

FEDERAL STATE AUTONOMOUS EDUCATIONAL INSTITUTION
OF HIGHER EDUCATION
ITMO UNIVERSITY

Report
on the practical task No. 4
*Algorithms for unconstrained nonlinear optimization. Stochastic and
metaheuristic algorithms*

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Goal

The use of stochastic and metaheuristic algorithms (Simulated Annealing, Differential Evolution, Particle Swarm Optimization) in the tasks of unconstrained nonlinear optimization and the experimental comparison of them with Nelder-Mead and Levenberg-Marquardt algorithms.

Formulation of the problem

Generate the noisy data (x_k, y_k) , where $k = 0, \dots, 1000$, according to the rule:

$$y_k = \begin{cases} -100 + \delta_k, & f(x_k) < -100, \\ f(x_k) + \delta_k, & -100 \leq f(x_k) \leq 100, \\ 100 + \delta_k, & f(x_k) > 100, \end{cases} \quad x_k = \frac{3k}{1000},$$
$$f(x) = \frac{1}{x^2 - 3x + 2},$$

where $\delta_k \sim N(0,1)$ are values of a random variable with standard normal distribution. Approximate the data by the rational function

$$F(x, a, b, c, d) = \frac{ax + b}{x^2 + cx + d}$$

by means of least squares through the numerical minimization of the following function:

$$D(a, b, c, d) = \sum_{k=0}^{1000} (F(x_k, a, b, c, d) - y_k)^2.$$

*To solve the minimization problem, use Nelder-Mead algorithm, Levenberg-Marquardt algorithm and **at least two** of the methods among Simulated Annealing, Differential Evolution and Particle Swarm Optimization. If necessary, set the initial approximations and other parameters of the methods. Use $\varepsilon = 0.001$ as the precision; at most 1000 iterations are allowed. Visualize the data and the approximants obtained **in a single plot**. Analyze and compare the results obtained (in terms of number of iterations, precision, number of function evaluations, etc.).*

Brief theoretical part

Metaheuristic algorithms are natural-inspired algorithms that solve an optimization problem by trial and error. Generally speaking, metaheuristic methods do not guarantee that a solution to the optimization problem will be found.

Within the framework of this laboratory work, the method of differential evolution is considered. Differential evolution is a metaheuristic algorithm that solves the optimization problem through the evolution of the population of agents, that is, possible solutions, creating new generations of agents by combining existing and a further selection of the best ones.

It is proposed to first generate noisy data generated by an essentially rational function with discontinuities on the segment $[0, 1]$, and then find a rational function F of the corresponding form approximating them. An extremely nonlinear problem of numerical optimization arises, where the chosen initial approximation can have a significant effect on the result.

Results

Minimization by Levenberg-Marquart method

The initial approximations a, b, c, d for the Levenberg-Marquart method are equal to $a = -1, b = 1, c = 1, d = 1$. The values of the coefficients after minimization are $a = 0.98$ and $b = 3.17, c = 1.09, d = 0.72$ number of function evaluations: 166, $\varepsilon = 0.01$. When choosing such initial approximations, the graph of the function was approximated as in Figure 1.

