[x^2+\frac{10x}{9}+\frac{287}{82}\right]$$

General solution $= C.F + P.I$

$$y(x) = Ae^{(4-\sqrt{7})x}+Be^{(4+\sqrt{7})x}+\frac{1}{18}\left[x^2+\frac{10x}{9}+\frac{287}{82}\right]$$

- **Solve $(D^3-3D^2+3D-1)y = e^{-x}sin2x.$ (L3)** **[8 Marks]**

Solution:

Auxiliary equation is $m^3-3m^2+3m-1 = 0$

$$(m-1)^3 = 0$$

$$(m-1)(m-1)(m-1) = 0$$

$$m = 1,1,1$$

The complimentary function $= (Ax^2 + Bx + C)e^{mx}$

$C.F = (Ax^2 + Bx + C)e^x$

The particular integral $= \dfrac{e^{-x} sin2x}{D^3 - 3D^2 + 3D - 1}$

$$= e^{-x}\left[\frac{sin3x}{(D-1)^3 - 3(D-1)^2 + 3(D-1) - 1}\right] (D \rightarrow D-1)$$

$$= e^{-x}\left[\frac{sin2x}{D^3 - 3D^2 + 3D - 1 - 3(D^2 - 2D + 1) + 3D - 3 - 1}\right]$$

$$= e^{-x}\left[\frac{sin2x}{D^3 - 6D^2 + 12D - 8}\right]$$

$$= e^{-x}\left[\frac{sin2x}{-4D + 24 + 12D - 8}\right] (D^2 = -4)$$

$$= e^{-x}\left[\frac{sin2x}{16 + 8D}\right] = \frac{e^{-x}}{8}\left[\frac{sin2x}{2 + D}\right]$$

Multiplying and dividing by $(2 - D)$

$$= \frac{e^{-x}}{8}\left[\frac{(2-D)sin2x}{(2-D)(2+D)}\right] = \frac{e^{-x}}{8}\left[\frac{(2-D)sin2x}{4 - D^2}\right]$$

$$= \frac{e^{-x}}{8}\left[\frac{2sin2x - Dsin2x}{4 - (-4)}\right] (D^2 = -4)$$

$$= \frac{e^{-x}}{8}\left[\frac{2sin2x - 2cos2x}{8}\right] = \frac{e^{-x}}{64}[sin2x - cos2x]$$

General solution $= C.F + P.I$

$$y(x) = (Ax^2 + Bx + C)e^x + \frac{e^{-x}}{64}[sin2x - cos2x]$$

Type 5

If X=x^nsinax or cosax

Step 1: Replace this formulae by Euler's expansion and do the fourth process.

Step 2:

$e^{i\theta}$=cos θ+isinθ

cosθ $= R.P[\ \ e^{i\theta}]$

sinθ =IP[$e^{i\theta}$]

- **Solve** $(\boldsymbol{D^2+4})\boldsymbol{y} = \boldsymbol{xsinx.}$ **(L3)** **[8 Marks]**

Solution:

Auxiliary equation $is\ m^2 + 4 = 0$

$$m^2 = -4$$

$m = 0 \pm 2i = \alpha \pm i\beta$

The complimentary function $= e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = 0\ and\ \beta = 2$$

$C.F = e^{0x}(Acos2x + Bsin2x)$

$C.F = Acos2x + Bsin2x$

The particular integral $= \dfrac{xsinx}{D^2+4}$

$$\left[Formula: \frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V\right]$$

$$P.I = x\left[\frac{sinx}{D^2+4}\right] - \frac{2D}{\{D^2+4\}^2}sinx$$

$$= x\left[\frac{sinx}{-1+4}\right] - \frac{2D}{\{-1+4\}^2}sinx \qquad (D^2 = -1)$$

$$= \frac{xsinx}{3} - \frac{2D(sinx)}{\{3\}^2} = \frac{xsinx}{3} - \frac{2cosx}{9}$$

$$= \frac{1}{9}(3xsinx - 2cosx)$$

General solution $= C.F + P.I$

$$y(x) = Acos2x + Bsin2x + \frac{1}{9}(3xsinx - 2cosx)$$

- **Solve $(D^2 - 2D + 1)y = xe^x cosx.$ (L3)** **[8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1,1$$

The complimentary function $= (Ax + B)e^{mx}$

$$C.F = (Ax + B)e^x$$

The particular integral $= \dfrac{xe^x cosx}{D^2 - 2D + 1}$

$$= e^x \left[\frac{xcosx}{(D+1)^2 - 2(D+1) + 1}\right] (D \to D+1)$$

$$= e^x \left[\frac{xcosx}{D^2 + 2D + 1 - 2D - 2 + 1}\right] = e^x \left[\frac{xcosx}{D^2}\right]$$

$$\left[Formula: \frac{1}{f(D)} xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V\right]$$

$$P.I = e^x \left[x\frac{1}{D^2}cosx - \frac{2D}{\{D^2\}^2}cosx\right]$$

$$= e^x \left[x\frac{1}{D}\int cosx\,dx - \frac{2}{D^3}cosx\right] = e^x \left[x\frac{1}{D}(sinx) - \frac{2}{D^2}\int cosx\,dx\right]$$

$$= e^x \left[x\int sinx\,dx - \frac{2}{D^2}(sinx)\right]$$

$$= e^x \left[-xcosx - \frac{2}{D}(-cosx)\right] = e^x[-xcosx + 2sinx]$$

General solution $= C.F + P.I$

$$y(x) = (Ax + B)e^x - e^x[xcosx - 2sinx]$$

- **Solve $(D^3 - 1)x = xsinx.$ (L3)** **[8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^3 - 1 = 0$

$$(m-1)(m^2+m+1) = 0$$

$m = 1,\ m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)}$$

$$m = \frac{-1 \pm i\sqrt{3}}{2} = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = \frac{-1}{2}\, and\ \beta = \frac{\sqrt{3}}{2}$$

$$\therefore C.F = Ae^{x} + e^{\frac{-x}{2}}\left(Bcos\frac{\sqrt{3}}{2}x + Csin\frac{\sqrt{3}}{2}x\right)$$

The particular integral $= \dfrac{xsinx}{D^3 - 1}$

$$\left[Formula\colon \frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V\right]$$

$$P.I = x\frac{1}{D^3-1}(sinx) - \frac{3D^2}{\{D^3-1\}^2}sinx$$

$$P.I = x\frac{1}{-D-1}sinx - \frac{3(-1)}{\{-D-1\}^2}sinx \quad (D^2 = -1)$$

$$= -x\frac{1}{D+1}sinx - \frac{3(-1)}{\{D+1\}^2}sinx$$

$$= -x\frac{(D-1)}{(D+1)(D-1)}sinx + \frac{3}{D^2+2D+1}sinx$$

$$= -x\frac{(D-1)}{D^2-1}sinx + \frac{3}{-1+2D+1}sinx \quad (D^2 = -1)$$

$$= -x\frac{(Dsinx - sinx)}{-1-1} + \frac{3}{2D}sinx$$

$$= -x\frac{(cosx - sinx)}{-2} + \frac{3}{2}(-cosx)$$

$$= \frac{x}{2}(cosx - sinx) - \frac{3}{2}(cosx)$$

$$General\ solution = C.F + P.I$$

$$y(x) = Ae^{x} + e^{\frac{-x}{2}}\left(Bcos\frac{\sqrt{3}}{2}x + Csin\frac{\sqrt{3}}{2}x\right) + \frac{x}{2}(cosx - sinx)$$

$$-\frac{3}{2}(cosx)$$

- **Solve $(\mathbf{D^2 + 9})\mathbf{x} = \mathbf{4cos(t + \frac{\pi}{3})}$ which is satisfied by $\mathbf{x = 0}$ when $\mathbf{t = 0}$ and by $\mathbf{x = 2}$ when $\mathbf{t = \frac{\pi}{6}}$. (L3)** **[8 Marks]**

<u>**Solution:**</u>

Auxiliary equation is $m^2 + 9 = 0$

$$m^2 = -9$$

$m = 0 \pm 3i = \alpha \pm i\beta$

The complimentary function $= e^{\alpha x}(Acos\beta x + Bsin\beta x)$

$$\alpha = 0\ and\ \beta = 3$$

$$\therefore\ C.F = e^{0x}(Acos3x + Bsin3x)$$

$$C.F = Acos3t + Bsin3t$$

The particular integral $= \dfrac{4cos\left(t + \frac{\pi}{3}\right)}{D^2 + 9}$

$$P.I = \frac{4cos\left(t + \frac{\pi}{3}\right)}{-1 + 9}(D^2 = -1)$$

$$= \frac{4cos\left(t + \frac{\pi}{3}\right)}{8} = \frac{1}{2}cos\left(t + \frac{\pi}{3}\right)$$

General solution $= C.F + P.I$

$$x(t) = Acos3t + Bsin3t + \frac{1}{2}co\,s\left(t + \frac{\pi}{3}\right) - - - (1)$$

$Given\ that\ , x\ =\ 0$ when $t\ =\ 0$

$$\Longrightarrow\ x(0) = Acos0 + Bsin0 + \frac{1}{2}co\,s\left(0+\frac{\pi}{3}\right) = 0$$

$$\Longrightarrow\ A + \frac{1}{2}co\,s\left(\frac{\pi}{3}\right) = 0$$

$$\Longrightarrow\ A = -\frac{1}{2}co\,s\left(\frac{\pi}{3}\right) = -\frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4}$$

$And, x(t) =\ 2$ when $t = \frac{\pi}{6}$

$$\Longrightarrow\ x\left(\frac{\pi}{6}\right) = Acos\left(\frac{\pi}{6}\right) + Bsin\left(\frac{\pi}{6}\right) + \frac{1}{2}co\,s\left(\frac{\pi}{6}+\frac{\pi}{3}\right) = 2$$

$$\Longrightarrow\ Acos3\left(\frac{\pi}{6}\right) + Bsin3\left(\frac{\pi}{6}\right) + \frac{1}{2}cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) = 2$$

$$\Longrightarrow\ Acos\left(\frac{\pi}{2}\right) + Bsin\left(\frac{\pi}{2}\right) + \frac{1}{2}cos\left(\frac{\pi}{2}\right) = 2$$

$$\Longrightarrow\ A(0) + B(1) + \frac{1}{2}(0) = 2$$

$$\Longrightarrow\ B = 2$$

$Substituting\ the\ values\ of\ A\ and\ B\ in\ equation\ (1), we\ have$

$$x(t) = -\frac{1}{4}cos3t + 2sin3t + \frac{1}{2}cos\left(t+\frac{\pi}{3}\right)$$

<u>Differential equations with variable co-officient</u>

Let x$=e^z$; z=log x

$x\frac{dy}{dx}$=D'y ; D'=$\frac{d}{dz}$

$x^2\frac{d^2y}{dx^2}$= (D'^{2}y– D')y

- **Solve $(x^2D^2 + \ xD)\ y = 0.$ (L3)**

  <u>Solution:</u>

  By the transformation $x = e^t$, we have

$t = logx,\ xD = D' ,\quad x^2D^2 = D' (D' - 1) \quad where\, D' = \frac{d}{dt}$

$$(D' (D' - 1) + D')y = 0$$

$$(D'^2 - D' + D')y = 0$$

$$D'^2 y = 0$$

$$\frac{d^2y}{dt^2} = 0$$

Integrating on both sides wrt t,we get,

$$\frac{dy}{dt} = A$$

Again Integrating on both sides wrt t,we get,

$$y(t) = At + B$$

$Put\ t = logx,\ then\ \ y(x)\ = A\,logx + B$

- **Solve $x^2y'' - 2xy' + 2y = 0$.(L3)**

Solution:

$If D = \frac{d}{dx},then\ (x^2D^2 - 2xD + 2)y = 0$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D' ,\quad x^2D^2 = D'(D' - 1) \quad where\, D' = \frac{d}{dt}$

$$(D' (D' - 1) - 2D' + 2)y = 0$$

$$(D'^2 - D' - 2D' + 2)y = 0$$

$$(D'^2 - 3D' + 2)y = 0$$

Auxiliary equation is $m^2 - 3m + 2 = 0$

$(m - 1)(m - 2) = 0, \qquad m = 1,2$

The complimentary function $= Ae^t + Be^{2t}$

$Put\ t = logx,\ then\ \ y(x)\ =\ Ae^{logx} + Be^{2logx} = Ax + Bx^2$

- **Solve $(x^2D^2 + xD + 1)\ y = 0$. (L3)**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D'\ ,\quad x^2D^2 = D'\ (D'-1)\quad whereD' = \frac{d}{dt}$

$$(D'\ (D'-1) + D' + 1)y = 0$$

$$(D'^2 - D' + D' + 1)y = 0$$

$$(D'^2 + 1)y = 0$$

Auxiliary equation is $m^2 + 1 = 0,\quad m^2 = -1,\ m = 0 \pm i = \alpha \pm i\beta$

The complimentary function $= e^{\alpha t}(Acos\beta t + Bsin\beta t)\alpha = 0\ and\ \beta = 1$

C.F $= e^{0t}(Acost + Bsint) = Acost + Bsint$

$Put\ t = logx,\ then\ \ y(x)\ =\ Acos(logx) + Bsin(logx)$

- **Find the particular integral of** $\boldsymbol{x^2y'' - 2xy' - 4y = x^4}$**. (L1)**

Solution:

$If\ D = \frac{d}{dx},then\ (x^2D^2 -\ 2xD - 4)y = x^4$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D'\ ,\quad x^2D^2 = D'\ (D'-1)\quad whereD' = \frac{d}{dt}$

$$(D'\ (D'-1) - 2D' - 4)y = (e^t)^4$$

$$(D'^2 - 3D' - 4)y = e^{4t}$$

The particular integral $= \dfrac{e^{4t}}{(D'^2 - 3D' - 4)} = \dfrac{e^{4t}}{16 - 12 - 4}\quad (D' = 4)$

Multiplying the numerator by t and differentiating denominator wrt D',

$$P.I = \frac{te^{4t}}{2D' - 3} = \frac{te^{4t}}{8 - 3}\quad (D' = 4) = \frac{te^{4t}}{5}$$

$$P.I = \frac{logxe^{4logx}}{5} = \frac{x^4logx}{5}$$

- **Solve** $(\boldsymbol{x^2D^2 + 2xD - 20})\ \boldsymbol{y = 0}$**. (L3)**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad where D' = \frac{d}{dt}$

$$(D'(D'-1) + 2D' - 20)y = 0$$

$$(D'^2 - D' + 2D' - 20)y = 0$$

$$(D'^2 + D' - 20)y = 0$$

Auxiliary equation

$$m^2 + m - 20 = 0, (m-4)(m+5) = 0, m = 4, -5$$

The complimentary function is $y(t) = Ae^{-5t} + Be^{4t}$

Put $t = logx$, *then* $y(x) = Ae^{-5logx} + Be^{4logx} = \frac{A}{x^5} + Bx^4$

- **Solve** $(\boldsymbol{x^2D^2 - xD + 4})\boldsymbol{y} = \boldsymbol{cos(logx)} + \frac{\boldsymbol{sin(logx)}}{\boldsymbol{x}}$**. (L3) [8 Marks]**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad Where\, D' = \frac{d}{dt}$

$$(D'(D'-1) - D' + 4)y = cos\,t + \frac{sint}{e^t}$$

$$(D'^2 - D' - D' + 4)y = cos\,t + e^{-t}sint$$

$$(D'^2 - 2D' + 4)y = cos\,t + e^{-t}sint$$

Auxiliary equation $(D' = m) = m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4 - 4(1)(4)}}{2(1)} = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3} = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha t}(Acos\beta t + Bsin\beta t)$

$$\alpha = 1\ and\ \beta = \sqrt{3}$$

$\therefore$ *C.F*$= e^t(Acos\sqrt{3}t + Bsin\sqrt{3}t)$

The particular integral $= \dfrac{\cos t + e^{-t}\sin t}{D'^2 - 2D' + 4}$

$$P.I_{\ 1} = \frac{\cos t}{D'^2 - 2D' + 4} = \frac{\cos t}{-1 - 2D' + 4}\,(D'^2 = -1)$$

$$= \frac{\cos t}{3 - 2D'}$$

$$\textit{Multiplying and dividing by } (3 + 2D'D')$$

$$= \frac{(3 + 2D')\cos t}{(3 + 2D')(3 - 2D')} = \frac{(3 + 2D')\cos t}{9 - 4D'^2}$$

$$= \frac{3\cos t + 2D'\cos t}{(9 + 4)} \quad (D'^2 = -1)$$

$$= \frac{1}{13}(3\cos t - 2\sin t)$$

$$P.I_{\ 2} = \frac{e^{-t}\sin t}{D'^2 - 2D' + 4}$$

$$= e^{-t}\left[\frac{\sin t}{(D' - 1)^2 - 2(D' - 1) + 4}\right](D' \to D' - 1)$$

$$= e^{-t}\left[\frac{\sin t}{D'^2 - 2D' + 1 - 2D' + 2 + 4}\right]$$

$$= e^{-t}\left[\frac{\sin t}{D'^2 - 4D' + 7}\right]$$

$$= e^{-t}\left[\frac{\sin t}{-1 - 4D' + 7}\right](D'^2 = -1)$$

$$= e^{-t}\left[\frac{\sin t}{6 - 4D'}\right] = \frac{e^{-t}}{2}\left[\frac{\sin t}{3 - 2D'}\right]$$

$$\textit{Multiplying and dividing by } (3 + 2D')$$

$$= \frac{e^{-t}}{2}\left[\frac{(3 + 2D')\sin t}{(3 + 2D')(3 - 2D')}\right] = \frac{e^{-t}}{2}\left[\frac{(3 + 2D')\sin t}{9 - 4D'^2}\right]$$

$$= \frac{e^{-t}}{2}\left[\frac{(3 + 2D')\sin t}{9 + 4}\right](D'^2 = -1)$$

$$= \frac{e^{-t}}{2}\left[\frac{3sint + 2D'\,sint}{13}\right] = \frac{e^{-t}}{26}[3sint + 2cost]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(t) = e^{t}\left(Acos\sqrt{3}t + Bsin\sqrt{3}t\right) + \frac{1}{13}(3cost - 2sint)$$
$$+\frac{e^{-t}}{26}[3sint + 2cost]$$

Put $t = logx$, *then*

$$y(x) = e^{logx}\left(Acos\left(\sqrt{3}logx\right) + Bsin\left(\sqrt{3}logx\right)\right)$$
$$+\frac{1}{13}\left(3cos(logx) - 2sin(logx)\right)$$
$$+\frac{e^{-logx}}{26}[3sin(logx) + 2cos(logx)]$$
$$y(x) = x\left(Acos\left(\sqrt{3}logx\right) + Bsin\left(\sqrt{3}logx\right)\right)$$
$$+\frac{1}{13}\left(3cos(logx) - 2sin(logx)\right)$$
$$+\frac{1}{26x}[3sin(logx) + 2cos(logx)]$$

- **Solve $(x^2D^2 + xD + 1)y = sin(2logx)\,sin(logx)$. (L3) [8 Marks]**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D' - 1)$ *Where* $D' = \frac{d}{dt}$

$(D'(D' - 1) + D' + 1)y = sin(2t)\,sin(t)$

$\left(D'^2 - D' + D' + 1\right)y = sin\,2t\,sint$

$\left(D'^2 + 1\right)y = sin\,2t\,sint$

Auxiliary equation $(D' = m) = m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$$\alpha = 0 \text{ and } \beta = 1$$

$\therefore$ C.F$= e^{0t}(A\cos t + B\sin t) = A\cos t + B\sin t$

The particular integral $= \dfrac{\sin 2t \sin t}{D'^2 + 1}$

$$\left[Formula:\ \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right]$$

$$P.I = \frac{\frac{1}{2}[\cos(2t-t) - \cos(2t+t)]}{D'^2 + 1} = \frac{1}{2}\left[\frac{\cos t - \cos 3t}{D'^2 + 1}\right]$$

$$P.I_1 = \frac{1}{2}\left[\frac{\cos t}{D'^2 + 1}\right] = \frac{1}{2}\left[\frac{\cos t}{-1 + 1}\right] (D'^2 = -1)$$

Multiplying the numerator by t and differentiating denominator wrt D', we have

$$P.I_1 = \frac{1}{2}\left[\frac{t\cos t}{2D'}\right] = \frac{t}{4}\int \cos t\, dt = \frac{t}{4}(\sin t)$$

$$P.I_2 = -\frac{1}{2}\left[\frac{\cos 3t}{D'^2 + 1}\right] = -\frac{1}{2}\left[\frac{\cos 3t}{-9 + 1}\right] (D'^2 = -9) = \frac{1}{16}[\cos 3t]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(t) = A\cos t + B\sin t + \frac{t}{4}(\sin t) + \frac{1}{16}[\cos 3t]$$

$$y(t) = A\cos t + B\sin t + \frac{1}{16}(4t\sin t + \cos 3t)$$

Put $t = \log x$, then

$$y(x) = A\cos(\log x) + B\sin(\log x) + \frac{1}{16}[4(\log x)\sin(\log x) + \cos(3\log x)]$$

- **Solve $x^2y'' + 4xy' + 2y = 6x$. (L3) [8 Marks]**

Solution:

$If\ D = \frac{d}{dx}, then\ (x^2D^2 + 4xD + 2)y = 6x$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad Where\,D' = \frac{d}{dt}$

$$(D'(D'-1) + 4D' + 2)y = 6e^t$$

$$(D'^2 - D' + 4D' + 2)y = 6e^t$$

$$(D'^2 + 3D' + 2)y = 6e^t$$

Auxiliary equation $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$m = -1, -2$

The complimentary function $= Ae^{m_1x} + Be^{m_2x}$

$$C.F = Ae^{-t} + Be^{-2t}$$

The particular integral $= \dfrac{6e^t}{D'^2 + 3D' + 2}$

$= \dfrac{6e^t}{1+3+2}(D' \to 1) = \dfrac{6e^t}{6} = e^t$

General solution $= C.F + P.I$

$y(t) = Ae^{-t} + Be^{-2t} + e^t$

$Put\ t = logx, then\ \ y(x) = Ae^{-logx} + Be^{-2logx} + e^{logx}$

$$= Ae^{logx^{-1}} + Be^{logx^{-2}} + e^{logx}$$

$= Ax^{-1} + Bx^{-2} + x = \dfrac{A}{x} + \dfrac{B}{x^2} + x$

- **Solve** $(\boldsymbol{x^2D^2 + xD - 9})\boldsymbol{y} = \frac{5}{x^2}$ **. (L3)** **[8 Marks]**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad Where\,D' = \frac{d}{dt}$

$$(D'(D'-1)+D'-9)y=\frac{5}{(e^t)^2}$$

$$\left(D'^2-D'+D'-9\right)y=5e^{-2t}$$

$$\left(D'^2-9\right)y=5e^{-2t}$$

Auxiliary equation $(D'=m)=m^2-9=0$

$$m^2=9$$

$$m=\pm 3$$

The complimentary function $=Ae^{m_1t}+Be^{m_2t}=Ae^{-3t}+Be^{3t}$

The particular integral $=\dfrac{5e^{-2t}}{D'^2-2D'+4}$

$$P.I=\frac{5e^{-2t}}{D'^2-9}$$

$$=\frac{5e^{-2t}}{4-9}==\frac{5e^{-2t}}{-5}=-e^{-2t}$$

General solution $=C.F+P.I$

$$y(t)=Ae^{-3t}+Be^{3t}-e^{-2t}$$

$Put\ t=logx,\ then\ y(x)=Ae^{-3logx}+Be^{3logx}-e^{-2logx}$

$$=Ae^{logx^{-3}}+Be^{logx^{3}}-e^{logx^{-2}}$$

$$y(x)=Ax^{-3}+Bx^3-x^{-2}=\frac{A}{x^3}+Bx^3-\frac{1}{x^2}$$

- **Solve $\mathbf{\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12logx}{x^2}}$.(L3)** **[8 Marks]**

Solution:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12logx}{x^2}$$

$Multiplying\ both\ sides\ by\ x^2$

$$x^2\frac{d^2y}{dx^2}+\frac{x^2}{x}\frac{dy}{dx}=\frac{x^2 12logx}{x^2}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = 12logx$$

$Let\ \frac{d}{dx} = D, then\ x^2D^2y + xDy = 12logx$

$$(x^2D^2 + xD)y = 12logx$$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad Where\ D' = \frac{d}{dt}$

$$(D'(D'-1) + D')y = 12t$$

$$(D'^2 - D' + D')y = 12t$$

$$D'^2y = 12t$$

Auxiliary equation is $m^2 = 0$ *(Where* $D' = m$)

$$m = 0,0$$

The complimentary function $= (At + B)e^{mt}$

C.F$= (At + B)e^{0t} = At + B$

The particular integral $= \dfrac{12t}{D'^2}$

Integrating wrt t,we get,

$$P.I = \frac{1}{D'}\int 12t\,dt = 12\left(\frac{t^2}{2}\right) = 6t^2$$

Again Integrating wrt t,we get,

$$= \int 6t^2dt = 6\left(\frac{t^3}{3}\right) = 2t^3$$

General solution $= C.F + P.I$

$y(t) = At + B + 2t^3$

Put $t = logx$, *then* $y(x) = A\,logx + B + 2(logx)^3$

- **Find the particular integral of** $\boldsymbol{x^2y'' + xy' + y = 4sin(logx)}$**. (L1)**

  Solution:

