2022-2023 Paper

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1.

The MME: $\theta = \bar{X}$, $\hat{\theta} = \bar{X}$

The MLE:

$$L = \prod \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$l = \log L = \sum x_i \log\theta + (n - \sum x_i)\log(1-\theta)$$

$$\frac{\partial l}{\partial \theta} = \frac{1}{\theta}\sum x_i - (n - \sum x_i)\frac{1}{1-\theta} = 0 \Rightarrow \frac{1}{\theta}\sum x_i = \frac{n-\sum x_i}{1-\theta}$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum x_i$$

2. a. $\bar{X} = \frac{\theta}{2}$, $\hat{\theta}_{MME} = 2\bar{X}$

b. $\hat{\theta}_{MLE} = X_{(n)}$, $\hat{\theta^2}_{MLE} = X^2_{(n)}$, where $X_{(n)}$ is the largest sample

c. $Var(2\bar{X}) = 4\, Var(\frac{1}{n}\sum x_i)$

$$= 4 \cdot \frac{1}{n^2} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n^2}$$

$$E(2\bar{X}) = 2E(\frac{1}{n}\sum x_i) = \frac{2}{n}\cdot E(\sum x_i) = \frac{2}{n}\cdot n \cdot \frac{\theta}{2} = \theta$$

The bias of $\hat{\theta}_{MME}$ is 0.

The MSE of $\hat{\theta}_{MME} = Var(\hat{\theta}_{MME}) + (bias)^2 + 0 = \frac{\theta^2}{3n^2}$

$\lim_{n\to\infty} MSE = 0$, which conclude that $\hat{\theta}_{MME}$ is a consistent estimator

d. $X_{(n)} = \max\{x_1 \cdots x_n\}$

The cdf of $X_{(n)}$ is $P(X_{(n)} \le x) = P(X_1 \le x) \times \cdots \times P(X_n \le x)$

$P(X \le x) = \int_0^x \frac{1}{\theta}dx = \frac{x}{\theta} \Rightarrow P(X_{(n)} \le x) = (\frac{x}{\theta})^n$.

The pdf of $X_{(n)} = d(\frac{x}{\theta})^n / dx = (\frac{1}{\theta})^n(n x^{n-1})$, $x \in (0, \theta)$

e. Yes. Because the likelihood function

$$L(x_1 \ldots x_n; \theta) = (\frac{1}{\theta})^n I(x), \text{ where } I(x) = \begin{cases} 1 & X_{(n)} \le \theta \\ 0 & X_{(n)} > \theta \end{cases}$$

By Factorization Theorem, $X_{(n)}$ is a sufficient statistic

3.

a. $\bar{X} = 5\theta$, $\hat{\theta}_{MME} = \frac{1}{5}\bar{X}$

b. $f = \frac{1}{\Gamma(5)\theta^5} x^4 e^{-\frac{x}{\theta}}$

$$L = \prod_{i=1}^{n} \frac{1}{\Gamma(5)\theta^5} x_i^4 e^{-\frac{\sum x_i}{\theta}}$$

$$l = \log L = \log\frac{1}{\Gamma(5)\theta^{5n}} + 4\log\sum x_i - \frac{1}{\theta}\sum x_i$$

$$= -\log\Gamma(5) - 5n\log\theta + 4\log\sum x_i - \frac{1}{\theta}\sum x_i$$

$$\frac{\partial l}{\partial\theta} = -\frac{5n}{\theta} - \sum x_i(-\frac{1}{\theta^2}) = 0 \Rightarrow \frac{5n}{\theta} = \frac{1}{\theta^2}\sum x_i \Rightarrow \hat{\theta}_{MLE} = \bar{X}/5$$

c.

$\frac{\partial}{\partial\theta} = -\frac{5n}{\theta} - \Sigma X_i \left(-\frac{1}{\theta^2}\right) = 0 \Rightarrow \frac{5n}{\theta} = \frac{1}{\theta^2}\Sigma X_i \Rightarrow \theta_{MLE} = \bar{X}/5$

c.

