1 STAMATICS WEEK-1 Home Assignment

Problem 1:

(a) Proof: Let A be an $m \times n$ matrix and x be an $n \times 1$ vector. We have
$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

So, $Ax = \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{bmatrix}$.

Now, let's take the derivative of Ax with respect to x:

$$\begin{aligned} \frac{d}{dx}[Ax] &= \frac{d}{dx} \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{dx}(\sum_{j=1}^n a_{1j}x_j) \\ \frac{d}{dx}(\sum_{j=1}^n a_{2j}x_j) \\ \vdots \\ \frac{d}{dx}(\sum_{j=1}^n a_{mj}x_j) \end{bmatrix} \\ &= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = A. \end{aligned}$$

So, $\frac{d}{dx}[Ax] = A$.

(b) Proof: Let A be an $n \times n$ matrix and x be an $n \times 1$ vector. We have
$x^T Ax = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

So, $x^T Ax = \begin{bmatrix} \sum_{i=1}^n x_i a_{i1} & \sum_{i=1}^n x_i a_{i2} & \cdots & \sum_{i=1}^n x_i a_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} x_j$.

Now, let's take the derivative of $x^T Ax$ with respect to x:

$$
\begin{aligned}
\frac{d}{dx}[x^T Ax] &= \frac{d}{dx}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j\right) \\
&= \frac{d}{dx}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j\right) \\
&= \left[\tfrac{d}{dx}\left(\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j\right)\right] \\
&= \left[\textstyle\sum_{i=1}^{n} a_{1i}x_i + \sum_{i=1}^{n} a_{i1}x_i \quad \sum_{i=1}^{n} a_{2i}x_i + \sum_{i=1}^{n} a_{i2}x_i \quad \cdots \quad \sum_{i=1}^{n} a_{ni}x_i + \sum_{i=1}^{n} a_{in}x_i\right] \\
&= \left[\textstyle\sum_{i=1}^{n}(a_{1i} + a_{i1})x_i \quad \sum_{i=1}^{n}(a_{2i} + a_{i2})x_i \quad \cdots \quad \sum_{i=1}^{n}(a_{ni} + a_{in})x_i\right] \\
&= x^T(A + A^T).
\end{aligned}
$$

Hence, $\frac{d}{dx}[x^T Ax] = x^T(A + A^T)$.

Problem 2: To find the dimension of the result when differentiating a matrix of dimension $m \times n$ with respect to a $k \times 1$ vector, we consider the following:

- The derivative with respect to a $k \times 1$ vector will introduce k new dimensions.
- Each element in the original matrix will be differentiated with respect to each element in the $k \times 1$ vector, resulting in $m \times (n \times k)$ new elements.
- Finally, the matrix is expanded by a factor of k in columns, however, the rows remain unchanged. Therefore, the resulting dimension will be $m \times (n \times k)$.

Problem 3:

(a): Given the vector $\mathbf{v} = \begin{bmatrix} 2\sin^2(x)\cos(y) \\ x^2 + 3e^y \end{bmatrix}$, we want to differentiate it with respect to the vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Let's denote $v_1(x, y) = 2\sin^2(x)\cos(y)$ and $v_2(x, y) = x^2 + 3e^y$. Then, the components of $\mathbf{v}$ are:

$$
v_1(x, y) = 2\sin^2(x)\cos(y)
$$
$$
v_2(x, y) = x^2 + 3e^y
$$

Now, let's find the partial derivatives of each component with respect to x and y:

For $v_1(x, y)$:

$$\frac{\partial v_1}{\partial x} = 4\sin(x)\cos(x)\cos(y)$$
$$\frac{\partial v_1}{\partial y} = -2\sin^2(x)\sin(y)$$

For $v_2(x, y)$:

$$\frac{\partial v_2}{\partial x} = 2x$$
$$\frac{\partial v_2}{\partial y} = 3e^y$$

