MATH 222: Week 3

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1 §13.3 Arc Length + Curvature

1.1 Arc Length

Given a parametric curve $\vec{r}(t)$ in $\mathbb{R}^3$, what is the distance travelled by the particle between some $t = a$ and $t = b$?

Suppose we partition this section of the curve into n sections of time. The differences in t between these sections might not be 1 so we write $\Delta t = \frac{b-a}{n}$. In this way b would equal $a + n\Delta t$. We could approximate the distance from $\vec{r}(a)$ to $\vec{r}(b)$ by adding together these the difference between $\vec{r}(a)$ and $\vec{r}(a+\Delta t)$, $\vec{r}(a+\Delta t)$ and $\vec{r}(a+2\Delta t)$, and so on. If we had three sections, the approximation would look something like this:

$$\sum_{i=0}^{2} \sqrt{(x(a+(i+1)\Delta t) - x(a+(i)\Delta t))^2 + (y(a+(i+1)\Delta t) - y(a+(i)\Delta t))^2}$$

I'll cut this short, but we can basically turn this into a Riemann sum by having n approach infinity. Then by using the Mean Value Theorem, we can find an actual equation for Arc Length (here denoted by L):

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

The professor has included more notes about how this is derived on MyCourses, so check that out if you're really, really, really interested for some reason.

For some $\vec{r}(t) =< f(t), g(t), h(t) >$ in $\mathbb{R}^3$, the formula is slightl different. We just need to add the third function.

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\ dt$$

This is also written as

$$L = \int_a^b \parallel \vec{r'}(t) \parallel\ \ dt$$

Example. *Compute arc length over $0 \leq t \leq 2\pi$ of the circular helix $\vec{r}(t) =< \cos t, \sin t, t >$*

Find $\vec{r'}(t)$ and its length

$$\vec{r'}(t) =< -\sin t, \cos t, 1 >$$

$$\parallel \vec{r'}(t) \parallel = \sqrt{(\sin(t))^2 + (\cos(t))^2 + 1} = \sqrt{2}$$

Then integrate this

$$\int_0^{2\pi} \sqrt{2}\ dt = 2\pi\sqrt{2} - 0\sqrt{2} = 2\pi\sqrt{2}$$

1.2 Curvature

Suppose we are on some interval of t where $\vec{r}(t)$ is a smooth function and $\vec{r'}(t) \neq 0$. In this setting we can discuss the curvature of the curve defined by $\vec{r}(t)$.

We think of curvature as the amount that the direction of the curve changes over a small step (the size of this step approaches 0)

Curvature depends solely on arc length. $<\cos(t), \sin(t)>$ is "drawn" more slowly than $<\cos(4t), \sin(4t)$ but they still have the same curvature because they're both circles with the same radius. We can define curvature as the rate of change of the unit tangent vector with respect to arc length, which would be written as this (where $K(t)$ is curvature at time t):

$$\begin{aligned} K(t) &= \parallel \frac{d\vec{T}}{ds} \parallel \\ &= \parallel \frac{d\vec{T}}{dt}\frac{dt}{ds} \parallel \\ &= \parallel \frac{d\vec{T}}{dt}\frac{1}{\parallel \vec{r'}(t) \parallel} \parallel \end{aligned}$$

In the last line we were able to replace $\frac{dt}{ds}$ with $\frac{1}{\|\vec{r'}(t)\|}$ because the rate in change of s (arc length) with respect to time t is $\parallel \vec{r'}(t) \parallel$ based on our equations for arc length. We just need to invert this to get $\frac{1}{\|\vec{r'}(t)\|}$.

By simplifying a bit more ($\frac{d\vec{T}}{dt} = \vec{T'}(t)$), we can write the equation for curvature at time t as

$$K(t) = \frac{\parallel \vec{T'}(t) \parallel}{\parallel \vec{r'}(t) \parallel}$$

Through a very long proof we also proved that there is an alternate equations for the curvature that may be easier to use in some cases.

$$K(t) = \frac{\parallel \vec{r'}(t) \times \vec{r''}(t) \parallel}{\parallel \vec{r'}(t) \parallel^3}$$

I will show the proof here (the book's much shorter version) but feel free to skip over this.

Proof. $K = \frac{\|\vec{r'}(t) \times \vec{r''}(t)\|}{\|\vec{r'}(t)\|^3}$

Since $\vec{T} = \frac{\vec{r}}{\|\vec{r'}\|}$ and $\parallel \vec{r'} = \frac{ds}{dt}$, we can write that

$$\vec{r'} = \parallel \vec{r'} \parallel \vec{T} = \frac{ds}{dt}\vec{T}$$

Using the product rule we can find that

$$\vec{r''} = \frac{d^2 s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T'}$$

Since the equation we wish to achieve has $\vec{r'} \times \vec{r''}$ in it, let's try to see what happens when we take the cross product.

$$\vec{r'} \times \vec{r''} = \frac{ds}{dt}\vec{T} \times \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T} \times \frac{ds}{dt}\vec{T'}$$

