37233 Linear Algebra

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Classes

- One two-hour lecture (p.w.)
- One two-hour tutorial/computer laboratory (p.w.)

Subject Assessments

- Weekly Tutorial Assignments (starting week 2): weight 15%
- Written Assignment: weight 25 %
- Final exam: weight 60%
  It is required to gain at least 40% at the final exam
- To pass the subject it is necessary to obtain at least 50% for the final combined mark

Teaching linear algebra

- Purely theoretical approach (abstract)
- Application-oriented approach (hands on approach)

Subject contents

- Fundamentals of Linear Algebra
- Applications of Linear Algebra
- Computational Methods

Software

- Wolfram Mathematica

Contents of Lecture 1

- What is the subject of Linear Algebra?
- Why do we need Liner Algebra?
- Applications of Linear Algebra

- A brief review
  - Linear systems of equations
  - Row reduction / elimination (Gauss–Jordan)
  - Determinants $\det \mathbf{A}$
  - Inverse of a matrix $\mathbf{A}^{-1}$

The subject of linear algebra

- Linear algebra is one of the essential parts of mathematics. In short, it is the study of linear equations.
- Linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b, \tag{1}$$

  where b and $a_1, a_2, \ldots a_n$ are real or complex numbers. The subscript n is any integer number.
- Equations

$$4x_1 - 5x_2 + 2 = x_1, \qquad x_2 = 2(\sqrt{6} - x_1) + x_3$$

  are linear because they can be arranged as in (1)

$$3x_1 - 5x_2 = -2, \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

- The equations

$$4x_1 - 5x_2 + 2 = x_1 \sin x_2, \quad x_2 = 2\sqrt{x_1} - 6$$

  are not linear.

Why do we need linear algebra?

- Linear models in science, engineering, economics, statistics . . .
- Many systems in real world behave in a linear manner over a significant parameter ranges, even though they are nonlinear
- Genuinely nonlinear problems can be often linearised — approximated by linear systems
- Natural phenomena are often described in terms of partial or ordinary differential equations. Solving these equations requires discretisation. This, in turn, leads to linear systems.

Applications of linear algebra

- Science
  - Physics
  - Chemistry
  - Biology
  - ...
- Engineering (mechanical, electrical, ...)
- Economics
- Statistics
- Big Data Analysis

Systems of linear equations

- A system of linear equations (a linear system) is a collection of one or more linear equations involving the same variables

$$\left\{\begin{array}{ccccccc} 2x_1 & - & x_2 & + & 1.5x_3 & = & 8, \\ x_1 & & & - & x_3 & = & -7. \end{array}\right.$$

  In general

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

- The solution of the system is a list of numbers $x_1, x_2, \dots, x_n$ that makes each equation a true statement.

System of linear equations

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ldots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

Set of coefficients a_{ij} is the matrix $\mathbf{A}[m \times n]$ of linear system

$$\mathbf{A} = \left[\begin{array}{cccc} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{array}\right],$$

The element a_{ij} located in i-th row and j-th column of $\mathbf{A}$.

Matrix representation of a linear system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In short, $\qquad \mathbf{Ax} = \mathbf{b}.$

Augmented matrix of a linear system

- For a linear system with m equations and n unknowns

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{array}\right.$$

an **augmented matrix** of the system is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

Systems of Linear Equations

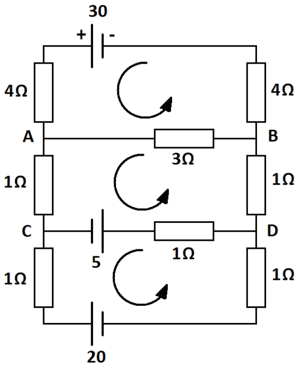
- Linear system naturally arise in network analysis
- Network is a set of branches through which something "flows"
  - Electrical wires (electricity flow)
  - Economic linkages (money flow)
  - Pipes through which oil, gas or water flows
  - Fibres through which information flows (Internet)
- Branches meet at nodes or junctions
- A numerical measure is the rate of flow through a branch
- Analysis of networks is based on linear systems

