BLACK-SCHOLES: THEORY AND APPLICATIONS

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1 Introduction

In the realm of finance, options are fundamental derivative instruments that grant their holders the right, but not the obligation, to buy (in the case of a call) or sell (in the case of a put) a specified financial asset at a predetermined price, known as the exercise price, on a future date. These financial instruments have gained popularity for their ability to offer investors increased flexibility and diversification of their portfolios, as well as opportunities for risk management. A call option gives its holder the right to buy the underlying asset at a predetermined price, while a put option gives the right to sell the underlying asset at the same price. Option holders pay a premium to acquire this right, which constitutes the price of the option. The primary objective of using options is to profit from anticipated fluctuations in the prices of underlying assets. For instance, an investor may purchase a call option on a stock in anticipation of its price increasing in the future, or buy a put option on a currency in anticipation of future depreciation. In the context of options, the terms "in-the-money," "out-of-the-money," and "at-the-money" are commonly used to describe the relative situation of the price of the underlying asset compared to the option's exercise price. An option is considered "in-the-money" if it would have a positive value if exercised immediately. Conversely, an option is "out-of-the-money" if it would have a zero value if exercised immediately. The option is "at-the-money" when the price of the underlying asset is equal to or very close to the option's exercise price. Understanding these concepts is essential for evaluating the potential profitability of an option and making informed decisions in options trading.

Before the emergence of the Black-Scholes model, several works laid the groundwork for options theory and contributed to the development of pricing models. One of the most notable precursors is the Bachelier model, proposed by Louis Bachelier in 1900. Although his model did not receive significant recognition at the time, it laid the initial foundations for the mathematical modeling of asset price movements, considering price movements as random processes. It was the publication in 1973 of the seminal article "The Pricing of Options and Corporate Liabilities" by Fischer Black and Myron Scholes that revolutionized the understanding of options. They developed a pricing model for european option that takes into account the underlying asset price, exercise price, option's lifespan, risk-free interest rate, and market volatility. Their model, complemented by the work of Robert Merton, resulted in the Black-Scholes model, which has become the cornerstone of options pricing and paved the way for numerous subsequent advances in quantitative finance.

Our work is structured as follows: Firstly, we will review the assumptions of the Black-Scholes model. Then, we will study some mathematical elements and essential mathematical properties necessary for the construction of the model, such as Ito's lemma or the various properties of Brownian motion. Next, we will delve into the analysis of the model's construction where we will examine elements such as the stochastic variations of the risky asset, the transition from risk-neutral probability to historical probability, or the construction of a risk-free portfolio with a determined quantity of stock and options. Following this, we will review the different sensitivities of the model, the Greek letters, and explain their relevance for decision-making in financial markets or for constructing a risk-free portfolio (Delta). Subsequently, after this theoretical first part, we will apply the Black-Scholes model to various options. We will present methods for estimating historical and implied volatility as well as their usefulness and limitations. Finally, we will attempt to observe the limitations of the Black-Scholes model through the realization of a volatility smile. We will conclude with a brief discussion regarding the extensions of this model, namely stochastic volatility models.

2 Black-Scholes model

2.1 Model hypothesis

The black-Scholes model hypothesis are the following:

1. The price of the underlying asset follows a geometric Brownian motion with constant volatility σ and a deterministic drift μ :
2. There are no arbitrage opportunities: This condition implies that it is not possible to make risk-free profit superior to the risk-free interest through simultaneous transactions involving different assets.
3. Time is a continuous function: This assumption assumes that time flows continuously, and the variations in asset prices can be observed at any point in the time continuum.
4. Short selling is possible: This means that it is allowed to sell assets not yet owned in the hope of buying them back at a lower price in the future, thereby making a profit if the price decreases.
5. There are no transaction costs: This assumption supposes that there are no fees associated with buying or selling assets, simplifying calculations and models.
6. There exists a known, constant risk-free interest rate: This implies that there is a constant interest rate for a given period that is free from risk and known in advance.
7. All underlying assets are perfectly divisible: This condition assumes that assets can be divided into arbitrarily small quantities, allowing for precise analysis of price variations.
8. In the case of a stock, it does not pay dividends between the option evaluation time and its expiration: This assumption states that stocks do not pay dividends during the period from when an option is evaluated until its expiration, thus simplifying the option valuation model.

2.2 Essential concepts

2.2.1 Continuous time discount rates

The first step to approach the Black-Scholes model which is continuous time model is to determine what is the discount rates in a continuous time model. To find this continuous discount rates, we just have to take the discount rates in discrete times and make its periodicity tend towards infinity.

Let r be the annual interest rate compound m times. The present value of 1$ payable in t years is:

$$\left(1+\frac{r}{m}\right)^{-mt}$$

So in continuous time we have:

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{-mt} = \lim_{m\to\infty} e^{ln\left(\left(1+\frac{r}{m}\right)^{-mt}\right)} = \lim_{m\to\infty} e^{-mt\times ln\left(1+\frac{r}{m}\right)} = e^{-mt\times\frac{r}{m}} = e^{-rt}$$

The discount rates of a t period time is e^{-rt} in continuous times.

2.2.2 Normal and Log-Normal distribution

Let be X a random variable. X follow the following normal distribution: $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. The density of X is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $W = aX + b$: $W \sim \mathcal{N}\left(a\mu + b, \sigma^2 a^2\right)$
If we have $Y = e^W$, Y follow a Log-Normal distribution.

$$Y \sim Log - \mathcal{N}\left(a\mu + b, a^2\sigma^2\right)$$

The expectation and the variance of Y are :

$$\mathbb{E}(Y) = e^{a\mu + b + \frac{a^2\sigma^2}{2}}$$

$$\mathbb{V}(Y) = e^{2(a\mu+b)+a^2\sigma^2}\left(e^{a^2\sigma^2} - 1\right)$$

Note n°1: If Y follow a Log-Normal distribution then ln(Y) follow a Normal distribution.

Note n°2: In continuous time, if the price variations of a risky asset follow a normal distribution, then the price of the asset follow a log-normal distribution.

2.2.3 Brownian motion and Geometric Brownian motion (GBM)

Let be a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, the stochastic process $(B_t)_{t\in\mathbb{R}^+}$ is a standard Brownian motion if and only if:

$$B_0 = 0$$
$$B_t \sim \mathcal{N}(0, t)$$
$$B_t - B_s \sim \mathcal{N}(0, t - s), \forall 0 \leq s \leq t$$
$$B_{t_1} - B_{s_1} \perp B_{t_2} - Bs_2, \forall 0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$$

The last line explain that the process had independent increments. Moreover $\forall s < t, B_s$ is $\mathcal{F}_t$-measurable.

A geometric Brownian motion with parameters (μ, σ) is a stochastic process $(S_t)_{t\in\mathbb{R}^+}$ such as for all t and for S_0 $\mathcal{F}_0$-measurable:

$$S_t = S_0 e^{(\mu-0,5\sigma^2)t+\sigma B_t}$$

<u>Note n°1:</u> S_t is of the form Ke^{aX+b}, with $b = (\mu - 0, 5\sigma^2)t$ and $a = \sigma^2 t$. S_t follow a Log-Normal distribution. Let's find the expectation and variance of S_t:

$$\mathbb{E}(S_t) = \mathbb{E}\left(S_0 e^{(\mu-0,5\sigma^2)t+\sigma B_t}\right) = S_0\mathbb{E}\left(e^{(\mu-0,5\sigma^2)t+\sigma B_t}\right) = S_0 e^{(\mu-0,5\sigma^2)t+0,5\sigma^2 t)} = S_0 e^{\mu t}$$

$$\mathbb{V}(S_t) = \mathbb{V}\left(S_0 e^{(\mu-0,5\sigma^2)t+\sigma B_t}\right) = S_0^2\mathbb{V}\left(e^{(\mu-0,5\sigma^2)t+\sigma B_t}\right) = S_0^2 e^{2\left(\mu-0,5\sigma^2\right)t+\sigma^2 t}\left(e^{t\sigma^2} - 1\right)$$
$$= S_0^2 e^{\mu t}\left(e^{\sigma^2 t} - 1\right)$$

<u>Note n°2:</u> S_t is the solution of the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

2.2.4 Ito's Lemma

The Ito formula, also known as Ito's lemma, is a powerful tool in stochastic calculus used for calculating derivatives of functions of stochastic processes. It generalizes the classic chain rule of differential calculus to stochastic processes. This notion is indispensable in Black-Scholes models to examine the payoff function of a call or a put which is a function of a stochastic process (GBM).

Let have the following Stochastic process S_t:

$$dS_t = \mu_t dt + \sigma dB_t$$

Where B_t is our Brownian Motion.

