MATH 222: Week 4

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1 §14.5 Chain Rule

The chain rule in 1-dimension is as follows:

For an equation $y = f(x(t))$

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Example. *If* $y = (x(t))^2$ *and* $x(t) = \ln 1 + t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2x\frac{1}{1+t} = \frac{2\ln 1 + t}{1+t}$$

Example. *Suppose* $f(x, y) = xy + x^2 + y$

$x(t) = \ln 1 + t,\ y(t) = e^{t^2}$

Turn $f(x, y)$ *into a function* $g(t)$ *with only the time parameter.*

$$g(t) = f(x(t), y(t)) = \ln 1 + te^{t^2} + (\ln 1 + t)^2 + e^{t^2}$$

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{e^{t^2}}{1+t} + 2t\ln 1 + te^{t^2} + \frac{2\ln 1 + t}{1+t} + 2te^{t^2}$$

Wherever we can, replace the values of $x(t),\ y(t)$ *with* $x(t),\ y(t)$*.*

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{y(t)}{1+t} + 2te^{t^2}x(t) + \frac{2}{1+t} + 2te^{t^2}$$

$$\begin{aligned}
&= x'(t)y(t) + y'(t)x(t) + 2x'(t)x(t) + y'(t) \\
&= x'(t)(y(t) + 2x(t)) + y'(t)(x(t) + 1) \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}
\end{aligned}$$

In the second to last line, we use the fact that differentiating f with respect to x gives $y + 2x$ and doing the same for y gives $x + 1$.

There are two possible cases for the chain rule. Suppose in both cases we have $z = f(x, y)$. In the first case we have $x = g(t)$ and $y = h(t)$. In this case z is a differentiable function of t.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second case, $x = g(s, t)$ and $y = h(s, t)$. Then z is a differentiable function of both s and t.

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. $z = e^x \cos(x + y),\ x = s^2 t,\ y = st^2$. *Find $\frac{dz}{ds}$ and $\frac{dz}{dt}$.*

$$\frac{\partial z}{\partial x} = e^x(\cos(x + y) - \sin(x + y))$$

$$\frac{\partial z}{\partial y} = -e^x \sin(x + y)$$

Then we need $\frac{dx}{ds}$, $\frac{dx}{dt}$, $\frac{dy}{ds}$, $\frac{dy}{dt}$

$$\frac{dx}{ds} = 2st,\ \frac{dx}{dt} = s^2$$

$$\frac{dy}{ds} = t^2,\ \frac{dy}{dt} = 2st$$

So now we can find the general equations for $\frac{dz}{ds}$ and $\frac{dz}{dt}$.

$$\frac{dz}{ds} = (e^x)(\cos(x + y) - \sin(x + y))(2st) + (-e^x)(\sin(x + y))(t^2)$$

$$\frac{dz}{dt} = (e^x)(\cos(x + y) - \sin(x + y))(s^2) + (-e^x)(\sin(x + y))(2st)$$

Example. *If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that $t\frac{dg}{ds} + s\frac{dg}{dt} = 0$.*

Based on f, $x(s,t) = s^2 - t^2$ and $y(s,t) = t^2 - s^2$. Therefore we can write that $g(s,t) = f(x(s,t), y(s,t))$. We need to find $\frac{dg}{dt}$ and $\frac{dg}{ds}$ and to do this we need need to differentiate $x(s,t)$, $y(s,t)$ by both s and t.

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\frac{dg}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

If we multiply the first equation all by t and the second all by s, we get

$$t\frac{dg}{ds} = -2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}$$

$$s\frac{dg}{dt} = 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}$$

Doing linear combination gives

$$t\frac{dg}{ds} + s\frac{dg}{dt} = 0$$

1.1 Chain rule and implicit functions

In 1 dimension, if we had an implicit function $F(x,y) = 0$ we would do the following to differentiate it.

$$\frac{dF}{dx}(x,y) = 0 \ \Rightarrow \ \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0$$

$\frac{dx}{dx}$ always equals 1, so we get an equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dy}}$$

Provided that $\frac{dF}{dy}$ doesn't equal 0.

We can apply this idea to an implicit function like $F(x,y,x) = 0$ as well

$$\frac{dF}{dx} = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} + \frac{dF}{dz}\frac{dz}{dx} = 0$$

Like before, $\frac{dx}{dx}$ is 1. In addition, because y is no longer a function of x we can say that

$\frac{dy}{dx} = 0$. y doesn't depend on x at all. However, z does depend on x so that stays put.

$$0 = \frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dz}}$$

1.2 Implicit function theorem

Suppose a function $F(x, y, z)$ is defined on a sphere around a point $(a, b, c) \in \mathbb{R}^3$ satisfying $F(a, b, c) = 0$.

If $F_x,\ F_y,\ F_z$ are continuous and $\frac{dF}{dz}$ evaluated at (a, b, c) does not equal 0, then in a neighborhood of (a, b, c) we have that the equation $F(x, y, z) = 0$ defines z as a function of $x,\ y$ near (a, b, c) in this neighborhood. In addition, this function is differentiable in this area and its partial derivatives $\frac{dz}{dx} = f_x(x, y),\ \frac{dz}{dy} = f_y(x,)$ are given by

$$\frac{dz}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dz}}$$

$$\frac{dz}{dy} = \frac{\frac{-dF}{dy}}{\frac{dF}{dz}}$$

This theorem is really a test that we use to determine if we can get a tangent plane at a given point on the function. If the slope is ∞ then our equations for $\frac{dz}{dx},\ \frac{dz}{dy}$ won't work because $\frac{dF}{dz}$ will be zero.

In clas we used the example of a simple circle $x^2 + y^2 = 1$, a circle of radius 1 centered at the origin. If we choose (a, b) to be somewhere in the middle of the top-right quadrant, in this neighborhood of (a, b) we can talk about the curve as a function. There is no point in this neighborhood where the slope is ∞. However, if we choose (a, b) to be $(1, 0)$ or $(-1, 0)$ we have a situation where the curve is not a function. We can tell this by looking at the graph of the circle but for more confusing curves we need the Implicit Function Theorem. If $f(a, b)$ has a slope of ∞ then since we're going to be dealing with the neighborhood of (a, b), there will be points in this neighborhood that fail the vertical line test. Then we can't find derivatives and tangent planes in this area since it's not a function here.

Example. *Let $F(x, y, z) = 0$ for $F(x, y, z) = x^3 + y^3 + z^3 + 6zyz - 9$. Show that around $(1, 1, 1)$ we can define z as a function of $x,\ y$. Find the values of $\frac{dz}{dy}$ and $\frac{dz}{dx}$ at $(1, 1, 1)$.*

First check if the point is on the surface. $1 + 1 + 1 + 6 - 9 = 0$, so it's on the surface.

Next we need to check that $\frac{dF}{dz}$ *at the point isn't 0.*

$$\frac{dF}{dz} = 3z^2 + 6xy$$

Evaluating this is the point gives 9, which isn't 0. Since F *is a polynomial we can also say that* F_x, F_y, F_z *are continuous .*

By the Implicit Function Theorem, near $(1, 1, 1)$ z *is a function of* $x,\ y$. *Now we need to find* $\frac{dz}{dy}$ *and* $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{-F_x}{F_z} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy}\Big|_{1,1,1} = -1$$

$$\frac{dz}{dy} = \frac{-F_y}{F_z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}\Big|_{1,1,1} = -1$$

2 §14.7 Maximum and Minimum Values