Investigating the Collision Between Two Globular Clusters

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This paper presents an investigation into collisions between globular clusters using the leapfrog finite difference approximation method to iteratively calculate the positions of the constituent stars across 1500 timesteps of size $dt = 0.15\ Myr$. A model of a single cluster is first developed and used to find suitable values for dt and the softening parameter, ε, which minimise the propagation of numerical errors. A head-on and a grazing collision between two globular clusters each containing 1000 stars is then simulated, with them conserving energy to within 0.116% and 0.160% respectively, suggesting that the model is somewhat reliable as it is not dominated by numerical errors. This is further supported when analysis implies that in each case, the model stabilises to a state in virial equilibrium. Drawbacks to this model become prevalent when one considers the long-term effect of the softening parameter as it causes the model to not be representative of reality, whilst also causing the energy conservation to deteriorate the longer the simulation runs due to the softening parameter preventing any stars from colliding.

1. Introduction

Globular clusters are centrally concentrated and gravitationally bound systems of stars [1]. Typically, globular clusters have radii which range from a couple of parsecs (pc) up to tens of parsecs [2], yet they can contain up to one million stars. As a result of this, they are very compact and have high stellar density, with their tightly bound spherical appearance being significantly shaped by the internal two-body gravitational interactions between their constituent stars.

Calculations of the net force acting on each star make use of Newton's law of gravitation which gives the force, F_i, on each body, i, due to each other body, j, as:

$$F_i = -Gm_i \sum_{j \neq i} m_j \frac{r_i - r_j}{\left[(r_i - r_j)^2 + \varepsilon^2\right]^{3/2}} \ , \qquad (1)$$

where G is the gravitational constant, m and r represent the masses and position vectors of the bodies in the system and ε represents the softening parameter. The equation was modified and ε was added to prevent any problems with infinities that may occur as the clusters collide due to the distance between stars having potential to reach zero. With the softening parameter, the stars can essentially not come within a distance of ε of each other. This has some effects on energy conservation as the more collisions that the softening parameter prevents, the more influential it is within the simulation. To find a suitable value of ε for this model, tests were conducted and details of these can be found in the Method.

Since globular clusters are systems of stars in equilibrium, the collision between two clusters should stabilise into a system which is approximately in virial equilbrium – a concept stating that for a system of particles in equilibrium, double the kinetic energy (T) divided by the potential energy (V) should be equal to minus unity. To calculate these energy values in a many-body system like a globular cluster, one must use,

$$T = \sum_i^n m_i v_i^2 \ , \qquad (2)$$

$$V = -Gm_i \sum_{j \neq i}^{n-1} \frac{m_j}{\left[(r_i - r_j)^2 + \varepsilon^2\right]^{1/2}} \ , \qquad (3)$$

where n is the total number of stars in the simulation.

Using equations (2) and (3), the virial equilbrium ratio can be found with its ideal value being -1,

$$\frac{2T}{V} = -1 \ . \qquad (4)$$

With this ratio, one can determine if a system is feasible in reality based on whether or not it yields a value close to the desired, -1.

Due to the nature of globular clusters, stars are gravitationally bound to the system and cannot leave the cluster unless they exceed a specific velocity known as the escape velocity, v_{esc}. This velocity is calculated using,

$$v_{esc} = \sqrt{\frac{2G \sum_i^n m_i}{R}} \ , \qquad (5)$$

where R is the radius of the cluster. This formula is not a robust method as it is difficult to track the total mass within the cluster radius due to some stars being ejected at high velocities when coming within short distances of other stars, however it provides a useful guide for analysis.

In this report, a simulation of a single globular cluster collapsing under its own gravity is used to determine suitable values for parameters required to collide two clusters together. Two clusters, each containing 1000 stars, were then subject to both a head-on and grazing collision. Initial and final positions are plotted as well as the energies of the system through time and analysis of the feasibility of the model is given. To close, conclusions are drawn about the results and the simulation procedure, before further work in this field is touched upon.