The Ito's Lemma say that if $f(S_t, t)$ is a $\mathcal{C}^2$ function, we have:

$$df\left(S_t,t\right)=\frac{\partial f\left(S_t,t\right)}{\partial t}dt+\frac{\partial f\left(S_t,t\right)}{\partial s}dS_t+\frac{1}{2}\frac{\partial^2 f\left(S_t,t\right)}{\partial s^2}d\left\langle S\right\rangle_t$$

Note n°1:

$$d\left\langle S\right\rangle_t=\left\langle dS_t,dS_t\right\rangle=d\left\langle \sigma_t B_t,\sigma_t B_t\right\rangle=\sigma^2 dt$$

2.3 Black-Scholes model

Before starting the development of the Black-Scholes model let's introduce some notation.

2.3.1 Notation

$V_{a,t}$ is the value of a risky asset a the time t.

$V_{c,t}=\Psi(V_{a,t},t)$ is the value of a call options a the time t. It is also a function of the time and the value of the risky asset underlying.

$V_{p,t}=\Upsilon(V_{a,t},t)$ is the value of a put options a the time t. It is also a function of the time and the value of the risky asset underlying.

T is the Maturity.

K is the strike .

r is the risk-free interest rate (interest rate of the cash).

$\mathbb{P}$ is the historic probability.

$\mathbb{Q}$ is the neutral risk probability.

2.3.2 Risky asset value under GBM

Let consider a filtered probability space $(\Omega,\mathcal{A},\mathcal{F},\mathbb{P})$. The return of the stock over dt follows the following stochastic differential equation (SDE). It is the Black and Scholes stochastic differential equation.

$$\frac{dV_{a,t}}{V_{a,t}}=\mu dt+\sigma dB_t$$

Here B_t is always a Brownian motion. This SDE explain that the stocks, over dt, as a deterministic return of μdt. The only alea source come from the Brownian motion B_t.

The expected return over dt over the risky asset return is:

$$\mathbb{E}\left(\frac{dV_{a,t}}{V_{a,t}}\right) = \mathbb{E}\left(\mu dt + \sigma dB_t\right) = \mu dt + \sigma d\mathbb{E}\left(B_t\right) = \mu dt$$

<u>Note n°1:</u> $B_t \sim \mathcal{N}(0,t)$ and $z \sim \mathcal{N}(0,1)$.

The variance of the asset return, who can be interpret as the volatility of the asset, over dt is:

$$\mathbb{V}\left(\frac{dV_{a,t}}{V_{a,t}}\right) = \sigma^2 dt$$

The solution of this SDE, is the following expression. It gives an explicit expression of $V_{a,t}$:

$$V_{a,t} = V_{a,0}e^{\left(\mu - 0,5\sigma^2\right)t + \sigma B_t} = V_{a,0}e^{\left(\mu - 0,5\sigma^2\right)t + \sigma\sqrt{t}z}$$

<u>Note n°2:</u> The asset value follow a geometric Brownian motion. Moreover, $V_{a,t}$ follow a Log-Normal distribution. as we find in the note n°1 of the section 2.1.3 the expectation and the variance of the asset value is the following:

$$\mathbb{E}(V_{a,t}) = S_0 e^{\mu t}$$

The expected asset return on a t time is $\mu\%$ compound continually.

$$\mathbb{V}(V_{a,t}) = V_{a,0}^2 e^{\mu t}\left(e^{\sigma^2 t} - 1\right)$$

<u>Note n°3:</u>We can also analyze the stochastic variation of the asset value $V_{a,t}$ to rediscovering the SDE:

$$dV_{a,t} = \frac{\partial V_{a,t}}{\partial B_t}dB_t + \frac{\partial V_{a,t}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V_{a,t}}{\partial B_t^2}dt$$

with:

$$\frac{\partial V_{a,t}}{\partial B_t} = V_{a,0}\sigma e^{\left(\mu - 0,5\sigma^2\right)t + \sigma B_t} = \sigma V_{a,t}$$

$$\frac{\partial^2 V_{a,t}}{\partial B_t^2} = \sigma^2 V_{a,t}$$

$$\frac{\partial^2 V_{a,t}}{\partial t} = \left(\mu - 0,5\sigma^2\right)V_{a,0}e^{\left(\mu - 0,5\sigma^2\right)t + \sigma B_t} = \left(\mu - 0,5\sigma^2\right)V_{a,t}$$

so now $dV_{a,t}$is:

$$dV_{a,t} = \sigma V_{a,t}dB_t + \left(\mu - 0,5\sigma^2\right)V_{a,t}dt + 0,5\sigma^2 V_{a,t} = \sigma V_{a,t}dB_t + \mu V_{a,t}dt$$

finally we find the SDE by dividing by $V_{a,t}$:

$$\frac{dV_{a,t}}{V_{a,t}} = \mu dt + \sigma dB_t$$

2.3.3 Neutral Risk Probability

All the development above was under the historic probability $\mathbb{P}$. But if we want to price an options or just study the the movement of the asset price in the future we have to work with the neutral risk probability $\mathbb{Q}$. So we have to construct this neutral risk probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$. We know that $\mathbb{Q}$ has to have some proprieties. Under $\mathbb{Q}$, the price of any discounted asset must be a martingale, so we must have:

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_{a,t}|\mathcal{F}_s\right) = \tilde{V}_{a,s}$$

$$\forall s < t$$

with:

$$\tilde{V}_{a,t} = e^{-rt}V_{a,t} = e^{-rt}V_{a,0}e^{\left(\mu-0,5\sigma^2\right)t+\sigma B_t} = V_{a,0}e^{\left(\mu-r-0,5\sigma^2\right)t+\sigma B_t}$$

the stochastic variation of the discounted asset is :

$$d\tilde{V}_{a,t} = \tilde{V}_{a,t}\left(\mu - r\right)dt + \sigma\tilde{V}_{a,t}dB_t$$

If $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, then the Radon-Nikodym theorem ensures the existence of a random variable Y_t that is $\mathcal{F}_T$-measurable, where:

$$Y_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}$$

Now, let's set:

$$W_t = B_t + \frac{\mu - r}{\sigma}t$$

With this change we have:

$$d\tilde{V}_{a,t} = \tilde{V}_{a,t}\left(\mu - r\right)dt + \sigma\tilde{V}_{a,t}dB_t =$$

$$\tilde{V}_{a,t}\left(\mu - r\right)dt + \sigma\tilde{V}_{a,t}\left(W_t - \frac{\mu - r}{\sigma}t\right)dt =$$

$$\tilde{V}_{a,t}\left(\mu - r\right)dt + \tilde{V}_{a,t}dW_t - \left(\mu - r\right)\tilde{V}_{a,t}dt =$$

$$\tilde{V}_{a,t}\sigma dW_t$$

If we manage to construct the probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ under which $W_t = B_t + \frac{\mu-r}{\sigma}t$ is a Brownian Movement, this probability will make the risky asset discounted martingale. The tools that will be useful to have this results is the Girsanov Theorem. The Girsanov theorem says that if we define Y_t like that:

$$Y_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-0,5\lambda^2-\lambda W_t}$$

The process W_t is a standard Brownian motion under $\mathbb{Q}$ for all $t \in [0, T]$.

Note n°1: $\lambda = \frac{\mu - r}{\sigma} t$

So now under $\mathbb{Q}$ we have:

$$V_{a,t} = V_{a,0} e^{\left(r - 0{,}5\sigma^2\right)t + \sigma W_t}$$

and the discounted asset price become:

$$\tilde{V}_{a,t} = V_{a,0} e^{-0{,}5\sigma^2 t + \sigma W_t}$$

Now let's see if under $\mathbb{Q}$ the discounted stock price is a martingale.

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_{a,t} | \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{Q}}\left(V_{a,0} e^{-0{,}5\sigma^2 t + \sigma W_t} | \mathcal{F}_s\right) =$$

$$V_{a,0} \mathbb{E}_{\mathbb{Q}}\left(e^{-0{,}5\sigma^2 s + \sigma W_s - 0{,}5\sigma^2 (t-s) + \sigma (W_t - W_s)} | \mathcal{F}_s\right) =$$

$$V_{a,0} e^{-0{,}5\sigma^2 s + \sigma W_s} \mathbb{E}_{\mathbb{Q}}\left(e^{-0{,}5\sigma^2 (t-s) + \sigma (W_t - W_s)} | \mathcal{F}_s\right) = \tilde{V}_{a,s}$$

by the in independent increment property and $\forall s < t$.

Note n°2: $W_t - W_s \sim \mathcal{N}(0, t-s)$

So now that we have our neutral risk probability we can start to analyze how an options is priced in the Black-Scholes model.

2.3.4 Options pricing: Call and Put

Under risk-neutral probability, it is possible to create a replication portfolio that has a return of r (risk-free rate) and carries no risk. To construct this replication portfolio, we needs to buy one risky asset and sell a quantity x_t of call options. This quantity x_t must be continuously adjusted based on variations in the underlying risky asset $V_{a,t}$. The value of the call option moves in the same direction as the value of the asset. Therefore, a negative quantity (short selling of the call option) is necessary to hedge against the risk inherent in the underlying asset.

