Solving linear systems on an industrial scale

- Matrix inversion $\mathbf{A}^{-1}$
- Brief revision: Gaussian reduction
- Matrix $\mathbf{A} = \mathbf{LU}$ decomposition (factorisation)
- General decomposition $\mathbf{PA} = \mathbf{LU}$
- Connection between Gaussian reduction and $\mathbf{LU}$ factorisation
  - Crout's method
  - Doolittle's method
  - Cholesky's method
- $\mathbf{LDU}$ decomposition

Matrix inversion method

- A system of n equations with n unknowns

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n. \end{cases}$$

  can be written in a matrix form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- So $\mathbf{Ax} = \mathbf{b}$ and therefore $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ (if $\mathbf{A}$ is invertible)
- Inversion is very slow for large matrices ($2n^3$ operations)

Revision: Gaussian reduction / elimination

Row operations that can be used:

- Swapping two rows
- Adding a multiple of one row to another
- Multiplying a row by a constant

With these operations, matrix is first reduced to echelon form (EF):

- All non-zero rows are above any rows of all zeroes.
- Each leading entry (first non-zero entry) of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeroes.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Revision: Gaussian reduction / elimination

The next step is to obtain a reduced echelon form (REF):

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

REF matrix, in addition to EF form, has the properties

- The leading entry in each non-zero row is 1
- The leading 1 is the only non-zero entry in its column

An echelon form is not unique, however reduced echelon form is:

Theorem: Any matrix is row-equivalent to one and only one matrix in reduced echelon form.

Revision: Gaussian reduction / elimination

Reduction into EF and then REF is achieved via standard steps:

1. Select the leftmost non-zero entry for the first pivot.
   If necessary, swap rows to move the first pivot to the first row.
2. Set the leftmost non-zero entry (in the upper row) as a pivot.
3. Add multiples of the pivot row to the rows below,
   to create zeros in all positions below the pivot.
4. Cover the row containing the pivot (and any rows above it).
   Apply the same steps 2–4 to the remaining sub-matrix.
   Repeat until there are no more non-zero rows to modify.
5. Beginning with the rightmost pivot and working upward and
   to the left, create zeroes above each pivot.
6. If a pivot is not 1, make it 1 by a row scaling operation.

Generally, it needs $2n^3/3$ operations (3 times faster than inversion).

Elementary matrices

Consider a general matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and the following examples of **elementary** matrices:

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \quad \mathbf{E}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

The actions of these elementary matrices over $\mathbf{A}$ are defined by matrix multiplications from the left:

$$\mathbf{E}_1\mathbf{A} \qquad \mathbf{E}_2\mathbf{A} \qquad \mathbf{E}_3\mathbf{A}$$

Elementary matrices

The action of the elementary matrix $\mathbf{E}_1$ over $\mathbf{A}$ is as follows:

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + k \cdot a_{31} & 0 \cdot a_{12} + 0 \cdot a_{22} + k \cdot a_{32} & 0 \cdot a_{13} + 0 \cdot a_{23} + k \cdot a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

This corresponds to an elementary row operation:

$$\mathbf{E}_1\mathbf{A} \quad \Leftrightarrow \quad \mathrm{R}_3 \to kR_3 \qquad \text{multiply row by a factor}$$

Elementary matrices

The action of the elementary matrix $\mathbf{E}_2$ over $\mathbf{A}$ is as follows:

$$\mathbf{E}_2\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} \\ 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This corresponds to an elementary row operation:

$$\mathbf{E}_2\mathbf{A} \quad \Leftrightarrow \quad \mathrm{R}_1 \leftrightarrow R_2 \qquad \text{swap two rows}$$

Elementary matrices

The action of the elementary matrix $\mathbf{E}_3$ over $\mathbf{A}$ is as follows:

$$\mathbf{E}_3\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} \\ 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} \\ k \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} & k \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} & k \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + ka_{11} & a_{32} + ka_{12} & a_{33} + ka_{13} \end{bmatrix}$$

This corresponds to an elementary row operation:

$\mathbf{E}_3\mathbf{A} \quad \Leftrightarrow \quad \mathrm{R}_3 \to (R_3 + kR_1) \qquad$ add a multiple of a row to another

