# Past Exam Paper 1

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2. (a) (I) $g_1(x) = x_1^2 + x_2^2 - 4 \quad g_2(x) = x_1 - 5x_2 + 5$

$\nabla^2 g_1(x) = 2I \succeq 0$ is convex

$\nabla^2 g_2(x) = 0$ is convex

Find $x_1 = 0$, $x_2 = 1$ as Slater point.

Slater Condition holds $\Rightarrow$ MFCQ holds at every feasible point

(II) The KKT is shown below

$$\begin{cases} \begin{bmatrix} 1 \\ 3x_2^2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -5 \end{bmatrix} = 0 & ① \\ \lambda_1 (x_1^2 + x_2^2 - 4) = 0 & ② \\ \lambda_2 (x_1 - 5x_2 + 5) = 0 & ③ \\ \lambda_1 \geq 0, \ \lambda_2 \geq 0 \end{cases}$$

If $\lambda_1 = 0$, $\lambda_2 = 0$ From ①, the result is unfeasible

If $\lambda_1 = 0$, $\lambda_2 > 0$ From ②, $x_1 = 0$, $x_2 = 1$

From ①, $3 - 5\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{3}{5}$

If $\lambda_1 > 0$, $\lambda_2 = 0$ From ①, $1 + 2\lambda_1 x_1 = 0 \Rightarrow x_1 = -\frac{1}{2\lambda_1}$

$3x_2^2 + 2\lambda_1 x_2 = 0$, $x_2 = \frac{-2\lambda_1 \pm \sqrt{4\lambda_1^2}}{6}$

$x_2 = 0$ or $x_2 = -\frac{4\lambda_1}{6} = -\frac{2}{3}\lambda_1$

If $x_2 = 0$, from ② $\left(-\frac{1}{2\lambda_1}\right)^2 \cdot \lambda_1 - 4\lambda_1 = 0$

$\Rightarrow \frac{1}{4\lambda_1^2} \cdot \lambda_1 - 4\lambda_1 = 0$, $\frac{1}{4\lambda_1} - 4\lambda_1 = 0$

$\Rightarrow 1 - 16\lambda_1^2 = 0$, $\lambda_1 = \frac{1}{4}$

If $x_2 = -\frac{2}{3}\lambda_1$, from ② $\lambda_1 \left(\frac{1}{4\lambda_1^2} + \frac{4}{9}\lambda_1^2 - 4\right) = 0$

$\Rightarrow \frac{1}{4\lambda_1^2} + \frac{4}{9}\lambda_1^2 - 4 = 0$, $\frac{1}{4} + \frac{4}{9}\lambda_1^4 - 4\lambda_1^2 = 0$

$\Rightarrow \frac{1}{4} + \frac{4}{9}t^2 - 4t = 0$. $t = \frac{4 \pm \sqrt{16 - \frac{4}{9}}}{2 \cdot \frac{4}{9}}$

$\Rightarrow \lambda_1 = \text{----}$

2. (b) $h(x) = \|Ax\|_2^2 - 1$. $g_i(x) = -x_i$

The MFCQ def is:

$\mu \nabla h(x) + \sum -\lambda_i \nabla g_i(x) = 0$, $\lambda_i g_i(x) = 0$, for $i \in I(x)$, $\lambda_i \geq 0$

$\Leftrightarrow \mu (A^T A x) + [-\lambda_1 \ \cdots \ -\lambda_n]^T = 0$, $\lambda_i g_i(x) = 0$, $\lambda_i \geq 0$

$\Leftrightarrow \mu (A^T A x) - [\lambda_1 \ \cdots \ \lambda_n]^T = 0$, $\lambda_i g_i(x) = 0$, $\lambda_i \geq 0$

$\Leftrightarrow \mu(A^TAx) - [\lambda_1 \dots \lambda_n]^T = 0, \ \lambda_i g_i(x) = 0, \ \lambda_i \geq 0$

$\Leftrightarrow \mu(A^TAx)x = [\lambda_1 \dots \lambda_n]^T x = 0$

$\Leftrightarrow \mu = 0, \ \lambda_i = 0 \Rightarrow$ MFCQ holds

3.(a)

