MATH 222: Week 1

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1 §11.1 Sequences

A sequence $\{a_n\}_{n=1}^{\infty}$ can be thought of as a an infinite list of numbers. Often they are generated by a function $a_n = f(n)$.

Example.

$$a_n = f(n) = \frac{n+1}{n}$$
$$a_1 = 2,\ a_2 = \frac{3}{2},\ a_3 = \frac{4}{3},\ ...$$

A sequence can also be defined recursively (like the Fibonacci sequence) but we won't use those much.

Thinking of a_n as a function $f(n) : N \to R$ can be useful when studying the limit for large n.

Definition. *A sequence converges to limit L written:*

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ $\exists$ a corresponding N such that $|a_n - L| < \varepsilon \forall N$

Definition. *We say that a sequence diverges as $n \to \infty$ if is not convergent. This includes if $\lim_{n\to\infty} a_n = \pm\infty$*

Example. *$\lim_{x\to\infty} sin(x)$ diverges because its value constantly changes. $\lim_{x\to\infty} e^x$ diverges as well because it keeps increasing up to $+\infty$. There are difference types of divergence.*

Example. *Prove that $a_n = (-1)^n$ diverges*
Suppose that $\lim_{n\to\infty} a_n = L$ exists. But this means that consecutive terms must get closer together, therefore

$$\lim_{n\to\infty} |a_n - a_{n+1}| = 0$$

if this assumption is true. However,

$$\lim_{n\to\infty} \left|(-1)^n - (-1)^{n+1}\right| = 2$$

and not 0. $\therefore a_n = (-1)^n$ diverges.

1.1 Proving a series converges

Theorem. *If* $\lim_{n\to\infty} f(x) = L$ *for* $x \in \mathbb{R}$ *and if* $f(n) = a_n$*, then* $\lim_{n\to\infty} a_n = L$

Remark. *If* $\lim_{x\to\infty} f(x)$ *diverges then this DOES NOT imply that* $a_n = f(n)$ *diverges.*

Example. $\lim_{n\to\infty} sin(\pi x)$ *diverges clearly, but* $\lim_{n\to\infty} sin(\pi x) = 0$ *and this sequence converges. When you change from* $\mathbb{R}$ *to* $\mathbb{N}$ *you might get different results.*

1.2 Sequence limit laws

Suppose $\{a_n\}, \{b_n\}$ are convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then:

$$\lim_{n\to\infty} (a_n + b_n) = A + B$$
$$\lim_{n\to\infty} (a_n - b_n) = A - B$$
$$\lim_{n\to\infty} c * a_n = c * A \ , \ c \in \mathbb{R}$$

Theorem. *If* $f(x)$ *is a continuous function and* $\lim_{n\to\infty} a_n = L$*, then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Example. $a_n = (\frac{1}{n})^{10}$
Take $f(x) = x^{10}$ *so then* $f(\frac{1}{n}) = (\frac{1}{n})^{10}$
Since $f(x) = x^{10}$ *is continuous:*

$$\lim_{n\to\infty} (\frac{1}{n})^{10} = \lim_{n\to\infty} f(\frac{1}{n}) = f(\lim_{n\to\infty} \frac{1}{n}) = f(0) = 0$$

1.3 Squeeze Theorem

If $a_n \leq b_n \leq c_n \ \forall n$ and $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} c_n = C$, then

$$A \leq \lim_{n\to\infty} b_n \leq C$$

Remark. $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$

2 §11.2 Series

Given a sequence $\{a_n\}_{n=1}^{\infty}$, we obtain an infinite series by adding up all if the terms of $\{a_n\}$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

Definition. ***Partial Sums***
The pth partial sum of $\sum\limits_{n=1}^{\infty} a_n$ *is* $s_p = \sum_{n=1}^{p} a_n = a_1 + a_2 + ... + a_p$. s_p *is the pth partial sum.*

Definition. ***Telescoping Series***
A telescoping series is a series where "consecutive" (not always) terms cancel so that's it's possible to write a closed form of the sum of the partial sum.