Elementary matrices

Other examples:

$$\mathbf{E}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{E}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{E}_6 = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, generally,

$$\begin{array}{cclr} e_{ii} = k & \Leftrightarrow & \mathrm{R}_i \to kR_i & \text{multiplies row } i \text{ by } k \\ \begin{cases} e_{ij} = e_{ji} = 1 \\ e_{ii} = e_{jj} = 0 \end{cases} & \Leftrightarrow & \mathrm{R}_i \leftrightarrow R_j & \text{swaps rows } i \text{ and } j \\ e_{ij} = k & \Leftrightarrow & \mathrm{R}_i \to (R_i + kR_j) & \text{adds } k \text{ times row } j \text{ to row } i \end{array}$$

Subsequent multiplication (from the left) $\mathbf{E}_m \ldots \mathbf{E}_2\, \mathbf{E}_1\, \mathbf{A}$ by a chain of appropriate elementary matrices eventually reduces $\mathbf{A}$ into REF.

LU factorisation (decomposition)

- Quite often, one needs to solve a number of linear systems $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$ for different $\mathbf{b}_i$ but with the same matrix $\mathbf{A}$.
- It would be inefficient to reduce $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$ to REF each time.
- **LU factorisation** provides a quicker method to solve the system $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$ for a number of vectors $\mathbf{b}_i$.
- If we can reduce a square matrix $\mathbf{A}$ to echelon form without row swaps, then it can be written as the product of an upper triangular matrix $\mathbf{U}$ and a lower triangular matrix $\mathbf{L}$:

$$\mathbf{A} = \mathbf{L}\mathbf{U}$$

(slightly more complicated if we need to also use row swaps).

LU factorisation

To solve the system $\mathbf{Ax} = \mathbf{b}$ we express $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix},$$

so the system can be written as: $\mathbf{Ax} = (\mathbf{LU})\mathbf{x} = \mathbf{L}(\mathbf{Ux}) = \mathbf{b}$.

Letting $\mathbf{Ux} = \mathbf{y}$ we get $\mathbf{L}(\mathbf{Ux}) = \mathbf{Ly} = \mathbf{b}$.

In this way, we obtain two equations to solve instead of one:

$$\begin{cases} \mathbf{Ly} = \mathbf{b} \\ \mathbf{Ux} = \mathbf{y} \end{cases}$$

however each of these is much quicker to solve.

LU factorisation

We solve $\mathbf{Ly} = \mathbf{b}$ first. This is easy because $\mathbf{L}$ is triangular:

$$\mathbf{Ly} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

We find the solution by forward substitution.

Then we can solve $\mathbf{Ux} = \mathbf{y}$. Also easy, because $\mathbf{U}$ is triangular:

$$\mathbf{Ux} = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

We solve this system by backward substitution.

LU factorisation

Summary: To solve the system $\mathbf{Ax} = \mathbf{b}$:

1. Obtain, if possible, matrices $\mathbf{L}$ and $\mathbf{U}$ such that $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix. Then $\mathbf{LUx} = \mathbf{b}$.
2. Assuming $\mathbf{y} = \mathbf{Ux}$, solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ using forward substitution.
3. Having obtained $\mathbf{y}$, solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$ using backward substitution.

The question now is, how to obtain the required LU factorisation.

Gaussian reduction and LU factorisation

Example: row-reduce a fun matrix $\mathbf{A} = \{1,\ 2,\ 3;\ 4,\ 5,\ 6;\ 7,\ 8,\ 9\}$

- To eliminate $a_{21} = 4$ we use $\mathbf{E}_{21}\big|_{-4}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

- To eliminate $a_{31} = 7$ we use $\mathbf{E}_{31}\big|_{-7}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

- To eliminate $a_{32} = -6$ we use $\mathbf{E}_{32}\big|_{-2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Gaussian reduction and LU factorisation

- These subsequent multiplications reduce $\mathbf{A}$ into echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

- Multiplication of such elementary matrices produces a low triangular matrix. Regarding the last equation as $\mathbf{L}^{-1}\mathbf{A} = \mathbf{U}$

where $\quad \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & -2 & 1 \end{bmatrix} \quad$ and so $\quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 15 & 2 & 1 \end{bmatrix}$

this result provides a factorisation $\mathbf{A} = \mathbf{LU}$.