This portfolio is composed as follows:

$$\Pi_t = V_{a,t} - x_t V_{c,t} = V_{a,t} - x_t \Psi(V_{a,t}, t)$$

On dt, Π_t varies by:

$$d\Pi_t = dV_{a,t} - x_t d\Psi(V_{a,t}, t)$$

Let's focus on $d\Psi(V_{a,t}, t)$. To do this we applied Ito's lemma:

$$d\Psi(V_{a,t}, t) = \frac{\partial \Psi(V_{a,t}, t)}{\partial t} dt + \frac{\partial \Psi(V_{a,t}, t)}{\partial V_{a,t}} dV_{a,t} + \frac{1}{2} \frac{\partial^2 \Psi(V_{a,t}, t)}{\partial V_{a,t}^2} V_{a,t}^2 \sigma^2 dt$$

Note n°1:

$$d\left\langle V_{a,t}\right\rangle_t = \left\langle dV_{a,t}, dV_{a;t}\right\rangle = \left\langle \sigma V_{a,t} dW_t, \sigma V_{a,t} dW_t\right\rangle = \sigma^2 V_{a,t}^2 dt$$

So now $d\Pi_t$ is:

$$d\Pi_t = dV_{a,t} - x_t\left(\frac{\partial\Psi(V_{a,t},t)}{\partial t}dt + \frac{\partial\Psi(V_{a,t},t)}{\partial V_{a,t}}dV_{a,t} + \frac{1}{2}\frac{\partial^2\Psi(V_{a,t},t)}{\partial V_{a,t}^2}V_{a,t}^2\sigma^2 dt\right)$$

By denoting the partials derivatives more compactly, we obtain:

$$d\Pi_t = dV_{a,t} - x_t\left(\partial\Psi_t dt + \partial\Psi_{V_{a,t}} dV_{a,t} + \frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2 V_{a,t}^2 dt\right)$$

$$d\Pi_t = \left(1 - x_t\partial\Psi_{V_{a,t}}\right)dV_{a,t} - x_t\left(\partial\Psi_t dt + \frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2 V_{a,t}^2 dt\right) \quad (1)$$

In the stochastic differential equation (1), the only source of randomness is $dV_{a,t}$. Therefore, for the portfolio to be risk-free, it is necessary to continuously nullify what multiplies $dV_{a,t}$ by adjusting the quantity xt at each moment. In other words, we want $\left(1 - x_t\partial\Psi_{V_{a,t}}\right) = 0$:

$$1 - x_t\partial\Psi_{V_{a,t}} = 0$$

$$x_t = \frac{1}{\partial\Psi_{V_{a,t}}} = \frac{\partial V_{a,t}}{\partial\Psi\left(V_{a,t},t\right)} = \frac{\partial V_{a,t}}{\partial V_{c,t}}$$

So now (1) become:

$$\left(1 - \frac{1}{\partial\Psi_{V_{a,t}}}\partial\Psi_{V_{a,t}}\right)dV_{a,t} - \frac{1}{\partial\Psi_{V_{a,t}}}\left(\partial\Psi_t dt + \frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2 V_{a,t}^2 dt\right) =$$

$$-\frac{1}{\partial\Psi_{V_{a,t}}}\left(\partial\Psi_t dt + \frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2 V_{a,t}^2 dt\right)$$

If, furthermore, the portfolio is risk-free, then at any time, under risk-neutral probability and according to the principle of absence of arbitrage, Π_t must have a return equal to the risk-free rate r. Therefore, on dt, we have:

$$d\Pi_t = r\Pi_t dt = r\left(V_{a,t} - x_t\Psi\left(V_{a,t},t\right)\right)dt$$

With $x_t = \frac{1}{\partial\Psi_{V_{a,t}}}$, we have:

$$d\Pi_t = r\Pi_t dt = r\left(V_{a,t} - \frac{1}{\partial\Psi_{V_{a,t}}}\Psi\left(V_{a,t},t\right)\right)dt \quad (2)$$

Now we equalise (1) and (2) (with always $x_t = \frac{1}{\partial\Psi_{V_{a,t}}}$) :

$$-\frac{1}{\partial\Psi_{V_{a,t}}}\left(\partial\Psi_t dt + \frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2 V_{a,t}^2 dt\right) = r\left(V_{a,t} - \frac{1}{\partial\Psi_{V_{a,t}}}\Psi\left(V_{a,t},t\right)\right)dt$$

We divide both side by dt:

$$-\frac{1}{\partial\Psi_{V_{a,t}}}\left(\partial\Psi_t+\frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2V_{a,t}^2\right)=r\left(V_{a,t}-\frac{1}{\partial\Psi_{V_{a,t}}}\Psi\left(V_{a,t},t\right)\right)$$

We multiply both side by $\partial\Psi_{V_{a,t}}$:

$$-\partial\Psi_t-\frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2V_{a,t}^2=r\partial\Psi_{V_{a,t}}V_{a,t}-\Psi(V_{a,t},t)$$

$$r\partial\Psi_{V_{a,t}}V_{a,t}-\Psi(V_{a,t},t)+\partial\Psi_t+\frac{1}{2}\partial^2\Psi_{V_{a,t}V_{a,t}}\sigma^2V_{a,t}^2=0 \tag{3}$$

Now, we write the equation (3) with the classical notation of the partials derivatives to obtain:

$$r\frac{\partial\Psi\left(V_{a,t},t\right)}{\partial V_{a,t}}V_{a,t}-\Psi\left(V_{a,t},t\right)+\frac{\partial\Psi\left(V_{a,t},t\right)}{\partial t}+\frac{1}{2}\frac{\partial^2\Psi\left(V_{a,t},t\right)}{\partial V_{a,t}^2}\sigma^2V_{a,t}^2=0$$

To find the solution of this second-order stochastic differential equation, that is, the expression of the call price, $V_{c,t}=\Psi\left(V_{a,t},t\right)$ at each moment, there are two main methods:

- the first one is to take the boundary conditions of the call option and solve the stochastic differential equation by making several changes of variables to reduce it to a heat equation type differential equation. This method is not addressed in our work.

- The second method consists of posing:

$$V_{c,0}=e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(V_{c,T}\right)$$

With,

$$V_{c,T}=\left(V_{a,T}-K\right)^+$$

we have:

$$V_{c,0}=e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\left(V_{a,T}-K\right)^+\right)$$

Note n°2:We can see that:

$$\mathbb{E}_{\mathbb{Q}}\left(\left(V_{a,T}-K\right)^+\right)=\mathbb{E}_{\mathbb{Q}}\left(\left(V_{a,T}-K\right)\mathbb{1}_{\{V_{a,T}>K\}}\right)$$

Note n°3: The range of price that the call can have is:

$$V_{a,T}\in[0,+\infty]$$

Let's focus on $\mathbb{E}_{\mathbb{Q}}\left(\left(V_{a,T}-K\right)\mathbb{1}_{\{V_{a,T}>K\}}\right)$. $V_{a,T}$ is a function of standard normal distribution we can note:

$$V_{a,T}=V_{a,0}e^{\left(r-0,5\sigma^2\right)T+\sigma B_T}=V_{a,0}e^{\left(r-0,5\sigma^2\right)T+\sigma\sqrt{T}z}=\phi(z)$$

Where $z \sim \mathcal{N}(0,1)$. So now we have:

$$\mathbb{E}_{\mathbb{Q}}\left(\left(\phi(z)-K\right)\mathbb{1}_{\{V_{a,T}>K\}}\right)=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{V_{a,T}>K\}}-K\mathbb{1}_{\{V_{a,T}>K\}}\right)=$$

$$\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{V_{a,T}>K\}}\right)-\mathbb{E}_{\mathbb{Q}}\left(K\mathbb{1}_{\{V_{a,T}>K\}}\right)=A-B$$

<u>Expression of B</u>:

$$\mathbb{E}_{\mathbb{Q}}\left(K\mathbb{1}_{\{V_{a,T}>K\}}\right)=K\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{V_{a,T}>K\}}\right)=K\mathbb{P}_{\mathbb{Q}}\left(V_{a,T}>K\right)=$$

$$K\mathbb{P}_{\mathbb{Q}}\left(V_{a,T}>K\right)=K\mathbb{P}_{\mathbb{Q}}\left(V_{a,0}e^{\left(r-0,5\sigma^2\right)T+\sigma\sqrt{T}z}>K\right)=K\mathbb{P}_{\mathbb{Q}}\left(e^{\left(r-0,5\sigma^2\right)t+\sigma\sqrt{T}z}>\frac{K}{V_{a,0}}\right)=$$

$$K\mathbb{P}_{\mathbb{Q}}\left(\left(r-0,5\sigma^2\right)t+\sigma\sqrt{T}z>ln\left(\frac{K}{V_{a,0}}\right)\right)=K\mathbb{P}_{\mathbb{Q}}\left(\sigma\sqrt{T}z>ln\left(\frac{K}{V_{a,0}}\right)-\left(r-0,5\sigma^2\right)T\right)$$

$$=K\mathbb{P}_{\mathbb{Q}}\left(z>\frac{ln\left(\frac{K}{V_{a,0}}\right)-\left(r-0,5\sigma^2\right)t}{\sigma\sqrt{T}}\right)$$

Because of the symmetry of the standard normal distribution we have:

$$K\mathbb{P}_{\mathbb{Q}}\left(z<\frac{ln\left(\frac{V_{a,0}}{K}\right)+\left(r-0,5\sigma^2\right)T}{\sigma\sqrt{T}}\right)=K\mathbb{P}_{\mathbb{Q}}\left(z<d_2\right)=KN\left(d_2\right)=B$$

$$d_2=\frac{ln\left(\frac{V_{a,0}}{K}\right)+\left(r-0,5\sigma^2\right)T}{\sigma\sqrt{T}}$$

and N(.) is the cumulative distribution function of the normal standard distribution.

