Monte Carlo SSA for extracting weak signals

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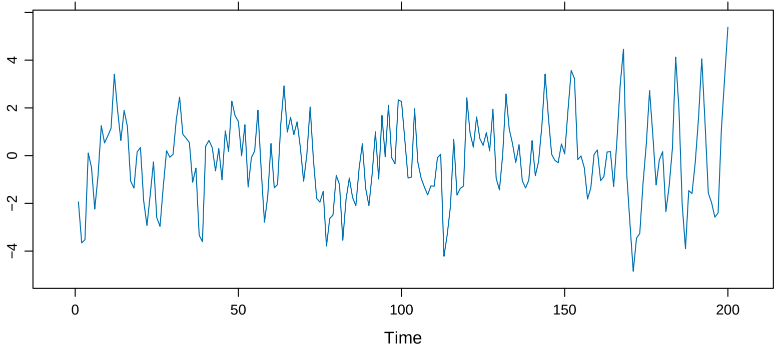
Is There a Signal?



Question: is it pure noise, or is there a signal?

Let $\mathsf{X} = (x_1, \ldots, x_N)$, $x_i \in \mathbb{R}$ be a time series.

Given: $\mathsf{X} = \mathsf{T} + \mathsf{H} + \mathsf{R}$, where T is a trend, H is a periodic component, and R is a noise.

Problems:

1. How to test for the presence of a signal $\mathsf{S} = \mathsf{T} + \mathsf{H}$?
2. How to extract the signal S, if present?

Methods:

1. Monte Carlo SSA (MC-SSA) [Allen and Smith, 1996] — tests $H_0 : \mathsf{S} = 0$.
2. Singular spectrum analysis (SSA) [Golyandina, Nekrutkin and Zhigljavsky, 2001].

Goal: to implement an algorithm for automatic signal extraction based on MC-SSA

For $\mathsf{X} = (x_1, \ldots, x_N)$, fix L $(1 < L < N)$.

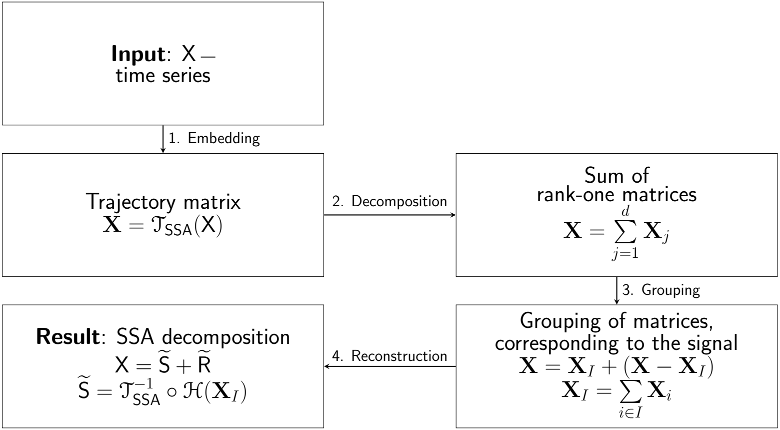
Embedding operator $\mathcal{T}_{\mathsf{SSA}}$:

$$\mathcal{T}_{\mathsf{SSA}}(\mathsf{X}) = \mathbf{X} = \begin{pmatrix} x_1 & x_2 & \cdots & x_K \\ x_2 & x_3 & \cdots & x_{K+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_L & x_{L+1} & \cdots & x_N \end{pmatrix},$$

where $K = N - L + 1$.

Hankelization operator $\mathcal{H}$ — averaging the matrix over its anti-diagonals.

Input: time series $\mathsf{X} = (x_1, \ldots, x_N)$.
Parameters: window length L, index set $I \subset \{1, \ldots, d\}$.
Output: signal estimate.



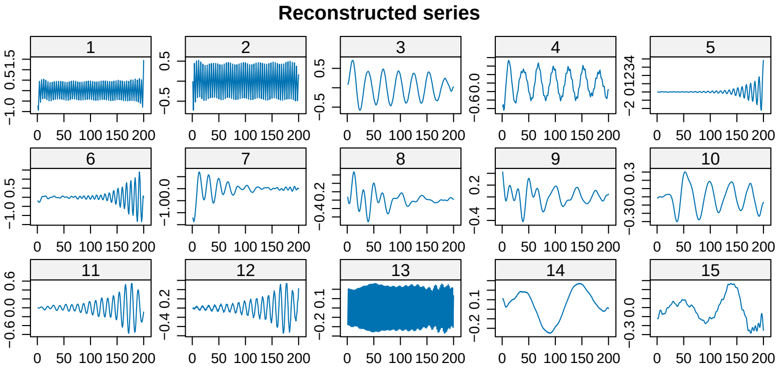


Figure: Elementary reconstructed components $(L = 100)$

Components corresponding to the signal: 1, 2, 5, 6 and 13.

