for $i = 1, \ldots, M$, where $\mathbf{u}_i$ is an *eigenvector* and λ_i is the corresponding *eigenvalue*. This can be viewed as a set of M simultaneous homogeneous linear equations, and the condition for a solution is that

$$|\mathbf{A} - \lambda_i \mathbf{I}| = 0, \tag{A.30}$$

which is known as the *characteristic equation*. Because this is a polynomial of order M in λ_i, it must have M solutions (though these need not all be distinct). The rank of $\mathbf{A}$ is equal to the number of non-zero eigenvalues.

Of particular interest are symmetric matrices, which arise as covariance matrices, kernel matrices, and Hessians. Symmetric matrices have the property that $A_{ij} = A_{ji}$, or equivalently $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$. The inverse of a symmetric matrix is also symmetric, as can be seen by taking the transpose of $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and using $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ together with the symmetry of $\mathbf{I}$.

In general, the eigenvalues of a matrix are complex numbers, but for symmetric matrices, the eigenvalues λ_i are real. This can be seen by first left-multiplying (A.29) by $(\mathbf{u}_i^\star)^{\mathrm{T}}$, where $\star$ denotes the complex conjugate, to give

$$(\mathbf{u}_i^\star)^{\mathrm{T}} \mathbf{A}\mathbf{u}_i = \lambda_i \, (\mathbf{u}_i^\star)^{\mathrm{T}} \mathbf{u}_i. \tag{A.31}$$

Next we take the complex conjugate of (A.29) and left-multiply by $\mathbf{u}_i^{\mathrm{T}}$ to give

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{A}\mathbf{u}_i^\star = \lambda_i^\star \mathbf{u}_i^{\mathrm{T}} \mathbf{u}_i^\star \tag{A.32}$$

where we have used $\mathbf{A}^\star = \mathbf{A}$ because we are considering only real matrices $\mathbf{A}$. Taking the transpose of the second of these equations and using $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, we see that the left-hand sides of the two equations are equal and hence that $\lambda_i^\star = \lambda_i$, and so λ_i must be real.

The eigenvectors $\mathbf{u}_i$ of a real symmetric matrix can be chosen to be orthonormal (i.e., orthogonal and of unit length) so that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = I_{ij} \tag{A.33}$$

where I_{ij} are the elements of the identity matrix $\mathbf{I}$. To show this, we first left-multiply (A.29) by $\mathbf{u}_j^{\mathrm{T}}$ to give

$$\mathbf{u}_j^{\mathrm{T}} \mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_j^{\mathrm{T}} \mathbf{u}_i \tag{A.34}$$

and hence, by exchanging the indices, we have

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j. \tag{A.35}$$

We now take the transpose of the second equation and make use of the symmetry property $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, and then subtract the two equations to give

$$(\lambda_i - \lambda_j) \, \mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0. \tag{A.36}$$

Hence, for $\lambda_i \neq \lambda_j$, we have $\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0$ so that $\mathbf{u}_i$ and $\mathbf{u}_j$ are orthogonal. If the two eigenvalues are equal, then any linear combination $\alpha \mathbf{u}_i + \beta \mathbf{u}_j$ is also an eigenvector