Definition. ***Geometric Series***
An infinite series is called geometric if $a_n = a * r^n$ *for* $a, r \in \mathbb{R}$

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ...$$

Theorem. *Geometric series* $\sum\limits_{n=1}^{\infty} ar^{n-1}$ *converges to* $\frac{a}{1-r}$ *if* $|r| < 1$

The proof of this would take a long time to type so I won't do it right now.

Example.

$$\begin{aligned}\sum_{n=2}^{\infty} ar^n &= ar^2 + ar^3_a r^4 + ... \\ &= r^2(a + ar + ar^2 + ...) \\ &= r^2 * \frac{a}{1-r} \\ &= \frac{ar^2}{1-r}\end{aligned}$$

Pay attention to where your geometric series starts!

3 §11.3 Integral Test

Suppose $f(x)$ is a continuous, positive, (eventually) decreasing function on $[1, +\infty)$ with $a_n = f(n)$. Then:

i if $\int_1^{\infty} f(x)dx$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is also convergent

ii if $\int_1^{\infty} f(x)dx$ is divergent, then $\sum\limits_{n=1}^{\infty} a_n$ is also divergent

This idea comes from Reimann sums. If the integral of a function is less than infinity, then the series must be less than infinity as well.

Example. $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ *is called the harmonic series.*

i) $f(x) = \frac{1}{x} > 0$, *so therefore* $\frac{1}{x}$ *is continuous*

ii) $f'(x) = \frac{-1}{x^2} < 0$, *so it's decreasing too*

iii) $\lim\limits_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim\limits_{t\to\infty} ln(t) - ln(1) \to \infty$

The integral is divergent, so the series is also divergent.

3.1 P-Test

By using the integral test on series of the form $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, we learn that if $p < 1$ the series diverges and if $p > 1$ the series converges.

Remark. *If* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=1}^{\infty} b_n$ *are convergent, then*

i $\sum\limits_{n=1}^{\infty} a_n + b_n = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$

ii $\sum\limits_{n=1}^{\infty} a_n - b_n = \sum\limits_{n=1}^{\infty} a_n - \sum\limits_{n=1}^{\infty} b_n$

Remark. *Suppose* $\sum\limits_{n=1}^{\infty} a_n$ *is convergent and* $\sum\limits_{n=1}^{\infty} b_n$ *is divergent. Then* $\sum\limits_{n=1}^{\infty} a_n + b_n$ *is divergent.*

4 §11.4 Comparison Test

4.1 Direct Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent and $a_n \leq b_n \forall n$, then $\sum a_n$ is convergent too

ii) If $\sum b_n$ is divergent and $a_n \leq b_n \forall n$, then $\sum a_n$ is divergent too

4.2 Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $Limn \to \infty \frac{a_n}{b_n} = c$ and c is non-zero and finite, then both $\sum a_n$ and $\sum b_n$ converge or diverge.

If you're a computer science student, you may recognize this as Big-Theta. The idea here is that if the limit of a function divided by another function is 0, then the function on the bottom must grow faster. If the limit is ∞, then the function on the top must grow faster. However, if both functions grow at roughly the same rate, the limit will be some constant c.

This is the relationship that the limit comparison test exploits. If the limit is c then a_n and b_n grow at roughly the same rate. For this reason if we have a series $\sum a_n$ that we want to test for convergence we can pick a series $\sum b_n$ that we know a lot about, such as the harmonic series, a p-series, etc. If we can show that a_n grows at the same rate as b_n we can say a_n must converge (or diverge) if b_n also converges (or diverges).

5 §11.5 Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ satisfied the following criteria:

i) $b_n \geq 0$

ii) $b_{n+1} \leq b_n \forall n \geq n_0$ (eventually)

iii) $\lim_{b_n} = 0$

then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Example. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+2}}$
Finding the first derivative of b_n gives $\frac{n}{(n^2+2)^{3/2}}$, which is less than 0 so b_n is decreasing, and the limit of b_n is 0. Therefore by AST, the series is convergent.