$\log f = \log \frac{1}{\Gamma(5)\theta^5} x^4 e^{-\frac{x}{\theta}} = -\log\Gamma(5) - 5\log\theta + 4\log x - \frac{x}{\theta}$

$\partial \log f / \partial\theta = -\frac{5}{\theta} + \frac{x}{\theta^2}$; $\partial^2 \log f / \partial\theta^2 = \frac{5}{\theta^2} - \frac{2x}{\theta^3}$

$I(\theta) = -E(\partial^2 \log f/\partial\theta^2) = -E(5/\theta^2 - 2x/\theta^3)$

$= -[5/\theta^2 - \frac{2EX}{\theta^3}] = \frac{5}{\theta^2}$

d. $E(\hat{\theta}_{MLE}) = E(5/\bar{X}) = 5/E(\bar{X}) = \theta$

So $\hat{\theta}_{MLE}$ is unbiased.

Compute C-R Bound: $\frac{1}{nI(\theta)} = \frac{1}{n}\cdot\frac{\theta^2}{5} = \frac{\theta^2}{5n}$

$Var(\hat{\theta}_{MLE}) = \frac{1}{25}\cdot\frac{1}{n^2}\cdot n\cdot 5\theta^2 = \frac{\theta^2}{5n}$ = C-R Low Bound

Hence, $\hat{\theta}_{MLE}$ is an efficient estimator of θ

4.

a. The pdf: $f(x;\theta) = \theta^x(1-\theta)^{1-x} = e^{\ln\theta^x(1-\theta)^{1-x}}$

$= e^{\ln\theta^x + \ln(1-\theta)^{1-x}} = e^{x\ln\theta + (1-x)\ln(1-\theta)}$

$= e^{x\ln\theta + \ln(1-\theta) - x\ln(1-\theta)} = e^{x(\ln\theta - \ln(1-\theta)) + \ln(1-\theta)}$

With the properties of exp family. let $K(x) = x$.

So $\sum_{i=1}^n x_i$ is a complete sufficient statistic for θ

b. $E(\sum_{i=1}^n X_i) = n\theta$, which means $E(\frac{1}{n}\sum_{i=1}^n X_i) = \theta$.

$\frac{1}{n}\sum_{i=1}^n X_i$ is a function about $\sum_{i=1}^n X_i$ and it is unbiased.

Hence, $\frac{1}{n}\sum_{i=1}^n X_i$ is the MVUE of θ

c. From b, $\frac{1}{n}\sum_{i=1}^n X_i$ is also the UMVUE of θ.

The UMVUE of $\tau(\theta) = e^2(\bar{X}(1-\bar{X}))$

5.

a. let $\theta_1 = \frac{1}{3}$ $\theta_2 := \theta < \frac{1}{3}$

$$\frac{L(x;\theta_1)}{L(x;\theta_2)} = \frac{\theta_1^{\Sigma X_i}(1-\theta_1)^{n-\Sigma X_i}}{\theta_2^{\Sigma X_i}(1-\theta_2)^{n-\Sigma X_i}} = \frac{\theta_1^{\Sigma X_i}(1-\theta_2)^{\Sigma X_i}}{(1-\theta_1)^{\Sigma X_i}\theta_2^{\Sigma X_i}} \le K\frac{(1-\theta_2)^n}{(1-\theta_1)^n}$$

$$\Rightarrow \left[\frac{\theta_1(1-\theta_2)}{\theta_2(1-\theta_1)}\right]^{\Sigma X_i} \le K\left(\frac{1-\theta_2}{1-\theta_1}\right)^n \Rightarrow \Sigma X_i \le C$$

The critical region: $C = \{\sum_{i=1}^n X_i \le C\}$, which means that it is a UMP Test

b. If $c = 0$, the critical region $X_1 + X_2 + X_3 + X_4 \le 0$

$f(x_i;\theta=\frac{1}{3}) = (\frac{1}{3})^{x_1+x_2+x_3+x_4}(\frac{2}{3})^{4-x_1-x_2-x_3-x_4}$

[illegible]

$$f(x_i;\theta=\tfrac{1}{3}) = \left(\tfrac{1}{3}\right)^{x_1+x_2+x_3+x_4}\left(\tfrac{2}{3}\right)^{4-x_1-x_2-x_3-x_4}$$

$$P\{x_1+x_2+x_3+x_4 \le 0\} = C_4^4 \cdot \left(\tfrac{2}{3}\right)^4 = \left(\tfrac{2}{3}\right)^4$$

c. As for b.

$$P\{x_1+x_2+x_3+x_4 \le 2\} = C_4^2 \left(\tfrac{1}{3}\right)^2 \cdot \left(\tfrac{2}{3}\right)^2 = 6 \cdot \frac{1}{3^2} \cdot \frac{4}{3^2} = \frac{24}{3^4}$$