$If D = \frac{d}{dx}$ *,then* $(x^2D^2 + xD + 1)y = 4sin(logx)$

By the transformation $x = e^t$*, we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'\,(D'-1)\quad whereD' = \frac{d}{dt}$

$$(D'\,(D'-1) + D' + 1)y = 4sint$$

$$({D'}^2 - D' + D' + 1)y = 4sint$$

$({D'}^2 + 1)y = 4sint$

The particular integral $= \dfrac{4sint}{({D'}^2+1)} = \dfrac{4sint}{(-1+1)}$

Multiplying the numerator by t and differentiating denominator wrt D',

$$P.I = \frac{4tsint}{2D'} = \frac{4t\int sint\,dt}{2}$$

$$= 2t(-cost) = -2tcost$$

Put $t = logx$*, then,* $P.I = -2(logx)cos(logx)$

- **Solve** $\boldsymbol{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4sin(logx).}$ **(L3)** **[8 Marks]**

Solution:

By the transformation $x = e^t$*, we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'\,(D'-1)\quad Where\,D' = \frac{d}{dt}$

$(D'\,(D'-1) + D' + 1)y = 4\,sin(t)$

$\left({D'}^2 - D' + D' + 1\right)y = 4\,sin\,t$

$\left({D'}^2 + 1\right)y = 4\,sint$

Auxiliary equation $(D' = m):\quad m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha t}(Acos\beta t + Bsin\beta t)$

$$\alpha = 0 \; and \; \beta = 1$$

$\therefore$ C.F$= e^{0t}(Acost + Bsint) = Acost + Bsint$

The particular integral $= \dfrac{4\,sint}{D'^2 + 1} = \dfrac{4\,sint}{D'^2 + 1} = \left[\dfrac{4\,sint}{-1 + 1}\right] \; (D'^2 = -1)$

Multiplying the numerator by t and differentiating denominator wrt D' , we have

$= 4t\left[\dfrac{sint}{2D'}\right] = 2t\displaystyle\int sin\,t\;dt = 2t(-cost)$

General solution $= C.F + P.I$

$y(t) = Acost + Bsint - 2tcost$

$Put\; t = logx,\; then$

$$y(x) = Acos(logx) + Bsin(logx) - 2(logx)cos(logx)$$

- **Solve** $(\boldsymbol{x^2D^2 - 2xD - 4})\boldsymbol{y} = \boldsymbol{32(logx)^2}$**. (L3)** **[8 Marks]**

Solution:

By the transformation $x = e^t$*, we have*

$t = logx,\;\; xD = D',\quad x^2D^2 = D'\,(D' - 1)\quad$ *Where* $D' = \frac{d}{dt}$

$(D'\,(D' - 1) - 2D' - 4)y = 32(t)^2$

$\left(D'^2 - D' - 2D' - 4\right)y = 32t^2$

$\left(D'^2 - 3D' - 4\right)y = 32t^2$

Auxiliary equation $(D' = m)\; is\;\; m^2 - 3m - 4 = 0$

$$(m - 4)(m + 1) = 0$$

$$m = -1{,}4$$

The complimentary function $= Ae^{m_1t} + Be^{m_2t}$

$$C.F = Ae^{-t} + Be^{4t}$$

The particular integral $= \dfrac{32t^2}{D'^2 - 3D' - 4}$

$$P.I = \frac{32t^2}{-4\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]} = \frac{32}{-4}\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} t^2$$

$$[Formula: (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 - \cdots]$$

$$= -8\left[1 + \left(\frac{D'^2 - 3D'}{4}\right) + \left(\frac{D'^2 - 3D'}{4}\right)^2 + \cdots\right] t^2$$

$$= -8\left[1 - \frac{3D'}{4} + \frac{D'^2}{4} + \frac{1}{16}(D'^4 - 6D'^3 + 9D'^2) + \cdots\right] t^2$$

$$D'\,(t^2) = 2t,\ \ D'^{\,2}(t^2\,) = 2,\ \ D'^{\,3}(t) = 0 \ldots,$$

$$= -8\left[1 - \frac{3D'}{4} + \left(\frac{1}{4} + \frac{9}{16}\right)D'^2\right] t^2$$

$$= -8\left[t^2 - \frac{3D't^2}{4} + \left(\frac{13}{16}\right)D'^2 t^2\right]$$

$$= -8\left[t^2 - \frac{3(2t)}{4} + \left(\frac{13}{16}\right)2\right] = -8\left[t^2 - \frac{3t}{2} + \left(\frac{13}{8}\right)\right]$$

$$= \frac{-8}{8}[8t^2 - 12t + 13] = -[8t^2 - 12t + 13]$$

General solution $= C.F + P.I$

$$y(t) = Ae^{-t} + Be^{4t} - [8t^2 - 12t + 13]$$

$Put\ t = logx,\ then$

$$y(x) = Ae^{-logx} + Be^{4logx} - [8(logx)^2 - 12(logx) + 13]$$

$$y(x) = Ax^{-1} + Bx^4 - [8(logx)^2 - 12(logx) + 13]$$

- **Solve $\boldsymbol{x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = sin(logx)}$. (L3)** **[8 Marks]**

Solution:

$Let\ \frac{d}{dx} = D, then\ x^2D^2y + 4xDy + 2y = sin(logx)$

$$(x^2D^2 + 4xD + 2)y = sin(logx)$$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad Where\ D' = \frac{d}{dt}$

$$(D'(D'-1) + 4D' + 2)y = sint$$

$\left(D'^2 - D' + 4D' + 2\right)y = sint$

$\left(D'^2 + 3D' + 2\right)y = sint$

Auxiliary equation $(D' = m)$; $\quad m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

The complimentary function $= Ae^{m_1t} + Be^{m_2t}$

$$C.F = Ae^{-t} + Be^{-2t}$$

The particular integral $= \dfrac{sint}{D'^2 + 3D' + 2}$

$= \dfrac{sint}{-1 + 3D' + 2}\ (D'^2 = -1) = \dfrac{sint}{3D' + 1}$

$$Multiplying\ and\ dividing\ by\ (3D' - 1)$$

$$= \frac{(3D'-1)sint}{(3D'-1)(3D'+1)} = \frac{(3D'-1)sint}{9D'^2 - 1}$$

$= \dfrac{3D'(sint) - sint}{9(-1) - 1} = \dfrac{3cost - sint}{-10}$

$= \dfrac{sint - 3cost}{10}$

General solution $= C.F + P.I$

$$y(t) = Ae^{-t} + Be^{-2t} + \frac{1}{10}(sint - 3cost)$$

$Put\ t = logx, then$

$$y(x) = Ae^{-logx} + Be^{-2logx} + \frac{1}{10}[sin(logx) - 3cos(logx)]$$

$$y(x) = Ax^{-1} + Bx^{-2} + \frac{1}{10}[sin(logx) - 3cos(logx)]$$

$$y(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{10}[sin(logx) - 3cos(logx)]$$

- **Solve $(x^2D^2 - 2xD + 4)y = x^2 + 2logx.$ (L3)** **[8 Marks]**

Solution:

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad$ *Where* $D' = \frac{d}{dt}$

$$(D'(D'-1) - 2D' + 4)y = (e^t)^2 + 2t$$

$$(D'^2 - 3D' + 4)y = e^{2t} + 2t$$

Auxiliary equation $(D' = m) = m^2 - 3m + 4 = 0$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm i\sqrt{7}}{2} = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha t}(Acos\beta t + Bsin\beta t)$

$$\alpha = \frac{3}{2}\ and\ \beta = \frac{\sqrt{7}}{2}$$

$$C.F = e^{\frac{3}{2}t}\left(Acos\frac{\sqrt{7}}{2}t + Bsin\frac{\sqrt{7}}{2}t\right)$$

The particular integral $= \dfrac{e^{2t} + 2t}{D'^2 - 3D' + 4}$

$$P.I_{\ 1} = \frac{e^{2t}}{D'^2 - 3D' + 4}$$

$$= \frac{e^{2t}}{2^2 - 3(2) + 4}\quad (D' = 2) = \frac{e^{2t}}{2}$$

$$P.I_{\ 2} = \frac{2t}{D'^2 - 3D' + 4} = \frac{2t}{4\left[1 + \frac{D'^2 - 3D'}{4}\right]}$$

$$= \frac{1}{2}\left[1 + \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} t$$

$$[Formula: (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots]$$

$$= \frac{1}{2}\left[1 - \left(\frac{D'^2 - 3D'}{4}\right) + \left(\frac{D'^2 - 3D'}{4}\right)^2 - \cdots\right] t$$

$$= \frac{1}{2}\left[1 + \frac{3D'}{4} - \frac{D'^2}{4} + \cdots\right] t$$

$$D'(t) = 1,\ \ D'^{\,2}(t\,) = 0,\ \ D'^{\,3}(t) = 0 \ldots,$$

$$= \frac{1}{2}\left[t + \frac{3D'(t)}{4} - \frac{D'^2(t)}{4} + \cdots\right] = \frac{1}{2}\left[t + \frac{3}{4}\right]$$

General solution $= C.F + P.I_{\ 1} + P.I_{\ 2}$

$$y(t) = e^{\frac{3}{2}t}\left(Acos\frac{\sqrt{7}}{2}t + Bsin\frac{\sqrt{7}}{2}t\right) + \frac{e^{2t}}{2} + \frac{1}{2}\left[t + \frac{3}{4}\right]$$

$Put\ t = logx,\ then$

$$y(x) = e^{\frac{3}{2}logx}\left(Acos(\frac{\sqrt{7}}{2}logx) + Bsin(\frac{\sqrt{7}}{2}logx)\right) + \frac{e^{2logx}}{2} + \frac{1}{2}\left[logx + \frac{3}{4}\right]$$

$$y(x) = x^{\frac{3}{2}}\left(Acos(\frac{\sqrt{7}}{2}logx) + Bsin(\frac{\sqrt{7}}{2}logx)\right) + \frac{x^2}{2} + \frac{1}{2}\left[logx + \frac{3}{4}\right]$$

- **Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^3 logx.$ (L3) [8 Marks]**

<u>Solution:</u>

$If = \frac{d}{dx}$, $then\ (x^2D^2 - 3xD + 5)y = x^3 logx$

$By\ the\ transformation\ x = e^t,\ we\ have$

$t = logx,\ xD = D',\quad x^2D^2 = D'\,(D' - 1)\quad Where\ D' = \frac{d}{dt}$

$$(D'(D'-1) - 3D' + 5)y = (e^t)^3(t)$$

$$(D'^2 - D' - 3D' + 5)y = te^{3t}$$

$$(D'^2 - 4D' + 5)y = te^{3t}$$

Auxiliary equation $(D' = m)$; $\quad m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{4 \pm i2}{2} = 2 \pm i = \alpha \pm i\beta$$

The complimentary function $= e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$$\alpha = 2 \ and \ \beta = 1$$

$\therefore$ C.F$= e^{2t}(A\cos t + B\sin t)$

The particular integral $= \dfrac{te^{3t}}{D'^2 - 4D' + 5}$

$$P.I = e^{3t}\left[\frac{t}{(D'+3)^2 - 4(D'+3) + 5}\right] \quad (D' \to D' + 3)$$

$$= e^{3t}\left[\frac{t}{D'^2 + 6D' + 9 - 4D' - 12 + 5}\right] = e^{3t}\left[\frac{t}{D'^2 + 2D' + 2}\right]$$

$$= \frac{e^{3t}}{2}\left[\frac{t}{1 + \left(\frac{D'^2 + 2D'}{2}\right)}\right] = \frac{e^{3t}}{2}\left[1 + \left(\frac{D'^2 + 2D'}{2}\right)\right]^{-1} t$$

$$[Formula: (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots]$$

$$= \frac{e^{3t}}{2}\left[1 - \left(\frac{D'^2 + 2D'}{2}\right) + \left(\frac{D'^2 + 2D'}{2}\right)^2 - \cdots\right] t$$

$$= \frac{e^{3t}}{2}\left[1 - \frac{2D'}{2} - \frac{D'^2}{2} + \cdots\right] t$$

$$D'(t) = 1, \ D'^2(t) = 0, \ D'^3(t) = 0 \ \ldots,$$

$$= \frac{e^{3t}}{2}\left[t - \frac{2D't}{2} - \frac{D'^2 t}{2} + \cdots\right] = \frac{e^{3t}}{2}[t - 1]$$

General solution $= C.F + P.I$

$$y(t) = e^{2t}(Acost + Bsint) + \frac{e^{3t}}{2}[t-1]$$

$Put\ t = logx,\ then$

$$y(x) = e^{2logx}[Acos(logx) + Bsin(logx)] + \frac{e^{3logx}}{2}[logx - 1]$$

$$y(x) = x^2[Acos(logx) + Bsin(logx)] + \frac{x^3}{2}[logx - 1]$$

- **Solve $x^2y'' + 4xy' + 2y = xlogx$ *such that* $x = 1, y = 0 = \frac{dy}{dt}$. (L3)**

[8 Marks]

Solution:

If $= \frac{d}{dx}$, *then* $(x^2D^2 + 4xD + 2)y = xlogx$

By the transformation $x = e^t$, *we have*

$t = logx,\ xD = D',\quad x^2D^2 = D'(D'-1)\quad$ *Where* $D' = \frac{d}{dt}$

$$(D'(D'-1) + 4D' + 2)y = e^t(t)$$

$$(D'^2 - D' + 4D' + 2)y = te^t$$

$$(D'^2 + 3D' + 2)y = te^t$$

Auxiliary equation $(D' = m)$; $\quad m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

The complimentary function $= Ae^{m_1t} + Be^{m_2t}$

$$C.F = Ae^{-t} + Be^{-2t}$$

The particular integral $= \dfrac{te^t}{D'^2 + 3D' + 2}$

$$P.I = e^t\left[\frac{t}{(D'+1)^2 + 3(D'+1) + 2}\right](D' \to D' + 3)$$

$$= e^t\left[\frac{t}{D'^2 + 2D' + 1 + 3D' + 3 + 2}\right] = e^t\left[\frac{t}{D'^2 + 5D' + 6}\right]$$

$$= \frac{e^t}{6}\left[\frac{t}{1+\left(\frac{D'^2+5D'}{6}\right)}\right] = \frac{e^t}{6}\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]^{-1} t$$

$$[Formula: (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots]$$

$$= \frac{e^t}{6}\left[1-\left(\frac{D'^2+5D'}{6}\right)+\left(\frac{D'^2+5D'}{6}\right)^2 - \cdots\right] t$$

$$= \frac{e^t}{6}\left[1-\frac{5D'}{6}-\frac{D'^2}{6}+\cdots\right] t$$

$$D'\,(t) = 1,\;\; D'^{\,2}(t\,) = 0,\;\; D'^{\,3}(t) = 0 \,...,$$

$$= \frac{e^t}{6}\left[t-\frac{5D't}{6}\right] = \frac{e^t}{6}\left[t-\frac{5}{6}\right]$$

$$\text{General solution} = C.F + P.I$$

$$y(t) = Ae^{-t} + Be^{-2t} + \frac{e^t}{6}\left[t-\frac{5}{6}\right]$$

$Put\ t = logx,\ then$

$$y(x) = Ae^{-logx} + Be^{-2logx} + \frac{e^{logx}}{6}\left[logx-\frac{5}{6}\right]$$

$$y(x) = Ax^{-1} + Bx^{-2} + \frac{x}{6}\left[logx-\frac{5}{6}\right]$$

$$y(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6}\left[logx-\frac{5}{6}\right] \_\_\_\_\_\_\_\_(1)$$

$$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{x}{6}\left(\frac{1}{x}\right) + \left[logx-\frac{5}{6}\right]\left(\frac{1}{6}\right)$$

$$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{1}{6} + \frac{1}{6}logx - \frac{5}{36} \qquad \_\_\_\_\_(2)$$

$Given\ that,\qquad x\ =\ 1, y\ =\ 0 = \frac{dy}{dx}$

From (1), $\quad y(1) = 0 \Longrightarrow \frac{A}{1} + \frac{B}{1} + \frac{1}{6}\left[log1 - \frac{5}{6}\right] = 0$

$(\because log1 = 0), A + B - \frac{5}{36} = 0 (or)\ A + B = \frac{5}{36}$ ____(3)

From (2), $\frac{dy}{dx}(at\ x = 1) = 0 \Longrightarrow -\frac{A}{1} - \frac{2B}{1} + \frac{1}{6} + \frac{1}{6}log1 - \frac{5}{36} = 0$

$\Longrightarrow -A - 2B + \frac{1}{6} - \frac{5}{36} = 0$

$$\Longrightarrow -A - 2B + \frac{1}{36} = 0\ (or)\ A + 2B = \frac{1}{36} \quad \text{\_\_\_\_}(4)$$

$$Solving\ for\ A\ and\ B\ from\ (3) and\ (4)$$

$(4) - (3) \Longrightarrow B = \frac{1}{36} - \frac{5}{36} = -\frac{4}{36} = -\frac{1}{9}$

$from(3) \quad A = \frac{5}{36} + \frac{1}{9} = \frac{9}{36} = \frac{1}{4}$

$Substituting\ the\ values\ of\ A\ and\ B\ in\ (1)$

$y(x) = \frac{1}{4x} - \frac{1}{9x^2} + \frac{x}{6}\left[logx - \frac{5}{6}\right]$

- **Solve $(x^2D^2 - 3xD + 4)y = x^2$ given that $y\,(1) = 1$ and $y'\,(1) = 0.$**

(L3)[8 Marks]

<u>Solution:</u>

By the transformation $x = e^t$, we have

$t = logx,\ xD = D',\quad x^2D^2 = D'(D' - 1) \quad Where\ D' = \frac{d}{dt}$

$$(D'(D' - 1) - 3D' + 4)y = (e^t)^2$$

$$(D'^2 - D' - 3D' + 4)y = e^{2t}$$

$$(D'^2 - 4D' + 4)y = te^t$$

Auxiliary equation $(D' = m);\ m^2 - 4m + 4 = 0$

$$(m - 2)(m - 2) = 0$$

$$m = 2,2$$

The complimentary function $= (At + B)e^{mt}$

$$C.F = (At + B)e^{2t}$$

The particular integral $= \dfrac{e^{2t}}{D'^2 - 4D' + 4}$

$$= \frac{e^{2t}}{4 - 8 + 4} \qquad (D' = 2)$$

Multiplying the numerator by t and differentiating denominator wrt D'

$$= \frac{te^{2t}}{2D' - 4} = \frac{te^{2t}}{4 - 4} \quad (D' = 2)$$

Again multiplying the numerator by t and differentiating denominator wrt D', we have

$$P.I = \frac{t^2 e^{2t}}{2}$$

General solution $= C.F + P.I$

$$y(t) = (At + B)e^{2t} + \frac{t^2 e^{2t}}{2}$$

Put $t = logx$, then

$$y(x) = (Alogx + B)e^{2logx} + \frac{(logx)^2 e^{2logx}}{2}$$

$$y(x) = (Alogx + B)x^2 + \frac{(logx)^2 x^2}{2} \qquad \_\_\_\_\_\_\_\_(1)$$

$$\frac{dy}{dx} = (Alogx + B)2x + \left(\frac{A}{x}\right)x^2 + \frac{1}{2}\left[(logx)^2 2x + 2logx\left(\frac{1}{x}\right)x^2\right]$$

$$= 2Axlogx + 2Bx + Ax + x(logx)^2 + xlogx \qquad \_\_\_\_\_\_\_\_(2)$$

Given that $y(1) = 1$ *and* $y'(1) = 0$

From (1), $y(1) = 1 \Longrightarrow (Alog1 + B)(1)^2 + \dfrac{(log1)^2(1)^2}{2} = 1$

$\Longrightarrow B = 1 \qquad (\because log1 = 0)$

From (2), $y'(1) = 0$

$$\Longrightarrow 2A(1)log1 + 2B(1) + A(1) + 1(log1)^2 + 1log1 = 0$$

$\Longrightarrow\ A + 2B = 0\ (\because\ log1 = 0)$

$\Longrightarrow A = -2(1) = -2$

Substituting the values of A and B in (1)

$$y(x) = (-2logx + 1)x^2 + \frac{(logx)^2 x^2}{2}$$

$$(or)\ y(x) = \frac{x^2}{2}[2(-2logx + 1) + (logx)^2]$$

UNIT-IV
THREE DIMENSIONAL ANALYTICAL GEOMETRY

Part A (2 Marks questions)

Direction cosines of a line

If α, β, γ be the angles which any line makes with the positive direction of the axes X,Y and Z ,then $cos\alpha, cos\beta, cos\gamma$ are called the direction cosines of the given line . They are denoted by (l, m, n).

Direction cosines of a line through origin

Let (x, y, z) be the co-ordinates of P and let OP=r

Let OP make angles α, β, γ with the positive X,Y,Z axes

D.C's of OP are

$$l = cos\alpha = \frac{x}{r}, m = cos\beta = \frac{y}{r}, n = cos\gamma = \frac{z}{r}$$

Result: Sum of squares of D.C's of a line is always one.

i.e.) $l^2 + m^2 + n^2 = 1$

Direction ratios of a line.

Numbers which are proportional to direction cosines of a line are called direction ratios (D.rs) .They are denoted by (a, b, c).

$a = kl, b = km, c = kn.$

Result: Direction ratios of line joining two points $(\boldsymbol{x_1, y_1, z_1})$ $\boldsymbol{and}$ $(\boldsymbol{x_2, y_2, z_2})$ **are** $(\boldsymbol{x_2 - x_1, y_2 - y_1, z_2 - z_1})$

Result: Relation between D.R's and D.C's

If (a, b, c) are D.R's and (l, m, n) are D.C's of a line, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \qquad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \qquad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Find direction cosines of the line joining (4,−1,6) and (−1,3,0).**

  Solution:

  The given points are (4, −1,6) and (−1,3,0).

  The direction ratios of the line joining are (4+1,-1-3,6-0) =(5,-4,6)

$$l = \frac{5}{\sqrt{25 + 16 + 36}} = \frac{5}{\sqrt{77}}$$

$$m = \frac{-4}{\sqrt{25 + 16 + 36}} = \frac{-4}{\sqrt{77}}$$

$$n = \frac{6}{\sqrt{25+16+36}} = \frac{6}{\sqrt{77}}$$

Direction cosines are $\left(\frac{5}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{6}{\sqrt{77}}\right)$

- **Find direction cosines of the line joining (7,1,0) and $(2, -3, 1)$.**

  Solution:

  The given points are $(7,1,0)$ and $(2, -3, 1)$.

  The direction ratios of the line joining are $(7-2, 1+3, 0-1) = (5, 4, -1)$

$$l = \frac{5}{\sqrt{25+16+1}} = \frac{5}{\sqrt{42}}$$
$$m = \frac{4}{\sqrt{25+16+1}} = \frac{4}{\sqrt{42}}$$
$$n = \frac{-1}{\sqrt{25+16+1}} = \frac{-1}{\sqrt{42}}$$

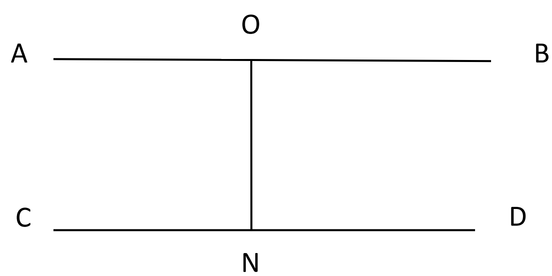
Direction cosines are $\left(\frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$

- **Find the direction cosines of a line perpendicular to the two lines whose direction ratios are (1,2,3) and $(-\mathbf{2}, \mathbf{1}, \mathbf{4})$.**

  Solution:

  As AB⊥ ON: $a + 2b + 3c = 0$

  CD⊥ ON: $-2a + b + 4c = 0$



$$\frac{a}{\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}} = \frac{b}{\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}}$$
$$\frac{a}{8-3} = \frac{b}{-6-4} = \frac{c}{1+4}$$

$$\frac{a}{5}=\frac{b}{-10}=\frac{c}{5}$$
$$\frac{a}{1}=\frac{b}{-2}=\frac{c}{1}$$
$$a=1, b=-2, c=1$$
$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{\sqrt{6}}$$
$$m=\frac{b}{\sqrt{a^2+b^2+c^2}}=\frac{-2}{\sqrt{1^2+(-2)^2+1^2}}=\frac{-2}{\sqrt{6}}$$
$$n=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{\sqrt{6}}$$
$\therefore$ D.C's of a line are $(l,m,n)=\left(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

- **Prove that the points A(3,1,3),B$(\mathbf{1},-\mathbf{2},-\mathbf{1})$ and C$(-\mathbf{1},-\mathbf{5},-\mathbf{5})$ are collinear.**

  Solution:

  The direction ratios of the line AB are $(1-3,-2-1,-1-3)$
  i.e.,$(-2,-3,-4)$
  The direction ratios of the line CB are $(-1-1,-5+2,-5+1)$
  i.e.,$(-2,-3,-4)$
  The direction ratios of the line AB and CB are the same
  $\Longrightarrow$AB and CB are parallel.
  $\therefore$A,B,C are collinear.

The projection of a line segment joining two points on a given line.

The projection of a line segment joining two points P(x_1,y_1,z_1) and Q(x_2,y_2,z_2) on a line whose direction cosines are (l,m,n) is

$$l(x_1-x_2)+m(y_1-y_2)+n(z_1-z_2)$$

<u>Angle between two lines</u> in terms of both DC's and DR's.

