**Due: 16th of September 2018 at 11:59pm**

# COMP 9020 – Assignment 2

Note: In your assignment, *how* you arrived at your solution is as important (if not more so) than the solution itself and will be assessed accordingly. There may be more than one way to find a solution, and your approach should contain enough detail to justify its correctness. Lecture content can be assumed to be common knowledge.

1. If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ are binary relations, the *composition* of $R_1$ and $R_2$ is the relation $R_1; R_2$ defined as:

   $R_1; R_2 := \{(a, c) : \text{There exists } b \in T \text{ such that } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$

   (a) If $f : S \to T$ and $g : T \to U$ are functions is $f; g$ a function?

   (b) If $R \subseteq S \times S$ is transitive, show that $R = R \cup (R; R)$. (*Hint: One way to show $A = B$ is to show $A \subseteq B$ and $B \subseteq A$. One of these directions is trivial.*)

   **Solution:**

   (a) Yes. If $f$ and $g$ are functions then for any $x \in S$ there is a unique $y \in T$ (namely $y = f(x)$) such that $(x, y) \in f$; and for any $y \in T$ there is a unique $z \in U$ (namely $z = g(y)$) such that $(y, z) \in g$. Therefore, for any $x \in S$ there is a unique $z \in U$ (namely $z = g(f(x))$ such that $(x, z) \in f; g$. In other words, $f; g = g \circ f$ where $\circ$ is function composition.

   (2 marks)

   (b) It is clear that $R \subseteq R \cup (R; R)$, so, following the hint, it is sufficient to show that $(R; R) \cup R \subseteq R$. For this, it is sufficient to show that $(R; R) \subseteq R$. Consider $(a, c) \in (R; R)$. From the definition of ; there is a $b$ such that $(a, b) \in R$ and $(b, c) \in R$. But then, by the transitivity of $R$, it follows that $(a, c) \in R$. So $(R; R) \subseteq R$ as required.

   (3 marks)

   Let $R \subseteq S \times S$ be any binary relation on a set $S$. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

   $$\begin{aligned} R^0 &:= R, \text{ and} \\ R^{i+1} &:= R^i \cup (R^i; R) \text{ for } i \geq 0 \end{aligned}$$

   (c) Prove that if $R^i = R^{i+1}$ for some $i$, then $R^i = R^j$ for all $j \geq i$.

(d) Prove that if $R^i = R^{i+1}$ for some $i$, then $R^k \subseteq R^i$ for all $k \geq 0$.

(e) If $|S| = n$, explain why $R^n = R^{n+1}$. (*Hint: Show that if* $(a,b) \in R^{n+1}$ *then* $(a,b) \in R^i$ *for some* $i < n+1$.)

**Solution:**

(c) Let $P(j)$ be the proposition that $R^j = R^i$. We will show that $P(j)$ holds for all $j \geq i$ by induction on $j$.

**Base case.** $P(i)$ is clearly true.

**Inductive case:** Assume $P(k)$ is true for some $k \geq i$. That is, $R^k = R^i$. Consider $R^{k+1}$:

$$\begin{aligned} R^{k+1} &:= R^k \cup (R^k; R) \quad &&\text{(definition of } R^{k+1}\text{)} \\ &= R^i \cup (R^i; R) &&\text{(IH)} \\ &= R^{i+1} &&\text{(definition of } R^{i+1}\text{)} \\ &= R^i &&\text{(definition of } i\text{)} \end{aligned}$$

So $P(k)$ implies $P(k+1)$. Therefore $P(j)$ is true for all $j \geq i$.

(5 marks)

(d) We have for all $j \geq 0$, $R^j \subseteq R^{j+1}$, so by the transitivity of $\subseteq$ (covered in lectures) we have that $R^k \subseteq R^i$ for all $k \leq i$. Question 1(c) established that $R^k \subseteq R^i$ for all $k > i$, so $R^k \subseteq R^i$ for all $k \geq 0$.

(2 marks)

(e) It suffices to show that $R^{n+1} \subseteq R^n$. If $(a,c) \in R^{n+1}$ then there exists $b_0, b_1, \ldots, b_n, b_{n+1} \in S$ such that $a = b_0$, $c = b_{n+1}$ and $(b_i, b_{i+1}) \in R$ for $0 \leq i \leq n$. As $S$ only has $n$ elements, there must be $i, j$ with $0 \leq i < j \leq n+1$ such that $b_i = b_j$. But this means that $(b_i, b_{j+1}) \in R$, and so $(a,c) \in R^{n+1-(j-i)}$. As $j > i$, $n+1-(j-1) \leq n$, so $(a,c) \in R^k$ for some $k \leq n$. From our earlier observation at the start of 1(d), $R^k \subseteq R^n$, so $(a,c) \in R^n$, as required.