Notations & Known Results: Monte Carlo SSA

Input: $\mathsf{X} = \mathsf{S} + \mathsf{R}$, where S is the signal and R is a realization of a zero-mean stationary process $\boldsymbol{\xi}$ with spectral density $f_{\boldsymbol{\theta}}$.

Parameters: window length L, normalized vectors corresponding to specific frequencies $W_1, \ldots, W_M \in \mathbb{R}^L$.

Test statistics:

$$\widehat{p}_k = \left\|\mathbf{X}^{\mathrm{T}} W_k\right\|^2.$$

Its distribution under H_0 is generally unknown and is estimated via Monte Carlo.

Multiple MC-SSA [Golyandina, 2023]: modification with multiple comparisons correction.

We chose sine waves with equidistant frequencies $\omega_k = k/(2L)$, $k = 1, \ldots, L$ as projection vectors.

Noise parameters $\boldsymbol{\theta}$ are generally unknown and must be estimated.

Parameter estimates can be obtained by maximizing the Whittle's likelihood [Whittle, 1953]:

$$\ell_W(\boldsymbol{\theta}) = -\frac{1}{m}\sum_{j=1}^{m}\left(\ln f_{\boldsymbol{\theta}}(\omega_j) + \frac{I_N(\omega_j)}{f_{\boldsymbol{\theta}}(\omega_j)}\right),$$

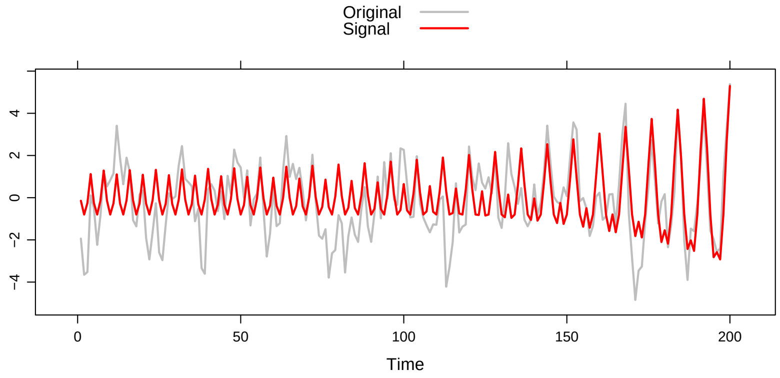
where $m = \lfloor (N-1)/2 \rfloor$, $f_{\boldsymbol{\theta}}$ is the spectral density of $\boldsymbol{\xi}$, I_N is the periodogram of the original series, and $\omega_j = j/N$.

Estimation can be done on part of the spectrum: let $J = \{j_1, \ldots, j_p\}$ be frequency indices we want to exclude when estimating parameters. Then $\ell_W(\boldsymbol{\theta})$ is computed only over indices $j \notin J$.

Example: Applying Monte Carlo SSA

$\mathsf{X} = \mathsf{S} + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is red noise with parameters $\phi = 0.7$ and $\sigma^2 = 1$, $N = 200$,

$$s_n = 0.075\, e^{0.02n} \cos(2\pi n/8) + 2\cos(2\pi n/4) + 0.2 \cdot (-1)^n.$$



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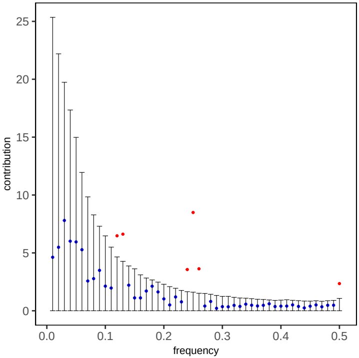


Figure: True noise model

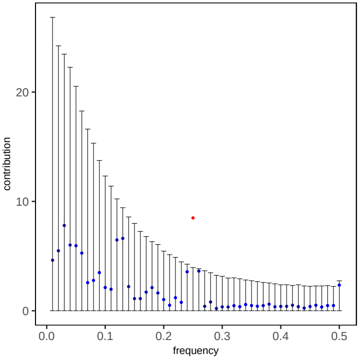


Figure: Estimated noise model

Problem: not all frequencies are detected during parameter estimation.