<u>Expression of A</u>:

$$A=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{V_{a,T}>K\}}\right)=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{V_{a,T}>K\}}\right)=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{z<d_2\}}\right)$$

$$d_2=-d$$

$$A=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{z>d\}}\right)$$

By translating the indicator function into boundary conditions of the integral, and thanks to the transfer theorem, we can rewrite A as:

$$A=\mathbb{E}_{\mathbb{Q}}\left(\phi(z)\mathbb{1}_{\{z>d\}}\right)=\int_d^{+\infty}\phi\left(z\right)f\left(z\right)dz$$

Where $f(z)$ is the density of the standard normal distribution.

$$\int_d^{+\infty}\phi\left(z\right)f\left(z\right)dz=\int_d^{+\infty}V_{a,0}e^{\left(r-0,5\sigma^2\right)T+\sigma\sqrt{T}z}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2}}dz=$$

$$V_{a,0}\int_{d}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{\left(r-0,5\sigma^2\right)T+\sigma\sqrt{T}z-0,5z^2}dz = V_{a,0}\int_{d}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{rT-\left(\frac{\sigma^2+2\sigma\sqrt{T}z-0,5z^2}{2}\right)}dz$$

We see that $\sigma^2+2\sigma\sqrt{T}z-0,5z^2=\left(z-\sigma\sqrt{T}\right)^2$.

$$= V_{a,0}\int_{d}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{rT-\frac{(z-\sigma\sqrt{T})^2}{2}}dz = V_{a,0}e^{rT}\int_{d}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{(z-\sigma\sqrt{T})^2}{2}}dz$$

Here, the function in the integral of the last equation is the density of a normal distribution with a mean of $\sigma\sqrt{T}$ and a variance of 1. Let's have:

$$X = z+\sigma\sqrt{T}$$

$$X \sim \mathcal{N}\left(\sigma\sqrt{T},1\right)$$

We write A as:

$$A = V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(X>d\right) = V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(z+\sigma\sqrt{T}>d\right) = V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(z>d-\sigma\sqrt{t}\right) =$$

$$V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(z<\sigma\sqrt{t}-d\right) = V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(z<\sigma\sqrt{T}+d_2\right)$$

$$d_1 = \sigma\sqrt{T}+d_2$$

$$A = V_{a,0}e^{rT}\mathbb{P}_{\mathbb{Q}}\left(z<d_1\right) = V_{a,0}e^{rT}N(d_1)$$

So now have found the expectation value of the call a the maturity T. Which is:

$$\mathbb{E}_{\mathbb{Q}}\left(V_{c,T}\right) = A-B = V_{a,0}e^{rT}N(d_1)-KN(d_2)$$

We can found the discounted value of the call a the time 0:

$$V_{c,0} = e^{-rT}\left(V_{a,0}e^{rT}N(d_1)-KN(d_2)\right) = V_{a,0}N(d_1)-e^{-rT}KN(d_2)$$

This last expression is the famous Black-Scholes formula.

More generally the value of the call at all time t is:

$$V_{c,t} = V_{a,t}N(d_1)-e^{-r(T-t)}KN(d_2)$$

note n°4: The T in the formula of $V_{c,0}$ is also replaced by (T-t) in the expression of d_1 and d_2.

Now, if we want to find the value of the put, we just need to go through the same reasoning, replacing the call's payoff function with that of the put. Here some interesting thing to see in the case of the put.

Firstly if we want a risk-free portfolio the quantity of put has to be:

$$x_t = \frac{\partial V_{a,t}}{\partial V_{p,t}}$$

With:

$$\frac{\partial V_{a,t}}{\partial V_{p,t}} < 0$$

Indeed, unlike the call, the value of the put varies in the opposite direction to the underlying asset. The rest of the stochastic differential equation stay unchanged.

Secondly, the payoff function of the put is at T:

$$Vp,T = (K - V_{a,T})^{+}$$

With:

$$V_{p;T} \in [0, K]$$

By the same development of the call we find that:

$$V_{p,0} = e^{-rT} K N\left(-d_2\right) - V_{a,0} N(-d_1)$$

and for all t we have:

$$V_{p,t} = e^{-r(T-t)} K N\left(-d_2\right) - V_{a,t} N(-d_1)$$

2.4 Sensibility : Greek's letters

The Greek letters in the Black-Scholes model allow for the calculation of the sensitivity of the value of the call or put to the model's parameter. The sensitivities of the price of an option to its various parameters are referred as the Greeks. we remind that the value of the call is :

$$V_{c,t} = V_{a,t} N(d_1) - e^{-r(T-t)} K N(d_2)$$

From now on, to simplify subsequent calculations, we introduce the following notations:

$$T - t = \tau$$

2.4.1 Delta

Delta (Δ) is the sensitivity of the price to the current value of the underlying asset:

$$\Delta_c = \frac{\partial V_{c,t}}{\partial V_{a,t}}$$

To start by using the derivative chain method and we can write this derivative like this:

$$\frac{\partial V_{c,t}}{\partial V_{a,t}} = \mathcal{N}(d_1) + V_{a,t} f\left(d_1\right) \frac{\partial d_1}{\partial V_{a,t}} - e^{-r\tau} K f\left(d_2\right) \frac{\partial d_2}{\partial V_{a,t}} \tag{4}$$

Where $f(.)$ is the density of the standard normal distribution and also the derivative of $\mathcal{N}(.)$.

To start let's manipulate a little bit the expression of d_1:

$$d_1 = \frac{ln\left(\frac{V_{a,t}}{K}\right) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} = \frac{ln\left(\frac{V_{a,t}}{K}\right) + ln\left(e^{\tau r}\right) + \tau\frac{\sigma^2}{2}}{\sigma\sqrt{\tau}} = \frac{ln\left(\frac{V_{a,t}}{Ke^{-\tau r}}\right) + \frac{\tau\sigma^2}{2}}{\sigma\sqrt{\tau}}$$

Now let's focus on $f(d_2)$:

$$f(d_2) = f\left(d_1 - \sigma\sqrt{\tau}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_1\sigma\sqrt{\tau}\right)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}+d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}}$$

Next, we substitute the d_1 in the second member of the addition in the exponential by his explicit form:

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}+d_1\sigma\sqrt{\tau}-\frac{\sigma^2\tau}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}+ln\left(\frac{V_{c,t}}{Ke^{-\tau r}}\right)+\frac{\tau\sigma^2}{2}-\frac{\tau\sigma^2}{2}} = \frac{V_{a,t}}{Ke^{-\tau r}}e^{-\frac{d_1^2}{2}}$$

So we can write $f(d_2)$ as:

$$f(d_2) = \frac{V_{a,t}}{Ke^{-\tau r}}f(d_1) \tag{5}$$

Secondly let's focus on $\frac{\partial d_2}{\partial V_{a,t}}$ and $\frac{\partial d_1}{\partial V_{a,t}}$:

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\frac{\partial d_2}{\partial V_{a,t}} = \frac{\partial d_1}{\partial V_{a,t}} \tag{6}$$

So now if we put (6) and (5) in to (4) we obtain:

$$\frac{\partial V_{c,t}}{\partial V_{a,t}} = \mathcal{N}(d_1) + V_{a,t}f\left(d_1\right)\frac{\partial d_1}{\partial V_{a,t}} - e^{-r\tau}K\frac{V_{a,t}}{Ke^{-r\tau}}f(d_1)\frac{\partial d_1}{\partial V_{a,t}}$$

$$= \mathcal{N}(d_1) + V_{a,t}f\left(d_1\right)\frac{\partial d_1}{\partial V_{a,t}} - V_{a,t}f\left(d_1\right)\frac{\partial d_1}{\partial V_{a,t}} = \mathcal{N}(d_1)$$

Finally we found that :

$$\Delta_c = \frac{\partial V_{c,t}}{\partial V_{a,t}} = \mathcal{N}(d_1)$$

The Delta measures how the price of the option changes when the price of the underlying asset fluctuates. The delta is very important for maintaining a risk-free portfolio. It indicates the amount of option that needs to be continuously adjusted to keep the portfolio risk-free. We refer to it as Delta-neutral hedging.