Gaussian reduction and LU factorisation

- The row reduction above can be expressed in matrix form as

$$\mathbf{E}_{32}\,\mathbf{E}_{31}\,\mathbf{E}_{21}\,\mathbf{A} = \mathbf{U}$$

- This is equivalent to

$$\mathbf{A} = \left(\mathbf{E}_{32}\,\mathbf{E}_{31}\,\mathbf{E}_{21}\,\right)^{-1}\mathbf{U} = \mathbf{E}_{21}^{-1}\,\mathbf{E}_{31}^{-1}\,\mathbf{E}_{32}^{-1}\,\mathbf{U} = \mathbf{L}\mathbf{U}$$

- So with a single row-reduction algorithm, we have obtained

$$\mathbf{L} = \mathbf{E}_{21}^{-1}\,\mathbf{E}_{31}^{-1}\,\mathbf{E}_{32}^{-1}$$

$$\mathbf{U} = \mathbf{E}_{32}\,\mathbf{E}_{31}\,\mathbf{E}_{21}\,\mathbf{A}$$

- The inverted elementary matrices however are very easy.

Gaussian reduction and LU factorisation

- Inverted matrices $\mathbf{E}_{ij}^{-1}$ are very easy to construct. Their actions just revert the original $\mathbf{E}_{ij}$ operation:

$$\mathbf{E}_{32}\big|_{-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad \mathbf{E}_{32}^{-1}\big|_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\mathbf{E}_{31}\big|_{-7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_{31}^{-1}\big|_{7} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E}_{21}\big|_{-4} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_{21}^{-1}\big|_{4} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- It's easy to see that $\mathbf{E}_{ij}\big|_{k}\ \mathbf{E}_{ij}^{-1}\big|_{-k} = \mathbf{E}_{ij}^{-1}\big|_{-k}\ \mathbf{E}_{ij}\big|_{k} = I$ where $\mathbf{I}$ is an identity matrix.

Gaussian reduction and LU factorisation

- If we also use row scaling, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

then the corresponding inverted matrix is also straightforward:

$$\mathbf{E}_{ii}^{-1}\Big|_{\frac{1}{k}} = \mathbf{E}_{ii}\Big|_{k}$$

$$\mathbf{E}_{22}\Big|_{-\frac{1}{3}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_{22}^{-1}\Big|_{-3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Permutation (row swapping)

- In general, Gaussian reduction process may require row swapping. This is done by permutation matrices.
- Permutation matrix $\mathbf{P}_{ij}$ is constructed from identity matrix by swapping rows i and j; and of course $\mathbf{P}_{ij} = \mathbf{P}_{ji}$:

$$\mathbf{P}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{P}_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{P}_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Quite naturally, an inverted permutation matrix equals to the original matrix,

$$\mathbf{P}_{ij}^{-1} = \mathbf{P}_{ji} = \mathbf{P}_{ij}$$

Gaussian reduction and LU factorisation

- Any elementary row operation on $\mathbf{A}$ is represented by pre-multiplication of $\mathbf{A}$ by a suitable elementary matrix.
- Hence Gaussian reduction is a sequence of pre-multiplications.
- Apart from row swaps, elementary matrices (and their inverses) are all lower triangular.
- Thus, Gaussian reduction is equivalent to $\mathbf{A} = \mathbf{LU}$ process.
- In case row swaps $\mathbf{PA}$ are required, these are performed first.
- So in the most general case, $\mathbf{PA} = \mathbf{LU}$

LU factorisation methods

- There is no unique way of factorising a matrix into a product of upper and lower triangular matrices $\mathbf{L}$ and $\mathbf{U}$. To get a unique decomposition, one can impose additional conditions.
- **Crout's** method implies that the diagonal elements of the upper triangular matrix $\mathbf{U}$ are equal to 1.
- **Doolittle's** method by contrast, requires that the diagonal elements of the lower triangular matrix $\mathbf{L}$ are equal to 1.

We will now have a look at these two methods in more detail.

Crout's method

For Crout's method, we have the following theorem on the existence of the decomposition:

Theorem

If, for an $n \times n$ matrix $\mathbf{A}$, all the n sub-matrices

$$\Delta_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}, \quad 1 \leqslant k \leqslant n, \quad \text{are invertible,}$$

then there exists a lower triangular matrix $\mathbf{L} = \{l_{ij}\}$ and an upper triangular matrix $\mathbf{U} = \{u_{ij}\}$ with $u_{ii} = 1 \ \forall i$, such that

$$\mathbf{A} = \mathbf{LU}.$$

Moreover, this factorisation is unique.