Solution: apply the test iteratively, extracting one harmonic at a time until $H_0 : \mathsf{S} = 0$ is no longer rejected.

For a series X of length N and $0 \leqslant \omega_1 \leqslant \omega_2 \leqslant 0.5$, define

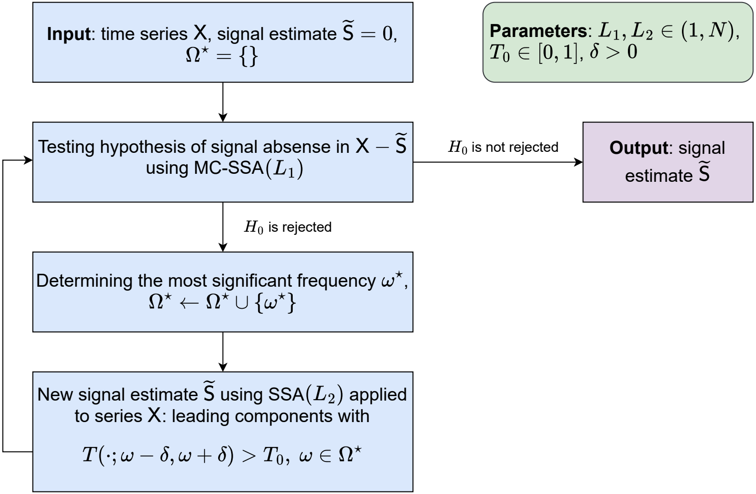
$$T(\mathsf{X}; \omega_1, \omega_2) = \frac{1}{\|\mathsf{X}\|^2} \sum_{k:\omega_1 \leqslant k/N \leqslant \omega_2} I_N(k/N),$$

where I_N is the periodogram of X.

$T(\mathsf{X}; \omega_1, \omega_2)$ may be interpreted as the fraction of total contribution of frequencies within $[\omega_1, \omega_2]$.

Let $\omega^\star$ be a significant frequency. Then we take $[\omega_1, \omega_2] = [\omega^\star - \delta, \omega^\star + \delta]$.

The autoMCSSA Algorithm



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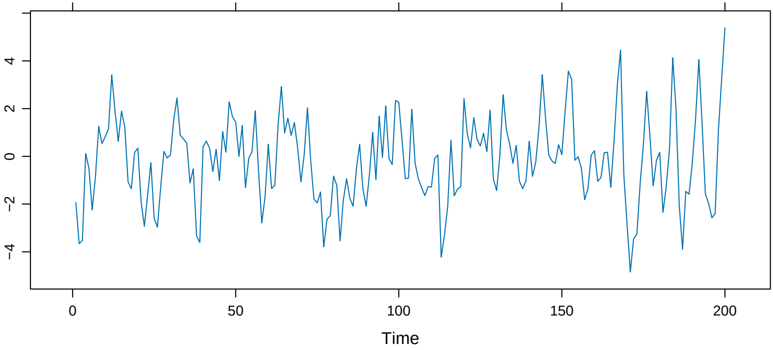

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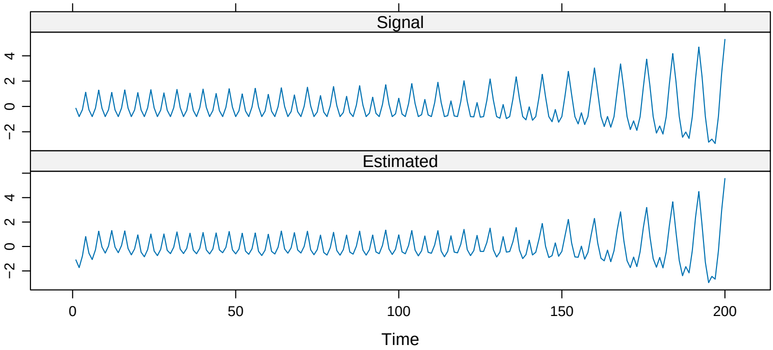

Parameters: $L_1 = 50$, $L_2 = 100$, $\delta = 1/80$, $T_0 = 0.5$.

Result: autoMCSSA correctly identified significant components (1, 2, 5, 6 and 13).

Conclusions

To sum up, we:

1. Implemented autoMCSSA, which automatically extracts a significant signal, plus a modification of the Whittle approach using part of the spectrum.

2. Found that autoMCSSA can extract signals whose SSA components are not necessarily dominant.

In the future, we plan to formulate a strategy for selecting autoMCSSA parameters.