Example: Electric circuits

Voltage drop across a resistor is given by Ohm's law $V = RI$

Kirchhoff's Voltage Law: *The algebraic sum of the IR voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.*

$$\sum_{i=1}^{N} R_i I_i = \sum_{i=1}^{M} V_i$$

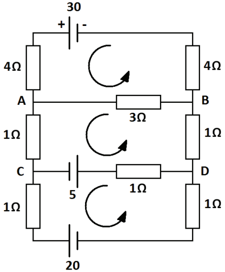


Example: Electric circuits

Loop 1: $4I_1 + 4I_1 + 3I_1 - 3I_2 = 30$

Loop 2: $-3I_1 + 6I_2 - I_3 = 5$

Loop 3: $-I_2 + 3I_3 = -25$



$$\begin{cases} 11I_1 - 3I_2 = 30 \\ -3I_1 + 6I_2 - I_3 = 5 \\ -I_2 + 3I_3 = -25 \end{cases}$$

The loop currents are $I_1 = 3\,\text{A}$, $I_2 = 1\,\text{A}$ and $I_3 = -8\,\text{A}$.

- The total current in the branch AB is $I_1 - I_2 = 3 - 1 = 2\,\text{A}$
- The current in branch CD is $I_2 - I_3 = 9\,\text{A}$

Example: Balancing chemical equations

- Chemical equations describe the quantities of substances consumed and produced by chemical reactions:

$$(x_1)\,\mathrm{C_3H_8} + (x_2)\,\mathrm{O_2} \quad \to \quad (x_3)\,\mathrm{CO_2} + (x_4)\,\mathrm{H_2O}$$

Balancing requires finding amounts x_1, x_2, x_3, x_4 such that the total amounts of carbon C, hydrogen H, and oxygen O atoms on the left match the corresponding numbers on the right.

$$\mathrm{C_3H_8} : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} \quad \mathrm{O_2} : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{CO_2} : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \mathrm{H_2O} : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The solution of this system is $x_1 = 1$, $x_2 = 5$, $x_3 = 3$, $x_4 = 4$.

Linear equations: graphical representation

Consider a system of linear equations

$$\begin{cases} x_1 & - & 2x_2 & = & -1, \\ -x_1 & + & 3x_2 & = & 3. \end{cases}$$

It has a unique solution $x_1 = 3$ and $x_2 = 2$.