6 §11.6 Absolute Convergence

Definition. *A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Definition. *A series* $\sum_{n=1}^{\infty} a_n$ *is called conditionally convergent if* $\sum_{n=1}^{\infty} |a_n|$ *is divergent but* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Example. *Determine if* $\sum_{n=1}^{\infty} \frac{sin(n)}{n^2}$ *is convergent or divergent*

This function isn't alternating because its period is 2π. *It's also not positive, so we can't use our other tests. We need to test for absolute convergence with the absolute value of* b_n.
$|sin(n)| \leq 1 \Rightarrow \left|\frac{sin(n)}{n^2}\right| \leq \frac{1}{n^2}$
This is a convergent p-series, therefore the series is absolutely convergent.

6.1 Ratio and Root Tests

Ratio Test For the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$:

i) if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

ii) if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent

iii) if $L = 1$, then the test is inconclusive

In theory this makes sense, because for a series to converge the next value in the series must be less than the previous and this relationship must be true as n reaches infinity.

Proof. Proof of part i:
Suppose $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, so that $\exists$ numbers k, r such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r < 1 \ \forall \ n \geq k$$

$$\Rightarrow |a_{n+1}| < r\,|a_n|$$

$$\Rightarrow |a_{n+2}| < r\,|a_{n+1}| < r^2\,|a_n|$$

$$\Rightarrow \sum |a_{n+k}| \leq \sum r^n\,|a_k|$$

□

Root Test For the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

i) if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

ii) if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent

iii) if $L = 1$, then the test is inconclusive

7 §11.8 Power Series

Definition. *A power series is a series of the form*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ...$$

Evaluating a power series at a number x gives a numerical series $\sum_{n=0}^{\infty} a_n$. Because x is a variable, the convergence of the series depends on the value of x.

We obtain a function $f(x) = \sum_{n=0}^{\infty} c_n x^n$, and the domain of this function is the set of all $x \in \mathbb{R}$ such that the power series converges to a finite number.

Example. *The power series where $c_n = 1$ is*
$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ...$
This is a geometric series. Geometric series converge if their value of r, the difference between the terms, is between -1 and 1. Therefore we can say that the series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ if $|x| < 1$

We say that $\sum_{n=0}^{\infty} x^n$ is the power series representation of $\frac{1}{1-x}$ on $|x| < 1$

A power series centered at $x = a$ is written

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

Notice that when we set $x = a$, the series equals c_0 because all other terms are multiplied by 0. **This gives the valuable information that a power series is always convergent at its center.**

Theorem. *For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities:*

i) The series converges at the center, where $x = a$

ii) The series converges for all $x \in \mathbb{R}$

iii) There exists a radius (abbreviated R) such that the series converges for $x - a < R$ and diverges for $x - a > R$

Power series that fit the third definition have both a **radius of convergence** and **interval of convergence**. Beginning at $x = a$, for values of x between $a + R$ and $a - R$, the series converges. The interval of convergence gives specific information about those endpoints. Sometimes the series converges for both, one, or neither of its endpoints. This is represented by a (or) if the endpoint doesn't result is convergence or a [or] if it does.

The **Ratio Test** has a lot of use with power series. It is used to find the ROC and IOC. This illustrates why.

Example. *Find the ROC and IOC of* $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$
Use the ratio test

$$\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}| < 1$$

$$\lim_{n\to\infty} |x \cdot \frac{n^n}{(n+1)^{n+1}}|$$

$$|x| \lim_{n\to\infty} |\frac{1}{n+1} \cdot (\frac{n}{n+1})^n| = 0$$

$\frac{1}{n+1}$ *has limit of 0 and* $(\frac{n}{n+1})^n$ *has a limit of* $\frac{1}{e}$*. The ratio test states that in order for the series to converge, the limit must be less than 1. In this case, no matter what the value of x is the limit will always be 0, which is less than 1. Therefore this power series converges for all values of x*
ROC$=\infty$ *and IOC*$=(-\infty, +\infty)$

8 §11.9 Representation

We stated previously that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. We can exploit this in order to represent a variety of functions as power series.