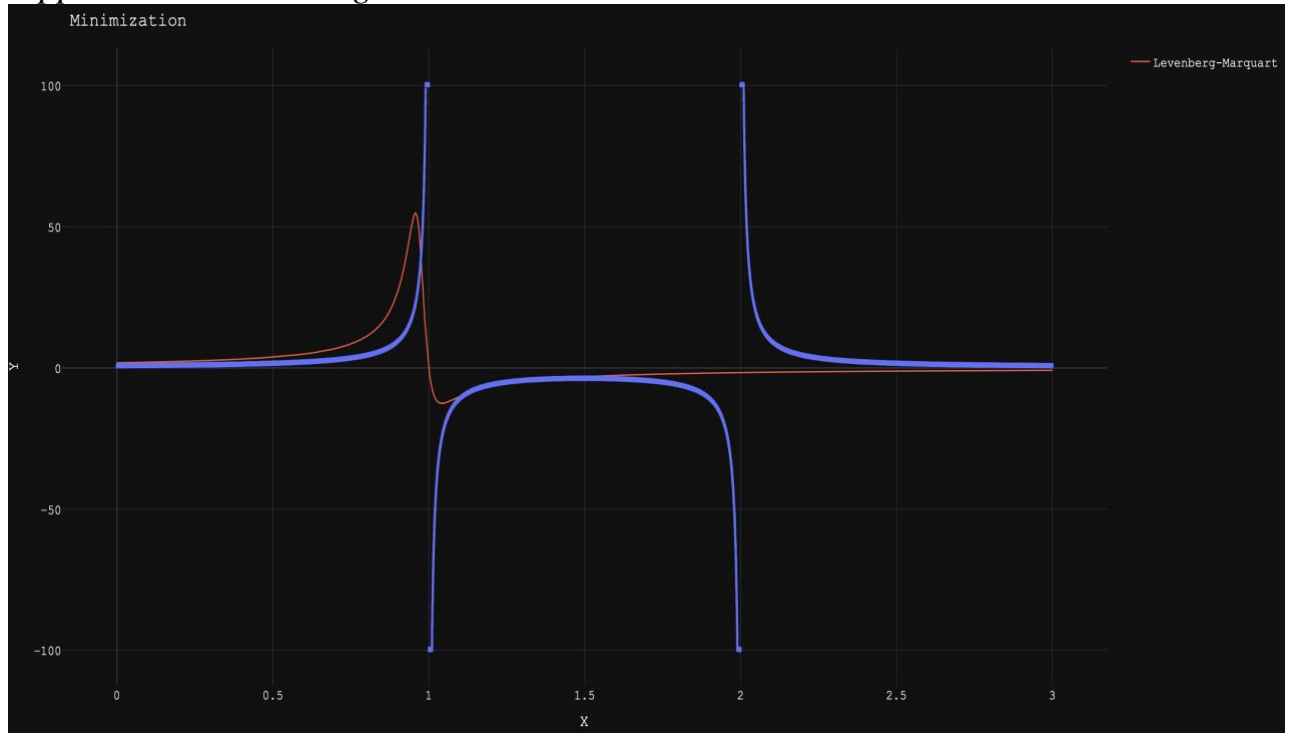


Figure 1: Rational function approximation of the least squares method with minimization by Levenberg-Marquart method

Minimization by Nelder-Mead method

The initial approximations a, b, c, d for the Nelder-Mead method are equal to $a = 0, b = 0, c = 0, d = 0$. The values of the coefficients after minimization are $a = -1.49$ and $b = 1.49, c = -2.00, d = 1.00$ number of function evaluations: 456, number of iteration: 259, $\varepsilon = 0.001$. When choosing such initial approximations, the graph of the function was approximated as in Figure 2.

