MATH 222: Week 4

Sarah Randall

Last updated: May 24, 2017

Contents

1 §14.5 Chain Rule

The chain rule in 1-dimension is as follows:

For an equation $y = f(x(t))$

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Example. *If* $y = (x(t))^2$ *and* $x(t) = \ln 1 + t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2x\frac{1}{1+t} = \frac{2\ln 1 + t}{1+t}$$

Example. *Suppose* $f(x, y) = xy + x^2 + y$

$x(t) = \ln 1 + t,\ y(t) = e^{t^2}$

Turn $f(x, y)$ *into a function* $g(t)$ *with only the time parameter.*

$$g(t) = f(x(t), y(t)) = \ln 1 + te^{t^2} + (\ln 1 + t)^2 + e^{t^2}$$

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{e^{t^2}}{1+t} + 2t\ln 1 + te^{t^2} + \frac{2\ln 1 + t}{1+t} + 2te^{t^2}$$

Wherever we can, replace the values of $x(t),\ y(t)$ *with* $x(t),\ y(t)$*.*

$$\begin{aligned}\frac{dg}{dt} = \frac{df}{dt} &= \frac{y(t)}{1+t} + 2te^{t^2}x(t) + \frac{2}{1+t} + 2te^{t^2} \\ &= x'(t)y(t) + y'(t)x(t) + 2x'(t)x(t) + y'(t)\end{aligned}$$

$$= x'(t)(y(t) + 2x(t)) + y'(t)(x(t) + 1)$$
$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second to last line, we use the fact that differentiating f with respect to x gives $y + 2x$ and doing the same for y gives $x + 1$.

There are two possible cases for the chain rule. Suppose in both cases we have $z = f(x, y)$. In the first case we have $x = g(t)$ and $y = h(t)$. In this case z is a differentiable function of t.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second case, $x = g(s, t)$ and $y = h(s, t)$. Then z is a differentiable function of both s and t.

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example. $z = e^x \cos(x + y),\ x = s^2 t,\ y = st^2$. *Find* $\frac{dz}{ds}$ *and* $\frac{dz}{dt}$.

$$\frac{\partial z}{\partial x} = e^x(\cos(x + y) - \sin(x + y))$$

$$\frac{\partial z}{\partial y} = -e^x \sin(x + y)$$

Then we need $\frac{dx}{ds}$, $\frac{dx}{dt}$, $\frac{dy}{ds}$, $\frac{dy}{dt}$

$$\frac{dx}{ds} = 2st,\ \frac{dx}{dt} = s^2$$

$$\frac{dy}{ds} = t^2,\ \frac{dy}{dt} = 2st$$

So now we can find the general equations for $\frac{dz}{ds}$ *and* $\frac{dz}{dt}$.

$$\frac{dz}{ds} = (e^x)(\cos(x + y) - \sin(x + y))(2st) + (-e^x)(\sin(x + y))(t^2)$$

$$\frac{dz}{dt} = (e^x)(\cos(x + y) - \sin(x + y))(s^2) + (-e^x)(\sin(x + y))(2st)$$

Example. *If* $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ *and* f *is differentiable, show that* $t\frac{dg}{ds} + s\frac{dg}{dt} = 0$.

Based on f, $x(s,t) = s^2 - t^2$ and $y(s,t) = t^2 - s^2$. Therefore we can write that $g(s,t) = f(x(s,t), y(s,t))$. We need to find $\frac{dg}{dt}$ and $\frac{dg}{ds}$ and to do this we need need to differentiate $x(s,t)$, $y(s,t)$ by both s and t.

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\frac{dg}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

If we multiply the first equation all by t and the second all by s, we get

$$t\frac{dg}{ds} = -2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}$$

$$s\frac{dg}{dt} = 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}$$

Doing linear combination gives

$$t\frac{dg}{ds} + s\frac{dg}{dt} = 0$$

1.1 Chain rule and implicit functions

In 1 dimension, if we had an implicit function $F(x,y) = 0$ we would do the following to differentiate it.

$$\frac{dF}{dx}(x,y) = 0 \;\Rightarrow\; \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0$$

$\frac{dx}{dx}$ always equals 1, so we get an equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dy}}$$