Gauge Theory

A Tour to Instanton and Monopole

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Preface

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1.3.1 Basic Definitions

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra.

Definition 1.3.1

A **principal** $G-$**bundle** is a triple (P, M, π) with

(1) P and M are smooth manifolds and $\pi : P \to M$ is a smooth surjective and submersion;

(2) G acts on P on the right such that $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$;

(3) G acts freely and transitively on each fibre $\pi^{-1}(m)$ for $m \in M$;

(4) For any $m \in M$, there exists a neighborhood U of m and diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times G$ such that $\mathrm{pr}_1 \circ \psi_U = \pi$, where $\mathrm{pr}_1 : U \times G \to U$ is the projection, and ψ_U is $G-$equivariant, i.e., $\psi_U(pg) = \psi_U(p) \cdot g$.

Example 1.3.1. *If $M_p \to M$ is a $p-$fold covering, then it is easy to prove that M_p is a principal $\mathbb{Z}_p-$bundle.*

Example 1.3.2. *For a vector spaces V, we denote $\mathrm{Fr}(V)$ as all the frames of V, i.e., the $\dim V = k$ tuple of linearly independent vetors. Easy to see, by given a basis $\{e_1, \cdots, e_k\}$ then $\mathrm{Fr}(V)$ can be identified with $\mathrm{GL}_k(\mathbb{R})$ by the transition maps from the tuple to the basis.*

Now for a rank k vector bundle $E \to M$, we define

$$\mathrm{Fr}(E) := \bigsqcup_{m\in M} \mathrm{Fr}(E_m),$$

then it is not hard to prove that $\mathrm{Fr}(E)$ is a principal $GL_k(\mathbb{R})-$bundle.

An navie question is that why we call such fibre bundle "principal"? It turns out that any vector bundle can be constructed by a principal bundle.

Definition 1.3.2: Associated Bundle

Let $\pi : P \to M$ be a principal $G-$bundle, then for any representation $\rho : G \to \mathrm{GL}(V)$, V is a vector space, we can constructed the **associated vector bundle** over M as

$$P \times_\rho V := P \times V/G,$$

where G acts on the right of $P \times V$ by $(p, v).g := (pg, \rho(g^{-1})v)$.

$\boxed{\textit{Remark.}}$ We will denote the element in $P \times_\rho V$ by $[p, v]$, where $[p, v] = [pg, \rho(g^{-1})v]$. And $\pi_\rho : P \times_\rho V \to M$ is given by $\pi_\rho([p, v]) := \pi(p)$.

To check that $P \times_\rho V$ is a vector bundle, it suffices to prove the local trivilization, i.e., $\pi_\rho^{-1}(U)$ is diffeomorphic to $U \times V$. Suppose $\psi_U : \pi^{-1}(U) \to U \times G$ is the local trivialization of P, then there exists a section $s := \psi_U^{-1}(\cdot, e) : U \to P$, hence we can define $(m, v) \mapsto [s(m), v]$.

Example 1.3.3. *From the frame bundle, we can construct various vector bundles related to E:*

- *consider* $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^k)$ *by* $\rho = \mathrm{id}$, *then* $\mathrm{Fr}(E) \times_\rho \mathbb{R}^k = E$*;*
- *consider* $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^k)$ *by* $\rho(g) = (g^\top)^{-1}$, *then* $\mathrm{Fr}(E) \times_\rho \mathbb{R}^k = E^*$*;*
- *consider* $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\otimes^l \mathbb{R}^k)$ *by* $\rho(g) = g \otimes \cdots \otimes g$, *more precisely,* $g(v_1 \otimes \cdots \otimes v_l) := (gv_1) \otimes \cdots \otimes (gv_l)$, *then* $\mathrm{Fr}(E) \times_\rho \otimes^l \mathbb{R}^k = E^{\otimes l}$*;*
- *consider* $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\oplus^l \mathbb{R}^k)$ *by* $\rho(g) = g \oplus \cdots \oplus g$, *more precisely,* $g(v_1, \cdots, v_l) := (gv_1, \cdots, gv_l)$, *then* $\mathrm{Fr}(E) \times_\rho \oplus^l \mathbb{R}^k = E^{\oplus l}$*;*
- *consider* $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\mathrm{End}(\mathbb{R}^k))$ *by* $\rho(g)A = gAg^{-1}$, *then* $\mathrm{Fr}(E) \times_\rho \mathrm{End}(\mathbb{R}^k) = \mathrm{End}(E)$. *Note that if we identify* $\mathrm{End}(\mathbb{R}^k)$ *with* $\mathbb{R}^k \otimes (\mathbb{R}^k)^*$, *then we can write* $\rho(g)$ *as* $g \otimes (g^\top)^{-1}$.

  (one can check the linear algebra fact above by $A = A^i_j E^j_i = A^i_j e_i \otimes e^j$, *then* $gAg^{-1} = g^i_k A^k_l G^l_j E^j_i$ *and* $(g \otimes (g^\top)^{-1})(e_k \otimes e^l) = ge_k \otimes (g^\top)^{-1} e^l$, *note that* $ge_k = g^i_k e_i$ *and* $(g^\top)^{-1} e^l = G^l_j e^j$ *where* $g^{-1} = (G^l_j)$ *since* $\langle (g^\top)^{-1} e^l, e_j \rangle = \langle e^l, g^{-1} e_j \rangle$*.)*

$\boxed{\textit{Remark.}}$ The examples given above can be quickly seen by the transtion maps, P can be reconstructed by $\bigsqcup_i (U_i \times G)/\sim$ with transition maps $\psi_{ij} : U_i \cap U_j \to G$ and $(m, g) \sim (m, \psi_{ij}(m) \cdot g)$. Then $P \times_\rho V$ can be reconstructed by $\bigsqcup_i (U_i \times V)/\sim'$ with transition maps $\psi'_{ij} = \rho \circ \psi_{ij} : U_i \cap U_j \to \mathrm{GL}(V)$ and $(m, v) \sim' (m, \psi'_{ij}(m) \cdot v)$.

Now for a general principal $G-$bundle P, to find a associated bundle, it suffices to find a representation of the Lie group, the most natural one is the following:

Example 1.3.4. *Consider adjoint representation* $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ *by*

$$\mathrm{ad}_g(\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(t\xi) g^{-1} = (C_g)_{*,e}(\xi),$$

where $C_g : G \to G$ *is given by* $C_g(h) = ghg^{-1}$.

Note that $\mathfrak{g}$ *is a* $\dim G$ *vector space, hence we can define the adjoint bundle of* P*:*

$$\boxed{\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g}.}$$

What we want to do is to define the connection on principal bundle P, a naive idea is to use the connection on ad$P$, i.e., some $\mathrm{d}_A : \Omega^*(\mathrm{ad}P) \to \Omega^{*+1}(\mathrm{ad}P)$. But now the problem is it is really abstract to see what $\Omega^*(\mathrm{ad}P)$ really is! So we need the follwing propositions:

Proposition 1.3.1

We have one to one correspondance:

$$\Gamma(P \times_\rho V) \cong C^\infty(P, V)^G,$$

where $C^\infty(P, V)^G$ is all the $G-$equivariant smooth map from P to V, i.e., the set of smooth map $\hat{s} : P \to V$ such that $\hat{s}(pg) = \rho(g^{-1})\hat{s}(p)$ for all $g \in G$ and $p \in P$.

Proof. Let $s \in \Gamma(P \times_\rho V)$ and $\hat{s} \in C^\infty(P, V)^G$, then

$(s \to \hat{s})$: note that $s : M \to P \times_\rho V$, then $s(m)$ has the form of $[p, v]$, then for each $p \in P$ with $\pi(p) = m$, there is a unique $\hat{s}(p) \in V$ such that

$$s(m) = [p, \hat{s}(p)],$$

we can also see that from this equation we define a smooth map $\hat{s} : P \to V$, and easy to see $[pg, \hat{s}(pg)] = s(m) = [p, \hat{s}(p)] = [pg, \rho(g^{-1})\hat{s}(p)]$, hence we have $\hat{s}(pg) = \rho(g^{-1})\hat{s}(p)$, which shows that $\hat{s}$ is $G-$equivariant.

$(\hat{s} \to s)$: for any $m \in M$, take any $p \in \pi^{-1}(m)$, then we define $s : M \to P \times_\rho V$ by $s(m) := [p, \hat{s}(p)]$, by $\hat{s}$ is $G-$equivariant then we know that s is well defined, i.e., independent on the choice of p. ♣

This proposition also has its main interest, we can use this identification to find the nowhere zero section of vector bundle. Note that if $s \in \Gamma(P \times_\rho V)$ is nowhere zero, then it is equivalently for any $p \in \pi^{-1}(m)$ we have $\widehat{s}(p) \neq 0$. Hence we have the nonwhere zero sction is $1-1$ coresponding to the $C^\infty(P, V \setminus \{0\})^G$, here $0 \in V$ denotes the zero element under $+$.

Here is a more acurate example:

Example 1.3.5. *We want to show that the Möbius line bundle L over S^1 is not the trivial one, i.e., there is no nowhere zero section of L.*

Note that L can be viewed as the associated bundle of $\mathbb{Z}/2\mathbb{Z}-$principal bundle of S^1, which is the connected $2-$fold covering of S^1, with the representation $\rho : \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}(\mathbb{R})$ given by $-1 \mapsto -1$.

Now if $f \in C^\infty(S^1, \mathbb{R} \setminus \{0\})^{\mathbb{Z}/2\mathbb{Z}}$, then it means that f is nowhere zero, and $f(-x) = f(x.(-1)) = \rho(-1)f(x) = -f(x)$, i.e., f is odd function, then by mean value theorem, f must have some zero points which is a contradiction.

More generally, we want to study the relation betwenn

$$\Omega^q(M; P \times_\rho V) \quad \text{and} \quad \Omega^q(P; V),$$

we need to give some restrictions on $\hat{a} \in \Omega^q(P; V)$.

Definition 1.3.3

For any $p \in P$, we define the **vertical subsapce** $\mathcal{V}_p \subset T_pP$ as $\mathcal{V}_p := \ker\left(\pi_{*,p} : T_pP \to T_{\pi(p)}M\right)$.

Remark. Actually, we have natural identification $\mathcal{V}_p = \ker \pi_{*,p} \cong \mathfrak{g}$, which is given by: for any $\xi \in \mathfrak{g}$, we have

$$K_\xi(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \cdot \exp(t\xi) \in T_pP,$$

easy to see $K_\xi(p) \in \mathcal{V}_p$, and since $\pi_{*,p}$ is surjective, hence $\dim \mathcal{V}_p = \dim P - \dim M = \dim G = \dim \mathfrak{g}$, then we know that $\xi \mapsto K_\xi$ is bijective.

Definition 1.3.4

- A $V-$valued $q-$form $a \in \Omega^q(P; V)$ is called $G-$**equivariant** if $R_g^* a = \rho(g^{-1})a$, where $R_g : P \to P$ is right multiplication by g.
- A $V-$valued $q-$form $a \in \Omega^q(P; V)$ is called **basic** if for any $v \in \mathcal{V} = \ker \pi_*$, we have $\iota_v a = 0$.

We denote $\boxed{\Omega^q_{\text{bas}}(P; V)^G}$ as all the $G-$equivariant and basic $V-$valued $q-$forms on P.

Clever readers may note that we actually have:

Proposition 1.3.2

We have one to one correspondance:

$$\Omega^q(M; P \times_\rho V) \cong \Omega^q_{\text{bas}}(P; V)^G.$$

Proof. Let $a \in \Omega^q(M; P \times_\rho V)$ and $\hat{a} \in \Omega^q(P; V)$, then

$(a \to \hat{a})$: To construct $\hat{a}$, for any $\widehat{v_1}, \cdots, \widehat{v_q} \in T_pP$, we need to define what $\widehat{a}(\widehat{v_1}, \cdots, \widehat{v_q}) \in V$ is. Hence a natural idea is

$$a_{\pi(p)}\left(\pi_{*,p}\widehat{v_1}, \cdots, \pi_{*,p}\widehat{v_q}\right) =: [p, \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})] \in (P \times_\rho V)_p\,, \tag{1.1}$$

this equation uniquely determines the $\widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q}) \in V$.

- check $\widehat{a}$ is **basic**: it is easy since if $\widehat{v} \in \mathcal{V}$, then $\pi_* \widehat{v} = 0$, then we get $\iota_v \widehat{a} = 0$.

- check $\widehat{a}$ is $G-$**equivariant**: i.e., $R_g^* \widehat{a}_{pg} = \rho(g)^{-1} \widehat{a}_p$, more precisely, it suffices to prove the following
$$[pg, R_g^* \hat{a}_{pg}(\widehat{v_1}, \cdots, \widehat{v_q})] = [pg, \rho(g^{-1}) \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})].$$
Note that
$$\begin{aligned} \text{LHS} &= [pg, R_g^* \hat{a}_{pg}(\widehat{v_1}, \cdots, \widehat{v_q})] \\ &= [pg, \widehat{a}_{pg} \left(R_{g*,p}\widehat{v_1}, \cdots, R_{g*,p}\widehat{v_q}\right)] \\ &= a_{\pi(pg)} \left(\pi_{*,pg} \left(R_{g*,p}\widehat{v_1}\right), \cdots, \pi_{*,pg} \left(R_{g*,p}\widehat{v_q}\right)\right) \\ &= a_{\pi(p)} \left(\pi_{*,p}\widehat{v_1}, \cdots, \pi_{*,p}\widehat{v_q}\right) \quad (\text{since } \pi \circ R_g = \pi) \\ &= [p, \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})] \\ &= [pg, \rho(g^{-1}) \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})] = \text{RHS}. \end{aligned}$$

$(\widehat{a} \to a)$: To construct a, for any $v_1, \cdots, v_q \in T_m M$, we need to assign an element of the form $[p, v]$ to $a(v_1, \cdots, v_q)$, where $p \in \pi^{-1}(m)$.

Choose any $p \in \pi^{-1}(m)$, and any lift $\widehat{v_i} \in T_p P$ of v_i, where $\pi_{*,p}\widehat{v_i} = v_i$. Such lift exists because π_* is surjective. hence we can define
$$a_m(v_1, \cdots, v_q) := [p, \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})] \in (P \times_\rho V)_p, \tag{1.2}$$
now we check this is well defined, i.e., the definition

- does not depend on the choice of lift $\widehat{v_i}$: If we have $\pi_{*,p}(\widehat{v_i}) = \pi_{*,p}(\widehat{v_i}') = v_i$, then since $\widehat{a}$ is <u>basic</u>, then we know that $\iota_{\widehat{v_i}} \widehat{a} = \iota_{\widehat{v_i}'} \widehat{a}$.

- does not depend on the choice of $p \in \pi^{-1}(m)$: this is not so hard, consider p and pg, from the discussion above, we can choose the lift of v_i as $\widehat{v_i}$ and $R_{g*,p}\widehat{v_i}$. Now we have
$$\begin{aligned} [pg, \widehat{a}_{pg}(R_{g*,p}\widehat{v_1}, \cdots, R_{g*,p}\widehat{v_q})] &= [pg, R_g^* \hat{a}_{pg}(\widehat{v_1}, \cdots, \widehat{v_q})] \\ &= [pg, \rho(g^{-1}) \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})] = [p, \widehat{a}_p(\widehat{v_1}, \cdots, \widehat{v_q})]. \end{aligned}$$

Finally, it is easy tp see $a \mapsto \widehat{a} \mapsto a'$ then actually $a' = a$, which is because (1.1) and (1.2) are same! Now we finish the proof. ♣

Remark. From now on, we will frquently use this identification, in most cases we will use $\widehat{\cdot}$ to denote the element in $\Omega^*(P; V)$ and $\pi^* : \Omega^*(M; P \times_\rho V) \to \Omega^*(P; V)$ to denote the identification, i.e., $\pi^* s = \widehat{s}$, $\pi^* a = \widehat{a}$. Actually from the (1.1) and (1.2) we can see that in local trivialization, the correspondence is really given by the pull-back π^*, so abuse of notation we use it for convinient.

Summary 1

What we should remember is that the corrsepondence:

- $\Gamma(P \times_\rho V) \cong C^\infty(P; V)^G$;
- $\Omega^q(M; P \times_\rho V) \cong \Omega^q_{\text{bas}}(P; V)^G$.

1.3.2 Connections and Curvatures

From now on, we focus on the principal $G-$bundle P and its adjoint bundle ad$P$.

Definition 1.3.5: Connection

A connection A on P is a $\mathfrak{g}-$valued $1-$form on P with the following properties: for $A \in \Omega^1(P; \mathfrak{g})$,

- $R_g^* A = \text{ad}_{g^{-1}}(A)$;
- $A(K_\xi) = \xi$, for any $\xi \in \mathfrak{g}$ and $K_\xi \in \mathcal{V} = \ker \pi_*$.

Remark. This definition looks like really abstract, but soon we will see that it is strongly related to the connection of vector bundle.

Similar as the connection space of vector bundle, we have

Proposition 1.3.3

For any principal bundle P the space of all connections $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\text{ad}P)$.

Proof. Under the identification of $\Omega^1(\text{ad}P) \cong \Omega^1_{\text{bas}}(P; \mathfrak{g})^G$, it suffices to prove that for any connections A, A', we have $A - A'$ is basic and $G-$equivariant, which are all clear by definition:

- $R_g^*(A - A') = R_g^* A - R_g^*(A') = \text{ad}_{g^{-1}}(A - A')$;
- $(A - A')(K_\xi) = \xi - \xi = 0$, hence basic.

Hence there exists $a \in \Omega^1(\text{ad}P)$ such that $\pi^* a = A - A'$, then we finish the proof. ♣

Now we want to study the relation between the connection on P and the connection on the associated vector bundle $P \times_\rho V$. Roughly speaking, a connection on the vector bundle E is somehow in $\Omega^1(\text{End}(E))$, so fibrewise we need to find a map from $\mathfrak{g}$ to $\text{End}(V)$.

A clever observation is that $\text{End}(V) = T_{\text{Id}}\text{GL}(V)$! Hence we have the map given by $\rho_{*,e}$:

$\mathfrak{g} \to \mathrm{End}(V)$. Then we have the action

$$A \cdot \widehat{\omega} := \rho_{*,e}(A) \wedge \widehat{\omega} \in \Omega^{q+1}(P; V),$$

where $A \in \Omega^1(P; \mathfrak{g})$ is the connection, and $\widehat{\omega} \in \Omega^q(P; V)$.