Let $h(y) = \ln(y^2+1)$

$$h''(y) = \left(\frac{2y}{y^2+1}\right)' = \left(\frac{2(y^2+1) - 2y \cdot 2y}{y^4+2y^2+1}\right) = \frac{2y^2-4y^2+2}{y^4+2y^2+1}$$

$$= \frac{-2y^2+2}{y^4+2y^2+1} \leq \frac{-2y^2+2}{2y^2+1} \leq m$$

$$\|\nabla^2 f(x)\|_2 = \|A^T \nabla^2 h(Ax-b) A\|_2 \leq m\|A^TA\|_2 \leq L \dots$$

(b) Compare $\frac{2}{L}$, $L$ ..

4. $t^2$ is convex, $\sum x_i^2$ is convex

$-\ln(t-5)$ is convex. $x_2+2x_3$ is a linear map

$q_\mu$ is convex

$$\nabla q_\mu = \left[2x_1 \quad 2x_2 - \mu\frac{1}{x_2+2x_3-5} \quad 2x_3 - \mu\frac{2}{x_2+2x_3-5} \quad 2x_4\right] = 0$$

$x_1 = 0$

$x_2 = \frac{\mu}{2} \cdot \frac{1}{x_2+2x_3-5} \qquad x_3 = \mu\frac{1}{x_2+2x_3-5} \qquad x_4 = 0$

$2x_2 = x_3$

$x_2 = \frac{\mu}{2} \cdot \frac{1}{x_2+4x_2-5} \Rightarrow x_2 = \frac{\mu}{2} \cdot \frac{1}{5x_2-5}$

$5x_2^2 - 5x_2 = \frac{\mu}{2}$, let $\mu \to 0 \quad x_2 = 1, \ x_3 = 2$

So the result is $(0 \ 1 \ 2 \ 0)$

5. (a)

Min $y_1 + y_2$

$y_1 \geq (|x_1|+1)^5 \quad y_2 \geq \sqrt{x_1^2+3x_2^2+x_3^2+1}$

$y_3 \geq |x_1|$ . $y_1 \geq (y_3+1)^5$

$y_1(y_3+1) \geq (y_3+1)^6 \qquad y_4 \geq (y_3+1)^2$

$y_1(y_3+1) \geq y_4^3$

$y_1 y_4 (y_3+1) \geq y_4^4 \qquad y_5 \geq y_4^2$

$y_1 y_4 (y_3+1) \geq y_5^2$

$y_1 y_4 (y_3+1) \geq y_4'$ $\qquad$ $y_5 \geq y_4$

$y_1 y_4 (y_3+1) \geq y_5^2$

(b) $b = [5 \quad 1]$

$A_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Dual $\quad \max \; b^T y$

$\quad$ s.t. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - y_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - y_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0$

$V_p = V_d$.

$tr(X^*)$ satisfy the primal question, $X^* \succ 0$

$tr(CX) = tr\left( \begin{matrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \cdot \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{matrix} \right) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 6$

If it is the optimal Sol. $b^T y = 6$

$\quad \begin{bmatrix} 5 \\ 1 \end{bmatrix} [y_1 \quad y_2] = 6 \qquad 5y_1 + y_2 = 6$

Dual s.t. $\begin{bmatrix} 2-y_1 & 1 & 0 & 0 \\ 1 & 2-y_1 & -\frac{y_2}{2} & 0 \\ 0 & -\frac{y_2}{2} & 2-y_1-y_2 & 0 \\ 0 & 0 & 0 & 1-y_1 \end{bmatrix}$ $y_1 \; 0$. $\qquad 1-y_1 \geq 0 \qquad y_1 \leq 1$

$\qquad 2-y_1-y_2 \geq 0, \quad y_1+y_2 \leq 2$

$\quad 5y_1+y_2 = 4y_1 + (y_1+y_2) \leq 4+2 \leq 6$

Let $y_1 = 1$, $y_2 = 1$, it achieve the optimal value

Consider max's s.t

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{vmatrix} = 0+0+0-\frac{1}{4} = -\frac{1}{4} < 0$

It is not feasible

So $x^*$ is not a optimal solution