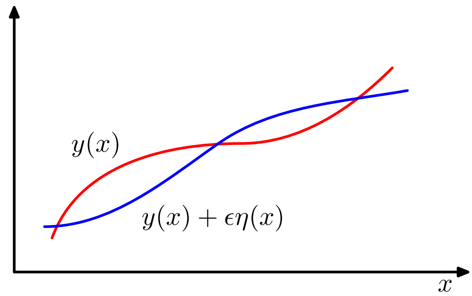
Figure B.1 A functional derivative can be defined by considering how the value of a functional $F[y]$ changes when the function $y(x)$ is changed to $y(x) + \epsilon\eta(x)$ where $\eta(x)$ is an arbitrary function of x.



$y(x)$, where $\eta(x)$ is an arbitrary function of x, as illustrated in Figure B.1. We denote the functional derivative of $F[y]$ with respect to $y(x)$ by $\delta F/\delta y(x)$ and define it by the following relation:

$$F[y(x) + \epsilon\eta(x)] = F[y(x)] + \epsilon \int \frac{\delta F}{\delta y(x)} \eta(x)\,\mathrm{d}x + \mathcal{O}(\epsilon^2). \tag{B.3}$$

This can be seen as a natural extension of (B.2) in which $F[y]$ now depends on a continuous set of variables, namely the values of y at all points x. Requiring that the functional be stationary with respect to small variations in the function $y(x)$ gives

$$\int \frac{\delta F}{\delta y(x)} \eta(x)\,\mathrm{d}x = 0. \tag{B.4}$$

Because this must hold for an arbitrary choice of $\eta(x)$, it follows that the functional derivative must vanish. To see this, imagine choosing a perturbation $\eta(x)$ that is zero everywhere except in the neighbourhood of a point $\widehat{x}$, in which case the functional derivative must be zero at $x = \widehat{x}$. However, because this must be true for every choice of $\widehat{x}$, the functional derivative must vanish for all values of x.

Consider a functional that is defined by an integral over a function $G(y, y', x)$, which depends on both $y(x)$ and its derivative $y'(x)$ and has a direct dependence on x:

$$F[y] = \int G\left(y(x), y'(x), x\right)\,\mathrm{d}x \tag{B.5}$$

where the value of $y(x)$ is assumed to be fixed at the boundary of the region of integration (which might be at infinity). If we now consider variations in the function $y(x)$, we obtain

$$F[y(x) + \epsilon\eta(x)] = F[y(x)] + \epsilon \int \left\{ \frac{\partial G}{\partial y}\eta(x) + \frac{\partial G}{\partial y'}\eta'(x) \right\} \mathrm{d}x + \mathcal{O}(\epsilon^2). \tag{B.6}$$

We now have to cast this in the form (B.3). To do so, we integrate the second term by parts and note that $\eta(x)$ must vanish at the boundary of the integral (because $y(x)$ is fixed at the boundary). This gives

$$F[y(x) + \epsilon\eta(x)] = F[y(x)] + \epsilon \int \left\{ \frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial G}{\partial y'}\right) \right\} \eta(x)\,\mathrm{d}x + \mathcal{O}(\epsilon^2) \tag{B.7}$$