which is easily proven by taking the transpose of (A.2) and applying (A.1).

A useful identity involving matrix inverses is the following:

$$(\mathbf{P}^{-1} + \mathbf{B}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{R}^{-1} = \mathbf{P}\mathbf{B}^{\mathrm{T}}(\mathbf{B}\mathbf{P}\mathbf{B}^{\mathrm{T}} + \mathbf{R})^{-1}, \tag{A.5}$$

which is easily verified by right-multiplying both sides by $(\mathbf{B}\mathbf{P}\mathbf{B}^{\mathrm{T}} + \mathbf{R})$. Suppose that $\mathbf{P}$ has dimensionality $N \times N$ and that $\mathbf{R}$ has dimensionality $M \times M$, so that $\mathbf{B}$ is $M \times N$. Then if $M \ll N$, it will be much cheaper to evaluate the right-hand side of (A.5) than the left-hand side. A special case that sometimes arises is

$$(\mathbf{I} + \mathbf{A}\mathbf{B})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{I} + \mathbf{B}\mathbf{A})^{-1}. \tag{A.6}$$

Another useful identity involving inverses is the following:

$$(\mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, \tag{A.7}$$

which is known as the *Woodbury identity*. It can be verified by multiplying both sides by $(\mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})$. This is useful, for instance, when $\mathbf{A}$ is large and diagonal and hence easy to invert, and when $\mathbf{B}$ has many rows but few columns (and conversely for $\mathbf{C}$), so that the right-hand side is much cheaper to evaluate than the left-hand side.

A set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ is said to be *linearly independent* if the relation $\sum_n \alpha_n \mathbf{a}_n = 0$ holds only if all $\alpha_n = 0$. This implies that none of the vectors can be expressed as a linear combination of the remainder. The rank of a matrix is the maximum number of linearly independent rows (or equivalently the maximum number of linearly independent columns).

A.2. Traces and Determinants

Square matrices have traces and determinants. The trace $\mathrm{Tr}(\mathbf{A})$ of a matrix $\mathbf{A}$ is defined as the sum of the elements on the leading diagonal. By writing out the indices, we see that

$$\mathrm{Tr}(\mathbf{A}\mathbf{B}) = \mathrm{Tr}(\mathbf{B}\mathbf{A}). \tag{A.8}$$

By applying this formula multiple times to the product of three matrices, we see that

$$\mathrm{Tr}(\mathbf{A}\mathbf{B}\mathbf{C}) = \mathrm{Tr}(\mathbf{C}\mathbf{A}\mathbf{B}) = \mathrm{Tr}(\mathbf{B}\mathbf{C}\mathbf{A}), \tag{A.9}$$

which is known as the *cyclic* property of the trace operator. It clearly extends to the product of any number of matrices. The determinant $|\mathbf{A}|$ of an $N \times N$ matrix $\mathbf{A}$ is defined by

$$|\mathbf{A}| = \sum (\pm 1) A_{1i_1} A_{2i_2} \cdots A_{Ni_N} \tag{A.10}$$

in which the sum is taken over all products consisting of precisely one element from each row and one element from each column, with a coefficient $+1$ or -1 according to whether the permutation $i_1 i_2 \ldots i_N$ is even or odd, respectively. Note that $|\mathbf{I}| = 1$,