# MATH 222: Week 4

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## Contents



# 1 §14.5 Chain Rule

The chain rule in 1-dimension is as follows:

For an equation $y = f(x(t))$

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

**Example.** *If* $y = (x(t))^2$ *and* $x(t) = \ln 1 + t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2x\frac{1}{1+t} = \frac{2\ln 1 + t}{1+t}$$

**Example.** *Suppose* $f(x, y) = xy + x^2 + y$
$x(t) = \ln 1 + t,\ y(t) = e^{t^2}$
*Turn* $f(x, y)$ *into a function* $g(t)$ *with only the time parameter.*

$$g(t) = f(x(t), y(t)) = \ln 1 + te^{t^2} + (\ln 1 + t)^2 + e^{t^2}$$

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{e^{t^2}}{1+t} + 2t\ln 1 + te^{t^2} + \frac{2\ln 1+t}{1+t} + 2te^{t^2}$$

*Wherever we can, replace the values of $x(t),\ y(t)$ with $x(t),\ y(t)$.*

$$\begin{aligned}\frac{dg}{dt} = \frac{df}{dt} &= \frac{y(t)}{1+t} + 2te^{t^2}x(t) + \frac{2}{1+t} + 2te^{t^2}\\ &= x'(t)y(t) + y'(t)x(t) + 2x'(t)x(t) + y'(t)\\ &= x'(t)(y(t)+2x(t)) + y'(t)(x(t)+1)\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\end{aligned}$$

*In the second to last line, we use the fact that differentiating $f$ with respect to $x$ gives $y+2x$ and doing the same for $y$ gives $x+1$.*

There are two possible cases for the chain rule. Suppose in both cases we have $z = f(x,y)$. In the first case we have $x = g(t)$ and $y = h(t)$. In this case $z$ is a differentiable function of $t$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second case, $x = g(s,t)$ and $y = h(s,t)$. Then $z$ is a differentiable function of both $s$ and $t$.

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

**Example.** $z = e^x\cos(x+y),\ x = s^2t,\ y = st^2$. *Find* $\frac{dz}{ds}$ *and* $\frac{dz}{dt}$.

$$\frac{\partial z}{\partial x} = e^x(\cos(x+y) - \sin(x+y))$$

$$\frac{\partial z}{\partial y} = -e^x\sin(x+y)$$

*Then we need* $\frac{dx}{ds},\ \frac{dx}{dt},\ \frac{dy}{ds},\ \frac{dy}{dt}$

$$\frac{dx}{ds} = 2st,\ \frac{dx}{dt} = s^2$$

$$\frac{dy}{ds} = t^2,\ \frac{dy}{dt} = 2st$$

*So now we can find the general equations for $\frac{dz}{ds}$ and $\frac{dz}{dt}$.*

$$\frac{dz}{ds} = (e^x)(\cos(x+y) - \sin(x+y))(2st) + (-e^x)(\sin(x+y))(t^2)$$

$$\frac{dz}{dt} = (e^x)(\cos(x+y) - \sin(x+y))(s^2) + (-e^x)(\sin(x+y))(2st)$$

**Example.** *If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and $f$ is differentiable, show that $t\frac{dg}{ds} + s\frac{dg}{dt} = 0$.*

*Based on $f$, $x(s,t) = s^2 - t^2$ and $y(s,t) = t^2 - s^2$. Therefore we can write that $g(s,t) = f(x(s,t), y(s,t))$. We need to find $\frac{dg}{dt}$ and $\frac{dg}{ds}$ and to do this we need need to differentiate $x(s,t)$, $y(s,t)$ by both $s$ and $t$.*

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\frac{dg}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

*If we multiply the first equation all by $t$ and the second all by $s$, we get*

$$t\frac{dg}{ds} = -2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}$$

$$s\frac{dg}{dt} = 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}$$

*Doing linear combination gives*

$$t\frac{dg}{ds} + s\frac{dg}{dt} = 0$$

## 1.1 Chain rule and implicit functions

In 1 dimension, if we had an implicit function $F(x,y) = 0$ we would do the following to differentiate it.

$$\frac{dF}{dx}(x,y) = 0 \;\Rightarrow\; \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0$$

$\frac{dx}{dx}$ always equals 1, so we get an equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dy}}$$

Provided that $\frac{dF}{dy}$ doesn't equal 0.