This may be represented graphically; in “Mathematica” type

```
ContourPlot[{x1-2x2==-1,-x1+3x2==3},{x1,0,6},{x2,0,4},Axes->True]
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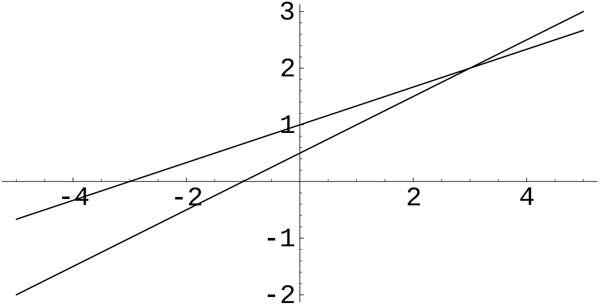
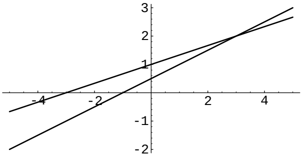


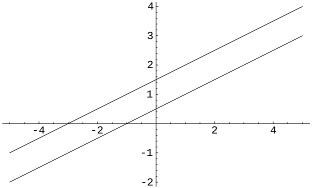
System 1: Unique solution

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 3x_2 = 3. \end{cases}$$



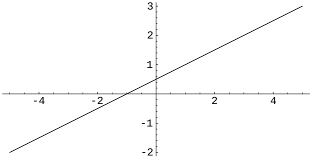
System 2: No solutions

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 2x_2 = 3. \end{cases}$$



System 3: Infinitely many solutions

$$\begin{cases} x_1 - 2x_2 = -1, \\ -x_1 + 2x_2 = 1. \end{cases}$$



Linear systems of equations

- A linear system may have
  - exactly one solution
  - no solutions
  - infinitely many solutions
- While it is easy to understand this in the simple case of two equations in two unknowns, for which we can visualise the system and its solutions graphically, it is harder in three dimensions, and impossible in higher dimensions.
- We need a general tool to understand when system of equations have solutions, and if so, whether the solution is unique.

Gaussian reduction / elimination

- Find all solutions of a linear system using Gaussian reduction First write the augmented matrix of linear system, e.g.

$$\begin{array}{rcrcrcr} & & 2x_2 & - & 8x_3 & = & 8 \\ x_1 & - & 2x_2 & + & x_3 & = & 0 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- There are only three **elementary row operations** to perform:
  - Adding a multiple of one row to another
  - Multiplying a row by a constant
  - Swapping two rows
- This process is called **row reduction**. The goal is to reduce the augmented matrix to **echelon form**:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

Example to solve:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

To bring this to triangular form we eliminate x_1 in equation 3. This we do with the operation

$$\text{Eq3} + 4 * \text{Eq1} \rightarrow \text{Eq3} \qquad \text{or} \qquad \text{R3} + 4 * \text{R1} \rightarrow \text{R3}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Next, we want to eliminate x_2 in equation 3. To simplify the arithmetics, we we'll factor R2.

$$\frac{1}{2} \times \text{Eq2} \rightarrow \text{Eq2} \qquad \text{or} \qquad \frac{1}{2} \times \text{R2} \rightarrow \text{R2}$$

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & 2x_2 & - & 8x_3 & = & 8 \\
 & & -3x_2 & + & 13x_3 & = & -9
\end{array}
\qquad
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{bmatrix}
$$

$$
\frac{1}{2} \times \text{Eq2} \to \text{Eq2} \qquad \text{or} \qquad \frac{1}{2} \times \text{R2} \to \text{R2}
$$

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & x_2 & - & 4x_3 & = & 4 \\
 & & -3x_2 & + & 13x_3 & = & -9
\end{array}
\qquad
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & -3 & 13 & -9
\end{bmatrix}
$$

Next, $\quad \text{Eq3} + 3 * \text{Eq2} \to \text{Eq3} \quad$ or $\quad \text{R3} + 3 * \text{R2} \to \text{R3}$

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & x_2 & - & 4x_3 & = & 4 \\
 & & & & x_3 & = & 3
\end{array}
\qquad
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{bmatrix}
$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The system of equations is now in a triangular form.

We can solve this directly or take a few elimination steps.

Solving directly:

$$\begin{array}{lcl} \text{Eq3} & \Rightarrow & x_3 = 3 \\ \text{Eq2} & \Rightarrow & x_2 = 4x_3 + 4 = 4 \times 3 + 4 = 16 \\ \text{Eq2} & \Rightarrow & x_1 = 2x_2 - x_3 + 0 = 2 \times 16 - 3 = 29 \end{array}$$

This is called **backward substitution**.

Gaussian reduction / elimination

- The alternative is to continue with the elimination.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now elimination above the diagonal:

$$\mathrm{Eq1} + 2 * \mathrm{Eq2} \to \mathrm{Eq1} \qquad \text{or} \qquad \mathrm{R1} + 2 * \mathrm{R2} \to \mathrm{R1}$$

$$\begin{array}{rcrcrcr} x_1 & & & - & 7x_3 & = & 8 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Gaussian reduction / elimination

$$\begin{array}{rcrcl} x_1 & & - & 7x_3 & = & 8 \\ & x_2 & - & 4x_3 & = & 4 \\ & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{lcl} \text{Eq1} + 7 * \text{Eq3} \rightarrow \text{Eq1} & \text{or} & \text{R1} + 7 * \text{R2} \rightarrow \text{R1} \\ \text{Eq2} + 4 * \text{Eq3} \rightarrow \text{Eq2} & \text{or} & \text{R2} + 4 * \text{R3} \rightarrow \text{R2} \end{array}$$

$$\begin{array}{ccccc} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Gaussian reduction / elimination

- Second example

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}$$

After the row reduction we get the reduced system in the form

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5/2 \end{array}$$

The **inconsistency** $0 = 5/2$ implies that this system does not have a solution.

Gaussian reduction / elimination

- Third example

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & -1/2 \end{array}$$

This reduces to

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 0 \end{array}$$

3 equations in 3 unknowns $\longrightarrow$ 2 equations in 3 unknowns
$\Rightarrow$ only two independent equations
No contradiction, but no unique solution (infinitely many)

Gaussian reduction / elimination

- Case 1: Consistent system, unique solution:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Case 2: Inconsistent system, no solution:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Case 3: Consistent system with infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- 1. System of equations is **consistent** if the solution is unique or there are infinitely many solutions.
  2. System of equations is **inconsistent** if it has no solutions.

Gaussian reduction / elimination

In the process of Gauss-Jordan elimination,
an augmented matrix is reduced to a diagonal form.

