Linear Algebra Tutorial 1

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$$\text{Q1}: \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$

- $\mathrm{A} + \mathrm{B} \neq;$

Dimensions[A] $\neq$ Dimensions[B]   [mathematica command, try it by yourself]

- $\mathrm{A} + \mathrm{B}^{\mathrm{T}};$

$$\mathrm{B}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$ ***Transpose[B]***

$$\mathrm{A} + \mathrm{B}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 2 \\ 3 & 2 \end{bmatrix}$$ ***A+B***

continue..

- $AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & & \\ & & \\ & & \end{bmatrix}$ $\boldsymbol{A}B = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & \\ & & \\ & & \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ & & \\ & & \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & & \\ & & \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & \\ & & \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & -1 \\ & & \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & -1 \\ 7 & & \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & -1 \\ 7 & 8 & \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & -1 \\ 7 & 8 & -3 \end{bmatrix}$ *A.B(period for multiplying)*

- $BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \\ & \end{bmatrix}$ $BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ & \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & \end{bmatrix}$ $BA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$ ***B.A***

- $AB^{\mathrm{T}} \neq$

Question 2 (a)

$$
\begin{aligned}
2x_1 - 4x_2 \qquad\quad + \quad x_4 + 7x_5 &= 11 \\
x_1 - 2x_2 - x_3 + \quad x_4 + 9x_5 &= 12 \\
-x_1 + 2x_2 + x_3 + 3x_4 - 5x_5 &= 16 \\
4x_1 - 8x_2 + x_3 - \quad x_4 + 6x_5 &= -2
\end{aligned}
$$

Q2(a) solution steps

- Augmented matrix

$$A=\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 1 & -2 & -1 & 1 & 9 & 12 \\ -1 & 2 & 1 & 3 & -5 & 16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix}$$

- Row operation to echelon form: from leftmost

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 1 & -2 & -1 & 1 & 9 & 12 \\ -1 & 2 & 1 & 3 & -5 & 16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix} \xRightarrow{R_2 \to R_2 + R_3} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ -1 & 2 & 1 & 3 & -5 & 16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix}$$

continue...

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ -1 & 2 & 1 & 3 & -5 & 16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix} \xRightarrow{R_3 \to 2R_3 + R_1} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix} \xRightarrow{R_4 \to R_4 - 2R_1} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 1 & -3 & -8 & -24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 4 & 4 & 28 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 1 & -3 & -8 & -24 \end{bmatrix} \xRightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 0 & 4 & 4 & 28 \end{bmatrix}$$

Echelon form.

Next step: Reduced Echelon Form

Continue…

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 0 & 4 & 4 & 28 \end{bmatrix} \xrightarrow{R_4 \to R_4/4} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 7 & -3 & 43 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 7R_4} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & -3 & -8 & -24 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_4} \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_4} \begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

Continue…

$$\begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 2 & 0 & -10 & -6 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} \begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Continue...

$$\begin{bmatrix} 2 & -4 & 0 & 0 & 6 & 4 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots : basic variables x_1, x_3, x_4

Non-pivots: free variables x_2, x_5

Solutions to the linear system:

$$x_1 = 2x_2 - 3x_5 + 2$$
$$x_3 = 5x_5 - 3$$
$$x_4 = -x_5 + 7$$

Solutions in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 5 \\ -1 \\ 1 \end{pmatrix} x_5 + \begin{pmatrix} 2 \\ 0 \\ -3 \\ 7 \\ 0 \end{pmatrix}$$

Mathematica command: RowReduce[A]

Question 2 (b)

$$
\begin{aligned}
2x_1 - 4x_2 \phantom{{}- x_3} + x_4 + 7x_5 &= 0 \\
x_1 - 2x_2 - x_3 + x_4 + 9x_5 &= 0 \\
-x_1 + 2x_2 + x_3 + 3x_4 - 5x_5 &= 0 \\
4x_1 - 8x_2 + x_3 - x_4 + 6x_5 &= 0
\end{aligned}
$$

Coefficients matrix is the same with (a), so do this by yourself.

