Quiz 9: Counting

Q1 How many different 9 letter words can be made by the using the letters in PINEAPPLE once each?

Answer: There are two approaches to this.

First, following strategies used in the lectures, we can try placing the P's in 3 of the 9 places (in $\binom{9}{3}$ ways), then place the E's in 2 of the 6 remaining spots (in $\binom{6}{2}$ ways), then place the remaining four letters (in 4! ways). Thus the total number of ways is $\binom{9}{3}\binom{6}{2}4! = \frac{9!}{3!2!}$.

A second approach is to first count the number of ways if we assume each of the P's and E's are distinguishable (9!) and then divide by the number of ways we have "overcounted": by assuming the P's are distinguishable, we have counted 3! duplicates, and by assuming the E's are distinguishable, we have counted 2! duplicates. So the total number of ways is $\frac{9!}{3!2!}$.

Q2 Let S be a set of size n. Which of the following gives the best asymptotic upper bound for the number of subsets of S of size k?

Answer: There are $\binom{n}{k}$ subsets of size k in a set of size n, so we are looking for an asymptotic bound for $\binom{n}{k}$. Note that $\binom{n}{k} \leq n^k$ because the number of ways of choosing k objects from n *without* replacement is going to be at most the number of ways of choosing k objects from n *with* replacement. In particular, $O(n^k)$ is going to be a better upper bound than $O(k^n)$ ($O(k^n)$ grows asymptotically faster than $O(n^k)$). Secondly, we know $\binom{n}{2} \in O(n^2)$, so $O(2^k)$ and $O(nk)$ are not suitable bounds even for $k = 2$. So the most appropriate upper bound for the number of subsets of S of size k is $O(n^k)$.

Q3 How many integers are there between 1 and 1000 which are divisible by 6 or 15 but not both?

Answer: Let $A_k = \{n \in [1, 1000] : k|n\}$. We are after $|A_6 \oplus A_{15}|$. Note that $A_6 \cap A_{15} = \{n \in [1, 1000] : 30|n\} = A_{30}$. From the lectures $|A_k| = \lfloor \frac{1000-1+1}{k} \rfloor$, so $|A_6| = 166$, $|A_{15}| = 66$, and $|A_{30}| = 33$. Now $A_6 \oplus A_{15} = (A_6 \cup A_{15}) \setminus (A_6 \cap A_{15})$, so

$$|A_6 \oplus A_{15}| \quad = \quad |A_6 \cup A_{15}| - |A_6 \cap A_{15}|$$

$$
\begin{aligned}
&= (|A_6| + |A_{15}| - |A_{30}|) - |A_{30}| \\
&= 166 + 66 - 33 - 33 \\
&= 166.
\end{aligned}
$$

Q4 How many sequences of $2n$ coin flips have exactly n heads and n tails?

Answer: We have $2n$ flips, and need to choose n of them to be heads. The remaining flips will be tails, so there are $\binom{2n}{n}$ possible sequences.

Q5 How many sequences of $2n$ coin flips contain no pair of consecutive heads (no HH) and no pair of consecutive tails (no TT)?

Answer: Note that a valid sequence is completely determined by the first flip: if it is heads then the sequence must proceed HTHTHT...; if it is tails then the sequence must be THTHTHT... Thus there are exactly 2 sequences that contain no pair of consecutive heads and no pair of consecutive tails.