<u>In terms of DC's</u>

The angle between the two lines in terms of direction cosines is given by

$$cos\theta=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{{l_1}^2+{m_1}^2+{n_1}^2}\sqrt{{l_2}^2+{m_2}^2+{n_2}^2}}$$

In terms of DR's

The angle between the two lines in terms of direction ratios is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Conditions for two lines to be parallel and perpendicular in terms of DC'.

The condition that the lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

The condition that the lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) are parallel is

$$l_1 = l_2, m_1 = m_2, n_1 = n_2$$

Conditions for two lines to be parallel and perpendicular in terms of DR's.

The conditions that the lines whose direction ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2) should be

(i) Perpendicular is $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) Parallel is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Prove that the points A(3,1,3),B$(1, -2, -1)$ and C$(-1, -5, -5)$ are collinear.**

  The direction ratios of the line AB are $(1 - 3, -2 - 1, -1 - 3)$
  i.e.,$(-2, -3, -4)$
  The direction ratios of the line CB are $(-1 - 1, -5 + 2, -5 + 1)$
  i.e.,$(-2, -3, -4)$
  The direction ratios of the line AB and CB are the same
  $\Rightarrow$AB and CB are parallel.
  $\therefore$A,B,C are collinear.

Symmetric form of a line

If (x_1, y_1, z_1) is any point lying on a straight line with D.R's (l, m, n), then the symmetric form of the equation is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- **Find the equation of the line passing through $(1, 2, 3)$ and $(-1, 0, 4)$.**

The line passes through $(1,2,3)$ & $(-1,0,4)$. The direction ratios of the line are $(-1-1, 0-2, 4-3) = (-2,-2,1)$.

The equations of the line are

$$\frac{x-1}{-1-1} = \frac{y-2}{0-2} = \frac{z-3}{4-3}$$
$$\frac{x-1}{-2} = \frac{y-2}{-2} = \frac{z-3}{1}$$

- **Find the equation of the line passing through $(0,-2,5)$ and $(7,0,-3)$.**

The line passes through $(0,-2,5)$ & $(7,0,-3)$. The direction ratios of the line are $(7-0, 0+2, -3-5)$.

The equations of the line are

$$\frac{x-0}{7-0} = \frac{y+2}{0+2} = \frac{z-5}{-3-5}$$
$$\frac{x}{7} = \frac{y+2}{2} = \frac{z-5}{-8}$$

- **Show that the following lines are perpendicular $\frac{x+1}{1} = \frac{y+2}{2} = \frac{z-3}{3}$ and $\frac{x-3}{1} = \frac{y-5}{4} = \frac{z-3}{-3}$.**

The direction ratios of the first line are $(1,2,3)$

The direction ratios of the second line are $(1,4,-3)$

$$\therefore \; l_1 l_2 + m_1 m_2 + n_1 n_2 = 1(1) + 2(4) + 3(-3)$$
$$= 1 + 8 - 9$$
$$= 0$$

$\therefore$ The given lines are perpendicular .

- **Find the value of k, if the lines $\frac{x-1}{-1} = \frac{y-3}{k} = \frac{z+1}{5}$ and $\frac{x+1}{-4} = \frac{y+1}{3} = \frac{z}{-k}$ are perpendicular.**

The direction ratios of the first line are $(-1, k, 5)$

The direction ratios of the second line are $(-4,3,-k)$

Given lines are perpendicular $\Longrightarrow \; l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$$\Longrightarrow (-1)(-4) + k(3) + 5(-k) = 0$$
$$\Longrightarrow 4 + 3k - 5k = 0$$
$$\Longrightarrow 4 - 2k = 0$$

$$\Longrightarrow 2k = 4$$
$$\Longrightarrow k = 2$$

- **Find the equation of the line joining the points $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ and $(\mathbf{7}, \mathbf{1}, -\mathbf{2})$.**

The equation of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are
$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$
The required equations are
$$\frac{x - 1}{1 - 7} = \frac{y - 2}{2 - 1} = \frac{z - 3}{3 + 2}$$
$$\frac{x - 1}{-6} = \frac{y - 2}{1} = \frac{z - 3}{5}$$

- **Find the symmetric form of the line jointly given by the planes $\mathbf{3x + 2y - z - 4 = 0}$ and $\mathbf{4x + y - 2z + 3 = 0}$. (L1)** **[8 Marks]**

Let the symmetrical form of the line be
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$
where (x_1, y_1, z_1) is any point on the line and (l, m, n) are direction ratios of line

To find a point on the line , put $z = 0$ in given equation
$$3x + 2y = 4 ------(1)$$
$$4x + y = -3 ------ (2)$$
$$(1) \times 4 \Rightarrow 12x + 8y = 16 --------(3)$$
$$(2) \times 3 \Rightarrow 12x + 3y = -9 -------- (4)$$
(3)−(4)⇒5y=25
$$\therefore y = 5$$
Put $y = 5$ in (1), we get,
$$3x + 2(5) = 4$$
$$3x = -6$$
$$\therefore x = -2$$
$\therefore$ Point (x_1, y_1, z_1) is $(-2,5,0)$

To find $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$:

The line is perpendicular to the normal of the planes ,
$$3x + 2y + z = 4 -------- (5)$$
$$4x + y - 2z = -3 ------(6)$$

Applying the condition of perpendicularity between the line and normals to planes (5) and (6) respectively ,we get

$$3l + 2m + n = 0$$

$$4l + m - 2n = 0$$

By using the rule of cross multiplication ,

$$\frac{l}{-4-1} = \frac{-m}{-6-4} = \frac{n}{3-8}$$

$$\frac{l}{-5} = \frac{-m}{-10} = \frac{n}{-5}$$

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

$(1, -2, 1)$ are proportional to l, m, n

Substituting the proportional values of l, m, n and the point (x_1, y_1, z_1) in symmetrical form of line equation we get ,

$$\frac{x+2}{1} = \frac{y-5}{-2} = \frac{z-0}{1}$$

- **Find the symmetric form of the line jointly given by the planes $3x - 2y + z - 1 = 0$ and $5x + 4y - 6z - 2 = 0$. [8 Marks]**

Let the symmetrical form of the line be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

where (x_1, y_1, z_1) is any point on line and l, m, n are direction ratios of line

To find a point on the line , put $z = 0$ in given equation

$$3x - 2y = 1 ------------(1)$$

$$5x + 4y = 2 ----------- (2)$$

$$(1) \times 4 \Rightarrow 12x - 8y = 4 ----(3)$$

$$(2) \times 2 \Rightarrow 10x + 8y = 4 ----(4)$$

$$(3)+(4) \Longrightarrow 22x = 8$$

$$\Longrightarrow x = \frac{8}{22}$$

$$\therefore x = \frac{4}{11}$$

Put $x = \frac{4}{11}$ in (1), we get,

$$3\left(\frac{4}{11}\right) - 2y = 1$$

$$\Rightarrow \quad -2y = 1 - \frac{12}{11}$$

$$\Rightarrow -2y = \frac{-1}{11}$$

$$\therefore y = \frac{1}{22}$$

$\therefore$ Point (x_1, y_1, z_1) is $\left(\frac{4}{11}, \frac{1}{22}, 0\right)$

To find $\boldsymbol{l, m, n}$:

The line is perpendicular to the normal of the planes ,

$$3x - 2y + z = -1 - - - - - - - - - - - - - - -(5)$$

$$5x + 4y - 6z = 2 - - - - - - - - - - - - - - (6)$$

Applying the condition of perpendicularity between the line and normals to planes (5) and (6) respectively ,we get

$$3l - 2m + n = 0$$

$$5l + 4m - 6n = 0$$

By using the rule of cross multiplication ,

$$\frac{l}{12 - 4} = \frac{-m}{-18 - 5} = \frac{n}{12 + 10}$$

$$\frac{l}{8} = \frac{-m}{-23} = \frac{n}{22}$$

(8,23,22) are proportional to l, m, n

Substituting the proportional values of l, m, n and the point (x_1, y_1, z_1) in symmetrical form of line equation we get ,

$$\frac{x - \frac{4}{11}}{8} = \frac{y - \frac{1}{22}}{23} = \frac{z}{22}$$

- **Find the foot, the length and equation of the perpendicular from the point (1,–2,2) to the line $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z+2}{1}$. [8 Marks]**

Let P be the given point (1,–2,2)

Let the foot of perpendicular be Q and

Let Q be $(2r + 1, 3r + 4, r - 2)$

(i.e),the general point on the given line L.

$\therefore$ The direction ratios of line PQ are $(2r+1-1, 3r+4+2, r-2-2)$

(i.e.,) $(2r, 3r+6, r-4)$

The direction ratios of the line L are (2,3,1)

Since L is perpendicular to PQ ,we get,

$$2(2r)+3(3r+6)+1(r-4)=0$$
$$\text{(i.e.,)} \quad 4r+9r+18+r-4=0$$
$$\Longrightarrow \quad 14r+14=0 \quad \Longrightarrow r=-1$$

$\therefore$ Q is $(2(-1)+1, 3(-1)+4, -1-2)=(-1,1,-3)$

The D.R's of PQ are $(-1-1, 1+2, -3-2)=(-2,3,-5)$

The equation of PQ are

$$\frac{x-1}{-2}=\frac{y-4}{3}=\frac{z+2}{-5}$$

Distance PQ =perpendicular distance $=\sqrt{4+9+25}=\sqrt{38}$

- **Find the foot, the length and equation of the perpendicular from the point $(1,0,-3)$ to the line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$. [8 Marks]**

Let the equation of the given line be

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} ----(1)$$

Any point on the line (1) is $\mathrm{P}(3r+2, 4r+3, 5r+4)$

The given point is $\mathrm{Q}(1,0,-3)$

Assume that the line PQ is perpendicular to the line (1) then P is the foot of perpendicular distance from the given point to the line (1)

Now the drs of the line PQ are $3r+2-1, 4r+3-0, 5r+4-(-3)$

$$(i.e.,) 3r+1, 4r+3, 5r+7$$

Also the drs of the line (1) are 3,4,5

The two lines with drs a_1, b_1, c_1 &a_2, b_2, c_2 and perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

Since the line and PQ are perpendicular ,we have

$$3(3r+1)+4(4r+3)+5(5r+7)=0$$
$$9r+3+16r+12+25r+35=0$$
$$50r+50=0$$
$$50r=-50$$

$$r = -1$$

Substitute the value $r = -1$ in (2), we get

$$P(3(-1) + 2, 4(-1) + 3, 5(-1) + 4)$$

(i.e.,)$P(-3 + 2, -4 + 3, -5 + 4)$

(i.e.,)$P(-1, -1, -1)$

which is the foot of the perpendicular ,to the line (1) from $(1,0,-3)$

Now PQ=$\sqrt{[1-(-1)]^2 + [0-(-1)]^2 + [-3-(-1)]^2}$

=$\sqrt{(1+1)^2 + (0+1)^2 + (-3+1)^2}$

=$\sqrt{2^2 + 1^2 + (-2)^2}$

=$\sqrt{9}$

=3

which is the perpendicular distance from the point $(1,0,-3)$ to the line (1)

Equation of PQ are

$$\frac{x-1}{-1-1} = \frac{y-0}{-1-0} = \frac{z+3}{-1+3}$$

$$\frac{x-1}{-2} = \frac{y-0}{-1} = \frac{z+3}{2}$$

<u>Skew lines</u>

Two lines which are neither parallel nor coplanar are called skew lines.

- **Find the equation and length of the shortest distance between the lines $\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$ and $\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$.** [**16** Marks]

<u>Solution:</u>

The equation of the first line be

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} = r - - - - - - - (1)$$

Any point on the line (1) be

$$\text{P}(3r+6, -r+7, r+4) - - - - - - (A)$$

The equation of the second line be

$$\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}=k-----------(2)$$

Any point on the line (2) be

$$\mathrm{Q}(-3k,2k-9,4k+2)-------(B)$$

Join P and Q and assume that PQ is perpendicular to both the lines (1) and (2).Then by the definition of the shortest distance , the length PQ is the length of the shortest distance and the equation of the line PQ is the equation of the shortest distance .

The direction ratios of the line PQ are

$$\big(3r+6-(3k),-r+7-(2k+9),r+4-(4k+2)\big)$$
$$\Longrightarrow (3r+6+3k,-r+7-2k+9,r+4-4k-2)$$
$$\Longrightarrow (3r+3k+6,-r-2k+16,r-4k+2)$$

Also the direction ratios of the line (1) are $(3,-1,1)$ and the direction ratios of the line (2) are $(-3,2,4)$

The two lines with direction ratios (a_1,b_1,c_1) and (a_2,b_2,c_2) are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

Since the line (1) and PQ are perpendicular ,we have

$$3(3r+3k+6)-1(-r-2k+16)+1(r-4k+2)=0$$
$$9r+9k+18+r+2k-16+r-4k+2=0$$
$$11r+7k=-4--------(3)$$

Since the line (2)and PQ are perpendicular ,we have

$$-3(3r+3k+6)+2(-r-2k+16)+4(r-4k+2)=0$$
$$-9r-9k-18-2r-4k+32+4r-16k+8=0$$
$$-7r-29\mathrm{k}+22=0$$
$$7r+29k=22-----------(4)$$

Solving (3) and (4) , we get

$$(3)\times 7 \Longrightarrow 77r+49k=-28 \quad ------(5)$$
$$(4)\times 11 \Longrightarrow 77r+319k=242------(6)$$
$$(6)-(5)\Longrightarrow 270k=270$$
$$\Longrightarrow k=1$$

Substituting the value $k=1$ in equation (3), we get,

$11r+7(1)=-4 \qquad \Longrightarrow 11r=-4-7 \qquad \Longrightarrow r=-1$

Substituting the value $r=-1$ in(A), we get

$$P(3(-1)+6,-(-1)+7,-1+4)$$
$$\Longrightarrow P(-3+6,1+7,-1+4)$$
$$\Longrightarrow P(3,8,3)$$

Substituting the value $k=1$ in (B), we get

$$Q(-3(1),2(1)-9,4(1)+2)$$
$$\Longrightarrow Q(-3,2-9,4+2)$$
$$\Longrightarrow Q(-3,-7,6)$$

Now , the length of the shortest distance

PQ=$\sqrt{[3-(-3)^2]+[8-(-7)]^2+[3-6]^2}$

$=\sqrt{(3+3)^2+(8+7)^2+(-3)^2}$

$=\sqrt{36+225+9}=\sqrt{270}=3\sqrt{30}$

The equation of the straight line passing through the points (x_1,y_1,z_1) , (x_2,y_2,z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

∴ The equation of the shortest distance through the points P(3,8,3) & Q$(-3,-7,6)$ is

$$\frac{x-3}{3-(-3)}=\frac{y-8}{8-(-7)}=\frac{z-3}{3-6}$$
$$\frac{x-3}{6}=\frac{y-8}{15}=\frac{z-3}{-3}$$

- **Find the equation and the length of the line of shortest distance between the lines $\frac{x-5}{3}=\frac{y-7}{-16}=\frac{z-3}{7}$ *and* $\frac{x-9}{3}=\frac{y-13}{8}=\frac{z-15}{-5}$. [16 Marks]**

<u>Solution:</u>

The equation of the first line be

$$\frac{x-5}{3}=\frac{y-7}{-16}=\frac{z-3}{7}=r------(1)$$

Any point on the line (1) be

$$\text{P}(3r+5,-16r+7,7r+3)--------(A)$$

The equation of the second line be

$$\frac{x-9}{3}=\frac{y-13}{8}=\frac{z-15}{-5}=k--------(2)$$

Any point on the line (2) be

$$Q(3k+9, 8k+13, -5k+15) -------- \quad (B)$$

Join P and Q and assume that PQ is perpendicular to both the lines (1) and (2).Then by the definition of the shortest distance , the length PQ is the length of the shortest distance and the equation of the line PQ is the equation of the shortest distance .

The direction ratios of the line PQ are

$$\big(3r+5-(3k+9), -16r+7-(8k+13), 7r+3-(-5k+15)\big)$$

$$(3r+5-3k-9, -16r+7-8k-13, 7r+3+5k-15)$$

$$(3r-3k-4, -16r-8k-6, 7r+5k-12)$$

Also the direction ratios of the line (1) are $(3,-16,7)$ and the direction ratios of the line (2) are $(3,8,-5)$

The two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Since the line (1)and PQ are perpendicular ,we have

$$3(3r-3k-4) - 16(-16r-8k-6) + 7(7r+5k-12) = 0$$

$$9r - 9k - 12 + 256r + 128k + 96 + 49r + 35k - 84 = 0$$

$$\Rightarrow 314r + 254k = 0 ---------- (3)$$

Since the line (2)and PQ are perpendicular ,we have

$$3(3r-3k-4) + 8(-16r-8k-6) - 5(7r+5k-12) = 0$$

$$-154r - 98k = 0$$

$$\Rightarrow 154r + 98k = 0 -----------(4)$$

Solving (3) and (4) ,we get ,

$r = 0, k = 0$ {there is no constant term in (3) & (4)}

substituting the value $r = 0$ in (A) we get

$$P(3(0)+5, -16(00+7, 7(0)+3)$$

(i.e.,) P(5,7,3)

substituting the value $k = 0$ in (A) we get

$$Q(3(0)+9, 8(0)+13, -5(0)+15)$$

(i.e.,) Q(9,13,15)

Now the length of the common perpendicular to the lines (1) & (2) is

$$PQ = \sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2}$$

$$= \sqrt{4^2 + 6^2 + 12^2}$$

$$= \sqrt{16 + 36 + 144}$$

$$= \sqrt{196} \quad = 14$$

$\therefore$ The equation of the straight line passing through the points $(x_1, y_1, z_1) \& (x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$\therefore$ The equation of the shortest distance through the points P(5,7,3) & Q(9,13,15) is

$$\frac{x-5}{9-5} = \frac{y-7}{13-7} = \frac{z-3}{15-3}$$

$$\frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12}$$

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

- **Find the equation and the length of the line of shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. [16 Marks]**

Solution:

The equation of the first line be

$$\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4} = r - - - - - - - - - - - - - (1)$$

Any point on the line (1) be

$$\text{P}(2r + 1, 3r + 2, 4r + 3) - - - - - - - - (A)$$

The equation of the second line be

$$\frac{x+1}{3} = \frac{y}{4} = \frac{z}{5} = k - - - - - - - - - - - - - - (2)$$

Any point on the line (2) be

$$\text{Q}(3k + 2, 4k + 4, 5k + 5) - - - - - - - - (\text{B})$$

Join P and Q and assume that PQ is perpendicular to both the lines (1) and (2).Then by the definition of the shortest distance , the length PQ is the length of the shortest distance and the equation of the line PQ is the equation of the shortest distance .

The direction ratios of the line PQ are

$$\big(2r + 1 - (3k + 2), 3r + 2 - (4k + 4), 4r + 3 - (5k + 5)\big)$$

$$(2r + 1 - 3k - 2, 3r + 2 - 4k - 4, 4r + 3 - 5k - 5)$$

$$(2r - 3k - 1, 3r - 4k - 2, 4r - 5k - 2)$$

Also the direction ratios of the line (1) are (2,3,4) and the direction ratios of the line (2) are (3,4,5)

The two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Since the line (1)and PQ are perpendicular ,we have

$$2(2r - 3k - 1) + 3(3r - 4k - 2) + 4(\,4r - 5k - 2\,) = 0$$

$$4r - 6k - 2 + 9r - 12k - 6 + 16r - 20k - 8 = 0$$

$$29r - 38k - 16 = 0 ---------(3)$$

Since the line (2)and PQ are perpendicular ,we have

$$3(2r - 3k - 1) + 4(3r - 4k - 2) + 5(\,4r - 5k - 2\,) = 0$$

$$6r - 9k - 3 + 12r - 16k - 8 + 20r - 25k - 10 = 0$$

$$38r - 50k - 21 = 0 --------------(4)$$

Solving (3) and (4) ,

$(3)\times 50 \Longrightarrow 1450r - 1900k - 800 = 0 ---------(5)$

$(4) \times 38 \Longrightarrow 1444r - 1990k - 798 = 0 ---------(6)$

$$(5) - (6) \Longrightarrow 6r - 2 = 0$$

$$\Longrightarrow r = \frac{1}{3}$$

substitute the value $r = \dfrac{1}{3}$ in (3) we get

$$\Longrightarrow \quad 29\left(\frac{1}{3}\right) - 38k - 16 = 0$$

$$\Longrightarrow \quad 38k = \frac{29}{3} - 16 \;\Longrightarrow\; 38k = \frac{29 - 48}{3}$$

$$\Longrightarrow 38k = -\frac{19}{3} \Longrightarrow k = -\frac{19}{3} \times \frac{1}{38} \;\Longrightarrow k = -\frac{1}{6}$$

substitute the value $r = \dfrac{1}{3}$ in (A), we get,

$$\mathrm{P}\left(2\left(\frac{1}{3}\right) + 1, 3\left(\frac{1}{3}\right) + 2, 4\left(\frac{1}{3}\right) + 3\right)$$

$$\Longrightarrow \mathrm{P}\left(\frac{2+3}{3}, 1 + 2\,, \frac{4+9}{3}\right) (\mathrm{i.e.,})\ \mathrm{P}\left(\frac{5}{3}, 3, \frac{13}{3}\right)$$

substitute the value $r = -\dfrac{1}{6}$ in (B), we get,

$$Q\left(3\left(-\frac{1}{6}\right) + 2, 4\left(-\frac{1}{6}\right) + 4, 5\left(-\frac{1}{6}\right) + 5\right)$$

$$\Rightarrow Q\left(\frac{-3+12}{6},\frac{-4+24}{6},\frac{-5+30}{6}\right)$$

$$\text{(i.e.,)}\text{Q}\left(\frac{9}{6},\frac{20}{6},\frac{25}{6}\right)\ or\ \text{Q}\left(\frac{3}{2},\frac{10}{3},\frac{25}{6}\right)$$

Now the length of the common perpendicular to the lines (1) & (2) is

$$\text{PQ} = \sqrt{\left(\frac{5}{3}-\frac{9}{6}\right)^2+\left(3-\frac{10}{3}\right)^2+\left(\frac{13}{3}-\frac{25}{6}\right)^2}$$

$$= \sqrt{\left(\frac{10-9}{6}\right)^2+\left(\frac{9-10}{3}\right)^2+\left(\frac{26-25}{6}\right)^2}$$

$$= \sqrt{\frac{1}{36}+\frac{1}{9}+\frac{1}{36}} = \sqrt{\frac{1+4+1}{36}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$$

$\therefore$ The equation of the straight line passing through the points (x_1, y_1, z_1) & (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$\therefore$The equation of the shortest distance through the points $\text{P}\left(\frac{5}{3},3,\frac{13}{3}\right)$ and $\text{Q}\left(\frac{3}{2},\frac{10}{3},\frac{25}{6}\right)$ is

$$\frac{x-\frac{5}{3}}{\frac{3}{2}-\frac{5}{2}} = \frac{y-3}{\frac{10}{3}-3} = \frac{z-\frac{13}{3}}{\frac{25}{6}-\frac{13}{3}}$$

$$\frac{3x-5}{3\left(\frac{9-10}{6}\right)} = \frac{y-7}{\left(\frac{10-9}{3}\right)} = \frac{3z-13}{3\left(\frac{25-26}{6}\right)}$$

$$\frac{3x-5}{-\frac{1}{2}} = \frac{y-7}{\frac{1}{3}} = \frac{3z-13}{3\left(\frac{-1}{6}\right)}$$

$$\frac{2(3x-5)}{-1} = \frac{3(y-7)}{1} = \frac{2(3z-13)}{-1}$$

$$\frac{6x-10}{-1} = \frac{3y-9}{1} = \frac{6z-26}{-1}$$

- **Find the equation and the length of the line of shortest distance between the lines $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{4}$ and $3x+2y-5z-6=0=2x-3y+z-3$.**

[16 Marks]

Solution:

The equation of the first line be

$$\frac{x+1}{3}=\frac{y-2}{2}=\frac{z}{4}---------(1)$$

The second line is the intersection of two planes

$$3x+2y-5z-6=0-----------------(2)$$

$$2x-3y+z-3=0-----------------(3)$$

We find a plane through the line of intersection of (2) and (3) and parallel to the line (1)

Any plane containing the second line(2) and (3) is of the form

$$(3x+2y-5z-6)+k(2x-3y+z-3)=0$$

$$(3+2k)x+(2-3k)y+(-5+k)z-6-3k=0----(4)$$

The plane (4) is parallel to the line (1)

$$\therefore 3(3+2k)+2(2-3k)+4(-5+k)=0$$

$$\Longrightarrow 9+6k+4-6k-20+4k=0$$

$$\Longrightarrow 4k=7 \quad \Longrightarrow k=\frac{7}{4}$$

The plane (4) is

$$(3x+2y-5z-6)+\frac{7}{4}(2x-3y+z-3)=0$$

$$\Longrightarrow 12x+8y-20z-24+14x-21y+7z-21=0$$

$$\Longrightarrow 26x-13y-13z-45=0-----------(5)$$

Take a point P(−1,2,0) on the line(1)

The required shortest distance =perpendicular distance from (−1,2,0) to the plane(5)

$$=\frac{26(-1)-13(2)-13(0)-45}{\sqrt{26^2+13^2+13^2}}$$

$$=\frac{-26-26-45}{\sqrt{676+169+169}}$$

$$=\frac{97}{\sqrt{676+169+169}}$$

$$=\frac{97}{\sqrt{1014}}$$

$$= \frac{97}{\sqrt{169 \times 6}} = \frac{97}{13\sqrt{6}}$$

The D.r's of SD line are $(26, -13, -13)$ (i.e.,)$(2, -1, -1)$

The SD line is the intersection of plane containing line(1) and the shortest line and plane containing the line (2) &(3) and shortest line

First any plane through (1) is of the form

$$a(x+1) + b(y-2) + cz = 0 \text{ ----------- } (6)$$

$$\text{with } 3a + 2b + 4c = 0 \text{ --------- } (7)$$

SD line is perpendicular to (6)

$$\therefore 2a - b - c = 0 \text{ ------------------ } (8)$$

Solving equation (7) and (8),we get,

(7)$\Longrightarrow$ $3a + 2b + 4c = 0$

(8)$\times 2 \Longrightarrow$ $4a - 2b - 2c = 0$ ----------------- (9)

(7)+(9) $\Longrightarrow 7a + 2c = 0$

(8) $\times 4 \Longrightarrow 8a - 4b - 4c = 0$ ----------------- (10)

(7)+(10) $\Longrightarrow 11a - 2b = 0$

$$\therefore 7a = -2c \ \& \ 11a = 2b$$

$$\Longrightarrow \frac{a}{2} = \frac{b}{11} = \frac{c}{-7}$$

$$\therefore a = 2, b = 11, c = -7$$

$\therefore$ (6)becomes $2(x+1) + 11(y-2) - 7(z) = 0$

$$2x + 11y - 7z + 2 - 22 = 0$$

$$2x + 11y - 7z - 20 = 0 \text{ --------------- } (11)$$

Any plane through the line(2) and (3) is of the form

$$(3x + 2y - 5z - 6) + k(2x - 3y + z - 3) = 0$$

$$\Longrightarrow (3 + 2k)x + (2 - 3k)y + (-5 + k)z - 6 - 3k = 0 \text{ ----- } (12)$$

The normal of the plane in (12) is perpendicular to the shortest line

$$\therefore \quad 2(3 + 2k) - 1(2 - 3k) - 1(5 + k) = 0$$

$$\Longrightarrow 6k + 9 = 0$$

$$\Longrightarrow k = \frac{-9}{6}$$

$$\Longrightarrow \ k = \frac{-3}{2}$$

$\therefore$ Equation (12) becomes

$$(3x + 2y - 5z - 6) - \frac{3}{2}(2x - 3y + z - 3) = 0$$

$$\Rightarrow 6x + 4y - 10z - 12 - 6x + 9y - 3z + 9 = 0$$

$$\Rightarrow 13y - 13z - 3 = 0$$

$\therefore$ The shortest line is $2x + 11y - 7z - 20 = 0 = 13y - 13z - 3$

PLANES

General form of a plane is $ax + by + cz + d_1 = 0$

Where (a, b, c) are D.R's of the normal to the plane

Intercept form of a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Where (a, b, c) are the x, y and z intercepts of the plane.