2. Method

To simulate the collision between two globular clusters, a model of a single star cluster had to first be developed. Mimicking the high central concentration of stars observed in globular clusters [1], a radial distribution of 1000 particles was randomly generated using spherical coordinates. This coordinate system was chosen as it allowed the spherical system to be efficiently created and naturally develop the required concentration gradient as the particle positions were calculated from randomised

radial distances, r, inclination angles, θ, and azimuthal angles, φ. In line with the aforementioned radii of observed globular clusters from literature [2], the radius of this system was set to be r = 20 pc = $R_{cluster}$, as this was on the lower end of typical radii, thus compensating for the low number of stars, yet it also allowed for the cluster to collapse on a slower timescale so star positions and velocities could be calculated more accurately.

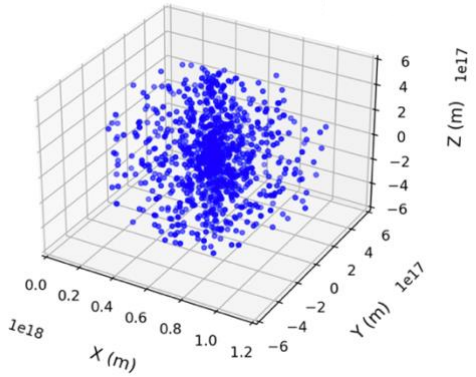

Figure 1: Initial state of a randomly generated, radial distribution of stars. The globular cluster has a radius $R_{cluster}$ = 20 pc and contains 1000 stars.

Once a suitable cluster had been generated, the model was extended to a two-cluster system using the method above to first generate the individual clusters, before adding a displacement in the x-direction to all stars in each cluster so that the separation between the two clusters was 40 pc.

After the initial conditions had been set and the two clusters generated, the clusters were made to collide with each other. To do this, one of the clusters (B) was given a non-zero initial velocity in the direction of the other cluster (A). The magnitude of this velocity was kept relatively low at 500 ms^{-1} despite clusters typically orbiting around the galactic centre at up to hundreds of kilometres per second [3]. To keep the model simple however, only the velocity difference was considered and so this was essentially the velocity of B in A's reference frame. The magnitude was decided via trial and error until the collision occurred at a suitable time which allowed the system to settle afterwards so that the final state could be properly investigated.

From here the system was allowed to evolve in time using the 'leapfrog finite difference approximation method'. The method was used instead of other, more accurate, integration schemes as it is lightweight, stable and does not produce systematic errors [4]. This scheme was used to calculate the positions and velocities of each star sequentially for a total number of 1500 timesteps – a value which was chosen via trial and error to ensure that there was both enough time for the two clusters to collide, and then stabilise after the collision. For this method to work correctly, the positions and velocities had to be calculated half a timestep out of phase using,

$$v_i^{n+1/2} = v_i^{n-1/2} + F_i \frac{dt}{m_i} \quad , \qquad (6)$$

$$x_i^{n+1} = x_i^n + v_i^{n+1/2} dt \quad , \qquad (7)$$

where x_i and v_i are the star positions and velocities respectively, dt is the size of each timestep, m_i is the mass of each star and F_i is the net force acting on each star respectively (as seen in equation (1)). For this simulation, every star was given a mass of one solar mass ($M_\odot$) as this meant that each star was close to the typical mass range of 0.3 – 0.8 $M_\odot$ for constituent stars in globular clusters [5], whilst also ensuring that the total mass of the cluster fell within the typical range of between 10^3 and 10^6 $M_\odot$ [6].

Using a randomly generated single cluster, different dt and ε values were tested to find out which combination would give the best energy conservation and virial equilibrium ratio over 1500 timesteps to ensure suitable values were chosen which minimised the numerical errors within the simulation of the collision of the two globular clusters. To make these comparisons, the overall percentage change in energy, $\frac{\Delta E}{E}$, and the virial equilibrium ratio, $\frac{2T}{V}$, were calculated for each different ε using $dt = 0.15\ Myr$, as seen in Table 1, and each different dt using $\varepsilon = 0.01\ R_{cluster}$, as seen in Table 2.

Table 1: Values of $\frac{\Delta E}{E}$ and $\frac{2T}{V}$ for different softening parameters, ε. The optimal ε appears to be 0.01 $R_{cluster}$ as this maintains a low percentage change in energy whilst also stabilising into a state closest to virial equilibrium $\left(\frac{2T}{V} = -1\right)$.