We can apply this idea to an implicit function like $F(x, y, x) = 0$ as well

$$\frac{dF}{dx} = \frac{dF}{dx}\frac{dx}{dx} + \frac{dF}{dy}\frac{dy}{dx} + \frac{dF}{dz}\frac{dz}{dx} = 0$$

Like before, $\frac{dx}{dx}$ is 1. In addition, because $y$ is no longer a function of $x$ we can say that $\frac{dy}{dx} = 0$. $y$ doesn't depend on $x$ at all. However, $z$ does depend on $x$ so that stays put.

$$0 = \frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dz}}$$

## 1.2 Implicit function theorem

Suppose a function $F(x, y, z)$ is defined on a sphere around a point $(a, b, c) \in \mathbb{R}^3$ satisfying $F(a, b, c) = 0$.

If $F_x,\ F_y,\ F_z$ are continuous and $\frac{dF}{dz}$ evaluated at $(a, b, c)$ does not equal 0, then in a neighborhood of $(a, b, c)$ we have that the equation $F(x, y, z) = 0$ defines $z$ as a function of $x,\ y$ near $(a, b, c)$ in this neighborhood. In addition, this function is differentiable in this area and its partial derivatives $\frac{dz}{dx} = f_x(x, y),\ \frac{dz}{dy} = f_y(x, )$ are given by

$$\frac{dz}{dx} = \frac{\frac{-dF}{dx}}{\frac{dF}{dz}}$$

$$\frac{dz}{dy} = \frac{\frac{-dF}{dy}}{\frac{dF}{dz}}$$

This theorem is really a test that we use to determine if we can get a tangent plane at a given point on the function. If the slope is $\infty$ then our equations for $\frac{dz}{dx},\ \frac{dz}{dy}$ won't work because $\frac{dF}{dz}$ will be zero.

In clas we used the example of a simple circle $x^2 + y^2 = 1$, a circle of radius 1 centered at the origin. If we choose $(a, b)$ to be somewhere in the middle of the top-right quadrant, in this neighborhood of $(a, b)$ we can talk about the curve as a function. There is no point in this neighborhood where the slope is $\infty$. However, if we choose $(a, b)$ to be $(1, 0)$ or $(-1, 0)$ we have a situation where the curve is not a function. We can tell this by looking at the graph of the circle but for more confusing curves we need the Implicit Function Theorem. If $f(a, b)$ has a slope of $\infty$ then since we're going to be dealing with the neighborhood of $(a, b)$, there will be points in this neighborhood that fail the vertical line test. Then we can't find derivatives and tangent planes in this area since it's not a function here.

**Example.** *Let $F(x, y, z) = 0$ for $F(x, y, z) = x^3 + y^3 + z^3 + 6zyz - 9$. Show that around $(1, 1, 1)$ we can define $z$ as a function of $x,\ y$. Find the values of $\frac{dz}{dy}$ and $\frac{dz}{dx}$ at $(1, 1, 1)$.*

*First check if the point is on the surface. $1 + 1 + 1 + 6 - 9 = 0$, so it's on the surface. Next we need to check that $\frac{dF}{dz}$ at the point isn't 0.*

$$\frac{dF}{dz} = 3z^2 + 6xy$$

*Evaluating this is the point gives 9, which isn't 0. Since $F$ is a polynomial we can also say that $F_x, F_y, F_z$ are continuous .*

*By the Implicit Function Theorem, near $(1, 1, 1)$ $z$ is a function of $x,\ y$. Now we need to find $\frac{dz}{dy}$ and $\frac{dz}{dx}$.*

$$\frac{dz}{dx} = \frac{-F_x}{F_z} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy}\Big|_{1,1,1} = -1$$
$$\frac{dz}{dy} = \frac{-F_y}{F_z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}\Big|_{1,1,1} = -1$$

## 1.3 Directional Derivative

**Definition.** *The directional derivative of $f(x, y)$ at $(x_0, y_0)$ in the direction of a unit vector $\vec{u} =< a, b >$ is*

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Let $g(h) = f(x + ah, y + bh)$ so that $g(0) = f(x, y)$.

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = D_{\vec{u}}f(x, y)$$

We can rewrite this using what we know about the chain rule

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

We can say that $x(h) = x_0 + ah$ and $y(h) = y_0 + bh$. From this we can also say that $\frac{dx}{dh} = a$ and $\frac{dy}{dh} = b$. Substitute these into this equation.

$$g'(h) = af_x(x + ah, y + bh) + bf_y(x + ah, y + bh)$$

Set $h = 0$

$$g'(0) = af_x(x, y) + bf_y(x, y)$$

Since we had previously that $g'(0) = D_{\vec{u}}f(x,y)$, we can set these equal and get

$$D_{\vec{u}}f(x,y) = af_x(x,y) + bf_y(x,y)$$

Another common way of writing this is

$$D_{\vec{u}}f(x,y) =< a,b > \cdot < f_x(x,y), f_y(x,y) >$$

$D_{\vec{u}}f$ is the rate of change of $f$ in direction $\vec{u}$. $\vec{u}$ must be a unit vector because we use its coordinates in our equation. If $\vec{u}$ isn't a unit vector, we won't get the correct answer.

