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1.

Solution.

$$P\left(c < \frac{X_{(n)}}{\theta} < 1\right) \Longleftrightarrow P(c\theta < X_{(n)} < \theta) = P(X_{(n)} < \theta) - P(X_{(n)} < c\theta)$$

$$P(X_{(n)} < \theta) = 1$$

$$P(X_{(n)} < c\theta) = P(X_{(1)} < c\theta) \cdot P(X_{(2)} < c\theta) \cdots P(X_{(n)} < c\theta)$$

$$= \left[\int_0^{c\theta} \frac{3x^2}{\theta^3} dx\right]^n = \left[\frac{1}{\theta^3} \cdot (c\theta)^3\right]^n$$

$$= c^{3n}$$

Then $P\left(c < \frac{X_{(n)}}{\theta} < 1\right) = 1 - c^{3n}$

2. $$\frac{L(\theta')}{L(\theta'')} = \frac{\prod\left[\frac{1}{2}^x \left(\frac{1}{2}\right)^{1-x}\right]}{\prod\left[\theta^x (1-\theta)^{1-x}\right]} = \frac{\left(\frac{1}{2}\right)^n}{\theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i}} \quad k$$

When $\theta < \frac{1}{2}$, $\left(\frac{1}{2}\right)^n \cdot \frac{1}{\theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i}} \le k \Longleftrightarrow$

$$\sum_{i=1}^n x_i (\log\theta - \log(1-\theta)) \ge -\log(2^n k) - n\log(1-\theta)$$

$$\sum_{i=1}^n x_i \le \frac{\log(2^n k) + n\log(1-\theta)}{\log(1-\theta) - \log\theta} := c$$

So, $C = \{(x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i \le c\}$ is a UMP region for testing $H_0: \theta = \frac{1}{2}$ versus $H_1: \theta < \frac{1}{2}$, it is a UMP test.

If $c = 1$, then $\sum_{i=1}^5 x_i \le 1$

$$\alpha = P\left(\sum_{i=1}^5 x_i \le 1 \mid \theta = \frac{1}{2}\right) = C_5^0 \left(\frac{1}{2}\right)^5 + C_5^1 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{32} + \frac{5}{32} = \frac{6}{32} = \frac{3}{16}$$

If $c = 0$, then $\sum_{i=1}^5 x_i \le 0$

$$\alpha = P\left(\sum_{i=1}^5 x_i \le 0 \mid \theta = \frac{1}{2}\right) = C_5^0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

3.

a. $E(\bar{X}) = E(k\bar{X}_1 + (1-k)\bar{X}_2) = kE(\bar{X}_1) + (1-k)E(\bar{X}_2)$

$= kE(\frac{1}{n}\sum_{i=1}^{n} X_i^1) + (1-k)E(\frac{1}{n}\sum_{i=1}^{n} X_i^2)$

$= k\mu + (1-k)\mu = \mu$

So $\bar{X}$ is unbiased for μ

b. $MSE(\bar{X}) = Var(\bar{X}) + [bias(\bar{X})]^2 = Var(\bar{X})$

$\lim_{n\to\infty} MSE(\bar{X}) = \lim_{n\to\infty} Var(\bar{X}) = \lim_{n\to\infty}[k^2 Var(\bar{X}_1)$
$+ (1-k)^2 Var(\bar{X}_2)] = \lim_{n\to\infty}[k^2 \cdot \frac{\sigma_1^2}{n} + (1-k)^2 \cdot \frac{\sigma_2^2}{n}]$

$= 0$

Thus, $\bar{X}$ is a consistent estimator for μ.

c. $\frac{dVar(\bar{X})}{dk} = 2k\frac{\sigma_1^2}{n} - 2(1-k)\cdot\frac{\sigma_2^2}{n}$

$\frac{d^2 Var(\bar{X})}{dk^2} = \frac{2\sigma_1^2}{n} + \frac{2\sigma_2^2}{n} > 0$.

When $\frac{dVar(\bar{X})}{dk} = 0$, it achieves its minimum.

Hence, $2k\frac{\sigma_1^2}{n} = (2-2k)\frac{\sigma_2^2}{n}$

$\Longleftrightarrow k = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$