

# MATH 222: Week 4

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## 1 §14.5 Chain Rule

The chain rule in 1-dimension is as follows:

For an equation  $y = f(x(t))$

$$\frac{dy}{dt} = \frac{df}{dx} \frac{dx}{dt}$$

**Example.** If  $y = (x(t))^2$  and  $x(t) = \ln 1 + t$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = 2x \frac{1}{1+t} = \frac{2 \ln 1 + t}{1+t}$$

**Example.** Suppose  $f(x, y) = xy + x^2 + y$

$$x(t) = \ln 1 + t, \quad y(t) = e^{t^2}$$

Turn  $f(x, y)$  into a function  $g(t)$  with only the time parameter.

$$g(t) = f(x(t), y(t)) = \ln 1 + te^{t^2} + (\ln 1 + t)^2 + e^{t^2}$$

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{e^{t^2}}{1+t} + 2t \ln 1 + te^{t^2} + \frac{2 \ln 1 + t}{1+t} + 2te^{t^2}$$

Wherever we can, replace the values of  $x(t)$ ,  $y(t)$  with  $x(t)$ ,  $y(t)$ .

$$\begin{aligned}\frac{dg}{dt} &= \frac{df}{dt} = \frac{y(t)}{1+t} + 2te^{t^2}x(t) + \frac{2}{1+t} + 2te^{t^2} \\ &= x'(t)y(t) + y'(t)x(t) + 2x'(t)x(t) + y'(t) \\ &= x'(t)(y(t) + 2x(t)) + y'(t)(x(t) + 1) \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}\end{aligned}$$

In the second to last line, we use the fact that differentiating  $f$  with respect to  $x$  gives  $y + 2x$  and doing the same for  $y$  gives  $x + 1$ .

There are two possible cases for the chain rule. Suppose in both cases we have  $z = f(x, y)$ . In the first case we have  $x = g(t)$  and  $y = h(t)$ . In this case  $z$  is a differentiable function of  $t$ .

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

In the second case,  $x = g(s, t)$  and  $y = h(s, t)$ . Then  $z$  is a differentiable function of both  $s$  and  $t$ .

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} \\ \frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}\end{aligned}$$

**Example.**  $z = e^x \cos(x + y)$ ,  $x = s^2t$ ,  $y = st^2$ . Find  $\frac{dz}{ds}$  and  $\frac{dz}{dt}$ .

$$\frac{\partial z}{\partial x} = e^x(\cos(x + y) - \sin(x + y))$$

$$\frac{\partial z}{\partial y} = -e^x \sin(x + y)$$

Then we need  $\frac{dx}{ds}$ ,  $\frac{dx}{dt}$ ,  $\frac{dy}{ds}$ ,  $\frac{dy}{dt}$

$$\frac{dx}{ds} = 2st, \quad \frac{dx}{dt} = s^2$$

$$\frac{dy}{ds} = t^2, \quad \frac{dy}{dt} = 2st$$

So now we can find the general equations for  $\frac{dz}{ds}$  and  $\frac{dz}{dt}$ .

$$\frac{dz}{ds} = (e^x)(\cos(x + y) - \sin(x + y))(2st) + (-e^x)(\sin(x + y))(t^2)$$

$$\frac{dz}{dt} = (e^x)(\cos(x+y) - \sin(x+y))(s^2) + (-e^x)(\sin(x+y))(2st)$$

**Example.** If  $g(s, t) = f(s^2 - t^2, t^2 - s^2)$  and  $f$  is differentiable, show that  $t \frac{dg}{ds} + s \frac{dg}{dt} = 0$ .

