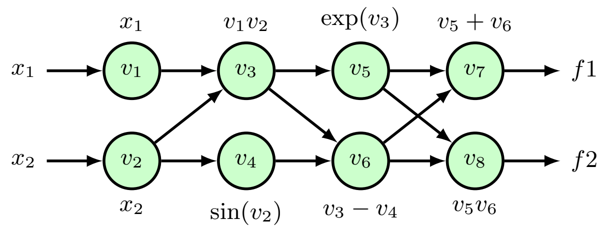

Figure 8.5 Extension of the example shown in Figure 8.4 to a function with two outputs f_1 and f_2.

equations (8.50) to (8.56), we obtain the following evaluation trace equations for the tangent variables

$$\dot{v}_1 = 1 \tag{8.58}$$
$$\dot{v}_2 = 0 \tag{8.59}$$
$$\dot{v}_3 = v_1\dot{v}_2 + \dot{v}_1 v_2 \tag{8.60}$$
$$\dot{v}_4 = \dot{v}_2 \cos(v_2) \tag{8.61}$$
$$\dot{v}_5 = \dot{v}_3 \exp(v_3) \tag{8.62}$$
$$\dot{v}_6 = \dot{v}_3 - \dot{v}_4 \tag{8.63}$$
$$\dot{v}_7 = \dot{v}_5 + \dot{v}_6. \tag{8.64}$$

We can summarize automatic differentiation for this example as follows. We first write code to implement the evaluation of the primal variables, given by (8.50) to (8.56). The associated equations and corresponding code for evaluating the tangent variables (8.58) to (8.64) are generated automatically. To evaluate the derivative $\partial f/\partial x_1$, we input specific values of x_1 and x_2 and the code then executes the primal and tangent equations, numerically evaluating the tuples $(v_i, \dot{v}_i)$ in sequence until we obtain $\dot{v}_5$, which is the required derivative.

Exercise 8.17

Now consider an example with two outputs $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ where $f_1(x_1, x_2)$ is defined by (8.49) and

$$f_2(x_1, x_2) = (x_1 x_2 - \sin(x_2)) \exp(x_1 x_2) \tag{8.65}$$

as illustrated by the evaluation trace diagram in Figure 8.5. We see that this involves only a small extension to the evaluation equations for the primal and tangent variables, and so both $\partial f_1/\partial x_1$ and $\partial f_2/\partial x_1$ can be evaluated together in a single forward pass. The downside, however, is that if we wish to evaluate derivatives with respect to a different input variable x_2 then we have to run a separate forward pass. In general, if we have a function with D inputs and K outputs then a single pass of forward-mode automatic differentiation produces a single column of the $K \times D$ Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_D} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_K}{\partial x_1} & \cdots & \dfrac{\partial f_K}{\partial x_D} \end{bmatrix}. \tag{8.66}$$