Quiz 2: Functions and Relations I

Q1 Consider $f : \mathbb{P} \to \mathbb{P}$ given by $f(x) = 2x + 1$. What is the inverse image of $\{1, 2, 3\}$, i.e. what is $f^{\leftarrow}(\{1, 2, 3\})$?

Answer: The values of x for which $2x + 1 \in \{1, 2, 3\}$ are 0, $\frac{1}{2}$, and 1. Of these, only $1 \in \mathbb{P}$, the domain of f. So $f^{\leftarrow}(\{1, 2, 3\}) = \{1\}$.

Q2 Let $A = \{a, b, c\}$ and consider $g : \mathrm{Pow}(A) \to \mathbb{N}$ given by $g(X) = |X|$. What is $\mathrm{Im}(g)$?

Answer: $g(\emptyset) = 0$, $g(\{a\}) = g(\{b\}) = g(\{c\}) = 1$, $g(\{a, b\}) = g(\{b, c\}) = g(\{a, c\}) = 2$, and $g(\{a, b, c\}) = 3$ so $\mathrm{Im}(g) = \{0, 1, 2, 3\}$.

Q3 Let $\Sigma = \{a, b\}$ and consider $f, g : \Sigma^* \to \Sigma^*$ given by

$$\begin{aligned} f(w) &= ww \\ g(w) &= awb \end{aligned}$$

What is $f \circ g(aba)$?

Answer: $f \circ g(aba) = f(g(aba)) = f(aabab) = aababaabab$

Q4 Suppose $f : S \to T$ and $g : T \to U$ are bijective. True or false: $g \circ f$ is always bijective.

Answer: If f and g are injective then: $g \circ f(x) = g \circ f(y)$ implies $g(f(x)) = g(f(y))$, which implies $f(x) = f(y)$ (because g is injective), which implies $x = y$ (because f is injective). So $g \circ f$ is injective.

If f and g are surjective then: for all $u \in U$ there is a $t \in T$ such that $g(t) = u$ and for all $t \in T$ there is an $s \in S$ such that $f(s) = t$. So, for all $u \in U$ there is an $s \in S$ such that $g \circ f(s) = g(f(s)) = u$. So $g \circ f$ is surjective.

Therefore, if f and g are bijective, then $g \circ f$ is (always) bijective.

Q5 Let $\Sigma = \{a, b\}$ and consider the relation $R \subseteq \Sigma^* \times \Sigma^*$ given by $(w, v) \in R$ if $\mathrm{length}(wv)$ is even. Which of the properties Reflexivity (R) and Transitivity (T) does R have?

Answer: For all $w \in \Sigma^*$, $\mathrm{length}(ww) = 2\mathrm{length}(w)$ is even, so $(w, w) \in R$. So R is reflexive (R).

If $\mathrm{length}(wv) = \mathrm{length}(w) + \mathrm{length}(v)$ is even and $\mathrm{length}(vu) = \mathrm{length}(v) + \mathrm{length}(u)$ is even then $\mathrm{length}(w)$, $\mathrm{length}(v)$ and $\mathrm{length}(u)$ are either all even, or all odd. In either case $\mathrm{length}(w) + \mathrm{length}(u) = \mathrm{length}(wu)$ is even, so $(w, u) \in R$. So R is transitive (T).