Equation of the plane passing through a point (x_1, y_1, z_1),

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- **Find the equation of the plane which bisects at right angles the join of $(1, 3, -2)$ and $(3, 1, 6)$.**

  Solution:

  Midpoint of the line P$(1,3,-2)$ & Q$(3,1,6)$= $\left[\frac{3+1}{2}, \frac{1+3}{2}, \frac{6-2}{2}\right]$=$(2,2,2)$

  Equation of the plane through the point$(2,2,2)$ is

  $$a(x - 2) + b(y - 2) + c(z - 2) = 0 \ --------- (1)$$

  As PQ is perpendicular to the plane ,

  D.r's of PQ=$(3 - 1, 1 - 3, 6 + 2)$

  $=(2, -2, 8) = (a, b, c)$

  Substituting the values of (a, b, c) in equation (1), we get,

  $$2(x - 2) - 2(y - 2) + 8(z - 2) = 0$$

  $$\Rightarrow 2x + 2y + 8z - 4 + 4 - 16 = 0$$

$$\Rightarrow 2x + 2y + 8z - 16 = 0$$
$$\Rightarrow x + y + 4z - 8 = 0$$

This is the required plane .

- **Find the equation of the plane through $(-1, 2, -3)$ and perpendicular to the line joining $(-3, 2, 4)$ and (5,4,1).**

  Solution:

  Direction ratio of the line joining is $= (-3 - 5, 2 - 4, 4 - 1) = (-8, -2, 3)$

  The plane is $-8x - 2y + 3z + k = 0 \quad - - - - - - - - (1)$

  As plane passes through $(-1,2,-3)$, we get,

$$-8(-1) - 2(2) + 3(-3) + k = 0$$
$$\Rightarrow 8 - 4 - 9 + k = 0$$
$$\Rightarrow -5 + k = 0$$
$$\Rightarrow k = 5$$

  Put $k = 5$ in(1), we get,

$$-8x - 2y + 3z + 5 = 0$$
$$\text{(or)} \qquad 8x + 2y - 3z - 5 = 0$$

- **Find the equation of the plane through (0,0,2),$(0, -1, 0)$ and $(-3, 0,0)$.**

  Solution:

  The equation of the required plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$=0

$$\Rightarrow \begin{vmatrix} x - 0 & y + 1 & z + 3 \\ 0 - 0 & -1 - 0 & 1 + 1 \\ 3 - 0 & 9 + 1 & 4 + 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 0 & y + 1 & z + 1 \\ 0 & -1 & 2 \\ 3 & 10 & 5 \end{vmatrix} = 0$$

$$\Rightarrow x(30 - 20) - (y + 1)(20 - 6) + (z + 1)(40 - 18) = 0$$
$$\Rightarrow 10x + (y + 1)(-14) + (z + 1)(22) = 0$$
$$\Rightarrow 10x - 14y - 14 + 22z + 22 = 0$$
$$\Rightarrow 10x - 14y + 22z + 8 = 0$$
$$\Rightarrow 5x - 7y + 11z + 4 = 0.$$

- **Find the equation of the plane passing through (1,2,3) and perpendicular to the joining (0,8,6)& $(1, -1, 7)$.**

  Solution:

The D.r's of the line joining of two points are $(0-1, 8+1, 6-7)$

$$(\text{i.e.,})(-1,9,-1)$$

Equation of the plane passing through the point (1,2,3) is

$$-1(x-1)+9(y-2)-1(z-3)=0$$

(since the plane passing through (x_1, y_1, z_1) and having (a,b,c) as D.r's of the normal of the plane is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$)

$$\Rightarrow -x+1+9y-18-z+3=0$$

$$\Rightarrow -x+9y-z+14=0$$

$$\Rightarrow x-9y+z+14=0$$

- **Find the equation of plane which passes through the points $(-1,2,1)$ and $(5,-4,1)$ and perpendicular to the plane $x+y-z+6=0$.**

[8 Marks]

Solution :

Any plane through the point $(-1,2,1)$ is of the form

$$a(x+1)+b(y-2)+c(z-1)=0 \quad ------(1)$$

where a, b, c are drs of the normal of the plane

Equation (1)Passes through the point $(5,-4,1)$

$$\Rightarrow 6a-6b=0 \quad ---------(2)$$

The plane $x+y-z+6=0$ is perpendicular to (1)

Their normals are perpendicular to each other .

$\Rightarrow$ using condition of perpendicularity ,$1.a+1.b+(-1)c=0$

$$(\text{i.e.,}) \quad a+b-c=0 \quad --------(3)$$

Solving (2) and (3) ,we have

$$\frac{a}{6-0}=\frac{-b}{-6-0}=\frac{c}{6+6}$$

$$\frac{a}{6}=\frac{b}{6}=\frac{c}{12}$$

$$\frac{a}{1}=\frac{b}{1}=\frac{c}{2}$$

The required equation of the plane is

$$1(x+1)+1(y-2)+2(z-1)=0$$

$$\Rightarrow x+y+2z+1-2-2=0$$

$$\Rightarrow x+y+2z-3=0$$

- **Find the equation of plane which passes through the points**

$(-\mathbf{1}, \mathbf{1}, \mathbf{1})$ **and** $(\mathbf{1}, -\mathbf{1}, \mathbf{1})$ **and perpendicular to the plane** $\boldsymbol{x + 2y + 2z - 5 = 0}$. **[8 Marks]**

<u>Solution :</u>

Any plane through the point $(-1,1,1)$ is of the form

$$a(x+1) + b(y-1) + c(z-1) = 0 - - - - - - (1)$$

where a, b, c are drs of the normal of the plane

Equation (1)Passes through the point$(1,-1,1)$

$$\Longrightarrow 2a - 2b = 0 - - - - - - - - - - - - - - (2)$$

The plane $x + 2y + 2z - 5 = 0$ is perpendicular to (1)

Their normals are perpendicular to each other .

$\Longrightarrow$ using condition of perpendicularity ,$1.a + 2.b + 2.c = 0$

$$\text{(i.e.,)} \quad a + 2b + 2c = 0 - - - - - - - - - (3)$$

Solving (2) and (3) ,we have

$$\frac{a}{-4-0} = \frac{b}{0-4} = \frac{c}{4+2}$$

$$\frac{a}{-4} = \frac{b}{-4} = \frac{c}{6}$$

$$\frac{a}{2} = \frac{b}{2} = \frac{c}{-3}$$

The required equation of the plane is

$$2(x+1) + 2(y-1) - 3(z-1) = 0$$

$$\Longrightarrow \quad 2x + 2y - 3z + 2 - 2 + 3 = 0$$

$$\Longrightarrow 2x + 2y - 3z + 3 = 0$$

- **Find the equation of plane which passes through the point** $(\mathbf{1}, \mathbf{0}, -\mathbf{2})$ **and perpendicular to the planes** $\boldsymbol{2x + y - z = 2}$ **and** $\boldsymbol{x - y - z = 3}$.

[8 Marks]

<u>Solution :</u>

Let the equation of required plane be

$$Ax + By + Cz + D = 0 - - - - - - - - - (1)$$

Equation (1) is passing through the point $(1,0,-2)$, substituting the point ,we get Equation (1) as

$$A - 2C + D = 0 - - - - - - - - - - - - (2)$$

Equation(1) is perpendicular to $2x + y - z = 2$

Applying the condition of perpendicularity,we get

$$2A + B - C = 0 ----------- (3)$$

Equation (1)is also perpendicular to $x - y - z = 3$

Again applying the condition of perpendicularity,we get

$$A - B - C = 5 -------(4)$$

Consider (3)&(4) and apply rule of cross multiplication

$$\frac{A}{-1-1} = \frac{B}{-1+2} = \frac{C}{-2-1}$$

$$\frac{A}{-2} = \frac{B}{1} = \frac{C}{-3}$$

Substitute A,B,C in (2), we get

$$A - 2C + D = 0$$

$$\Rightarrow \quad -2 - 2(-3) + D = 0$$

$$\Rightarrow \quad -2 + 6 + D = 0$$

$$\Rightarrow 4 + D = 0 \quad \Rightarrow D = -4$$

Substitute $D = -4$ in (1), we get

$$-2x + y - 3z - 4 = 0$$

(or) $2x$ −y+3z+4=0

- **Find the angle between the planes $3x + 4y - 5z = 9$and $2x + 6y + 6z = 7$.**

  <u>Solution:</u>

  The angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}}$$

∴ The angle between the planes $3x + 4y - 5z = 9$and $2x + 6y + 6z = 7$ is given by

$$\cos\theta = \frac{3(2) + 4(6) + (-5)6}{\sqrt{3^2 + 4^2 + (-5)^2}\sqrt{2^2 + 6^2 + 6^2}}$$

$$= \frac{6 + 24 - 30}{\sqrt{9 + 16 + 25} \ \sqrt{4 + 36 + 36}} = \frac{0}{\sqrt{50}\ \sqrt{76}} = 0 \ \Rightarrow \ \theta = \frac{\pi}{2}$$

- **Find the angle between the planes $\mathbf{2x - y + z + 7 = 0}$ and $\mathbf{x + y + 2z - 11 = 0}$.**

  <u>Solution:</u>

  The angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}}$$

  $\therefore$ The angle between the planes $2x - y + z + 7 = 0$ and $x + y$ +2z-11=0 is given by

$$\cos\theta = \frac{2(1) + (-1)(1) + 1(2)}{\sqrt{2^2 + (-1)^2 + 1^2}\sqrt{1^2 + 1^2 + 2^2}}$$

$$= \frac{2 - 1 + 2}{\sqrt{4 + 1 + 1}\ \sqrt{1 + 1 + 4}} = \frac{3}{\sqrt{6}\ \sqrt{6}} = \frac{3}{6} = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

- **Find the point where the line $\mathbf{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}}$ meets the plane $\mathbf{x + y + z = 15}$.**

  <u>Solution:</u>

  Any point on the line is of the form

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$$

  (i.e.,)$(1 + 2r, 2 + 3r, 3 + 4r)$ ------------------------------(1)

  If this point be the meeting point , it lies on the plane

$$\therefore (1 + 2r) + (2 + 3r) + (3 + 4r) = 15$$
$$\implies 9r + 9 = 15$$
$$\implies 9r = 15 - 9$$
$$\implies 9r = 9$$
$$\implies r = 1$$

  Substituting $r = 1$ in equation (1), we get ,the point (3,5,7).

- **Find the equation of plane which contains the line $\mathbf{\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}}$ and perpendicular to the plane $\mathbf{x + 2y + z = 12}$.** **[8 Marks]**

  <u>Solution:</u>

The equation of the plane containing the given line

$\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ is

$$a(x-1)+b(y+1)+c(z-3)=0 \text{ ------- }(1)$$

and $$2a-b+4c=0 \text{ --------- }(2)$$

If this plane is perpendicular to the plane $x+2y+z=12$, we get

$$a+2b+c=0 \text{ --------- }(3)$$

(By condition of perpendicularity of two planes)

Solving (2) and (3) we get

$$\frac{a}{-1-8}=\frac{b}{4-2}=\frac{c}{4+1}$$

$$\frac{a}{-9}=\frac{b}{2}=\frac{c}{5}$$

$$a=-9, b=2, c=5$$

Substituting these values of a, b, c in (1) we get

$$-9(x-1)+2(y+1)+5(z-3)=0$$

$$-9x+9+2y+2+5z-15=0$$

$$\Longrightarrow \quad 9x-2y-5z+4=0$$

which is the required equation of the plane.

- **Find the equation of plane which contains the line $\frac{x+1}{2}=\frac{y-1}{3}=\frac{z+2}{-1}$ and passes through the point $(1,2,-1)$.** **[8 Marks]**

Solution:

The given line is

$$\frac{x+1}{2}=\frac{y-1}{3}=\frac{z+2}{-1} \text{ ----------- } (1)$$

The equation of any plane containing the line (1) is

$$a(x+1)+b(y-1)+c(z+2)=0 \text{ --------- }(2)$$

and $$2a+3b-c=0 \text{ ----------- }(3)$$

If the plane (2) passes through the point $(1,2,-1)$ then we have

$$2a+b+c=0 \qquad \text{ ------------- } (4)$$

Solving (3) & (4) by cross rule multiplication we get ,

$$\frac{a}{3+1}=\frac{b}{-2-2}=\frac{c}{2-6}$$

$$\frac{a}{4}=\frac{b}{-4}=\frac{c}{-4}$$

$$a = 4, b = -4, c = -4$$

Substituting these values of a, b, c in (2) we get ,

$$4(x+1) - 4(y-1) - 4(z+2) = 0$$

$$\Rightarrow \quad 4x - 4y - 4z = 0$$

$$\Rightarrow \quad x - y - z = 0$$

which is the required equation of the plane .

<u>Condition For Coplanarity</u>

Two lines $\dfrac{x-x_1}{l_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$, $\dfrac{x-x_2}{l_2} = \dfrac{y-y_2}{m_2} = \dfrac{z-z_2}{n_2}$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

- **Prove that the four points $(1,2,3)$, $(-2,4,-3)$, $(4,5,7)$ and (4,10,5) are coplanar.**

<u>Solution:</u>

The equation of the plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) &(x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Using the points $(1,2,3)$, $(-2,4,-3)$, $(4,5,7)$ in the determinant we get

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -2-1 & 4-2 & -3-3 \\ 4-1 & 5-2 & 7-3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -3 & 2 & -6 \\ 3 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(8+18) - (y-2)(-12+18) + (z-3)(-9-6) = 0$$

$$\Rightarrow (x-1)(26) - (y-2)(6) + (z-3)(-15) = 0$$

$$\Rightarrow 26x - 6y - 15z - 26 + 12 + 45 = 0$$

$$\Rightarrow 26x - 6y - 15z + 31 = 0 \quad ---------(1)$$

Substituting the fourth point (4,10,5) in (1), we get

$$\begin{aligned} 26(4) - 6(10) - 15(5) + 31 &= 104 - 60 - 75 + 31 \\ &= 135 - 135 \\ &= 0 \end{aligned}$$

$\therefore$ The equation (1) is satisfied

Hence the four points are coplanar.

- **Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar. Also find their point of intersection and the plane in which they lie.**

[8Marks]

Solution:

Let the equation of the given lines be

$$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3} = r - - - - - - -(1)$$

$$\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1} - - - - - - - - - -(2)$$

Two lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$, $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Now, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -1-(-3) & -1-(-5) & -1-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}$

$$= \begin{vmatrix} -1+3 & -1+5 & -8 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 4 & -8 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 2(-3+15) - 4(-2+12) - 8(10-12) \\ &= 2(12) - 4(10) - 8(-2) \\ &= 24 - 40 + 16 \ = 0 \end{aligned}$$

$\therefore$ The given lines are coplanar .

The equation of the plane containing the lines is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - (-3) & y - (-5) & z - 7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x + 3 & y + 5 & z - 7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$$

$$\Longrightarrow (x+3)(-3+15) - (y+5)(-2+12) + (z-7)(10-12) = 0$$

$$\Longrightarrow (x+3)(12) - (y+5)(10) + (z-7)(-2) = 0$$

$$\Longrightarrow 12x + 36 - 10y - 50 - 2z + 14 = 0$$

$$\Longrightarrow 12x - 10y - 2z = 0$$

$$\Longrightarrow 6x - 5y - z = 0$$

which is the required equation of the plane.

Any point on the first line is $\mathrm{P}(2r-3, 3r-5, -3r+7)$

If this point P lies on the second line we have

$$\frac{2r-3+1}{4} = \frac{3r-5+1}{5} = \frac{-3r+7+1}{-1}$$

$$\Rightarrow \frac{2r-2}{4} = \frac{3r-4}{5}$$

$$\Rightarrow 5(2r-2) = 4(3r-4)$$

$$\Rightarrow 10r - 10 = 12r - 16$$

$$\Rightarrow \quad 12r - 10r = -10 + 16$$

$$\Rightarrow 2r = 6 \Rightarrow r = 3$$

substituting the value of r in the point P, we get

$$\mathrm{P}(2(3)-3, 3(3)-5, -3(3)+7)$$

$$\mathrm{P}(6-3, 9-5, -9+7)$$

$\mathrm{P}(3,4,-2)$ which is the required point of intersection.

- **Show that the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ are coplanar. Also find their point of intersection and the plane in which they lie.**

[8 Marks]

<u>Solution:</u>

Let the equation of the given lines be

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}=r --------(1)$$

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}----------(2)$$

Two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$
are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Now, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=\begin{vmatrix} 1-4 & -1-(-3) & -10-(-1) \\ 1 & -4 & 7 \\ 2 & -3 & 8 \end{vmatrix}$

$$=\begin{vmatrix} -3 & -1+3 & -10+1 \\ 1 & -4 & 7 \\ 2 & -3 & 8 \end{vmatrix}$$

$$=\begin{vmatrix} -3 & 2 & -9 \\ 1 & -4 & 7 \\ 2 & -3 & 8 \end{vmatrix}$$

$$=-3(-32+21)-2(8-14)-9(-3+8)$$

$$=-3(-11)-2(-6)-9(5)$$

$$=33+12-45=0$$

$\therefore$ The given lines are coplanar .

The equation of the plane containing the lines is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-4 & y-(-3) & z-(-1) \\ 1 & -4 & 7 \\ 2 & -3 & 8 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-4 & y+3 & z+1 \\ 1 & -4 & 7 \\ 2 & -3 & 8 \end{vmatrix}=0$$

$$\Longrightarrow (x-4)(-32+21)-(y+3)(8-14)+(z+1)(-3+8)=0$$

$$\Longrightarrow (x-4)(-11)-(y+3)(-6)+(z+1)(5)=0$$

$$\Rightarrow -11x + 44 + 6y + 18 + 5z + 5 = 0$$
$$\Rightarrow -11x + 6y + 5z + 67 = 0$$
$$\Rightarrow 11x + 6y - 5z - 67 = 0$$

which is the required equation of the plane.

Any point on the first line is $P(r + 4, -4r - 3, 7r - 1)$

If this point P lies on the second line we have

$$\frac{r + 4 - 1}{2} = \frac{-4r - 3 + 1}{-3} = \frac{7r - 1 + 10}{8}$$
$$\Rightarrow \frac{r + 3}{2} = \frac{-4r - 2}{-3}$$
$$\Rightarrow -3(r + 3) = 2(-4r - 2)$$
$$\Rightarrow -3r - 9 = -8r - 4$$
$$\Rightarrow 8r - 3r = 9 - 4 \quad \Rightarrow 5r = 5 \quad \Rightarrow \quad r = 1$$

substituting the value of r in the point P, we get

$$P(1+4, -4(1) - 3, 7(1) - 1)$$

$P(5, -7, 6)$ which is the required point of intersection.

- **Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are coplanar. Find their common point and equation of the plane in which they lie.**

[8 Marks]

<u>Solution:</u>

Let the equation of the given lines be

$$\frac{x - 5}{4} = \frac{y - 7}{4} = \frac{z + 3}{-5} = r - - - - - - -(1)$$
$$\frac{x - 8}{7} = \frac{y - 4}{1} = \frac{z - 5}{3} - - - - - - - - (2)$$

Two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$, $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Now, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 3(12+5) + 3(12+35) + 8(4-28)$$

$$= 51 + 141 - 192 = 0$$

∴ The given lines are coplanar .

The equation of the plane containing the lines is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (x-5)(12+5) - (y-7)(12+35) + (z+3)(4-28) = 0$$

$$\Rightarrow (x-5)(17) + (y-7)(-47) + (z+3)(-24) = 0$$

$$\Rightarrow 17x - 47y - 24z - 85 + 329 - 72 = 0$$

$$\Rightarrow 17x - 47y - 24z + 172 = 0$$

which is the required equation of the plane.

Any point on the first line is $\mathrm{P}(4r+5, 4r+7, -5r-3)$

If this point P lies on the second line we have

$$\frac{4r+5-8}{7} = \frac{4r+7-4}{1} = \frac{-5r-3-5}{3}$$

$$\Rightarrow \quad \frac{4r-3}{7} = \frac{4r+3}{1} = \frac{-5r-8}{3}$$

Taking the first two ratios we have

$$\Rightarrow \quad \frac{4r-3}{7} = \frac{4r+3}{1}$$

$$\Rightarrow \quad 4r-3 = 7(4r+3)$$

$$\Rightarrow \quad 4r-3 = 28r+21$$

$$\Rightarrow \quad 28r-4r = -21-3$$

$$\Rightarrow \quad 24r = -24 \quad \Rightarrow r = -1$$

substituting the value of r in the point P, we get

$$\mathrm{P}(-4+5, -4+7, 5-3)$$

P(1,3,2) which is the required point of intersection.

- **Show that the lines** $\frac{x-4}{5}=\frac{y-3}{-2}=\frac{z-2}{-6}$ *and* $\frac{x-3}{4}=\frac{y-2}{-3}=\frac{z-1}{-7}$ **are coplanar. Find their point of intersection and the equation of the plane in which they lie.** **[8 Marks]**

Solution:

Let the equation of the given lines be

$$\frac{x-4}{5}=\frac{y-3}{-2}=\frac{z-2}{-6}=r---------(1)$$

$$\frac{x-3}{4}=\frac{y-2}{-3}=\frac{z-1}{-7}----------(2)$$

Two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Now, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=\begin{vmatrix} 3-4 & 2-3 & 1-2 \\ 5 & -2 & -6 \\ 4 & -3 & -7 \end{vmatrix}$

$$=\begin{vmatrix} -1 & -1 & -1 \\ 5 & -2 & -6 \\ 4 & -3 & -7 \end{vmatrix}$$

$$=-1(14-18)+1(-35+24)-1(-15+8)$$

$$=-1(-4)-11-1(-7)$$

$$=4-11+7=0$$

$\therefore$ The given lines are coplanar .

The equation of the plane containing the lines is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-4 & y-3 & z-2 \\ 5 & -2 & -6 \\ 4 & -3 & -7 \end{vmatrix}=0$$

$$\Rightarrow (x-4)(14-18)-(y-3)(-35+24)+(z-2)(-15+8)=0$$

$$\Rightarrow (x-4)(-4)+(y-3)(11)+(z-2)(-7)=0$$
$$\Rightarrow \quad -4x+11y-7z+16-33+14=0$$
$$\Rightarrow \quad -4x+11y-7z-3=0$$
$$\Rightarrow \quad 4x-11y+7z+3=0$$

which is the required equation of the plane.