Based on  $f$ ,  $x(s, t) = s^2 - t^2$  and  $y(s, t) = t^2 - s^2$ . Therefore we can write that  $g(s, t) = f(x(s, t), y(s, t))$ . We need to find  $\frac{dg}{dt}$  and  $\frac{dg}{ds}$  and to do this we need need to differentiate  $x(s, t)$ ,  $y(s, t)$  by both  $s$  and  $t$ .

$$\frac{dg}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\frac{dg}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

If we multiply the first equation all by  $t$  and the second all by  $s$ , we get

$$t \frac{dg}{ds} = -2st \frac{\partial f}{\partial x} + 2st \frac{\partial f}{\partial y}$$

$$s \frac{dg}{dt} = 2st \frac{\partial f}{\partial x} - 2st \frac{\partial f}{\partial y}$$

Doing linear combination gives

$$t \frac{dg}{ds} + s \frac{dg}{dt} = 0$$

## 1.1 Chain rule and implicit functions

In 1 dimension, if we had an implicit function  $F(x, y) = 0$  we would do the following to differentiate it.

$$\frac{dF}{dx}(x, y) = 0 \Rightarrow \frac{dF}{dx} \frac{dx}{dx} + \frac{dF}{dy} \frac{dy}{dx} = 0$$

$\frac{dx}{dx}$  always equals 1, so we get an equation for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = \frac{-\frac{dF}{dx}}{\frac{dF}{dy}}$$

Provided that  $\frac{dF}{dy}$  doesn't equal 0.

We can apply this idea to an implicit function like  $F(x, y, x) = 0$  as well

$$\frac{dF}{dx} = \frac{dF}{dx} \frac{dx}{dx} + \frac{dF}{dy} \frac{dy}{dx} + \frac{dF}{dz} \frac{dz}{dx} = 0$$

Like before,  $\frac{dx}{dx}$  is 1. In addition, because  $y$  is no longer a function of  $x$  we can say that

$\frac{dy}{dx} = 0$ .  $y$  doesn't depend on  $x$  at all. However,  $z$  does depend on  $x$  so that stays put.

$$0 = \frac{dF}{dx} + \frac{dF}{dz} \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{-\frac{dF}{dx}}{\frac{dF}{dz}}$$

## 1.2 Implicit function theorem

Suppose a function  $F(x, y, z)$  is defined on a sphere around a point  $(a, b, c) \in \mathbb{R}^3$  satisfying  $F(a, b, c) = 0$ .

If  $F_x, F_y, F_z$  are continuous and  $\frac{dF}{dz}$  evaluated at  $(a, b, c)$  does not equal 0, then in a neighborhood of  $(a, b, c)$  we have that the equation  $F(x, y, z) = 0$  defines  $z$  as a function of  $x, y$  near  $(a, b, c)$  in this neighborhood. In addition, this function is differentiable in this area and its partial derivatives  $\frac{dz}{dx} = f_x(x, y), \frac{dz}{dy} = f_y(x, y)$  are given by

$$\frac{dz}{dx} = \frac{-\frac{dF}{dx}}{\frac{dF}{dz}}$$

$$\frac{dz}{dy} = \frac{-\frac{dF}{dy}}{\frac{dF}{dz}}$$

This theorem is really a test that we use to determine if we can get a tangent plane at a given point on the function. If the slope is  $\infty$  then our equations for  $\frac{dz}{dx}, \frac{dz}{dy}$  won't work because  $\frac{dF}{dz}$  will be zero.

In class we used the example of a simple circle  $x^2 + y^2 = 1$ , a circle of radius 1 centered at the origin. If we choose  $(a, b)$  to be somewhere in the middle of the top-right quadrant, in this neighborhood of  $(a, b)$  we can talk about the curve as a function. There is no point in this neighborhood where the slope is  $\infty$ . However, if we choose  $(a, b)$  to be  $(1, 0)$  or  $(-1, 0)$  we have a situation where the curve is not a function. We can tell this by looking at the graph of the circle but for more confusing curves we need the Implicit Function Theorem. If  $f(a, b)$  has a slope of  $\infty$  then since we're going to be dealing with the neighborhood of  $(a, b)$ , there will be points in this neighborhood that fail the vertical line test. Then we can't find derivatives and tangent planes in this area since it's not a function here.

**Example.** Let  $F(x, y, z) = 0$  for  $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 9$ . Show that around  $(1, 1, 1)$  we can define  $z$  as a function of  $x, y$ . Find the values of  $\frac{dz}{dy}$  and  $\frac{dz}{dx}$  at  $(1, 1, 1)$ .

First check if the point is on the surface.  $1 + 1 + 1 + 6 - 9 = 0$ , so it's on the surface.