Crout's method

3×3 case: We want to obtain **A** as a product of **L** (lower triangular) and **U** (upper triangular) where the diagonal elements of **U** are equal to 1:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

To find the u_{ij} and l_{ij} we multiply the **L** and **U** matrices:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

Crout's method

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

This provides the following equations for the entries of $\mathbf{L}$ and $\mathbf{U}$:

$$l_{11} = a_{11}, \qquad l_{21} = a_{21}, \qquad l_{31} = a_{31}$$

$$\begin{aligned} l_{11}u_{12} &= a_{12} & \Rightarrow \quad & u_{12} = a_{12}/l_{11} \\ l_{21}u_{12} + l_{22} &= a_{22} & \Rightarrow \quad & l_{22} = a_{22} - l_{21}u_{12} \\ l_{31}u_{12} + l_{32} &= a_{32} & \Rightarrow \quad & l_{32} = a_{32} - l_{31}u_{12} \\ l_{11}u_{13} &= a_{13} & \Rightarrow \quad & u_{13} = a_{13}/l_{11} \\ l_{21}u_{13} + l_{22}u_{23} &= a_{23} & \Rightarrow \quad & u_{23} = (a_{23} - l_{21}u_{13})/l_{22} \\ l_{31}u_{13} + l_{32}u_{23} + l_{33} &= a_{33} & \Rightarrow \quad & l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \end{aligned}$$

Crout's method

The extension of this method to an $n \times n$ matrix is straightforward.

Algorithm:

For $\mathbf{A} = \mathbf{LU}$ decomposition, matrix elements are determined by

$$u_{ii} = 1 \qquad i = 1, 2, \ldots, n$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad i \geqslant j = 1, 2, \ldots, n$$

$$u_{ij} = \frac{1}{l_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right) \qquad i < j = 2, 3, \ldots, n.$$

Crout's method (example)

Decompose the following matrix using Crout's method:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 2 & 2 \end{pmatrix}$$

$$l_{11} = a_{11} = 2, \qquad l_{21} = a_{21} = 4, \qquad l_{31} = a_{31} = 3$$

$$u_{12} = a_{12}/l_{11} = -\frac{1}{2} \qquad u_{13} = a_{13}/l_{11} = \frac{1}{2}$$

$$l_{22} = a_{22} - l_{21}u_{12} = 5 \qquad l_{32} = a_{32} - l_{31}u_{12} = \frac{7}{2}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = -\frac{3}{5} \qquad l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = \frac{13}{5}$$

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 3 & 7/2 & 13/5 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{LU}$$

Doolittle's method

3×3 case: We want to obtain **A** as a product of **L** (lower triangular) and **U** (upper triangular), now with the diagonal elements of **L** equal to 1:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

To find the u_{ij} and l_{ij} we multiply the **L** and **U** matrices:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

Doolittle's method

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

This provides the following equations for the entries of **L** and **U**:

$$u_{11} = a_{11}, \qquad u_{12} = a_{12}, \qquad u_{13} = a_{13}$$

$$\begin{aligned} l_{21}u_{11} = a_{21} \quad &\Rightarrow \quad l_{21} = a_{21}/u_{11} \\ l_{21}u_{12} + u_{22} = a_{22} \quad &\Rightarrow \quad u_{22} = a_{22} - l_{21}u_{12} \\ l_{21}u_{13} + u_{23} = a_{23} \quad &\Rightarrow \quad u_{23} = a_{23} - l_{21}u_{13} \\ l_{31}u_{11} = a_{31} \quad &\Rightarrow \quad l_{31} = a_{31}/u_{11} \\ l_{31}u_{12} + l_{32}u_{22} = a_{32} \quad &\Rightarrow \quad l_{32} = (a_{32} - l_{31}u_{12})/u_{22} \\ l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \quad &\Rightarrow \quad u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \end{aligned}$$

Doolittle's method

The extension of this method to $n \times n$ is also straightforward.