Compare this result with (a), check the difference.

Question 2 (c)

$$
\begin{aligned}
x_1 - x_2 - 3x_3 + 8x_4 &= -2 \\
3x_1 \qquad\quad - 3x_3 + 9x_4 &= -1 \\
x_1 + x_2 + x_3 - 2x_4 &= 1
\end{aligned}
$$

Q2(c) solution steps

- Augmented matrix

$$A=\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 3 & 0 & -3 & 9 & -1 \\ 1 & 1 & 1 & -2 & 1 \end{bmatrix}$$

- Row operation to echelon form: from leftmost

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 3 & 0 & -3 & 9 & -1 \\ 1 & 1 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_3} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -3 & -6 & 15 & -4 \\ 1 & 1 & 1 & -2 & 1 \end{bmatrix}$$

continue…

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -3 & -6 & 15 & -4 \\ 1 & 1 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -3 & -6 & 15 & -4 \\ 0 & 2 & 4 & -10 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -3 & -6 & 15 & -4 \\ 0 & 2 & 4 & -10 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2/3,\ R_3 \to R_3/2} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 1 & 2 & -5 & 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 1 & 2 & -5 & 3/2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

Echelon form.

Next step: Reduced Echelon Form

Continue...

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 0 & 0 & 0 & 1/6 \end{bmatrix} \xRightarrow{R_3 \to 6R_3} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & -4/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{R_2 \to R_2 + 4/3 R_3} \begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 & 8 & -2 \\ 0 & -1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & -1 & -3 & 8 & 0 \\ 0 & -1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -3 & 8 & 0 \\ 0 & -1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & -1 & -3 & 8 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

continue...

$$\begin{bmatrix} 1 & -1 & -3 & 8 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivots : basic variables x_1, x_2

Non-pivots: free variables x_3, x_4

Solutions to the linear system:

$$x_1 = x_3 - 3x_4$$
$$x_2 = -2x_3 + 5x_4$$
$$0 = 1$$

No solution due to the third equation.

Question 2 (d)

$$
\begin{aligned}
-x_1 + x_2 + x_3 &= 9 \\
2x_1 + x_2 - x_3 &= -10 \\
3x_1 - 2x_3 &= -19 \\
-x_1 + 2x_2 - 3x_3 &= -10
\end{aligned}
$$

Q2(d) solution steps

- Augmented matrix

$$A=\begin{bmatrix} -1 & 1 & 1 & 9 \\ 2 & 1 & -1 & -10 \\ 3 & 0 & -2 & -19 \\ -1 & 2 & -3 & -10 \end{bmatrix}$$

- Row operation to echelon form: from leftmost

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 2 & 1 & -1 & -10 \\ 3 & 0 & -2 & -19 \\ -1 & 2 & -3 & -10 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 3 & 0 & -2 & -19 \\ -1 & 2 & -3 & -10 \end{bmatrix}$$

continue…

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 3 & 0 & -2 & -19 \\ -1 & 2 & -3 & -10 \end{bmatrix} \xRightarrow{R_3 \to R_3 + 3R_1} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 3 & 1 & 8 \\ -1 & 2 & -3 & -10 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 3 & 1 & 8 \\ -1 & 2 & -3 & -10 \end{bmatrix} \xRightarrow{R_4 \to R_4 - R_1} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & -4 & -19 \end{bmatrix}$$

continue...

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & -4 & -19 \end{bmatrix} \xRightarrow{R_3 \to R_3 - \mathrm{R}_2} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & -19 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & -19 \end{bmatrix} \xRightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & -4 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 1 & -4 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_3 \to 3R_3 - R_2} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & -13 & -65 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form.
Next step: Reduced Echelon Form

Continue…

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & -13 & -65 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3/(-13)} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 9 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} -1 & 1 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

continue..

$$\begin{bmatrix} -1 & 1 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_2 \to R_2/3} \begin{bmatrix} -1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \to R_1 \text{-} R_2} \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \to R_1 \times (-1)} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots : basic variables x_1, x_2, x_3

Non-pivots: free variables $none$

Solutions to the linear system: x_1 = -3; $x_2 = 1$; $x_3 = 5$

Mathematica: Solve[{-x+y+z==0,2x+y-z==-10,3x-2z==-19,-x+2y-3z==-10}]

Question 4

$$Ax = b$$

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Question 4(a)

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 4 & 5 & 5 & 0 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

Row Reducing

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 4 & 5 & 5 & 0 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \xRightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \xRightarrow{R_3 \to R_3 - (3/2)R_1} \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & -5 & -12 & -3/2 & 0 & 1 \end{array}\right)$$

continue...