Next we need to check that  $\frac{dF}{dz}$  at the point isn't 0.

$$\frac{dF}{dz} = 3z^2 + 6xy$$

Evaluating this at the point gives 9, which isn't 0. Since  $F$  is a polynomial we can also say that  $F_x, F_y, F_z$  are continuous.

By the Implicit Function Theorem, near  $(1, 1, 1)$   $z$  is a function of  $x, y$ . Now we need to find  $\frac{dz}{dy}$  and  $\frac{dz}{dx}$ .

$$\begin{aligned}\frac{dz}{dx} &= \frac{-F_x}{F_z} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} \Big|_{1,1,1} = -1 \\ \frac{dz}{dy} &= \frac{-F_y}{F_z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy} \Big|_{1,1,1} = -1\end{aligned}$$

### 1.3 Directional Derivative

**Definition.** The directional derivative of  $f(x, y)$  at  $(x_0, y_0)$  in the direction of a unit vector  $\vec{u} = \langle a, b \rangle$  is

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Let  $g(h) = f(x_0 + ah, y_0 + bh)$  so that  $g(0) = f(x_0, y_0)$ .

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = D_{\vec{u}}f(x_0, y_0)$$

We can rewrite this using what we know about the chain rule

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh}$$

We can say that  $x(h) = x_0 + ah$  and  $y(h) = y_0 + bh$ . From this we can also say that  $\frac{dx}{dh} = a$  and  $\frac{dy}{dh} = b$ . Substitute these into this equation.

$$g'(h) = af_x(x_0 + ah, y_0 + bh) + bf_y(x_0 + ah, y_0 + bh)$$

Set  $h = 0$

$$g'(0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$$

Since we had previously that  $g'(0) = D_{\vec{u}}f(x_0, y_0)$ , we can set these equal and get

$$D_{\vec{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$$

Another common way of writing this is

$$D_{\vec{u}}f(x, y) = \langle a, b \rangle \cdot \langle f_x(x, y), f_y(x, y) \rangle$$

$D_{\vec{u}}f$  is the rate of change of  $f$  in direction  $\vec{u}$ .  $\vec{u}$  must be a unit vector because we use its coordinates in our equation. If  $\vec{u}$  isn't a unit vector, we won't get the correct answer.

**Example.** Compute the directional derivative of  $f(x, y) = x^2 + 2y^2 + y$  at  $(1, 1)$  in direction  $\langle 1, 2 \rangle$

To find  $a, b$  we need to divide the coordinates of  $\vec{u}$  by the vector's length.

$$a = \frac{1}{\|\vec{u}\|} = \frac{\sqrt{5}}{5}$$

$$b = \frac{2}{\|\vec{u}\|} = \frac{2\sqrt{5}}{5}$$

Now if we just find  $f_x$  and  $f_y$ , we can find the directional derivative.

$$D_{\vec{u}} = \left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle \cdot \langle 2x, 4y + 1 \rangle$$

Plug in the point we were given.

$$\left\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle \cdot \langle 2, 5 \rangle = \frac{12\sqrt{5}}{5}$$

This is the directional derivative at  $(1, 1)$  in direction  $\langle 1, 2 \rangle$ .

A question that naturally follows from this is for what  $\vec{u} = \langle a, b \rangle$  do we get the largest directional derivative? If we want to maximize  $\langle a, b \rangle \cdot \langle f_x(x, y), f_y(x, y) \rangle$  we should look at an identity we learned earlier in the course.

$$\langle a, b \rangle \cdot \langle f_x(x, y), f_y(x, y) \rangle = \|\langle a, b \rangle\| \|\langle f_x, f_y \rangle\| \cos(\theta)$$

We already know that the length of  $\langle a, b \rangle$  is 1.  $\langle f_x, f_y \rangle$  has a fixed sized.  $\cos(\theta)$  is largest at  $\theta = 0$ . Therefore the directional derivative is largest when  $\langle a, b \rangle$  is parallel to  $\langle f_x, f_y \rangle$  (when the angle between them is 0). Therefore  $D_{\vec{u}}f$  is maximized when  $\langle a, b \rangle = \frac{\langle f_x, f_y \rangle}{\|\langle f_x, f_y \rangle\|}$ .

## 2 §14.7 Maximum and Minimum Values