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I affirm that I have adhered to the honor code on this assingment.

Hello again, scientist! I'll write in italics, and problems for you will always be in ***bold****. As a general rule, I expect you to do at least as much writing as I do. Code should be part of your solution, but I expect variables to be clear and explanation to involve complete sentences. Cite your sources; if you work with someone in the class on a problem, that's an extremely important source.*

Problem 6.2.

Here's another nice trick. If a set of differentiable functions is linearly dependent, then the derivatives of those functions satisfy the same linear dependency (because differentiation is a linear function!).

```
syms x;
f = x;
g = x^2
A = [f g];
W = [A; diff(A)]
det(W)
```

```
g =

x^2


W =

[ x, x^2]
[ 1, 2*x]


ans =

x^2
```

Since the determinant of W is not always 0 (it's 0 at 0, but it's not the zero function, because when you plug in x=1 you don't get 0!), the matrix W is invertible, so its columns are linearly independent, which means that the functions f and g are linearly independent. The function det(W) is called the Wronskian, which is a shame because Wronski was not a great person.

Prove that the following set of functions is linearly independent. *You'll need to set up a 4x4 Wronskian for this one: https://en.wikipedia.org/wiki/Wronskian. (If you've taken discrete, it's not a huge leap to imagine that you could use a proof by induction to show that {1, x, ... , x^n} is linearly independent. That's not part of the problem though.)*

```
a = 1; % this is the function a(x) = 1, not the number 1
b = x; c = x^2; d = x^3;

B = [a,b,c,d];
W_2 = [B; diff(B); diff(B,2);diff(B,3)];
```

```
det(W_2)
```

```
ans =

12
```

The determinant of the 4 x 4 Wronskian W_2 is 12. As this is not 0, the matrix W_2 is invertible, so its columns are linearly independent. Therefore, the functions a(x), b(x), c(x), and d(x) are linearly independent.

In the late 1800s, when the first modern analysis books were being written, it was regularly claimed by some really smart people that if the Wronskian of two functions was the zero function then the functions must be linearly dependent. Peano (1899) proved that this was wrong:

```
f = x^2; g = x*abs(x);
```

Prove that the Wronskian of {f,g} is the 0 function. Prove that {f,g} is linearly independent.

```
C = [f,g];
W_3 = [C;diff(C)];
det(W_3);
subs(det(W_3),x,0)
subs(det(W_3),x,1)

D = [f;g];
subs(D,x,-2)
subs(D,x,2)
```

```
ans =

0


ans =

0


ans =

  4
 -4


ans =

 4
 4
```

The determinant of the 2 x 2 Wronskian W_3 is 0. However, when substituting for x in the matrix D, the ratio of 4 to -4 is not the same as 4 to 4, so the multiple relating f(x) to g(x) is NOT the same for all real values of x. Therefore, f(x) and g(x) are linearly independent.

Not part of the problem, but a nice follow-up: you can use the Wronskian to show that if {a_1, ... , a_n} is a set of n distinct real numbers, then the functions {(a_1)^x, ... (a_n)^x}. Proof outline: compute the Wronskian matrix, plug in x=0, recognize that this is the transpose of a matrix you've seen in a previous MATLAB assignment.

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