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Solution homework exercise 7.3:

Statement: Let $A \in \mathbb{R}^{n\times n}$ *be invertible and* $x, y \in \mathbb{R}^n$. $A + xy^T$ *is invertible if and only if* $y^T A^{-1}x \neq -1$ *and in this case*

$$(A + xy^T)^{-1} = A^{-1} - \frac{A^{-1}xy^T A^{-1}}{1 + y^T A^{-1}x}.$$

Proof: Consider first the case that $x = 0$. Then

$$A + xy^T = A$$

and thus $A + xy^T$ is invertible because A is invertible. Furthermore, $x = 0$ implies

$$y^T A^{-1}x = 0 \neq -1.$$

Consider now the case $x \neq 0$. Assume that $A + xy^T$ is invertible. Then $(A + xy^T)A^{-1}$ is invertible as well and

$$0 \neq (A + xy^T)A^{-1}x = x + xy^T A^{-1}x = x(1 + y^T A^{-1}x).$$

This implies $1 + y^T A^{-1}x \neq 0$ and thus $y^T A^{-1}x \neq -1$.

Assume now that $y^T A^{-1}x \neq -1$. Then

$$\begin{aligned}
(A + xy^T)\left(A^{-1} - \frac{A^{-1}xy^T A^{-1}}{1 + y^T A^{-1}x}\right) &= AA^{-1} + xy^T A^{-1} - A\frac{A^{-1}xy^T A^{-1}}{1 + y^T A^{-1}x} - xy^T\frac{A^{-1}xy^T A^{-1}}{1 + y^T A^{-1}x} \\
&= I + xy^T A^{-1} - \frac{xy^T A^{-1}}{1 + y^T A^{-1}x} - \frac{xy^T A^{-1}xy^T A^{-1}}{1 + y^T A^{-1}x} \\
&= I + xy^T A^{-1} - \frac{xy^T A^{-1}}{1 + y^T A^{-1}x} - y^T A^{-1}x\frac{xy^T A^{-1}}{1 + y^T A^{-1}x} \\
&= I + xy^T A^{-1} - (1 + y^T A^{-1}x)\frac{xy^T A^{-1}}{1 + y^T A^{-1}x} \\
&= I + xy^T A^{-1} - xy^T A^{-1} = I.
\end{aligned}$$

Hence $(A + xy^T)$ is invertible and $A^{-1} - \frac{A^{-1}xy^T A^{-1}}{1+y^T A^{-1}x}$ is its inverse. $\square$