Softening Parameter, ε	$\frac{\Delta E}{E}$ **(%)**	$\frac{2T}{V}$
1 $R_{cluster}$	-2.60 x 10^{-5}	-0.718
0.1 $R_{cluster}$	-1.65 x 10^{-3}	-0.871
0.01 $R_{cluster}$	-0.0663	-1.09
0.001 $R_{cluster}$	28.2	-0.691
0.0001 $R_{cluster}$	225	-4.57

Table 2: Values of $\frac{\Delta E}{E}$ and $\frac{2T}{V}$ for different timesteps, dt. The optimal value is $dt = 0.15\ Myr$, as this maintains a low percentage change in energy whilst also stabilising into a state closest to virial equilibrium, $\left(\frac{2T}{V} = -1\right)$.

Timestep, dt (Myr)	$\frac{\Delta E}{E}$ (%)	$\frac{2T}{V}$
0.015	-5.39 x 10^{-4}	-1.19
0.15	0.280	-1.05
0.75	47.3	-1.19
1.5	70.3	-1.25
7.5	264	-16.7

To further investigate errors and support the tabulated values, both $\frac{\Delta E}{E}$ and $\frac{2T}{V}$ were plotted at each timestep. These can be seen in Figures 2 and 3, and combined with Tables 1 and 2 to infer the optimal combination which minimises the propagation of numerical errors.

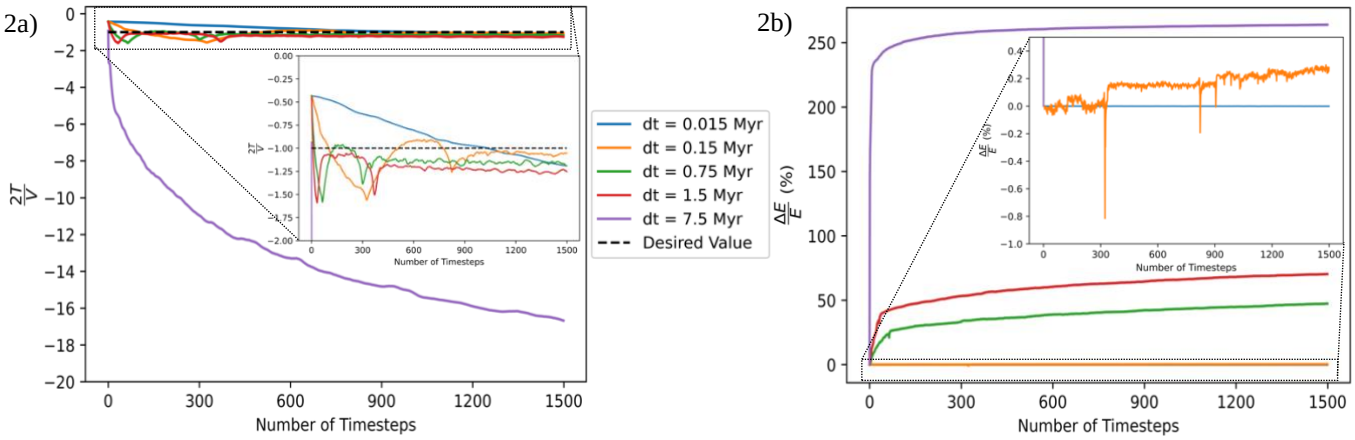

Figure 2: Graphs used to minimise the numerical errors associated with the size of the timestep dt. The dt values associated with each line in the two graphs can be found in the legend in the centre. For this simulation, ε was set as $0.01\ R_{cluster}$.

a) The virial equilibrium ratio, $\frac{2T}{V}$, of the cluster plotted at each timestep throughout the simulation where T and V are the kinetic and potential energies of the system, respectively. As seen in the zoomed-in section, the $dt = 0.15\ Myr$ line converges closest to the desired value of $\frac{2T}{V} = -1$, suggesting that the numerical errors were less prevalent for said timestep.

b) The percentage change in energy, $\frac{\Delta E}{E}$, of the cluster plotted at each timestep throughout the simulation. This graph seems to have a clear trend in that the magnitude of $\frac{\Delta E}{E}$ grows with the size of the timestep. The $dt = 0.015\ Myr$ and $dt = 0.15\ Myr$ lines display changes much closer to zero compared to the others, suggesting that the numerical errors were less prevalent for these timesteps, with $dt = 0.015\ Myr$ having such small deviations from zero that they cannot be seen on the zoomed-in section.