Row operations that can be used:

- Adding a multiple of one row to another
- Multiplying a row by a constant
- Swapping two rows

Row reduction and echelon form

- The first non-zero element in a row is called the **leading element** of the row
- The reduction of a matrix to echelon form occurs via a sequence of row operations
- The matrix in echelon form must have the following properties
  - All non-zero rows are above any rows of all zeroes.
  - Each **leading entry** of a row is in a column to the right of the **leading entry** of the row above it.
  - All entries in a column below a leading entry are zeroes.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduction and echelon form

- **Echelon form (EF) and reduced echelon form (REF)**

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- **REF** matrix, in addition to **EF** form, has the properties
  - The **leading entry** in each non-zero row is 1
  - The leading 1 in a column is the only non-zero entry in its column.

Matrices in EF and REF forms

- Scheme of EF form ($\blacksquare$ is non-zero number, $*$ is any number)

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

- Scheme of REF form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Matrices in EF and REF forms

- The echelon form of a matrix (**EF**) is not unique, however reduced echelon form **REF** is unique.
- **Theorem**: Each matrix is row equivalent to one and only one matrix in **reduced echelon form REF**.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- A **pivot position** corresponds to leading 1 in REF. A **pivot column** is a column that contains a pivot position.

The row reduction to EF and REF forms

- **Step 1**: Begin with the leftmost non-zero column.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Step 2**: Select nonzero entry in the pivot column as a pivot. If necessary, interchange rows.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **Step 3**: Use row replacement operation to create zeros in all positions below the pivot (here, use $R_2 \to R_2 - R_1$).

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The EF form ("forward phase")

- **Step 4**. Cover the row containing the pivot (and any rows above it). Apply all steps 1-3 to the remaining sub-matrix. Repeat until there are no more non-zero rows to modify.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we divide the second row by 2)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(now, we use $R_3 \to R_3 - 3R_2$)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The REF form ("backward phase")

- $$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- **Step 5**. Beginning with the rightmost pivot and working upward and to the left, create zeroes above each pivot. If pivot is not 1, make it 1 by a scaling operations.
  Row operation $R_1 \to R_1/3$ leads to

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

then, $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - 2R_3$ lead to

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The REF form ("backward phase")

- $$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

  now, $R_1 \to R_1 + 3R_2$. This finally brings us to the REF form

  $$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- Note there are 3 equations for 5 variables.
- Variables with pivots are called **basic variables**: x_1, x_2, x_5.
- Variables without pivots are called **free variables**: x_3, x_4.
- In the final solution basic variables x_1, x_2, x_5 must be expressed in terms of free variables x_2, x_4.

- $$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

- The solution is

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_5 &= 4 \end{aligned}$$

x_1, x_2, x_5 (basic) are expressed in terms of x_3, x_4 (free).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2x_3 - 3x_4 \\ -7 + 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Homogeneous versus inhomogeneous linear systems

- A linear system can be written as $\mathbf{Ax} = \mathbf{b}$.
- If $\mathbf{b} \neq \mathbf{0}$, the system is called **inhomogeneous**
- If $\mathbf{b} = \mathbf{0}$, the system is called **homogeneous** $(\mathbf{Ax} = 0)$
- For the inhomogeneous system from the previous example the explicit linear system is

$$\left\{ \begin{array}{cccccl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array} \right.$$

- The corresponding homogeneous system reads

$$\left\{ \begin{array}{cccccl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = 0 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 0 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 0 \end{array} \right.$$

Homogeneous linear systems

- For the homogeneous linear system

$$\begin{cases} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = 0 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 0 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 0 \end{cases}$$

the augmented matrix and its REF are as follows

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 0 \\ 3 & -7 & 8 & -5 & 8 & 0 \\ 3 & -9 & 12 & -9 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- So the solution is

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_5 &= 0 \end{aligned}$$

Basic variables x_1, x_2, x_5 are expressed in terms of free x_3, x_4.

Inhomogeneous linear systems

- **Theorem**: Suppose the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set is, all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}$ where $\mathbf{v}$ is any solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.
- In our examples, a particular solution of the inhomogeneous system $\mathbf{p}$ and the solution set of the homogeneous system $\mathbf{v}$ were

$$\mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

so the solution set of the inhomogeneous system is indeed

$$\mathbf{w} = \mathbf{p} + \mathbf{v}$$