Algorithm: For $\mathbf{A} = \mathbf{LU}$ decomposition, matrix elements are

- For $i = 1, 2, \ldots, n$: $\qquad l_{ii} = 1 \qquad$ (diagonal of **L**);
- For $k = 1, 2, \ldots, n$:
  - Diagonal elements of **U**:
    $$u_{kk} = a_{kk} - \sum_{m=1}^{k-1} l_{km} u_{mk}$$
  - k-th column of **L**:
    $$l_{ik} = \frac{1}{u_{kk}} \left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk} \right), \qquad k \leqslant i \leqslant n$$
  - k-th row of **U**:
    $$u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}, \qquad k \leqslant j \leqslant n$$

Doolittle's method (example)

Use Doolittle's **LU** factorisation to find the solution for system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$u_{11} = a_{11} = 1, \qquad u_{12} = a_{12} = -2, \qquad u_{13} = a_{13} = 1,$$
$$l_{21} = a_{21}/u_{11} = 0, \qquad l_{31} = a_{31}/u_{11} = -4,$$
$$u_{22} = a_{22} - l_{21}u_{12} = 2, \qquad u_{23} = a_{23} - l_{21}u_{13} = -8,$$
$$l_{32} = (a_{32} - l_{31}u_{12})/u_{22} = -3/2,$$
$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 1$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -\frac{3}{2} & 1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Doolittle's method (example)

So $\mathbf{A} = \mathbf{L}\mathbf{U}$ with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -\frac{3}{2} & 1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can split $\mathbf{A}\mathbf{x} = \mathbf{b}$ into solving $\mathbf{L}\mathbf{y} = \mathbf{b}$ and then $\mathbf{U}\mathbf{x} = \mathbf{y}$.

The $\mathbf{L}\mathbf{y} = \mathbf{b}$ equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

With forward substitution we obtain $y_1 = 0,\ y_2 = 8,\ y_3 = 3$

Doolittle's method (example)

Now we use $\mathbf{U}\mathbf{x} = \mathbf{y}$ to find $\mathbf{x}$:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix}$$

With backward substitution we get $x_3 = 3$, $x_2 = 16$, $x_1 = 29$.

So the final solution is

$$\mathbf{x} = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}.$$

LU factorisation

- Both the Doolittle's and Crout's methods are easy to code and are quite reliable.
- However, for certain matrices, these algorithms can fail.
- In such cases, one can usually swap suitable rows of the matrix and obtain an **LU** decomposition of the permuted matrix:

$$\mathbf{PA} = \mathbf{LU}$$

(swapping rows is an elementary row operation, so this does not change the solution).

An easy criterion for **LU** decomposition

Definition: A square ($n \times n$) matrix $\mathbf{A}$ is **diagonally dominant** if for each $i = 1, 2, \ldots, n$

$$|a_{ii}| > \sum_{j=1,\ j \neq i}^{n} |a_{ij}| \,.$$

Theorem: If an $n \times n$ matrix $\mathbf{A}$ is diagonally dominant, then:

- $\mathbf{A}$ is non-singular (and therefore, is invertible);
- There exist $n \times n$ matrices $\mathbf{L}$ and $\mathbf{U}$ which are lower- and upper-triangular matrices respectively, satisfying $\mathbf{A} = \mathbf{LU}$.

Note: This is a **sufficient**, but not **necessary** condition.

Check these examples: $\begin{pmatrix} -5 & 3 & 1 \\ 2 & -9 & 4 \\ 3 & 2 & 6 \end{pmatrix}$ $\begin{pmatrix} 6 & 2 & -2 \\ 9 & 4 & -5 \\ 3 & 3 & -7 \end{pmatrix}$

Choleski method (pre-requisites)

In many applications of linear algebra, matrices have certain special properties that help solving the associated problems.

For example, **sparse matrices**, in which most of the elements are equal to zero, can be treated by special methods.

Another common type of matrix is a **symmetric** matrix: $a_{ij} = a_{ji}$.

Symmetric matrices are equal to their own transpose: $\mathbf{A}^T = \mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

Choleski method (pre-requisites)

An efficient method for finding the **LU** decomposition of a symmetric **positive definite** matrix is due to Choleski.

Definition: A square $(n \times n)$ matrix $\mathbf{A}$ is **positive definite** if

$$\mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x} > 0 \qquad \forall\ \mathbf{x} \neq \mathbf{0}$$

Remark: A symmetric matrix is positive definite if and only if all its eigenvalues are positive.

If $\mathbf{A}$ is a symmetric matrix which is strictly diagonally dominant, and if $a_{ii} > 0\ \ \forall\ i = 1, 2, \ldots, n$, then $\mathbf{A}$ is positive definite.

For a symmetric matrix, $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$, thus $\mathbf{LU} = (\mathbf{LU})^{\mathsf{T}} = \mathbf{U}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}$.

This suggests that we can decompose $\mathbf{A}$ uniquely in the form $\mathbf{A} = \mathbf{U}^{\mathsf{T}}\mathbf{U}$ where $\mathbf{U}$ is an upper triangular matrix.