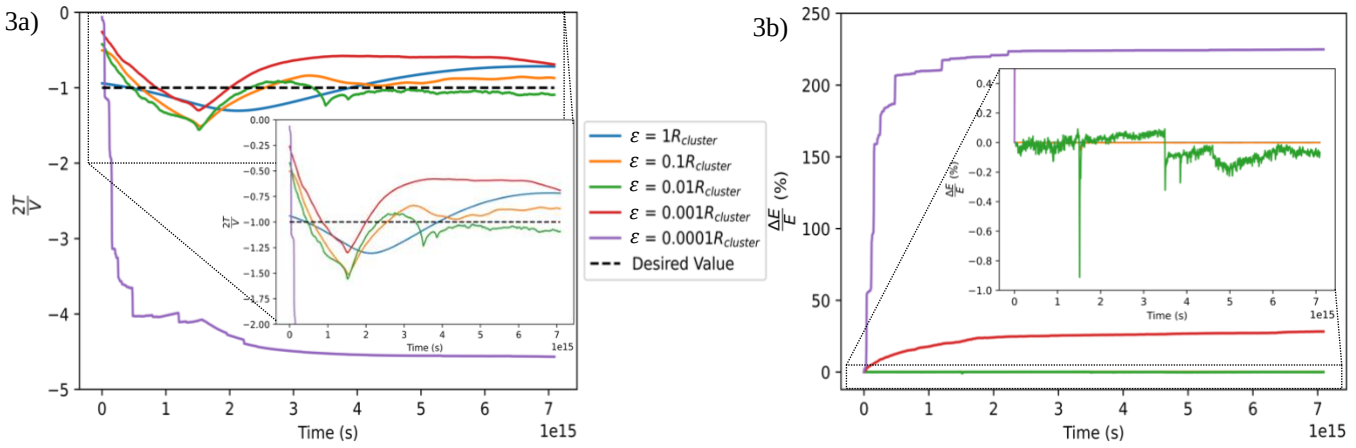

Figure 3: Graphs used to minimise the numerical errors associated with the size of the softening parameter, ε. The ε values associated with each line in the two graphs can be found in the legend in the centre. For this simulation, dt was set as $0.15\ Myr$.

a) The virial equilibrium ratio, $\frac{2T}{V}$, of the cluster plotted against each timestep throughout the simulation where T and V are the kinetic and potential energies of the system respectively. As seen in the zoomed-in section, the $\varepsilon = 0.01\ R_{cluster}$ line converges closest to the desired value of $\frac{2T}{V} = -1$, suggesting that numerical errors were less prevalent for said softening parameter.

b) The percentage change in energy, $\frac{\Delta E}{E}$, of the cluster plotted at each timestep throughout the simulation. This graph seems to have a clear trend in that the magnitude of $\frac{\Delta E}{E}$ grows as the size of the softening parameter decreases. The $\varepsilon = 1\ R_{cluster}$, $0.1\ R_{cluster}$ and $0.01\ R_{cluster}$ lines display changes much closer to zero compared to the others, suggesting that numerical errors were less prevalent for these timesteps, with both the $\varepsilon = 1\ R_{cluster}$ and $0.1\ R_{cluster}$ seeming to have such small deviations from zero that they cannot be seen in the zoomed-in portion of the graph.

From the two figures and tables, the optimal dt and ε values were chosen to be $0.15\,Myr$ and $0.01\,R_{cluster}$ respectively as these both converged to a state that was closest to virial equilibrium whilst also yielding one of the lowest percentage changes in energy, each conserving energy to within 0.5%. The softening parameter was also chosen to be the smallest of the values yielding the low numerical errors because it prevents stars from being within its specified magnitude away from each other. Therefore, to ensure one's model was as close to reality as possible, the context of this parameter was considered rather than strictly choosing the parameter which gave the best energy conservation.

Once the dt and ε had been chosen to minimise the error propagation, the two-cluster collision simulation was run for 1500 timesteps and the evolution was analysed. Throughout the simulation, the positions and velocities of each particle were tracked which allowed the percentage energy change, virial equilibrium ratio, average velocities of particles and other results to be calculated.

3. Results & Discussion

To begin the investigation, B was given an initial velocity of 500 ms^{-1} in the x-direction towards A to cause a head on collision between the two clusters. As a first insight into the evolution of this system, the initial and final states of the two clusters were compared.