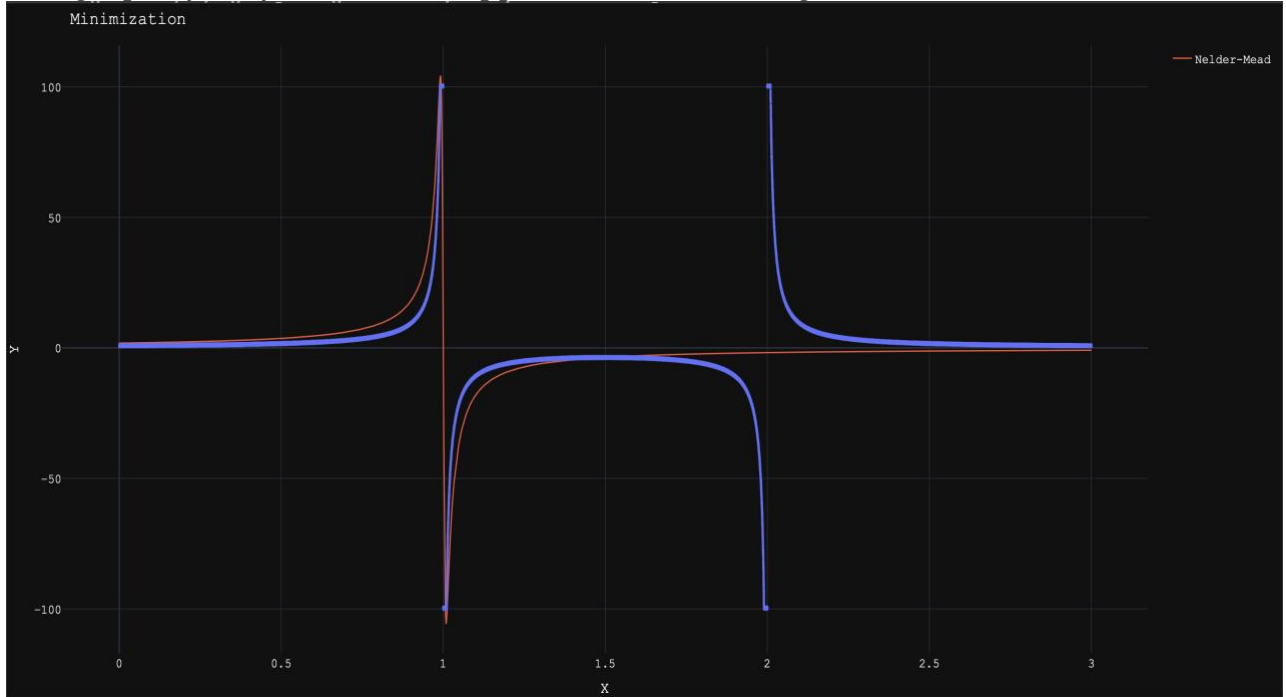


Figure 2: Rational function approximation of the least squares method with minimization by Nelder-Mead method

Minimization by Differential Evolution method

The initial approximations a, b, c, d for the Differential Evolution method in bounds $a = (-2,2), b = (-2,2), c = (-2,2), d = (-2,2)$. The values of the coefficients after minimization are $a = -0.99$ and $b = 1.00, c = -2.00, d = 1.00$ number of function evaluations: 8825, number of max iteration: 144, $\varepsilon = 0.001$. When choosing such initial approximations, the graph of the function was approximated as in Figure 3.

