Similarly

$$\frac{\partial}{\partial x}(\mathbf{AB}) = \frac{\partial \mathbf{A}}{\partial x}\mathbf{B} + \mathbf{A}\frac{\partial \mathbf{B}}{\partial x}. \tag{A.20}$$

The derivative of the inverse of a matrix can be expressed as

$$\frac{\partial}{\partial x}\left(\mathbf{A}^{-1}\right) = -\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x}\mathbf{A}^{-1} \tag{A.21}$$

as can be shown by differentiating the equation $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ using (A.20) and then right-multiplying by $\mathbf{A}^{-1}$. Also

$$\frac{\partial}{\partial x}\ln|\mathbf{A}| = \mathrm{Tr}\left(\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial x}\right), \tag{A.22}$$

which we shall prove later. If we choose x to be one of the elements of $\mathbf{A}$, we have

$$\frac{\partial}{\partial A_{ij}}\mathrm{Tr}\left(\mathbf{AB}\right) = B_{ji} \tag{A.23}$$

as can be seen by writing out the matrices using index notation. We can write this result more compactly in the form

$$\frac{\partial}{\partial \mathbf{A}}\mathrm{Tr}\left(\mathbf{AB}\right) = \mathbf{B}^{\mathrm{T}}. \tag{A.24}$$

With this notation, we have the following properties:

$$\frac{\partial}{\partial \mathbf{A}}\mathrm{Tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right) = \mathbf{B}, \tag{A.25}$$

$$\frac{\partial}{\partial \mathbf{A}}\mathrm{Tr}(\mathbf{A}) = \mathbf{I}, \tag{A.26}$$

$$\frac{\partial}{\partial \mathbf{A}}\mathrm{Tr}(\mathbf{ABA}^{\mathrm{T}}) = \mathbf{A}(\mathbf{B}+\mathbf{B}^{\mathrm{T}}), \tag{A.27}$$

which can again be proven by writing out the matrix indices. We also have

$$\frac{\partial}{\partial \mathbf{A}}\ln|\mathbf{A}| = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}, \tag{A.28}$$

which follows from (A.22) and (A.24).

A.4. Eigenvectors

For a square matrix $\mathbf{A}$ of size $M \times M$, the eigenvector equation is defined by

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \tag{A.29}$$