# Quiz 10: Probability

Q1 Suppose we have three urns with black and white balls distributed as follows:

- Urn A has 10 black balls and 20 white balls
- Urn B has 10 black balls and 10 white balls
- Urn C has 10 black balls and 1 white ball.

Suppose we choose an urn (uniformly at random) and draw a ball (uniformly at random) from that urn. What is the probability that the ball is white?

**Answer:** Let $A$, $B$, and $C$ be the (mutually exclusive) events that Urn A, B, C (respectively) was chosen, and let $W$ be the event that a white ball is drawn. We have:

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

and

$$P(W|A) = \frac{2}{3} \quad P(W|B) = \frac{1}{2} \quad P(W|C) = \frac{1}{11}.$$

It follows that:

$$P(W \cap A) = \frac{2}{9} \quad P(W \cap B) = \frac{1}{6} \quad P(W \cap C) = \frac{1}{33}$$

and so

$$P(W) = P(W \cap A) + P(W \cap B) + P(W \cap C) = \frac{2}{9} + \frac{1}{6} + \frac{1}{33} = \frac{83}{198}.$$

Q2 Suppose we roll three six-sided dice with the following numbers on them:

- Die A: 1,1,6,6,8,8
- Die B: 2,2,4,4,9,9
- Die C: 3,3,5,5,7,7

Let A,B,C be the random variables denoting the numbers shown by Die A,B,C (respectively). Which of the following statements are not true:

(a) $\frac{1}{2} < P(A < B)$

(b) $\frac{1}{2} < P(B < C)$

(c) $\frac{1}{2} < P(A < C)$

**Answer:** onsider the sample spaces when we roll dice X and Y:

| | $(A,B)$ | | | $(B,C)$ | | | $(A,C)$ | |
|---|---|---|---|---|---|---|---|---|
| (1,2) | (1,4) | (1,9) | (2,3) | (2,5) | (2,7) | (1,3) | (1,5) | (1,7) |
| (6, 2) | (6, 4) | (6,9) | (4, 3) | (4,5) | (4,7) | (6, 3) | (6, 5) | (6,7) |
| (8, 2) | (8, 4) | (8,9) | (9, 3) | (9, 5) | (9, 7) | (8, 3) | (8, 5) | (8, 7) |

The outcomes where the second player scores higher than the first have been highlighted. From this we conclude that:

$$P(A < B) = P(B < C) = \frac{5}{9} \quad \text{and} \quad P(A < C) = \frac{4}{9},$$

So (c) is the correct answer.

**Comment:** The dice A,B,C form a set of Non-transitive dice – a set of dice where any one die will always be beaten by at least one other with probability greater than 1/2.

Q3 Suppose we choose an $n$ letter word at random (uniformly) from $\Sigma^n$ where $\Sigma = \{a, b, c\}$. What is the probability that the letters of the word are in (increasing) alphabetical order?

**Answer:** There are $|\{a, b, c\}|^n = 3^n$ $n$-letter words in $\Sigma^n$, so we just need to count the number of words that have the letters in increasing alphabetical order. Such a word is of the form $aa \ldots abb \ldots bcc \ldots c$ where we have 0 or more $a$'s, 0 or more $b$'s and 0 or more $c$'s (but $n$ letters in total). So the number of alphabetically ordered words can be seen as the same as the number of ways of distributing $n$ indistinguishable balls into 3 distinguishable boxes (with 0 or more balls per box) – the number of balls in each box corresponds to the number of letters in the word. There are therefore $\binom{n+3-1}{3-1} = \frac{(n+2)(n+1)}{2}$ alphabetically ordered words, so the probability that the letters are alphabetically ordered is $\frac{(n+2)(n+1)}{2.3^n}$.

Q4 Suppose two normal, six-sided dice are rolled, a red die and a black die.

Let

- $A$ be the event that both dice show 4 or higher;
- $B$ be the event that the red die shows 4 or lower; and
- $C$ be the event that the black die shows 4.

True or false: $A$ and $(B \cap C)$ are independent events.

**Answer:** There are:

- 36 outcomes in the sample space $\Omega = \{(r,b) \,:\, 1 \le r,b \le 6\}$
- 9 outcomes in $A = \{(r,b) \,:\, 4 \le r,b \le 6\}$,
- 4 outcomes in $B \cap C = \{(r,4) \,:\, 1 \le r \le 4\}$, and
- 1 outcome in $A \cap (B \cap C) = \{(4,4)\}$.

So

$$\begin{aligned} P(A).P(B \cap C) &= \frac{9}{36}.\frac{4}{36} \\ &= \frac{1}{36} \\ &= P(A \cap (B \cap C)). \end{aligned}$$

So $A$ and $(B \cap C)$ are independent.

Q5 You flip a fair coin twice. What is the probability you flipped two heads given that one of your flips was a head?

**Answer:** The sample space is $\Omega = \{HH, HT, TH, TT\}$. Let $A$ be the event "two heads", i.e. $A = \{HH\}$, and $P(A) = \frac{1}{4}$. Let $B$ be the event "one flip is a head", so $B = \{HH, HT, TH\}$ and $P(B) = \frac{3}{4}$. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$