These last two equations can also be written in the form

$$\mathbf{A} = \sum_{i=1}^{M} \lambda_i \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}} \tag{A.45}$$

$$\mathbf{A}^{-1} = \sum_{i=1}^{M} \frac{1}{\lambda_i} \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}}. \tag{A.46}$$

If we take the determinant of (A.43) and use (A.12), we obtain

$$|\mathbf{A}| = \prod_{i=1}^{M} \lambda_i. \tag{A.47}$$

Similarly, taking the trace of (A.43), and using the cyclic property (A.8) of the trace operator together with $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$, we have

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{M} \lambda_i. \tag{A.48}$$

We leave it as an exercise for the reader to verify (A.22) by making use of the results (A.33), (A.45), (A.46), and (A.47).

A matrix $\mathbf{A}$ is said to be *positive definite*, denoted by $\mathbf{A} \succ 0$, if $\mathbf{w}^{\mathrm{T}}\mathbf{A}\mathbf{w} > 0$ for all non-zero values of the vector $\mathbf{w}$. Equivalently, a positive definite matrix has $\lambda_i > 0$ for all of its eigenvalues (as can be seen by setting $\mathbf{w}$ to each of the eigenvectors in turn and noting that an arbitrary vector can be expanded as a linear combination of the eigenvectors). Note that having all positive elements does not necessarily mean that a matrix is that positive definite. For example, the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{A.49}$$

has eigenvalues $\lambda_1 \simeq 5.37$ and $\lambda_2 \simeq -0.37$. A matrix is said to be *positive semidefinite* if $\mathbf{w}^{\mathrm{T}}\mathbf{A}\mathbf{w} \geqslant 0$ holds for all values of $\mathbf{w}$, which is denoted $\mathbf{A} \succeq 0$ and is equivalent to $\lambda_i \geqslant 0$.

The condition number of a matrix is given by

$$\mathrm{CN} = \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{1/2} \tag{A.50}$$

where $\lambda_{\max}$ is the largest eigenvalue and $\lambda_{\min}$ is the smallest eigenvalue.