SoSe 2020 Kiel University
Dr. Joscha Reimer Department of Computer Science

Solution homework exercise 6.3:

Statement: Consider the globalized Newton-Method with $c \in (0, 1]$ *applied to*

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto |x|^p$$

with $p > 2$*. Let* $(x_k)_{k\in\mathbb{N}_0}$ *be the sequence of iterates.*

a) *If* $x_k > 0$*, then* $d_k = -\frac{1}{p-1}x_k$*, for all* $k \in \mathbb{N}_0$*.*

b) *If* $x_0 > 0$ *and* $\rho_k = 1$ *for all* $k \in \mathbb{N}_0$*, then* $x_k > x_{k+1} > 0$ *for all* $k \in \mathbb{N}_0$*.*

c) *If* $x_0 > 0$ *and* $\rho_k = 1$ *for all* $k \in \mathbb{N}_0$*, then* $(\rho_k)_{k\in\mathbb{N}_0}$ *is an efficient step-sizes sequence.*

d) *If* $x_0 > 0$ *and* $\rho_k = 1$ *for all* $k \in \mathbb{N}_0$*, then* $(x_k)_{k\in\mathbb{N}_0}$ *converges Q-linear but not Q-superlinear to a global minimizer of* f*.*

e) *We can not apply the convergence result for the globalized Newton method.*

Proof: a) Let $k \in \mathbb{N}_0$ and assume $x_k > 0$. Then

$$\nabla f(x_k) = p(x_k)^{p-1} \tag{1}$$

and

$$\nabla^2 f(x_k) = p(p-1)(x_k)^{p-2} \tag{2}$$

because $p > 2$. Thus $\nabla^2 f(x_k)$ is invertible and the Newton direction

$$\begin{aligned} d_k &= -\nabla^2 f(x_k)^{-1}\nabla f(x_k) && \text{(def. of Newton direction)} \\ &= -\left(p(p-1)(x_k)^{p-2}\right)^{-1} p(x_k)^{p-1} && ((1),(2)) \\ &= -\tfrac{1}{p-1}x_k \end{aligned} \tag{3}$$

is well defined. Furthermore $\nabla f(x_k) \neq 0$ and

$$\begin{aligned} -\frac{\nabla f(x_k)^T d_k}{\|\nabla f(x_k)\|\|d_k\|} &= -\frac{(p(x_k)^{p-1})(-\frac{1}{p-1}x_k)}{|p(x_k)^{p-1}|\ |-\frac{1}{p-1}x_k|} && ((1),(3)) \\ &= 1 \\ &\geq c. \end{aligned}$$

Thus the Newton direction d_k is gradient-related and is chosen by the globalized Newton-Method due to its definition.

b) Assume $x_0 > 0$, $x_k > 0$ and $\rho_k = 1$ for a $k \in \mathbb{N}_0$. Then

$$\begin{aligned} x_{k+1} &= x_k + \rho_k d_k && \text{(def. of method)} \\ &= x_k + d_k && (\rho_k = 1) \\ &= x_k + \left(-\tfrac{1}{p-1}x_k\right) && \text{(part a))} \\ &= \left(1 - \tfrac{1}{p-1}\right) x_k. \end{aligned} \tag{4}$$

Since $p > 2$, this implies $x_k > x_{k+1} > 0$. Thus if $x_0 > 0$ and $\rho_k = 1$ for all $k \in \mathbb{N}_0$, then $x_k > x_{k+1} > 0$ for all $k \in \mathbb{N}_0$.

c) Assume $x_0 > 0$ and $\rho_k := 1$ for all $k \in \mathbb{N}_0$. Define

$$c_S := \frac{1 - \left(1 - \frac{1}{p-1}\right)^p}{p^2 (x_0)^{p-2}}.$$

Then $c_S > 0$ since $p > 2$ and $x_0 > 0$. Let $k \in \mathbb{N}_0$. Then $x_k > 0$ due to part b) and thus

$$\begin{aligned}
\left(\frac{\nabla f(x_k)^T d_k}{\|d_k\|}\right)^2 &= \left(\frac{p(x_k)^{p-1}\left(-\frac{1}{p-1}x_k\right)}{\left|-\frac{1}{p-1}x_k\right|}\right)^2 && ((1),\ \text{part a)}) \\
&= \left(p(x_k)^{p-1}\right)^2 \\
&= p^2 (x_k)^{2p-2}.
\end{aligned}$$

Furthermore $x_{k+1} > 0$ due to part b) and thus

$$\begin{aligned}
& f(x_k) - f(x_k + \rho_k d_k) - c_S \left(\frac{\nabla f(x_k)^T d_k}{\|d_k\|}\right)^2 \\
=& (x_k)^p - \left(x_k - \tfrac{1}{p-1}x_k\right)^p - c_S p^2 (x_k)^{2p-2} && \text{(def. of } f\text{, part a) and b) and eq. above)} \\
=& \left(1 - \left(1 - \tfrac{1}{p-1}\right)^p - c_S p^2 (x_k)^{p-2}\right)(x_k)^p && \text{(factoring out)} \\
>& \left(1 - \left(1 - \tfrac{1}{p-1}\right)^p - c_S p^2 (x_0)^{p-2}\right)(x_k)^p && (c_S > 0, p > 2 \text{ and } x_k > x_0) \\
=& 0 && \text{(def. of } c_S).
\end{aligned}$$

Thus $(\rho_k)_{k\in\mathbb{N}_0}$ is an efficient step-sizes sequence due to the definition of efficient step-sizes.

d) $x^* := 0$ is the only global minimizer of f. Consider the k-th Q-factor for $k \in \mathbb{N}_0$:

$$\begin{aligned}
q_k &= \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} && \text{(def. of } q_k) \\
&= \frac{\|x_{k+1}\|}{\|x_k\|} && (x^* = 0) \\
&= \frac{\|\left(1 - \frac{1}{p-1}\right)x_k\|}{\|x_k\|} && ((4)) \\
&= \left|1 - \tfrac{1}{p-1}\right| \frac{\|x_k\|}{\|x_k\|} \\
&= \left|1 - \tfrac{1}{p-1}\right| && .
\end{aligned}$$

Thus

$$\begin{aligned}
Q((x_k)_{k\in\mathbb{N}_0}) &= \limsup_{k\to\infty} q_k && \text{(def. of } Q) \\
&= \limsup_{k\to\infty} \left|1 - \tfrac{1}{p-1}\right| && \text{(eq. above)} \\
&= \left|1 - \tfrac{1}{p-1}\right| && \text{(constant sequence)} \\
&\in (0,1) && (p > 2).
\end{aligned}$$

Hence $(x_k)_{k\in\mathbb{N}_0}$ converges Q-linear but not Q-superlinear to a global minimizer of f.

e) We can not apply the convergence result for the globalized Newton method because f is not differentiable in x^* or $\nabla^2 f(x^*) = 0$ and is thus not positive definite.

$\square$