So, the Jacobian matrix $\mathbf{J}$ representing the partial derivatives of $\mathbf{v}$ with respect to $\mathbf{x}$ is:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 4\sin(x)\cos(x)\cos(y) & -2\sin^2(x)\sin(y) \\ 2x & 3e^y \end{bmatrix}$$

And the resulting vector after differentiating $\mathbf{v}$ with respect to $\mathbf{x}$ is:

$$\frac{d\mathbf{v}}{d\mathbf{x}} = \mathbf{J} = \begin{bmatrix} 4\sin(x)\cos(x)\cos(y) & -2\sin^2(x)\sin(y) \\ 2x & 3e^y \end{bmatrix}$$

(b): Given the vectors $\mathbf{v}_1 = \begin{pmatrix} 3x^2y + xyzw \\ \sin(x^2 + yw - z) \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, we want to differentiate $\mathbf{v}_1$ with respect to $\mathbf{v}_2$.

Let's denote $v_{1,1}(x, y, z, w) = 3x^2y + xyzw$ and $v_{1,2}(x, y, z, w) = \sin(x^2 + yw - z)$. Then, the components of $\mathbf{v}_1$ are:

$$v_{1,1}(x, y, z, w) = 3x^2y + xyzw$$
$$v_{1,2}(x, y, z, w) = \sin(x^2 + yw - z)$$

Now, let's find the partial derivatives of each component with respect to x, y, z, and w:

For $v_{1,1}(x, y, z, w)$:

$$\frac{\partial v_{1,1}}{\partial x} = 6xy + yzw$$
$$\frac{\partial v_{1,1}}{\partial y} = 3x^2 + xzw$$
$$\frac{\partial v_{1,1}}{\partial z} = xyw$$
$$\frac{\partial v_{1,1}}{\partial w} = xyz$$

For $v_{1,2}(x, y, z, w)$:

$$\frac{\partial v_{1,2}}{\partial x} = 2x\cos(x^2 + yw - z)$$
$$\frac{\partial v_{1,2}}{\partial y} = w\cos(x^2 + yw - z)$$
$$\frac{\partial v_{1,2}}{\partial z} = -\cos(x^2 + yw - z)$$
$$\frac{\partial v_{1,2}}{\partial w} = y\cos(x^2 + yw - z)$$

So, the Jacobian matrix $\mathbf{J}$ representing the partial derivatives of $\mathbf{v}_1$ with respect to $\mathbf{v}_2$ is:

$$\mathbf{J} = \begin{pmatrix} 6xy + yzw & 3x^2 + xzw & xyw & xyz \\ 2x\cos(x^2 + yw - z) & w\cos(x^2 + yw - z) & -\cos(x^2 + yw - z) & y\cos(x^2 + yw - z) \end{pmatrix}$$

And the resulting vector after differentiating $\mathbf{v}_1$ with respect to $\mathbf{v}_2$ is:

$$\frac{d\mathbf{v}_1}{d\mathbf{v}_2} = \mathbf{J} = \begin{pmatrix} 6xy + yzw & 3x^2 + xzw & xyw & xyz \\ 2x\cos(x^2 + yw - z) & w\cos(x^2 + yw - z) & -\cos(x^2 + yw - z) & y\cos(x^2 + yw - z) \end{pmatrix}$$

Bonus: PROBLEM -4: To solve for $\frac{d}{dx}[e^{\beta^\top x}]$, where β is an $n \times 1$ vector with components $\beta_1, \beta_2, \ldots, \beta_n$, and x is an $n \times 1$ vector with components $x_1, x_2, \ldots, x_n$, we'll use the chain rule.

Given $y = e^{\beta^\top x}$, we have $u = \beta^\top x$.

First, differentiate $y = e^u$ with respect to u:

$$\frac{dy}{du} = e^u$$

Next, differentiate $u = \beta^\top x$ with respect to x:

$$\frac{du}{dx} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{\beta^\top x} \cdot \beta^T$$

So, the solution is:

$$\frac{d}{dx}[e^{\beta^\top x}] = e^{\beta^\top x} \cdot \beta^T$$