

# Demand Estimation

## *MIXTAPE SESSION*

Jeff Gortmaker and Ariel Pakes



## Last Class

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- Let's go over your first coding exercise.

# Unrealistic Substitution Patterns

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- Last week we derived the own-price elasticity. What about the cross-price one?

$$\eta_{jkt} = \frac{\partial \log q_{jt}}{\partial \log p_{kt}} = \frac{\partial q_{jt}}{\partial p_{kt}} \frac{p_{kt}}{q_{jt}} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = -\alpha \cdot p_{kt} \cdot s_{kt}$$

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- Doesn't depend on the characteristics of  $j$ !  
→ Independence of Irrelevant Alternatives (IIA) property.



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- Most industrial organization examples are about cereals or automobiles.
- There are two options: buying a car or a blue bus. Each has a 50% market share.
- Introduce a second bus, but it's red. Pure logit (IIA) predicts 33% market shares.
  - In your exercise, consumers substituted *proportionally* from each cereal.
- In reality, we'd expect the car to still have 50% and each bus to have 25%.
  - In your exercise, we'd hope for more substitution from more similar cereals.

# Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

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- This will allow 50% of consumers to really like cars and 50% to really like buses.
  - When a new bus is introduced, this doesn't really affect the car-lovers' choice.
- Want  $\mu_{ijt}$  to dominate logit substitution from convenient but unrealistic  $\varepsilon_{ijt}$ .
  - Want to add multiple dimensions of heterogeneity that really matter in our setting.



# Random Coefficients

$$u_{ijt} = x'_{jt}\beta + \xi_{jt} + \varepsilon_{ijt}$$

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  - For simplicity, I'll just let  $x_{jt}$  denote all characteristics, including prices  $p_{jt}$ .
- Intuitively, we want to replace  $\beta$  with *random coefficients*  $\beta_{it}$ .
  - *Random* in that they're drawn from a distribution of consumer types  $i \in \mathcal{I}_t$ .
  - For  $x_{jt} = \text{car}_{jt}$  and  $\mathcal{I}_t = \{\text{car-lovers}, \text{bus-lovers}\}$ , want  $\beta_{it} \gg 0$  for car-lovers.

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- Most common specification is  $\beta_{it} \sim N(\beta + \Pi y_{it}, \Sigma \Sigma')$ .
  - $\Pi$  shifts preferences according to “observed” demographics  $y_{it} \sim \text{census}$ .
  - $\Sigma$  shifts preferences according to “unobserved” preferences  $\nu_{it} \sim N(0, I)$ .
  - $\Sigma$  is the *Cholesky root* of the variance matrix. Usually diagonal with standard deviations.

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- In practice, we implement random coefficients by making a new dataset.
  - In PyBLP lingo, “product data” rows are  $(j, t)$ ’s, and new “agent data” rows are  $(i, t)$ ’s.

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- In your coding exercise, you’ll just draw  $|\mathcal{I}_t| = 100$  types per market.
  - Draw  $\nu_{it} \sim N(0, I)$  from a random number generator.
  - Draw  $y_{it}$  from census data on demographics: income, etc.
  - Each type is equally-likely, so use equal sampling weights  $w_{it} = 1/|\mathcal{I}_t|$ .

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  - Each type is equally-likely, so use equal sampling weights  $w_{it} = 1/|\mathcal{I}_t|$ .
- The goal is to have a dataset that reflects the *distribution* of individuals.
  - Realism aside, this allows us to address distributional concerns.
  - E.g. will a tax or price change affect high- or low-income individuals differently?

# Roadmap

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Coding Exercise 2



# From Linear Regression to GMM

$$\log \frac{s_{jt}}{s_{0t}} = \delta_{jt} = x'_{jt} \beta + \xi_{jt}$$

- In your exercise, you estimated  $\beta$  by running the above regression.
  - Again, let  $x_{jt}$  include price, a constant, any other characteristics.
  - Let  $z_{jt}$  include our price IV and exogenous characteristics in  $x_{jt}$ .