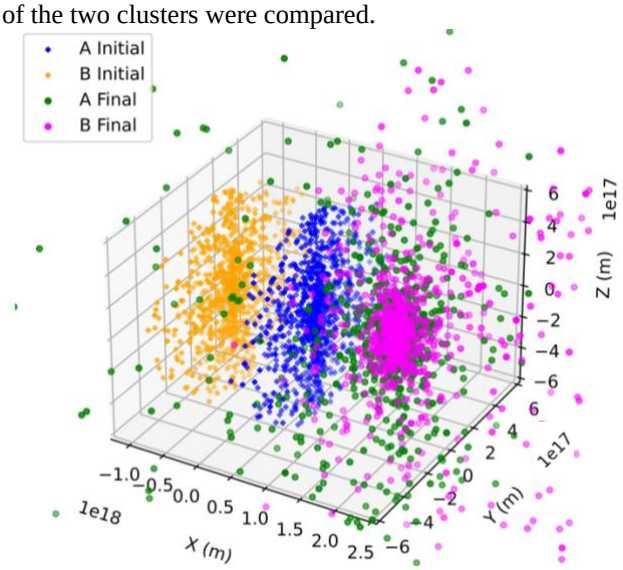

Figure 4: The final and initial states of clusters A and B after B collides with A with a velocity of 500 ms^{-1} in the positive x-direction. Each cluster began with 1000 stars and the system was evolved for 0.225 Gyr, with positions and velocities of each star being calculated every 0.15 Myr for 1500 timesteps.

As seen in Figure 4, the final state is very different to the initial state with there being just a single cluster containing most of the stars left over, rather than the two separate clusters. Moreover, many of the stars can be seen at very large radii, suggesting that they have been ejected at high velocities.

To further investigate this, the energy profile of the system was observed which can be found below in Figure 5.

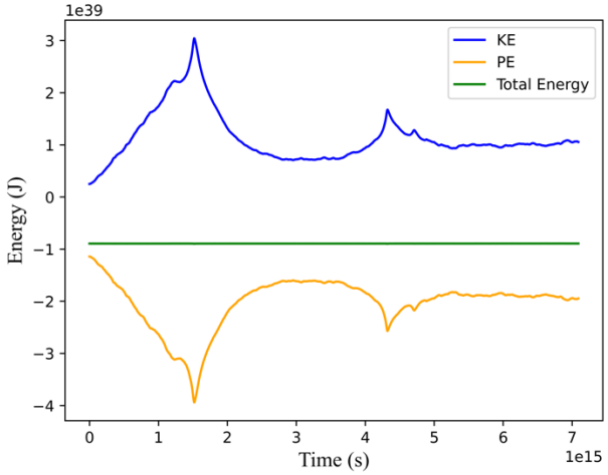

Figure 5: Kinetic energy (KE), potential energy (PE) and total energy of the system every 0.15 Myr for 1500 timesteps. As desired, the total energy of the system seems to remain relatively constant throughout the simulation with the overall percentage change being 0.116% (3SF). Furthermore, the magnitude of the PE seems to be approximately twice the magnitude of the KE for the entire duration of the simulation, with $\frac{2T}{V}$ giving a value of -1.08 (3SF) at the final timestep.

Through analysing Figure 5, one can see that there are two significant peaks in the kinetic and potential energies at two separate timesteps. The first of these peaks, at approximately 1.5 x 10^{15} s, coincides with the initial collision between the two clusters. Here, most of the stars are still retained within the two clusters as they each begin to collapse under their own gravity and therefore, the collision causes the maximum number of stars to be as close as possible to each other. This hypothesis is supported by Figure 5 displaying a large peak in both the kinetic and potential energies at this epoch because as the clusters collide, and the distance between stars becomes much smaller, the magnitude of the PE increases (due to the r dependence of PE seen in equation (3)) and the magnitude of the KE increases (due to the r dependence of the force in equation (1) combined with the proportionality of the force with the velocity in equation (6) which is used to calculate the KE in equation (2)). At this point, the maximum distance between any two stars was calculated to be 28.5 pc meaning that, on average, the star density within the resultant cluster was 0.021 stars/pc^3.

A second peak in the energies can be found at approximately 4.3 x 10^{15} s. This peak is the result of the new, large cluster collapsing under its own gravity. The magnitude of this peak is much lower than the initial peak as there is a reduced number of stars collapsing here than earlier due to the initial collision ejecting many of the stars.