Proposition 1.3.4

For any connection A on P, we can define the connection d_A on $P \times_\rho V$ as the following: for $\omega \in \Omega^q(P \times_\rho V)$, and $\widehat{\omega} = \pi^*\omega \in \Omega^q_{\text{bas}}(P; V)^G$, we have

$$\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega} \in \Omega^{q+1}_{\text{bas}}(P; V)^G,$$

hence we can define $\mathrm{d}_A\omega \in \Omega^{q+1}(P \times_\rho V)$ by using

$$\pi^*\mathrm{d}_A\omega := \mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}.$$

Proof. We prove it in the following steps:

- check $\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}$ is <u>$G-$equivariant</u>: note that

$$\begin{aligned} R_g^*\left(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}\right) &= \mathrm{d}(R_g^*\widehat{\omega}) + R_g^*A \cdot R_g^*\widehat{\omega} \\ &= \rho(g^{-1})\mathrm{d}\widehat{\omega} + \rho_{*,e}\left(\mathrm{ad}_{g^{-1}}A\right) \wedge \rho(g^{-1})\widehat{\omega} \\ &= \rho(g^{-1})\mathrm{d}\widehat{\omega} + \rho(g^{-1})\rho_{*,e}(A)\rho(g) \wedge \rho(g^{-1})\widehat{\omega} \\ &= \rho(g^{-1})\left(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}\right), \end{aligned}$$

  where the last second equality comes from that ad is Lie algebra homomorphism.

- check $\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}$ is <u>basic</u>: note that

$$\iota_{K_\xi}\left(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}\right) = \mathcal{L}_{K_\xi}\widehat{\omega} - \mathrm{d}\left(\iota_{K_\xi}\widehat{\omega}\right) + A(K_\xi) \cdot \widehat{\omega} = \mathcal{L}_{K_\xi}\widehat{\omega} + \xi \cdot \widehat{\omega},$$

  where we use Cartan magic formula $\mathcal{L}_X = \mathrm{d}\iota_X + \iota_X\mathrm{d}$ and $\iota_{K_\xi}\widehat{\omega} = 0$ since basic. Note that

$$\mathcal{L}_{K_\xi}\widehat{\omega} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R^*_{\exp(t\xi)}\widehat{\omega} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(-t\xi))\widehat{\omega} = \rho_{*,e}(-\xi)\widehat{\omega} = -\xi \cdot \widehat{\omega},$$

  where we use $\widehat{\omega}$ is $G-$equivariant,hence we get $\iota_{K_\xi}\left(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}\right) = 0$, hence basic.

- check the defined d_A is a connection, i.e., satisfies the Lebniz rule: note that

$$\mathrm{d}(\widehat{f}\widehat{\omega} + A \cdot (\widehat{f}\widehat{\omega})) = \mathrm{d}\widehat{f} \wedge \widehat{\omega} + \widehat{f}\left(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}\right),$$

  then we have $\mathrm{d}_A(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}_A\omega$ by applying π^*.

Finally we finish the proof. ♣

Remark. Again we can see that somehow : "$\mathrm{d}_A = \mathrm{d} + A$".

Now we want to define the curvature, recall in the vector bundle case, locally we have $\mathrm{d}_A = \mathrm{d} + A$ where A is a $1-$from matrix, then we have curvature $F_A = \mathrm{d}A + A \wedge A$, here $A \wedge A$ is the mixture of matrix multiplication and wedge product.

However, in the principal bundle case, $A \in \Omega^1(P;\mathfrak{g})$, and we have no direct multiplication in Lie algebra $\mathfrak{g}$, but luckily, we have Lie bracket! So to define the curvature, we need to firstly define

Definition 1.3.6: Super Lie Bracket

For $\omega \in \Omega^p(P;\mathfrak{g})$ and $\eta \in \Omega^q(P;\mathfrak{g})$, we have supper Lie bracket $[\omega \wedge \eta] \in \Omega^{p+q}(P;\mathfrak{g})$ by take Lie bracket for the Lie algebra part and wedge product for the form part.

Remark. Under the identification of $\Omega^*(\mathrm{ad}P)$ and $\Omega^*_{\text{bas}}(P;\mathfrak{g})^G$, we also have supper Lie bracket operation on $\Omega^*(\mathrm{ad}P)$.

Supper Lie bracket has some trivial properties, for $\omega \in \Omega^p(P;\mathfrak{g})$, $\eta \in \Omega^q(P;\mathfrak{g})$, $\gamma \in \Omega^r(P;\mathfrak{g})$, we have

- supper commutativity:
$$[\omega \wedge \eta] = (-1)^{pq+1}[\eta \wedge \omega];$$
- supper Jacobi identity:
$$(-1)^{pr}[\omega \wedge [\eta \wedge \gamma]] + (-1)^{qp}[\eta \wedge [\gamma \wedge \omega]] + (-1)^{rp}[\gamma \wedge [\omega \wedge \eta]] = 0.$$

Now we can define the curvature through the proposition below:

Proposition 1.3.5

For $A \in \mathcal{A}(P)$, we have
$$\widehat{F_A} := \mathrm{d}A + \frac{1}{2}[A \wedge A] \in \Omega^2_{\text{bas}}(P;\mathfrak{g})^G.$$

Proof. It is stiil need two steps:

- check $\widehat{F_A}$ is $G-$equivariant: note that
$$R_g^*\widehat{F_A} = \mathrm{d}(R_g^*A) + \frac{1}{2}[R_g^*A \wedge R_g^*A],$$
and $f \in \mathrm{GL}(\mathfrak{g})$, we have $f([\xi,\eta]) = [f(\xi), f(\eta)]$, then it is clear.

- check $\widehat{F_A}$ is basic: note that

$$\begin{aligned}\iota_{K_\xi}\widehat{F_A} &= \mathcal{L}_{K_\xi}A + \frac{1}{2}[A(K_\xi), A] - \frac{1}{2}[A, A(K_\xi)] \\ &= \mathcal{L}_{K_\xi}A + \frac{1}{2}[\xi, A] - \frac{1}{2}[A, \xi],\end{aligned}$$

by supper commutativity, we have $\frac{1}{2}[\xi, A] - \frac{1}{2}[A, \xi] = [\xi, A]$.Now note that

$$\mathcal{L}_{K_\xi}(A) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R^*_{\exp(t\xi)}A = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{ad}_{\exp(-t\xi)}A = \mathrm{ad}_{*,e}(-\xi)(A).$$

Recall the differential of adjoint map is actually the Lie bracket, i.e., $\mathrm{ad}_{*,e}(\xi)(\eta) = [\xi, \eta]$, then we know that $\mathcal{L}_{K_\xi}(A) = -[\xi, A]$, hence $\iota_{K_\xi}\widehat{F_A} = 0$, i.e., $\widehat{F_A}$ is basic.

Finally we finish the proof. ♣

Remark. Note that if $\mathfrak{g}$ is matrix Lie algebra, then we have $\frac{1}{2}[A \wedge A] = A \wedge A$, one can check this by using local coordinates $A = A^{ij} \otimes E_{ij}$ where A^{ij} is the $1-$form, then we have $[A \wedge A] = (A^{ij} \wedge A^{kl}) \otimes [E_{ij}, E_{kl}] = (A^{ij} \wedge A^{jk} - A^{jk} \wedge A^{ij}) \otimes E_{ik} = 2(A^{ij} \wedge A^{jk}) \otimes E_{ik} = 2A \wedge A$.

Definition 1.3.7: Curvature

For $A \in \mathcal{A}(P)$, we have the curvature $F_A \in \Omega^2(\mathrm{ad}P)$ determined by $\pi^* F_A = \widehat{F_A}$.

Now consider the associated vector bundle $P \times_\rho V$, we already have the connection d_A, recall we define the curvature on this vector bundle by some $\mathrm{d}_A^2 = F_A \in \Omega^2(\mathrm{End}(P \times_\rho V))$, does these two definitions conicidence? Actually yes!

Proposition 1.3.6

For $F_A \in \Omega^2(\mathrm{ad}P)$, $\omega \in \Omega^q(P \times_\rho V)$, we have

$$\mathrm{d}_A \circ \mathrm{d}_A(\omega) = F_A \cdot \omega,$$

here the multipliation is given by $F_A \cdot \omega := \rho_{*,e}(F_A) \wedge \omega$.

Proof. Note that we have

$$\begin{aligned}\pi^*\left(\mathrm{d}_A \circ \mathrm{d}_A(\omega)\right) &= \mathrm{d}(\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}) + A \cdot (\mathrm{d}\widehat{\omega} + A \cdot \widehat{\omega}) \\ &= \mathrm{d}A \cdot \widehat{\omega} - A \cdot \mathrm{d}\widehat{\omega} + A \cdot \mathrm{d}\widehat{\omega} + A \cdot (A \cdot \widehat{\omega}) \\ &= \mathrm{d}A \cdot \widehat{\omega} + \rho_{*,e}(A) \wedge \rho_{*,e}(A) \wedge \widehat{\omega},\end{aligned}$$

note that

$$\frac{1}{2}[A \wedge A] \cdot \widehat{\omega} = \frac{1}{2}\rho_{*,e}([A \wedge A]) \wedge \widehat{\omega} = \frac{1}{2}[\rho_{*,e}(A) \wedge \rho_{*,e}(A)] \wedge \widehat{\omega},$$

recall we have said that for matrix supper Lie bracket $\frac{1}{2}[\rho_{*,e}(A) \wedge \rho_{*,e}(A)] = \rho_{*,e}(A) \wedge \rho_{*,e}(A)$, hence we know the identity holds. ♣

Finally to sum up this section, we calculate a precise example:

Example 1.3.6. *Consider the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, where $\pi : S^{2n+1} \subset \mathbb{C}^{n+1} \to \mathbb{C}P^n$ by $\pi(z_0, \cdots, z_n) := [z_0 : \cdots : z_n]$, and each fibre is given by $\lambda(z_0, \cdots, z_n)$ where $|\lambda| = 1$. Hence S^{2n+1} can be viewed as a principal $S^1 = \mathrm{U}(1)-$bundle.*

Note that $\mathfrak{u}(1) \cong \mathrm{i}\mathbb{R}$, hence for any $\mathrm{i}a \in \mathfrak{u}(1)$, we have $K_{ia}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tia) \cdot z = ia \cdot z \in T_zS^{2n+1}$. Recall $T_zS^{2n+1} = \{v \in \mathbb{C}^{n+1} : \mathrm{Re}\langle z, v\rangle = 0\}$, hence we can define the connection form $A \in \Omega^1(S^{2n+1}; \mathrm{i}\mathbb{R})$ by

$$A_z(v) := \langle \mathrm{i}z, v\rangle \mathrm{i}, \quad v \in T_zS^{2n+1}$$

easy to see since $\mathrm{Re}\langle z, v\rangle = 0$, hence $\langle \mathrm{i}z, v\rangle \in \mathbb{R}$, then A is well defined. And $A_z(K_{\mathrm{ia}}) = \langle iz, iaz\rangle \mathrm{i} = \mathrm{i}a$, because $\langle \lambda X, \mu Y\rangle = \lambda\bar{\mu}\langle X, Y\rangle$.

Now to calculate the curvature, we move on the special case of $n = 1$, then using the coordinate of $\mathbb{R}^4$, we have $T_xS^3 = \mathrm{span}\{v_1, v_2, v_3\}$, where

$$v_1 = (-x_1, x_0, -x_3, x_2), \quad v_2 = (-x_2, x_3, x_0, -x_1), \quad v_3 = (-x_3, -x_2, x_1, x_0),$$

note that $x = (z_0, z_1)$ where $z_0 = x_0 + \mathrm{i}x_1$, $z_1 = x_2 + \mathrm{i}x_3$, hence the vertical space is generated by

$$K_{\mathrm{i}}(z) = \mathrm{i}z = (\mathrm{i}x_0 - x_1, \mathrm{i}x_2 - x_3) = (-x_1, x_0, -x_3, x_2) = v_1,$$

hence we have the connction form is $A_x(v) = (v_1, v)\mathrm{i}$, here $(,)$ denotes the Euclidean inner product, then we have

$$A_x = (-x_1\mathrm{d}x_0 + x_0\mathrm{d}x_1 - x_3\mathrm{d}x_2 + x_2\mathrm{d}x_3)\mathrm{i} \in \Omega^1(S^3; \mathrm{i}\mathbb{R}).$$

Then we have the curvature

$$\widehat{F_{A,x}} = \mathrm{d}A_x + \frac{1}{2}[A_x \wedge A_x] = \mathrm{d}A_x = 2(\mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3)\mathrm{i},$$

where we use the Lie bracket in $\mathfrak{u}(1)$ is trivial. Easy to see

$$\widehat{F_{A,x}}(v_2, v_3) = 2\mathrm{i}.$$

It can be shown that the quotient metric of $S^3/\mathrm{U}(1)$ yields the round metric on $S^2\left(\frac{1}{2}\right)$, more precisely, given by

$$(z_0, z_1) \mapsto \left(z_0\overline{z_1}, \frac{1}{2}(|z_0|^2 - |z_1|^2)\right),$$

one can also prove that π_*v_2 and π_*v_3 are orthogonal basis, then from curvature on $S^2\left(\frac{1}{2}\right)$ has the identity

$$F_A(\pi_*v_2, \pi_*v_3) = \pi^*F_A(v_2, v_3) = \widehat{F_A}(v_2, v_3) = 2\mathrm{i},$$

we know that $F_A = 2\mathrm{i} \cdot \mathrm{Vol}_{S^2(1/2)}$*, then we know that*

$$\int_{S^2} F_A = 2\mathrm{i} \cdot 4\pi \cdot \frac{1}{4} = 2\pi\mathrm{i}, \quad \Rightarrow \int_{S^2} \frac{\mathrm{i}}{2\pi} F_A = -1,$$

then we know that $c_1(S^3) = -1$*, if we view* S^3 *as the* S^1-*bundle over* S^2*.*

Remark. For more details of characteristic class of principal bundle, one could refer next section.

1.3.3 Gauge Group

Gauge group is one of the most improtant definitions of gauge theory, since it reflects the symmetry of the system we consider. In most problem, we need to modulo the symmetries, hence only consider the moduli space.

Definition 1.3.8: Gauge group

For principal $G-$bundle $P \to M$, the **gauge group** of P is

$$\mathcal{G}_P := \{\psi \in \mathrm{Diff}(P) : \pi \circ \psi = \pi, \quad \psi(pg) = \psi(p)g\}.$$

Note that $\psi(p)$ lies in the same fibre of p, and G acts on P transitively, hence there exists $\widehat{\psi} : P \to G$ such that

$$\psi(p) = p \cdot \widehat{\psi}(p).$$

Note that $\psi(pg) = \psi(p)g$, hence we have

$$pg \cdot \widehat{\psi}(pg) = \psi(pg) = \psi(p)g,$$

then we know that $\widehat{\psi}(pg) = g^{-1}\widehat{\psi}(p)g$. Hence we know that the map

$$P \to P \times G, \quad p \mapsto \left(p, \widehat{\psi}(p)\right)$$

satifies $pg \mapsto \left(pg, g^{-1}\widehat{\psi}(p)g\right)$, which descends to a map $M \to P \times_{\mathrm{Ad}} G =: \mathrm{Ad}P$, where

$$P \times_{\mathrm{Ad}} G := P \times G/G, \quad (p, g) \sim \left(ph, \mathrm{Ad}_{h^{-1}} g\right),$$

here $\mathrm{Ad}_h g = hgh^{-1}$. Hence from a gauge transformation ψ, we get a section of $\mathrm{Ad}P$, the converse is similar, hence we have

$$\boxed{\mathcal{G}_P \cong \Gamma(\mathrm{Ad}P).}$$

Remark. Easy to see, if P is trivial, then we have $\mathrm{Ad}P$ is also trivial, hence $\mathcal{G}_P \cong \Gamma(\mathrm{Ad}P) = C^\infty(M, G)$.

Example 1.3.7. *If we consider the abealian gauge theory, i.e., G is abealian, then we have $g^{-1}\widehat{\psi}(p)g = \widehat{\psi}(p)$, hence we know that $\mathrm{Ad}P \cong M \times G$, then we also have $\mathcal{G}_P \cong \Gamma(\mathrm{Ad}P) = C^\infty(M, G)$.*

Definition 1.3.9

We have natural action of $\mathcal{G}_P$ on $\mathcal{A}_P \subseteq \Omega^1(P; \mathfrak{g})^G$, the connetion space of P, which is given by pull-back:

$$\mathcal{G}_P \times \mathcal{A}_P \to \mathcal{A}_P, \quad (\psi, A) \mapsto \psi^* A.$$

1.4 Chern-Weil Theory II

In this section, we do the similar thing as we do in the previous section, i.e., use special polynomial and curvature to define characteristic classes.

1.4.1 ad-invariant polynomial

Definition 1.4.1

We call $p : \mathfrak{g} \to \mathbb{C}$ is an ad-invariant homogeneous polynomial of degree k if it satisfies

- $p(\mathrm{ad}_g \xi) = p(\xi)$ for any $\xi \in \mathfrak{g}$ and $g \in G$;
- $p(\lambda\xi) = \lambda^k p(\xi)$ for any $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{g}$.

Remark. More precisely, given a basis $\xi_1, \cdots, \xi_n$ of $\mathfrak{g}$, then we have $p(x_1\xi_1 + \cdots + x_n\xi)$ is a polynomial of degree k in $x_1, \cdots, x_n$.

Remark. To be more precise, we can view p as a special case $\widetilde{p} : \mathrm{Sym}^k\mathfrak{g} \to \mathbb{C}$, the $k-$symmetric multilinear ad$-$invariant function, i.e., $\widetilde{p}$ has the properties that $\widetilde{p}(\xi_1, \cdots, \xi_k) = \widetilde{p}(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)})$. And then we have $p(\xi) := \widetilde{p}(\xi, \cdots, \xi)$.

