Economics 5330, Fall 2016

Review Notes: Vector Autoregressive Models

N.B.: I produced these notes originally for my advanced econometrics class. The presentation uses some advanced concepts and is expressed in matrix form. But for those would like to have a brief overview at a slightly deeper level on Vector Autoregressive Models this set of notes might prove helfpul.

This is a review of Vector Autoregressive Models (VARs). The purpose is to provide a review and to fix notation and ideas. We will assume, unless stated otherwise, that we are dealing with a stationary VAR(p) process. These short notes are not meant to replace your classroom notes, but rather to supplement, summarize, and to correct notation. I hope they are helpful to you.

Reduced-Form VARs

A reduced-form VAR(p) consists of K endogenous variables, $y_t = (y_{1t}, y_{2t}, \ldots, y_{Kt})'$, and can be written as:

$$y_t = \nu + A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + u_t$$

For $K = 2$ (bivariate) a VAR(p) model in matrix form is the following:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \times \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \cdots + \begin{bmatrix} a_{11}^{(p)} & a_{12}^{(p)} \\ a_{21}^{(p)} & a_{22}^{(p)} \end{bmatrix} \times \begin{bmatrix} y_{1t-p} \\ y_{2t-p} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

We can obtain the variance-covariance matrix as $E[u_t u_t'] = \Omega$.

Note: A VAR model is a Seemingly Unrelated Regressions (SUR) model with each equation containing the same regressors. In this special case we noted that there is no additional efficiency gained by systems estimation (using GLS). So the optimal estimation strategy is to apply OLS equation-by-equation to VAR system. Estimating the VAR system in this form is particularly convenient.

Vector Moving Average Form of the Reduced-Form VAR

We noted that every stationary VAR can be represented as an infinite-order Vector Moving Average (VMA) model. The model in this form is given by the following:

$$y_t = \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i}$$

in which $\mu = \sum_{i=0}^{\infty} \Phi_i \nu$.

Note: Typically the model is estimated in the VAR(p) form and "inverted" to the VMA form.

To go from the VAR(p) form to the VMA form one applies the following recursions:

$$\Phi_s = \sum_{j=1}^{s} \Phi_{s-j} A_j, \quad \text{for } s = 1, 2, \ldots,$$

in which $\Phi_0 = I_K$ and $A_j = 0$ for $j > p$.

We noted that we desire the VMA form of the model because:

- It is most convenient to calculate variances in this form because the VMA is a linear combination of indenpendent terms. Thus, there are no covariance terms in the calculations.
- We would like to interpret the ϕ_{ij} elements of the Φ_i matrices as dynamic multipliers and conduct impulse response analysis.

Structural Vector Autoregressive Models (SVARs)

We noted that before we could interpret the ϕ_{ij}'s as dynamic multipliers that we typically need to impose some structure on the VAR model. That is because unless the variance-covariance matrix, Ω, is diagonal, the ϕ_{ij} cannot be cleanly interpreted as an exogenous unexpected shock to the u_{it}.

We specify an SVAR(p) as follows:

$$By_t = \gamma + \Gamma_1 y_{t-1} + \Gamma_2 y_{t-2} + \cdots + \Gamma_p y_{t-p} + \epsilon_t$$

The structural variance-covariance matrix is D, which is diagonal. This gives us the following mapping from reduced-form to structural parameters:

$$\begin{aligned} &\nu = B^{-1}\gamma \\ &A_i = B^{-1}\Gamma_i, \quad \text{for i = 1, 2, \ldots, p} \\ &\Omega = B^{-1} D B^{-1\prime} \end{aligned}$$

Notes: Typically, without some kind of restrictions the structural parameters are not identified by the reduced-form parameters.

For $K = 2$ (bivariate) an SVAR(p) model in matrix form is the following:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} \gamma_{11}^{(1)} & \gamma_{12}^{(1)} \\ \gamma_{21}^{(1)} & \gamma_{22}^{(1)} \end{bmatrix} \times \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \cdots + \begin{bmatrix} \gamma_{11}^{(p)} & \gamma_{12}^{(p)} \\ \gamma_{21}^{(p)} & \gamma_{22}^{(p)} \end{bmatrix} \times \begin{bmatrix} y_{1t-p} \\ y_{2t-p} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

One way that we discussed to impose some structure on a VAR is to estimate a VAR in Sim's triangular form (for an SVAR(1) model):

$$
\begin{aligned}
y_{1t} &= \gamma_1 + \gamma_{11}^{(1)} y_{1t-1} + \gamma_{12}^{(1)} y_{2t-1} + \epsilon_{1t} \\
y_{2t} &= \gamma_2 - b_{21} y_{1t} + \gamma_{21}^{(1)} y_{1t-1} + \gamma_{22}^{(1)} y_{2t-1} + \epsilon_{2t}
\end{aligned}
$$

We noted that the algebra of OLS will guarantee that the variance-covariance matrix of the model in this form will be diagonal. The structural matrix B from this model will look like the following:

$$
B = \begin{bmatrix} 1 & 0 \\ b_{21} & 1 \end{bmatrix}
$$

Note: Sim's triangular form imposes a zero restriction on the parameter b_{12}. This is enough structure to allow us to identify the structural parameters.

We note that the triangular form imposes a recursive causal order. That is, the variable y_{1t} has a contemporaneous effect on the variable y_{2t}, but y_{2t} does not have a contemporaneous effect on y_{1t}. With the model in this form we could proceed to estimatation equation-by-equation with OLS.

Identification of the SVAR from the Reduced-Form VAR and the Cholesky Decomposition

The triangular matrix decomposition (a special case of the Cholesky decomposition) decomposes a positive semi-definite matrix as follows:

$$
\Omega = T \Lambda T'
$$

For which,

$$
T = \begin{bmatrix} 1 & 0 \\ t_{21} & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
$$

We can use this result to identify the SVAR model as follows:

- Estimate each equation of the reduced-form VAR with OLS.
- Apply the triangular decomposition to the estimated variance-covariance matrix.

This gives us the following psuedo-structural VAR:

$$
\begin{aligned}
T^{-1}y_t &= T^{-1}\nu + T^{-1}A_1 y_{t-1} + \cdots + T^{-1}A_p y_{t-p} + T^{-1}u_t \\
By_t &= \gamma + \Gamma_1 y_{t-1} + \cdots + \Gamma_p y_{t-p} + \epsilon_t
\end{aligned}
$$

Which one can see is equivalent to the SVAR model. The important thing to note here is that the order of the variables as they enter the reduced-form VAR model, $y_t = (y_{1t}, y_{2t})'$, determines the recursive causal ordering of the identified SVAR.

Structural Vector Moving Average Form

Just as with the reduced-form VAR, the stationary SVAR model implies an infinite-order structural vector moving average form:

$$
\begin{aligned}
y_t &= \sum_{i=0}^{\infty} \Theta_i \gamma + \sum_{i=0}^{\infty} \Theta_i \epsilon_{t-i} \\
&= \mu + \sum_{i=0}^{\infty} \Theta_i \epsilon_{t-i}
\end{aligned}
$$

The Θ_j matrices can be obtained from the following recursions:

$$
\Theta_j = \Phi_j B^{-1}, \quad \text{for } i = 1, 2, \ldots,
$$

Now we can interpret the θ_{ij} parameters as dynamic multipliers and conduct policy analysis with impulse response functions and forecast error variance decompositions.