For the put Δ is:

$$\Delta_p = \frac{\partial V_{p,t}}{\partial V_{a,t}} = -\mathcal{N}(-d_1)$$

2.4.2 Gamma

Gamma (Γ) is the sensitivity of the option's delta to a change in the price of the underlying asset:

$$\Gamma_c = \frac{\partial^2 V_{c,t}}{\partial V_{a,t}^2} = \frac{\partial \Delta}{\partial V_{a,t}}$$

$$\frac{\partial \Delta}{\partial V_{a,t}} = f\left(d_1\right) \frac{\partial d_1}{\partial V_{a,t}}$$

Here $\frac{\partial d_1}{\partial V_{a,t}}$ is equal to:

$$\frac{\partial d_1}{\partial V_{a,t}} = \frac{1}{\sigma\sqrt{\tau}} \cdot \frac{1}{K} \cdot \frac{K}{V_{a,t}} = \frac{1}{\sigma\sqrt{\tau}} \cdot \frac{1}{V_{a,t}}$$

So finally we have:

$$\Gamma_c = \frac{f\left(d_1\right)}{V_{a,t}\sigma\sqrt{\tau}}$$

If the Gamma is low, the delta varies only slightly, and Delta hedging don't need to be modified often.However, if the Gamma is high, it is often necessary to reconsider the number of asset in the portfolio to keep it Delta-neutral. Indeed, if we consider discrete-time hedging, the stronger the gamma, the higher the risk of error is . The error risk is proportional to the adjustment frequency of the portfolio. To cut in half the error risk, the portfolio needs to be rebalanced four times as often. Another option to mitigate this risk is delta-gamma hedging, where a sufficiently liquid asset is added to the portfolio to counter this risk.

The Gamma of the put is the same as the Gamma of the call:

$$\Gamma_c = \Gamma_p = \frac{\partial^2 V_{p,t}}{\partial V_{a,t}^2} = \frac{f\left(d_1\right)}{V_{a,t}\sigma\sqrt{\tau}}$$

2.4.3 Theta

Theta (Θ) is the sensitivity of the price to the elapsed time (τ):

$$\Theta_c = -\frac{\partial V_{c,t}}{\partial \tau}$$

Firstly we can write this derivative as:

$$\frac{\partial V_{c,t}}{\partial \tau} = V_{a,t} f(d_1) \frac{\partial d_1}{\partial \tau} - \left(-re^{-r\tau} K\mathcal{N}(d_2) + e^{-r\tau} K f(d_2) \frac{\partial d_2}{\partial \tau}\right) \tag{7}$$

We remember that $d_2 = d_1 - \sigma\sqrt{\tau}$ so if we take derivative on the two side we obtain:

$$\frac{\partial d_2}{\partial \tau} = \frac{\partial d_1}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}} \tag{8}$$

with $\frac{\partial d_2}{\partial \tau}$ equal to:

$$\frac{\partial d_1}{\partial \tau} = -\frac{\sqrt{\tau}}{2\sigma} ln\left(\frac{V_{a,t}}{K}\right) + \frac{r + \frac{\sigma^2}{2}}{2\sigma\sqrt{\tau}} \tag{9}$$

So now we compile (7) by taking the expression found in (8) and (9). We also write $f(d_2)$ with the expression found in (5) to finally obtain:

$$\frac{\partial V_{c,t}}{\partial \tau} = re^{-r\tau} K\mathcal{N}(d_2) + V_{a,t} f(d_1)\frac{\partial d_1}{\partial \tau} - e^{r\tau} K \frac{V_{a,t}}{Ke^{-r\tau}} f(d_1)\left(\frac{\partial d_1}{\partial \tau} - \frac{\sigma}{2\sqrt{\tau}}\right)$$

$$\frac{\partial V_{c,t}}{\partial \tau} = re^{-r\tau} K\mathcal{N}(d_2) + V_{a,t} f(d_1)\frac{\sigma}{2\sqrt{\tau}}$$

$$\Theta_c = -re^{-r\tau} K\mathcal{N}(d_2) - V_{a,t} f(d_1)\frac{\sigma}{2\sqrt{\tau}}$$

The theta reflects the evolution of the value of a call option over time.

The theta of the put is:

$$\Theta_p = -\frac{\partial V_{p,t}}{\partial \tau} = \frac{V_{a,t}}{V_{p,t}} = re^{-r\tau} K\mathcal{N}(-d_2) - V_{a,t} f(d_1)\frac{\sigma}{2\sqrt{\tau}}$$

2.4.4 Vega

Vega ($\mathcal{V}$) is the sensitivity of the price to volatility:

$$\mathcal{V}_c = \frac{\partial V_{c,t}}{\partial \sigma}$$

Firstly, we can this derivative like that:

$$\frac{\partial V_{c,t}}{\partial \sigma} = V_{a,t} f(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-r\tau} K f(d_2)\frac{\partial d_2}{\partial \sigma}$$

by taking the expression of d_2 relative to d_1 and derivating on the two side we obtain:

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}$$

Finally if we substitute $f(d_2$ by the expression found in (5) we have:

$$\mathcal{V}_c = V_{a,t} f(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-r\tau} K f(d_1)\frac{V_{a,t}}{Ke^{-r\tau}}\left(\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}\right)$$

$$\mathcal{V}_c = f(d_1) V_{a,t}\sqrt{\tau}$$

Vega is a very important Greek. It gives the relation between the price of an option and the volatility of the underlying. More the underlying is volatile more the option will be expensive. Moreover the volatility is a parameter really difficult to estimate. Usually, if the volatility is high, the calibration of the model will be difficult and the risk of error will be high.

The Vega of the call is the same as the put:

$$\mathcal{V}_c = \frac{\partial V_{p,t}}{\partial \sigma} = \mathcal{V}_p = f(d_1) V_{a,t}\sqrt{\tau}$$

2.4.5 Rho

Rho (ρ) is the sensitivity of the price to the interest rate (r):

$$\rho_c = \frac{\partial V_{c,0}}{\partial r}$$

firstly we can write this derivative as follow:

$$\frac{\partial V_{c,t}}{\partial r} = V_{a,t} f(d_1) \frac{\partial d_1}{\partial r} - \left(-\tau e^{-r\tau} K \mathcal{N}(d_2) + e^{-r\tau} K f(d_2) \frac{\partial d_2}{\partial r} \right) \tag{10}$$

with:

$$\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r}$$

and:

$$\frac{\partial d_1}{\partial r} = \frac{\tau}{\sigma \sqrt{\tau}} = \frac{\sqrt{\tau}}{\sigma}$$

So now if we write (10) with the results found above and using (5) we have:

$$V_{a,t} f(d_1) \frac{\partial d_1}{\partial r} - \left(-\tau e^{-r\tau} K \mathcal{N}(d_2) + e^{-r\tau} K \frac{V_{a,t}}{K e^{-r\tau}} f(d_1) \frac{\partial d_1}{\partial r} \right)$$

To finally obtain:

$$\rho_c = \frac{\partial V_{c,t}}{\partial r} = \tau e^{-r\tau} K \mathcal{N}(d_2)$$

This Greek is the less important of the Greek because usually the price of an option is less sensitive to changes in the risk-free interest rate than to changes in other parameters.

For the put Rho is:

$$\rho_p = \frac{\partial V_{p,t}}{\partial r} = -\tau K e^{r\tau} \mathcal{N}(-d_2)$$

Note n°1: If we take the equation (3) with the Greek notation we have:

$$\frac{1}{2} V_{a,t}^2 \sigma^2 \Gamma + r V_{a,t} \Delta - r V_{c,t} + \Theta = 0$$

Note n°2: There is a lot of other Greeks like the second orders Greeks or third orders Greeks.

Note n°3: The Greeks are very powerful tools for analyzing and making decisions when it comes to managing options. It helps in making decisions and establishing investment strategies.

3 Application of the Black and Scholes model

Having covered the theoretical aspects of the Black-Scholes model, we're now moving on to its real-world application. we're now looking how it works in practical situations. In the upcoming section, we'll apply the model to an financial asset. This shift from theory to application allows us to see how the Black-Scholes model plays out in the dynamic realm of financial markets. In this section, we will follow a specific plan to apply the Black-Scholes model to a concrete case. First, we will present the chosen call option and underlying asset for our study. Next, we will delve into how to select the risk-free interest rate. We will then proceed to examine two methods for estimating volatility: the historical method and the implied method. Regarding the latter, we will explore the concept of "volatility smile" and discuss potential limitations of the model.

3.1 The choice of the options and the underlying

In our application of the Black-Scholes model, we have selected a European call option with a one-year maturity, expiring on February 28, 2025. The strike for this option is set at 80$.

Our first parameters are therefore:

$$T = 365$$

$$K = 80$$

$$V_{a,0} = 83,4$$

Note n°1: The units for the time is the day and for the price the USD.

Note n°2:All the prices used on a given date are the closing market prices.