(4 marks)

In the above sequence, $R^n$ is defined to be the *transitive closure* of $R$, denoted $R^*$ (closely related to the $^*$ operator used to describe the set of all words over an alphabet).

(f) Show that $R^*$ is transitive.

**Solution:**

(f) Suppose $(a, b) \in R^*$ and $(b, c) \in R^*$. Then there exists $b_1, \ldots, b_k$ (where $k \leq n$) and $c_1, \ldots, c_r$ (where $r \leq n$) such that the following pairs are all in $R$:

$$(a, b_1), (b_i, b_{i+1}) \text{ for } 1 \leq i < k, (b_k, b), (b, c_1), (c_i, c_{i+1}) \text{ for } 1 \leq i < r, (c_r, c).$$

This means that $(a, c) \in R^{k+r+1}$. From 1(d) and 1(e), $R^{k+r+1} \subseteq R^n$, so $(a, c) \in R^n = R^*$. Hence $R^*$ is transitive.

(4 marks)

(20 marks)

2. The following table describes several subjects and the students taking them:

| **Potions** | **Charms** | **Herbology** | **Astronomy** | **Transfiguration** |
|---|---|---|---|---|
| Harry | Ron | Harry | Hermione | Hermione |
| Ron | Luna | George | Neville | Fred |
| Malfoy | Ginny | Neville | Seamus | Luna |

You have been tasked to create an examination timetable for these subjects, and your goal is to find the *smallest number* of timeslots needed so that all subjects can be examined, without any conflicts occurring (i.e. no students having to take two or more exams at the same time).

(a) Explain how this can be formulated as a graph-based problem. That is, describe what the vertices and edges would be, and how to relate the given problem to a common graph problem.

(b) For this problem in particular determine the minimum number of timeslots required.

**Solution:**

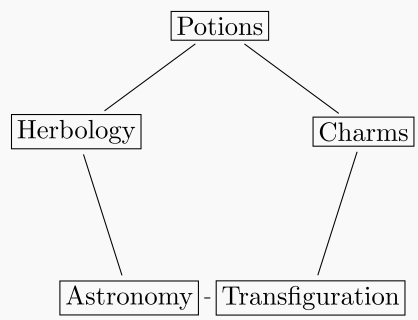
(a) Define a graph $G$ as follows:

- The set of vertices correspond to the subjects
- An edge between a pair of vertices (subjects) if they have a student in common (i.e. the subjects are in *conflict*)

The *chromatic number* of $G$ then indicates the minimum number of timeslots required: each colour in a valid colouring corresponds to a choice of timeslot. Subjects that are in conflict share an edge so they will have to have a different colour.

(3 marks)

(b) For this problem, the associated graph would be:




This graph, being an odd-length cycle, has chromatic number 3, so the minimum number of timeslots is three. An example allocation might be:

| Timeslot 1 | Timeslot 2 | Timeslot 3 |
|---|---|---|
| Potions | Charms | Herbology |
| Transfiguration | Astronomy | |

(2 marks)

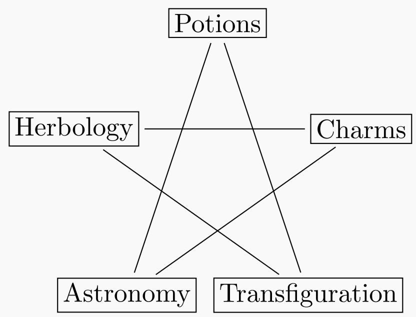
(c) Suppose instead your goal was to determine the *largest number* of subjects that can be examined at the same time without conflicts. How do your answers to (a) and (b) change?

(10 marks)

**Solution:**

(c) Use the same set of vertices, but put an edge between vertices if the corresponding subjects are *not* in conflict. The *clique number* of this graph then indicates the largest number of subjects that are mutually not in conflict – i.e. the largest number of subjects that can be examined at the same time.

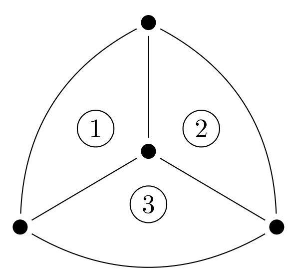
For the problem at hand, the graph becomes:




Which has clique number 2, implying that the maximum number of subjects assessible in any single timeslot is two. For example Potions and Transfiguration could be assessed at the same time.