Ejection of stars can be inferred from observing both the average distance of stars from the centre of mass (COM) of the system and the average velocity of stars at each timestep.

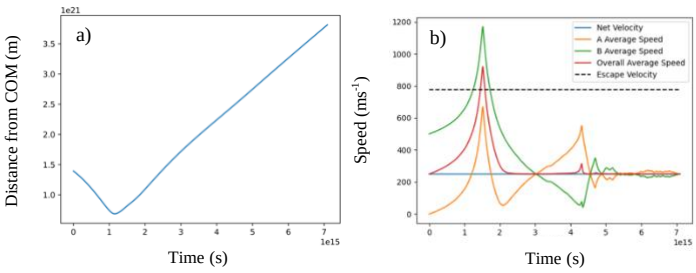

Figure 6: a) Average distance of stars from the COM of the system plotted at each timestep.
b) Average speed of stars in the systems plotted at each timestep. The average speed for stars belonging to A and B have been plotted separately as well as the overall average of these. Furthermore, the escape velocity is plotted as a dotted line, and this has been calculated using equation (5). Finally, the net velocity of the system has also been plotted in blue.

An initial trough can be seen in Figure 6a at a similar time to the spike observed earlier in the KE and PE. After this trough, the distance seems to follow a linear trend suggesting that this average distance from the COM is dominated by the ejected stars which are moving away at high enough velocities that they appear constant and are not affected by the gravitational pull of the cluster. This is supported by Figure 6b as the average velocity of the system at this same epoch is higher than the escape velocity (780 ms^{-1}) which implies that many stars are ejected from the system as a result of the collision. Typically, globular clusters have escape velocities of order of 20-30 kms^{-1} [7] which is much greater than the escape velocity of the cluster in this simulation, further highlighting the problems that this simulation faces when being directly compared to reality due to the lower number of stars used. The net velocity line in Figure 6b is constant throughout the entire simulation with the overall change being 2.39 x 10^{-13}%, thus implying that momentum was conserved which supports the reliability of this model.

Since a head-on collision is an extremely rare event, a second simulation was run to instead model a grazing collision where B's initial velocity projected it to hit the top of A (i.e., B moved on a projection angled at approximately 14º above the line along the x-axis which initially connected the centres of the two clusters). Through doing this, one hoped to further test the accuracy of the model developed.

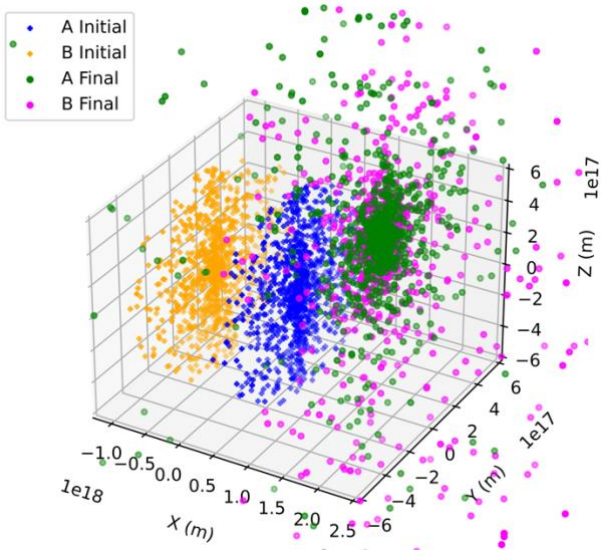

Figure 7: The final and initial states of clusters A and B after B collides with A with a velocity projected at an angle of approximately 14º above the initial x-axis separation between the centres of the two clusters. Each cluster began with 1000 stars and the system was evolved for 0.225 Gyr, with positions and velocities of each star being calculated every 0.15 Myr for 1500 timesteps.

As seen in Figure 7, the final state for a grazing collision between the clusters is very similar to that for a head-on collision, with the only noticeable difference visually being that the resultant cluster is now higher in the XZ plane due to B initially starting with a velocity which had a positive z-component. The final position of the centre of mass of the resultant cluster lies at an angle of 12.5º above the line initially connecting the centres of the two clusters. This is a more acute angle than the initial projection which is expected as the gravitational force from A should cause the stars in B to be forced downwards slightly as the larger cluster is formed.

