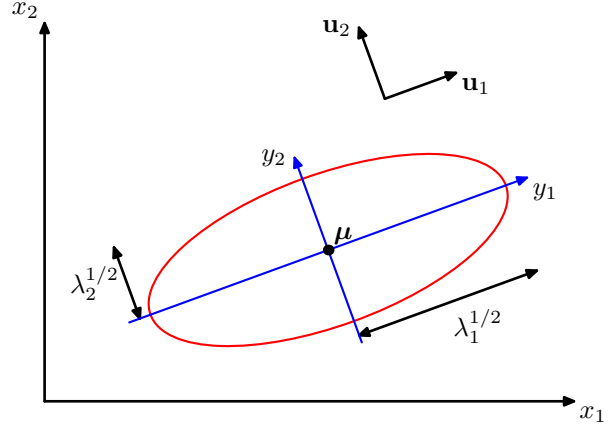


Figure 3.3 The red curve shows the elliptical surface of constant probability density for a Gaussian in a two-dimensional space $\mathbf{x} = (x_1, x_2)$ on which the density is $\exp(-1/2)$ of its value at $\mathbf{x} = \boldsymbol{\mu}$. The axes of the ellipse are defined by the eigenvectors \mathbf{u}_i of the covariance matrix, with corresponding eigenvalues λ_i .



which case the distribution is singular and is confined to a subspace of lower dimensionality. If all the eigenvalues are non-negative, then the covariance matrix is said to be *positive semidefinite*.

Now consider the form of the Gaussian distribution in the new coordinate system defined by the y_i . In going from the \mathbf{x} to the \mathbf{y} coordinate system, we have a Jacobian matrix \mathbf{J} with elements given by

$$J_{ij} = \frac{\partial x_i}{\partial y_j} = U_{ji} \quad (3.36)$$

where U_{ji} are the elements of the matrix \mathbf{U}^T . Using the orthonormality property of the matrix \mathbf{U} , we see that the square of the determinant of the Jacobian matrix is

$$|\mathbf{J}|^2 = |\mathbf{U}^T|^2 = |\mathbf{U}^T| |\mathbf{U}| = |\mathbf{U}^T \mathbf{U}| = |\mathbf{I}| = 1 \quad (3.37)$$

and, hence, $|\mathbf{J}| = 1$. Also, the determinant $|\boldsymbol{\Sigma}|$ of the covariance matrix can be written as the product of its eigenvalues, and hence

$$|\boldsymbol{\Sigma}|^{1/2} = \prod_{j=1}^D \lambda_j^{1/2}. \quad (3.38)$$

Thus, in the y_j coordinate system, the Gaussian distribution takes the form

$$p(\mathbf{y}) = p(\mathbf{x}) |\mathbf{J}| = \prod_{j=1}^D \frac{1}{(2\pi\lambda_j)^{1/2}} \exp \left\{ -\frac{y_j^2}{2\lambda_j} \right\}, \quad (3.39)$$

which is the product of D independent univariate Gaussian distributions. The eigenvectors therefore define a new set of shifted and rotated coordinates with respect to which the joint probability distribution factorizes into a product of independent distributions. The integral of the distribution in the \mathbf{y} coordinate system is then

$$\int p(\mathbf{y}) \, d\mathbf{y} = \prod_{j=1}^D \int_{-\infty}^{\infty} \frac{1}{(2\pi\lambda_j)^{1/2}} \exp \left\{ -\frac{y_j^2}{2\lambda_j} \right\} \, dy_j = 1 \quad (3.40)$$