Any point on the first line is $P(5r+4, -2r+3, -6r+2)$

If this point P lies on the second line we have

$$\frac{5r+4-3}{4}=\frac{-2r+3-2}{-3}=\frac{-6r+2-1}{-7}$$

$$\frac{5r+1}{4}=\frac{-2r+1}{-3}=\frac{-6r+1}{-7}$$

Taking the first two ratios we have

$$\Rightarrow \frac{5r+1}{4}=\frac{-2r+1}{-3}$$
$$\Rightarrow -3(5r+1)=4(-2r+1)$$
$$\Rightarrow -15r-3=-8r+4$$
$$\Rightarrow -15r+8r=4+3$$
$$\Rightarrow -7r=7 \quad \Rightarrow r=-1$$

substituting the value of r in the point P, we get

$$P(-5+4, 2+3, 6+2)$$

$P(-1,5,8)$ which is the required point of intersection.

<u>**Distance Between The Parallel Planes**</u>

The distance between the parallel planes $ax+by+cz+d_1=0$,

$ax+by+cz+d_2=0$ is given by $D=\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}$

- **Find the distance between the parallel planes $3x+6y+2z=22$ and $3x+6y+2z=27$.**

<u>**Solution:**</u>

The given parallel planes are

$$3x+6y+2z=22$$
$$3x+6y+2z=27$$

The distance between the parallel planes $ax+by+cz+d_1=0$,

$ax + by + cz + d_2 = 0$ is given by $D = \frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}}$

$\therefore$ The distance between the given parallel plane is

$$D = \frac{-22+27}{\sqrt{3^2+6^2+2^2}} = \frac{5}{\sqrt{9+36+4}} = \frac{5}{\sqrt{49}} = \frac{5}{7}$$

- **Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$.**

Solution:

The given parallel planes are

$$2x - 2y + z + 3 = 0 \quad , 4x - 4y + 2z + 5 = 0 - - - - - - (1)$$

Since the equations of the parallel planes differ by the constant term only,

$\therefore$ Equation(1) can be rewritten as

$$2x - 2y + z + 3 = 0 \quad , 2x - 2y + z + \frac{5}{2} = 0 - - - - - - (2)$$

The distance between the parallel planes $ax + by + cz + d_1 = 0$, $ax + by + cz + d_2 = 0$ is given by

$$D = \frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$The distance between the given parallel plane is

$$D = \frac{3 - \frac{5}{2}}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{6-5}{2\sqrt{4+4+1}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

- **Find the image of the point (1,3,4) in the plane $2x - y + z + 3 = 0$**

[8 Marks]

Solution:

Let the given point be A(1,3,4)

Let the required image be B(a,b,c)

The midpoint of the line joining A and B is

$$C\left[\frac{a+1}{2}, \frac{b+3}{2}, \frac{c+4}{2}\right]$$

By the definition of image ,

C lies on the plane $2x - y + z + 3 = 0 - - - - - - - - (1)$

$$\therefore \quad 2\left[\frac{a+1}{2}\right] - \left[\frac{b+3}{2}\right] + \left[\frac{c+4}{2}\right] + 3 = 0$$

$$\Rightarrow \frac{2a+2-b-3+c+4+6}{2}=0$$

$\Rightarrow 2a-b+c+9=0 - - - - - - - - - - - - -(2)$

Also the direction ratios of the normals to the plane (1) are $(2,-1,1)$

The direction ratios of the line AB are $(a-1, b-3, c-4)$

By the definition of image,

The line AB and equation(1) are parallel

$\therefore$ Their D.R's are proportional,

$$\Rightarrow \quad \frac{a-1}{2}=\frac{b-3}{-1}=\frac{c-4}{1}=r$$

$$\Rightarrow a=2r+1\ , b=-r+3\ , c=r+4 - - - - - - -(3)$$

$\therefore$ From (2) , $2(2r+1) — (-r+3)+r+4+9=0$

$$\Rightarrow 4r+2+r-3+r+4+9=0$$

$$\Rightarrow 6r+12=0$$

$$\Rightarrow r=-2$$

Substituting the value of r in (3),we get

$$a=2(-2)+1=-4+1=-3$$

$$b=-(-2)+3=2+3=5$$

$$c=-2+4=2$$

$\therefore$ The required image is B(−3,5,2)

<u>SPHERE</u>

The equation of a sphere whose centre is (a,b,c) and radius is 'r' is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

The general equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Centre of the sphere is $C(-u,-v,-w)$

Radius of the sphere is $r=\sqrt{u^2+v^2+w^2-d}$

- **Find the equation of the sphere whose centre is $(1,-2,3)$ and which passes through the origin.**

  <u>Solution:</u>

  The equation of a sphere whose centre is (a,b,c) and radius is 'r' is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad -------(1)$$

Here $a=1, b=-2, c=3$ $\quad ----------------(2)$

Radius (r) =length of OA

$$=\sqrt{(1-0)^2+(-2-0)^2+(3-0)^2}$$

$$=\sqrt{1+4+9}$$

$$=\sqrt{14} \quad ----------------(3)$$

Substituting (2) and (3) in (1), we get

$$\Longrightarrow (x-1)^2+(y+2)^2+(z-3)^2= (\sqrt{14})^2$$

$$\Longrightarrow x^2+1-2x+y^2+4-4y+z^2+9-6z=14$$

$$\Longrightarrow x^2+y^2+z^2-2x-4y-6z+14-14=0$$

$$\Longrightarrow x^2+y^2+z^2-2x-4y-6z=0$$

- **Find the sphere having the points $(2, -3, 4)$ and $(-1, 5, 7)$ as the ends of its diameter.**

<u>**Solution:**</u>

$$S: (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$\Longrightarrow (x-2)(x+1)+(y+3)(y-5)+(z-4)(z-7)=0$$

$$\Longrightarrow x^2-2x+x-2+y^2+3y-5y-15+z^2-4z-7z+28=0$$

$$\Longrightarrow x^2-x-2+y^2-2y-15+z^2-11z+28=0$$

$$\Longrightarrow x^2+y^2+z^2-x-2y-11z+11=0$$

- **Find the centre and radius of the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$.**

<u>**Solution:**</u>

The given equation of the sphere is

$$x^2+y^2+z^2-4x+2y-6z+5=0.$$

By the general equation of the sphere,

$$2u=-4, 2v=2, 2w=-6, d=5$$

$$u=-2\ , v=1, w=-3\ , d=5$$

Now , the centre of the sphere is $C(-u, -v, -w)$

$\therefore$ The centre is $C(2, -1, 3)$

The radius of the sphere is $r=\sqrt{u^2+v^2+w^2-d}$

$$=\sqrt{2^2+(-1)^2+3^2-5}$$

$$=\sqrt{4+1+9-5}=\sqrt{14-5}=\sqrt{9}=3$$

- **Find the centre and radius of the sphere $2x^2+2y^2+2z^2-2x+4y-6z-1=0$.**

Solution:

The given equation of the sphere is

$$2x^2+2y^2+2z^2-2x+4y-6z-1=0$$

Dividing by 2 on both sides ,we get

$$x^2+y^2+z^2-x+2y-3z-\frac{1}{2}=0 ------(1)$$

By the general equation of the sphere,

$$2u=-1,\quad 2v=2,\quad 2w=-3,\qquad d=-\frac{1}{2}$$

$$u=-\frac{1}{2}\ ,\quad v=1,\qquad w=-\frac{3}{2}\ ,\qquad d=-\frac{1}{2}$$

Now , The centre of the sphere is $C(-u,-v,-w)$

$\therefore$ The centre is $C\left(-\frac{1}{2},1,-\frac{3}{2}\right)$

The radius of the sphere is $r=\sqrt{u^2+v^2+w^2-d}$

$$=\sqrt{\left(-\frac{1}{2}\right)^2+(1)^2+\left(-\frac{3}{2}\right)^2+\frac{1}{2}}$$

$$=\sqrt{\frac{1}{4}+1+\frac{9}{4}+\frac{1}{2}}\ =\sqrt{\frac{1+4+9+2}{4}}\ =\sqrt{\frac{16}{4}}\ =\sqrt{4}\ =2$$

- Find the equation of the sphere which passes through the points $(0,0,0),(0,1,-1),(-1,2,0)$ and $(1,2,3)$. **.(L1)** [8 Marks]

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 ------(1)$$

(1) passes through (0,0,0) we get $d=0$

i.e., $x^2+y^2+z^2+2ux+2vy+2wz=0 ------(2)$

(2) passes through $(0,1,-1)$ we get

$$0+1+1+0+2v-2w=0$$

$$\Rightarrow\qquad 2+2v-2w=0$$

$$\Rightarrow\quad v-w=-1 --------(3)$$

(2) passes through $(-1,2,0)$ we get

$$1 + 4 + 0 - 2u + 4v + 0 = 0$$
$$\Longrightarrow \quad 5 - 2u + 4v = 0$$
$$\Longrightarrow 2u - 4v = 0 - - - - - - (4)$$

(2) passes through (1,2,3) we get

$$1 + 4 + 9 + 2u + 4v + 6w = 0$$
$$\Longrightarrow 14 + 2u + 4v + 6w = 0$$
$$\Longrightarrow 2u + 4v + 6w = -14$$
$$\Longrightarrow u + 2v + 3w = -7 - - - - - - - - - -(5)$$

Solving (3) ,(4) and (5) we get

$$2\times (5) \Longrightarrow 2u + 4v + 6w = -14 - - - - - - -(6)$$
$$(4)+(6) \Longrightarrow 4u + 6w = -9 \; - - - - - - - - - -(7)$$
$$6\times (3) \Longrightarrow 6v - 6w = -6 \quad - - - - - - - - - -(8)$$
$$(7)+(8) \Longrightarrow 4u + 6v = -15 \quad - - - - - - - - - -(9)$$
$$(4) \times 2 \Longrightarrow 4u - 8v = 10 \quad - - - - - - - - - -(10)$$
$$(9) - (10) \Longrightarrow 14v = -25$$
$$\Longrightarrow v = -\frac{25}{14}$$

From (3), we get, $-w = -1 + \dfrac{25}{14}$

$$\Longrightarrow -w = \frac{11}{14} \quad \Longrightarrow w = \frac{-11}{14}$$

From (4) we get, $2u = 5 + 4v$

$$= 5 - \frac{100}{14}$$
$$= -\frac{30}{14}$$
$$\Longrightarrow \; u = -\frac{15}{14}$$

Therefore the equation of the sphere is

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$$
$$7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$$

which is the required equation of the sphere.

- **Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and also find the point of contact.**
[16 Marks]

Solution:

The sphere is

$$x^2+y^2+z^2-2x-4y+2z-3=0 --------(1)$$

The plane is

$$2x-2y+z+12=0 \quad --------------(2)$$

Centre of (1) is $(-u,-v,-w)=(1,2,-1)$

and Radius $=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+1+3} \quad =\sqrt{9}=3$

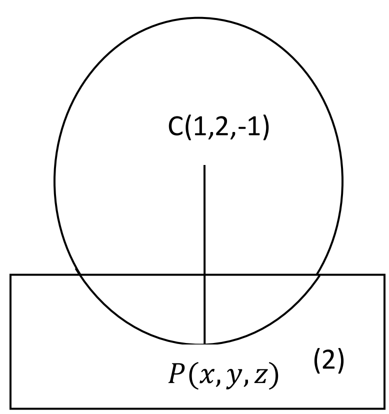
Therefore ,Length of perpendicular from $(-u,-v-w)$ to the plane

$$lx+my+nz=p \text{ is } =\frac{|-lu-mv-nw-p|}{\sqrt{(l^2+m^2+n^2)}}$$

Length of perpendicular from (1,2,−1) on plane (2)

$$=\frac{|2-4-1+12|}{\sqrt{4+4+1}} =\frac{9}{3}=3(\text{radius of the sphere})$$

The plane (2) touches the sphere (1)



Let $P(x,y,z)$ be the point of contact . Then CP is normal to the plane (2).

∴ D.R 's of CP are $(2,-2,1)$.Also CP passes through $C(1,2,-1)$

Hence the equations of CP are

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r$$

Any point on this line may be taken as

$$(2r+1,-2r+2,r-1) \quad -----------(3)$$

If this point lies on(2) , it will be the point of contact.

This point lies on plane (2) , if

$$2(2r+1)-2(,-2r+2)+(r-1)+12=0$$

$$\Longrightarrow \quad 4r+2+4r-4+r-1+12=0$$

$$\Longrightarrow \quad 9r+9=0 \quad \Longrightarrow r=-1$$

$\therefore$ From (3) ,The point of contact is $(-1,4,-2)$

- **Find the equation of the sphere which passes through the points $(\mathbf{1},\mathbf{0},\mathbf{0})$, (0,1,0) ,(0,0,1) and centre on the plane $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}=\mathbf{6}$.**

[8 Marks]

Solution:

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \; ----- (1)$$

As centre $c(-u,-v,-w)$ on the plane $x+y+z=6$

$$\Longrightarrow -u-v-w=6 \; -------------(2)$$

[A point on the sphere satisfies the equation of the sphere]

Substitute the point (1,0,0) , (0,1,0) ,(0,0,1) in (1)

$$1+2u+d=0,1+2v+d=0,1+2w+d=0$$

$$\Longrightarrow 2u=-1-d,2v=-1-d,2w=-1-d$$

$$\Longrightarrow \; u=-\frac{1}{2}-\frac{d}{2},v=-\frac{1}{2}-\frac{d}{2},w=-\frac{1}{2}-\frac{d}{2} \quad -------(3)$$

Put (3) in (1) $\Longrightarrow \; \frac{1}{2}+\frac{d}{2}+\frac{1}{2}+\frac{d}{2}+\frac{1}{2}+\frac{d}{2}=6$

$$\frac{3}{2}+\frac{3d}{2}=6$$

$$\Longrightarrow \quad \frac{3d}{2}=6-\frac{3}{2}$$

$$\Longrightarrow \frac{3d}{2}=\frac{9}{2}\Longrightarrow \; d=\frac{9}{2}\times\frac{2}{3} \quad \Longrightarrow \quad d=3$$

Substitute $d=3$ in (3),we get,

$$u=-\frac{1}{2}-\frac{3}{2}=-\frac{4}{2}=-2$$

$$v = -\frac{1}{2} - \frac{3}{2} = -\frac{4}{2} = -2$$
$$w = -\frac{1}{2} - \frac{3}{2} = -\frac{4}{2} = -2$$
Substitute the values $u = -2, v = -2\ , w = -2$ in (1), we get,
$$x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$$
which is the required equation of the sphere.

- **Find the equation of the sphere which passes through the circle $x^2 + y^2 + z^2 = 5$ and $x + 2y + 3z = 3$ and touches the plane $4x + 3y + 15 = 0$** **[8 Marks]**

Solution:

The equation of the sphere through the circle

S≡ $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$\pi \equiv ax + by + cz + d = 0$ is given by $S + \lambda\pi = 0$

∴ The equation of the sphere through the given circle is
$$x^2 + y^2 + z^2 - 5 + \lambda(x + 2y + 3z - 3) = 0$$
$$\Longrightarrow x^2 + y^2 + z^2 - 5 + \lambda x + 2\lambda y + 3\lambda z - 3\lambda = 0$$
$$\Longrightarrow x^2 + y^2 + z^2 + \lambda x + 2\lambda y + 3\lambda z - 3\lambda - 5 = 0 - - - - - - (1)$$
Here $2u = \lambda, 2v = 2\lambda, 2w = 3\lambda, d = -3\lambda - 5$
$$\Longrightarrow u = \frac{\lambda}{2}\ , v = \lambda\ , w = \frac{3\lambda}{2}, d = -3\lambda - 5$$
∴ Centre of the sphere (1) is $C(-u, -v, -w)$
$$\therefore\ C\left(\frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}\right)$$
Also the perpendicular distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0\ is$
$$d = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$
∴ The perpendicular distance from the centre $C\left(\frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}\right)$ to the

plane $4x+3y-15=0$ is

$$d=\frac{4\left(\frac{-\lambda}{2}\right)+3(-\lambda)-15}{\sqrt{4^2+3^2}}$$

$$=\frac{-\frac{4\lambda}{2}-3\lambda-15}{\sqrt{16+9}}=\frac{-\frac{4\lambda}{2}-3\lambda-15}{5}$$

$$=\frac{-4\lambda-6\lambda-30}{2}\times\frac{1}{5}\quad=\frac{-10\lambda-30}{10}=-\lambda-3\ =-(\lambda+3)$$

Now , the radius of the sphere (1) is

$$R=\sqrt{u^2+v^2+w^2-d}$$

$$=\sqrt{\left[\frac{-\lambda}{2}\right]^2+[-\lambda]^2+\left[\frac{-3\lambda}{2}\right]^2-(-3\lambda-5)}$$

$$=\sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+3\lambda+5}$$

$$=\sqrt{\frac{\lambda^2+4\lambda^2+9\lambda^2+12\lambda+20}{4}}$$

$$R=\frac{\sqrt{14\lambda^2+12\lambda+20}}{2}$$

Given that the plane $4x+3y-15=0$ touches the required sphere.

$\therefore$ The perpendicular distance from the centre of the sphere (1) to the plane $4x+3y-15=0$ equal to the radius of the sphere (1)

$$\Longrightarrow -(\lambda+3)=\frac{\sqrt{14\lambda^2+12\lambda+20}}{2}$$

$$\Longrightarrow (\lambda+3)^2=\frac{14\lambda^2+12\lambda+20}{4}$$

$$\Longrightarrow 4(\lambda^2+6\lambda+9)=14\lambda^2+12\lambda+20$$

$$\Rightarrow 4\lambda^2 + 24\lambda + 36 = 14\lambda^2 + 12\lambda + 20$$

$$\Rightarrow 14\lambda^2 + 12\lambda + 20 - 4\lambda^2 - 24\lambda - 36 = 0$$

$$\Rightarrow 5\lambda^2 - 6\lambda - 8 = 0$$

$$\Rightarrow 5\lambda^2 - 10\lambda + 4\lambda - 8 = 0$$

$$\Rightarrow 5\lambda(\lambda - 2) + 4(\lambda - 2) = 0$$

$$\Rightarrow (5\lambda + 4)(\lambda - 2) = 0 \quad \Rightarrow \lambda = 2, \frac{-4}{5}$$

Put $\lambda = 2$ in (1) ,we get

$$x^2 + y^2 + z^2 - 5 + 2(x + 2y + 3z - 3) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 5 + 2x + 4y + 6z - 6 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 5 + 2x + 4y + 6z - 11 = 0$$

Put $\lambda = \frac{-4}{5}$ in (1) , we get

$$x^2 + y^2 + z^2 - 5 - \frac{4}{5}(x + 2y + 3z - 3) = 0$$

$$\frac{5(x^2 + y^2 + z^2) - 25 - 4x - 8y - 12z + 12}{5} = 0$$

$$\Rightarrow \quad 5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 13 = 0$$

- **Find the equation of the sphere whose centre is at $(6, -1, 2)$ and touching the plane $2x + y + 2z = 2$.** **[8 Marks]**

Solution:

The equation of the sphere having centre at (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 - - - - - - - - - . (1)$$

Given that centre $(a, b, c) = (6, -1, 2)$

$$\Rightarrow a = 6, b = -1, c = 2$$

Since the sphere (1) touches the plane $2x + y + 2z = 2$, the radius of the sphere (1) is equal to the perpendicular distance from the centre of the sphere to the plane .

$$\therefore r = \frac{2(6) - (-1) + 2(2) - 2}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{12 + 1 + 4 - 2}{\sqrt{4 + 1 + 4}} = \frac{15}{3} = 5$$

$$\therefore (1) \Longrightarrow (x-2)^2 + (y—1)^2 + (z-2)^2 = 5^2$$

$$\Longrightarrow x^2 - 12x + 36 + y^2 + 2y + 1 + z^2 - 4\,z + 4 = 25$$

$$\Longrightarrow x^2 + y^2 + z^2 - 12x + 2y - 4z + 16 = 0$$

which is the required equation of the sphere .

- **Show that the two spheres $\boldsymbol{x^2 + y^2 + z^2 - 2x - 4y - 4z = 0}$ and $\boldsymbol{x^2 + y^2 + z^2 + 10x + 2z + 10 = 0}$ touch each other .Find their point of contact.** **[8 Marks]**

<u>**Solution:**</u>

Let $s_1 \equiv \; x^2 + y^2 + z^2 - 2x - 4y - 4z = 0$

Here $2u = -2$,$2v = -4$,$2w = -4$, $d = 0$

$$\Longrightarrow u = -1, v = -2, w = -2\,, d = 0$$

Centre $c_1(-u, -v, -w)$

i.e., $c_1(1,2,2)$

$$\text{Radius } R_1 = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{1 + 4 + 4 - 0} \;= \sqrt{9} = 3$$

Let $s_2 \equiv \quad x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$

Here $2u = 10$,$2v = 0$,$2w = 2$,$d = 10$

$$\Longrightarrow u = 5\,, v = 0\,, w = 1\,, d = 10$$

$\therefore$ centre $c_2(-u, -v, -w)$

i.e., $c_2(-5,0,-1)$

$$\text{Radius } R_2 = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{(-5)^2 + 0 + (-1)^2 - 10}$$

$$= \sqrt{25 + 0 + 1 - 10} = \sqrt{16} = 4$$

Distance between the centres c_1& c_2

$$= \sqrt{(1+5)^2 + (2+0)^2 + (2+1)^2}$$

$$= \sqrt{6^2 + 2^2 + 3^2}$$

$$= \sqrt{36+4+9} = \sqrt{49} = 7$$

Sum of the radii =3+4=7

$\therefore$ The distance between the centres c_1&c_2 is equal to the sum of the radii R_1&R_2

$\therefore$ The given spheres touch externally .

Now the point of contact is the point which divides the line joining c_1&c_2 in the ratio3: 4 internally

$\therefore$ The point of contact is

$$\left[\frac{4(1)+3(-5)}{3+4}, \frac{4(2)+3(0)}{3+4}, \frac{4(2)+3(-1)}{3+4}\right]$$

$$\Rightarrow \left[\frac{4-15}{7}, \frac{8}{7}, \frac{8-3}{7}\right] \Rightarrow \left[\frac{-11}{7}, \frac{8}{7}, \frac{5}{7}\right]$$

- **Show that the two spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 18x - 24y - 40z + 225 = 0$ touch each other .Find their point of contact . [8 Marks]**

Solution:

Let $s_1 \equiv x^2 + y^2 + z^2 - 25 = 0$

Here $2u = 0$,$2v = 0$,$2w = 0$, $d = -25$

$$\Rightarrow u = v = w = 0 , d = -25$$

Centre $c_1(-u, -v, -w)$

i.e., $c_1(0,0,0,)$

Radius $R_1 = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{0+0+0-(-25)} = \sqrt{25} = 5$$

Let $s_2 \equiv x^2 + y^2 + z^2 - 18x - 24y - 40z + 225 = 0$

Here $2u = -18$,$2v = -24$,$2w = -40$,$d = 225$

$$\Longrightarrow u = -9 \,, v = -12 \,, w = -20$$

$\therefore$ centre $c_2(-u, -v, -w)$

i.e., $c_2(9,12,20)$

Radius $R_2 = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{9^2 + 12^2 + 20^2 - 225} = \sqrt{81 + 144 + 400 - 225} = \sqrt{400} = 20$$

Distance between the centres c_1& c_2 $= \sqrt{(9-0)^2 + (12-0)^2 + (20-0)^2}$

$$= \sqrt{9^2 + 12^2 + 20^2} = \sqrt{81 + 144 + 400} = \sqrt{625} = 25$$

Sum of the radii =20+5=25

$\therefore$ The distance between the centres c_1&c_2 is equal to the sum of the radii R_1& R_2

$\therefore$ The given spheres touch externally .

Now the point of contact is the point which divides the line joining c_1&c_2 in the ratio 5: 20 internally

$\therefore$ The point of contact is

$$\left[\frac{20 \times 0 + 5 \times 9}{5 + 20}, \frac{20 \times 0 + 5 \times 12}{5 + 20}, \frac{20 \times 0 + 5 \times 20}{5 + 20}\right]$$

$$\Longrightarrow \quad \left[\frac{45}{25}, \frac{60}{25}, \frac{100}{25}\right] \quad \Longrightarrow \quad \left[\frac{9}{5}, \frac{12}{5}, 4\right]$$

UNIT-V
VECTOR CALCULUS

Vector calculus deals with variable vectors which are varying in magnitude or direction or both. A physical quantity, that is a function of the points in space is called a scalar function (denoted by ∅) or a vector function(denoted by $\underset{F}{\rightarrow}$) according as the quantity is a scalar or a vector.

- Temperature at any point and electric potential are examples of a scalar function.
- Velocity of a moving particle and gravitational force are examples of a vector function.

Vector Differential operator - ∇

∇ is an operator which can be operated on both scalar and vector function. It is known as Del.

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

- When **∇** is operated on a scalar function, It gives the gradient ∇∅.
- When **∇** is operated on a vector function, It gives the Divergence and Curl.

Gradient of a scalar point function.

Let ∅ $(x$, y, z) be a scalar point function defined in a certain region of space.

$$\nabla\emptyset = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\emptyset \quad = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

is defined as the gradient of ∅ and shortly defined as Grad ∅.