Example. *Here are a few examples of rewriting functions in this way*

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

true when $|x| < 1$

$$\frac{1}{1-2x} = \sum_{n=1}^{\infty}(2x)^n$$

true when $|2x| < 1 \;\Rightarrow\; |x| < \frac{1}{2}$

$$\frac{1}{1+x^4} = \frac{1}{1-(-x)^4} = \sum_{n=0}^{\infty}(-1)^n x^{4n}$$

true when $|x^4| < 1 \;\Rightarrow\; |x| < 1$

Since we can do this, we are also able to write the power series for functions whose integrals or derivative ressemble $\frac{1}{1-x}$ such as $\ln x$ and $\arctan x$. We do this by finding the power series for the integral or derivative, then (respective) differentiate or integrate that series to find the power series of the original function.

9 §11.10 Taylor and Maclaurin Series

Taylor and Maclaurin series allow us to find a power series representation of any given function $f(x)$ even if its derivative or antiderivative isn't close to the form $\frac{1}{1-x}$.

Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n,\ |x-a| < R$

$$f(a) = c_0 + 0 \;\Rightarrow\; c_0 = f(a)$$

$$f'(a) = \frac{d}{dx}(c_0 + c_1(x-a) + c_2(x-a)^2 + ...) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ...$$
$$\Rightarrow\; f'(a) = c_1 + 0 + 0 + ... \Rightarrow\; f'(a) = c_1$$

From this above pattern we can deduce that:

$$f''(a) = 2c_2 \Rightarrow\; c_2 = \frac{f''(a)}{2}$$

$$f'''(a) = 3!c_3 \Rightarrow\; c_3 = \frac{f'''(a)}{3!}$$

Therefore the power series (Taylor series) that represents/equals $f(x)$ can be written as:

$$\sum_{n=0}^{\infty}\frac{f^n(a)(x-a)^n}{n!}$$

There are three main Maclaurin series that are important to know/memorize:

e^x

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$

$\sin x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$\cos x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

All of these converge to the true value of the functions they represent. This can we verified using the ratio test.

9.1 Remainders

Suppose we only compute a finite number of terms from a Taylor/Maclaurin series. How far is this "finite" series from the true value?

Suppose we deconstruct the standard form of a Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!} = \sum_{n=0}^{p} \frac{f^n(a)(x-a)^n}{n!} + \sum_{n=p+1}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

As long as the second term of this is small, $f(x)$ can be approximated using just the first term. Since the series must converge in order to equal the function and convergent series have terms that generally decrease in size as n becomes larger, for many values of p we will have an acceptable amount of error

For convenience's sake:

$$T_p(x) = \sum_{n=0}^{p} \frac{f^n(a)(x-a)^n}{n!}, \ R_p(x) = \sum_{n=p+1}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

Theorem. *If $f(x) = T_p(x) + R_p(x)$ where $T_p(x)$ is the pth degree Taylor polynomial of $f(x)$ centered at a and $\lim_{p\to\infty} R_p(x) = 0$ on $|x-a| < ROC$, then $f(x)$ equals its Taylor series on $|x-a| < ROC$*

Theorem.

$$|R_p(x)| \leq \frac{f^{p+1}(z)(x-a)^{p+1}}{(p+1)!}$$

Where z equals the value between $x-a$ and $x+a$ that causes this expression to have its largest possible value (and therefore the largest possible difference from the true value of the function $f(x)$)

The proof of the $p=1$ case of this involves calling $f''(z) = M$ (for $x = max(x-a, x+a)$) and then integrating this twice.

$$\int_a^x f''(t)\ dt \leq \int_a^x M\ dt \ \Rightarrow f'(x) - f'(a) \leq M(x-a)$$

$$\int_z^x f'(t) - f'(a)\ dt \leq \int_a^x M(t-a)\ dt \Rightarrow f(x) - f(a) - f'(a)(x-a) \leq \frac{M(x-a)^2}{2}$$

$$\Rightarrow |f(x) - T_1(x)| \leq \frac{M(x-a)^2}{2}$$

However, the LHS equals $|R_1(x)|$

$$\Rightarrow |R_p(x)| \leq \frac{f^{p+1}(z)(x-a)^{p+1}}{(p+1)!}$$

This is known as the Lagrange form of the remainder

$$R_p(x) = \frac{f^{p+1}(z)(x-a)^{p+1}}{(p+1)!}$$

Example. *Find a Taylor polynomial which approximates $\sin\frac{1}{10}$ to 4 decimal places.*