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- Our exclusion restriction implies the moment condition  $\mathbb{E}[\xi_{jt} \cdot z_{jt}] = 0$ .
- We'd get the exact same  $\hat{\beta}$  by optimizing the following GMM objective:

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} g(\beta)' W g(\beta) \quad \text{where} \quad g(\beta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt} - x'_{jt}\beta) \cdot z_{jt}$$

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$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp[\delta_{jt} + \mu_{ijt}(\Sigma, \Pi)]}{1 + \sum_{k \in \mathcal{J}_t} \exp[\delta_{kt} + \mu_{ikt}(\Sigma, \Pi)]} \quad \text{for all } j \in \mathcal{J}_t$$

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  - PyBLP will take care of this, but see [Conlon and Gortmaker \(2020\)](#) if interested.
- [BLP's \(1995\)](#) big advancement was how to incorporate flexible preference heterogeneity.
  - Built on simulation estimator advancements ([Pakes and Pollard, 1989](#); [McFadden, 1989](#)).

# The BLP Estimator

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} g(\theta)Wg(\theta)' \quad \text{where} \quad g(\theta) = \frac{1}{N} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}_t} (\delta_{jt}(\Sigma, \Pi) - x'_{jt}\beta) \cdot z_{jt}$$

- BLP estimation consists of two nested loops.
  1. In the “outer” loop, we optimize over  $\theta = (\beta, \Sigma, \Pi)$ .
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  - Get  $\hat{\beta}$  by running an IV regression of  $\delta_{jt}(\Sigma, \Pi)$  on  $x_{jt}$ , like in the pure logit exercise.
- What about the GMM weighting matrix  $W$ ?
  - If you’re just-identified ( $\dim z_{jt} = \dim \theta$ ), it doesn’t matter. You’ll get a zero objective.
  - Otherwise, you may want to repeat optimization with optimal the two-step GMM  $\hat{W}$ .

# Roadmap

Preference Heterogeneity

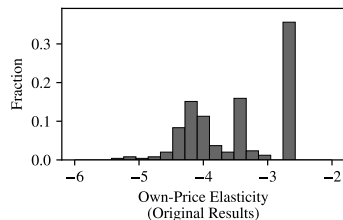
Mixed Logit Estimation

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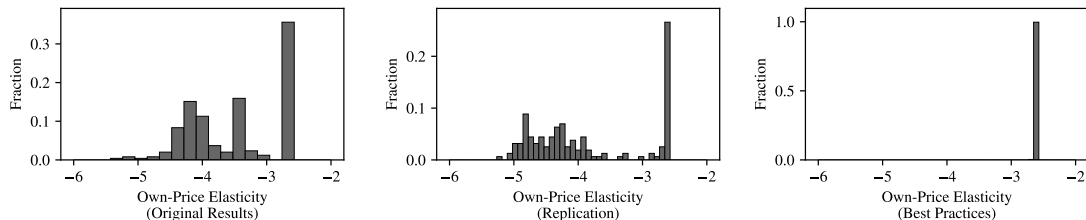
Coding Exercise 2

# Motivation for Numerical Best Practices



- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers ([Knittel and Metaxoglou, 2014](#)).

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- Variation in BLP estimates across different optimization algorithms and starting values has disillusioned some researchers ([Knittel and Metaxoglou, 2014](#)).
- But there are some numerical best practices that you can follow to avoid these kinds of issues ([Conlon and Gortmaker, 2020](#)).
  - They're likely to be useful for most computation-heavy structural estimation, not just BLP!

# Nonlinear Optimization

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- Set **box constraints**  $\theta \in [\underline{\theta}, \bar{\theta}]$  to preclude unrealistic and unstable guesses of  $\theta$ .
  - E.g. huge  $\Sigma$  values can make the inner loop unstable.
  - Economic intuition and initial estimates will give a sense for reasonable bounds.

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- Check that 3-5 **different starting values**  $\theta \sim U(\underline{\theta}, \bar{\theta})$  give the same  $\hat{\theta}$ .
  - For 2-step GMM, do this twice, once for each step (6-10 jobs total).
  - If you have access to a cluster, each can be a separate job, run in parallel.