- **Find grad $\emptyset$ at the point $(1,-2,-1)$ when $\emptyset = 3x^2y - y^3z^2$.**

<u>**Solution:**</u>

Let $\emptyset = 3x^2y - y^3z^2$

$$\text{Grad}\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$\frac{\partial\emptyset}{\partial x} = 6xy \;;\; \frac{\partial\emptyset}{\partial y} = 3x^2 - 3y^2z^2 \;;\; \frac{\partial\emptyset}{\partial z} = -2y^3z$$

$$\text{Grad}\,\emptyset = \vec{i}\,6xy + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}\,(-2y^3z)$$

$$\text{Grad}\emptyset_{(1,-2,-1)} = -12\,\vec{i} - 9\vec{j} - 16\vec{k}$$

- **If $\vec{r}$ is the position vector of the point (x,y,z), prove that $\nabla(r) = \left(\frac{1}{r}\right)\vec{r}$.**

<u>**Solution:**</u>

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}\,| = r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Partially differentiating, with respect to x, y, z we get,

$$2r\frac{\partial r}{\partial x} = 2x \;;\; 2r\frac{\partial r}{\partial y} = 2y \;;\; 2r\frac{\partial r}{\partial z} = 2z$$

$$\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = \frac{\vec{r}}{r}$$

- **Find the unit vector normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.**

<u>**Solution:**</u>

Unit vector normal to the surface is $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|}$

where $\hat{n}$ is the unit in the direction of the normal.

$\vec{n} = \nabla\emptyset$ at the given point.

Let $\emptyset = xy^3z^2 - 4$

$$\vec{n} = \nabla\emptyset = \vec{i}(y^3z^2) + \vec{j}(3xy^2z^2) + \vec{k}(2xy^3z)$$

$$\nabla\emptyset_{(-1,-1,2)} = -4\vec{\imath} - 12\vec{\jmath} + 4\vec{k}$$

$$|\nabla\emptyset| = \sqrt{16+144+16} = \sqrt{176} = 4\sqrt{11}$$

$$\therefore \ |\vec{n}| = 4\sqrt{11}$$

$$\hat{n} = \frac{-4\vec{i} - 12\vec{j} + 4\vec{k}}{4\sqrt{11}} = \frac{-\vec{\imath} - 3\vec{\jmath} + \vec{k}}{\sqrt{11}}$$

- **Find the angle between the normals to the surface $xy^3z^2 = 4$ at the points $(-1,-1,2)$ and $(4,1,-1)$.** **[8 Marks]**

Solution:

we know that, $\quad \mathrm{Cos}\theta = \dfrac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}|.\overline{|n_2|}}$

Let $\vec{n_1} = \nabla\phi$ at $(-1,-1,2)$

$\vec{n_2} = \nabla\phi$ at $(4,1,-1)$ where $\emptyset = xy^3z^2 - 4$

$$\nabla\phi = \vec{\imath}(y^3z^2) + \vec{\jmath}(3xy^2z^2) + \vec{k}(2xy^3\mathrm{z})$$

$\nabla\phi$ at $(-1,-1,2) = -4\vec{\imath} - 12\vec{\jmath} + 4\vec{k}$

$$|\nabla\emptyset| = |\vec{n_1}| = \sqrt{(16+144+16)} = \sqrt{176}$$

$\nabla\phi$ at $(4,1,-1) = \vec{\imath} + 12\vec{\jmath} - 8\vec{k}$

$$|\nabla\emptyset| = |\vec{n_2}| = \sqrt{(1+144+64)} = \sqrt{209}$$

$$\therefore \mathrm{Cos}\theta = \frac{(-4\vec{\imath} - 12\vec{\jmath} + 4\vec{k}).(\vec{\imath} + 12\vec{\jmath} - 8\vec{k})}{\sqrt{176}.\sqrt{209}}$$

$$= \frac{-4 - 144 - 32}{\sqrt{176}.\sqrt{209}}$$

$$= \frac{-180}{\sqrt{176}.\sqrt{209}} = \frac{-45}{\sqrt{19}.11}$$

- **Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(-1, -2, 1)$.** **[8 Marks]**

<u>**Solution:**</u>

we know that, $\quad \text{Cos}\theta = \dfrac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}|.|\overrightarrow{n_2}|}$

$\overrightarrow{n_1} = \nabla\phi_1$ at $(-1, -2, 1)$

$\overrightarrow{n_2} = \nabla\phi_2$ at $(-1, -2, 1)$

Let $\phi_1 = xy^2z - 3x - z^2$

$\nabla\phi_1 = \vec{i}(y^2z - 3) + \vec{j}(2xyz) + \vec{k}(xy^2 - 2z)$

$\nabla\phi_1$ at $(-1, -2, 1) = \vec{i} - 4\vec{j} - 6\vec{k}$

$|\nabla\phi_1| = \sqrt{(1 + 16 + 36)} = \sqrt{53}$

Let $\phi_2 = 3x^2 - y^2 + 2z = 1$

$\nabla\phi_2 = \vec{i}\,(6x) + \vec{j}\,(-2y) + \vec{k}(2)$

$\nabla\phi_2$ at $(-1, -2, 1) = -6\,\vec{i} + 4\vec{j} + 2\vec{k}$

$|\nabla\phi_2| = \sqrt{(36 + 16 + 4)} = \sqrt{56}$

$$\therefore \quad \text{Cos}\theta = \frac{\left(\vec{i} - 4\vec{j} - 6\vec{k}\right).(-6\,\vec{i} + 4\vec{j} + 2\vec{k})}{\sqrt{53}\,\sqrt{56}}$$

$$= \frac{-6 - 16 - 12}{\sqrt{53}\,\sqrt{56}} = \frac{-34}{\sqrt{53}\,\sqrt{56}}$$

- **Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6, 4, 3).** **[8 Marks]**

<u>**Solution:**</u>

we know that, $\quad \text{Cos}\theta = \dfrac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}|.|\overrightarrow{n_2}|}$

$\overrightarrow{n_1} = \nabla\phi_1$ at (6,4,3)

$\overrightarrow{n_2} = \nabla\phi_2$ at (6,4,3)

Let $\phi_1 = x^2 - y^2 - z^2 - 11$

$\nabla\phi_1 = \vec{\imath}\,(2x) + \vec{\jmath}\,(-2y) + \vec{k}(-2z)$

$\nabla\phi_1$ at(6, 4, 3)$= 12\,\vec{\imath} - 8\vec{\jmath} - 6\vec{k}$

$|\nabla\phi_1| = \sqrt{(144 + 64 + 36)} = \sqrt{244}$

Let $\phi_2 = xy + yz - zx - 18$

$\nabla\phi_2 = \vec{\imath}\,(y - z) + \vec{\jmath}\,(x + z) + \vec{k}(y - x)$

$\nabla\phi_2$ at $(-1, -2, 1) = \vec{\imath} + 9\vec{\jmath} - 2\vec{k}$

$|\nabla\phi_2| = \sqrt{(1 + 81 + 4)} = \sqrt{86}$

$$\therefore \quad \text{Cos}\theta = \frac{\left(12\,\vec{\imath} - 8\vec{\jmath} - 6\vec{k}\right).\left(\vec{\imath} + 9\vec{\jmath} - 2\vec{k}\right)}{\sqrt{244}\,\sqrt{86}}$$

$$= \frac{12 - 72 + 124}{\sqrt{244}\,\sqrt{86}}$$

$$= \frac{-24}{\sqrt{5246}}$$

- **Find the values of $\boldsymbol{\lambda}$ and μ if the surfaces $\boldsymbol{\lambda x^2 - \mu yz = (\lambda + 2)x}$ and $\boldsymbol{4x^2y + z^3 = 4}$ cut orthogonally at the point $\boldsymbol{(1, -1, 2)}$.** [8 Marks]

Solution:

Let $\lambda x^2 - \mu yz - (\lambda + 2)x = 0$ ----------(1) and

$4x^2y + z^3 - 4 = 0$------------------------(2)

To prove that two surfaces cut orthogonally, we have to prove that

$$\overrightarrow{n_1}.\overrightarrow{n_2} = 0, \quad \text{since } \theta = \frac{\pi}{2}$$

$$\overrightarrow{n_1} = \nabla\phi_1 \text{ at } (1,-1,2)$$

Let $\phi_1 = \lambda\, x^2 - \mu yz - (\lambda + 2)x$

$$\nabla\phi_1 = \vec{\imath}\left(2\lambda x - (\lambda + 2)\right) + \vec{\jmath}(-\mu z) + \vec{k}(-\mu y)$$

$$\nabla\phi_1 \text{ at } (1,-1,2) = \vec{\imath}(2\lambda - \lambda - 2) + \vec{\jmath}(-2\mu) + \vec{k}(\mu)$$

$$\nabla\phi_2 = \vec{\imath}(8xy) + \vec{\jmath}(4x^2) + \vec{k}(3\,z^2) = \overrightarrow{n_2}$$

$$\nabla\phi_2 \text{ at } (1,-1,2) = -8\vec{\imath} + 4\vec{\jmath} + 12\vec{k}$$

$$\overrightarrow{n_1}.\overrightarrow{n_2} = -8(2\lambda - \lambda - 2) - 8\mu + 12\mu = 0\text{--------------}(3)$$

Since $(1,-1,2)$ is the point of intersection of two surfaces ,it lies in (1)

$$\lambda + 2\mu = (\lambda + 2) \Rightarrow \lambda + 2\mu - \lambda - 2 = 0 \Rightarrow 2\mu - 2 = 0$$

$$\Rightarrow \mu = 1$$

Substitute μ=1 in (3),we get,

$$-8(\lambda - 2) - 8 + 12 = 0 \Rightarrow -8\lambda + 16 + 4 = 0$$

$$\Rightarrow \lambda = \frac{5}{2}$$

- **Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$.** **[8 Marks]**

Solution:

Let $\emptyset = 2xz^2 - 3xy - 4x - 7$

$$\nabla\emptyset = \vec{\imath}(2z^2 - 3y - 4) + \vec{\jmath}(-3x) + \vec{k}(4xz)$$

$$\nabla\emptyset \text{ at } (1,-1,2) = 7\vec{\imath} - 3\vec{\jmath} + 8\vec{k}$$

$\nabla\emptyset$ at $(1,-1,2)$ is a vector in the direction of the normal to the surface $\emptyset = c$.

D.R.'s of the normal to the surface $\emptyset = c$ at the point $(1,-1,2)$ and having the line whose D.R.'s are $(7,-3,8)$ as a normal .

Equation of the tangent plane is $7(x-1) - 3(y+1) + 8(z-2) = 0$

$$\Longrightarrow \qquad 7x - 3y + 8z - 26 = 0.$$

Directional derivative of a scalar point function.

A scalar quantity that is a function of the position of a point in space is called a scalar point function.

- **Find the directional derivative of $\emptyset = xy + yz + zx$ at the point (1, 2, 3) along the x-axis.**

  Solution:

  $\nabla\emptyset = \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(y+x)$.

  Along the x- axis $y = z = 0$

  Directional derivative $= \dfrac{\nabla\emptyset.\vec{a}}{|\vec{a}|}$, where $\vec{a}$ is the given unit vector.

  $\vec{a} = \vec{i}$

  $\therefore |\vec{a}| = \sqrt{1} = 1$

  $$\therefore \frac{\nabla\emptyset.\vec{a}}{\vec{a}} at\ (1,2,3) = \frac{\left(5\,\vec{i} + 4\vec{j} + 3\vec{k}\right).\vec{i}}{1} = 5$$

- **Find the direction in which the directional derivative of $\emptyset = x^2y^2z^4$ from (3, 1,−2) is maximum.**

  Solution:

  $\nabla\emptyset = \vec{i}(2xy^2z^4) + \vec{j}(2x^2yz^4) + \vec{k}(4x^2y^2z^3)$.

  $\nabla\emptyset_{(3,1,-2)} = \vec{i}(6\times1\times16) + \vec{j}(2\times9\times16) + \vec{k}[(4\times9\times(-8)]$

  $= 96\vec{i} + 288\,\vec{j} - 288\,\vec{k} = 96(\vec{i} + 3\vec{j} - 3\vec{j})$

  Max. directional derivative $|\nabla\emptyset| = 96\sqrt{19}$

$$= \frac{96(\vec{\imath} + 3\vec{\jmath} - 3\vec{\jmath})}{96\sqrt{19}}$$

$$= \frac{\vec{\imath} + 3\vec{\jmath} - 3\vec{\jmath}}{\sqrt{19}}$$

- **Find the maximum directional derivative of $\emptyset = x^3y^2z$ at the point (1, 1, 1).**

<u>**Solution:**</u>

Max. directional derivative of $\emptyset$=$|\nabla\emptyset|$

$\nabla\emptyset = \vec{\imath}(3x^2y^2 z) + \vec{\jmath}(2x^3yz) + \vec{k}(x^3y^2)$

$\nabla\emptyset_{(1,1,1)} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$

$|\nabla\emptyset| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

- **Prove that $\nabla r^n = nr^{n-2}\vec{r}$.**

<u>**Solution:**</u>

Let $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$r^2 = x^2 + y^2 + z^2$ --------------(1)

$$\nabla r^n = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)r^n$$

$$= \vec{\imath}\frac{\partial}{\partial x}(r^n) + \vec{\jmath}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{\imath}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{\jmath}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\vec{\imath}\frac{\partial r}{\partial x} + \vec{\jmath}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

From (1), $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$

$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla r^n = nr^{n-1}\left(\vec{\imath}\frac{x}{r} + \vec{\jmath}\frac{y}{r} + \vec{k}\frac{z}{r}\right) = nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

- **Find the directional derivative of the function $\emptyset = xy^2 + yz^2$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x\, logz - y^2 + 4 = 0$ at the point $(-1, 2, 1)$.**

Solution:

Directional derivative is given by $\nabla\emptyset$

$$\nabla\emptyset = \vec{\imath}\frac{\partial\emptyset}{\partial x} + \vec{\jmath}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= \vec{\imath}(y^2) + \vec{\jmath}(2xy + z^2) + \vec{k}(2yz)$$

$$\nabla\emptyset_{at(2,-1,1)} = \vec{\imath} + \vec{\jmath}(-4+1) + \vec{k}(-2)$$

$$= \vec{\imath} - 3\vec{\jmath} - 2\vec{k}$$

Now we have the equation of the surface $x\, logz - y^2 + 4 = 0$ is identified with $\psi(x, y, z)$ =c.

$$\therefore\ \psi(x, y, z) = x\, logz - y^2 + 4 \Longrightarrow c = -4$$

The direction of the normal to this surface is the same as that of $\nabla\psi$.

$$\nabla\psi = logz\,\vec{\imath} - 2y\vec{\jmath} + \frac{x}{z}\vec{k}$$

$\nabla\psi$ at $(-1,2,1) = -4\vec{\jmath} - \vec{k} = b$(say)

Directional derivative of ϕ in the direction of $\vec{b}$

$$\frac{\nabla\emptyset.\vec{b}}{|\vec{b}|} = \frac{[(\vec{\imath} - 3\vec{\jmath} - 2\vec{k}).(-4\vec{\jmath} - \vec{k})]}{\sqrt{17}} = \frac{14}{\sqrt{17}}$$

- **Find the directional derivative of $\emptyset = xy^2 + yz^3$ at the point P $(2, -1, 1)$ in the direction of PQ where Q is the point (3, 1, 3).**

Solution:

Given that $\emptyset = xy^2 + yz^3$

$$\nabla\emptyset = \vec{\imath}(y^2) + \vec{\jmath}(2xy + z^3) + \vec{k}(3yz^2)$$

$$\nabla\emptyset_{\ (2,1,-1)} = \vec{\imath} + \vec{\jmath}(-4+1) + \vec{k}(-3)$$

$$= \vec{\imath} - 3\vec{\jmath} - 3\vec{k}$$

Directional derivative of $\emptyset$ in the direction of $\overrightarrow{PQ}$=component or projection of directional derivative of $\nabla\emptyset$ along

$$\overrightarrow{PQ} = \frac{\nabla\emptyset\,.\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = (-3\vec{i} + \vec{j} - 3\vec{k}) - (-2\vec{i} + \vec{j} - \vec{k}) = -\vec{i} - 2\vec{k}$$

$$|\overrightarrow{PQ}| = \sqrt{5}$$

$$\therefore\ \overrightarrow{PQ} = \frac{(-\vec{i} - 2\vec{k}).(\vec{i} - 3\vec{j} - 3\vec{k})}{\sqrt{5}} = \frac{(-1+0+6)}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \text{ units}$$

- **Find the directional derivative of $\nabla.(\nabla\phi)$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\emptyset = x^2y^2z^2$.** **[8 Marks]**

<u>Solution:</u>

$$\nabla\phi = \vec{i}(2xy^2z^2) + \vec{j}(2x^2yz^2) + \vec{k}(2x^2y^2z)$$

$$\nabla.(\nabla\phi) = \left(\vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}\right).\left[\vec{i}(2xy^2z^2) + \vec{j}(2x^2yz^2) + \vec{k}(2x^2y^2z)\right]$$

$$= 2y^2z^2 + 2x^2z^2 + 2x^2y^2$$

$$= 2(x^2z^2 + y^2z^2 + x^2y^2)$$

$$\nabla.(\nabla\phi) \text{ at } (1,-2,1) = 2(1+4+4) = 18$$

The direction of the normal to the surface is the same as that of $\nabla\psi$.

$$\psi(x,y,z) = xy^2z - 3x - z^2$$

$$\nabla\psi = \vec{i}(y^2z - 3) + \vec{j}(2xyz) + \vec{k}(xy^2 + 2z)$$

$$\nabla\psi_{(1,2,-1)} = \vec{i}(-4-3) - \vec{j}(2) + \vec{k}(4-2)$$

$$= -7\vec{i} - 2\vec{j} + 2\vec{k}$$

$$|\nabla\psi| = \sqrt{(-7)^2 + (-2)^2 + 2^2} = \sqrt{57}$$

$$\text{Directional derivative} = \frac{\nabla.(\nabla\phi).\nabla\psi}{|\nabla\psi|} = \frac{18.\left(\overline{-7\,i} - 2\vec{j} + 2\vec{k}\right)}{\sqrt{57}}$$

$$= \frac{-126\vec{\imath} - 36\vec{\jmath} + 36\vec{k}}{\sqrt{57}}$$

Define Divergence of a vector.

If $\vec{F}(x,y,z)$ is a differentiable vector point function defined at each point (x,y,z) in some region of space ,then the divergence of $\vec{F}$,denoted as div$\vec{F}$ is defined as ,

$$\text{div}\,\overrightarrow{F} = \nabla.\overrightarrow{F}$$

$$= \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{F}$$

$$= \vec{\imath}\frac{\partial \vec{F}}{\partial x} + \vec{\jmath}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$$

- **If** $\vec{F} = 3xyz^2\vec{i} + 2xy^3\vec{j} - x^2yz\vec{k}$ **, find** $\nabla.\overrightarrow{\boldsymbol{F}}$ **at the point (1,−1,1).**

Solution:

$$\text{div}\,\overrightarrow{F} = \nabla.\overrightarrow{F} = \vec{\imath}\frac{\partial \vec{F}}{\partial x} + \vec{\jmath}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$$

$$= (3yz^2 + 6xy^2 - x^2y)$$

$$= \nabla.\vec{F}_{(1,-1,1)} = 3(-1)(1) + 6(1)(1) - 1(-1) = 4$$

Define Solenoidal vector

If $\overrightarrow{F}$ is a vector such that $\nabla.\overrightarrow{F}=0$ at all points in a region, then it is said to be a Solenoidal vector in that region.

- **Show that** $\vec{F} = (x+2y)\vec{i} + (y+3z)\vec{j} + (x-2z)\vec{k}$ **is solenoid.**

Solution:

To prove that $\overrightarrow{F}$ is solenoidal, we have, $\nabla.\overrightarrow{F}=0$

$$\nabla.\overrightarrow{F} = \frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(y+3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2 = 0$$

$\therefore \overrightarrow{F}$ is solenoidal.

- **Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal. (L6)**

  <u>Solution:</u>

  To prove that $\vec{F}$ is solenoidal, we have

  $$\nabla.\vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = 0$$

- **Find the value of λ, so that $\vec{F} = \lambda y^4 z^2 \vec{i} + 4x^3 z^2 \vec{j} + 5x^2 y^2 \vec{k}$ may be solenoidal. (L1)**

  <u>Solution:</u>

  $\vec{F}$ is solenoidal ,if $\nabla.\vec{F}=0$

  $$\Rightarrow \nabla.\vec{F} = \frac{\partial}{\partial x}(\lambda y^4 z^2) + \frac{\partial}{\partial y}(4x^3 z^2) + \frac{\partial}{\partial z}(5x^2 y^2)$$
  $$= 0$$

  Since all the values are zero , λ can take any value.

- **If $\nabla\emptyset$ is solenoidal, then find $\nabla^2\emptyset$**

  <u>Solution:</u>

  Since $\nabla\emptyset$ is solenoidal, $\nabla.\nabla\emptyset = 0$

  $$\Rightarrow \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}\right) = 0$$
  $$\Rightarrow \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0$$
  $$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\emptyset = 0$$
  $$\Rightarrow \nabla^2\emptyset = 0$$

<u>Define Curl of a vector</u>

If $\vec{F}(x, y, z)$ is a differentiable vector point function defined at each point (x, y, z) in some region of space,then the curl of $\vec{F}$,denoted as curl $\vec{F}$ is defined as ,curl$\vec{F}$ or rot$\vec{F}$. It is given by,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

<u>Define irrotational vector</u>

If curl$\vec{F} = 0$ (i.e) ∇x $\vec{F}$=0, then $\vec{F}$ is said to be irrotational.

- **Show that** $\vec{F}=(\sin y+z)\vec{i}+(x\cos y-z)\vec{j}+(x-y)\vec{k}$ is irrotational.

  $\vec{F}$ is said to be irrotational if $\nabla\text{x}\,\vec{F}=0$

$$\nabla\text{x}\,\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (siny+z) & (xcosy-z) & (x-y)\end{vmatrix}=0$$

$$=\vec{i}(-1+1)-\vec{j}(1-1)+\vec{k}(cosy-cosy)$$

$\therefore$ $\vec{F}$ is irrotational.

- **Find the values of a , b , c so that the vector**

  $\vec{F}=(x+y+az)\vec{i}+(bx+2y-z)\vec{j}+(-x+cy+2z)\vec{k}$ **may be irrotational.**

  <u>**Solution:**</u>

  $\vec{F}$ is said to be irrotational if $\nabla\text{x}\,\vec{F}=0$

$$\nabla\text{x}\,\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (x+y+az) & (bx+2y-z) & (-x+cy+2z)\end{vmatrix}=0$$

$$\Rightarrow \vec{i}(c+1)-\vec{j}(-1-a)+\vec{k}(b-1)=0$$

$$\Rightarrow c+1=0\,;a+1=0;\ b-1=0$$

$$\Rightarrow c=-1\,;\ a=-1\,;\ b=1$$

- **Prove that the gradient of any scalar point function is irrotational.**

  <u>**Solution:**</u>

Let $\nabla\emptyset$ be the gradient of a scalar point function $\emptyset$ then to prove that it is irrotational ,we have $\nabla x\nabla\emptyset=0$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \emptyset}{\partial x} & \frac{\partial \emptyset}{\partial y} & \frac{\partial \emptyset}{\partial z} \end{vmatrix} = 0$$

- **If $\overrightarrow{F}$ is irrotational, prove that it is conservative.**

<u>**Solution:**</u>

Let $\overrightarrow{F} = \nabla\emptyset$

$$\int_A^B \overrightarrow{F}.d\overrightarrow{r} = \int_A^B \nabla\emptyset.d\vec{r}$$

$$= \int_A^B \left(\vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \overrightarrow{k}\frac{\partial\emptyset}{\partial z}\right).\left(dx\vec{i} + dy\vec{j} + dz\overrightarrow{k}\right)$$

$$= \int_A^B \left(\frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz\right) = \int_A^B d\emptyset$$

$$= (\emptyset)_A^B \quad = \emptyset(B) - \emptyset(A)$$

$\therefore \overrightarrow{F}$ is conservative.

