Appendix C. Lagrange Multipliers

Lagrange multipliers, also sometimes called *undetermined multipliers*, are used to find the stationary points of a function of several variables subject to one or more constraints.

Consider the problem of finding the maximum of a function $f(x_1, x_2)$ subject to a constraint relating x_1 and x_2, which we write in the form

$$g(x_1, x_2) = 0. \tag{C.1}$$

One approach would be to solve the constraint equation (C.1) and thus express x_2 as a function of x_1 in the form $x_2 = h(x_1)$. This can then be substituted into $f(x_1, x_2)$ to give a function of x_1 alone of the form $f(x_1, h(x_1))$. The maximum with respect to x_1 could then be found by differentiation in the usual way, to give the stationary value $x_1^\star$, with the corresponding value of x_2 given by $x_2^\star = h(x_1^\star)$.

One problem with this approach is that it may be difficult to find an analytic solution of the constraint equation that allows x_2 to be expressed as an explicit function of x_1. Also, this approach treats x_1 and x_2 differently and so spoils the natural symmetry between these variables.

A more elegant, and often simpler, approach introduces a parameter λ called a Lagrange multiplier. We shall motivate this technique from a geometrical perspective. Consider a D-dimensional variable $\mathbf{x}$ with components $x_1, \ldots, x_D$. The constraint equation $g(\mathbf{x}) = 0$ then represents a $(D-1)$-dimensional surface in $\mathbf{x}$-space as indicated in Figure C.1.

First note that at any point on the constraint surface, the gradient $\nabla g(\mathbf{x})$ of the constraint function is orthogonal to the surface. To see this, consider a point $\mathbf{x}$ that lies on the constraint surface along with a nearby point $\mathbf{x} + \boldsymbol{\epsilon}$ that also lies on the surface. If we make a Taylor expansion around $\mathbf{x}$, we have

$$g(\mathbf{x} + \boldsymbol{\epsilon}) \simeq g(\mathbf{x}) + \boldsymbol{\epsilon}^{\mathrm{T}} \nabla g(\mathbf{x}). \tag{C.2}$$

Because both $\mathbf{x}$ and $\mathbf{x} + \boldsymbol{\epsilon}$ lie on the constraint surface, we have $g(\mathbf{x}) = g(\mathbf{x} + \boldsymbol{\epsilon})$ and hence $\boldsymbol{\epsilon}^{\mathrm{T}} \nabla g(\mathbf{x}) \simeq 0$. In the limit $\|\boldsymbol{\epsilon}\| \to 0$, we have $\boldsymbol{\epsilon}^{\mathrm{T}} \nabla g(\mathbf{x}) = 0$, and because $\boldsymbol{\epsilon}$ is

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