We want $R_p(\frac{1}{10})$ to be less than $\frac{1}{10000}$. We must solve the inequality

$$\frac{f^{p+1}(z)x^{p+1}}{(p+1)!} \leq \frac{1}{10000}$$

There are a few routes you can take with a question like this. I have decided to start at $p=2$ and see if it will make the inequality true. The 3rd derivative of $\sin x$ is $-\cos x$.

$$\frac{-\cos{(z)}x^3}{3!} \leq \frac{1}{10000}$$

$$\frac{-\cos{(z)}(\frac{1}{10})^3}{6} \leq \frac{1}{10000}$$

Multiply each side by $(\frac{1}{10})^{-3}$.

$$|\frac{-1}{6}| \leq \frac{1}{10}$$

It doesn't look like $p = 2$ *will work, let's try* $p = 3$. *The 4th derivative of* $\sin x$ *is* $\sin x$.

$$\frac{\sin{(z)} x^4}{4!} \leq \frac{1}{10000}$$

$$\frac{\sin{(z)} (\frac{1}{10})^4}{24} \leq \frac{1}{10000}$$

$$\frac{\sin{(z)}}{24} \leq 1$$

The only way for this to not be true is if $\sin z = \sin \frac{1}{10} > 24$ *(*z *will be* $\frac{1}{10}$ *because that's the value of* z *that makes the function larges in this range). This is impossible because* $\sin$ *has an upper bound of* 1. *Therefore we need a polynomial of at least degree* 3.
A polynomial that approximates $\sin \frac{1}{1}$ *to* 4 *decimal places is* $x - \frac{x^3}{3!}$. *The true value of* $\sin \frac{1}{10}$ *is* 0.0998334166 *and our approximation gives* $0.0998\overline{3}$

Example. *Find the Maclaurin series of* $F(x) = \int e^{-x^2}$

The only way to do this is with power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; \forall \; x \in \mathbb{R}$$

$$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Now we just need to integrate this by using the rule that $\int x^n = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \int e^{-x^2} = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} + C$$

9.2 Limits using power series

Try to compute $\lim\limits_{x \to 0} \frac{\sin x^2}{x}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Now rewrite the limit using the long-form of the series.

$$\lim_{x \to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^10}{5!} - ...}{x}$$

Divide it all by x

$$\lim_{x \to 0} x - \frac{x^5}{3!} + \frac{x^9}{5!} - ... = 0$$

A hint for problems like these is to center the series at whatever the value of x is approaching (0, in this case)

9.3 $\times$ and $\div$ of Power Series

Theorem. *Suppose we have two power series*

$$f(x) = \sum a_n (x-a)^n, \ |x-a| < R_1$$

$$g(x) = \sum b_n (x-a)^n, \ |x-a| < R_2$$

Then:

$$f(x) + g(x) = \sum (a_n + b_n)(x-a)^n, \ |x-a| < min(R_1, R_2)$$

$$f(x)g(x) = (\sum a_n (x-a)^n)(\sum b_n (x-a)^n), \ |x-a| < min(R_1, R_2)$$

And, if $g(a) \neq 0$:

$$\frac{f(x)}{g(x)} = \frac{\sum a_n (x-a)^n}{\sum b_n (x-a)^n}$$

for $|x-a| < r$, for some r

Example. *For example, to find the first three non-zero terms of $f(x) = \sin x \cos x$ we would have to find the sum of all multiples of terms in their series that had the three smallest degrees. In this case:*

$$(x - \frac{x^3}{3!} + \frac{x^5}{5!})(1 - \frac{x^2}{2!} + \frac{x^4}{4!})$$

$$= x - x^3(\frac{1}{6} + \frac{1}{2}) + x^5(\frac{1}{120} + \frac{1}{24} + \frac{1}{12})$$