The grazing collision also showed almost identical trends for the energies as the head-on collision did, with the maximum distance between any two stars at the first collapse being 31.0 pc. This is a slightly larger distance than in the head-on case and so resulted in a slightly smaller maximum stellar density within the resultant cluster of 0.016 stars/pc^3.

The maximum densities of the resultant clusters in both cases are much less than even a below average star density of observed globular clusters which literature gives as 0.1 stars/pc^3 [8]. This is one of the main limitations of this model. Due to the low number of stars used in this simulation, the largest density of a cluster achieved was approximately 5% of the observed density meaning that this model falls short when trying to assess timescales for collapse and evaporation of the clusters [9].

When considering the energy conservation and virial equilibrium ratio for this type of collision, the simulation yields consistent values with the head on collision, maintaining a virial equilibrium ratio of -1.09 (3SF) and a percentage change in energy of 0.160% (3SF). With both collisions yielding very similar values for both energy

conservation and the virial equilibrium ratio, the model's reliability is reinforced despite the use of both a below average number of stars for a globular cluster and a softening parameter.

4. Conclusions

To conclude, a simulation of the collision of two globular clusters was developed and the result was a single cluster which in the case of the head-on collision had a minimum radius of 28.5 pc (a 42.5% increase compared to the initial clusters) and in the case of a grazing collision had a minimum radius of 31.0 pc (a 55% increase compared to the initial clusters). This cluster then collapsed under its own gravity and many of the stars were fired out at velocities larger than the escape velocity of the resultant cluster (780 ms^{-1}) suggesting that if the cluster was left for a long enough time, it would eventually evaporate [9] as this process of ejecting stars would occur continuously.

The results from the simulation can somewhat be relied upon as for each collision, the energy was conserved to within a fifth of a percent which implies that numerical errors did not propagate or dominate within the simulation. Moreover, the system stabilised into a state that was very close to virial equilibrium, with the head-on and grazing collisions having virial equilibrium ratios of -1.08 and -1.09 respectively. Finally, when analysing the net momentum of the cluster in the head-on collision case, the percentage difference between the initial and final state was just 2.39 x 10^{-13}%, further supporting the notion that numerical errors within this simulation were kept to a minimum.

The main source of error in this simulation was due to the softening parameter. By preventing any collisions between stars, which would otherwise occur approximately every ten thousand years within a single cluster [10], the softening parameter greatly limits the quality of the model. The longer the simulation is run for the more collisions the softening parameter is masking and not only does the energy conservation of the model become worse, but also the resultant system strays further and further from what would naturally occur in reality.

As a result of this, to improve this model, alongside having more stars present within the simulation to make the cluster densities similar to reality, one would look to develop a method of simulating collisions between stars and look to eliminate the requirement of a softening parameter. Once this was developed, the model could then be extended to investigate claims that globular clusters seem to have some internal rotation [11]. This would cause the clusters to be spheroidal in shape rather than the spherical models in this report. With this, a collision between two rotating clusters could be observed and the rotational dynamics of the ejected stars and resultant cluster could then be analysed. Finally, observations suggest that some globular clusters may have small black holes at their centres [12], thus opening another avenue which could be explored using this model providing a black hole type entity was simulated.

References

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Appendix A

Collision of Two Globular Clusters Simulation Videos

1. Head-on collision between two globular clusters with its associated energy graph

URL:
https://youtu.be/sBM5RH_C0Dc

2. Grazing collision between two globular clusters with its associated energy graph

URL:
https://youtu.be/X9EYruq3x6c

Scientific Summary for a General Audience

Globular clusters are huge groups of up to one million stars which are held together by the force of gravity attracting stars within them to each other. For this report, two clusters were simulated before being collided with each other in two different ways. The first method of collision was a head-on collision where one cluster hit the other cluster directly and the second method was a grazing collision, where one cluster skimmed the top of the other cluster. To keep the simulation simple, there was a 'softening parameter' implemented which had the purpose of preventing any stars from actually colliding with each other.

After these collisions were simulated, the results of the simulation were analysed and showed that from the two clusters colliding, a single, much larger cluster remained with some stars travelling away from this cluster at extremely high velocities due to the gravitational force they had felt during the collision being so large. Some error analysis was then done to assess how reliable the simulation was and whether or not the results are applicable to real systems with the conclusion being that the major flaw in this model was that actual collisions between individual stars are forbidden.