We know that $\vec{T} \times \vec{T}$ will be 0.

$$\vec{r'} \times \vec{r''} = (\frac{ds}{dt})^2(\vec{T} \times \vec{T'})$$

Next we will take the magnitude of both sides but we can also simplifying the cross product. Since $\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \| \vec{b} \| \sin\theta$ and $\vec{T}$ is perpendicular to $\vec{T'}$, $\theta = \frac{\pi}{2}$. $\sin(\frac{\pi}{2}) = 1$. Therefore the cross product can be rewritten.

$$\| \vec{r'} \times \vec{r''} \| = (\frac{ds}{dt})^2 \| \vec{T} \| \| \vec{T'} \|$$

$\vec{T}$ is defined to be a unit vector, so its magnitude is 1.

$$\| \vec{r'} \times \vec{r''} \| = (\frac{ds}{dt})^2 \| \vec{T'} \|$$

Solving for $\| \vec{T'} \|$ gives

$$\| \vec{T'} \| = \frac{\| \vec{r'} \times \vec{r''} \|}{(ds/dt)^2} = \frac{\| \vec{r'} \times \vec{r''} \|}{\| \vec{r'} \|^2}$$

Finally, if we divide all of this by $\| \vec{r'} \|$ in order to get the equation for curvature on one side $\frac{\|\vec{T'}\|}{\|\vec{r'}\|}$, we can see that our theorem is true.

$$K(t) = \frac{\| \vec{T'} \|}{\| \vec{r'} \|} = \frac{\| \vec{r'} \times \vec{r''} \|}{\| \vec{r'} \|^3}$$

So there are two equivalent ways of finding an equation for curvature. □

Example. *Compute the curvature of $\vec{r}(t) =< t\cos(t), t\sin(t), t >$ at $t = \frac{\pi}{2}$*

We need $\vec{r'}$, $\vec{r''}$, and $\| \vec{r'} \|$

$$\begin{aligned} \vec{r'}(t) &=< -t\sin(t) + \cos(t), t\cos(t) + \sin(t), t > \\ \vec{r'}(\frac{\pi}{2}) &=< -\frac{\pi}{2}, 1, 1 > \\ \| \vec{r'}(\frac{\pi}{2}) \| &= \| < -\frac{\pi}{2}, 1, 1 > \| = \sqrt{\frac{\pi^2}{4} + 2} \end{aligned}$$

$$\begin{aligned}\vec{r''}(t) &=< -t\cos(t) - \sin(t) - \sin(t), -t\sin(t) + \cos(t) + \cos(t), 0 > \\ &=< -t\cos(t) - 2\sin(t), -t\sin(t) + 2\cos(t), 0 > \\ \vec{r''}(\frac{\pi}{2}) &=< -2, -\frac{\pi}{2}, 0 >\end{aligned}$$

Now that we have all the piece we need, we just plug it all into the curvature equation.

$\vec{r'}(\pi/2) \times \vec{r''}(\pi/2) = \begin{vmatrix} i & j & k \\ -\frac{\pi}{2} & 1 & 1 \\ -2 & -\frac{\pi}{2} & 0 \end{vmatrix} =< \frac{\pi}{2}, -2, \frac{\pi^2}{4} + 2 >$ *Our equation for curvature is therefore*

$$K(\frac{\pi}{2}) = \frac{\| < \frac{\pi}{2}, -2, \frac{\pi^2}{4} + 2 > \|}{(\frac{\pi^2}{4} + 2)^{3/2}} = \frac{\sqrt{\frac{\pi^2}{4} + 4 + (\frac{\pi^2}{4} + 2)^2}}{(\frac{\pi^2}{4} + 2)^{3/2}}$$

You can do the insane amount of computation if you'd like, but it's sufficient to just plug this into a calculator.

1.3 Curvature of functions

The concept of curvature can also be applied to functions $y = f(x)$ since a parametrization of a function like this is $\vec{r}(x) =< x, f(x), 0 >$

$$\vec{r'}(x) =< 1, f'(x), 0 >$$

$$\vec{r''}(x) =< 0, f''(x), 0 >$$

$\vec{r'}(x) \times \vec{r''}(x) = \begin{vmatrix} i & j & k \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = k(f''(x))$ Therefore $\| \vec{r'} \times \vec{r''} \| = |f''(x)|$. The curvature of any function $y = f(x)$ can be written as

$$K(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

Example. *Find $K(x)$ for $f(x) = x^2$*

$f'(x) = 2x$ and $f''(x) = 2$. Therefore $K(x)$ is

$$K(x) = \frac{2}{(1 + 4x^2)^{3/2}}$$

1.4 Normal and Binormal Vectors

Earlier we learned that $\vec{T}(t)\cdot\vec{T'}(t) = 0$, which implies that the two vectors are perpendicular.

Let $\vec{N}(t) = \frac{\vec{T'}(t)}{\|\vec{T'}(t)\|}$. We call this the "unit outward normal". $N(t)$ indicates the direction in which the curve given by $\vec{r}(t)$ is turning.

