MATH 222: Week 4

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1 §14.5 Chain Rule

The chain rule in 1-dimension is as follows:

For an equation $y = f(x(t))$

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Example. *If* $y = (x(t))^2$ *and* $x(t) = \ln 1 + t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 2x\frac{1}{1+t} = \frac{2\ln 1 + t}{1+t}$$

Example. *Suppose* $f(x, y) = xy + x^2 + y$
$x(t) = \ln 1 + t,\ y(t) = e^{t^2}$
Turn $f(x, y)$ *into a function* $g(t)$ *with only the time parameter.*

$$g(t) = f(x(t), y(t)) = \ln 1 + te^{t^2} + (\ln 1 + t)^2 + e^{t^2}$$

$$\frac{dg}{dt} = \frac{df}{dt} = \frac{e^{t^2}}{1+t} + 2t\ln 1 + te^{t^2} + \frac{2\ln 1 + t}{1+t} + 2te^{t^2}$$

Wherever we can, replace the values of $x(t),\ y(t)$ *with* $x(t),\ y(t)$.

$$\begin{aligned}\frac{dg}{dt} = \frac{df}{dt} &= \frac{y(t)}{1+t} + 2te^{t^2}x(t) + \frac{2}{1+t} + 2te^{t^2} \\ &= x'(t)y(t) + y'(t)x(t) + 2x'(t)x(t) + y'(t) \\ &= x'(t)(y(t) + 2x(t)) + y'(t)(x(t) + 1)\end{aligned}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second to last line, we use the fact that differentiating f with respect to x gives $y + 2x$ and doing the same for y gives $x + 1$.

There are two possible cases for the chain rule. Suppose in both cases we have $z = f(x, y)$. In the first case we have $x = g(t)$ and $y = h(t)$. In this case z is a differentiable function of t.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

In the second case, $x = g(s, t)$ and $y = h(s, t)$. Then z is a differentiable function of s and t.

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$