Solution homework exercise 7.2:

Statement: Let $H \in \mathbb{R}^{n\times n}$ be symmetric and positive definite and $y, s \in \mathbb{R}^n$ with $y^T s > 0$. Then

$$A = H + \frac{yy^T}{y^T s} - \frac{Hs(Hs)^T}{s^T H s}$$

is well defined, symmetric and positive definite.

Proof: It is assumed that $y^T s > 0$. This implies $s \neq 0$. Hence, $s^T H s \neq 0$ because H is positive definite. Thus A is well defined. Moreover A is symmetric because $A^T = A$ because H is symmetric.

Next we prove the positive definiteness of A. Let $x \in \mathbb{R}^n$ with $x \neq 0$. Define

$$\langle \cdot, \cdot \rangle_H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (x, y) \mapsto x^T H y.$$

This is a scalar product because H is symmetric and positive definite. Hence the Cauchy-Schwarz-inequality implies

$$(\langle x, s\rangle_H)^2 \begin{cases} < \langle x, x\rangle_H \langle s, s\rangle_H & \text{if } x \text{ and } s \text{ are linearly independent} \\ \leq \langle x, x\rangle_H \langle s, s\rangle_H & \text{otherwise} \end{cases}. \tag{1}$$

For $x \in \mathbb{R}^n$ with $x \neq 0$,

$$\begin{aligned} x^T A x &= x^T \left(H + \frac{yy^T}{y^T s} - \frac{Hs(Hs)^T}{s^T H s} \right) x && \text{(def. of } A) \\ &= x^T H x + \frac{x^T y y^T x}{y^T s} - \frac{x^T H s (Hs)^T x}{s^T H s} \\ &= x^T H x + \frac{(x^T y)^2}{y^T s} - \frac{(x^T H s)^2}{s^T H s} \\ &= \langle x, x\rangle_H + \frac{(x^T y)^2}{y^T s} - \frac{(\langle x, s\rangle_H)^2}{\langle s, s\rangle_H} && \text{(def. of } \langle \cdot, \cdot \rangle_H). \end{aligned} \tag{2}$$

If x and s are linearly independent, (1) and (2) implies

$$\begin{aligned} x^T A x &> \langle x, x\rangle_H + \frac{(x^T y)^2}{y^T s} - \frac{\langle x, x\rangle_H \langle s, s\rangle_H}{\langle s, s\rangle_H} && ((1) \text{ and } (2)) \\ &= \frac{(x^T y)^2}{y^T s} \\ &\geq 0 && (y^T s > 0). \end{aligned}$$

If x and s are linearly dependent, then $\lambda \in \mathbb{R}$ exists with $x = \lambda s$. Because $x \neq 0$, this implies $\lambda \neq 0$. Thus (1) and (2) implies

$$\begin{aligned}
x^T A x &\geq \langle x, x\rangle_H + \frac{(x^T y)^2}{y^T s} - \frac{\langle x, x\rangle_H \langle s, s\rangle_H}{\langle s, s\rangle_H} && \text{((1) and (2))} \\
&= \frac{(x^T y)^2}{y^T s} \\
&= \frac{(\lambda s^T y)^2}{y^T s} && (x = \lambda s) \\
&= \lambda^2 y^T s && (s^T y = y^T s) \\
&> 0 && (\lambda \neq 0 \text{ and } y^T s > 0).
\end{aligned}$$

Thus

$$x^T A x > 0$$

in either case. This means that A is positive definite. □