Quiz 5: Logic I

Q1 Which of the following is logically equivalent to: $\varphi = \neg(A \leftrightarrow B)$:

Answer: Consider the valuation $v(A) = T$, $v(B) = F$. Then $v(\varphi) = T$. However $v(\neg(A \to B) \wedge \neg(B \to A)) = F$ and $v(\neg A \leftrightarrow \neg B) = F$ and $v((A \vee B) \wedge (\neg A \wedge \neg B)) = F$. Examination of the other three possible valuations shows that $(A \wedge \neg B) \vee (\neg A \wedge B)$ is logically equivalent to φ.

Q2 Which of the following is a well-formed propositional formula (allowing for conventional omissions):

- $p \leftrightarrow q \leftrightarrow r$
- $p \wedge q \leftrightarrow r \vee s$
- $p \to q \wedge q \to p$
- $p \vee q \vee r$

Answer: Because $\wedge$ and $\vee$ bind more tightly than $\leftrightarrow$ we can add parentheses as follows:

- $p \leftrightarrow q \leftrightarrow r$
- $(p \wedge q) \leftrightarrow (r \vee s)$
- $p \to (q \wedge q) \to p$
- $p \vee q \vee r$

Of these, only the second meets the definition of an unambiguous propositional formula.

Q3 Suppose φ and ψ are logically equivalent. Which of $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ are tautologies?

Answer: For any valuation v,

- if $v(\varphi) = T$ then $v(\psi) = T$ so $v(\varphi \to \psi) = v(\varphi \leftrightarrow \psi) = T$
- otherwise $v(\varphi) = F$, so $v(\psi) = F$ and hence $v(\varphi \to \psi) = v(\varphi \leftrightarrow \psi) = T$.

So both $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ evaluate to T for all valuations. So both are tautologies.

Q4 Suppose $\theta, \psi \models \neg\varphi$. Which of the following does NOT hold:

- $\theta, \varphi, \psi \models \bot$
- $\models \psi \to (\theta \to \neg\varphi)$
- $\theta, \varphi \models \neg\psi$
- $\theta \models \varphi \to \neg\psi$
- $\varphi \models \neg\theta \wedge \neg\psi$

Answer: Consider the following valuations:

	θ	ψ	φ	$\psi \to (\theta \to \neg\varphi)$	$\neg\psi$	$\varphi \to \neg\psi$	$\neg\theta \wedge \neg\psi$
v_1 :	F	F	F	T	T	T	T
v_2 :	F	F	T	T	T	T	T
v_3 :	F	T	F	T	F	T	F
v_4 :	F	T	T	T	F	F	F
v_5 :	T	F	F	T	T	T	F
v_6 :	T	F	T	T	T	T	F
v_7 :	T	T	F	T	F	T	F

Note that because $\theta, \psi \models \neg\varphi$ there is no valuation v that has $v(\theta) = v(\psi) = v(\varphi) = T$. In particular, even though there is no valuation v such that $v(\bot) = T$, it is the case that $v(\bot) = T$ for all valuations such that $v(\theta) = v(\psi) = v(\varphi) = T$ (because there are no such valuations). Thus $\theta, \varphi, \psi \models \bot$. For the other options, we see that

- for all valuations $v(\psi \to (\theta \to \neg\varphi)) = T$ so $\psi \to (\theta \to \neg\varphi)$ is a tautology;
- for $v \in \{v_6\}$: $v(\neg\psi) = T$, so $\theta, \varphi \models \neg\psi$;
- for $v \in \{v_5, v_6, v_7\}$: $v(\varphi \to \neg\psi) = T$, so $\theta \models \varphi \to \neg\psi$; and
- for $v = v_4$ (or $v = v_6$): $v(\neg\theta \wedge \neg\psi) = F$ but $v(\varphi) = T$, so $\varphi \not\models \neg\theta \wedge \neg\psi$.

Q5 True or false: $((p \to (q \to \bot)) \to \bot) \leftrightarrow (p \wedge q)$ is a tautology.

Answer: Observe that $p \to \bot$ is logically equivalent to $\neg p$, so the left-side is logically equivalent to $\neg(p \to \neg q)$ which is logically equivalent to $\neg(\neg p \vee \neg q)$ and this is logically equivalent to $(p \wedge q)$. It follows by the theorem given in lectures that $((p \to (q \to \bot)) \to \bot) \leftrightarrow (p \wedge q)$ is a tautology.