**Example.** *Compute the directional derivative of* $f(x,y) = x^2 + 2y^2 + y$ *at* $(1,1)$ *in direction* $< 1,2 >$

*To find* $a,\ b$ *we need to divide the coordinates of* $\vec{u}$ *by the vector's length.*

$$a = \frac{1}{\| \vec{u} \|} = \frac{\sqrt{5}}{5}$$

$$b = \frac{2}{\| \vec{u} \|} = \frac{2\sqrt{5}}{5}$$

*Now if we just find* $f_x$ *and* $f_y$*, we can find the directional derivative.*

$$D_{\vec{u}} =< \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} > \cdot < 2x, 4y+1 >$$

*Plug in the point we were given.*

$$< \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} > \cdot < 2,5 >= \frac{12\sqrt{5}}{5}$$

*This is the directional derivative at* $(1,1)$ *in direction* $< 1,2 >$.

## 1.4 The Gradient Vector

A question that naturally follows from this is for what $\vec{u} =< a,b >$ do we get the largest directional derivative? If we want to maximize $< a,b > \cdot < f_x(x,y), f_y(x,y) >$ we should look at an identity we learned earlier in the course.

$$< a,b > \cdot < f_x(x,y), f_y(x,y) >= \|< a,b >\| \ \|< f_x, f_y >\| \cos(\theta)$$

We already know that the length of $< a, b >$ is 1. $< f_x, f_y >$ has a fixed sized. $\cos(\theta)$ is largest at $\theta = 0$. Therefore the directional derivative is largest when $< a, b >$ is parallel to $f_x, f_y >$ (when the angle between them is 0). Therefore $D_{\vec{u}} f$ is maximized when $< a, b >= \frac{< f_x, f_y >}{\| < f_x, f_y > \|}$.

We call this vector in the direction of the maximum change if you start at some point $x_0, y_0$ the gradient vector:

$$\nabla f(x_0, y_0) =< f_x(x_0, y_0), f_y(x_0, y_0) >$$

We don't necessarily need this to be a unit vector, so we won't divide the vector by its length here like it would be in the directional derivative.

**Example.** *Find the direction of the maximum derivative for $f(x, y) = x^2 + 2y^2 + y$ and find the value of that max derivative.*

*The direction of the max derivative will be the gradient vector. Find $f_x$, $f_y$ and plug them into our gradient vector equation.*

$$\nabla f =< 2x, 4y + 1 > \Rightarrow \nabla f(1, 1) =< 2, 5 >$$

*This vector gives the direction (we don't really care that it's not a unit vector). However, to find the max value of the directional derivative we'll need to do:*

$$D_{\vec{u}} f(x, y) =< a, b > \cdot < f_x(x, y), f_y(x, y) >$$

*In this case, $< a, b >$ is $< \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} >$ since we want the directional derivative in this direction. We already know that $< f_x, f_y >$ is $< 2, 5 >$ because we already calculated these values.*

$$D_{\nabla f} f(1, 1) = \frac{< 2, 5 >}{\| < 2, 5 > \|} \cdot < 2, 5 >= \frac{\| < 2, 5 > \|^2}{\| < 2, 5 > \|} = \sqrt{29}$$

This gives us valuable information. The maximum directional derivative (directional derivative calculated at $\nabla f$) will be

$$D_{\nabla f} f = \frac{< f_x, f_y >}{\| < f_x, f_y > \|} \cdot < f_x, f_y >= \frac{\| < f_x, f_y > \|^2}{\| < f_x, f_y > \|} = \| < f_x, f_y > \|$$

We can do this because of the rule we learned early in the course that the dot product of a vector and itself equals the magnitude squared of that vector.

Here is an example of a type of problem that might make more sense. As a brief note, the directional derivative in 3D is as follows:

$$D_{\vec{u}}f =< u_1, u_2, u_3 > \cdot \nabla f(x, y, z)$$

**Example.** *Suppose the temperature at a point* $(x, y, z)$ *in space is given by* $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$ *in degrees Celsius. In what direction does the temperature increase fastest when starting from* $(1, 1, -2)$*? What is the rate of increase?*

# 2 §14.7 Maximum and Minimum Values

# 3 §14.8 Lagrange Multipliers