SoSe 2020 Kiel University
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Solution homework exercise 6.1:

Statement: Define

$$x_k := \begin{cases} \left(\frac{1}{3}\right)^k & \text{if } k \text{ is odd} \\ \left(\frac{1}{4}\right)^k & \text{if } k \text{ is even} \end{cases} \qquad \text{for all } k \in \mathbb{N}.$$

$(x_k)_{k\in\mathbb{N}}$ is Q-sublinearly convergent and R-linearly convergent.

Proof: $(x_k)_{k\in\mathbb{N}}$ converges to 0. Define

$$x^* := \lim_{k\to\infty} x_k = 0$$

Let $k \in \mathbb{N}$. If k is odd,

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = \frac{\left(\frac{1}{4}\right)^{k+1}}{\left(\frac{1}{3}\right)^k} = \frac{1}{4}\left(\frac{3}{4}\right)^k.$$

Furthermore

$$\lim_{k\to\infty} \frac{1}{4}\left(\frac{3}{4}\right)^k = 0.$$

If k is even,

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = \frac{\left(\frac{1}{3}\right)^{k+1}}{\left(\frac{1}{4}\right)^k} = \frac{1}{3}\left(\frac{4}{3}\right)^k.$$

Moreover

$$\lim_{k\to\infty} \frac{1}{3}\left(\frac{4}{3}\right)^k = \infty.$$

Hence

$$\limsup_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = \infty \geq 1.$$

Thus $(x_k)_{k\in\mathbb{N}}$ is Q-sublinearly convergent.

If k is odd,

$$\sqrt[k]{\frac{\|x_k - x^*\|}{\|x_0 - x^*\|}} = \sqrt[k]{\frac{\left(\frac{1}{3}\right)^k}{\frac{1}{3}}} = \frac{1}{3}\sqrt[k]{3}.$$

Furthermore

$$\lim_{k\to\infty} \frac{1}{3}\sqrt[k]{3} = \frac{1}{3}.$$

If k is even,

$$\sqrt[k]{\frac{\|x_k - x^*\|}{\|x_0 - x^*\|}} = \sqrt[k]{\frac{\left(\frac{1}{4}\right)^k}{\frac{1}{3}}} = \frac{1}{4}\sqrt[k]{3}.$$

Furthermore
$$\lim_{k\to\infty} \frac{1}{4}\sqrt[k]{3} = \frac{1}{4}.$$
Hence
$$\limsup_{k\to\infty} \sqrt[k]{\frac{\|x_k - x^*\|}{\|x_0 - x^*\|}} = \frac{1}{3} \in (0,1).$$
Thus $(x_k)_{k\in\mathbb{N}}$ is R-linearly convergent. $\square$