Quiz 7: Induction and Recursion

Q1 Suppose you are trying to prove that the square of every natural number has remainder 0 or 1 on division by 4. If $P(n)$ is the proposition:

$$P(n) : \ n^2 = 0 \text{ or } 1 \pmod 4$$

Which of the following would be sufficient to prove the result by induction:

(a) $P(0)$ and $P(1)$ and $\forall k \geq 0,\ P(k) \to P(k+2)$

(b) $P(0)$ and $\forall k \geq 0,\ P(k^2) \to P((k+1)^2)$

(c) $P(0)$ and $\forall k \geq 0,\ P(k) \to P(k+2)$ and $\forall k \geq 0,\ P(k+1) \to P(k+3)$

(d) $P(0)$ and $\forall k \geq 0,\ P(k^2) \to P(k^2+1)$

Answer: (a) This is the correct answer: From $P(0)$ and $P(k) \to P(k+2)$ we have that $P(n)$ holds for all even n; and from $P(1)$ and $P(k) \to P(k+2)$ we have that $P(n)$ holds for all odd n. So $P(n)$ holds for all natural numbers.

(b) This fails because from $P(0)$ and $P(k^2) \to P((k+1)^2)$, we can establish $P(1)$, $P(4)$, $P(9)$, etc. Although we can show that $P(n)$ holds for infinitely many natural numbers, there are some missing: for example, we cannot establish $P(2)$ holds.

(c) This fails because from $P(0)$ and $P(k) \to P(k+2)$ we can establish that $P(n)$ holds for all even n. $P(k+1) \to P(k+3)$ says that we could establish $P(n)$ for all odd n *if* we could establish $P(1)$, but we cannot establish that from $P(0)$ and the two inductive steps.

(d) This fails because from $P(0)$ and $P(k^2) \to P(k^2+1)$ we can establish $P(1)$ and $P(2)$ and that is all – there is no k other than $k = 0, 1$ such that $P(k^2)$ has been established.

Q2 The *dual* of a propositional formula is obtained by replacing $\top$ with $\bot$, $\bot$ with $\top$, $\wedge$ with $\vee$, and $\vee$ with $\wedge$. Recursively, if $\mathtt{PF}$ is the set of propositional formulas over Prop, we define $\mathsf{dual} : \mathtt{PF} \to \mathtt{PF}$ as follows:

- $\mathsf{dual}(\top) = \bot$; $\mathsf{dual}(\bot) = \top$;
- $\mathsf{dual}(p) = p$ for all $p \in \mathsf{Prop}$;
- $\mathsf{dual}(\neg\varphi) = \neg\mathsf{dual}(\varphi)$;
- $\mathsf{dual}(\varphi \wedge \psi) = \mathsf{dual}(\varphi) \vee \mathsf{dual}(\psi)$;

- $\mathsf{dual}(\varphi \vee \psi) = \mathsf{dual}(\varphi) \wedge \mathsf{dual}(\psi)$.

You may take it as given that if φ is logically equivalent to ψ, then $\mathsf{dual}(\varphi)$ is logically equivalent to $\mathsf{dual}(\psi)$. Which of the following is logically equivalent to $\mathsf{dual}(p \to q)$?

Answer: $p \to q$ is logically equivalent to $\neg p \vee q$. So

$$\begin{aligned}
\mathsf{dual}(p \to q) &\equiv \mathsf{dual}(\neg p \vee q) \\
&= \mathsf{dual}(\neg p) \wedge \mathsf{dual}(q) \\
&= \neg\mathsf{dual}(p) \wedge q \\
&= \neg p \wedge q \\
&\equiv \neg(\neg\neg p \vee \neg q) \\
&\equiv \neg(p \vee \neg q) \\
&\equiv \neg(q \to p).
\end{aligned}$$

Q3 As before, let $\mathtt{PF}$ be the set of well-formed formulas over Prop. Define $\mathsf{flip} : \mathtt{PF} \to \mathtt{PF}$ recursively as follows:

- $\mathsf{flip}(\top) = \top$; $\mathsf{flip}(\bot) = \bot$;
- $\mathsf{flip}(p) = \neg p$ for all $p \in \mathsf{Prop}$;
- $\mathsf{flip}(\neg\varphi) = \neg\mathsf{flip}(\varphi)$;
- $\mathsf{flip}(\varphi \wedge \psi) = \mathsf{flip}(\varphi) \wedge \mathsf{flip}(\psi)$;
- $\mathsf{flip}(\varphi \vee \psi) = \mathsf{flip}(\varphi) \vee \mathsf{flip}(\psi)$.