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & -5 & -12 & -3/2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - (5/3) R_2} \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & 0 & -1/3 & 11/6 & -5/3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & 0 & -1/3 & 11/6 & -5/3 & 1 \end{array}\right) \xrightarrow{R_3 \to (-3) \times R_3} \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right) \xrightarrow{R_2 \to R_2 + 7R_3} \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & 0 & -81/2 & 36 & -21 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & -3 & 0 & -81/2 & 36 & -21 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right) \xrightarrow{R_1 \to R_1 - 6R_3} \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 34 & -30 & 18 \\ 0 & -3 & 0 & -81/2 & 36 & -21 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

continue...

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 34 & -30 & 18 \\ 0 & -3 & 0 & -81/2 & 36 & -21 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right) \xRightarrow{R_2 \to R_2 \times (-\frac{1}{3})} \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 34 & -30 & 18 \\ 0 & 1 & 0 & 27/2 & -12 & 7 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 34 & -30 & 18 \\ 0 & 1 & 0 & 27/2 & -12 & 7 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right) \xRightarrow{R_1 \to R_1 - 4R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -20 & 18 & -10 \\ 0 & 1 & 0 & 27/2 & -12 & 7 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -20 & 18 & -10 \\ 0 & 1 & 0 & 27/2 & -12 & 7 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right) \xRightarrow{R_1 \to R_1/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 9 & -5 \\ 0 & 1 & 0 & 27/2 & -12 & 7 \\ 0 & 0 & 1 & -11/2 & 5 & -3 \end{array}\right)$$

Inverse of A:

$$A^{-1} = \begin{pmatrix} -10 & 9 & -5 \\ 27/2 & -12 & 7 \\ -11/2 & 5 & -3 \end{pmatrix}$$

Inverse[A] or RowReduce[A|I]

Question 4(b) : matrix of cofactors

$$\mathrm{B} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix}$$

Calculate cofactors one by one:
delete the row and column indexed by subscripts of cofactor

$$A_{11} = (-1)^{1+1}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 5 & 5 \\ 1 & -3 \end{pmatrix} = 5 \times (-3) - 5 \times 1 = -20$$

$$A_{12} = (-1)^{1+2}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = -\det\begin{pmatrix} 4 & 5 \\ 3 & -3 \end{pmatrix} = -(4 \times (-3) - 5 \times 3) = 27$$

$$A_{13} = (-1)^{1+3}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix} = 4 \times 1 - 5 \times 3 = \text{-}11$$

continue..

$$A_{21} = (-1)^{2+1}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = -\det\begin{pmatrix} 4 & 6 \\ 1 & -3 \end{pmatrix} = -(4 \times (-3) - 6 \times 1) = 18$$

$$A_{22} = (-1)^{2+2}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 2 & 6 \\ 3 & -3 \end{pmatrix} = 2 \times (-3) - 6 \times 3 = -24$$

$$A_{23} = (-1)^{2+3}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = -\det\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = -(2 \times 1 - 3 \times 4) = 14$$

$$A_{31} = (-1)^{3+1}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 4 & 6 \\ 5 & 5 \end{pmatrix} = 4 \times 5 - 5 \times 6 = -10$$

$$A_{32} = (-1)^{3+2}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = -\det\begin{pmatrix} 2 & 6 \\ 4 & 5 \end{pmatrix} = -(2 \times 5 - 4 \times 6) = 14$$

continue..

$$A_{33} = (-1)^{3+3}\det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = \det\begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix} = 2 \times 5 - 4 \times 4 = -6$$