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- Check that 3-5 **different starting values**  $\theta \sim U(\underline{\theta}, \bar{\theta})$  give the same  $\hat{\theta}$ .
- Prefer using **gradient-based algorithms** for “smooth” problems like BLP.
  - Avoid derivative-free methods like Nelder-Mead/simplex, which tend to work worse.
  - I prefer trust-region algorithms, e.g. SciPy’s `trust-constr` or Knitro if you have it.

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- Try to terminate on **strict first-order conditions**, e.g.  $\|\text{gradient}\|_{\infty} < 1\text{e-}8$ .
  - Inner loop should be tighter to prevent error “bubbling up.” PyBLP default is very tight.
  - Can also check second-order conditions, i.e. Hessian eigenvalues are positive.

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- Prefer using **gradient-based algorithms** for “smooth” problems like BLP.
- Try to terminate on **strict first-order conditions**, e.g.  $\|\text{gradient}\|_{\infty} < 1\text{e-}8$ .
- **Configure your optimizer!** Defaults may not work for your setting.

# Numerical Integration

$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})}$$

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- Sometimes there are only a few types that we can integrate exactly.
  - E.g. high- and low-income types  $i \in \{1, 2\}$  with known shares  $w_{1t}$  and  $w_{2t} = 1 - w_{1t}$ .

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- Sometimes there are only a few types that we can integrate exactly.
- But usually we approximate the distribution with **Monte Carlo** integration.
  - Use a random number generator (RNG) to draw  $|\mathcal{I}_t| \approx 1,000$  of  $(\nu_{it}, y_{it})$ 's per market.
  - Even better than your default RNG are **quasi-Monte Carlo** sequences.
  - I recommend scrambled Halton sequences. R: **Owen (2017)**. Python: SciPy or PyBLP.

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- But usually we approximate the distribution with **Monte Carlo** integration.
- If you just need a few  $\nu_{it} \sim N(0, I)$ 's, try out **Gauss-Hermite quadrature**.
  - 10-100× fewer carefully-chosen  $(w_{it}, \nu_{it})$ 's that do just as well as Monte Carlo.
  - Chosen to exactly integrate a polynomial expansion of the integrand.



# Numerical Integration

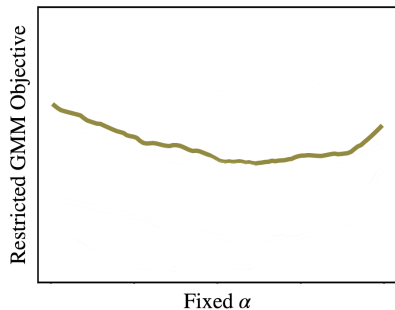
$$s_{jt} = \sum_{i \in \mathcal{I}_t} w_{it} \cdot \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})} \approx \int \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_{k \in \mathcal{J}_t} \exp(\delta_{kt} + \mu_{ikt})} dF(\mu_{it})$$

- Individual types  $i$  are typically an *approximation* to a population distribution.
- Sometimes there are only a few types that we can integrate exactly.
- But usually we approximate the distribution with **Monte Carlo** integration.
- If you just need a few  $\nu_{it} \sim N(0, I)$ 's, try out **Gauss-Hermite quadrature**.
- **Keep increasing  $|\mathcal{I}_t|$**  until your estimates stabilize across draws/starting values.