- **Find the values of a , b , c so that the vector** $\overrightarrow{F} = (x+y+az)\vec{i} + (bx+2y-z)\vec{j} + (-x+cy+2z)\vec{k}$ **may be irrotational.**

<u>**Solution:**</u>

To prove that $\overrightarrow{F}$ is conservative , $\nabla x \overrightarrow{F}=0$

$$\nabla \text{ x } \overrightarrow{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+yz) & (xz-3) & (xy) \end{vmatrix}$$

$$=\vec{i}(x-x) - \vec{j}(y-y) + \overrightarrow{k}(z-z) = 0.$$

- **If** $\overrightarrow{F} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$**, show that $\overrightarrow{F}$ is perpendicular to curl$\overrightarrow{F}$.**

<u>**Solution:**</u>

To prove that $\overrightarrow{F}$ is perpendicular we have to prove that

$\vec{F}.\,\mathrm{curl}\,\vec{F}=0.$

To find $\mathrm{curl}\vec{F}$,

$$\nabla\,\mathrm{x}\,\vec{F}=\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -(x+y) \end{vmatrix}$$

$$=\vec{\imath}(-1-0)-\vec{\jmath}(-1-0)+\vec{k}\,(0-1)$$

$$=-\vec{\imath}+\vec{\jmath}-\vec{k}$$

$$\vec{F}.\,\mathrm{curl}\,\vec{F}=\left[(x+y+1)\vec{\imath}+\vec{\jmath}-(x+y)\,\vec{k}\right].\left[(-\vec{\imath}+\vec{\jmath}-\vec{k}\,)\right]$$

$$=-x-y-1+1+x+y=0$$

- **If** $\vec{F}=z\vec{i}+x\vec{j}+y\vec{k}$**, prove that curl (curl $\vec{F}$) = 0.**

<u>**Solution:**</u>

$$\mathrm{curl}\,\vec{F}=\nabla\mathrm{x}\vec{F}=\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$=\vec{\imath}(1-0)-\vec{\jmath}(0-1)+\vec{k}(1-0)$$

$$=\vec{\imath}+\vec{\jmath}+\vec{k}$$

$$\mathrm{curl}\,(\mathrm{curl}\vec{F})=\nabla\,\mathrm{x}\,\mathrm{Curl}\,\vec{F}.$$

$$=\nabla\,\mathrm{x}\,(\vec{\imath}+\vec{\jmath}+\vec{k})$$

$$=\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{vmatrix}=0$$

- **If** $\boldsymbol{\vec{F}}=\boldsymbol{3x^2\vec{\imath}+5xy^2\vec{\jmath}+xyz^3\vec{k}}$ **, find** $\nabla(\nabla.\boldsymbol{\vec{F}})$ **and** $\nabla.(\nabla\mathrm{x}\boldsymbol{\vec{F}})$ **at the point (1,2,3).** **[8 Marks]**

<u>**Solution:**</u>

(i) $\nabla.\vec{F}=\left(\vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left(3x^2\vec{\imath}+5xy^2\vec{\jmath}+xyz^3\vec{k}\,\right)$

$$=6x+10xy+3xyz^2$$

$$\nabla(\nabla.\vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(6x + 10xy + 3xyz^2)$$

$$= (6 + 10y + 3yz^2)\,\vec{i} + (10x + 3xz^2)\vec{j} + (6xyz)$$

$$\nabla(\nabla.\vec{F})_{(1,2,3)} = (6 + 20 + 18)\,\vec{i} + (10 + 27)\,\vec{j} + 36\,\vec{k}$$

$$= 44\,\vec{i} + 37\vec{j} + 36\,\vec{k}$$

(ii) $$\nabla \mathrm{x} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 & 5xy^2 & xyz^3 \end{vmatrix}$$

$$= \vec{i}\,(xz^3 - 0) - \vec{j}(yz^3 - 0) + \vec{k}(5y^2 - 0)$$

$$= xz^3\vec{i} - yz^3\vec{j} + 5y^2\vec{k}$$

$$\nabla.(\nabla \mathrm{x} \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(xz^3\vec{i} - yz^3\vec{j} + 5y^2\vec{k}\right)$$

$$= z^3 - z^3 + 0 = 0$$

- **If $u = x^2yz$ and $v = xy - 3z^2$, find (i) $\nabla.(\nabla u \text{ x } \nabla v)$ and (ii) ∇ x $(\nabla u \text{ x } \nabla v)$ at the point (1,1,0).** **[8 Marks]**

<u>**Solution:**</u>

(i) $\nabla.(\nabla u \text{ x } \nabla v)$

$$\nabla u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2yz)$$

$$= \vec{i}2xyz + \vec{j}x^2z + \vec{k}x^2y$$

$$\nabla v = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy - 3z^2)$$

$$= \vec{i}(y) + \vec{j}\,(x) - \vec{k}(6z)$$

$$(\nabla u \text{ x } \nabla v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2xyz & \mathrm{x^2z} & x^2y \\ y & x & -6z \end{vmatrix}$$

$$= \vec{\imath}(-6x^2z^2 - x^3y) - \vec{\jmath}(-12xyz^2 - x^2y^2) + \vec{k}(2x^2yz - x^2zy)$$

$\therefore \quad \nabla.(\nabla u \times \nabla v)$

$$= \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left[\vec{\imath}(-6x^2z^2 - x^3y) + \vec{\jmath}(12xyz^2 + x^2y^2) + \vec{k}(2x^2yz - x^2zy)\right]$$

$$= \frac{\partial}{\partial x}(-6x^2z^2 - x^3y) + \frac{\partial}{\partial y}(-12xyz^2 - x^2y^2) + \frac{\partial}{\partial z}(2x^2yz - x^2zy)$$

$$= (-12xz^2 - 3x^2y) - (12xz^2 + 2x^2y) + (2x^2y - x^2y)$$

$$= (-12xz^2 - 3x^2y) - (12xz^2 + 2x^2y) + (x^2y)$$

$$\nabla.(\nabla u \times \nabla v)_{(1,1,0)} = -3 - 2 + 1 = -4$$

(ii) $\nabla \times (\nabla u \times \nabla v)$

$$(\nabla u \times \nabla v) = \vec{\imath}(-6x^2z^2 - x^3y) + \vec{\jmath}(12xyz^2 + x^2y^2) + \vec{k}(x^2yz)$$

$$\nabla \times (\nabla u \times \nabla v) \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (-6x^2z^2 - x^3y) & (12xy\,z^2 + x^2\,y^2) & (x^2yz) \end{vmatrix}$$

$$= \vec{\imath}(x^2z - 24xyz) - \vec{\jmath}(2xyz + 12x^2z) + \vec{k}(12yz^2 + 2xy^2 + x^3)$$

$$\nabla \times (\nabla u \times \nabla v)_{(1,1,0)} = 3\vec{k}$$

Work Done By The Force

- Work done by the force $\vec{F}$ through the displacement d$\vec{r}$ is $\vec{F}.d\vec{r}$
- Work done by the force $\vec{F}$ through the displacement d$\vec{r}$ along the curve C is $\int_c \vec{F}.d\vec{r}$
- If the work done by a force does not depend on the path C, but only on the end points of C, then the force $\vec{F}$ is said to be conservative.
- If $\vec{F}$ is conservative vector, then there corresponds a scalar point function $\emptyset$ such that $\vec{F} = \nabla\emptyset$.

- $\emptyset$ is called the scalar potential of $\vec{F}$.
- If $\vec{F}$ is irrotational,then $\vec{F}$ is conservative
- **Find the work done by the force $\vec{F} = x\vec{i} + 2y\vec{j}$ when it moves a particle on the curve 2y = x^2 from (0, 0) to (2, 2).**

Solution:

Work done by a force $\vec{F} = \int_c \vec{F}.d\vec{r}$

$$\vec{F}.d\vec{r} = (x\,\vec{i} + 2y\vec{j}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$= xdx + 2ydy$$

$$\int_c \vec{F}.d\vec{r} = \int_0^2 xdx + 2ydy$$

$$2y = x^2 \Rightarrow 2dy = 2xdx$$

$$= \int_0^2 xdx + \frac{x^2}{2}2xdx$$

$$= \int_0^2 xdx + x^3dx$$

$$= \left(\frac{x^2}{2} + \frac{x^4}{4}\right)_0^2 = 2 + 4 = 6$$

- **Find the scalar point function whose gradient is $(y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$**

Solution:

Given that

$$\nabla\emptyset = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$$

$$\Rightarrow \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$$

Comparing and equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ on both sides and integrating partially with respect to x, y, z respectively we get

$$\frac{\partial \emptyset}{\partial x} = y^2 - 2xyz^3 \quad \Longrightarrow \emptyset = y^2x - 2\frac{x^2}{2}yz^3 + f(y,z)$$

$$= y^2x - x^2yz^3 + f(y,z) ---------- (1)$$

$$\frac{\partial \emptyset}{\partial y} = 3 + 2xy - x^2z^3 \quad \Longrightarrow \emptyset = 3y + 2x\frac{y^2}{2} - x^2z^3y + f(x,z)$$

$$= 3y + xy^2 - x^2z^3y + f(x,z) ---------- (2)$$

$$\frac{\partial \emptyset}{\partial z} = 6z^3 - 3x^2yz^2 \quad \Longrightarrow \emptyset = 6\frac{z^4}{4} + 3x^2y\frac{z^3}{3} + f(x,y)$$

$$= 3\frac{z^4}{2} + x^2yz^3 + f(x,y) -------- (3)$$

Combining (1),(2),(3) we get,

$$\emptyset = y^2x - x^2yz^3 + 3y + 3\frac{z^4}{2} + C$$

- **If $\nabla\emptyset = 2xyz^3\,\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$, find $\emptyset(x,y,z)$ given that $\emptyset(1,-2,2) = 4.$** **[8 Marks]**

Solution: Given that

$$\nabla\emptyset = \vec{\imath}\frac{\partial \emptyset}{\partial x} + \vec{\jmath}\frac{\partial \emptyset}{\partial y} + \vec{k}\frac{\partial \emptyset}{\partial z} = 2xyz^3\,\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$$

Comparing and equating the coefficients of $\vec{\imath}, \vec{\jmath}, \vec{k}$ on both sides and integrating partially with respect to x, y, z respectively we get ,

$$\frac{\partial \emptyset}{\partial x} = 2xyz^3 \quad \Longrightarrow \emptyset = 2\frac{x^2}{2}yz^3 + f(y,z)$$

$$= x^2yz^3 + f(y,z) \qquad ---------- (1)$$

$$\frac{\partial \emptyset}{\partial y} = x^2z^3 \quad \Longrightarrow \emptyset = x^2yz^3 + f(x,z) \qquad ---------- (2)$$

$$\frac{\partial \emptyset}{\partial z} = 3x^2yz^2\vec{k} \implies \emptyset = \frac{3\,x^2yz^3}{3} + f(x,y)$$

$$= x^2yz^3 + f(x,y) \qquad ----------(3)$$

Combining (1),(2),(3) we get, $\emptyset = x^2yz^3 + \text{C}$

$$\emptyset(1,-2,2) = 4 \implies \emptyset = -16 + \text{C} \implies \text{C} = 16$$

$$\therefore \quad \emptyset = x^2yz^3 + 16$$

- **Prove that $\vec{F} = 3yz\vec{\imath} + 2zx\vec{\jmath} + 4xy\vec{k}$ is not irrotational, but $(x^2yz^3)\vec{F}$ is irrotational.** **[8 Marks]**

Solution:

Given that $\vec{F} = 3yz\vec{\imath} + 2zx\vec{\jmath} + 4xy\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\text{yz} & 2\text{z}x & 4xy \end{vmatrix}$$

$$= \vec{\imath}(4x - 2x) - \vec{\jmath}(4y - 3y) + \vec{k}(2z - 3z)$$

$$= 2x\vec{\imath} - y\vec{\jmath} - \vec{k} \neq 0$$

$$\implies \vec{F} \text{ is not irrotational.}$$

$$(x^2yz^3)\vec{F} = (x^2yz^3)\left(3yz\vec{\imath} + 2zx\vec{\jmath} + 4xy\vec{k}\right)$$

$$= 3x^2y^2z^4\vec{\imath} + 2x^3yz^4\vec{\jmath} + 4x^3y^2z^3\vec{k}$$

$$\nabla \times (x^2yz^3)\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\,x^2y^2\text{z}^4 & 2x^3y\text{z}4 & 4x^3y^2\text{z}^3 \end{vmatrix}$$

$$= \vec{\imath}(8x^3y\text{z}^3 - 8x^3y\text{z}^3) - \vec{\jmath}(12x^2y^2\text{z}^3 - 12x^2y^2\text{z}^3) + \vec{k}(6\,x^2y\text{z}^4 - 6\,x^2y\text{z}^4)$$

$$= 0$$

$\Longrightarrow\ (x^2yz^3)\overrightarrow{F}$ is irrotational.

- **Show that $\overrightarrow{F} = (z^2 + 2x + 3y)\overrightarrow{\imath} + (3x + 2y + z)\overrightarrow{\jmath} + (y + 2zx)\overrightarrow{k}$ is irrotational, but not solenoidal. Find also its scalar potential.**

[8 Marks]

Solution:

(i) To prove that $\overrightarrow{F}$ is irrotational,we have to prove $\nabla \times \overrightarrow{F} = 0$

$$\nabla \times \overrightarrow{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (z^2 + 2x + 3y) & (3x + 2y + z) & (y + 2zx) \end{vmatrix}$$

$$= \vec{\imath}(1 - 1) - \vec{\jmath}(2z - 2z) + \vec{k}(3 - 3) = 0$$

(ii) To prove that $\vec{F}$ is not solenoidal, we have to prove $\nabla.\overrightarrow{F} \neq 0$.

$$\nabla.\overrightarrow{F} = \frac{\partial}{\partial x}(z^2 + 2x + 3y) + \frac{\partial}{\partial y}(3x + 2y + z) + \frac{\partial}{\partial z}(y + 2zx)$$

$$= 2 + 2 + 2x = 4 + 4x \neq 0.$$

(iii)To find its scalar potential ,

since $\nabla \times \overrightarrow{F}$ is irrotational $\overrightarrow{F} = \nabla\emptyset \Longrightarrow \emptyset$ is a scalar potential.

$$\overrightarrow{F} = (z^2 + 2x + 3y)\overrightarrow{\imath} + (3x + 2y + z)\overrightarrow{\jmath} + (y + 2zx)\overrightarrow{k} = \nabla\emptyset$$

$$\nabla\emptyset = \overrightarrow{\imath}\frac{\partial\emptyset}{\partial x} + \overrightarrow{\jmath}\frac{\partial\emptyset}{\partial y} + \overrightarrow{k}\frac{\partial\emptyset}{\partial z} = (z^2 + 2x + 3y)\overrightarrow{\imath} + (3x + 2y + z)\overrightarrow{\jmath} + (y + 2zx)\overrightarrow{k}$$

Comparing and equating the coefficients of $\overrightarrow{\imath}, \overrightarrow{\jmath}, \vec{k}$ on both sides and integrating partially with respect to x, y, z respectively we get ,

$$\frac{\partial\emptyset}{\partial x} = z^2 + 2x + 3y \quad \Longrightarrow \emptyset = z^2x + 2\frac{x^2}{2} + 3xy + c_1$$

$$= z^2x + x^2 + 3xy + c_1 \qquad ----------(1)$$

$$\frac{\partial \emptyset}{\partial y} = 3x + 2y + z \quad \Longrightarrow \emptyset = 3xy + 2\frac{y^2}{2} + zy + c_2$$

$$= 3xy + y^2 + zy + c_2 - - - - - - - - -(2)$$

$$\frac{\partial \emptyset}{\partial z} = y + 2zx \Longrightarrow \emptyset = yz + 2\frac{z^2}{2}x + c_3$$

$$= yz + z^2x \ + c_3 \qquad - - - - - - - - - - (3)$$

Combining (1),(2),(3) we get,

$$\emptyset = x^2 + y^2 + z^2 + z^2x + 3xy + zy + \mathrm{C}$$

If C is a simple closed curve and $\vec{r} = \vec{\imath} + y\vec{\jmath} + z\vec{k}$,

prove that $\int_c \vec{F}.d\vec{r}$= 0.

Solution:

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$d\vec{r} = dx\vec{\imath} \ + \ dy\vec{\jmath} + dz\vec{k}$$

By Stoke's thorem, $\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \mathrm{x}\, \vec{F}).\hat{n}\, ds$

$$\text{If } \vec{F} = \vec{r} = \nabla \mathrm{x}\, \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

$$\therefore \int_C \vec{r}.d\vec{r} = 0.$$

- **Evaluate $\int_C \vec{r}.d\vec{r}$, where C is the line $y = x$ in the xy-plane from(1,1) to(2,2).**

Solution:

$$\vec{r}.d\vec{r} = xdx + ydy + zdz$$

$$\int \vec{r}.d\vec{r} = \int_1^2 (xdx + ydy + zdz) \text{, if } y = x \text{ then } z = 0 \text{ in the } xy - \text{plane}$$

C

$$= 2\left(\frac{x^2}{2}\right)_1^2 = 3$$

- **Evaluate** $\int_C (x\,dy - y\,dx)$ **where C is the circle** $\boldsymbol{x^2 + y^2 = a^2}$**. (L6)**

<u>**Solution:**</u>

$$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2 \Rightarrow y = \sqrt{(a^2 - x^2)}$$

$$y = \sqrt{(a^2 - x^2)} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{\frac{-1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{(a^2 - x^2)}}$$

$$\Rightarrow dy = \frac{-xdx}{\sqrt{(a^2 - x^2)}}$$

$$\int_c (xdy - ydx) = \int_0^a \left[\frac{x(-x)dx}{\sqrt{(a^2 - x^2)}} - \sqrt{(a^2 - x^2)}\,dx\right]$$

$$= -\int_0^a \left[\frac{(a^2 - a^2 + x^2)\,dx}{\sqrt{(a^2 - x^2)}}\right] - \left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= -\left\{\int_0^a \left(\frac{x^2 - a^2}{\sqrt{(a^2 - x^2)}}\right) dx + \int_0^a \left(\frac{a^2}{\sqrt{(a^2 - x^2)}}\right) dx\right\} - \frac{\pi a}{4}$$

$$= \int_0^a \left(\frac{a^2 - x^2}{\sqrt{(a^2 - x^2)}}\right) dx + a^2 \int_0^a \left(\frac{dx}{\sqrt{(a^2 - x^2)}}\right) - \frac{\pi a}{4}$$

$$= \int_0^a \left(\sqrt{(a^2 - x^2)}\right) dx + a^2\left(\sin^{-1}\frac{x}{a}\right)_0^a - \frac{\pi a}{4}$$

$$= \frac{\pi a}{4} - \frac{\pi a}{4} + \frac{\pi a^2}{2} = \frac{\pi a^2}{2}$$

- **Find the constants a, b, c so that**

  $$\boldsymbol{\vec{F} = (x + 2y + az)\vec{\imath} + (bx - 3y - z)\vec{\jmath} + (4x + cy + 2z)\vec{k}}$$

  may be irrotational. For these values of a, b, c, find its scalar potential also. **[8 Marks]**

Solution:

Since $\vec{F}$ is irrotational , $\nabla \times \vec{F}=0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\Longrightarrow \quad \vec{i}(c-1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$

$$\Rightarrow \; c-1=0 \Rightarrow c=1; \; 4-a=0 \Rightarrow a=4; \; b-2=0 \Rightarrow b=2$$

$$\therefore \vec{F} = (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x+y+2z)\vec{k}$$

To find the scalar potential ,let $\vec{F}=\nabla\emptyset$

$$\therefore \; \nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x+y+2z)\vec{k}$$

Comparing and equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ on both sides and integrating partially with respect to x, y, z respectively we get ,

$$\frac{\partial\emptyset}{\partial x} = x+2y+4z \Longrightarrow \emptyset = \frac{x^2}{2} + 2xy + 4zx + f(y,z)$$

$$----------(1)$$

$$\frac{\partial\emptyset}{\partial y} = 2x-3y-z \Longrightarrow \emptyset = 2xy - 3\frac{y^2}{2} - zy + f(x,z)$$

$$----------(2)$$

$$\frac{\partial\emptyset}{\partial z} = 4x+y+2z \Longrightarrow \emptyset = 4xz + zy + 2\frac{z^2}{2} + f(x,y)$$

$$= 4xz + zy + z^2 + f(x,y) -------(3)$$

Combining (1),(2),(3) we get,

$$\emptyset = 4xz + yz + z^2 - 3\frac{y^2}{2} + 2xy + \frac{x^2}{2} + \mathrm{C}$$

- **If $\emptyset = xy + yz + zx$ and $\vec{F} = x^2y\vec{\imath} + y^2z\vec{\jmath} + z^2x\vec{k}$, find $\vec{F}$.grad$\emptyset$ and $\vec{F}$ x grad$\emptyset$ at the point $(3, -1, 2)$. [8 Marks]**

Solution:

(i) To find $\vec{F}$.grad$\emptyset$ we are given that $\emptyset = xy + yz + zx$ and

$$\vec{F} = x^2y\vec{\imath} + y^2z\vec{\jmath} + z^2x\vec{k}$$

$$\text{grad}\emptyset = \vec{\imath}\frac{\partial\emptyset}{\partial x} + \vec{\jmath}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= \vec{\imath}(y+z) + \vec{\jmath}(z+x) + \vec{k}(x+y)$$

$$\vec{F}.\text{grad}\emptyset = \left(x^2y\vec{\imath} + y^2z\vec{\jmath} + z^2x\vec{k}\right).\left[\vec{\imath}(y+z) + \vec{\jmath}(z+x) + \vec{k}(x+y)\right]$$

$$= x^2y(y+z) + y^2z(z+x) + z^2x(x+y)$$

$$\vec{F}.\text{grad}\emptyset_{(3,-1,2)} = -9 + 10 + 24 = 25$$

(ii) To find $\vec{F}$ x grad$\emptyset$

$$\vec{F} \text{ x grad}\emptyset = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x^2y & y^2z & z^2x \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \vec{\imath}[(y^2z)(x+y) - (z^2x)(z+x)] - \vec{\jmath}[(x^2y)(x+y) - (z^2x)(z+x)] + \vec{k}[(x^2y)(z+x) - (y^2z)(y+z)]$$

$$\vec{F} \text{ x grad}\emptyset_{(3,-1,2)} = \vec{\imath}(4-60) - \vec{\jmath}(-18-12) + \vec{k}(-45-2)$$

$$= \vec{\imath}(-56) - \vec{\jmath}(-30) + \vec{k}(-47)$$

- **Find the work done by the force $\vec{F} = (2xy + z^3)\vec{\imath} + x^2\vec{\jmath} + 3xz^2\vec{k}$, when it moves the particle from $(1, -2, 1)$ to (3, 1, 4) along any path.**

Solution:

To evaluate the work done by a force ,the equation of the path and the terminal points must be given .As the equation of the path is not given, we guess that the given force $\overrightarrow{F}$ is conservative .Let us verify whether $\vec{F}$ is conservative (i.e.) irrotational.

$$\nabla \times \overrightarrow{F} \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^3) & (x^2) & (3xz^2) \end{vmatrix} = 0$$

$\therefore$ $\overrightarrow{F}$ is irrotational.

Since $\overrightarrow{F}$ is irrotational let $\overrightarrow{F}=\nabla\emptyset$

$$\overrightarrow{F}=\nabla\emptyset \Longrightarrow \vec{\imath}(2xy + z^3)+\vec{\jmath}(x^2) + \vec{k}(3xz^2)=\nabla\emptyset$$

Comparing and equating the coefficients of $\vec{\imath}, \vec{\jmath}, \vec{k}$ on both sides and integrating partially with respect to x, y, z respectively we get,

$$\frac{\partial\emptyset}{\partial x} = 2xy + z^3 \Longrightarrow \emptyset = 2\frac{x^2}{2}y + z^3x + c_1$$

$$= x^2y + z^3x + c_1 ---------- (1)$$

$$\frac{\partial\emptyset}{\partial y} = x^2 \qquad \Longrightarrow \emptyset = x^2y + c_2 ---------- (2)$$

$$\frac{\partial\emptyset}{\partial z} = 3xz^2 \qquad \Longrightarrow \emptyset = 3x\frac{z^3}{3} + c_3 = xz^3 + c_3 ----(3)$$

Combining (1),(2),(3) we get,

$$\emptyset = x^2y + xz^3 + \mathrm{C}$$

$$\text{Work done } = \int_{\mathrm{C}} \overrightarrow{F}\,.\,d\overrightarrow{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} \vec{F}.d\vec{r} = \int_{(1,-2,1)}^{(3,1,4)} \nabla\emptyset.d\vec{r} = \int_{(1,-2,1)}^{(3,1,4)} d\emptyset = (\emptyset)_{(1,-2,1)}^{(3,1,4)}$$

$$= (x^2y + xz^3)_{(1,-2,1)}^{(3,1,4)} = 9 + (3X64) - (-2+1) = 202$$

<u>**State Green's theorem in a plane**</u>

If C is a simple closed curve enclosing a region R in the xy-plane and P(x,y), Q(x, y) and its first order partial derivatives are continuous in R ,

Then,

$$\oint_C (\mathrm{P}dx + \mathrm{Q}dy) = \iint_R \left(\frac{\partial \mathrm{Q}}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where C is inscribed in the anticlockwise direction

- **Verify Green's theorem in the plane for the integral** $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ **,where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$. [16 Marks]**

<u>**Solution:**</u>

By Green's theorem, we have,

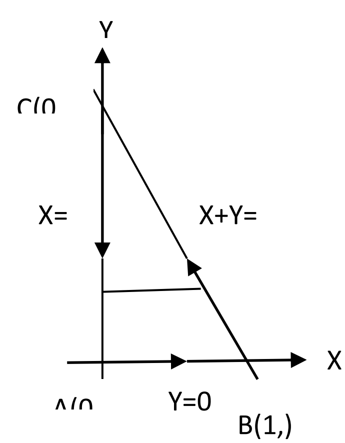
$$\oint_C (\mathrm{P}dx + \mathrm{Q}dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Where C is inscribed in the anticlockwise direction.

$$\text{R.H.S.} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Here $\mathrm{P} = 3x^2 - 8y^2$; $\mathrm{Q} = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \quad ; \quad \frac{\partial Q}{\partial x} = -6y$$



$$\text{R.H.S.} = \int_0^1 \int_0^{1-y} (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 10y\,(x)_0^{1-y}\,dy$$

$$= \int_0^1 10y\,(1-y)dy$$

$$= \int_0^1 (10y - 10y^2)dy$$

$$= \left[10\frac{y^2}{2} - 10\frac{y^2}{2}\right]_0^1 = \frac{5}{3}$$

$$\text{L. H. S.} = \int_{\text{c}} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int 3x^2 dx + \int [(3(1-y)^2 - 8y^2)(-dy) + 4y - 6y(1-y)]\,dy + \int 4ydy$$

Along $\overrightarrow{AB}$ Along $\overrightarrow{BC}$ Along $\overrightarrow{CA}$

$y = 0.\,dy = 0$ $x + y = 1 \Rightarrow x = 1 - y \Rightarrow dx = -dy$ $x = 0; dx = 0$

$$= \int_0^1 3x^2 dx + \int_0^1 [-(3(1-y)^2 - 8y^2) + (4y - 6y + 6y^2)]dy + \int_1^0 4ydy$$

$$= \left(\frac{3x^3}{3}\right)_0^1 + \int_0^1 \{[-3(1 + y^2 - 2y) + 8y^2] + (6y^2 - 2y)\}dy - 4\left(\frac{y^2}{2}\right)_0^1$$

$$= 1 + \int_0^1 (-3 - 3\,y^2 + 6y + 8y^2 + 6y^2 - 2y)dy - 2$$

$$= -1 + \int_0^1 (11\,y^2 + 4y - 3)dy$$

$$= -1 + \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3\text{y}\right)_0^1 \quad = -1 + \left(\frac{11-3}{3}\right) = \frac{5}{3}$$

$\therefore$ LHS $=$ RHS

Hence the theorem is verified.