3.2 Risk free interest

In the Black-Scholes model, the choice of the risk-free interest rate is crucial as it directly influences the pricing of options. The risk-free rate represents the theoretical return an investor would expect from an absolutely risk-free investment over a certain period. It serves as a foundation for discounting future cash flows to their present value. In the context of the Black-Scholes model, the risk-free rate is used to discount the expected payoff of the option to its present value. The selection of an appropriate risk-free rate is essential for accurate option pricing. One commonly used risk-free rate is the Federal Reserve's target interest rate. The Federal Reserve, as the central bank of the United States, sets the target federal funds rate, which is the interest rate at which depository institutions lend balances to other depository institutions. Using the Federal Reserve's target interest rate as the risk-free rate in the Black-Scholes model can be used when valuing options on assets denominated in US dollars or in the context of US financial markets. Currently, this rate ranges between 5.25% and 5.5%. Throughout our work, the choice of the interest rate will fall within this range.

3.2.1 Maximum like hood estimator of $\hat{\sigma}$

The first step to find the historic volatility estimator $\hat{\sigma}$ is to use regularly spaced data with a range of δ. In our case the range will be a day. We introduce :

$$V_{a,t+\delta} = V_{a,t} e^{\mu\delta - \sigma(W_{t+\delta} - W_t)}$$

We introduce also u which is :

$$u_t = ln\left(\frac{V_{a,t+\delta}}{V_{a,t}}\right) = ln\left(\frac{V_{a,t} e^{\mu\delta - \sigma(W_{t+\delta} - W_t)}}{V_{a,t}}\right) = \mu\delta - \sigma\left(W_{t+\delta} - W_t\right)$$

Note n°1: u_t can be interpret as the approximation of the relative variation of the risky asset over the period δ.

Note n°2:We remember that $W_{t+\delta} - W_t \sim \mathcal{N}(0, \delta)$. So we can conclude that:

$$u_t \sim \mathcal{N}\left(\mu\delta, \sigma^2\delta\right)$$

Note n°3: We remember also one specificity of the Brownian motion which the independents increments propriety (2.1.3). So over multiple time period we have multiple u_i who are independent. So we have a sample of u_i who are independent and identically distributed. We can made statistical inference on this sample to estimate $\hat{\sigma}$.

Let have $u_1; ...; u_n$ i.i.d where $u_i \sim \mathcal{N}\left(\mu\delta, \sigma^2\delta\right)$, The like hood function of this sample is:

$$L\left(u_1; ...; u_n; \sigma; \mu\right) = \prod_{i=1}^{n} f\left(u_i\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\delta}\sigma} e^{-\frac{1}{2}\left(\frac{u_i - \mu\delta}{\sigma\sqrt{\delta}}\right)^2} =$$

$$\left(\sqrt{2\pi\delta}\sigma\right)^{-n} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i - \mu\delta}{\sigma\sqrt{\delta}}\right)^2}$$

The log-like hood function is:

$$l\left(u_1; ...; u_n; \sigma; \mu\right) = ln\left(L\left(u_1; ...; u_n; \sigma; \mu\right)\right) = ln\left(\left(\sqrt{2\pi\delta}\sigma\right)^{-n} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i - \mu\delta}{\sigma\sqrt{\delta}}\right)^2}\right) =$$

$$-n.ln\left(\sqrt{2\pi\delta}\sigma\right) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u_i - \mu\delta}{\sigma\sqrt{\delta}}\right)^2$$

We want to find the maximum like-hood estimator of $(\hat{\mu}; \hat{\sigma})$ which is the solution of the following problem:

$$\underset{\mu;\sigma}{\operatorname{argmax}} \quad l\left(u_1; ...; u_n; \sigma; \mu\right)$$

The necessary first orders conditions are:

$$\begin{cases} \frac{\partial l}{\partial \mu}|_{\hat{\mu}=\mu} = 0 \\ \frac{\partial l}{\partial \sigma}|_{\hat{\sigma}=\sigma} = 0 \end{cases}$$

Firstly let's focus on $\frac{\partial l}{\partial \mu}$:

$$\frac{\partial l}{\partial \mu} = \sum_{i=1}^{n} u_i - \mu\delta$$

The estimator of $\hat{\mu}$ is:

$$-\sum_{i=1}^{n} u_i - \hat{\mu}\delta = 0$$

$$\hat{\mu} = \frac{1}{n\delta}\sum_{i=1}^{n} u_i$$

Now we put $\hat{\mu}$ in $\frac{\partial l}{\partial \sigma}$ to find $\hat{\sigma}$:

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3\delta}\sum_{i=1}^{n}(u_i - \hat{\mu}\delta)^2$$

So the estimator $\hat{\sigma}$ is:

$$-\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3\delta}\sum_{i=1}^{n}(u_i - \hat{\mu}\delta)^2 = 0$$

$$\frac{1}{n}\sum_{i=1}^{n}(u_i - \hat{\mu}\delta)^2 = \hat{\sigma}^2\delta$$

$$\hat{\sigma} = \sqrt{\frac{\frac{1}{n}\sum_{i=1}^{n}(u_i - \hat{\mu}\delta)^2}{\delta}}$$

Note n°4:We can substitute the δ from our results. In our case δ is equal to the length of one period (1 day) and is equal to 1.

The maximum like-hood estimators are so:

$$(\hat{\mu};\hat{\sigma}) = \left(\frac{1}{n}\sum_{i=1}^{n} u_i; \sqrt{\frac{1}{n}\sum_{i=1}^{n}(u_i - \hat{\mu})^2}\right)$$

And the historic volatility estimator is finally:

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(u_i - \hat{\mu})^2}$$

Now let's use this estimators to our cases with real data.

3.2.2 Estimation of the model with the historic volatility

In this section, we will apply the Black-Scholes model to the following option: a call option with the underlying asset being Brent Crude Oil. The maturity is set to 1 year. The strike price is \$ 83. The value on March 16, 2024, is \$85.33. The risk-free interest rate is 5.5%. So the parameters are:

$$V_{a,0} = 85.33$$

$$K = 83$$

$$r = 0.055$$

$$T = 1$$

To apply the Black-Scholes model to the option presented above, we estimated historical volatility based on the historical price of this underlying asset. Our data is sourced from *boursorama.com*. The statistical estimation technique is the same as that presented in the previous section. In the table below are the estimates of historical volatility at various given dates. So now that we have all

Historical data from the last :	**Historical Volatility :**
3 months	0.01384
6 months	0.01879
1 year	0.01902
3 years	0.02888
10 years	0.02528

the information necessary to calculate the price of the option, we can compute all the parameters in the Black-Scholes formula. For example, for the 3 months old historic volatility we have:

$$d_2 = \frac{ln\left(\frac{85.33}{83}\right) + 1 \times \left(0.055 - 0.5 \times 0.1383^2\right)}{0.1383 \times \sqrt{1}} = 0.4945217$$

$$d_1 = 0.1383\sqrt{1} + d_2 = 0.6328217$$

$$V_{c,0} = 85.33 \times \mathcal{N}(0.6328217) - e^{0.055\times 1} \times 83 \times \mathcal{N}(0.4945217) = 6.771671$$

So in this case the Black-Scholes model says that the value of the call today is 6.77\$.
For the other volatility estimation, the Black Scholes model says that:

Historical data from the last :	**Historical Volatility:**	**Call value:**
3 Months	0.01384	6.771671
6 Months	0.01879	6.768956
1 Year	0.01902	6.768622
5 Years	0.02888	6.737097
10 Years	0.02528	6.751294

3.2.3 Which historic volatility?

Generally, it is advisable to take data over a period ranging from 90 to 180 days. However, this should be juxtaposed with several other factors. If the option's maturity is much longer, for example, two years, it would be reasonable to take historical data over a slightly larger range. For example, over 1 to 2 years.

<u>note n°1:</u>In our case, the choice of historical volatility is delicate because oil cannot be considered as an asset like others. It is subject to much stronger systemic risks. These risks can arise from very varied causes such as geopolitical context or even macroeconomic conditions.

3.3 Estimation with implicit volatility

Implicit volatility is another method of calculating volatility. In this method, the option price is considered known. It is the result of market forces and is regarded as an equilibrium price. By considering this, the only unknown in the Black-Scholes formula becomes σ. That is to say, at a given option price (with all other parameters known), we obtain a value for σ. This value is the implied volatility. When estimating implied volatility, the Black-Scholes formula is no longer used as an option pricing formula but rather as a method for estimating volatility and risk of the underlying asset. This implied volatility can be seen as the anticipated future volatility by the market.

$$V_{c,T}^{OB}(t, V_{a,t}, T, K) = V_{c,T}^{BS}(t, V_{a,t}, T, K, \sigma_{Impl})$$

$V_{c,T}^{OB}$ is the price observed on the market.
$V_{c,T}^{BS}$ is the price which is given by the black-Scholes formula.
σ_{Impl} is the implicit volatility.