Determinants

- Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $\det \mathbf{A} = ad - bc$ is the determinant of the 2×2 matrix.

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties (these properties hold for any $n \times n$) matrix

- $$\begin{vmatrix} a & b \\ c+e & d+f \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ e & f \end{vmatrix}$$

- $$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- $$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinants

- Determinant of an identity (unitary) matrix is 1

$$\det \mathbf{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

- If two rows of $\mathbf{A}$ are same, then $\det \mathbf{A} = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

- The elementary row operation of subtraction a multiple of one row from another row leaves the determinant unchanged

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c \pm ka & d \pm kb \end{vmatrix}$$

- Determinant with a zero row is zero

$$\begin{vmatrix} 0 & 0 \\ d & c \end{vmatrix} = 0$$

Determinants

- **Definition**: Let $\mathbf{A}$ be $n \times n$ matrix and a_{ij} an element of $\mathbf{A}$. The **cofactor** of a_{ij} denoted by A_{ij}, is the $(n-1) \times (n-1)$ determinant obtained by
  1. deleting the i-th row and j-th column of $\mathbf{A}$ and
  2. multiplying the resulting matrix determinant by $(-1)^{(i+j)}$.
- The determinant obtained by deleting the i-th row and j-th column of $\mathbf{A}$ is called a minor of a_{ij}.
- For a 3×3 matrix: $\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants

- Let $\mathbf{A}$ be $n \times n$ matrix. Then the determinant of $\mathbf{A}$ is the number $a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$.
- Expansion along i-th row is $\det \mathbf{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$
- For a 3×3 matrix expansion along the second row reads $\det \mathbf{A} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.
- So

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$-a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Inverse of a matrix

- **Definition**: An $n \times n$ matrix $\mathbf{A}$ is said to be invertible if there is an $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{CA} = \mathbf{I} \quad \text{and} \quad \mathbf{AC} = \mathbf{I},$$

  where $\mathbf{I}$ is a unitary $n \times n$ matrix. Then $\mathbf{C}$ is an inverse of $\mathbf{A}$.

- $\mathbf{C}$ is uniquely determined by $\mathbf{A}$.
  Indeed, suppose $\mathbf{B}$ is another inverse of $\mathbf{C}$. Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

  This unique inverse $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$, so that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \text{and} \quad \mathbf{AA}^{-1} = \mathbf{I}.$$

- A non-invertible matrix is called a **singular** matrix.
  An invertible matrix is called a **non-singular** matrix.

- Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Then

$$\mathbf{AC} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{CA} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Theorem:**

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det \mathbf{A} = ad - bc \neq 0$

then $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of a matrix

- If $\det \mathbf{A} = ad - bc = 0$, then $\mathbf{A}$ is not invertible.
- The matrix is invertible if and only if $\det \mathbf{A} \neq 0$.
- **Theorem:**
  If $\mathbf{A}$ is an invertible $n \times n$ matrix then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
- **Theorem**:

  a) If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

  b) If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices, then so is $\mathbf{AB}$, and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Inverse of a matrix

- $$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n. \end{cases}$$

  The system can be written in the matrix form

  $$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

  In short $\mathbf{Ax} = \mathbf{b}$

  $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Inverse of a matrix

- So the solution of $\mathbf{Ax} = \mathbf{b}$ is (given $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{Ix} = \mathbf{x}$)

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}, \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

- One of the way to find $\mathbf{A}^{-1}$ is using the adjoint matrix $\mathbf{A}^{adj}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}(\mathbf{A}^c)^T = \frac{1}{\det \mathbf{A}}\mathbf{A}^{adj}.$$

$$\mathbf{A}^c = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

$\mathbf{A}^{-1}$ by Gauss-Jordan elimination

- **Theorem:** A $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ is row-equivalent to identity matrix $\mathbf{I}$, and in this case any sequence of elementary row operations that reduces $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ into $\mathbf{A}^{-1}$.
- This gives an algorithm of finding $\mathbf{A}^{-1}$.
- Row reduce the augmented matrix $[\mathbf{A}\ \mathbf{I}]$. If $\mathbf{A}$ is row equivalent to $\mathbf{I}$, then $[\mathbf{A}\ \mathbf{I}]$ is row equivalent to $[\mathbf{I}\ \mathbf{A}^{-1}]$. Otherwise $\mathbf{A}$ does not have an inverse.
- In practice $\mathbf{A}^{-1}$ is seldom computed directly ($2N^3$ operations). Row reduction is faster and often more accurate.
- Example: Find inverse of a matrix, if it exists

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

- We form the extended augmented matrix

$$\left[\begin{array}{cccccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$\left[\begin{array}{cccccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

- $R_3 \to R_3/2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

- So $\mathbf{A} \sim \mathbf{I}$, and $\mathbf{A}$ is invertible, $\mathbf{A}^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$