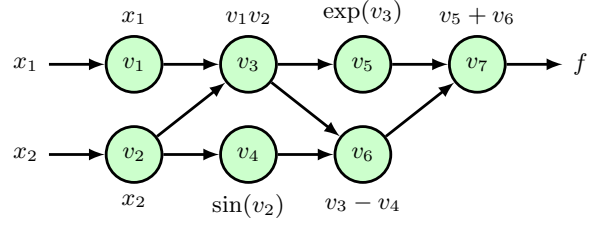**Figure 8.4** Evaluation trace diagram showing the steps involved in the numerical evaluation of the function (8.49) using the primal equations (8.50) to (8.56).

automatically by the software environment. Instead of simply doing forward propagation to compute $\{z_i\}$, the code now propagates tuples $(z_i, \dot{z}_i)$ so that variables and derivatives are evaluated in parallel. The original function is generally defined in terms of elementary operators consisting of arithmetic operations and negation as well as transcendental functions such as exponential, logarithm, and trigonometric functions, all of which have simple formulae for their derivatives. Using these derivatives in combination with the chain rule of calculus allows the code used to evaluate gradients to be constructed automatically.

As an example, consider the following function, which has two input variables:

$$f(x_1, x_2) = x_1 x_2 + \exp(x_1 x_2) - \sin(x_2). \tag{8.49}$$

When implemented in software, the code consists of a sequence of operations that can be expressed as an *evaluation trace* of the underlying elementary operations. This trace can be visualized in the form of a graph, as shown in Figure 8.4. Here we have defined the following primal variables

$$v_1 = x_1 \tag{8.50}$$
$$v_2 = x_2 \tag{8.51}$$
$$v_3 = v_1 v_2 \tag{8.52}$$
$$v_4 = \sin(v_2) \tag{8.53}$$
$$v_5 = \exp(v_3) \tag{8.54}$$
$$v_6 = v_3 - v_4 \tag{8.55}$$
$$v_7 = v_5 + v_6. \tag{8.56}$$

Now suppose we wish to evaluate the derivative $\partial f/\partial x_1$. We define the tangent variables by $\dot{v}_i = \partial v_i/\partial x_1$. Expressions for evaluating these can be constructed automatically using the chain rule of calculus:

$$\dot{v}_i = \frac{\partial v_i}{\partial x_1} = \sum_{j \in \text{pa}(i)} \frac{\partial v_j}{\partial x_1} \frac{\partial v_i}{\partial v_j} = \sum_{j \in \text{pa}(i)} \dot{v}_j \frac{\partial v_i}{\partial v_j}. \tag{8.57}$$

where $\text{pa}(i)$ denotes the set of *parents* of the node i in the evaluation trace diagram, that is the set of variables with arrows pointing to node i. For example, in Figure 8.4 the parents of node v_3 are nodes v_1 and v_2. Applying (8.57) to the evaluation trace