Recall for matrix Lie algebra, ad_g is just the similar, hence there are natural examples such as determinat and trace:

Example 1.4.1. *For $\mathfrak{g} = \mathfrak{u}(r) = \{A \in \mathbb{C}^{r\times r} : A + A^{\mathsf{H}} = 0\}$, then we can consider*

$$p_k(\xi) := \mathrm{i} \cdot \mathrm{tr}(\xi^d).$$

Example 1.4.2. *For $\mathfrak{g} = \mathfrak{u}(r) = \{A \in \mathbb{C}^{r\times r} : A + A^{\mathsf{H}} = 0\}$, we can define the polynomials c_i of degree i by the following equation*

$$\det\left(\lambda I_r + \frac{\mathrm{i}}{2\pi}\xi\right) = \lambda^r + c_1(\xi)\lambda^{r-1} + \cdots + c_{r-1}(\xi)\lambda + c_r(\xi). \tag{1.3}$$

Easy to see by linear algebra, we have

- $c_1(\xi) = \dfrac{\mathrm{i}}{2\pi}\mathrm{tr}\xi;$
- $c_2(\xi) = \dfrac{1}{2}\left\{\mathrm{tr}\left(\dfrac{\mathrm{i}}{2\pi}\xi\right) \cdot \mathrm{tr}\left(\dfrac{\mathrm{i}}{2\pi}\xi\right) - \mathrm{tr}\left(\left(\dfrac{\mathrm{i}}{2\pi}\xi\right)^2\right)\right\};$
- $\cdots$
- $c_r(\xi) = \det\left(\dfrac{\mathrm{i}}{2\pi}\xi\right) = \left(\dfrac{\mathrm{i}}{2\pi}\right)^r \det\xi.$

Remark. Note that $\xi \in \mathfrak{u}(r)$ hence $\overline{\xi} + \xi^{\mathsf{T}} = 0$, then we have

$$\overline{\det\left(\lambda I_r + \frac{\mathrm{i}}{2\pi}\xi\right)} = \det\left(\lambda I_r - \frac{\mathrm{i}}{2\pi}\overline{\xi}\right) = \det\left(\lambda I_r + \frac{\mathrm{i}}{2\pi}\xi^{\mathsf{T}}\right) = \det\left(\lambda I_r + \frac{\mathrm{i}}{2\pi}\xi\right),$$

hence we actually have $c_i : \mathfrak{g} \to \mathbb{R}$.

1.4.2 Chern-Weil Theorem

Now we can use ad-invariant polynomial and curvature $\widehat{F_A} \in \Omega^2(\mathrm{ad}P) \cong \Omega^2_{\mathrm{bas}}(P;\mathfrak{g})^G$ to define the characteristic classes.

Note that $\widehat{F_A}$ is nothing but a spectial $\mathfrak{g}-$valued two form since $\Omega^2(P;\mathfrak{g}) = \Omega^2(P) \otimes \mathfrak{g}$. So if p is an ad-invariant degree k homogeneous polynomial, then we can use p to eat the $\mathfrak{g}$ part, then we have $p(\widehat{F_A}) \in \Omega^{2k}(P)$. Similarly, we can fibrewisely define $p(F_A) \in \Omega^{2k}(M)$.

Proposition 1.4.1

$\pi^* : \Omega^k(M) \to \Omega^k_{\mathrm{bas}}(P)$ is isomorphism, and $\pi^* p(F_A) = p(\widehat{F_A})$.

Remark. The above proposition is almost trivial, we omit the details. But it is powerful in the proof of the properties of $p(F_A)$ because we cannot globally express $p(F_A)$ as $\mathfrak{g}-$valued. So it is often easier to prove the properties use forms on P, then use isomorphism π^* to derive what we want.

Theorem 1.4.1: Chern-Weil

We have

(1) $p(F_A) \in \Omega^{2k}(M)$ is closed;

(2) $[p(F_A)] \in H^{2k}(M)$ is independent on the chocise of connection A.

The key equation we will use in the proof is the following

Proposition 1.4.2: Bianchi Identity

We have $\mathrm{d}\widehat{F_A} = [\widehat{F_A} \wedge A]$.

Proof. Note that $\mathrm{d}\widehat{F_A} = \mathrm{d}\left(\mathrm{d}A + \frac{1}{2}[A \wedge A]\right) = \frac{1}{2}[\mathrm{d}A \wedge A] - \frac{1}{2}[A \wedge \mathrm{d}A] = [\mathrm{d}A \wedge A]$, since by supper Jacobi we have $[[A \wedge A] \wedge A] = 0$, then we finish. ♣

Now we can start the proof of Chern-Weil theorem :

Proof. (1) To prove $\mathrm{d}(p(F_A)) = 0$, note that $\mathrm{d}\pi^* = \pi^*\mathrm{d}$ and π^* is isomorphism, it suffices to prove $p(\widehat{F_A})$ is closed. Using $p(\widehat{F_A}) = \widetilde{p}(\widehat{F_A}, \cdots, \widehat{F}_A)$,

$$\begin{aligned}\mathrm{d}p(\widehat{F_A}) &= \widetilde{p}(\mathrm{d}\widehat{F_A}, \cdots, \widehat{F_A}) + \cdots + \widetilde{p}(\widehat{F_A}, \cdots, \mathrm{d}\widehat{F}_A) \\ &= \widetilde{p}([\widehat{F_A} \wedge A], \cdots, \widehat{F_A}) + \cdots + \widetilde{p}(\widehat{F_A}, \cdots, [\widehat{F_A} \wedge A]).\end{aligned}$$

Note that $\widetilde{p}$ only eats $\mathfrak{g}$ part hence it suffices to prove the following: for any $\xi, \xi_1, \cdots, \xi_k \in \mathfrak{g}$, we have

$$\boxed{\widetilde{p}([\xi, \xi_1], \cdots, \xi_k) + \cdots \widetilde{p}(\xi_1, \cdots, [\xi, \xi_k]) = 0.} \tag{1.4}$$

Recall $[\xi, \xi_i] = \mathrm{ad}_{*,e}(\xi)(\xi_i) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{ad}_{\exp(t\xi)}(\xi_i)$, then we have

$$\begin{aligned}&\widetilde{p}([\xi, \xi_1], \cdots, \xi_k) + \cdots \widetilde{p}(\xi_1, \cdots, [\xi, \xi_k]) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{p}\left(\mathrm{ad}_{\exp(t\xi)}(\xi_1), \cdots, \mathrm{ad}_{\exp(t\xi)}(\xi_k)\right) \quad \underline{\text{by multilinear}} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{p}(\xi_1, \cdots, \xi_k) \quad \underline{\text{by ad-invaiant}} \\ &= 0,\end{aligned}$$

then we show that $p(F_A)$ is closed.

(2) Thanks to π^* again, it suffices to prove that for different connections $A_0, A_1 \in \mathcal{A}(P)$, we have $p(\widehat{F_{A_1}}) - p(\widehat{F_{A_0}}) = \mathrm{d}\omega$ for some $\omega \in \Omega^{2k-1}(P)$.

Note that $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\mathrm{ad}P) \cong \Omega^1_{\text{bad}}(P; \mathfrak{g})^G$, so we can let $A_t = A_0 + ta \in \mathcal{A}(P)$ for $a = A_1 - A_0 \in \Omega^1(\mathrm{ad}P)$. Then we have

$$\begin{aligned}\widehat{F_{A_t}} &= \mathrm{d}A_t + \frac{1}{2}[A_t \wedge A_t] \\ &= \mathrm{d}A_0 + t\left(\mathrm{d}a + \frac{1}{2}[a \wedge A_0] + \frac{1}{2}[A_0 \wedge a]\right) + \frac{t^2}{2}[a \wedge a],\end{aligned}$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{F_{A_t}} = \mathrm{d}a + [a \wedge A_0] + t[a \wedge a],$$

hence we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}p(\widehat{F_{A_t}}) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{p}(\widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) \\ &= \widetilde{p}\left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}\right) + \cdots + \widetilde{p}\left(\widehat{F_{A_t}}, \cdots, \frac{\mathrm{d}}{\mathrm{d}t}\widehat{F_{A_t}}\right) \\ &= k \cdot \widetilde{p}\left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}\right) \\ &= k \cdot \widetilde{p}(\mathrm{d}a + [a \wedge A_0] + t[a \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}).\end{aligned}$$

Let $\omega_t := \widetilde{p}(a, \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}})$, then we have

$$
\begin{aligned}
&\widetilde{p}(\mathrm{d}a + [a \wedge A_0] + t[a \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) \\
=& \widetilde{p}(\mathrm{d}a, \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) + \widetilde{p}([a \wedge A_0], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) + \widetilde{p}(t[a \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) \\
=& \mathrm{d}\omega_t - \widetilde{p}(a, \mathrm{d}\widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) - \cdots - \widetilde{p}(a, \widehat{F_{A_t}}, \cdots, \mathrm{d}\widehat{F_{A_t}}) \\
& + \widetilde{p}([a \wedge A_0], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) + \widetilde{p}(t[a \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) \\
=& \mathrm{d}\omega_t \quad \underline{\text{use } \mathrm{d}\widehat{F_{A_t}} = -[A_0 \wedge \widehat{F_{A_t}}] - t[a \wedge \widehat{F_{A_t}}]} \\
& + \left\{\widetilde{p}([A_0 \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) + \widetilde{p}(a, [A_0 \wedge \widehat{F_{A_t}}], \cdots, \widehat{F_{A_t}}) + \cdots + \widetilde{p}(a, \widehat{F_{A_t}}, \cdots, [A_0 \wedge \widehat{F_{A_t}}])\right\} \\
& + t\left\{\widetilde{p}([a \wedge a], \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) + \widetilde{p}(a, [a \wedge \widehat{F_{A_t}}], \cdots, \widehat{F_{A_t}}) + \cdots + \widetilde{p}(a, \widehat{F_{A_t}}, \cdots, [a \wedge \widehat{F_{A_t}}])\right\} \\
=& \mathrm{d}\omega_t \quad \underline{\text{use (1.4)}},
\end{aligned}
$$

hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\widehat{F_{A_t}}) = k \cdot \mathrm{d}\omega_t.$$

Now let $\omega := \int_0^1 k\omega_t = \int_0^1 k\widetilde{p}(a, \widehat{F_{A_t}}, \cdots, \widehat{F_{A_t}}) \in \Omega^{2k-1}(P)$, we have

$$p(\widehat{F_{A_1}}) - p(\widehat{F_{A_0}}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} p(\widehat{F_{A_t}}) = \mathrm{d}\omega,$$

then we finish the proof. ♣

Recall we have defined polynomials c_i in example 1.4.2, then we can define

Definition 1.4.2: Chern Class

Let $P \to X$ be a $\mathrm{U}(r)-$principal bundle with connection $A \in \mathcal{A}(P)$, then we call $c_i(P) := [c_i(F_A)] \in \mathrm{H}^{2i}_{\mathrm{dR}}(X)$ the $\underline{i\text{–th Chern class}}$.

More precisely, we have

- $c_1(P) = \frac{\mathrm{i}}{2\pi}\mathrm{tr}F_A$;
- $c_2(P) = \frac{1}{2}\left\{\mathrm{tr}\left(\frac{\mathrm{i}}{2\pi}F_A\right) \cdot \mathrm{tr}\left(\frac{\mathrm{i}}{2\pi}F_A\right) - \mathrm{tr}\left(\left(\frac{\mathrm{i}}{2\pi}F_A\right)^2\right)\right\}$.

Especially, for $\mathrm{SU}(r)-$bundle, then F_A is locally $\mathfrak{su}(r)-$valued, then $\mathrm{tr}(F_A) = 0$, then we have in this case $c_1(P) = 0$, and $c_2(P) = \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A)$.

1.5 Functional and Special Connections

In this section, we consider some special bundles and connections. More precisely, we will study the functionals over connection space, and find their critical points. These will give us some interesting equations and invariants for manifolds.

1.5.1 Yang-Mills Functional and ASD Connections

Throughout this section, we will consider the $\mathrm{SU}(2)-$bundle P over compact oriented Riemanian four manifold X. For the curvature $F_A \in \Omega^2(\mathrm{ad}P)$, locally, $F_A|_U \in \Omega^2(U) \otimes \mathfrak{su}(2)$, hence we can define the inner product $\langle , \rangle$ on $\Omega^2(\mathrm{ad}P)$ by

$$\langle \alpha, \beta \rangle \mathrm{dvol}_X := \mathrm{tr}(\alpha \wedge *\beta^{\mathsf{H}}) = -\mathrm{tr}(\alpha \wedge *\beta),$$

which is combined the inner products $\omega \wedge *\eta$ on $\Omega^2(U)$ and $\mathrm{tr}(AB^{\mathsf{H}})$ on $\mathfrak{su}(2)$.

Hence we can define the very first interesting functional

Definition 1.5.1: Yang-Mills Functional

For the connection $A \in \mathcal{A}(P)$, we define the **Yang-Mills functional** of A as

$$\mathcal{YM}(A) := \int_X |F_A|^2 \mathrm{dvol}_X := \int_X \langle F_A, F_A \rangle \mathrm{dvol}_X.$$

Now we can calculate the critial points of $\mathcal{YM}$ on $\mathcal{A}(P)$. Note that $\mathcal{A}(P)$ is affine space modelled on $\Omega^1(\mathrm{ad}P)$, then if A is the critical point, then for any $a \in \Omega^1(\mathrm{ad}P)$, we have $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{YM}(A + ta) = 0$.

Proposition 1.5.1

For any $a \in \Omega^1(\mathrm{ad}P)$, $F_{A+a} = F_A + \mathrm{d}_A a + \frac{1}{2}[a \wedge a]$, here $\mathrm{d}_A : \Omega^*(\mathrm{ad}P) \to \Omega^{*+1}(\mathrm{ad}P)$ is defined in proposition 1.3.2.

Proof. Still thanks to the isomorphism of π^*, we note that

$$\widehat{F_{A+a}} = \mathrm{d}A + \mathrm{d}a + \frac{1}{2}[A \wedge A] + [A \wedge a] + \frac{1}{2}[a \wedge a],$$

note that $\pi^* \mathrm{d}_A a = \mathrm{d}a + A \cdot a = \mathrm{d}A + \mathrm{ad}_{*,e}(A) \wedge a = \mathrm{d}a + [A \wedge a]$, hence we have $\widehat{F_{A+a}} = \widehat{F_A} + \pi^* \mathrm{d}_A a + \frac{1}{2}[a \wedge a]$, then we know the proof. ♣

Now we have

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathcal{YM}(A+ta)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_X|F_{A+ta}|^2\mathrm{dvol}_X\\
=&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_X\left(|F_A|^2+2t\langle F_A,\mathrm{d}_Aa\rangle+o(t^2)\right)\mathrm{dvol}_X\\
=&2\int_X\langle F_A,\mathrm{d}_Aa\rangle\mathrm{dvol}_X=2\int_X\langle\mathrm{d}_A^*F_A,a\rangle\mathrm{dvol}_X,
\end{aligned}$$

where d_A^* is the dual operator of d_A.

Hence if A is the critical point, then we have $\boxed{\mathrm{d}_A^*F_A=0}$, which is the **Yang-Mills equations**.

In general, this equation is not so easy to study, luckily, we can use a little calculation to find the minimum of Yang-Mills functional.

To do this, we need the decomposition of $\Omega^2(X)$ through the Hodge star $*$, reacll $*^2=(-1)^{p(n-p)}$, hence $*:\Omega^2(X)\to\Omega^2(X)$ satisfies $*^2=1$, then we have $\Omega^2(X)=\Omega^2_+(X)\oplus\Omega^2_-(X)$ by the eigenvalue of $*$.

This decomposition is local, so we can get the similar result as $\Omega^2(\mathrm{ad}P)=\Omega^2_+(\mathrm{ad}P)\oplus\Omega^2_-(\mathrm{ad}P)$, hence $F_A=:F_A^++F_A^-$, where $*F_A^+=F_A^+$ and $*F_A^-=-F_A^-$, then we have

$$\begin{aligned}
c_2(P)&=\frac{1}{8\pi^2}\mathrm{tr}\left(F_A^+\wedge F_A^++F_A^+\wedge F_A^-+F_A^-\wedge F_A^++F_A^-\wedge F_A^-\right)\\
&=\frac{1}{8\pi^2}\mathrm{tr}\left(-F_A^+\wedge*(F_A^+)^{\mathsf{H}}+F_A^+*(\wedge F_A^-)-F_A^-\wedge*(F_A^+)^{\mathsf{H}}+F_A^-\wedge*(F_A^-)^{\mathsf{H}}\right)\\
&=\frac{1}{8\pi^2}\left\{-|F_A^+|^2+|F_A^-|^2-\langle F_A^+,F_A^-\rangle+\langle F_A^-,F_A^+\rangle\right\}\mathrm{dvol}_X\\
&=\frac{1}{8\pi^2}\left(|F_A^-|^2-|F_A^+|^2\right)\mathrm{dvol}_X.
\end{aligned}$$

Note that $\langle F_A^+,F_A^-\rangle=0$, we have

$$\mathcal{YM}(A)=\int_X|F_A|^2\mathrm{dvol}_X=\int_X\left(|F_A^+|^2+|F_A^-|^2\right)\mathrm{dvol}_X,$$

hence we know that

$$\mathcal{YM}(A)=8\pi^2\int_Xc_2(P)+2\int_X|F_A^+|^2\mathrm{dvol}_X,$$

then we know that

$$\mathcal{YM}(A)\geq8\pi^2\langle c_2(P),[X]\rangle=:\kappa(P),$$

where $\kappa(P)$ is a topological invariant independent on the choice of connection.

Hence we know that the Yang-Mills functional has the minmum $\kappa(P)$, which we also called the **instanton** of P. And the equality holds if and only $F_A^+=0$.

Definition 1.5.2

For $A \in \mathcal{A}(P)$, we said A is

- **Yang-Mills connctions**, if $\mathrm{d}_A^* F_A = 0$;
- **ASD connetions**, if $F_A^+ = 0$.

From the discussion above we know that if A is ASD then $\mathcal{YM}(A)$ attains the minmum, hence A is also the critial point, i.e., A is Yang-Mills. Then we have

$$\{\text{ASD conncetions}\} \subseteq \{\text{Yang-Mills Connections}\}, \quad F_A^+ = 0 \Rightarrow \mathrm{d}_A^* F_A = 0.$$

Remark. One can try to prove $F_A^+ = 0 \Rightarrow \mathrm{d}_A^* F_A = 0$ directly.

1.5.2 Chern-Simons Functional and Flat Connections

Similar as ASD connections, we study the $\mathrm{SU}(2)$ bundle over four manifolds. Now we want to consider three dimensional case.

Suppose Y is an oriented closed three manifold, and $P \to Y$ is a $\mathrm{SU}(2)$ bundle. Note that $\mathrm{BSU}(2) = \mathbb{H}P^\infty$, and its celluar has dimension $4k$, hence $f : Y \to \mathbb{H}P^\infty$ is homotopic to constant, then we know that P is always the trivial bundle!

