

where we have used the result (2.51) for the normalization of the univariate Gaussian. This confirms that the multivariate Gaussian (3.26) is indeed normalized.

3.2.2 Moments

We now look at the moments of the Gaussian distribution and thereby provide an interpretation of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The expectation of \mathbf{x} under the Gaussian distribution is given by

$$\begin{aligned}\mathbb{E}[\mathbf{x}] &= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp \left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \right\} \mathbf{x} d\mathbf{x} \\ &= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp \left\{ -\frac{1}{2}\mathbf{z}^T \boldsymbol{\Sigma}^{-1} \mathbf{z} \right\} (\mathbf{z} + \boldsymbol{\mu}) d\mathbf{z}\end{aligned}\quad (3.41)$$

where we have changed variables using $\mathbf{z} = \mathbf{x} - \boldsymbol{\mu}$. Note that the exponent is an even function of the components of \mathbf{z} , and because the integrals over these are taken over the range $(-\infty, \infty)$, the term in \mathbf{z} in the factor $(\mathbf{z} + \boldsymbol{\mu})$ will vanish by symmetry. Thus,

$$\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}, \quad (3.42)$$

and so we refer to $\boldsymbol{\mu}$ as the mean of the Gaussian distribution.

We now consider second-order moments of the Gaussian. In the univariate case, we considered the second-order moment given by $\mathbb{E}[x^2]$. For the multivariate Gaussian, there are D^2 second-order moments given by $\mathbb{E}[x_i x_j]$, which we can group together to form the matrix $\mathbb{E}[\mathbf{x}\mathbf{x}^T]$. This matrix can be written as

$$\begin{aligned}\mathbb{E}[\mathbf{x}\mathbf{x}^T] &= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp \left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \right\} \mathbf{x}\mathbf{x}^T d\mathbf{x} \\ &= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp \left\{ -\frac{1}{2}\mathbf{z}^T \boldsymbol{\Sigma}^{-1} \mathbf{z} \right\} (\mathbf{z} + \boldsymbol{\mu})(\mathbf{z} + \boldsymbol{\mu})^T d\mathbf{z}\end{aligned}\quad (3.43)$$

where again we have changed variables using $\mathbf{z} = \mathbf{x} - \boldsymbol{\mu}$. Note that the cross-terms involving $\boldsymbol{\mu}\mathbf{z}^T$ and $\boldsymbol{\mu}^T \mathbf{z}$ will again vanish by symmetry. The term $\boldsymbol{\mu}\boldsymbol{\mu}^T$ is constant and can be taken outside the integral, which itself is unity because the Gaussian distribution is normalized. Consider the term involving $\mathbf{z}\mathbf{z}^T$. Again, we can make use of the eigenvector expansion of the covariance matrix given by (3.28), together with the completeness of the set of eigenvectors, to write

$$\mathbf{z} = \sum_{j=1}^D y_j \mathbf{u}_j \quad (3.44)$$