In addition, let $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$. We call this the binormal vector. Since $\vec{B}(t)$ is orthogonal to both $\vec{T}(t)$ and $\vec{N}(t)$ and these vectors are in the same plane, $\vec{B}(t)$ comes "out" of the page and is orthogonal to that plane.

Vectors B and N form the normal plane to curve C at point P where $t = a$. The normal plane contains all vectors that are perpendicular to T. Vectors T and N form the osculating plane to curve C at point P. The osculating plane contains all vectors that are perpendicular to B.

Example. *Find the equation for the normal plane at $t = \frac{\pi}{2}$ for $\vec{r}(t) =< \cos(t), \sin(t), t^2 >$*

$$\vec{T} = \frac{r'}{\parallel \vec{r'} \parallel} = \frac{< -\sin(t), \cos(t), 2t >}{\sqrt{1+4t^2}}$$

$$\vec{T}(\frac{\pi}{2}) = \frac{< -1, 0, \pi >}{\sqrt{1+\pi^2}}$$

Since $\vec{T}$ is the normal to our plane, we can find the equation of this plane using $\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0$.

$$\frac{1}{\sqrt{1+\pi^2}} < -1, 0, \pi > \cdot((x, y, z) - (0, 1, \frac{\pi^2}{4})) = 0$$

$$\frac{-1}{\sqrt{1+\pi^2}}(x) + \frac{\pi}{\sqrt{1+\pi^2}}(z - \frac{\pi^2}{4}) = 4$$

This is the equation of the normal plane to the curve at $t = \frac{\pi}{2}$.

2 §13.4 Acceleration Vector

If $\vec{r}(t)$ denotes position, $\vec{r'}(t) = \vec{v}(t)$ denotes the velocity vector. Also, $\vec{r''}(t) = \vec{a}(t)$ denotes the acceleration vector.

The magnitude of the velocity vector is speed, and it is written as $\nu(t)$

Since $\vec{a}(t) = \vec{v}(t)$, then

$$\int_{t_0}^{t} \vec{a}(\tau)\ d\tau = \int_{t_0}^{t} \vec{v'}(\tau)\ d\tau = \vec{v}(t) - \vec{v}(t_0)$$

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(\tau)\ d\tau$$

If we can integrate $\vec{a}(t)$ and we have the initial velocity, we can find $\vec{v}(t)$

2.1 Tangential and normal components of acceleration

It is often useful to be able to break up the acceleration vector into its components. We'll start with the definition of $\vec{T}(t)$.

$$\vec{T} = \frac{\vec{r'}}{\| \vec{r'} \|} = \frac{\vec{v}}{\| \vec{v} \|} = \frac{\vec{v}}{\nu}$$

From here we can see that

$$\vec{v} = \nu\vec{T}$$

If we differentiate both sides of this with respect to t, we get that

$$\vec{a} = \vec{v'} = \nu'\vec{T} + \nu\vec{T'}$$

Using the equation for curvature and replacing $\| \vec{r'} \|$ with ν, we can solve for $\| \vec{T'} \|$ and get $\| \vec{T'} \| = K\nu$. We can substitute this into the equation for $\vec{N}$ and see that

$$\vec{N} = \frac{\vec{T'}}{\| \vec{T'} \|}$$

$$\vec{T'} = \vec{N} \| \vec{T'} \|$$

Substituting this into the equation we were originally working on gives

$$\vec{a} = \nu'\vec{T} + K\nu\vec{N}$$

We are able to write $\vec{a}$ as a combintion of a scalar multiple of the unit tangent and a scalar multiple of the unit normal. When talking about the components of the acceleration vector, we call the component in the direction of $\vec{T}$ a_T, which equals ν'. The component in the direction of $\vec{N}$ is called a_N and it equals $K\nu^2$.

One interesting thing that this information implies is that the acceleration vector will always be in the osculating plane, since $\vec{B}$ is not one of the components of $\vec{a}$.

For convenience sake, since we'll often be computing $\vec{r}$, $\vec{r'}$, $\vec{r''}$ it would be nice to have equations for these components in terms of these vectors. The book goes into more deal why they do what they do to derive these equations, but here they are:

$$a_T = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{\| \vec{r'}(t) \|}$$

$$a_N = \frac{\| \vec{r'}(t) \times \vec{r''}(t) \|}{\| \vec{r'}(t) \|}$$

3 §14.1 Multivariable Functions

Definition. *A function of two variables is a rule that assigns to each ordered pair (x, y) in a set D the unqiue real number $f(x, y)$. We write that $z = f(x, y)$*

The domain of $f(x, y)$ is the set of all $x, y) \in \mathbb{R}^2$ where $f(x, y)$ is a well-defined real number. This means that $f(x, y)$ cannot output nothing nor can it output more than one value.

4 §14.2 Limits and Continuity

5 §14.3 Partial Derivatives

6 §14.4 Tangent planes, linear approximation