From now on, we consider $P = Y \times \mathrm{SU}(2)$ and then $\mathrm{ad}P = Y \times \mathfrak{su}(2)$, then the connection space $\mathcal{A}_P$ of P is an affine space modelled on $\Omega^1(Y; \mathrm{ad}P) \cong \Omega^1(Y; \mathfrak{su}(2))$.

Definition 1.5.3: Chern-Simons Functional

Suppose $A \in \mathcal{A}_P$ is a connection on P, then we call $\mathcal{CS} : \mathcal{A}_P \to \mathbb{R}$:

$$\mathcal{CS}(A) := \frac{1}{8\pi^2} \int_Y \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right)$$

is the **Chern-Simons functional** of Y.

Note that any oriented closed three manifold Y is null-bordant, i.e., there exists (simply connected) smooth $4-$manifold X such that $\partial X = Y$, so people are sometimes used to define the Chern-Simons functional as

$$\mathcal{CS}(A) := \frac{1}{8\pi^2} \int_X \mathrm{tr}\left(F_A \wedge F_A\right),$$

here we extend the bundle and connection trivially on X.

Remark. Maybe this definition is more natural, since one can easily see that this is just the modification of the second chern class $c_2(P)$. But one should note that here we do NOT have $\mathcal{CS}(A)$ equals to $c_2(X \times \mathrm{SU}(2))$ because X is not closed!

From the proposition below, we know that these two definitions equal:

Proposition 1.5.2

For $Y = \partial X$ and SU(2) connection A on X, we have

$$\mathrm{tr}\,(F_A \wedge F_A) = \mathrm{dtr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right)$$

on X, hence by Stokes theorem we have the desired result.

Proof. Locally we have $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A] = \mathrm{d}A + A \wedge A$, since this is matrix lie algebra. Then we have on the one hand

$$\begin{aligned} \mathrm{tr}\,(F_A \wedge F_A) &= \mathrm{tr}\,((\mathrm{d}A + A \wedge A) \wedge (\mathrm{d}A + A \wedge A)) \\ &= \mathrm{tr}\,(\mathrm{d}A \wedge \mathrm{d}A + 2\mathrm{d}A \wedge A \wedge A) + \mathrm{tr}\,(A \wedge A \wedge A \wedge A) \\ &= \mathrm{tr}\,(\mathrm{d}A \wedge \mathrm{d}A + 2\mathrm{d}A \wedge A \wedge A)\,, \end{aligned}$$

where note that $\mathrm{tr}(A^4) = \mathrm{tr}(A \wedge A^3) = -\mathrm{tr}(A^3 \wedge A) = -\mathrm{tr}(A^4)$, hence $\mathrm{tr}(A^4) = 0$.

On the other hand, we have

$$\begin{aligned} &\mathrm{dtr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right) \\ &= \mathrm{tr}\left(\mathrm{d}A \wedge \mathrm{d}A + \frac{2}{3}\mathrm{d}A \wedge A \wedge A - \frac{2}{3} A \wedge \mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge \mathrm{d}A\right) \\ &= \mathrm{tr}\,(\mathrm{d}A \wedge \mathrm{d}A + 2\mathrm{d}A \wedge A \wedge A)\,, \end{aligned}$$

hence we know they are equal, then we finish the proof. ♣

When we have a functional, the first thing we need to do is to check whether it is invariant under the gauge transformation, if it does, then we can define the functional down to the moduli space.

Note that for $P = Y \times \mathrm{SU}(2)$, then the gauge group $\mathcal{G}_P \cong \Gamma(\mathrm{Ad}P) = \Gamma(P) = C^\infty(Y, \mathrm{SU}(2))$. Note that $\mathrm{SU}(2) \cong S^3$, hence for any $g \in \mathcal{G}_P$, we have the mapping degree $\deg g$. Then an interesting result is

Proposition 1.5.3

For $g \in \mathcal{G}_P$, then we have

$$\mathcal{CS}(g^* A) - \mathcal{CS}(A) = \deg g \in \mathbb{Z},$$

where $g^* A$ is the pull back connection.

1.5.3 Digression: What is Floer Homology?

2

Analysis Backgroud

2.1 Elliptic Differential Operators

2.1.1 Differential Operators

Fisrst of all, we fix some basic notations:

- X is a smooth manifold, E, F are smooth real/complex vector bundles over X, then we have
  - $\Gamma(E) = C^\infty(X, E)$, the smooth sections of E;
  - $\Omega^k(X, E) = \Gamma(\wedge^k T^*X \otimes E)$, the $E-$valued k form on X.
- $n-$tuple of non-negative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$, we set $|\alpha| = \sum_{k=1}^{n} \alpha_k$.
  - for each $\xi \in \mathbb{R}^n$, set $\xi^n = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$;
  - in local coordinates $(x_1, \cdots, x_n)$ on X, we define the differential operators D^α by
$$D^\alpha := (-\mathrm{i})^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} = (-\mathrm{i})^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Definition 2.1.1

A **differential operator of order m** on X is a linear map $P : \Gamma(E) \to \Gamma(F)$ such that in the local coordinate and also the local trivialization $(U, (x_1, \cdots, x_n))$ of X, for $E|_U \cong U \times \mathbb{C}^p$, $F|_U \cong U \times \mathbb{C}^q$, we have locally P can be written as

$$P(x) = \sum_{|\alpha| \leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad x \in U, \tag{2.1}$$

where each $A_\alpha(x)$ is a $q \times p-$ matrix of smooth complex-valued fuctions and where $A^\alpha \neq 0$ for some $|\alpha| = m$. A **real differential operator of order m** is defined similarly with $\mathbb{C}$ replaced by $\mathbb{R}$.

Note that here we use column vector, so the matrix is $q \times p$. Now

1. if we change local trivialization of $E|_U$ and $F|_U$, i.e., consider $g_E : U \to \mathrm{GL}(p, \mathbb{C})$ and $g_F : U \to \mathrm{GL}(q, \mathbb{C})$, then under this trivialization, we have
$$P = \sum_{|\alpha| \leq m} \underline{A}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} = g_F \left(\sum_{|\alpha| \leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \right) g_E^{-1},$$

where $\underline{A}_\alpha$'s are again $q \times p-$matrices of smooth functions of $x \in U$ and where

$$\underline{A}_\alpha = g_F A^\alpha g_E^{-1}, \quad \text{for } |\alpha| \leq m. \tag{2.2}$$

2. if we change of local coordinates $\widetilde{x} = \widetilde{x}(x)$ on U, then using the fact that

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial \widetilde{x}_k}{\partial x_j} \frac{\partial}{\partial \widetilde{x}_k},$$

and we suppose

$$P = \sum_{|\alpha| \leq m} \widetilde{A}^\alpha(\widetilde{x}) \frac{\partial^{|\alpha|}}{\partial \widetilde{x}^\alpha}$$

where

$$\widetilde{A}^\alpha = \sum_{|\beta|=m} A^\beta \left[\frac{\partial \widetilde{x}}{\partial x}\right]_\beta^\alpha \quad \text{for } |\alpha| = m \tag{2.3}$$

and where $[\partial\widetilde{x}/\partial x]_*^*$ denotes the symmetrization of the m^{th} tensor power of the Jacobian matrix $(\partial\widetilde{x}_i/\partial xj)$.

Hence from 1,2 above, we know that the coefficients $\{\mathrm{i}^m A^\alpha\}_{|\alpha|=m}$ represent a well-defined section $\sigma(P)$ of the bundle $(\mathrm{Sym}^m TX) \otimes \mathrm{Hom}(E, F)$, where $\mathrm{Sym}^m TX$ denotes the symmetric tensor product.

Definition 2.1.2

The section $\sigma(P) \in \Gamma((\mathrm{Sym}^m TX) \otimes \mathrm{Hom}(E, F))$ is called the **principal symbol** of the differential operator P.

Recall that for a vector space V, the space $\mathrm{Sym}^m V$ is canonically isomorphic to the space of homogeneous polynomial functions of degree m on V^*. Hence, for each cotangent vector $\xi \in T_x^* X$, the principal symbol gives an element

$$\sigma_\xi(P) : E_x \to F_x.$$

If we fix local coordinates and trivializations as before, then for $\xi = \sum_{k=1}^{n} \xi_k \mathrm{d}x_k$,

$$\sigma_\xi(P) = \mathrm{i}^m \sum_{|\alpha|=m} \xi^\alpha A^\alpha(x).$$

It is now possible to present the fundamental concepts of this note:

Definition 2.1.3

Let P be a differential operator of order m over a manifold X. Then P is **elliptic** if for each non-zero cotangent vector $\xi \in T^*X$, the principal symbol $\sigma_\xi(P) : E_x \to F_x$ is invertible.

Example 2.1.1. *Let $E = F$ be the trivialized line bundle and consider the Laplace operator $\Delta : C^\infty(X) \to C^\infty(X)$ of Riemannian manifold (X, g), then in local coordinates $(x_1, \cdots, x_n)$ we have*

$$\begin{aligned}\Delta f &= \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{\det g} g^{ji} \frac{\partial f}{\partial x_i} \right) \\ &= \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \text{lower order terms.}\end{aligned}$$

Hence for $\xi = \sum \xi_k \mathrm{d}x_k$, we have

$$\sigma_\xi(\Delta) = \mathrm{i}^2 \sum_{i,j=1}^{n} g^{ij} \xi_i \xi_j = -|\xi|^2$$

which is certainly invertible (as a linear map $\mathbb{C} \to \mathbb{C}$) for $\xi \neq 0$.

Example 2.1.2. *Consider $E = \Omega^{\text{odd}}(X) := \bigoplus_{k \text{ odd}} \Omega^k(X)$, and $F = \Omega^{\text{even}}(X)$ similarly and $\mathrm{d} + \mathrm{d}^* : \Omega^{even}(X) \to \Omega^{\text{odd}}(X)$ the differential operator.*

Proposition 2.1.1

Let $P.P' : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(L)$ be differential operators over X, then for all $\xi \in T^*X$ and for all $t, t' \in \mathbb{R}$, one has that

$$\sigma_\xi(tP + t'P') = t\sigma_\xi(P) + t'\sigma_\xi(P'), \quad \sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P).$$

2.1.2 Sobolev Spaces and Sobolev Theorems

Let E be a hermitian vector bundle with connection d_A on a compact Riemannian manifold X, i.e.,

$$\mathrm{d}_A : \Omega^*(X, E) \to \Omega^{*+1}(X, E).$$

Definition 2.1.4

or any $k \in \mathbb{N}$, we define the norm

$$||u||_k^p := \sum_{j=0}^{k} \int_X |\underbrace{\mathrm{d}_A \circ \mathrm{d}_A \cdots \circ \mathrm{d}_A}_{j \text{ times}} u|^p, \quad u \in \Gamma(E) \tag{2.4}$$

called the **basic Sobolev knorm** on $\Gamma(E)$. The completion of this norm is the **Sobolev space** $L_p^k(E)$.

Proposition 2.1.2

A differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m extends to a bounded linear map $P : L_k^p(E) \to L_{k-m}^p(F)$ for all $k \geq m$.

Proof. For any $u \in L_k^p(E) \setminus \Gamma(E)$, then we have $\{u_i\} \subseteq \Gamma(E)$ such that $||u_i - u||_k^p \to 0$. Note that X is compact, hence there is constant $C(P)$ such that for any $v \in \Gamma(E)$,

$$|Pv| \leq C(P) \cdot \sum_{j=0}^{m} |\nabla^j v|.$$

Now we consider $\{Pu_i\} \subseteq \Gamma(F)$, then we have

$$\begin{aligned}
||Pu_i - Pu_j||_{k-m}^p &= \sum_{l=0}^{k-m} \int_X |\nabla^l \circ P(u_i - u_j)|^p \\
&\leq \sum_{l=0}^{k-m} \int_X C(P) \cdot \sum_{t=0}^{m} |\nabla^{l+t}(u_i - u_j)|^p \\
&\leq C'(P,k) \sum_{s=0}^{k} \int_X |\nabla^s(u_i - u_j)|^p = C'(P,k) \cdot ||u_i - u_j||_k^p,
\end{aligned}$$

hence we know that we can define $Pu = \lim_{i \to \infty} Pu_i \in L_{k-m}^p(F)$, and it is trivially linear by definition. And from the discussion above, we know that $C'(P,k)$ is an upper bound. ♣

3
Topology of Three Manifolds

4 Topology of Four Manifolds

4.1 Classifications of Four Manifolds

4.1.1 Homotopy Types

4.1.2 Homeomorphism Types

4.1.3 Diffeomorphism Types

4.2 Complex Surfaces

4.2.1 Algebraic surfaces

4.2.2 The blow-up process

Blow-up is a powerful tool to construct new complex surfaces, desingularization and reduces intersections. In algebraic geometry, blow-up is also a so-called birational map, and actually every birational map is the compositions of sevaral blow-ups and blow-downs.

Wo will not step too far in the algebraic geometry, the main purpose of this section is try to explian the interesting topology behind the blow-ups.

We start with the standard model:

Definition 4.2.1: Blow-up of $\mathbb{C}^2$

We define the **blow-up of** $\mathbb{C}^2$ **at the point** 0, is rougly speaking, the space collect all lines through 0 marked with thier "directions". More precisely, it is denoted as

$$\mathrm{Bl}_0(\mathbb{C}^2) := \{(\ell, z) \in \mathbb{C}P^1 \times \mathbb{C}^2 : z \in \ell\} \subset \mathbb{C}P^1 \times \mathbb{C}^2.$$

Easy to see, $\mathrm{Bl}_0(\mathbb{C}^2)$ is nothing but $\mathcal{O}(-1)$, the tautological line bundle over $\mathbb{C}P^1$.

Why we call it "blow-up at 0"? Here is the reason:

Proposition 4.2.1

Consider the projection map $\mathrm{pr}_2 : \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$, which is given by $(\ell, z) \mapsto z$, then note that $\mathrm{pr}_2^{-1}(0) = \{(\ell, 0) : \ell \in \mathbb{C}P^1\} \cong \mathbb{C}P^1$, also the zero section of $\mathcal{O}(-1)$, then we have the restriction

$$\mathrm{pr}_2 : \mathrm{Bl}_0(\mathbb{C}^2) - \mathbb{C}P^1 \to \mathbb{C}^2 - \{0\}$$

is a biholomorphic.

Remark. We can see that blow up at one point is just replace this point with the "recorder of direction" $\mathbb{C}P^1$.

Proof. This is directly from $(\ell, z) \mapsto z$, and the line across $z \neq 0$ is unique. ♣

Now we can define the blow up of a complex surface of a point at p, this is just defined by locally identified the neighborhood of p with $\mathbb{C}^2$:

Definition 4.2.2: Complex Blow-up

Let S be a complex surface, $p \in S$, we define $\mathrm{Bl}_p(S)$ the **(complex) blow-up of** S **at the point** p as follows:

- take a neighborhood U of p with a biholomorphic $\varphi : U \to \mathbb{C}^2$, and $\varphi(p) = 0$,

- consider the biholomorphic $\psi : \mathbb{C}^2 - \{0\} \to \mathrm{Bl}_0(\mathbb{C}^2) - \mathbb{C}P^1$,

now we take

$$\mathrm{Bl}_p(S) := (S - U) \cup_{\varphi\circ\psi} \mathrm{Bl}_0(\mathbb{C}^2),$$

easy to see $\mathrm{Bl}_p(S)$ is still a complex surface.

Remark. Rougly speaking, same as blow-up of $\mathbb{C}^2$, we have

$$S = (S - U) \cup (\mathbb{C}^2 - \{0\}) \cup \{p\}$$
$$\mathrm{Bl}_p(S) = (S - U) \cup \left(\mathrm{Bl}_0(\mathbb{C}^2) - \mathbb{C}P^1\right) \cup \mathbb{C}P^1,$$

blow-up just replace p with $\mathbb{C}P^1 = \mathbb{P}(T_pS)$.

Definition 4.2.3: Complex blow-down

Easy to see, we can extend $\mathrm{pr}_2 : \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ naturally to a holomorphic map $\sigma : \mathrm{Bl}_p(S) \to S$ through the identification in the remark above. σ is called the **(complex) blow-down**. Easy to see $\sigma : \mathrm{Bl}_p(S) - \sigma^{-1}(p) \to S - \{p\}$ is biholomorphic.

Remark. σ just contracts the total $\sigma^{-1}(p) \cong \mathbb{C}P^1$ to a point p.

Definition 4.2.4: Exceptinal curve

For the blow-down $\sigma : \mathrm{Bl}_p(S) \to S$, we call $\sigma^{-1}(p)$ is the **exceptional curve** (or **-1 curve**) of $\mathrm{Bl}_p(S)$. The latter name comes from $[\sigma^{-1}(p)] \cdot [\sigma^{-1}(p)] = -1$, since we can identify $\sigma^{-1}(p)$ as the zero section of $\mathcal{O}(-1)$.

Exercise 1. Prove that $c_1(\mathrm{Bl}_p(S)) = \sigma^* c_1(S) - \mathrm{PD}.(E)$, where $E := [\sigma^{-1}(p)] \in H_2(\mathrm{Bl}_p(S); \mathbb{Z})$.
Hint: Easy to see $c_1(\mathrm{Bl}_p(S)) = \sigma^* c_1(S) + k\mathrm{PD}.(E)$ for some k, then eat E at both sides and use adjunction formla. For a algebraic geometry proof, one can ref Huybrechts' book, prop 2.5.5.

From present, it may seem really abstract! So let's go back to topology–we will show that $\mathrm{Bl}_p(S)$ is nothing but $S\#\overline{\mathbb{C}P^2}$!

We start with a lemma:

Lemma 4.2.1. *$\overline{\mathbb{C}P^2} - \{P\}$ is orientation preserving diffeomorphic to $\mathcal{O}(-1)$, where $P \in \overline{\mathbb{C}P^2}$ is arbitrary.*

Proof. It suffices to prove $\overline{\mathbb{C}P^2} - \{P\}$ admits a complex line bundle structure over $\mathbb{C}P^1$ and this bundle isomorphic to $\mathcal{O}(-1) \to \mathbb{C}P^1$.

Choose arbitrary line $L_0 \cong \mathbb{C}P^1$ in $\overline{\mathbb{C}P^2} - \{P\}$, then for any $Q \in \overline{\mathbb{C}P^2} - \{P\}$, there exists unique complex line L_{PQ} passing through P, Q, then we define

$$\pi : \overline{\mathbb{C}P^2} - \{P\} \to L_0, \quad L_{PQ} \cap L_0,$$

and the fibre of $\pi(Q)$ is just $L_{PQ} - \{P\} \cong \mathbb{C}P^1 - \{P\} \cong \mathbb{C}$, hence we determines a bundle structure.

