Quiz 6: Logic II

Q1 Let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and $B = \{1, 2, 4, 8\}$. Define:

- $a \vee b = \mathrm{lcm}(a, b)$
- $a \wedge b = \gcd(a, b)$
- $a' = 30/a$ for $a \in A$; $b' = 8/b$ for $b \in B$

Which of A and B equipped with these operations is a Boolean Algebra?

Answer: A is a Boolean algebra. This is seen easiest by associating each element of A with the set of its prime divisors – a subset of $\{2, 3, 5\}$. $\vee$ then corresponds to union (since the lcm of a and b will be the product of primes in the union of the sets of primes making up a and b), $\wedge$ corresponds to intersection, and $(\cdot)'$ corresponds to set complementation (w.r.t. $\{2, 3, 5\}$). In other words $(A, \vee, \wedge, ', 1, 30)$ is *isomorphic* to the Boolean Algebra $(\mathrm{Pow}(\{2, 3, 5\}), \cup, \cap, \cdot^c, \emptyset, \{2, 3, 5\})$.

B is not a Boolean algebra. This can be seen with a violation of the complementation law: $(2' \vee 2) = (4 \vee 2) = 4$ whereas $(1' \vee 1) = (8 \vee 1) = 8$. So there is no unique "1" element, so B cannot be a Boolean Algebra.

Q2 True or false: $\neg(p \to (q \wedge r)), (\neg r \vee p) \models \neg q$?

Answer: The valuation which maps $p \mapsto T$, $q \mapsto T$, $r \mapsto F$ will map $\neg(p \to (q \wedge r))$ to T and $(\neg r \vee p)$ to T but $\neg q$ to F. So there is a valuation which sets all the formulae on the left to T, but the formula on the right to F, so $\neg q$ is not a logical consequence of $\neg(p \to (q \wedge r)), (\neg r \vee p)$. So the answer is **false**.

Q3 The *parity* function, $\mathsf{parity}(x, y, z)$, is a 3-ary Boolean function that returns T if and only if an odd number of x, y, z are T. How many clauses are there in the canonical DNF for parity?

Answer: The canonical DNF can be read off the truth table for the function: each row which evaluates to T corresponds to a clause in the canonical DNF. For parity this is the truth table:

x	y	z	parity
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

and the resulting DNF will be:

$$\mathsf{parity}(x, y, z) = xy\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\,\overline{y}\,\overline{z}.$$

So there are **four** clauses in the canonical DNF.

Q4 How many clauses are there in an optimal DNF for parity (i.e. what is the minimum number of covering rectangles in a Karnaugh map for parity)?

Answer: The Karnaugh map for parity looks like:

	xy	$\overline{x}y$	$\overline{x}\,\overline{y}$	$x\overline{y}$
z	$+$		$+$	
$\overline{z}$		$+$		$+$

We cannot use any rectangle other than 1×1 rectangles to cover the $+$'s, so the minimum number of covering rectangles (and hence the minimum number of clauses in an optimal DNF) is **four**.

Q5 Which of the following propositional formulae is in CNF:

A: $\neg x \wedge y \wedge z$

B: $\neg x \vee y \vee z$

Answer: Recall: a formula is in CNF if it is a conjunction of CNF-clauses, where a CNF-clause is a disjunction of one or more literals, and a literal is a propositional variable or the negation of a propositional variable.

So, A is a formula in CNF with 3 clauses (one literal per clause) and B is a formula in CNF with 1 clause (containing three literals). So **both** A and B are in CNF.

We note that A and B are also formulae in DNF [A having one DNF-clause, B having three DNF-clauses].