<u>Note n°1</u>: The inversion of the Black-Scholes formula is not possible analytically. However, σ_{Impl} can be approximated numerically using the Newton-Raphson algorithm.

So let's take an call options with petrol WTI on underlying with the following parameters:

-The spot price of the underlying at t_0(16/03/2024) is $V_{a,0} = 80.99\$$.
-Maturity: 16/05/2024 (2 month of maturity), so T=0.1205 (44/365)
-Strike : K=83\$
-Call price : $V_{c,0} = 1,902$ \$
-Risk free interest: r=5,5%.

The Black-Scholes formula is in this case:

$$1,902 = 80.99N(d_1) - e^{-0.055\times0.1205}KN(d_2)$$

With:

$$d_2 = \frac{ln\left(\frac{80.99}{83}\right) + \left(0.055 - 0,5\sigma_{Impl}^2\right) \times 0.1205}{\sigma_{Impl}\sqrt{0.1205}}$$

$$d_1 = \sigma_{Impl}\sqrt{0.1205} + d_2$$

So now on R we can calculate the implicit volatility.(all the script are available in the appendix sections). We found that:

$$\sigma_{Impl} = 0.291$$

So logically if we compile the Black-Scholes with all this parameters we will have the price found on the market.

This volatility can be compared to historical volatility. If historical volatility is lower than implied volatility, it can be interpreted as a signal that the market anticipates an increase in volatility. Conversely, if historical volatility is higher than implied volatility, then the market anticipates a decrease in volatility.

Now, if we want to delve deeper into the analysis of implied volatility, we can calculate implied volatility for options with the same parameters and only vary the strike . The Black-Scholes model is a model with constant volatility. This means that regardless of the distance between the spot price and the strike , the volatility should remain the same. In other words, the option price should adjust through market forces in such a way that the implied volatility remains constant regardless of the difference between the spot price and the strike . The next section will verify these results and may potentially lead to drawing conclusions about the accuracy of the assumptions and the limitations of the Black-Scholes model.

3.4 Smile with implicit volatility

The "volatility smile" is a phenomenon observed in financial markets where options exhibit different implied volatilities depending on the exercise prices. In other words, rather than having a constant volatility for all options on a given underlying asset, the volatility smile depicts a curve shaped like a smile, with generally higher volatility for "in-the-money" and "out-of-the-money" options compared to "at-the-money" options. The volatility smile contradicts the assumptions of the Black-Scholes model (constant volatility). So now let's see if we can reproduce this results.

In the case of a call option,we remember that the call option is "in-the-money" when, compared to the current spot price, there would be interest in exercising the call option. In other words, the option is "in-the-money" when the spot price of the underlying asset is lower than the strike price. The call option is "at-the-money" when the spot price of the underlying asset is similar to the strike price. And the call option is "out-of-the-money" when there is no interest in exercising the option, meaning when the spot price of the underlying asset is higher than the strike price.

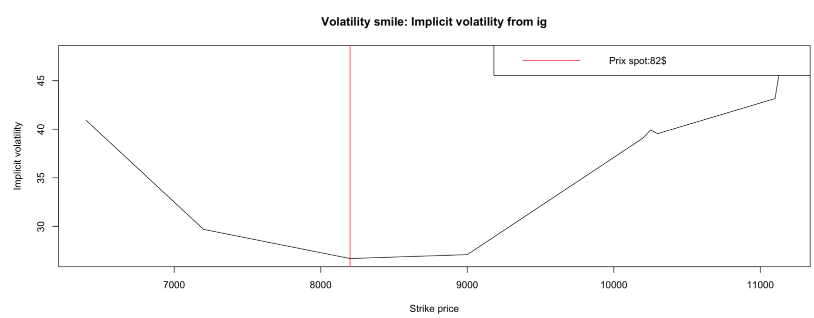
For the study of the volatility smile, we have chosen to use a call option on WTI crude oil with a maturity date of April 17, 2024. The data is sourced from *ig.com*and is available in the appendix. Additionally, an implied volatility index was available with the data. We will analyze the volatility smile using the calculation provided by *ig.com* and with our calculation of implied volatility derived from the Black-Scholes model.

The spot price is 82$.

3.4.1 Smile with implicit volatility calculated by *ig.com*

In this case, we collected various data regarding call options on the same underlying asset with the same maturity. The only parameter that varied was the strike . Each option had a market-set price and an directly calculated implicit volatility. We gathered all this data to plot the implicit volatility of each option on the ordinate axis and the associated strike on the abscissa axis.

On this graph, we can clearly observe the volatility smile. The more the un-

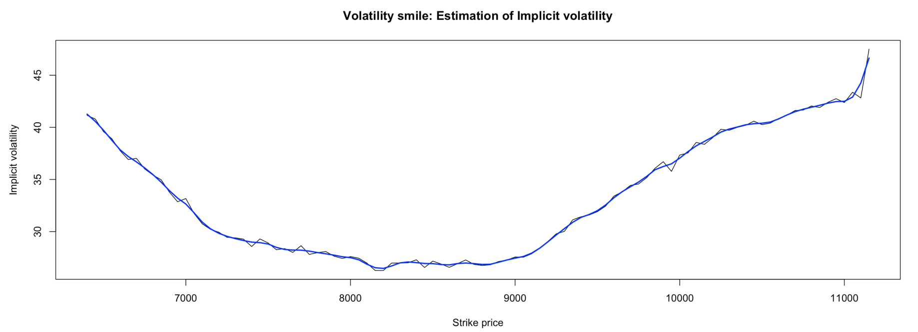


derlying is in or out of the money, the higher the implied volatility. To the right of the red line, the options are out-the-money, and to the left the options are in-of-the-money. Here, we clearly see the limitations of the Black-Scholes model. If the Black-Scholes model were perfectly adhered to, we should see a horizontal line.

Note n°1:The volatility is in % and the strike is in cents.

3.4.2 Smile with implicit volatility calculated by ourselves

In this section, we aim to reproduce the previous results using identical options. However, we calculate the implied volatility ourselves using the method developed in Section 3.3, based on the Black-Scholes model. (Scripts and data available in the appendix.)



<u>note n°2</u>: Here the price is in cents and the volatility is in %.

We notice the same trend on both graphs, there is an increase in implied volatility as we move away from the spot price. However we not have found exactly the same results as *ig.com*. The major factor that explain this are the following:

-The pricing model used for calculating implied volatility might not have been the same and the calibration of the parameters also (like for example the choice of the risk-free interest rate). We have no information about Ig's option pricing model used for implied volatility calculation. So it is entirely conceivable to obtain slightly different estimates of implied volatility.

This inconsistency underscores the complexity of the underlying market dynamics and the limitations of the models used. Further analysis and refinement of our methods may provide deeper insights into the behavior of implied volatility across different option prining model.

3.5 Extension of the Black-Scholes model: Stochastic volatility

To resolve this weakness, several extensions of the Black-Scholes model have been developed to better capture the phenomena observed in this section. These refinements aimed to impart a more realistic nature to volatility by making it not constant but stochastic. These stochastic volatility models were notably introduced by Hull and White (1987). In this model, volatility is now a function of time and is governed by a diffusion process given by:

$$dV_{a,t} = \mu V_{a,t} dt + \sqrt{\sigma_t} V_{a,t} dB_t$$

$$d\sigma_t = \alpha\left(\sigma_t\right) dt + \beta\left(\sigma_t\right) dW_t$$

$$d\left\langle B, W\right\rangle_t = \rho dt$$

The last equation explains that the two Brownian motions are correlated. The correlation coefficient is ρ and is generally negative. In the Heston model the function $\alpha(\sigma_t)$ and $\beta(\sigma_t)$ are given and the volatility diffusion process are the following:

$$d\sigma_t = \kappa(\theta - \sigma_t)\,dt + \xi\sqrt{\sigma_t}dW_t$$

κ is the mean reversion rate.
θ is the long term variance.
ξ is the volatility of the volatility.

In econometric literature, to study these phenomena of stochastic volatility, we can use autoregressive conditional heterosdedasticity (ARCH) and generalized autoregressive conditional heterosdedasticity (GARCH) models. It is important to notice that these models are discrete-time.

4 Conclusion

In conclusion, the Black-Scholes model has significantly contributed to the advancement of financial theory by offering an innovative approach to evaluate financial options. Its revolutionary proposition has enhanced the understanding of market behavior and provided valuable tools for financial risk management. One of the main contributions of the Black-Scholes model lies in its ability to quantify the price of an option based on various parameters, such as the underlying asset price, volatility, time remaining until option expiration, and the risk-free interest rate. This mathematical approach has opened new perspectives in understanding market mechanisms and has enabled investors to make more informed decisions. Moreover, extensions of the Black-Scholes model have further enriched financial theory by introducing more sophisticated concepts for evaluating and managing risks. Variants of the initial model, such as the Black-Scholes-Merton model or the Heston models, have been developed to account for additional factors such as dividends or volatility variations. Additionally, the Black-Scholes model has paved the way for other approaches to modeling financial markets, such as jump-diffusion models or stochastic volatility models, which aim to better capture phenomena observed in real markets. However, it is essential to recognize that the Black-Scholes model has certain limitations. It is based on a number of simplifying assumptions, such as market efficiency and price continuity, which may not always hold true in real-world situations. Furthermore, the model only applies to European options. In conclusion, while the Black-Scholes model has represented a significant advancement in financial theory and continues to play a central role in the practice of option valuation, it is important to consider it judiciously and acknowledge its inherent limitations. A thorough understanding of its theoretical foundations and practical implications is essential for effective decision-making in financial contexts.