True or false: $\mathsf{dual}(\varphi)$ is logically equivalent to $\neg\mathsf{flip}(\varphi)$?

Answer: True. Proof is part of Assignment 3.

Q4 Suppose $T(n)$ is defined recursively as follows:

- $T(0) = 1$;
- $T(n) = T(n-1) + 2n$

Which of the following is a valid formula for $T(n)$?

(a) $T(n) = 2n + 1$
(b) $T(n) = 2^{n+1} - 1$
(c) $T(n) = n^2 + n + 1$
(d) $T(n) = n^3 - 2n^2 + 3n + 1$

Answer: We can compute the first few values of $T(n)$ to eliminate the possibilities. Observe that $T(0) = 1$, $T(1) = 3$, $T(2) = 7$ (eliminating (a)), and $T(3) = 13$ (eliminating (b) and (d)); so (c) is the only possibility by elimination. We can show that it is the case by unrolling the definition of $T(n)$, or, more formally, by induction.

Let $P(n)$ be the proposition that $T(n) = n^2 + n + 1$. We will show that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case. $T(0) = 1$ (by definition) and $0^2 + 0 + 1 = 1$ so $P(0)$ holds.

Inductive case. Assume $P(k)$ holds for $k \geq 0$. That is, $T(k) = k^2 + k + 1$. Then we have:

$$\begin{array}{rcll} T(k+1) & = & T(k) + 2(k+1) & \text{(definition of } T\text{)} \\ & = & (k^2 + k + 1) + 2(k+1) & \text{(IH)} \\ & = & k^2 + 3k + 3 & \\ & = & (k^2 + 2k + 1) + (k+1) + 1 & \\ & = & (k+1)^2 + (k+1) + 1 & \end{array}$$

So $P(k+1)$ holds. That is, $P(k) \to P(k+1)$.
So, by the principle of Mathematical Induction: $P(n)$ holds for all $n \in \mathbb{N}$.

Q5 Suppose $f, g : \{a, b\}^* \to \{a, b\}^*$ are defined recursively as follows:

- $f(\lambda) = a$
- $g(\lambda) = b$
- $f(aw) = f(w)g(w)$
- $f(bw) = g(w)f(w)$
- $g(aw) = g(bw) = f(w)$

What is $f(aab)$?

Answer:

$$\begin{array}{rll} f(aab) & = f(ab) \cdot g(ab) & \text{(Definition of } f(aw)\text{)} \\ & = f(b) \cdot g(b) \cdot g(ab) & \text{(Definition of } f(aw)\text{)} \\ & = g(\lambda) \cdot f(\lambda) \cdot g(b) \cdot g(ab) & \text{(Definition of } f(bw)\text{)} \\ & = b \cdot f(\lambda) \cdot g(b) \cdot g(ab) & \text{(Definition of } g(\lambda)\text{)} \\ & = b \cdot a \cdot g(b) \cdot g(ab) & \text{(Definition of } f(\lambda)\text{)} \\ & = b \cdot a \cdot f(\lambda) \cdot g(ab) & \text{(Definition of } g(bw)\text{)} \\ & = b \cdot a \cdot a \cdot g(ab) & \text{(Definition of } f(\lambda)\text{)} \\ & = b \cdot a \cdot a \cdot f(b) & \text{(Definition of } g(aw)\text{)} \\ & = b \cdot a \cdot a \cdot b \cdot a & \text{(Definition of } f(b)\text{, computed earlier)} \end{array}$$