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I affirm that I have adhered to the honor code on this assignment

Hello again, scientist! I'll write in italics, and problems for you will always be in ***bold****. As a general rule, I expect you to do at least as much writing as I do. Code should be part of your solution, but I expect variables to be clear and explanation to involve complete sentences. Cite your sources; if you work with someone in the class on a problem, that's an extremely important source.*

Problem 6.3.

The Wronskian is kind of a special little trick; the following method is a much more general and powerful method for checking linear independence in function space.

```
syms t;
a1 = 1; a2 = t; a3 = t^2;
A = double(subs([a1 a2 a3],t,[1;2;3]))
```

```
A =

     1     1     1
     1     2     4
     1     3     9
```

What I've done here is to evaluate the vectors 1, t, t^2 at the points 1, 2, 3. This is called an alternant matrix: https://en.wikipedia.org/wiki/Alternant_matrix. ***I strongly recommend skimming that article.*** *Note: there's nothing special about 1, 2, 3. More on that in a bit. The idea here is the same as the Wronskian: if two functions are linearly dependent, then they satisfy the same linear dependence at every input. The alternant probes the functions at different values.*

```
det(A)
```

```
ans =

     2
```

Since det(A) is not 0, A is invertible. That means the columns of A are linearly independent, which proves that {1,t,t^2} is linearly independent! Cool, right?

Okay, so first let's go back to what I was saying about 1,2,3 not being special.

```
b1 = t^2 - 3*t + 2;
b2 = t^3 - 3*t^2 + 2*t;

B = double(subs([b1 b2],t,[1;2]))
```

```
B =
```

```
     0     0
     0     0
```

The rank of B is 0, but b1 and b2 are linearly independent. ***Prove that.*** *(Try some different numbers.)*

```
B_new = double(subs([b1 b2],t,[3;4]));
det(B_new)


ans =

   12.0000
```

B_new is an alternant matrix that evaluates b1 and b2 at 3 & 4 rather than 1 & 2. B_new is row equivalent to the identity matrix in R^2 and therefore is rank 0. The determinant of this matrix is nonzero, so the columns of B_new are linearly independent. Therefore, b1 & b2 are linearly independent.

It's way more interesting when functions are linearly dependent.

```
c1 = sin(t)^2;
c2 = cos(t)^2;
c3 = 1;

C = double(subs([c1 c2 c3],t,[1;2;3]))
det(C) % this is close enough to 0 to be round-off error
rref(C)


C =

    0.7081    0.2919    1.0000
    0.8268    0.1732    1.0000
    0.0199    0.9801    1.0000


ans =

  -1.0278e-17


ans =

     1     0     1
     0     1     1
     0     0     0
```

Use the above computation to write c3 as a linear combination of c1 and c2. What trigonometric identity have you just rediscovered?

c3 = c2 + c1. We have rediscovered the trigonometric identiy sin^2 + cos^2 = 1.

Okay but that's super-boring, right? Let's do some real heavy calculus. (By which I mean that we're about to destroy pretty much every difficult integral in Calculus II.)

```
f = cos(t)^4;
```

Finding the integral of f is an absolute nightmare in Calc II: you have to use the power-reduction formula, foil, then use the power-reduction formula again! That's atrocious. Let's cheat.

```
d0 = 1;
d1 = cos(t);
d2 = cos(2*t);
d3 = cos(3*t);
d4 = cos(4*t);
```

Prove that {d0, d1, d2, d3, d4} are linearly independent.

```
D = double(subs([d0 d1 d2 d3 d4],t,[1;2;3;4;5]))
det(D)
```

```
D =

    1.0000    0.5403   -0.4161   -0.9900   -0.6536
    1.0000   -0.4161   -0.6536    0.9602   -0.1455
    1.0000   -0.9900    0.9602   -0.9111    0.8439
    1.0000   -0.6536   -0.1455    0.8439   -0.9577
    1.0000    0.2837   -0.8391   -0.7597    0.4081


ans =

    1.0992
```

The matrix D is the alternant matrix of d0...d4 evaluated at 1,2,3,4, and 5. The determinant of D is nonzero, so the columns of D are linearly independent. Therefore, d0...d4 are linearly independent.

Prove that f is in the span of {d0, ... ,d4}. Write f as a linear combination of d0, ... d4.

```
spantest = double(subs([d0 d1 d2 d3 d4 f],t,[1;2;3;4;5]))
rref(spantest)
```

```
spantest =

    1.0000    0.5403   -0.4161   -0.9900   -0.6536    0.0852
    1.0000   -0.4161   -0.6536    0.9602   -0.1455    0.0300
    1.0000   -0.9900    0.9602   -0.9111    0.8439    0.9606
    1.0000   -0.6536   -0.1455    0.8439   -0.9577    0.1825
    1.0000    0.2837   -0.8391   -0.7597    0.4081    0.0065


ans =

    1.0000         0         0         0         0    0.3750
         0    1.0000         0         0         0    0.0000
         0         0    1.0000         0         0    0.5000
         0         0         0    1.0000         0    0.0000
```

```
         0         0         0         0    1.0000    0.1250
```

The matrix spantest is an augmented matrix of d0...d4 and f. The row reduced form of the system is consistent, so f is in the span of {d0...d4}. F can be written as (0.375 * d0) + (0.5 * d2) + (0.125 * d4). This is equivalent to (0.375 * 1) + (0.5 * cos(2*x)) + (0.375 * cos(4*x)).

Using your answer, integrate f by hand. *It should be easy now, no trig required.*

The resulting integral is equal to (sin(4x) + 8 sin(2x) + 12x) / 32.

That was probably just a fluke, right?

```
g = (t^4 - 2*t^3 + 4)/(t-1)^2;

h0 = 1;
h1 = t;
h2 = t^2;
h3 = 1/(t-1);
h4 = 1/(t-1)^2;
```

Repeat for g and {h0, ... , h4}. *You've just reinvented partial fractions. Boom.*

```
E = double(subs([h0 h1 h2 h3 h4],t,[2;3;4;5;6]));
det(E)

spantest_2 = double(subs([h0 h1 h2 h3 h4 g],t,[2;3;4;5;6]))
rref(spantest_2)
```

```
ans =

   -0.0200


spantest_2 =

    1.0000    2.0000    4.0000    1.0000    1.0000    4.0000
    1.0000    3.0000    9.0000    0.5000    0.2500    7.7500
    1.0000    4.0000   16.0000    0.3333    0.1111   14.6667
    1.0000    5.0000   25.0000    0.2500    0.0625   23.6875
    1.0000    6.0000   36.0000    0.2000    0.0400   34.7200


ans =

     1     0     0     0     0    -1
     0     1     0     0     0     0
     0     0     1     0     0     1
     0     0     0     1     0    -2
     0     0     0     0     1     3
```

The matrix E is the alternant matrix of h0...h4 evaluated at 2,3,4,5, and 6. The determinant of E is nonzero, so the columns of E are linearly independent. Therefore, h0...h4 are linearly independent. The second computation creates a matrix spantest_2 that is an augmented system of h0...h4 and g. The row reduced form of this matrix results in g being expressed as (-1 * h0) + (1 * h2) + (-2 * h3) + (3 * h4), which is equivalent to (-1) + (t^2) + (-2 * (1/(t-1))) + (3 * (1/(t-1)^2)). The resulting integral is equal to (x^3/3) - x - (3/(x-1)) - 2 * ln(abs(x-1)).

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