2023 - 2024

1. f_μ is convex

$\nabla f_\mu(x) = [\, 2x_1 + \frac{\mu}{8-x_1+3x_3} \quad 2x_2 \quad 2x_3 - \frac{3\mu}{8-x_1+3x_3} \,]$

Let $\nabla f_\mu(x) = 0$

$\Rightarrow 2x_1 + \frac{\mu}{8-x_1+3x_3} = 0 . \quad 2x_1 = -\frac{\mu}{8-x_1+3x_3}$ ①

$2x_2 = 0 \Rightarrow x_2 = 0$

$2x_3 - \frac{3\mu}{8-x_1+3x_3} = 0 . \quad 2x_3 = \frac{3\mu}{8-x_1+3x_3}$ ②

From ①.② $-3(2x_1) = 2x_3 , \ -3x_1 = x_3$

Consider ① $2x_1 = -\frac{\mu}{8-x_1-9x_1}$

$2x_1 = -\frac{\mu}{8-10x_1}$

$16x_1 - 20x_1^2 + \mu = 0 \Rightarrow x_1 = \frac{-16 \pm \sqrt{16^2+80\mu}}{-40}$

Let $\mu \to 0$, $\boxed{\begin{matrix} x_1 = 0 \\ \downarrow \\ x_3 = 0 \end{matrix}}$ or $x_1 = \frac{32}{40}$, $x_3 = -\frac{32 \cdot 3}{40}$

global minimizer

2.

(a) $\sum_{i=1}^{3} \sqrt{8+x_i^2} \geq \sum_{i=1}^{3} \sqrt{x_i^2} \geq \sum_{i=1}^{3} |x_i| \geq C_1 \sum_{i=1}^{n} \|x\|_2$ By Armijo Rule

$C_1 \|x\|_2 \leq f(x^k) \leq f(x^{k-1}) - \sigma\alpha_{k-1} \|\nabla f(x^{k-1})\|_2^2 \leq \cdots \leq f(x^0)$

$\|x\|_2 \leq \frac{1}{c} f(x^0) \Rightarrow \{x^k\}$ is bounded

(b) Let $h_1(y) = \ln(5+y^2)$. $h_1''(y) = \left(\frac{2y}{5+y^2}\right)' = \frac{2(5+y^2)-4y^2}{(5+y^2)^2} = \frac{-2y^2+10}{25+10y^2+y^4}$

$\leq \frac{10}{25} = \frac{2}{5} = 0.4$

Let $h_2(y) = \sqrt{8+y^2}$

$= \left((8+y^2)^{0.5}\right)'' = \left(y(8+y^2)^{-0.5}\right)' = \left(\frac{y}{\sqrt{8+y^2}}\right)' = \frac{1}{\sqrt{8+y^2}} + y \cdot 2y \cdot (8+y^2)^{-1.5} \cdot (-0.5)$

$= \frac{1}{\sqrt{8+y^2}} - y^2(8+y^2)^{-1.5} \leq \frac{1}{\sqrt{8}} = 0.35\ldots$

$\left(\sqrt{8+x^2}\right)''$

$= \left(\frac{x}{\sqrt{8+x^2}}\right)'$

$= \frac{1}{\sqrt{8+x^2}} - \cdots$

$\leq \frac{1}{\sqrt{8}}$

$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$ $\|A\|_2^2 = \lambda_{max}(\underline{AA^T}) =$

$AA^T = \begin{bmatrix} 6 & -4 \\ -4 & 10 \end{bmatrix}$ $(6-\lambda)(10-\lambda) - 16 = 0$

$\lambda^2 - 16\lambda + 60 - 16 = 0 \quad \lambda^2 - 16\lambda + 44 = 0$

$\lambda = \frac{16 \pm \sqrt{16^2 - 4 \cdot 44}}{2} = \frac{16 \pm \sqrt{80}}{2}$. $\lambda_{max} = \frac{16+\sqrt{80}}{2}$

$\|\nabla^2 f(x)\| \leq \|A\|_2^2 \cdot 0.4 + 0.35 \approx 5.03398$. $\frac{2}{L} = 0.37461$

$1.176 = \pi \approx 0.37433 < 0.37461 \Rightarrow$ a stationary point of f

3. (a) $g_1(x) = x_1^2 + 2x_2^2 - 5 \quad g_2(x) = 2x_1 + x_2 - 3$

$\nabla^2 g_1(x) = 22 \succeq 0 \quad \nabla^2 g_2(x) = 0$ convex

3. (a) $g_1(x) = x_1^2 + 2x_2^2 - 5 \qquad g_2(x) = 2x_1 + x_2 - 3$

$\nabla^2 g_1(x) = 2I \succ 0 \qquad \nabla^2 g_2(x) = 0 \qquad$ Convex

$(1, 1)$ is a Slater point

MFCQ holds every feasible point

KKT

$$\begin{cases} \begin{bmatrix} -3 \\ -3x_2^2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0 \\ \lambda_1 (x_1^2 + 2x_2^2 - 5) = 0 \\ \lambda_2 (2x_1 + x_2 - 3) = 0 \\ \lambda_1, \lambda_2 \geq 0 \end{cases}$$

If there are two contraints inactive

$\lambda_1 = \lambda_2 = 0$

$\begin{bmatrix} -3 \\ -3x_2^2 \end{bmatrix} = 0$. $3 = 0$ infeasible

(b) ① MFCQ holds in Ω. Consider in S

5.

$$A = \begin{bmatrix} 5 & 0 & & 0 & & 0 & -2 \\ 0 & 3 & & & & -1 & 0 \\ & & 2 & & 0 & & \\ 0 & & & 2 & & & 0 \\ & & 0 & & 2 & & \\ 0 & -1 & & 0 & & 3 & 0 \\ -2 & 0 & & & & 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \Rightarrow \lambda = 7 \text{ or } 3$$

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad \lambda = 4 \text{ or } 2$$

$\lambda(A) = 7, 4, 3, 2$

By Luenberger Theorem

$$(x^4 - x^*) A^T (x^4 - x^*) \leq \left(\frac{\lambda_{2023-4} - \lambda_1}{\lambda_{2023-4} + \lambda_1} \right)^2 (x^0 - x^*)^T A (x^0 - x^*)$$

$$(x^4 - x^*) A^T (x^4 - x^*) \leq 0$$

$$\Rightarrow \| x^4 - x^* \|_2^2 \leq 0 \Rightarrow x^4 = x^*$$