MATH 222: Week 3

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Last updated: May 23, 2017

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1 §13.3 Arc Length + Curvature

1.1 Arc Length

Given a parametric curve $\vec{r}(t)$ in $\mathbb{R}^3$, what is the distance travelled by the particle between some $t = a$ and $t = b$?

Suppose we partition this section of the curve into n sections of time. The differences in t between these sections might not be 1 so we write $\Delta t = \frac{b-a}{n}$. In this way b would equal $a + n\Delta t$. We could approximate the distance from $\vec{r}(a)$ to $\vec{r}(b)$ by adding together these the difference between $\vec{r}(a)$ and $\vec{r}(a + \Delta t)$, $\vec{r}(a + \Delta t)$ and $\vec{r}(a + 2\Delta t)$, and so on. If we had three sections, the approximation would look something like this:

$$\sum_{i=0}^{2} \sqrt{(x(a+(i+1)\Delta t) - x(a+(i)\Delta t))^2 + (y(a+(i+1)\Delta t) - y(a+(i)\Delta t))^2}$$

I'll cut this short, but we can basically turn this into a Riemann sum by having n approach infinity. Then by using the Mean Value Theorem, we can find an actual equation for Arc Length (here denoted by L):

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

The professor has included more notes about how this is derived on MyCourses, so check that out if you're really, really, really interested for some reason.

For some $\vec{r}(t) =< f(t), g(t), h(t) >$ in $\mathbb{R}^3$, the formula is slightl different. We just need to add the third function.

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\ dt$$

This is also written as

$$L = \int_a^b \parallel \vec{r'}(t) \parallel\ dt$$

Example. *Compute arc length over* $0 \leq t \leq 2\pi$ *of the circular helix* $\vec{r}(t) =< \cos t, \sin t, t >$

Find $\vec{r'}(t)$ *and its length*

$$\vec{r'}(t) =< -\sin t, \cos t, 1 >$$

$$\parallel \vec{r'}(t) \parallel = \sqrt{(\sin(t))^2 + (\cos(t))^2 + 1} = \sqrt{2}$$

Then integrate this

$$\int_0^{2\pi} \sqrt{2}\ dt = 2\pi\sqrt{2} - 0\sqrt{2} = 2\pi\sqrt{2}$$

1.2 Curvature

Suppose we are on some interval of t where $\vec{r}(t)$ is a smooth function and $\vec{r'}(t) \neq 0$. In this setting we can discuss the curvature of the curve defined by $\vec{r}(t)$.

We think of curvature as the amount that the direction of the curve changes over a small step (the size of this step approaches 0)

Curvature depends solely on arc length. $< \cos(t), \sin(t) >$ is "drawn" more slowly than $< \cos(4t), \sin(4t)$ but they still have the same curvature because they're both circles with the same radius.

2 §13.4 Acceleration Vector

3 §14.1 Multivariable Functions

4 §14.2 Limits and Continuity

5 §14.3 Partial Derivatives

6 §14.4 Tangent planes, linear approximation