Substituting these value in:

$$\mathrm{B} = \begin{pmatrix} -20 & 27 & -11 \\ 18 & -24 & 14 \\ -10 & 14 & -6 \end{pmatrix} \quad \mathrm{B}^{\mathrm{T}} = \begin{pmatrix} -20 & 18 & -10 \\ 27 & -24 & 14 \\ -11 & 14 & -6 \end{pmatrix}$$

Transpose[B]

1st row expanding to calculate determinant of A

$$\det(A) = \det\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 5 \\ 3 & 1 & -3 \end{pmatrix} = 2A_{11} + 4A_{12} + 6A_{13}$$

Det [A]

$$= 2 \times (-20) + 4 \times 27 + 6 \times (-11) = 2$$

$$A^{-1} = \frac{1}{\det(\mathrm{A})} B^T = \begin{pmatrix} -10 & 9 & -5 \\ 27/2 & -12 & 7 \\ -11/2 & 5 & -3 \end{pmatrix}$$

Mathematica: product of a constant and a matrix: a B (notice the space between them)

Question 4(c) :use inverse matrix to solve

$Ax = b$

$$x = A^{-1}b = \begin{pmatrix} -10 & 9 & -5 \\ 27/2 & -12 & 7 \\ -11/2 & 5 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{bmatrix} -7 \\ \\ \end{bmatrix}$$

$$= \begin{pmatrix} -10 & 9 & -5 \\ 27/2 & -12 & 7 \\ -11/2 & 5 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{bmatrix} -7 \\ \frac{21}{2} \\ \\ \end{bmatrix}$$

$$= \begin{pmatrix} -10 & 9 & -5 \\ 27/2 & -12 & 7 \\ -11/2 & 5 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{bmatrix} -7 \\ \frac{21}{2} \\ -\frac{9}{2} \end{bmatrix}$$

Inverse [A].b

Question 4(d) : Gaussian elimination

augmented matrix

$$[A\ b] = \begin{bmatrix} 2 & 4 & 6 & 1 \\ 4 & 5 & 5 & 2 \\ 3 & 1 & -3 & 3 \end{bmatrix}$$

row operations

$$\begin{bmatrix} 2 & 4 & 6 & 1 \\ 4 & 5 & 5 & 2 \\ 3 & 1 & -3 & 3 \end{bmatrix} \xRightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 3 & 1 & -3 & 3 \end{bmatrix} \xRightarrow{R_3 \to R_3 - 3/2R_1} \begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & -5 & -12 & 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & -5 & -12 & 3/2 \end{bmatrix} \xRightarrow{R_3 \to R_3 - \frac{5}{3}R_2} \begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & -\frac{1}{3} & 3/2 \end{bmatrix}$$

Echelon form.
Next step: Reduced Echelon Form
Continue...

$$\begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & -\frac{1}{3} & 3/2 \end{bmatrix} \xRightarrow{R_3 \to -3R_3} \begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xRightarrow{R_2 \to R_2 + 7R_3} \begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & 0 & -\frac{63}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 & 1 \\ 0 & -3 & 0 & -\frac{63}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xRightarrow{R_1 \to R_1 - 6R_3} \begin{bmatrix} 2 & 4 & 0 & 28 \\ 0 & -3 & 0 & -\frac{63}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xRightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 2 & 4 & 0 & 28 \\ 0 & 1 & 0 & \frac{21}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 & 28 \\ 0 & 1 & 0 & \frac{21}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xRightarrow{R_1 \to R_1 - 4R_2} \begin{bmatrix} 2 & 0 & 0 & -14 \\ 0 & 1 & 0 & \frac{21}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xRightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & \frac{21}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix}$$

continue..

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & \frac{21}{2} \\ 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix}$$

Pivots : basic variables x_1, x_2, x_3

Non-pivots: free variables none

Solutions to
the linear system:

$$x_1 = -7$$
$$x_2 = \frac{21}{2}$$
$$x_3 = -\frac{9}{2}$$

RowReduce[A b]

- Solve others by yourself.

- Should you have any questions, email me!