Now note that $Q_{\overline{\mathbb{C}P^2}} = (-1)$, hence $L_0 \cdot L_0 = -1$, then we know this bundle has $c_1 = -1$, hence we know the bundle $\overline{\mathbb{C}P^2} - \{P\} \to L_0$ is isomorphic to $\mathcal{O}(-1) \to \mathbb{C}P^1$. ♣

Remark. there are several topological discriptions of $\mathcal{O}(-1) \to \mathbb{C}P^1$:

- view $\mathcal{O}(-1)$ as the disk bundle, then it is the associated vector bundle of the Hopf fibration $S^3 \to \mathbb{C}P^1$, and actually the boudary of $\mathcal{O}(-1)$ is S^3;
- Note that $\overline{\mathbb{C}P^1} \hookrightarrow \overline{\mathbb{C}P^2}$ has self-intersection -1, hence we can identify the tubular neighborhood $N\overline{\mathbb{C}P^1}$ of $\overline{\mathbb{C}P^1}$ as $\mathcal{O}(-1)$, then from the lemma above we know that $\partial N\overline{\mathbb{C}P^1} = S^3$ bounds a ball on the other side.

Exercise 2. Use the remark above, prove that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is diffeomorphic to $S^2 \widetilde{\times} S^2$, and then calculate $\pi_3(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$.
Hint: It is well known that the sphere bundle over S^2 is classified by $\pi_1\left(\mathrm{Diff}^+(S^2)\right) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$, and here $S^2 \widetilde{\times} S^2$ denotes the non-trivial bundle.

Now we see that $\mathrm{Bl}_0(\mathbb{C}^2) \cong \mathcal{O}(-1) \cong \overline{\mathbb{C}P^2} - \{P\} \cong \overline{\mathbb{C}P^2} - \mathbb{D}^4$, where $\mathbb{D}^4$ is a small $4-$ball around P, hence we have

$$\begin{aligned} \mathrm{Bl}_p(S) &= (S - U) \cup \mathrm{Bl}_0(\mathbb{C}^2) \\ &= (S - U) \cup_{S^3} \left(\overline{\mathbb{C}P^2} - \mathbb{D}^4\right) \cong S \# \overline{\mathbb{C}P^2}. \end{aligned}$$

Remark. Conversely, we can use the construction of complex blow up to give $S \# \overline{\mathbb{C}P^2}$ a complex structure.

The above discussion motivates us to generalize the definition of blow-up:

Definition 4.2.5: Topological blow-up

For a smooth connected oriented $4-$manifold X, the connected sum $X' := X \# \overline{\mathbb{C}P^2}$ is called the **(topological) blow-up** of X.
The $\overline{\mathbb{C}P^1}$ in the $\overline{\mathbb{C}P^2}$ summand is called the **exceptinal sphere**, with homology class

denoted as

$$E := [\overline{\mathbb{C}P^1}] \in H_2(X';\mathbb{Z}) \cong H_2(\overline{\mathbb{C}P^2};\mathbb{Z}) \oplus H_2(X;\mathbb{Z}),$$

then easy to see $E \cdot E = -1$ and $E \cdot H_2(X;\mathbb{Z}) = 0$. Consider the quotient map $\sigma : X' \to X$, contracts the whole $\overline{\mathbb{C}P^1}$ to a point, it is called the **(topological) blow-down**.

It is natural to ask given a complex surface (or a smooth $4-$manifold generally), whether there is a blow-down of it, the existence of exceptinal sphere is the only obstruction:

Theorem 4.2.1

Suppose that the smooth complex surface S contains a rational curve C (i.e., a complex submanifold biholomorphic to $\mathbb{C}P^1$) with $[C] \cdot [C] = -1$, i.e., C is exceptional. There there exists complex surface T such that S is biholomorphic to a blow-up of T.

This theorem is too technical to give a detailed proof here, one can refel Barth's *Compact Complex Surfaces* of Griffth, Harris's book. We rather focus on the following topological version:

Theorem 4.2.2

- If there exists exceptional sphere Σ_- in smooth $4-$manifold X', i.e., with $\Sigma_- \cdot \Sigma_- = -1$, then there exists smooth $4-$manifold X such that X' is diffeomorphic to $X\#\overline{\mathbb{C}P^2}$.
- If there exists co-exceptional sphere Σ_+ in smooth $4-$manifold X', i.e., with $\Sigma_+ \cdot \Sigma_+ = 1$, then there exists smooth $4-$manifold X such that X' is diffeomorphic to $X\#\mathbb{C}P^2$.

Proof. For Σ_- case, consider the tubular neighborhood $N\Sigma_-$ of Σ_-, which is an oriented $2-$plane bundle over Σ_- with $e(N\Sigma_-)[\Sigma_-] = [\Sigma_-] \cdot [\Sigma_-] = -1$, hence it is diffeomorphic to $\mathcal{O}(-1) \to \mathbb{C}P^1$. Recall we have proved that $\mathcal{O}(-1)$ is orientation preservating diffeomorphic to $\overline{\mathbb{C}P^2} - \mathbb{D}^4$, hence we have $N\Sigma_-$ is orientation preserving diffeomorphic to $\overline{\mathbb{C}P^2} - \mathbb{D}^4$, and also $\partial(X' - N\Sigma_-) \cong S^3$.

Now take $X = (X - N\Sigma_-) \cup_{S^3} \mathbb{D}^4$, then easy to see

$$\begin{aligned} X' &= \left(X' - N\Sigma_-\right) \cup N\Sigma_- \\ &\cong \left(X' - N\Sigma_-\right) \cup \left(\overline{\mathbb{C}P^2} - \mathbb{D}^4\right) \\ &\cong (X - \mathbb{D}^4) \cup \left(\overline{\mathbb{C}P^2} - \mathbb{D}^4\right) = X\#\overline{\mathbb{C}P^2}, \end{aligned}$$

as desired. Now for the Σ_+ case, only need to reverse the orientation of X. ♣

Remark. Note that the co-exceptional shpere case has NO holomorphic counterpart, beacuse $S\#\mathbb{C}P^2$ will never be complex! Since we will prove in the latter section that *smooth* $4-$*manifold* X *admits an almost complex structure iff* $b_1(X) - b_2^+(X)$ *is odd.*

Remark. The exceptional sphere is similar as the **essential sphere** in three manifold, i.e., the sphere cannot bounds a ball. If a three manifold $Y \neq S^2 \times S^1$ contains an essential sphere, then Y can be decomposed as a non-trivial connected sum.

As the introduction of this section said, we always take the manifolds get from blow-up/blow-down are equivalent (birational equivalent), so this motivates us to define

Definition 4.2.6: Minimal complex surface

A complex surface is called **minimal** if it is NOT the blow up of other complex surfaces, equivalently, it does NOT contain any rational exceptional sphere.

Example 4.2.2. *Note that* $Q_{\mathbb{C}P^2} = (1)$, *hence it is minimal.*

Example 4.2.3. *Easy to see if the complex surface* S *has even intersection form, then* S *is minimal, because for every homology class* α, $Q_S(\alpha, \alpha)$ *is even, hence can never be* -1.

From this we know that $\mathbb{C}P^1 \times \mathbb{C}P^1$, *and* S_d *for* d *even are minimal, here* S_d *denotes the degree* d *algebraic surfaces in* $\mathbb{C}P^3$ *as before.*

Remark. Actually for all $d \geq 4$, S_d is minimal, but it takes more effort.

Since the blow-down operation reduces the b_2, so it can be repeated at most finite many times. Starrting from S and blow-down along the rational exceptional sphere as much as we can, we get a minimal complex surface S_{min}, which is called a **minimal model** of S.

Remark. One should note that the minimal model is not unique, one can prove that $(S^2 \times S^2)\#\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ are biholomorphic, but they have minimal model as $S^2 \times S^2$ and $\mathbb{C}P^2$ respectively.

Remark. Study the minimal model is a huge task, which is called **Minimal Model Program (MMP)**.

¶ Proper transform

Now we can use blow-up to reduce the intersection of curves. Informally, suppose C_1, C_2 are complex curves in complex surface S intersecting each othere transversely (and only) in p.

Now we consider the blow-up at P, $\sigma : \mathrm{Bl}_p(S \to S)$, then note the closure

$$\widetilde{C_i} := \overline{\sigma^{-1}\left(C_i - \{p\}\right)},$$

it is easy to see $\widetilde{C_1}$ and $\widetilde{C_2}$ will be disjoint in $\mathrm{Bl}_p(S)$.

This motivates us to define:

Definition 4.2.7: Proper transform

Let smooth surface Σ embedded into the smooth $4-$manifold X and blow-up a point $p \in \Sigma$, denote the blow-down by $\sigma : X' := X\#\overline{\mathbb{C}P^2} \to X$, then

- the inverse image $\Sigma' := \sigma^{-1}(\Sigma) \subset X$ is called the **total transform** of Σ,
- the closeure $\widetilde{\Sigma} := \overline{\sigma^{-1}\left(\Sigma - \{p\}\right)}$ is the **proper transform** of Σ.

Remark. In algebraic geometry, they sometimes call proper transform as *strict transform*.

Assume now that we have two surfaces $\Sigma_1, \Sigma_2 \subset X$ intersecting each other transversally at p, and that the sign of this intersection is $+1$. A more "differential topological" description of the blow-up process and the proper transforms can be given in the following way:

By blowing up p —as we already saw earlier — we just replace a 4-ball neighborhood D of p with

$$\mathcal{O}(-1) \cong \overline{\mathbb{C}P^2} - D',$$

where D' is a 4-ball neighborhood of a point $q \in \overline{\mathbb{C}P^2}$.

The proper transforms of Σ_1 and Σ_2 can easily be seen as follows: We can choose two lines L_1, L_2 going through q in $\overline{\mathbb{C}P^2}$ in such a way that the pairs

$$(D, D \cap (\Sigma_1 \cup \Sigma_2)) \quad \text{and} \quad (D', D' \cap (L_1 \cup L_2))$$

correspond via a diffeomorphism f that reverses the orientations of D, Σ_1 and Σ_2.

Remark. This is because $\Sigma_1 \cdot \Sigma_2 = +1$ while $L_1 \cdot L_2 = -1$.

If we use the restriction of f to glue $X - D^\circ$ and $\overline{\mathbb{C}P^2} - (D')^\circ$ along their boundaries, the proper transforms $\widetilde{\Sigma}_i$ are equal to $\Sigma_i \# L_i$ (with the orientations corresponding correctly).

Since $L_1 - \{q\}$ and $L_2 - \{q\}$ are disjoint, the intersection point $p \in \Sigma_1 \cap \Sigma_2$ has disappeared from the intersection $\widetilde{\Sigma_1} \cap \widetilde{\Sigma_2}$.

Remark. Note that in the case of complex curves in a complex surface we did not hypothesize that the intersection is positive; the reason is that the transverse intersection of two complex curves in a complex surface is always positive.

$\boxed{\textit{Remark.}}$ If we take Σ_2 as the little transverse perturbtion of Σ_1, then the proper transform just reduces the self intersection number by 1.

$\boxed{\textit{Remark.}}$ If the intersection is negative, then we can do similar topologically-coblow-up with $\mathbb{C}P^2$, but in most cases this is not as good as we wish, because connected sum with $\mathbb{C}P^2$ is a really "bad" way, it will make SW invariants identically vanish for example. This also why we like and always assume *positive intersections*.

In summary, we have

Proposition 4.2.2: topological discription of proper transform

Proper transform at a (smooth) point $p \in \Sigma$, for the result $\widetilde{\Sigma}$, we have

$$\widetilde{\Sigma} = \Sigma \# L \subset X \# \overline{\mathbb{C}P^2},$$

where $L \cong \mathbb{C}P^1$, hence $\widetilde{\Sigma}$ has the same genus as Σ, and

$$[\widetilde{\Sigma}] = [\Sigma] - E,$$

where $E := [\overline{\mathbb{C}P^1}] \in H_2(\overline{\mathbb{C}P^2}; Z)$.

Proof. Note that $[\widetilde{\Sigma}] = [\Sigma \# L] = [\Sigma] + [L] = [\Sigma] + [\mathbb{C}P^1] = [\Sigma] - [\overline{\mathbb{C}P^1}] = [\Sigma] - E$. ♣

$\boxed{\textit{Remark.}}$ one can from this again see that $[\widetilde{\Sigma_1}] \cdot [\widetilde{\Sigma_2}] = ([\Sigma_1] - E) \cdot ([\Sigma_2] - E) = [\Sigma_1] \cdot [\Sigma_2] - 1$.

$\boxed{\textit{Remark.}}$ Carefully thinking why here L has the orientation $\mathbb{C}P^1$, we need to follow back the orientation preserving identification of $\overline{\mathbb{C}P^2} - D'$ with $\mathcal{O}(-1) \to \mathbb{C}P^1$, where we view $L \subset \overline{\mathbb{C}P^2}$ as somehow the compactification of the fibre of $\mathcal{O}(-1)$, note that the orientation of $\mathcal{O}(-1)$ is given by the product orientation of $\mathbb{C}$ and $\mathbb{C}P^1$, then we know that the orientation of L is indeced from the orientation of $\mathbb{C}$, as desired.

Exercise 3. Suppose C is a complex curve, then $\left(C \times \mathbb{C}P^1\right) \# \overline{\mathbb{C}P^2}$ and $\left(C \widetilde{\times} \mathbb{C}P^1\right) \# \overline{\mathbb{C}P^2}$ are diffeomorphic.

$\boxed{\text{Hint:}}$ Exercise 3.4.30 of Gompf's book.

¶ Desingularization of curves

We have talked about the proper transform at the point of embedded surface, now we can turn to the immersed surface with singularities, we can use the proper transform to do the desingularization.

If at the singular point with multiplicity m, then $[\widetilde{\Sigma}] = [\Sigma] - mE$, we can finaly make the immersed surface be embedded after proper transform at the whole singularities.

FACT: generically $f : \Sigma \to X$ is immersed with only transverse double points.

Ref Kirby 2.3, GH P506

4.2.3 Obstruction of almost complex structure

4.3 Kirby Calculus

4.4 Trisection and Rational Blow-down

4.5 Appendix: $h-$corbordism

Part II

Siberg Witten Invariants

5

Spin Geometry

5.1 Clifford Modules and Dirac Operators

5.1.1 Vector spaces case

The motivation of Dirac operators is to find the square root of the following Laplaican:

$$\Delta : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \to C^\infty(\mathbb{R}^n, \mathbb{C}^m), \quad f \mapsto -\left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\right) f,$$

here $f = (f_1, \cdots, f_m)^\mathsf{T}$, with Hermitian inner product

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f \cdot \overline{g}\mathrm{dvol}.$$

Easy to see, this Laplacian has the following properties:

- self-adjoint, i.e., $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$;
- positive definite, i.e., $\langle \Delta f, f\rangle = \langle \nabla f, \nabla f\rangle \geq 0$.

$\boxed{\text{Dirac's Question:}}$ Does there exists differential operator $\not{D} : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \to C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ such that $\not{D}$ is self-adjoint, i.e., $\langle \not{D} f, g\rangle = \langle f, \not{D} g\rangle$ and $\Delta = \not{D}^2$?

Example 5.1.1. *If $n = 1$, then for any m, we have $\not{D} := \sqrt{-1}\frac{\partial}{\partial x}$, then easy to see $\not{D}^2 = -\frac{\partial^2}{\partial x^2} = \Delta$, and*

$$\langle \not{D} f, g\rangle = \langle \sqrt{-1}\frac{\partial f}{\partial x}, g\rangle = -\sqrt{-1}\langle f, \frac{\partial g}{\partial x}\rangle = \langle f, \sqrt{-1}\frac{\partial}{\partial x} g\rangle = \langle f, \not{D} g\rangle.$$

Example 5.1.2. *If $n = 2$, then consider $m = 2$ case, note that*

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)\left(-\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right),$$

hence we have

$$\not{D} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}\frac{\partial}{\partial y}$$

is desired differential operator.

Now generally, given n, for what m we can find such operator $\not{D}$? To be more precise, if such $\not{D}$ exists, then there exists $A_i \in \mathrm{End}(\mathbb{C}^m)$ for $1 \leq i \leq n$ such that

$$\not{D} = \sum_{i=1}^n A_i \frac{\partial}{\partial x_i}$$

satisfies:

- $\not{D}^2 = \Delta$, note that

$$\not{D}^2 = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right) = \sum_{i=1}^{n} A_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}(A_iA_j + A_jA_i)\frac{\partial^2}{\partial x \partial y},$$

where we use A_i's are constant matrices then $\frac{\partial A_i}{\partial x_j} = 0$. Then we have

$$\boxed{A_i \cdot A_i = -1, \quad A_i \cdot A_j + A_j \cdot A_i = 0, \quad i \neq j.} \tag{5.1}$$

- $\not{D}$ is self-adjoint, note that

$$\langle \not{D} f, g\rangle = \left\langle \sum_{i=1}^{n} A_i \frac{\partial f}{\partial x_i}, g \right\rangle = -\left\langle f, \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A_i^{\mathsf{H}} g)\right\rangle = \left\langle f, \sum_{i=1}^{n} -A_i^{\mathsf{H}} \frac{\partial f}{\partial x_i}\right\rangle,$$

then we have $\boxed{A_i = -A_i^{\mathsf{H}}}$.

So now Dirac's question seems to be a linear algebra exercise! But more accuratly, to find the matirx A_i is equivalnet to a representation problem from the algebra satisfing (5.1). To be more precise, we introduce

Definition 5.1.1: Clifford Algebra

Let (V, g) be a real Euclidean space with orthognormal basis $\{e_1, \cdots, e_n\}$, we define the Clifford algebra of V, $\mathrm{Cl}(V)$ is a real algebra generated by e_i modulo the relation $e_i^2 = -1$ and $e_ie_j + e_je_i = 0$.

More precisely, $\mathrm{Cl}(V) = \mathcal{T}(V)/\mathcal{I}$, where $\mathcal{T}(V)$ is the tensor algebra $\bigotimes_{n\geq 0} V^{\otimes n}$, and the ideal $\mathcal{I}$ is generated by $\{v \otimes v + |v|^2 : v \in V\}$.

Easy to see, as a real vector space $\mathrm{Cl}(V)$ has dimension 2^n, with basis

$$e_{i_1}e_{i_2}\cdots e_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \quad 0 \leq k \leq n,$$

hence from $e_ie_j = -e_je_i$, we know that $\mathrm{Cl}(V) \cong \wedge^* V$ as vector spaces.