(5 marks)

3. Given a plane-drawing (i.e. no crossing edges) of a *connected* planar graph $G$, a *face* is a region that is enclosed by edges. For example, the following plane-drawing of $K_4$ has 3 faces (labelled 1,2,3):




(a) How many edges must a connected graph with $n$ vertices and 1 face have?

(b) By examining several planar graphs, come up with an equation that relates the number of vertices ($n$), the number of edges ($m$) and the number of faces ($f$) of a plane-drawing of a planar graph.

(c) Prove, by induction on $f$ or otherwise, that your formula is correct. *Hint: What happens if you delete an edge of a plane-drawing that doesn't disconnect the graph?*

(10 marks)

**Solution:**

(a) A connected, planar graph with $n$ vertices and 1 face contains a single cycle. Deleting an edge in that cycle creates a tree, which we know from lectures contains $n-1$ edges. Therefore the original graph has $n$ edges. (2 marks)

(b) A simple equation would be $n - m + f = 1$ (or $n - m + f = 2$ if we consider the "exterior" region as a face). This is known as **Euler's formula**. (2 marks)

(c) Let $P(f)$ be the proposition that for all $n$, if there is a connected planar graph with $n$ vertices and $f$ faces then it has $n + f - 1$ edges. We will show that $P(f)$ holds for all $f \in \mathbb{N}$.

**Base case ($f = 0$)** A planar, connected graph with 0 faces is, as observed above, a tree. Therefore it has $n-1 = n+f-1$ edges. So $P(0)$ holds.

**Inductive case** Assume $P(f)$ holds for $f \geq 0$. Consider a planar, connected graph $G$ with $n$ vertices and $f + 1$ faces. Let $G'$ be the graph that results from removing one edge that is adjacent to a face (since there are $f + 1 \geq 1$ faces, there is such an edge). We observe that:

- $G'$ is planar since removing edges does not make a planar graph non-planar.
- $G'$ is connected since the edge removed was part of a cycle: the cycle that defined the face it was adjacent to
- $G'$ has $n$ vertices
- $G'$ has one fewer faces than $G$, i.e. $G'$ has $f$ faces: the face that was adjacent to the removed edge is now "merged with" the face (or exterior) on the other side of the removed edge.

It follows from the inductive hypothesis that $G'$ has $n + f - 1$ edges, so $G$ has $n + f = n + (f + 1) - 1$ edges.

Thus $P(f)$ implies $P(f + 1)$; and so $P(f)$ holds for all $f \in \mathbb{N}$. (6 marks)

4. Extend the syntactical definition of propositional formulae to include the connective $\circ$:

   - If $\varphi$ and $\psi$ are propositional formulae, then $(\varphi \circ \psi)$ is a propositional formula.

   Given a truth valuation $v : Prop \to \mathbb{B}$, define the semantics for $\circ$ as

   $$v(\varphi \circ \psi) \; = !\big(v(\varphi) \,\&\, v(\psi)\big)$$

   (a) Draw the truth table for $(p \circ q) \circ (p \circ q)$. Give a logically equivalent formula.

   (b) For each of the following formulae, give a logically equivalent formula that only uses $\circ$ and propositional variables. Justify your answer.

      i. $\neg p$

      ii. $p \vee q$

      iii. $p \to q$

      iv. $p \leftrightarrow q$

**Solution:**

(a) Here is the truth table for $(p \circ q) \circ (p \circ q)$ and for $p \wedge q$ showing that they are logically equivalent.

| $p$ | $q$ | $p \circ q$ | $(p \circ q) \circ (p \circ q)$ | $p \wedge q$ |
|---|---|---|---|---|
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |

((2 marks)

(b) Here are some more truth tables for various combinations of $p$, $q$ and $\circ$:

| $p$ | $q$ | $p \circ p$ | $q \circ q$ | $(p \circ p) \circ (q \circ q)$ | $p \circ (q \circ q)$ | $((p \circ p) \circ (q \circ q)) \circ (p \circ q)$ |
|---|---|---|---|---|---|---|
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |

This shows that:

i. $\neg p$ is logically equivalent to $p \circ p$

ii. $p \vee q$ is logically equivalent to $(p \circ p) \circ (q \circ q)$

iii. $p \to q$ is logically equivalent to $p \circ (q \circ q)$, and

iv. $p \leftrightarrow q$ is logically equivalent to $((p \circ p) \circ (q \circ q)) \circ (p \circ q)$. Alternatively, it is logically equivalent to $(p \to q) \wedge (q \to p)$, which islogically equivalent to $(A \circ B) \circ (A \circ B)$ where $A = p \circ (q \circ q)$ and $B = q \circ (p \circ p)$.

(2 marks each)

(10 marks)