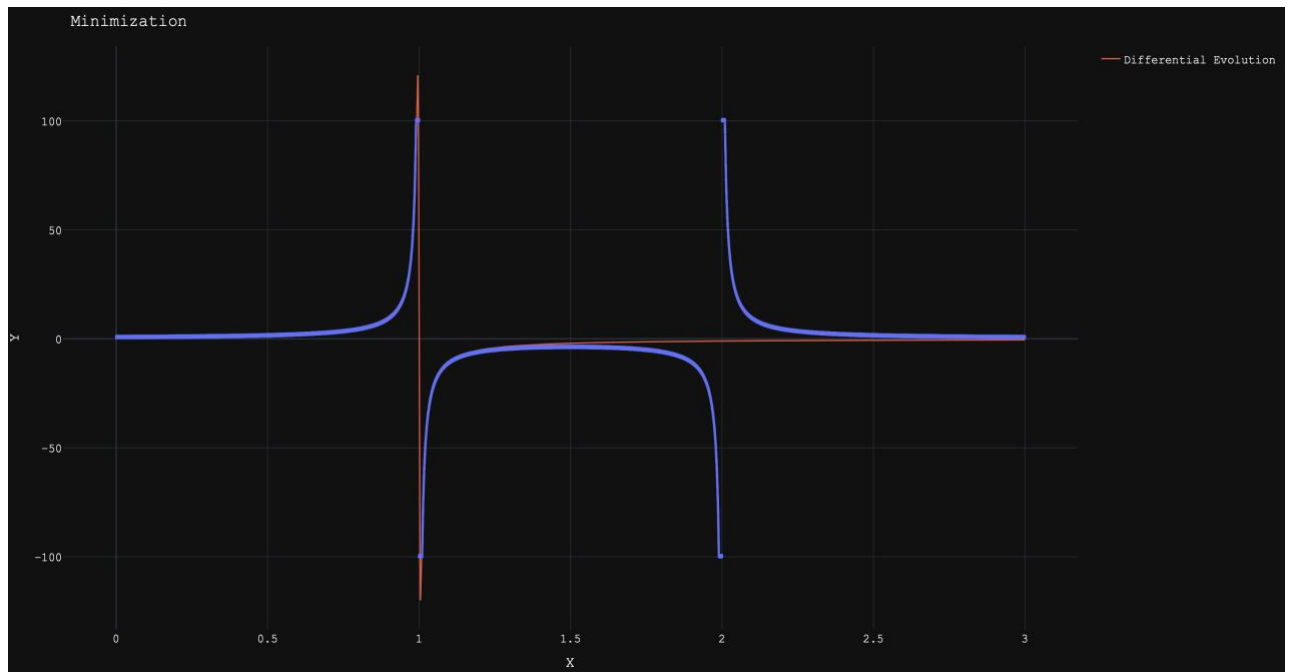


Figure 3: Rational function approximation of the least squares method with minimization by Differential Evolution method

Minimization results on the one plot

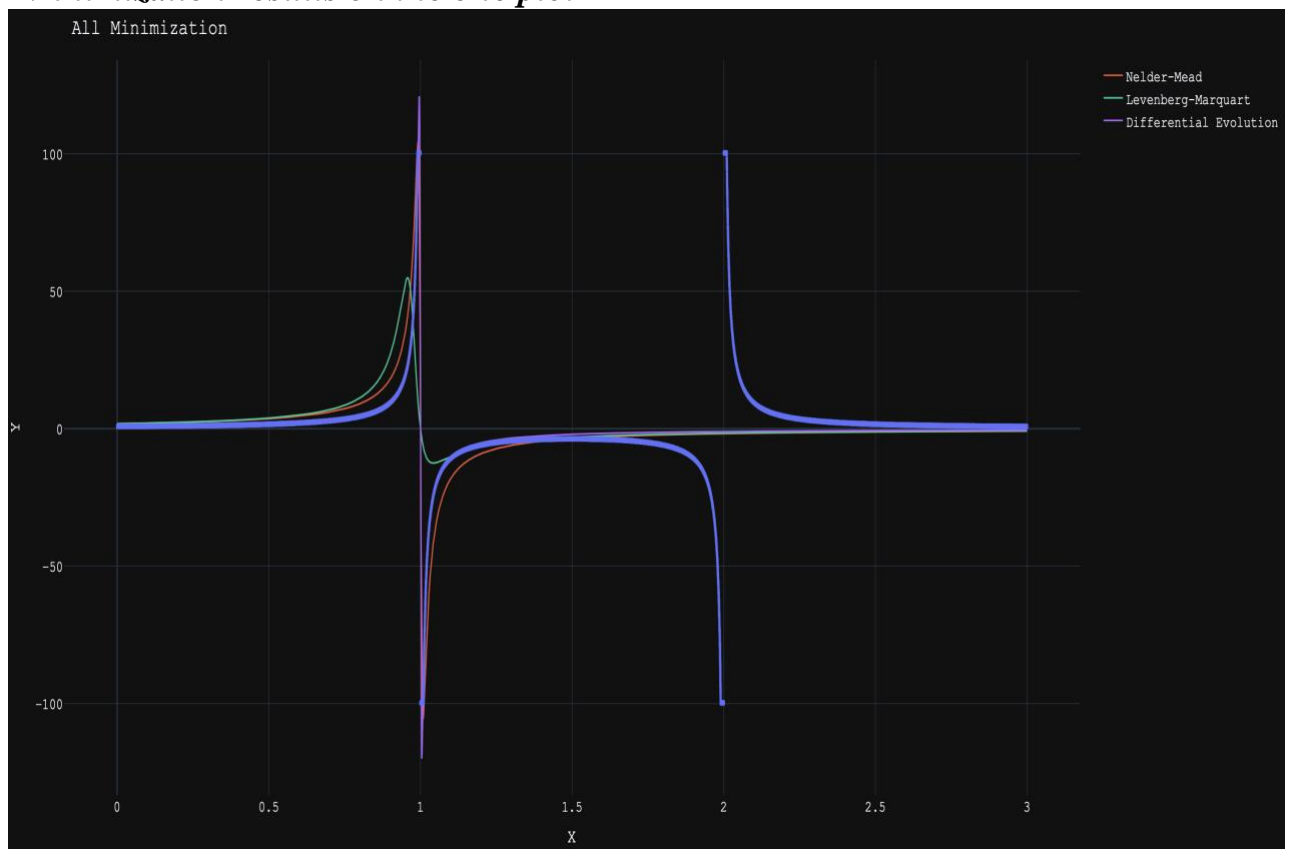


Figure 4: Rational function approximation of the least squares method with minimization by Differential Evolution, Nelder-Mead and Levenberg-Marquart methods