Example 5.1.3. *We have:*

- $\mathrm{Cl}(\mathbb{R}) \cong \mathbb{C}$, *since it is generated by* $1, e_1$ *and* $e_1^2 = -1$;
- $\mathrm{Cl}(\mathbb{R}^2) \cong \mathbb{H}$, *since it is generated by* $1, e_1, e_2, e_1e_2$, *corresponding to* $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, *the basis of* $\mathbb{H}$.

Hence we can turn the problem of finding A_i into finding the representation of Clifford algebra, which to be more accurate, we introduce the following definition :

Definition 5.1.2: Clifford Module

A Clifford module of (V, g) is (S, γ) where S is an complex Hermitian $\langle , \rangle$ vector space, and $\gamma : V \to \mathrm{End}(S)$ is the Clifford multiplication such that

- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)\mathrm{Id}$;
- $\gamma(v) = -\gamma(v)^*$.

Actually from the relation above, we can extend $\gamma : \mathrm{Cl}(V) \to \mathrm{End}(S)$, i.e., a complex representation of Clifford algebra of V.

Example 5.1.4. *For $V = \mathbb{R}^2$, we have seen that $S = \mathbb{C}^2$ is a Clifford module by given*

$$\gamma(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Example 5.1.5. *For $V = \mathbb{R}^3$, we can take $S = \mathbb{C}^2$ and $\gamma(e_i) = B_i$, the Pauli matrix, i.e.,*

$$B_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 5.1.6. *For $V = \mathbb{R}^4$, we can take $S = \mathbb{C}^4$ with*

$$\gamma(e_0) = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}, \quad \gamma(e_i) = \begin{pmatrix} O & B_i \\ B_i & O \end{pmatrix}, \quad i = 1, 2, 3.$$

Generally, we have the following

Theorem 5.1.1

Suppose V is an n dimensional Euclidean space, then

- if $n = 2k$, then there exitsts only one finite dimension irreducible Clifford module (S, γ) up to isomorphism, and $\dim_{\mathbb{C}} S = 2^k$;
- if $n = 2k + 1$, then there exitsts only two finite dimension irreducible Clifford modules (S, γ), $(S, -\gamma)$ up to isomorphism, and $\dim_{\mathbb{C}} S = 2^k$.

For the proof, one can refer Lawson's *Spin Geometry*. And here the isomorphism we mean that

Definition 5.1.3: Clifford Module Isomorphism

We call two Clifford module for V, (S, γ) and (S', γ') are isomorphism if there exists a linear isometry $f : S \to S'$ such that $f \circ \gamma(v) = \gamma'(v) \circ f$ for all $v \in V$.

A easy result is

Proposition 5.1.1: Schur

Let (S, γ) be an irreducible Clifford module for V, then every automorphism of (S, γ) is of the form $\lambda \mathrm{Id}$ for some $\lambda \in S^1$, i.e., $\mathrm{Aut}(S, \gamma) \cong S^1$.

Proof. Since $f : (S, \gamma) \to (S, \gamma)$ is isometry, hence f is a unitary transformation, hence can be diagonalized with all eigenvalues have unit norm. Suppose S_i is the eigenspace of f with eigenvalue λ_i.

Since for any $v \in V$, $\gamma(v) \circ f = f \circ \gamma(v)$, then we know that S_i can be viewed as a submodule, hence by irreducibilty, we know that $S_i = S$, i.e., $f = \lambda \mathrm{Id}$ for some $\lambda \in S^1$. ♣

Now given Clifford module S for V, then we have the Dirac operator $\not{D} : C^\infty(V, S) \to C^\infty(V, S)$:

$$\not{D} = \sum_{i=1}^{n} \gamma(e_i) \frac{\partial}{\partial e_i},$$

where $\{e_i\}$ is the orthogonormal basis of V, then by the previous discussion, we know that $\not{D}$ is self-adjoint and $\not{D}^2 = \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial e_i^2}$.

5.1.2 Vector bundles case

Now we want to extend the discussion the above to the vector bundle, i.e., define the Dirac operator $\not{D} : \Gamma(S) \to \Gamma(S)$ for vector bundle S.

Throught this section, we suppose (M, g) is a closed oriented Riemanian manifold.

Definition 5.1.4: Clifford Modules

An Hermitian/Riemanian $\langle , \rangle$ vector bundle $E \to M$ is called a Clifford module for M if there exists a bundle map $\gamma : TM \to \mathrm{End}(E)$ such that for any $p \in M$, (E_p, γ) is a Clifford module of (T_pM, g).
More precisely, the bundle map $\gamma : TM \to \mathrm{End}(E)$ has the following properties: for any $u, v \in \mathfrak{X}(M)$,

- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)\mathrm{Id}$;
- $\gamma(v) = -\gamma(v)^*$, i.e., $\langle \gamma(v)\sigma_1, \sigma_2\rangle = -\langle \sigma_1, \gamma(v)\sigma_2\rangle$.

Remark. Easy to see, the Clifford multiplication $\gamma : TM \to \mathrm{End}(S)$ can be naturally extended to the bundle map $\gamma : \mathrm{Cl}(TM) \to \mathrm{End}(S)$, here $\mathrm{Cl}(TM) := \cup_{p\in M}\mathrm{Cl}(T_pM)$, one can check it is a vector bundle.

Example 5.1.7. *Suppose (S, γ) is a Clifford module of V, then for any open subset $U \subseteq V$, we have the bundle $U \times S \to S$ is trivially a Clifford module for manifold U.*

Definition 5.1.5: Dirac Bundle

If $E \to M$ is a Clifford module, then it is further called a Dirac bundle if there exists a connection d_A on E such that

- d_A is unitary/Euclidean, i.e., $\mathrm{d}\langle \sigma_1, \sigma_2\rangle = \langle \mathrm{d}_A\sigma_1, \sigma_2\rangle + \langle \sigma_1, \mathrm{d}_A\sigma_2\rangle$;
- d_A is compatible with the Clifford multipication, i.e.,

$$\mathrm{d}_A(\gamma(v)\sigma) = \gamma\left(\mathrm{d}_\nabla v\right)\sigma + \gamma(v)\mathrm{d}_A\sigma,$$

  here $\mathrm{d}_\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ is the unique Levi-Civita connection on TM.

Remark. For short, we may directly wirte $v \cdot \sigma$ to replace $\gamma(v)\sigma$.

Now for Dirac bundle, we can define the desired Dirac operator:

Definition 5.1.6: Dirac Operator

Let $E \to M$ be a Dirac bundle with connection d_A, then we have

$$\not{D}_A : \Gamma(E) \xrightarrow{\mathrm{d}_A} \Gamma(T^*M \otimes E) \xrightarrow{g} \Gamma(TM \otimes E) \xrightarrow{\gamma} \Gamma(E)$$

is the Dirac operator of E.

Remark. Here we use g as the isomorphism $T^*M \to TM$ given by music isomoprhism $g : \omega \mapsto \omega^\sharp$, where for any $X \in \mathfrak{X}(M)$, $g(\omega^\sharp, X) = \omega(X)$.

Proposition 5.1.2

Locally if $\{e_i\}$ is an orthognormal basis of $TM|_U$, then for any $\sigma \in \Gamma(E)|_U$, we have

$$\not{D}_A\sigma = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i}\sigma,$$

here $\nabla^A_X\sigma := \mathrm{d}_A\sigma(X)$ as defined in before.

Proof. Note that locally, suppose $\{e^i\}$ is the dual basis of $\{e_i\}$, more precisely $e^i = e_i^\flat$. Then we have

$$\mathrm{d}_A\sigma = e^i \otimes \mathrm{d}_A\sigma(e_i) = e^i \otimes \nabla^A_{e_i}\sigma \in \Gamma(T^*M \otimes E)|_U,$$

hence

$$g \circ \mathrm{d}_A\sigma = \sum_{i=1}^n e_i \otimes \nabla^A_{e_i}\sigma,$$

then we finally have

$$\not{D}_A\sigma = \gamma \circ g \circ \mathrm{d}_A\sigma = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i}\sigma,$$

which is as desired. ♣

Recall the vector spaces' version Dirac operator, we want it is self-adjoint and is the square root of Laplacian. We will see these two properties in the following.

¶ Self-adjoint of Dirac operators

Proposition 5.1.3

Let $E \to M$ be a Dirac bundle with Dirac operator $\not{D}_A : \Gamma(E) \to \Gamma(E)$, then $\not{D}_A$ is self-adjoint, i.e., for any $\sigma_1, \sigma_2 \in \Gamma(E)$, we have

$$(\not{D}_A\sigma_1, \sigma_2) = (\sigma_1, \not{D}_A\sigma_2),$$

here $(,)$ is the Hermitian/Euclidean inner product on $\Gamma(E)$, induced by $\langle , \rangle$, more precisely, $(\sigma, \tau) := \int_M \langle \sigma, \tau \rangle \mathrm{dvol}_M$.

Proof. We calculate locally at the point $p \in M$, hence we can choose normal coordinates around p such that there exists local orthorgonormal basis $\{e_i\}$ of TM such that $(\nabla_{e_i} e_j)_p = 0$, since the normal coordinates guarantee that $\Gamma_{ij}^k(p) = 0$. Hence we have at p,

$$\begin{aligned}
\langle \not{D}_A \sigma_1, \sigma_2 \rangle &= \sum_{i=1}^n \langle e_i \cdot \nabla_{e_i}^A \sigma_1, \sigma_2 \rangle \\
&= \sum_{i=1}^n -\langle \nabla_{e_i}^A \sigma_1, e_i \cdot \sigma_2 \rangle \quad \underline{\text{Since connection is unitary/Euclidean}} \\
&= \sum_{i=1}^n -e_i \langle \sigma_1, e_i \cdot \sigma_2 \rangle + \langle \sigma_1, \nabla_{e_i}^A (e_i \cdot \sigma_2) \rangle \\
&= \sum_{i=1}^n -e_i \langle \sigma_1, e_i \cdot \sigma_2 \rangle + \langle \sigma_1, e_i \cdot \nabla_{e_i}^A \sigma_2 \rangle \quad \underline{\text{since} (\nabla_{e_i} e_j)_p = 0} \\
&= \langle \sigma_1, \not{D}_A \sigma_2 \rangle + \sum_{i=1}^n -e_i \langle \sigma_1, e_i \cdot \sigma_2 \rangle.
\end{aligned}$$

Now for the $-e_i \langle \sigma_1, e_i \cdot \sigma_2 \rangle$ term, recall for $X \in \mathfrak{X}(M)$, we have

$$\mathrm{div} X := \sum_{i=1}^n g(\nabla_{e_i} X, e_i),$$

hence at the point p, we have

$$\mathrm{div} X = \sum_{i=1}^n e_i g(X, e_i),$$

so we can define X by given for any $Y \in \mathfrak{X}(M)$, $g(X, Y) := -\langle \sigma_1, Y \cdot \sigma_2 \rangle$, hence we have at p then holds for general

$$\langle \not{D}_A \sigma_1, \sigma_2 \rangle = \langle \sigma_1, \not{D}_A \sigma_2 \rangle + \mathrm{div} X,$$

then from divergence theorem $\int_M \mathrm{div} X \mathrm{dvol}_M = 0$, we finish the proof. ♣

¶ Example of Dirac bundle

So far, we have not seen any examples of non-trivial Clifford modules and Dirac bundle, we will construct a lot examples in the next section after we introduce spin and spin^c structure.

In this small section, we consider the easiest example: As before (M, g) is a closed oriented Riemmanian manifod. Let

$$E := \wedge T^* M = \bigoplus_{p=0}^n \wedge^p T^* M,$$

then we claim there is a Clifford multiplication:

$$\gamma : TM \otimes E \to E, \quad v \otimes \alpha \mapsto v^\flat \wedge \alpha - \iota_v \alpha,$$

here $v^\flat \in T^*M$ is given by $v^\flat(X) := g(v, X)$, the music isomorphism.

Now locally fix the orthognol basis $\{e_i\}$ of TM and $\{e^i\}$ of T^*M, then we define the induced inner product on E is given by for $\omega = \omega_I e^I$, $\eta = \eta_J e^J$, then $g(\omega, \eta) = \omega_I \eta_J \delta^{IJ}$, here $\delta^{IJ} = 1$ iff $I = J$ and equals 0 otherwise.

¶Check γ is a Clifford multiplication:

-

Proposition 5.1.4

Consider the Levi-Civita connection on E, then we have locally

$$\mathrm{d} = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i}, \quad \mathrm{d}^* = -\sum_{i=1}^{n} \iota_{e_i} \circ \nabla_{e_i}.$$

Then it is easy to see the Dirac operator on E is given by

$$\not{D}\alpha = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\alpha = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i}\alpha - \iota_{e_i}\nabla_{e_i}\alpha = (\mathrm{d} + \mathrm{d}^*)\alpha,$$

that is, we have

$$\boxed{\not{D}^2 = (\mathrm{d} + \mathrm{d}^*)^2 = \Delta.}$$

For the details, refer Spin Geometry P124.

¶ Dirac operator VS Laplacian— —Weitzenbock formula

Now we see the relation between Dirac operators and Lalacian, before do that, we need firstly define what Laplacian we mean!

Definition 5.1.7: Connetion Laplacian

Let $E \to M$ has unitary/Euclidean connection $\mathrm{d}_A = \nabla^A$, then we have connection Laplacian $\Delta^A : \Gamma(E) \to \Gamma(E)$, which is defined locally by

$$\Delta^A \sigma := -\sum_{i=1}^{n} \left(\nabla^A_{e_i} \circ \nabla^A_{e_i} \sigma - \nabla^A_{\nabla_{e_i} e_i} \sigma \right),$$

where $\{e_i\}$ is orthognol basis.

Remark. Here the $\nabla^A_{\nabla_{e_i} e_i}$ term is to make the definition does not depend on the choice of $\{e_i\}$ and hecne defined globally. One can check this fact, nothing but well known Riemanian geomtry.

Proposition 5.1.5

For any $\sigma_1, \sigma_2 \in \Gamma(E)$, we have

$$\int_M \langle \Delta^A \sigma_1, \sigma_2 \rangle \mathrm{dvol}_M = \int_M \langle \nabla^A \sigma_1, \nabla^A \sigma_2 \rangle \mathrm{dvol}_M = \int_M \langle \sigma_1, \Delta^A \sigma_2 \rangle \mathrm{dvol}_M,$$

hence we know that Δ^A is self-adjoint and positive definite.

Proof. Again, we do calculation pointwise, at $p \in M$, we choose local orthogonal basis $\{e_i\}$ such that $(\nabla_{e_i} e_j)_p = 0$, then at p, we have

$$\begin{aligned} \langle \Delta^A \sigma_1, \sigma_2 \rangle &= \sum_{i=1}^{n} -\langle \nabla^A_{e_i} \nabla^A_{e_i} \sigma_1, \sigma_2 \rangle \\ &= \sum_{i=1}^{n} -e_i \langle \nabla^A_{e_i} \sigma_1, \sigma_2 \rangle + \langle \nabla^A_{e_i} \sigma_1, \nabla^A_{e_i} \sigma_2 \rangle \\ &= \mathrm{div} X + \langle \nabla^A \sigma_1, \nabla^A \sigma_2 \rangle, \end{aligned}$$

where X is defined by $g(X, Y) := -\langle \nabla^A_Y \sigma_1, \sigma_2 \rangle$, and note that $\nabla^A \sigma = \nabla^A_{e_i} \sigma \otimes e^i$, and the inner product on $\Omega^1(E)$ is given by tensorwise.

Then we from divergence theorem again to finish the proof. ♣

Now from Dirac operator $\not{D}_A : \Gamma(E) \to \Gamma(E)$, and the connection Laplacian $\Delta^A : \Gamma(E) \to \Gamma(E)$, what is the relation between $\not{D}_A^2$ and Δ^A?

Unlike the trivial case, curvature enters!

Proposition 5.1.6: Weitzenbock Formula

Let $E \to M$ be a Dirac bundle with connection d_A and Dirac operator $\not{D}_A$, then we have

$$\not{D}_A^2 = \Delta^A + \frac{1}{2}\gamma(F_A),$$

where $F_A \in \Omega^2(\mathrm{End}(E))$ is the curvature of d_A.

Remark. Here $\gamma : \Gamma(\wedge^2 T^*M \otimes \mathrm{End}(E)) \to \Gamma(\mathrm{End}(E))$ is natural, where locally choosen $\{e_i\}$ and dual $\{e^i\}$, we have $F_A = \sum_{i,j=1}^{n} e^i \wedge e^j \otimes F_A(e_i, e_j)$, then we have

$$\gamma(F_A) = \sum_{i,j=1}^{n} \gamma(e^i \wedge e^j) \circ F_A(e_i, e_j) = 2\sum_{i<j} \gamma(e_i)\gamma(e_j) \circ F_A(e_i, e_j).$$

Remark. One should recall that $F_A(e_i, e_j)\sigma = \nabla^A_{e_i}\nabla^A_{e_j}\sigma - \nabla^A_{e_j}\nabla^A_{e_i}\sigma - \nabla^A_{[e_i,e_j]}\sigma$.

Proof. Again[2], we do calculation pointwise, at $p \in M$, we choose local orthogonal basis $\{e_i\}$ such that $(\nabla_{e_i}e_j)_p = 0$, and we have $[e_i, e_j]_p = \left(\nabla_{e_i}e_j - \nabla_{e_j}e_i\right)_p = 0$, then at p, we have

$$\begin{aligned}
\not{D}_A^2\sigma &= \left(\sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\right)\left(\sum_{j=1}^{n} e_j \cdot \nabla^A_{e_j}\right)\sigma \\
&= \sum_{i,j=1}^{n} e_i \cdot \nabla^A_{e_i}\left(e_j \cdot \nabla^A_{e_j}\sigma\right) \\
&= \sum_{i,j=1}^{n} e_i \cdot e_j \cdot \nabla^A_{e_i}\nabla^A_{e_j}\sigma \quad \underline{\text{since}(\nabla_{e_i}e_j)_p = 0} \\
&= -\sum_{i=1}^{n} \nabla^A_{e_i}\nabla^A_{e_i}\sigma + \sum_{i<j} e_i \cdot e_j \left(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i}\right)\sigma \\
&= \Delta^A\sigma + \frac{1}{2}\gamma(F_A)\sigma \quad \underline{\text{since}(\nabla_{e_i}e_j)_p = [e_i, e_j]_p = 0,}
\end{aligned}$$

then we finish the proof. ♣

Now we apply this Weitzenbock formula to the $E = \wedge T^*M$ case and deduce some interesting results.

