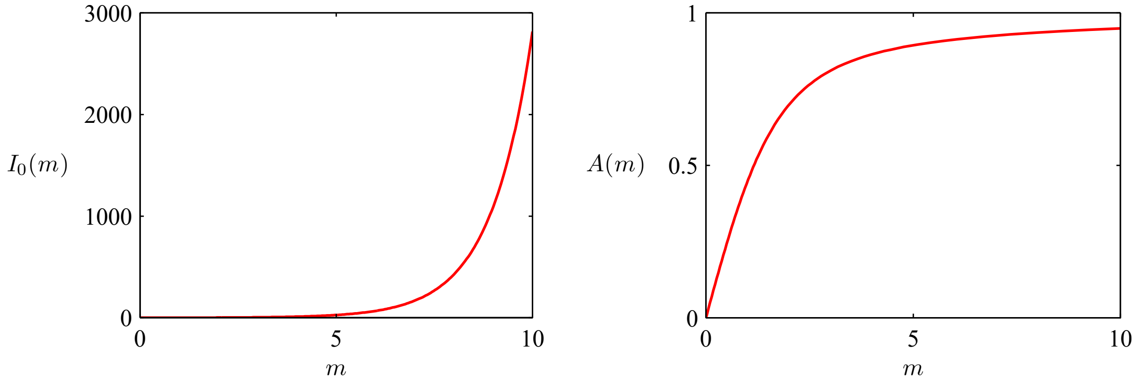

Figure 3.12 Plot of the Bessel function $I_0(m)$ defined by (3.130), together with the function $A(m)$ defined by (3.136).

$$\theta_0^{\mathrm{ML}} = \tan^{-1}\left\{\frac{\sum_n \sin\theta_n}{\sum_n \cos\theta_n}\right\}, \tag{3.134}$$

which we recognize as the result (3.119) obtained earlier for the mean of the observations viewed in a two-dimensional Cartesian space.

Similarly, maximizing (3.131) with respect to m and making use of $I_0'(m) = I_1(m)$ (Abramowitz and Stegun, 1965), we have

$$A(m_{\mathrm{ML}}) = \frac{1}{N}\sum_{n=1}^{N}\cos(\theta_n - \theta_0^{\mathrm{ML}}) \tag{3.135}$$

where we have substituted for the maximum likelihood solution for θ_0^{ML} (recalling that we are performing a joint optimization over θ and m), and we have defined

$$A(m) = \frac{I_1(m)}{I_0(m)}. \tag{3.136}$$

The function $A(m)$ is plotted in Figure 3.12. Making use of the trigonometric identity (3.128), we can write (3.135) in the form

$$A(m_{\mathrm{ML}}) = \left(\frac{1}{N}\sum_{n=1}^{N}\cos\theta_n\right)\cos\theta_0^{\mathrm{ML}} + \left(\frac{1}{N}\sum_{n=1}^{N}\sin\theta_n\right)\sin\theta_0^{\mathrm{ML}}. \tag{3.137}$$

The right-hand side of (3.137) is easily evaluated, and the function $A(m)$ can be inverted numerically. One limitation of the von Mises distribution is that it is unimodal. By forming *mixtures* of von Mises distributions, we obtain a flexible framework for modelling periodic variables that can handle multimodality.

For completeness, we mention briefly some alternative techniques for constructing periodic distributions. The simplest approach is to use a histogram of observations in which the angular coordinate is divided into fixed bins. This has the virtue of