# What Typically Goes Wrong

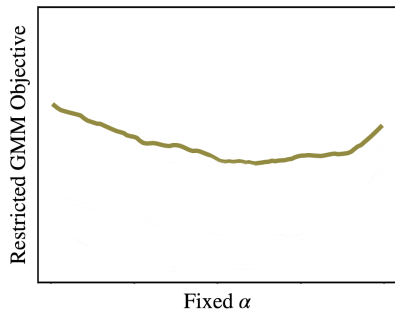
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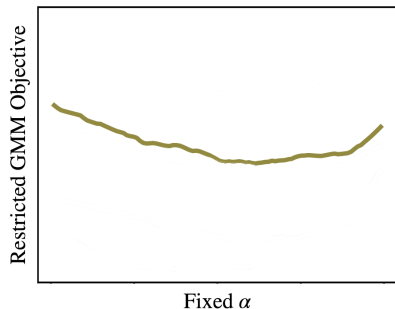
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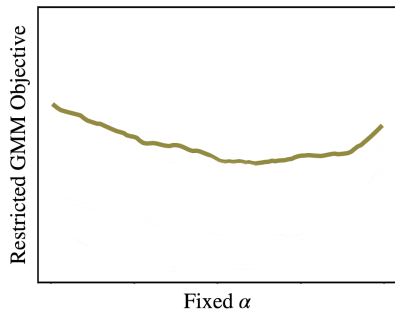
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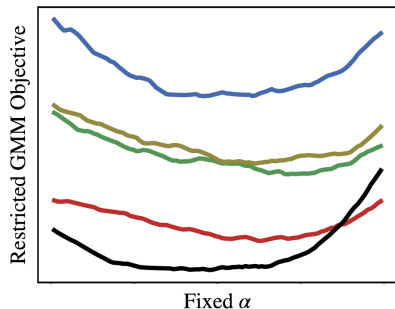
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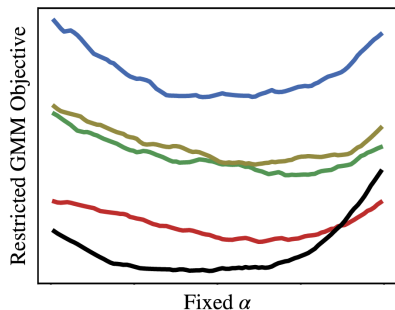
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- Different instruments give different objectives.
  - Even if they're all valid, some may be weaker.
  - Weaker means flatter and harder to optimize.





# Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

# Adding Instruments

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- Later, adding more can help with weakness and testing exclusion restrictions.

# Linear Regression Approximation

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- Let's use our stronger intuition about linear regression to think about instruments!

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- If we set  $\sigma = \pi = 0$  like on day 1, we get our familiar pure logit regression.
  - Use the same IV as before to target  $\beta$ : if  $x_{jt} = p_{jt}$ , a price IV; if exogenous,  $x_{jt}$  itself.

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$$\log \frac{s_{jt}}{s_{0t}} \approx \beta x_{jt} + \sigma^2 d_{jt}^x + \xi_{jt} \quad \text{where} \quad d_{jt}^x = \left( \frac{x_{jt}}{2} - \sum_{k \in \mathcal{J}_t} s_{kt} x_{kt} \right) x_{jt}$$

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  - We want **cross-market demographic variation**, otherwise  $m_t^y x_{jt}$  is collinear with  $x_{jt}$ .
  - Can technically identify  $\pi$  from higher-order variation, e.g. in variance  $v_t^y$ .

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- In your exercise, you’ll target  $(\beta, \sigma, \pi)$  with  $z_{jt} = (x_{jt}, \sum_{k \neq j} (x_{jt} - x_{kt})^2, m_t^y x_{jt})$ .  
→ If  $x_{jt} = p_{jt}$ , can replace  $x_{jt}$  with fitted values  $\hat{p}_{jt}$  from the price IV’s first stage.

# Optimal Instruments

- There are many valid instruments that satisfy exclusion restrictions  $\mathbb{E}[\xi_{jt} \mid z_{jt}] = 0$ .  
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- Can be a bit tricky to compute, but with PyBLP it's just one line of code.
  - In practice, can update your IVs along with your weighting matrix for a second GMM step.

# Roadmap

Preference Heterogeneity

Mixed Logit Estimation

Numerical Best Practices

Differentiation Instruments

Coding Exercise 2

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- If you have time, try the supplemental exercises.
  - Numerical integration alternatives.
  - Optimal weights and instruments.
  - Supply-side restrictions.

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