State Stoke's theorem

If S is an open two sided surface bounded by a simple closed curve C and if $\vec{F}$ is a vector point function with continuous first order partial derivatives on S , then

$$\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \text{x} \vec{F}).\hat{n}\, ds$$

where C is described in the anti clockwise direction as seen from the positive tip of the outward drawn normal at any point on S.

- **Use Stoke's theorem to prove that $\nabla \times \nabla\phi = 0$.**

Solution:

Here $\nabla\emptyset$ is called the grad$\emptyset$ =0.

By Stoke's thorem, $\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \text{x} \vec{F}).\hat{n}\, ds$

By taking $\vec{F} =$ grad$\emptyset$,we get

$$\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \text{x} \vec{F}).\hat{n}\, ds$$

$$= \iint_S curl(\text{grad}\emptyset).\overrightarrow{ds}$$

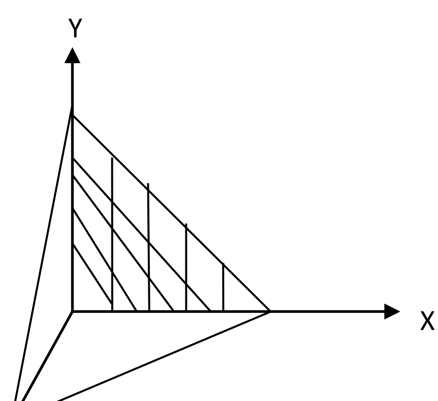
$$= \oint_C (\text{ grad}\phi).\overrightarrow{dr}$$

$$= \oint (\text{ d}\phi) = 0$$

- **Evaluate $\iint_S \vec{A}\, d\vec{s}$, where $\vec{A} = 12x^2y\vec{\imath} - 3yz\vec{\jmath} + 2z\vec{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant. [8 Marks]**

Solution:

Given integral I= $\iint_S \vec{A}.\hat{n}\overrightarrow{ds}$ where $\hat{n}$ is the



the surface S given by $\emptyset = c$

Let $\emptyset = x + y + z - 1$

$$\nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= \vec{i}(1) + \vec{j}(1) + \vec{k}(1)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

$$\therefore \quad \mathrm{I} = \iint_S \left(12x^2y\vec{i} - 3yz\vec{j} + 2z\vec{k}\right).\left(\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}\right) ds$$

$$= \frac{1}{\sqrt{3}}\iint (12x^2y - 3yz + 2z)ds$$

To convert the surface integral as a double integral ,we project the surface S on the xy-plane .Then $dscos\gamma$ =dA, where γ is the angle between the surface S and the xy-plane (i.e.)the angle between $\hat{n}$ and $\vec{k}$ and therefore

$cos\gamma = \hat{n}.\vec{k}$

$$ds = \frac{dA}{\hat{n}\,.\,\vec{k}} = \frac{dxdy}{\frac{1}{\sqrt{3}}}$$

(since the projection of S on the xy plane is ΔOAB).

$$\therefore \quad \mathrm{I} = \iint [12x^2y - 3y(1 - x - y) + 2(1 - x - y)]\, dxdy.$$

(Since the above integral is a double integral ,z is written in terms of x & y).

$$\therefore \mathrm{I} = \int_0^1 \int_0^{1-y} (12x^2y + 3xy + 3y^2 - 5y - 2x + 2)dxdy$$

$$=\left(\int_0^1 12\frac{x^3}{3}y-3y\frac{x^2}{2}+3y^2x-5yx-2\frac{x^2}{2}+2x\right)_0^{1-y} dy$$

$$=\int_0^1\left[4(1-y)^3y-3y\frac{(1-y)^2}{2}+3y^2(1-y)-5y(1-y)-2\frac{(1-y)^2}{2}\right.$$

$$\left.+2(1-y)_0^{1-y}\right]dy$$

$$=\int_0^1\left[4y(1-y^3-3y+3y^2)-\frac{3}{2}y(1+y^2-2y)+3y^2-3y^3-5\mathrm{y}+5y^2\right.$$

$$\left.-1-\ y^2+2y+2-2y\right]dy$$

$$=\int_0^1\left[4\mathrm{y}-4\mathrm{y}^4-12y^2+12y^3-\ \frac{3}{2}y-\ \frac{3}{2}y^3+7y^2-3y^3-5y+1\right]\mathrm{dy}$$

$$=\int_0^1\left(-4y^4+9y^3-\frac{3}{2}y^3-5y^2-y+1\right)\mathrm{dy}$$

$$=\left(-4\frac{y^5}{5}+\frac{15}{2}\frac{y^4}{4}-5\frac{y^3}{3}-\frac{y^2}{2}+y\right)_0^1$$

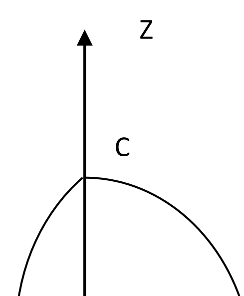
$$=-\ \frac{4}{5}+\frac{15}{8}-\ \frac{5}{3}-\ \frac{1}{2}+1=\frac{49}{120}$$

- **Evaluate $\iint_S \vec{F}.\,d\vec{s}$, where $\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$ and S is the part of the sphere**

$x^2+y^2+z^2=1$ that lies in the first octant. **[8 Marks]**

<u>Solution:</u>

Given integral I= $\iint_S \vec{F}\,.\,\hat{n}$ds,where $\hat{n}$ is the unit normal to the surface



S given by $\emptyset = x^2 + y^2 + z^2 - 1$

$\nabla\emptyset = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$= \frac{x\vec{\imath} + y\vec{\jmath} + z\vec{k}}{\sqrt{4(x^2 + y^2 + z^2}}$$

$$= x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\mathrm{I} = \iint_{S} (yz\vec{\imath} + zx\vec{\jmath} + xy\vec{k}).(x\vec{\imath} + y\vec{\jmath} + z\vec{k})ds$$

$$= \iint_{s} 3xyz\ ds = \iint_{R} 3xyz\ \frac{dxdy}{\hat{n}.\vec{k}}$$

where R is the region in the xy-plane bounded by the circle $x^2 + y^2$=1 and lying in 1st quadrant.

$$\mathrm{I} = \iint 3xyz \frac{dxdy}{z}$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} 3\,xydxdy$$

R

$$= \int_0^1 3y\left(\frac{x^2}{2}\right)_0^{\sqrt{1-y^2}} dy = \frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)_0^1 = \frac{3}{8}$$

R

- **Verify Stoke's theorem for $\vec{F} = (2xy - x^2)\vec{\imath} - (x^2 - y^2)\vec{\jmath}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x$ and $x^2 = y$.**

[16 Marks]

Solution:

By Stoke's thorem, $\int\limits_{C} \vec{F}.d\vec{r} = \iint\limits_{S} (\nabla \mathrm{x}\,\vec{F}).\hat{n}\,ds$

R.H.S.:

$$(\nabla \mathrm{x}\,\vec{F}) = \mathrm{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy - x^2) & -(x^2 - y^2) & 0 \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}(0) + \vec{k}(-2x - 2x) = -4x\,\vec{k}$$

$$\therefore \quad \iint\limits_{S} (\nabla \mathrm{x}\,\vec{F}).\hat{n}\,ds = \iint\limits_{R} -4x\,\vec{k}.\hat{n}\,ds - \iint\limits_{R} 4x\,\vec{k}.\vec{k}\,dxdy$$

[since the given region lies in the xy-plane $\hat{n}=\vec{k}$,is the unit normal to xy-plane and $ds = dxdy$]

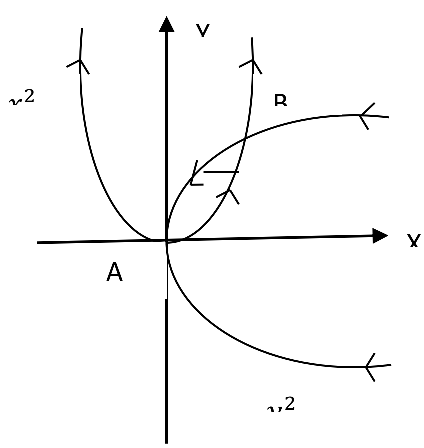
$$\therefore \quad \iint\limits_{S} (\nabla \mathrm{x}\,\vec{F}).\hat{n}\,ds = -\iint\limits_{R} 4x\,dxdy = -\int_0^1 \int_{y^2}^{\sqrt{y}} (4x$$

$$= \int_0^1 4\left(\frac{x^2}{2}\right)_{y^2}^{\sqrt{y}} dy$$

$$= -\int_0^1 2(x^2)_{y^2}^{\sqrt{y}}\,dy = -2\int_0^1 (y - y^4)dy$$

$$= -2\left[\left(\frac{y^2}{2} - \frac{y^5}{5}\right)\right]_0^1$$

$$= -2\left(\frac{1}{2} - \frac{1}{5}\right) = -\frac{3}{5}$$



L.H.S.

$$\int_C \vec{F}.d\vec{r} = \int [(2xy - x^2)\vec{\imath} - (x^2 - y^2)\vec{\jmath}].(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

Along AB, $x^2 = y;\ 2xdx = dy$

$$+\int [(2xy - x^2)\vec{\imath} - (x^2 - y^2)\vec{\jmath}].(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

Along BA, $y^2 = x;\ 2ydy = dx$

$$= \int_0^1 [(2x.x^2 - x^2) - (x^2 - x^4)]2xdx + \int_1^0 [(2y.y^2 - y^4)2ydy - (y^4 - y^2)dy]$$

$$= \int_0^1 [(2\,x^3 - x^2) - (2x^3 - 2x^5)]dx - \int_0^1 [(4\,y^4 - 2y^5) - (y^4 - y^2)]dy$$

$$= \int_0^1 (2x^5 - x^2)dx - \int_0^1 (3\,y^4 - 2y^5 + y^2)dy$$

$$= \left(2\frac{x^6}{6} - \frac{x^3}{3}\right)_0^1 - \left(3\frac{y^5}{5} - 2\frac{y^6}{6} + \frac{y^3}{3}\right)_0^1$$

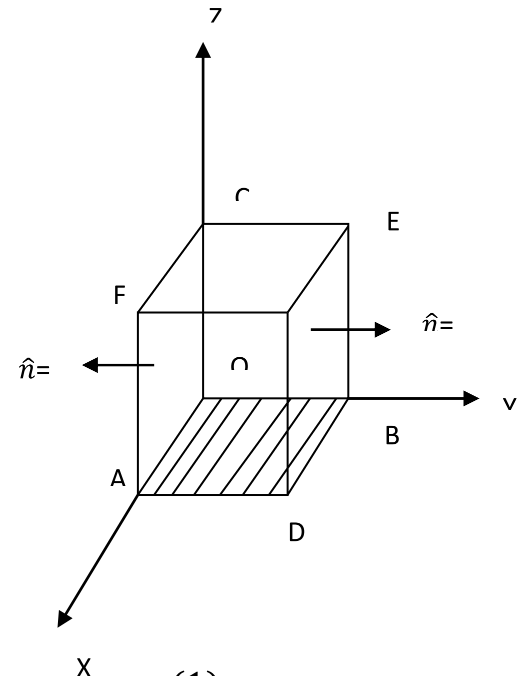
$$= \left(\frac{1}{3} - \frac{1}{3}\right) - \left(\frac{3}{5} - \frac{1}{3} + \frac{1}{3}\right) = -\frac{3}{5}$$

$\therefore$ LHS $=$ RHS

Hence verified.

- **Verify Stoke's theorem for $\vec{F} = xy\vec{i} - 2yz\,\vec{j} - zx\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes $\boldsymbol{x = 0, x = 1}$, $\boldsymbol{y = 0, y = 2}$ *and* $\boldsymbol{z = 3}$ above the $\boldsymbol{xy}$-plane. [16 Marks]**

Solution:



By Stoke's thorem, $\int\limits_{C} \vec{F}.d\vec{r} = \iint\limits_{S} (\nabla \text{x} \vec{F}).\hat{n}\, ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xy) & -2yz & -zx \end{vmatrix}$$

$$= 2y\vec{i} + z\vec{j} - x\vec{k}$$

$\therefore$ Stoke's theorem takes the form

$$\int\limits_{c} (xydx - 2yzdy - zxdz) = \iint\limits_{s} (2y\vec{i} + z\vec{j} - x\vec{k}).ds \quad ----(1)$$

The open cuboid S is made of the five faces $x = 0, x = 1, y = 0, y = 2$ and $z = 3$ and is bounded by the rectangle OADB lying in the xy- plane .

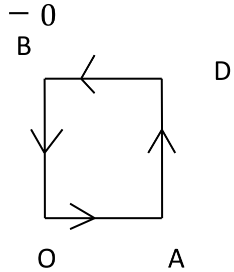
$$\text{L.H.S. of (1)} = \int\limits_{OADB} (xydx - 2yzdy - zxdz)$$

Since the boundary C lies on the plane $z = 0$,substitute $z = 0$ in the above equation ,we get,

$$\int\limits_{OADB} xydx = \int\limits_{OA} + \int\limits_{AD} + \int\limits_{DB} + \int\limits_{BO} (xydx)$$

$$y = 0; dy = 0 \qquad x = 1; dx = 0 \qquad y = 2; dy = 0 \qquad x = 0; dx = 0$$

$$= \int_0^1 2xdx = -1$$



R.H.S. of (1)

$$= \iint_{x=0;\ \hat{n}=-\vec{i}} + \iint_{x=1;\ \hat{n}=\vec{i}} + \iint_{y=0;\ \hat{n}=-\vec{j}} + \iint_{y=2;\ \hat{n}=\vec{j}} + \iint_{z=3;\ \hat{n}=\vec{k}} [(2y\vec{i} + z\vec{j} - x\vec{k}).\hat{n}\, ds]$$

$\hat{n}$ is unit normal at any point of the concerned surface.For example at any point of the plane surface $y = 0$, the outward drawn normal is parallel to the y-axis, but in opposite in direction.

$\therefore$ $\hat{n}$ at any point of $y = 0$ is equal to $-\vec{j}$. Similarly, $\hat{n}$ for $y = 2$ is equal to $\vec{j}$ and so on.

Using the relevant values of $\hat{n}$ and simplifying the integrand ,we have

$$\text{R. H. S. of (1)} = -\iint_{x=0} 2y dS + \iint_{x=1} 2y dS - \iint_{y=0} z dS + \iint_{y=2} z dS - \iint_{z=3} x dS$$

$$= -\int_0^3\int_0^2 2y\, dydz + \int_0^3\int_0^2 2y\, dydx - \int_0^1\int_0^3 z\, dxdz + \int_0^1\int_0^3 z\, dxdz - \int_0^2\int_0^1 x\, dxdy$$

$$= -\int_0^2\int_0^1 x\, dxdy \quad \text{(since all the integrals themselves cancel each other.)}$$

$$= -\int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy = -1$$

$\therefore$ LHS = RHS

Hence the theorem is verified.

- **Verify Stoke's theorem for $\vec{F} = -y\vec{i} + 2yz\vec{j} + y^2\vec{k}$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the**

xy-plane. **[16 Marks]**

Solution:

By stoke's theorem,

$$\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \mathrm{x}\, \vec{F}).\hat{n}\, \mathrm{ds}$$

$$\text{In R.H.S. } \operatorname{curl} \vec{F} = \nabla \mathrm{x}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (-y) & 2yz & y^2 \end{vmatrix}$$

$$= \vec{i}(2y - 2y) - \vec{j}(0 - 0) + \vec{k}(0 + 1) = \vec{k}$$

$$\iint (\nabla \mathrm{x}\, \vec{F}).\hat{n}\, \mathrm{ds} = \iint \vec{k}.\hat{n}ds, \text{where } \hat{n} = \frac{\nabla \emptyset}{|\nabla \emptyset|}$$

where $\emptyset = x^2 + y^2 + z^2 - a^2$

Now $\nabla \emptyset = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}; |\nabla \emptyset| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4a^2} = 2a$

$$\therefore \hat{n} = \frac{\nabla \emptyset}{|\nabla \emptyset|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2a} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\therefore \text{RHS} = \iint_S \vec{k}.\hat{n} = \iint_S \vec{k}.\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right] ds = \iint_S \frac{z}{a}\mathrm{ds} = \iint_R \frac{z}{a}\frac{dxdy}{\hat{n}.\vec{k}}$$

$$= \iint_R \left(\frac{z}{a}\right)\frac{dxdy}{\left(\frac{z}{a}\right)} = \iint_R dxdy = \pi a^2 \qquad \left[\text{since } \hat{n}.\vec{k} = \left(\frac{z}{a}\right)\right]$$

(since it is the double integral over the variables x , y and $z = 0$and also $\iint dxdy \; gives$ the area of the circle $x^2 + y^2 = a^2$ of radius "a" is πa^2)

$$\therefore \text{ LHS} \int_C \vec{F}.d\vec{r} = \int_{x^2+y^2=a^2} -ydx$$

(because it is given that C is the circular boundary on the xy-plane so that $z = 0 \Rightarrow x^2 + y^2 = a^2$ is the circle of radius "a" whose parametric equations are given by $x = acos\theta$; $y = asin\theta$)

$$\int_0^{2\pi} a^2 \sin^2 \theta d\theta = \int_0^{2\pi} a^2 \left(\frac{1 - cos2\theta}{2}\right) d\theta = \frac{a^2}{2}\left(\theta - \frac{sin2\theta}{2}\right)_0^{2\pi}$$

$$= \frac{a^2}{2}\left[\left(2\pi - \frac{sin4\pi}{2}\right) - 0\right] = \frac{a^2}{2}(2\pi - 0) = \pi a^2$$

$\therefore$ LHS = RHS

Hence the theorem is proved.

<u>State Gauss Divergence theorem</u>

If S is a closed curve enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint \vec{F}.d\vec{s} = \iiint div\,\vec{F} dv$$

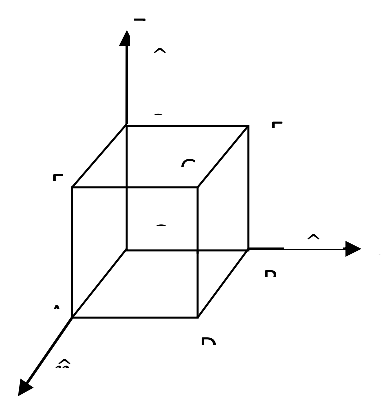
- **Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, where S is the surface of the cuboid formed by the planes**

$x = 0, x = a, y = 0, y = b, z = 0$ and $z = c$. [**16 Marks**]

<u>Solution:</u>

By Gauss divergence theorem,

$$\iint_S \vec{F}.d\vec{s} = \iiint_V div\,\vec{F} dv \quad ------(1)$$



where S is made up of 6 planes surfaces

Face	Unit normal	Eqn.of face	ds
OBEC	$-\vec{i}$	$x=0$	$dydz$
ADGF	$\vec{i}$	$x = a$	$dydz$
OAFC	$-\vec{j}$	$y=0$	$dxdz$
GEBD	$\vec{j}$	$y=b$	$dxdz$
OADB	$-\vec{k}$	z=0	$dxdy$
CEGF	$\vec{k}$	z=c	$dxdy$

LHS of (1) $\iint \vec{F}.d\vec{s} = \iint \vec{F}.\hat{n}\ ds$

$$= \iint_{OBEC} + \iint_{ADGF} + \iint_{OAFC} + \iint_{GEBD} + \iint_{OADB} + \iint_{CEGF} \left[\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right).\hat{n}ds\right]$$

$$= -\iint_{x=o} x^2 ds + \iint_{x=a} x^2 ds - \iint_{y=o} y^2 ds + \iint_{y=b} y^2 dS - \iint_{z=0} z^2 ds + \iint_{z=c} z^2\, ds$$

$$= 0 + \int_0^c \int_0^b a^2\, dy\, dz - 0 + \int_0^c \int_0^a b^2\, dxdz - 0 + \int_0^b \int_0^a c^2\, dxdy$$

$$= \int_0^c a^2 [(y)_0^b]\, dz + \int_0^c b^2\, [(x)_0^a dz + \int_0^b c^2\, [(x)_0^a]\mathrm{dy}$$

$$= a^2 b[(z)_0^c] + b^2 a[(z)_0^c] + c^2 a[(y)_0^b]$$

$$= a^2 bc + b^2 ac + c^2 ab = \mathrm{abc(a + b + c)}.$$

$$\text{R.H.S.} \iiint div\, \vec{F} dv = \iiint (2x + 2y + 2z)\, dxdydz$$

V V

$$= \int_0^c \int_0^b \int_0^a (2x + 2y + 2z)\, dxdydz$$

$$= \int_0^c \int_0^b \left[\frac{2x^2}{2} + 2xy + 2zx\right]_0^a dydz$$

$$= \int_0^c \left[\int_0^b (a^2 + 2ay + 2az)\, dy\right] dz$$

$$= \int_0^c \left[a^2 y + 2a\left(\frac{y^2}{2}\right) + 2azy\right]_0^b]dz$$

$$= \int_0^c (a^2 b + a(b^2) + 2azb)\, dz$$

$$= \left[a^2 bz + a(b^2 z) + 2a\left(\frac{z^2}{2}\right) b\right]_0^c$$

$$= a^2 bc + b^2 ac + c^2 ab = abc(a + b + c)$$

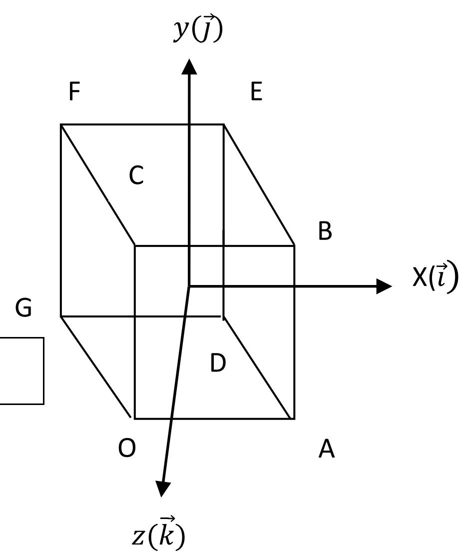
$\therefore$ LHS = RHS

- **Verify divergence theorem for $\vec{F} = x^2 \vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $\boldsymbol{x = \pm 1, y = \pm 1}$ and $\boldsymbol{z = \pm 1}$.** **[16 Marks]**

Solution:

By Gauss divergence theorem,

$$\iint_S \vec{F}.d\vec{s} = \iiint_V div\,\vec{F}\, dv$$



Face	Unit normal	Eqn.of face	ds

ABED	$\vec{i}$	x=1	$dydz$
OCFG	$-\vec{i}$	$x = -1$	$dydz$
CBEF	$\vec{j}$	y=1	$dxdz$
AOGD	$-\vec{j}$	y=-1	$dxdz$
OABC	$\vec{k}$	z=1	$dxdy$
GDEF	$-\vec{k}$	z=-1	d$x$dy

LHS: $\iint_{ABED} + \iint_{OCFG} + \iint_{CBEF} + \iint_{AOGD} + \iint_{OABC} + \iint_{GDEF} (x^2\vec{i} + z\vec{j} + yz\vec{k}).\overrightarrow{ds}$ where $\overrightarrow{ds}=\hat{n}.ds$)

$$= \iint_{x=1} + \iint_{x=-1} + \iint_{y=1} + \iint_{y=-1} + \iint_{z=1} + \iint_{z=-1} (x^2\vec{i} + z\vec{j} + yz\vec{k}).\hat{n}.ds$$

$$= \iint dydz - \iint dydz \iint \mathrm{d}x\mathrm{dz} - \iint \mathrm{d}x\mathrm{dz} + \iint \mathrm{d}x\mathrm{dy} - \iint \mathrm{d}x\mathrm{dy}$$

$= 0$

R.H.S.

In, $\iiint_V div\,\vec{F}\,dv,$ we have, $div\vec{F} = \nabla.\vec{F} = \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{F}}{\partial z} = 2x + y$

$$\therefore \iiint_V (2x + y)\ dxdydz = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x + y)dxdydz.$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(2\frac{x^2}{2} + xy\right)_{-1}^{1} dydz$$

$$= \int_{-1}^{1} \left\{ \int_{-1}^{1} [(1+y)-(1-y)]dy \right\} dz$$

$$= \int_{-1}^{1} \int_{-1}^{1} 2ydydz$$

$$= \int_{-1}^{1} \left(2\frac{y^2}{2}\right)_{-1}^{1} dz = 0$$

$\therefore \mathrm{LHS} = \mathrm{RHS}$

Hence the theorem is verified.


*Me
Semester Vara Pothu Padida..!