Quiz 8: Big-Oh notation

Q1 Suppose $f, g, h, k : \mathbb{N} \to \mathbb{R}$ are such that $f(n) \in O(h(n))$ and $g(n) \in O(k(n))$. Which of the following are true:

A: $f(n)g(n) \in O(h(n)k(n))$

B: $f(n)/g(n) \in O(h(n)/k(n))$

Answer: $f(n) \in O(h(n))$ means that there exists c_1, n_1 such that $f(n) \leq c_1.h(n)$ for $n \geq n_1$; and $g(n) \in O(k(n))$ means that there exists c_2, n_2 such that $g(n) \leq c_2.k(n)$ for $n \geq n_2$. This means that for $n \geq \max\{n_1, n_2\}$ we have $f(n)g(n) \leq c_1c_2h(n)k(n)$, so $f(n)g(n) \in O(h(n)k(n))$.

On the other hand, we can't say $f(n)/g(n) \in O(h(n)/k(n))$: for example consider $f(n) = h(n) = n^3$, $g(n) = n$, and $k(n) = n^2$. We have $f(n) \in O(h(n))$ and $g(n) \in O(k(n))$ but $f(n)/g(n) = n^2$ and this is not $O(h(n)/k(n)) = O(n)$.

So **only A is true**.

Q2 Let F denote the set of all functions from $\mathbb{N}$ to $\mathbb{R}$. Define relations $R, S \subseteq F \times F$ as follows:

- $(f, g) \in R$ if $f \in O(g)$
- $(f, g) \in S$ if $f \in \Theta(g)$

Which of the following statements is true:

A: R is a partial order

B: S is an equivalence relation

Answer: R is reflexive and transitive but it is not anti-symmetric: for example $n \in O(2n)$ and $2n \in O(n)$ but $n \neq 2n$. S is reflexive, symmetric, and transitive so it is an equivalence relation. So **only B is true**.

Q3 Which of the following statements is always true:

A: For any graph G with vertices V, and edges E, $|E| \in O(|V|^2)$

B: For any tree T with vertices V, and edges E, $|E| \in O(|V|)$

Answer: In any (undirected) graph with n vertices, the number of edges is at most $n(n-1)/2 \in O(n^2)$. In any tree with n vertices, the number of edges is precisely $n-1 \in O(n)$. So **both A and B are true**.

Q4 Suppose $T(n)$ is defined as follows:

- $T(1) = 1$
- $T(n) = 8T(n/4) + 2n^2$

Which of the following provides the best upper bound for the asymptotic complexity of $T(n)$?

Answer: We can apply the Master Theorem here, with $d = 4$, $\alpha = \frac{3}{2}$, and $\beta = 2$. So we are in Case 3: meaning $T(n) \in O(n^\beta) = O(n^2)$.

Q5 Order the following functions in increasing asymptotic complexity:

- $2n.\log(n) + 3n^2$
- $\sqrt{7n^2 + 3n + 1}$
- $(2^{1.5})^{\log(n)}$
- $4n^{\log(\log(n))}$
- $n^2/\log(n)$

We have the following observations:

- $2n.\log(n) + 3n^2 \in \Theta(n^2)$
- $\sqrt{7n^2 + 3n + 1} = (7n^2 + 3n + 1)^{1/2} \in \Theta(n)$
- $(2^{1.5})^{\log(n)} = (2^{\log(n)})^{1.5} = n^{1.5} \in \Theta(n^{1.5})$
- $4n^{\log(\log(n))} \in \Omega(n^3)$ because for sufficiently large n, $\log(\log(n)) > 3$.
- For any constant c, for sufficiently large n, $n^2/\log(n) > c.n^{1.6}$ and $n^2/\log(n) < c.n^2$. So $n^2/\log(n) \in \Omega(n^{1.6})$ and $n^2 \notin O(n^2/\log(n))$.

This means that the ordering of the functions in increasing asymptotic complexity is:

1. $\sqrt{7n^2 + 3n + 1}$
2. $(2^{1.5})^{\log(n)}$
3. $n^2/\log(n)$
4. $2n.\log(n) + 3n^2$
5. $4n^{\log(\log(n))}$