Corollary 5.1.1

If ω is a $1-$form on M, then we have

$$\Delta\omega = \Delta^\nabla\omega + \mathrm{Ric}(\omega),$$

here $\Delta = \not{D}^2 = (\mathrm{d}+\mathrm{d}^*)^2$, Δ^∇ is the connection Laplacian for the Levi-Civita connection.

$\boxed{\textit{Remark.}}$ Here Ric $\in \mathrm{End}(T^*M)$ is defined by

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega^\sharp, e_i)e^i = \sum_{j=1}^n g(R(e_j, e_i)\omega^\sharp, e_j)e^i$$

Proof. For the Levi-Civita connection d_∇, and corresponding curvature F_∇, then from Witzenbock formula, we have

$$\Delta\omega = \Delta^\nabla\omega + \frac{1}{2}\gamma(F_\nabla)\omega.$$

Note that

$$\begin{aligned}\frac{1}{2}\gamma(F_\nabla)\omega &= \sum_{i<j} e_i \cdot e_j \circ F_\nabla(e_i, e_j)\omega \\ &= \sum_{i<j} e_i \cdot e_j \cdot g(R(e_i, e_j)\omega^\sharp, e_k)e^k = \sum_{i<j} g(R(e_i, e_j)\omega^\sharp, e_k)e_i \cdot e_j \cdot e^k,\end{aligned}$$

hence we have

$$\begin{aligned}\left(\frac{1}{2}\gamma(F_\nabla)\omega\right)^\sharp &= \frac{1}{2}\sum_{i,j,k=1}^n g(R(e_i, e_j)\omega^\sharp, e_k)e_i \cdot e_j \cdot e_k \\ &= \frac{1}{6}\sum_{i\neq j\neq k\neq i}\left(\underbrace{g(R(e_i, e_j)\omega^\sharp, e_k) + g(R(e_k, e_i)\omega^\sharp, e_j) + g(R(e_j, e_k)\omega^\sharp, e_i)}_{\text{by Bianchi identity }=0}\right) e_i \cdot e_j \cdot e_k \\ &\quad + \frac{1}{2}\sum_{i,j} g(R(e_i, e_j)\omega^\sharp, e_i)e_i \cdot e_j \cdot e_i + \frac{1}{2}\sum_{i,j} g(R(e_i, e_j)\omega^\sharp, e_j)e_i \cdot e_j \cdot e_j \\ &= \frac{1}{2}\sum_{i,j} g(R(e_i, e_j)\omega^\sharp, e_i)e_j + \frac{1}{2}\sum_{i,j} g(R(e_i, e_j)\omega^\sharp, e_j)(-e_i) \\ &= \sum_{i,j} g(R(e_i, e_j)\omega^\sharp, e_i)e_j = \mathrm{Ric}(e_j, \omega^\sharp)e_j.\end{aligned}$$

Then we have

$$\frac{1}{2}\gamma(F_\nabla)\omega = \left(\mathrm{Ric}(e_j, \omega^\sharp)e_j\right)^\flat = \mathrm{Ric}(e_j, \omega^\sharp)e^j = \mathrm{Ric}(\omega),$$

which is the desired result. ♣

Corollary 5.1.2

If (M, g) admits positive Ricci curvature, i.e., $\mathrm{Ric}(v, v) > 0$ for all $v \in TM$ and $v \neq 0$, then we have $b_1(M) = 0$.

Proof. By Hodge's theorem, we have $H^1_{\mathrm{dR}}(M) \cong \mathcal{H}^1(M) = \{\omega \in \Omega^1(M) : \Delta\omega = 0\}$, hence take any $\omega \in \mathcal{H}^1(M)$, then by Weitzenbock formula, we have

$$0 = \Delta\omega = \Delta^\nabla\omega + \mathrm{Ric}(\omega),$$

note that $g(\mathrm{Ric}(\omega), \omega) = \mathrm{Ric}(\omega^\sharp, \omega^\sharp) \geq 0$, hence we have

$$0 = \int_M g(\Delta^\nabla \omega, \omega) + g(\mathrm{Ric}(\omega), \omega)\mathrm{dvol}_M = \int_M g(\nabla\omega, \nabla\omega) + \mathrm{Ric}(\omega^\sharp, \omega^\sharp)\mathrm{dvol}_M \geq 0,$$

hence we know that $\omega^\sharp = 0$, then $\omega = 0$, then we finish the proof. ♣

¶ Dirac operators as differential operators

Easy to see $\not{D}_A : \Gamma(E) \to \Gamma(E)$ can be viewd as a differential operator, since locally we can write

$$\nabla^A = \mathrm{d} + A,$$

where A is some $1-$form matrix, hence we have

$$\not{D}_A = \sum_{i=1}^n e_i \cdot \nabla^A_{e_i} = \sum_{i=1}^n e_i \cdot \frac{\partial}{\partial e_i} + \text{ zero order terms.}$$

Hence for any $\xi \in T^*M$ and locally can be written as $\xi = \xi_i e^i$, then we have the principal symbol of $\not{D}_A$ is

$$\sigma_\xi(\not{D}_A) = \sum_{i=1}^n e_i \cdot (\sqrt{-1}\xi_i) = \sqrt{-1}\xi^\flat \cdot,$$

here $\xi^\flat \cdot$ denotes the Clifford multiplication.

Now note that

$$\sigma_\xi(\not{D}_A^2) = \sigma_\xi(\not{D}_A) \circ \sigma_\xi(\not{D}_A) = -\xi^\flat \cdot \xi^\flat \cdot = g(\xi^\flat, \xi^\flat)\mathrm{Id},$$

hence we know that for all $\xi \neq 0$, we have $\sigma_\xi(\not{D}_A)$ is invertible, hence we know that the Dirac operator is a self-adjoint, ellptic differential operator of order 1.

Then $\not{D}_A$ can be extened as a Fredholm operator from $L^p_{k+1}(E) \to L^p_k(E)$, note that $\not{D}_A$ is self-adjoint, hence we have

$$\mathrm{index}(\not{D}_A) = \dim \mathrm{Ker}\not{D}_A - \dim \mathrm{coker}\not{D}_A = 0.$$

5.2 Spin Structures and Spinor Bundles

In this section, we construct more examples of Clifford modules and Dirac bundles, i.e, we will introduce the so called spin manifolds, and their related spinor bundles.

Our final goal is to prove that these spinor bundles are Dirac bundles, hence we have Dirac operators. And then we can state the famous Atiyah-Singer index theorem with some interesting applications.

5.2.1 Spin groups and spin structures

¶ Introduction

It is well-known that $\pi_1(\mathrm{SO}(k)) \cong \mathbb{Z}/2\mathbb{Z}$ if $k \geq 3$, which is because we have fibration $\mathrm{SO}(k-1) \to \mathrm{SO}(k) \to S^{k-1}$. Then it is natural to ask: what is the double covering(or equivalently the universal covering) of $\mathrm{SO}(k)$?

Definition 5.2.1: Spin group — —the very first def

For all $k \geq 2$, $\mathrm{Spin}(k)$ is the Lie group which is the connected double covering of $\mathrm{SO}(k)$.

Example 5.2.1. *It is well-known that* $\mathrm{SO}(3) \cong \mathbb{R}P^3$*, then we have* $\mathrm{Spin}(3) \cong S^3$*, more precisely, we have* $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$.

Now we can state the definition of the spin structure for a real Riemanian vector bundle $E \to M$ with $\mathrm{rank}E = k$. Since E is Riemanian, we can reduce its frame bundle $\mathrm{Fr}(E)$ to a principal $\mathrm{SO}(k)-$bundle $P_{\mathrm{SO}}(E)$, by choosing the oriented orthogonal basis.

Definition 5.2.2: Spin structure

Suppose $k \geq 2$, then a spin structure on E is a principal $\mathrm{Spin}(k)-$bundle $P_{\mathrm{spin}}(E)$ together with a double covering bundle map

$$\xi : P_{\mathrm{spin}}(E) \to P_{\mathrm{SO}}(E),$$

such that $\xi(p.g) = \xi(p).\rho(g)$, for any $g \in \mathrm{Spin}(k)$ and $\rho : \mathrm{Spin}(k) \to \mathrm{SO}(k)$ is the double covering map.

Before we step further, why we consider such spin structures? Here is a general principle: If a Lie group G has better connectivity, then there are less obstructions for the existence of sections for the principal $G-$bundle. More precisely:

Proposition 5.2.1

Suppose $F \to P \to X$ is a fibration, $\pi_i(F) = 0$ for $i = 1, \cdots, k-1$ and X is a CW complex with dim$X \leq k$, then there is a section $s : X \to P$.

Proof. We can construct the section by induction on cells, suppose X has skeltons $X^0 \subset X^1 \subset \cdots \subset X^k$, then if we already have constructed $s : X^l \to P$, now we want to extend s to $X^{l+1} \to P$, then locally at a $(l+1)-$cell $\mathbb{D}^{l+1}$, the problem is we already have $s : \partial\mathbb{D}^{l+1} = S^l \to F$, can we extend to $s : \mathbb{D}^{l+1} \to F$, then it is easy to see if $\pi_l(F) = 0$ then there is no obstructions. Then we finish the proof. ♣

So now we can see that what we do to consider spin structures is somehow to want have better connectivity:

$$\underset{\pi_0 \neq 0}{\mathrm{GL}(k, \mathbb{R})} \overset{\text{reduce}}{\longrightarrow} \underset{\pi_0 = 0, \pi_1 \neq 0}{\mathrm{SO}(k)} \overset{\text{lift}}{\longrightarrow} \underset{\pi_0 = 0, \pi_1 = 0}{\mathrm{Spin}(k)} .$$

Now a natural question is that can we do further? i.e., to replace Spin(k) by some Lie group G with better connectivity than simply connected?

The answer is NO, since we have

- Simply connected Lie group has $\pi_2 = 0$, one can find the proof in *Morse Theory* of Milnor.
- If Lie group G has $\pi_i = 0$ for $i \leq 3$, then G is conctractible.

Definition 5.2.3

A closed oriented manifold is called spin if its tangent bundle TM admits a spin sturcture.

Example 5.2.2. *All closed oriented manifold with dimesnion ≤ 3 are spin, then we can prove that any oriented three manifold Y is parallezible, i.e., has trivial tangent bundle.*

This is because Spin$(3) \cong S^3$, *hence* $\pi_2(\mathrm{Spin}(3)) = 0$, *then by proposition above, we know that the lift of the frame bundle of* TY, *the* $P_{\text{spin}}(TY)$ *has a global section hence trivial, then we know its associated tangent bundle is trivial.*

Now we have take a glimpse of the spin structures, but we have not even see what is Spin(k)! Our tasks in the following are :

- Give the spin groups a more accurate description;
- Discuss the existence and classification of spin structures.

¶ Spin groups

To define spin groups more accurately, we need Clifford algebras again! Recall for a real Euclidean vertor space (V, g), we have the Clifford algebra is $\mathrm{Cl}(V) = \mathcal{T}(V)/\mathcal{I}$, where $\mathcal{T}(V)$ is the tensor algebra of V and $\mathcal{I}$ is the ideal generated by $v \otimes v + g(v, v)$.

From now on, we take $V = \mathbb{R}^n$ and g is the standard inner product, and wirte Cl_n to denote $\mathrm{Cl}(\mathbb{R}^n)$.

Easy to see Cl_n has natural $\mathbb{Z}/2\mathbb{Z}-$graded, $\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1$, where

$$\mathrm{Cl}_n^i := \mathrm{span}\{e_{i_1} \cdots e_{i_k} | k \equiv i \mod 2\},$$

and it is easy to see $\mathrm{Cl}_n^i \cdot \mathrm{Cl}_n^j \subseteq \mathrm{Cl}_n^{i+j}$.

Exercise 1. Prove that $\mathrm{Cl}_n^0 \cong \mathrm{Cl}_{n-1}$.
$\boxed{\text{Hint:}}$ consider $f : \mathbb{R}^{n-1} \to \mathrm{Cl}_n^0$, $f(e_i) := e_n \cdot e_i$, then extends naturally to $\widetilde{f} : \mathrm{Cl}_{n-1} \to \mathrm{Cl}_n^0$.

Now we can define the spin groups (again):

Definition 5.2.4

- we have $\underline{\mathrm{Pin}(n)} := \{v_1 \cdots v_k : v_i \in V = \mathbb{R}^n, |v_i| = 1\}$;
- then we have $\underline{\mathrm{Spin}(n)} := \mathrm{Pin}(n) \cap \mathrm{Cl}_n^0 = \{v_1 \cdots v_{2k} : v_i \in V, |v_i| = 1\}$.

Example 5.2.3. *For $n = 2$, $\mathrm{Spin}(2) = \{v_1 v_2 : v_i \in \mathbb{R}^2, |v_i| = 1\} \subset \mathrm{Cl}_2 \cong \mathbb{C}$, hence we have $\mathrm{Spin}(2) \cong S^1$.*

Example 5.2.4. *For $n = 3$, note that $\mathrm{Cl}_3^0 \cong \mathrm{Cl}_2 \cong \mathbb{H}$, then we have $\mathrm{Spin}(3) \subseteq \mathbb{H}$, since the element in $\mathrm{Spin}(3)$ has unit norm, then we have $\mathrm{Spin}(3) \subseteq \mathrm{Sp}(1)$ the unit quaternion. We actually have $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$, but this is not easy to see directly from this algebraic definition.*

Now we start the proof of the most important property of spin groups— —it is the double covering of $\mathrm{SO}(n)$, before do that, we need some prelimanaries:

Definition 5.2.5: Twisted adjoint representation

Let $\mathrm{Cl}_n^\times$ denotes the invertivle elements of Cl_n, then we have the <u>twisted adjoint representation</u>:

$$\widetilde{\mathrm{Ad}} : \mathrm{Cl}_n^\times \to \mathrm{GL}(\mathrm{Cl}_n), \quad \varphi \mapsto \widetilde{\mathrm{Ad}}_\varphi := (x \mapsto \alpha(\varphi) x \varphi^{-1}),$$

where $\alpha|_{\mathrm{Cl}_n^i} := (-1)^i \mathrm{Id}$.

Remark. Easy to see $V - \{0\} \subset \mathrm{Cl}_n^\times$, since for $v \neq 0$, we have $v^{-1} = -\frac{v}{g(v,v)}$.

Proposition 5.2.2

If $v \in V - \{0\}$, then $\widetilde{\mathrm{Ad}}_v \in \mathrm{O}(V) \subset \mathrm{End}(V)$, more precisely,

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{g(v,w)}{g(v,v)}v,$$

i.e., the mirror reflection along the hyperplane $v^\perp = \{u \in V : g(u,v) = 0\}$.

Proof. This is direct calculation, by definition

$$\widetilde{\mathrm{Ad}}_v(w) = -v \cdot w \cdot v^{-1} = -v \cdot w \cdot \frac{-v}{g(v,v)},$$

then from $w \cdot v = -v \cdot w - 2g(v,w)$, then we finish the proof. ♣

Note that $\widetilde{\mathrm{Ad}}_{\varphi_1 \cdot \varphi_2} = \widetilde{\mathrm{Ad}}_{\varphi_1} \circ \widetilde{\mathrm{Ad}}_{\varphi_2}$, hence we have $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(n) \to \mathrm{O}(n)$, then restrict to $\mathrm{Spin}(n)$, we have the representation

$$\boxed{\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n), \quad \rho =: \widetilde{\mathrm{Ad}}|_{\mathrm{Spin}(n)}.}$$

Remark. Here we use the fact that $\det \widetilde{\mathrm{Ad}}_v = -1$, hence when $\varphi \in \mathrm{Spin}(n)$, then $\det \widetilde{\mathrm{Ad}}_\varphi = (-1)^{2k} = 1$, where $\varphi = v_1 \cdots v_{2k}$.

Theorem 5.2.1

There is a exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \to 1,$$

here $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ is naturally lies in $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^0$. In fact, $\mathrm{Spin}(n)$ is the connected double covering of $\mathrm{SO}(n)$ for all $n \geq 2$.

Proof. We prove it in the following:

- i is clearly injective; ρ is surjective is provided by a well-known result in linear algebra: $\mathrm{SO}(n)$ is generated by the products of even compositions of mirror reflections.
- Easy to see $\widetilde{\mathrm{Ad}}_{\pm 1}x = (\pm 1) \cdot x \cdot (\pm 1)^{-1} = x$, then $\mathbb{Z}/2\mathbb{Z} \subseteq \mathrm{Ker}\rho$.
- Now we want to prove $\mathrm{Ker}\rho = \mathbb{Z}/2\mathbb{Z}$. Suppose $\rho(\psi) = \mathrm{Id}$ for $\psi \in \mathrm{Spin}(n)$, i.e., for any $v \in V$, we have $\psi \cdot v = v \cdot \psi$. Suppose $\psi = v_1 \cdots v_{2k}$, then use the standard basis $\{e_i\}$, ψ can be writen as $a_0 + e_1 a_1$, here $a_i \in \mathrm{Cl}_n^i$ are polynomials of $e_2, \cdots, e_n$.

Now $\psi \cdot e_1 = a_0 e_1 + e_1 a_1 e_1 = a_0 e_1 - e_1^2 a_1 = a_0 e_1 + a_1$, and $e_1 \cdot \psi = e_1 a_0 - a_1 = a_0 e_1 - a_1$, hence we know $a_1 = 0$. Then by induction we can show that ψ does not contain $\{e_i\}$, i.e., ψ is just a scalar. Since $|\psi| = |v_1 \cdots v_{2k}| = 1$, then we have $\psi = \pm 1$, as desired.

- Now it suffices to prove $\text{Spin}(n)$ is connected for $n \geq 2$, and it is nessary to prove there is a path connecting ± 1. Note that

$$\gamma(t) := \cos t + e_1 \cdot e_2 \sin t = \left(e_1 \cos \frac{t}{2} + e_2 \sin \frac{t}{2}\right) \cdot \left(-e_1 \cos \frac{t}{2} + e_2 \sin \frac{t}{2}\right) \in \text{Spin}(n).$$

Finally, we finish the proof. ♣

$\boxed{\textit{Remark.}}$ Similarly, we have exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \text{pin}(n) \xrightarrow{\widetilde{\text{Ad}}} \text{O}(n) \to 1.$$

Example 5.2.5. *Note that we have* $\text{Spin}(4) \subset \text{Cl}_4^0 \cong \text{Cl}_3 \cong \mathbb{H} \oplus \mathbb{H}$, *then it is not "hard" to see* $\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1)$.

