The Natural Exponential Function

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Introduction

This note is a quick introduction to the *natural exponential function*.

See the following wikipedia articles for more details:

- https://en.wikipedia.org/wiki/Exponential_function
- https://en.wikipedia.org/wiki/E_(mathematical_constant)

Exponential Functions

Exponential functions take the following form:

$$f(x) = b^x$$

in which the parameter b is called the base and the variable x occurs as an exponent.

One particularly helpful exponential function is called the *natural exponential function*.

Brief Review of Compounding Relations

Recall from your introductory finance courses that we use the following formula to calculate periodic compounding:

$$\left(1 + \frac{r}{n}\right)^{n \times T}$$

We get more and more frequent compounding as n increases for some interest rate r and time horizon T. What happens as $n \to \infty$? We can use calculus to solve this problem.

But first, let's just use a simple computer experiment to investigate what happens. Let's look at a simplified version of the above formula:

$$\left(1 + \frac{1}{x}\right)^{x}$$

For this investigation let's first set up a user-defined function in `R` for the compounding function:

```
f <- function(x)
{
  result <- (1 + (1/x))^x
  return(result)
}
```

Now, let's use the function for larger and larger values of x. Let's increase by powers of 10 to speed things up:

```r
powers <- 1:8
base <- 10

for(i in powers)
{
  value <- f(base^i)
  print(value)
}
```

```
## [1] 2.593742
## [1] 2.704814
## [1] 2.716924
## [1] 2.718146
## [1] 2.718268
## [1] 2.71828
## [1] 2.718282
## [1] 2.718282
```

From this computer experiment we can see that the values converge to a certain value, which we will denote as $e \approx 2.718282$. The number e is called ***Euler's number*** for the mathematician Euler who first discovered it.

This is a simple way to intuitively check the result from calculus for continuous compounding:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n\times T} = e^{r\times T}$$

What is Natural about the Natural Exponential Function?

So what is *natural* about the *natural exponential function*? It's that as we let the value of x go to infinity we get the exponential function with e as it's base. It's thus a natural choice for the base, and we call it the natural exponential function. For finance this is indeed a natural choice as it gives continuous compounding.

It is also mathematically convenient to calculate the future value of a continuouslly compounded rate, r and maturity T as:

$$FV = e^{r\times T}$$

where T is the maturity of the cash flows (expressed in annual terms, i.e. one year is $T = 1$).

We can also get present values that are continuouslly compounded as follows:

$$PV = e^{-r\times T}$$