Conclusions

In the course of the laboratory work use of stochastic and metaheuristic algorithms Differential Evolution in the tasks of unconstrained nonlinear optimization and the experimental comparison of them with Nelder-Mead and Levenberg-Marquardt algorithms. The Levenberg-Marquardt method performed worse than others, but if you remove the restrictions for epsilon, the graph will be similar to the results of other methods. The Nelder-Mead and Levenberg-Marquardt methods have similar results, but the number of evaluations of the function is very different (see Figure 4). The result is also strongly influenced by restrictions and the choice of initial approximations.

Appendix

[Code Link](#)

▼ LAB 4

```
1 import random
2 import numpy as np
3
4 def fx(x):
5     return 1/(x**2-3*x+2)
6
7
8 def y(x,delta):
9     if fx(x) < -100:
10         return -100 + delta
11
12     if abs(fx(x)) <= 10**2:
13         return fx(x) + delta
14
15     if fx(x) > 100:
16         return 100 + delta
17
18 xk = np.zeros(1000)
19 yk = np.zeros(1000)
20 delta = np.random.sample(1000)
21
22 for k in range(0,1000):
23     xk[k] = (3*k)/1000
24     yk[k] = y(xk[k],delta[k])
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```

1 plot_func(xk,yk,"data","data")

1 def rsm(x,n):
2     return (a * xk + b)/(xk**2+c*xk+d)

1 from scipy import optimize
2 def f1(z,*args):
3     a,b,c,d = z
4     return (a * xk + b)/(xk**2+c*xk+d)
5
6 def f(z,*args):
7     a,b,c,d = z
8     return ((f1(z)-yk)**2)
9
10 res_lm = optimize.least_squares(f,[-1,1,1,1],method='lm',xtol = 0.001)
11 res_lm.x
12 # coef = optimize.curve_fit(lsm, xk, yk, method=None)
13 # a = coef[0][0]
14 # b = coef[0][1]
15 # print(a)
16 # print(b)
17

1 res_lm.nfev

1 def f1(z,*args):
2     a,b,c,d = z
3     return (a * xk + b)/(xk**2+c*xk+d)
4
5 def f(z,*args):
6     a,b,c,d = z
7     return np.sum(((f1(z)-yk)**2))
8
9 x0=np.zeros(4)
10 res_nm = optimize.minimize(f,x0,args=(xk,yk), method='Nelder-Mead', tol=0.001, options
11 res_nm.x

1 res_nm.nit

1 from scipy.optimize import rosen, differential_evolution
2 bounds = [(-2,2), (-2, 2), (-2, 2), (-2, 2)]
3 result = differential_evolution(f, bounds, maxiter=1000, tol=0.001)
4 result.x
5

1 result.nfev

1 fig, ax = plt.subplots()

```

```
2 plt.plot(xk, rsm(xk,res_lm.x[0],res_lm.x[1],res_lm.x[2],res_lm.x[3]), label = Levenb
3 plt.plot(xk, rsm(xk,res_nm.x[0],res_nm.x[1],res_nm.x[2],res_nm.x[3]), label = "Nelder
4 plt.plot(xk, rsm(xk,result.x[0],result.x[1],result.x[2],result.x[3]), label = 'Differ
5 ax.scatter(xk, yk)
6 # if flag:
7 # plt.plot(additionalX, additionalY, label = additionallabel)
8 # ax.set_title(title,fontsize = fs)
9 ax.set_xlabel('X', fontsize=26)
10 ax.set_ylabel('Y', fontsize=26)
11 plt.tick_params(axis='both', which='major', labelsize=26)
12 fig.set_figwidth(15)
13 fig.set_figheight(15)
14 # plt.plot(x,y)
15 # plt.axis([-0.1, 1.01, 0, 2.3])
16 handles, labels = ax.get_legend_handles_labels()
17 ax.legend(handles, labels, prop={'size': 26} )
18 plt.show()
```