Choleski method

Obtaining Cholesky decomposition is straightforward:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \mathbf{U}^{\mathsf{T}}\mathbf{U} = \begin{pmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11}^2 & u_{11}u_{12} & u_{11}u_{13} \\ u_{11}u_{12} & u_{12}^2 + u_{22}^2 & u_{12}u_{13} + u_{22}u_{23} \\ u_{11}u_{13} & u_{12}u_{13} + u_{22}u_{23} & u_{13}^2 + u_{23}^2 + u_{33}^2 \end{pmatrix}$$

From here, we easily find the elements of the **U** matrix:

$$u_{11} = \sqrt{a_{11}}, \quad u_{12} = a_{12}/u_{11}, \quad u_{13} = a_{13}/u_{11},$$

$$u_{22} = \sqrt{a_{22} - u_{12}^2}, \quad u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}},$$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2}.$$

Choleski method

The extension of $\mathbf{A} = \mathbf{U}^\mathsf{T}\mathbf{U}$ decomposition to $n \times n$ is as follows:

$$u_{11} = \sqrt{a_{11}}$$

$$\text{For} \quad j = 2, 3, \dots, n: \qquad u_{1j} = \frac{a_{1j}}{u_{11}}$$

$$\text{For} \quad i = 2, 3, \dots, n: \qquad u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$\left\{ \begin{array}{l} \text{For} \quad i = 2, 3, \dots, n \\ \qquad j = i+1, i+2, \dots, n \end{array} \right. : \qquad u_{ij} = \frac{1}{u_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj} \right)$$

$$\text{For} \quad i > j: \qquad u_{ij} = 0$$

Choleski method (example)

Let us consider an example with symmetric $a_{ij} = a_{ji}$ matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 26 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 55 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 26 \end{bmatrix} = \mathbf{U}^\mathsf{T}\mathbf{U} = \begin{bmatrix} u_{11}^2 & u_{11}u_{12} & u_{11}u_{13} \\ u_{11}u_{12} & u_{12}^2 + u_{22}^2 & u_{12}u_{13} + u_{22}u_{23} \\ u_{11}u_{13} & u_{12}u_{13} + u_{22}u_{23} & u_{13}^2 + u_{23}^2 + u_{33}^2 \end{bmatrix}$$

$$u_{11} = \sqrt{1} = 1, \quad u_{12} = 2/u_{11} = 2, \quad u_{13} = 3/u_{11} = 3,$$

$$u_{22} = \sqrt{5 - u_{12}^2} = 1, \quad u_{23} = (10 - u_{12}u_{13})/u_{22} = 4,$$

$$u_{33} = \sqrt{26 - u_{13}^2 - u_{23}^2} = 1.$$

Choleski method (example)

The decomposed form $\mathbf{Ax} = \mathbf{U}^\mathsf{T}\mathbf{Ux} = \mathbf{b}$ then reads

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 55 \end{bmatrix}$$

We can then solve it as $\mathbf{Ux} = \mathbf{y}$ and $\mathbf{U}^\mathsf{T}\mathbf{y} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 55 \end{bmatrix}$$

Then: $\quad y_1 = 10, \quad y_2 = 26 - 2y_1 = 6, \quad y_3 = 55 - 3y_1 - 4y_2 = 1.$

$$\mathbf{y} = \begin{bmatrix} 10 \\ 6 \\ 1 \end{bmatrix}$$

Choleski method (example)

Now we can solve $\mathbf{Ux} = \mathbf{y}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 1 \end{bmatrix}$$

Then: $\quad x_3 = 1, \quad x_2 = 6 - 4x_3 = 2, \quad x_1 = 10 - 3x_3 - 2x_2 = 3$ and

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 26 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 55 \end{bmatrix} = \mathbf{b}$$

LDU decomposition

- If $\mathbf{A}$ is a square matrix which can be reduced to row echelon form without row swaps, then $\mathbf{A}$ can be factorised uniquely as $\mathbf{A} = \mathbf{LDU}$, where $\mathbf{L}$ is lower triangular, $\mathbf{U}$ upper triangular, and $\mathbf{D}$ is a strictly diagonal matrix.

- This is called an LDU-decomposition of matrix $\mathbf{A}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying the last two matrices leads to Doolittle's method; multiplying the first two matrices leads to Crout's method.

Next lecture

Friday 6 April

9:00

CB 06.03.022

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