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6 Appendix

```
####################IMPLICIT VOLATILITY########################

install.packages('RQuantLib', type="binary")
library(RQuantLib)


iv= EuropeanOptionImpliedVolatility(type="call", value=1.902,
underlying=79.29,strike=83, dividendYield=0, riskFreeRate=0.055,
maturity=0.12054, volatility=0.2)


###################VOLATILITY SMILE############################

data.smilev2 <- read.csv("~/Desktop/M1 ES/TER
/DONNEE TER /data smilev3.csv", sep=";")
cn= c('igimplvol','callprice','strikeprice')
colnames(data.smilev2)=cn

lim=c(0,200)
plot(data.smilev2[,3],data.smilev2[,1], main = "Volatility
smile: Implicit volatility from ig", xlab="Strike price",
ylab="Implicit volatility",type='l')
abline(v = 8200, col = "red",ylim=lim)
legend("topright", legend = "Prix spot:82$", col = "red", lty = 1)

#
data.smilev2$callprice= data.smilev2$callprice/100
data.smilev2$strikeprice=data.smilev2$strikeprice/100


calculate_implied_volatility2 <- function(x, y) {
  tryCatch({
    volatility <- EuropeanOptionImpliedVolatility(type = "call",
                                                  value = x,
                                                  underlying = 82,
                                                  strike = y,
                                                  dividendYield = 0,
                                                  riskFreeRate = 0.055,
                                                  maturity = 0.0630,
                                                  volatility = 0.001)
    return(volatility)
  }, error = function(e) {
    return(NA)
```

```
  })
}

resultats_implvol <- numeric(nrow(data.smilev2))

# Boucle for pour calculer les volatilites implicites
for (i in 1:nrow(data.smilev2)) {
  resultats_implvol[i] <- calculate_implied_volatility2
  (data.smilev2[i, 2], data.smilev2[i, 3])
}

data.smilev2$resultats_implvol <- resultats_implvol

plot(data.smilev2[,3],data.smilev2[,4], main =
"Volatility smile: Estimation
Implicit volatility", xlab="Strike price",
ylab="Implicit volatility",type='l')
abline(v = 82, col = "red",ylim=lim)
legend("topright", legend = "Prix spot:82$", col = "red", lty = 1)
spline_fit <- smooth.spline(data.smilev2[,3], smile)
lines(spline_fit , col = "blue", lwd = 2)


###########Volatilite historique1###############################

#cours 6M

BRENT.6M <- read.delim("~/Desktop/M1 ES/TER/DONNEE TER /BRENT 6M.txt")

cours6M = BRENT.6M$clot
u6M =log(cours6M[2:length(cours6M)] / cours6M[1:(length(cours6M)-1)])
meanu6M = mean(u6M)
sigma6M= sqrt((sum((u6M-meanu6M)^2))/length(u6M))

#cours 3M

BRENT.3M <- read.delim("~/Desktop/M1 ES/TER/DONNEE TER /BRENT 3M.txt")

cours3M = BRENT.3M$clot

u3M = log(cours3M[2:length(cours3M)] / cours3M[1:(length(cours3M)-1)])
meanu3M = mean(u3M)
sigma3M= sqrt((sum((u3M-meanu3M)^2))/length(u3M))

#Cours 10A
```

```
BRENT.10A <- read.delim("~/Desktop/M1 ES/TER/DONNEE TER /BRENT 10A.txt")

cours10A = BRENT.10A$clot
u10A = log(cours10A[2:length(cours10A)] / cours10A[1:
(length(cours10A)-1)])
meanu10A = mean(u10A)
sigma10A= sqrt((sum((u10A-meanu10A)^2))/length(u10A))

#Cours 1A

BRENT.1A <- read.delim("~/Desktop/M1 ES/TER/DONNEE TER /BRENT 1 AN.txt")

cours1A = BRENT.1A$clot
u1A = log(cours1A[2:length(cours1A)] / cours1A[1:(length(cours1A)-1)])
meanu1A = mean(u1A)
sigma1A= sqrt((sum((u1A-meanu1A)^2))/length(u1A))

#Cours5A

BRENT.5A <- read.delim("~/Desktop/M1 ES/TER/DONNEE TER /BRENT 5ANS.txt")
cours5A = BRENT.5A$clot
u5A = log(cours5A[2:length(cours5A)] / cours5A[1:(length(cours5A)-1)])
meanu5A = mean(u5A)

#Formule Black-Scholes

BSM = function(x){
  d1 <- ((log(85.33/83)+(0.055+x^2/2)*1/(x*sqrt(1))))
  d2 <- d1-x*sqrt(1)
  C <- 85.33*pnorm(d1,0,1)-83*exp(-1*0.055*(1))*pnorm(d2,0,1)
return(C)
}

vectsigm=c(0.01383 , 0.01878 ,0.01902 , 0.02887 , 0.02528)
resultatsBSM <- sapply(vectsigm, BSM)


 ##########DATA############
 igimplvol callprice strikeprice
1      40.9000     1808.4          6400
2      40.2000     1758.8          6450
3      39.5000     1709.2          6500
4      38.8000     1659.6          6550
5      38.1000     1610.1          6600
6      37.4000     1560.6          6650
7      36.7000     1511.2          6700
```

8	36.0000	1461.8	6750
9	35.3000	1412.4	6800
10	34.6000	1363.2	6850
11	33.9000	1313.9	6900
12	33.2000	1264.8	6950
13	32.5000	1215.8	7000
14	31.8000	1166.8	7050
15	31.1000	1118.0	7100
16	30.4000	1069.2	7150
17	29.7000	1020.6	7200
18	29.5500	973.8	7250
19	29.4000	927.4	7300
20	29.2500	881.6	7350
21	29.1000	836.4	7400
22	28.9500	791.9	7450
23	28.8000	748.1	7500
24	28.6500	705.1	7550
25	28.5000	662.1	7600
26	28.3500	621.0	7650
27	28.2000	580.9	7700
28	28.0500	541.8	7750
29	27.9000	504.0	7800
30	27.7500	467.3	7850
31	27.6000	432.0	7900
32	27.4500	398.1	7950
33	27.3000	365.5	8000
34	27.1500	334.5	8050
35	27.0000	304.9	8100
36	26.8500	276.9	8150
37	26.7000	250.5	8200
38	26.7250	227.3	8250
39	26.7500	205.7	8300
40	26.7750	185.6	8350
41	26.8000	167.1	8400
42	26.8250	150.0	8450
43	26.8500	134.3	8500
44	26.8750	119.9	8550
45	26.9000	106.8	8600
46	26.9250	94.9	8650
47	26.9500	84.1	8700
48	26.9750	74.4	8750
49	27.0000	65.7	8800
50	27.0250	57.9	8850
51	27.0500	50.8	8900
52	27.0750	44.2	8950
53	27.1000	38.7	9000

54	27.6000	35.9	9050
55	28.1000	33.4	9100
56	28.6000	31.1	9150
57	29.1000	29.1	9200
58	29.6000	27.3	9250
59	30.1000	25.6	9300
60	30.6000	24.2	9350
61	31.1000	22.8	9400
62	31.6000	21.6	9450
63	32.1000	20.5	9500
64	32.6000	19.4	9550
65	33.1000	18.5	9600
66	33.6000	17.7	9650
67	34.1000	16.9	9700
68	34.6000	16.2	9750
69	35.1000	15.5	9800
70	35.6000	14.9	9850
71	36.1000	14.4	9900
72	36.6000	13.9	9950
73	37.1000	13.4	10000
74	37.6000	13.0	10050
75	38.1000	12.6	10100
76	38.6000	12.2	10150
77	39.1000	11.8	10200
78	39.9250	11.1	10250
79	39.5500	10.5	10300
80	39.7775	9.9	10350
81	40.0000	9.4	10400
82	40.2250	8.9	10450
83	40.4500	8.4	10500
84	40.6750	8.0	10550
85	40.9000	7.6	10600
86	41.1250	7.2	10650
87	41.3500	6.9	10700
88	41.5750	6.6	10750
89	41.8000	6.3	10800
90	42.0250	6.0	10850
91	42.2500	5.8	10900
92	42.4750	5.6	10950
93	42.7000	5.4	11000
94	42.9250	5.2	11050
95	43.1500	5.0	11100
96	47.7750	5.0	11150