To be more precise, we can identify $\text{Spin}(4)$ *with* $\text{SU}_+(2) \times \text{SU}_-(2)$, *and* $\mathbb{R}^4 = \mathbb{C}^2$, *hence we have*

$$\rho : \text{Spin}(4) \to \text{SO}(4), \quad \begin{pmatrix} A_+ & O \\ O & A_- \end{pmatrix} \mapsto \left(x \mapsto A_- x A_+^{-1}\right),$$

with kernel $\mathbb{Z}/2\mathbb{Z} = \{(\text{Id}, \text{Id}), (-\text{Id}, -\text{Id})\}$.

Now for convienience of further discussions, let's talk about $\rho_{*,e} : \mathfrak{spin}(n) \to \mathfrak{so}(n)$. Since ρ is covering map, then we have $\rho_{*,e}$ gives the isomorphism betwenn these two Lie algebras. Thanks to Clifford algebra, we can write this isomorphicm explicitly:

Firstly, note that $\mathfrak{so}(n) = \{A \in \text{End}(\mathbb{R}^n) : A + A^\mathsf{T} = 0\}$, hence it is generated by skew-symmetric matrices $E_{ij} - E_{ji}$. Note that we can view $\wedge^2 V \cong \mathfrak{so}(n)$ by

$$e_i \wedge e_j : V \to V, \quad (e_i \wedge e_j)(x) := g(e_i, x)e_j - g(e_j, x)e_i.$$

Note that under this identification, $e_i \wedge e_j$ is nothing but $E_{ij} - E_{ji}$.

Now for $A = (A_{ij}) \in \mathfrak{so}(n)$, we have

$$A = \sum_{i,j=1}^n A_{ij} E_{ij} = \frac{1}{2} \sum_{i<j} A_{ij} e_i \wedge e_j.$$

Proposition 5.2.3

$\rho_{*,e}^{-1} : \mathfrak{so}(n) \to \mathfrak{spin}(n) \subset \mathrm{Cl}_n$ is given by

$$\boxed{\rho_{*,e}^{-1}(A) = \rho_{*,e}^{-1}\left(\frac{1}{2}\sum_{i<j} A_{ij} e_i \wedge e_j\right) = \frac{1}{4}\sum_{i<j} A_{ij} e_i \cdot e_j.}$$

Remark. Here $\mathfrak{spin}(n) \subset \mathrm{Cl}_n$ is because Cl_n is vector space, hence we have $T_e\mathrm{Cl}_n = \mathrm{Cl}_n$.

Proof. We firstly prove that $\mathfrak{spin}(n) = \mathrm{span}\{e_ie_j : i < j\}$. In fact, on the one hand consider

$$\gamma(t) := \cos t + e_1 \cdot e_2 \sin t = \left(e_1 \cos \frac{t}{2} + e_2 \sin \frac{t}{2}\right) \cdot \left(-e_1 \cos \frac{t}{2} + e_2 \sin \frac{t}{2}\right) \in \mathrm{Spin}(n),$$

with $\gamma(0) = e$, $\gamma'(0) = e_i \cdot e_j$, then we have $\mathrm{span}\{e_ie_j : i < j\} \subseteq \mathfrak{spin}(n)$. Now by both have dimension $\dfrac{n(n-1)}{2}$, we know they are equal.

Noe note that

$$\rho_{*,e}(e_ie_j)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \widetilde{\mathrm{Ad}}_{\gamma(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\gamma(t) \cdot x \cdot \gamma(t)^{-1}) = \gamma'(0)x - x\gamma'(0).$$

Now suppose $x = x^ie_i$, then we further have

$$e_ie_jx - e_ie_jx = 2g(e_i, x)e_j - 2g(e_j, x)e_i = 2(e_i \wedge e_j)(x).$$

Hence we have $\rho_{*,e}(e_ie_j) = 2e_i \wedge e_j$, then we finish the proof. ♣

¶ Spin Structures

Note that we have seen what is the spin groups, we now want to turn to the next tasks:

- Discuss the existence and classification of spin structures.

Recall the Reimanian vector bundle E admits spin structures if $P_{\mathrm{SO}}(E)$ admits a lift to $P_{\mathrm{spin}}(E)$.

Now fix a good cover of local trivialization of $E \to M$, $\{U_\alpha\}$i.e., all finite intersections of $\{U_\alpha\}$ are contractible, then we know that $P_{\mathrm{SO}}(E)$ is determined by transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(k)$ where $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ satisfies the cocycle condition.

Such a collection $\{g_{\alpha\beta}\}$ defines a Cech cocycle hence an element $[\{g_{\alpha\beta}\}] \in \check{H}^1(\{U_\alpha\}, \mathrm{SO}(k))$. Easy to see two cocycle $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ determines the same principal bundle iff $[\{g_{\alpha\beta}\}] = [\{h_{\alpha\beta}\}] \in \check{H}^1(\{U_\alpha\}, \mathrm{SO}(k))$, and since refinement of $\{U_\alpha\}$ doesn't change the cocycle, hence we have

$$\varinjlim \check{H}^1(\{U_\alpha\}, \mathrm{SO}(k)) = H^1(M, \mathrm{SO}(k))$$

classfies all principal bundles.

Remark. Actually, the discussion above holds for all compact Lie group G, i.e., we have

$$\{\text{Principal } G - \text{bundle over } M\}/\text{iso} \cong H^1(M,G) \cong [M, K(G,1)] = [M, BG],$$

where BG is the classifying space we have discussed.

Now back to spin case, locally, since $U_\alpha \cap U_\beta$ is simply connected, then there exists $\widetilde{g}_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \text{Spin}(k)$, such that:

$$\begin{tikzcd} & \text{Spin}(k) \arrow[d, "\rho"] \\ U_\alpha \cap U_\beta \arrow[ur, dashed, "\widetilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] & \text{SO}(k) \end{tikzcd}$$

But the cocycle condition may not hold, we only have

$$\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1,$$

which implies that $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \in \{\pm 1\} \cong \text{Ker}\rho$. Hence easy to see

Proposition 5.2.4

E admits spin structures iff there exists $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Spin}(k)$ such that $\{\tilde{g}_{\alpha\beta}\}$ satifies the cocycle condition.

So to measure the failure, we define

$$\omega = \{\omega_{\alpha\beta\gamma}\}, \quad \omega_{\alpha\beta\gamma} := \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}/2\mathbb{Z}.$$

Proposition 5.2.5

We have $\omega \in C(\{U_\alpha\}; \mathbb{Z}/2\mathbb{Z})$ is cocycle, i.e., $\delta\omega = 1$, hence we have $[\omega] \in H^2(M; \mathbb{Z}/2\mathbb{Z})$.

Proof. Note that $\omega_{\alpha\beta\gamma} = \omega_{\alpha\beta\gamma}^{-1} \in \mathbb{Z}/2\mathbb{Z}$ is scalar, hence we have $\omega_{\alpha\beta\gamma} = \omega_{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)}$, where σ is any permutation. And note that $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\alpha} = 1$, we can directly calculate that

$$\begin{aligned}
(\delta\omega)_{\alpha\beta\gamma\eta} &= \omega_{\beta\gamma\eta}\omega_{\alpha\gamma\eta}^{-1}\omega_{\alpha\beta\eta}\omega_{\alpha\beta\gamma}^{-1} \\
&= \widetilde{g}_{\eta\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\eta} \cdot \widetilde{g}_{\eta\gamma}\widetilde{g}_{\gamma\alpha}\widetilde{g}_{\alpha\eta} \cdot \widetilde{g}_{\eta\alpha}\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\eta} \cdot \omega_{\alpha\beta\gamma} \\
&= \widetilde{g}_{\eta\beta} \cdot \widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha}\widetilde{g}_{\alpha\beta} \cdot \widetilde{g}_{\beta\eta} \cdot \omega_{\alpha\beta\gamma} \\
&= \widetilde{g}_{\eta\beta} \cdot \omega_{\alpha\beta\gamma} \cdot \widetilde{g}_{\beta\eta} \cdot \omega_{\alpha\beta\gamma} = 1,
\end{aligned}$$

where the last equality holds because $\omega_{\alpha\beta\gamma}$ is a scalar, then we finish the proof. ♣

Definition 5.2.6: Second Stiefel-Whitney class

$w_2(E) := [\omega] \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is called the **second Stiefel-Whiteny class**.

To sum up and show the definition above is well, we need to prove the following:

Theorem 5.2.2: Obstruction and Classification of spin structures

1) $w_2(E)$ is independent on the choice of the lift $\{\widetilde{g}_{\alpha\beta}\}$;

2) E admits a $\boxed{\text{spin structure}}$ if and only if $\boxed{w_2(E) = 0 \in H^2(M; \mathbb{Z}/2\mathbb{Z})}$;

3) If $w_2(E) = 0$, then we have $\{\text{distinct spin structures on } E\}/\text{iso} \cong H^1(M; \mathbb{Z}/2\mathbb{Z})$.

$\boxed{\textit{Remark.}}$ Here $w_2(E) = 0$ is under the meaning of $\mathbb{Z}/2\mathbb{Z} = \{0, \bar{1}\}$, one should note the differnece we use above as $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$.

Proof. 1) Let $\{\widetilde{g}'_{\alpha\beta}\}$ be another lif, then there exists $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}/2\mathbb{Z}$ such that $\widetilde{g}'_{\alpha\beta} = f_{\alpha\beta}\widetilde{g}_{\alpha\beta}$, then we have $\omega'_{\alpha\beta\gamma} = f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha}\omega_{\alpha\beta\gamma}$, hence we have

$$\omega'_{\alpha\beta\gamma}\omega^{-1}_{\alpha\beta\gamma} = (\delta f)_{\alpha\beta\gamma}, \quad \text{i.e., } [\omega'] = [\omega] \in H^2(M; \mathbb{Z}/2\mathbb{Z}).$$

2) If E admits a spin structure, then we can choose lift $\widetilde{g}_{\alpha\beta}$ such that $\omega_{\alpha\beta\gamma} = 1$, then $w_2(E) = [\omega] = 0 \in H^2(M; \mathbb{Z}/2\mathbb{Z})$.

Now if $w_2(E) = 0$, then $[\omega] = 0$ we know that ω is a coboundary, hence $\omega = \delta f$, then we can modify the origin lift $\widetilde{g}_{\alpha\beta}$ by $\widetilde{g}'_{\alpha\beta} := f_{\alpha\beta}\widetilde{g}_{\alpha\beta}$. Then we have

$$\omega'_{\alpha\beta\gamma} = f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha}\omega_{\alpha\beta\gamma} = \omega^2_{\alpha\beta\gamma} = 1,$$

i.e., $\{\widetilde{g}'_{\alpha\beta}\}$ satisfies cocycle condition, hence gives a spin structures.

3) Roughly speaking, the $[\{f_{\alpha\beta}\}] \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ determines the different spin structures, we omit the details here. ♣

Example 5.2.6. *The manifold M is spin iff $w_2(M) = 0$, by Wu's formula, we have for oriented manifold with dimension ≤ 3, we have $w_2(M) = 0$ hence spin.*

Example 5.2.7. *Four closed oriented simply connected smooth four manifold X, TFAE:*

- *X is spin;*
- *the normal bundle of every compact oriented surface Σ embedded in X has even Euler class;*
- *the intersection form of X is even, i.e., for any $[\Sigma] \in H_2(X; \mathbb{Z})$, $[\Sigma] \cdot [\Sigma]$ is even.*

The last two equivalent is trivial since we have Hopf index theorem $[\Sigma] \cdot [\Sigma] = e(N\Sigma)[\Sigma]$, and any $a \in H_2(X;\mathbb{Z})$, a can be represented by a embedded oriented surface.

For the first two, note that for $i : \Sigma \hookrightarrow X$, we have $i^* w_2(X) = w_2(T\Sigma \oplus N\Sigma) = w_2(N\Sigma)$ since Σ is spin, then we have

$$w_2(X)\,(i_*[\Sigma]) = i^* w_2(X)[\Sigma] = w_2(N\Sigma)[\Sigma] = e(N\Sigma)[\Sigma] \mod 2,$$

note that $H_2(X;\mathbb{Z})$ is generated by $i_*[\Sigma]$, then we have $w_2(X) = 0$ if and only if $e(N\Sigma)[\Sigma]$ is even for any Σ.

Example 5.2.8. *$\mathbb{C}P^2$ is not spin, since its intersection form is (1); $S^2 \times S^2, T^4$ are spin.*

5.2.2 Spinor bundles and Dirac operators

From now on, we assume M is spin manifod, and P_{spin} is the lift of TM. Then we can use the repesentation of the Clifford algebra to construct the so called spinor bundle, which we will soon prove that is a Dirac bundle for M.

Firstly, let's recall the result of Clifford representation: For any $n \in \mathbb{N}$, let $\Delta_n := \mathbb{C}^{[\frac{n}{2}]}$, we have

- If $n = 2k$, there exists only irreducible representation (Δ_n, ρ_n) of Cl_n;
- If $n = 2k+1$, there exists two irreducible representation (Δ_n, ρ_n) and $(\Delta_n, -\rho_n)$ of Cl_n.

Restrict this representation to $\text{Spin}(n) \subset \text{Cl}_n$, we have the representation $\rho_n : \text{Spin}(n) \to \text{End}(\Delta_n)$, we call it is **spinor representation**.

Definition 5.2.7

Let M be spin manifold, then the associated complex vector bundle

$$S := P_{\text{spin}} \times_{\rho_n} \Delta_n$$

is called the **spinor bundle**. The section of S is called **spinors**.

We now want to prove that $S \to M$ is a Dirac bundle of M, we prove in two steps: first prove it admits Clifford multiplication, hence is a Clifford module. Then find a connection on S compatible with Clifford multiplication, hence a Dirac bundle.

Firstly

Theorem 5.2.3

The spinor bundle S is the Clifford module of M.

Proof. We need to construct a bundle map $\gamma : \text{Cl}(TM) \otimes S \to S$, note that we have $\rho_n : \text{Cl}_n \otimes \Delta_n \to \Delta_n$, and $\text{Cl}(TM) = P_{\text{spin}} \times_{\text{Ad}} \text{Cl}_n$, $S = P_{\text{spin}} \times_{\rho_n} \Delta_n$, hence

$$\text{Cl}(TM) \otimes S = P_{\text{spin}} \times_{\text{Ad} \otimes \rho_n} (\text{Cl}_n \otimes \Delta_n)\,.$$

We can also construct the map $\tilde{\gamma} : P_{\text{spin}} \times (\text{Cl}_n \otimes \Delta_n) \to P_{\text{spin}} \times \Delta_n$ by $(p, v \otimes \psi) \mapsto (p, v \cdot \psi)$, here $v \cdot \psi := \rho_n(v)\psi$. Then to prove the $\tilde{\gamma}$ can be reduced to the Clifford multiplication on the associated bundle, it suffices to prove that $\tilde{\gamma}$ is $\text{Spin}(n)-$equivariant.

More precisely, we want for any $g \in \text{Spin}(n)$, the following diagram commutative:

$$\begin{tikzcd} P_{\text{spin}} \times (\mathrm{Cl}_n \otimes \Delta_n) \arrow[r, "\tilde{\gamma}"] \arrow[d, "\cdot g"'] & P_{\text{spin}} \times \Delta_n \arrow[d, "\cdot g"] \\ P_{\text{spin}} \times (\mathrm{Cl}_n \otimes \Delta_n) \arrow[r, "\tilde{\gamma}"'] & P_{\text{spin}} \times \Delta_n \end{tikzcd}$$

From $(p, v\otimes\psi)$, on the top we have $(p, v\otimes\psi) \mapsto (p, v\cdot\psi) \mapsto (p.g, g\cdot v\cdot\psi)$, here $g\cdot v := \rho_n(g)\psi$, and on the bottom we have $(p, v\otimes\psi) \mapsto (p, gvg^{-1} \otimes g\cdot\psi) \mapsto (p, gvg^{-1}\cdot g\psi) = (p, g\cdot v\cdot\psi)$, then we know the digram commutes and we finish the proof. ♣

Exercise 2. Prove that $\mathrm{Cl}(TM) = P_{\text{spin}} \times_{\text{Ad}} \mathrm{Cl}_n$.

Now to prove S is a Dirac bundle, we need to construct a connection on S. Since S is the associated bundle of P_{spin}, it suffices to give a connection on P_{spin}.

Recall a connection of P_{spin} lies in $\Omega^1(\mathrm{P}_{\text{spin}}, \mathfrak{spin}(n))$. Recall we have $\xi : P_{\text{spin}} \to P_{\text{SO}}$ is a double covering map, and $\rho_{*,e} : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ gives the isomorphism, we can construct a connection from P_{SO} by pullback and identify $\mathfrak{spin}(n)$ with $\mathfrak{so}(n)$!

Now the most natural/canonical connection on P_{SO} is the Levi-Civita connection

$$\widehat{\omega} \in \Omega^1(P_{\text{SO}}, \mathfrak{so}(n)),$$

hence we can define:

Definition 5.2.8: Spin connection

The **spin connection** on P_{spin} is $\xi^*\widehat{\omega} \in \Omega^1(P_{\text{spin}}, \mathfrak{spin}(n))$, here we use $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$.

Recall from the connection on principal bundle, we can define the connection on associted bundle $S = P_{\text{spin}} \times_{\rho_n} \Delta_n$, the ∇^S, by

$$\pi^*\nabla^S\psi := \mathrm{d}\widehat{\psi} + \xi^*\widehat{\omega} = \mathrm{d}\widehat{\psi} + \rho_{n*,e}\left(\xi^*\widehat{\omega}\right)\widehat{\psi}.$$

It is really abstract! But once we recall some basic Riemanian geometry soon, we can write down the explict expression of ∇^S.

Locally on $U \subseteq M$, choose a orthognol basis $\{e_i\}$, we have connection form

$$\nabla e_i = \sum_{i=1}^{n} \omega_{ji} e_i, \quad \omega_{ji} \in \Omega^1(U),$$

and since ∇ is Levi-Civita, then we have $\omega = (\omega_{ij})$ the connection $1-$from matrix is skew-symmetric, and no surprinsingly, $\pi^*\omega = \widehat{\omega}$.

Recall $e_i \wedge e_j$ forms all skew-symmetric matrices on $TM|_U$, where

$$(e_i \wedge e_j)(v) := g(e_i, v)e_j - g(e_j, v)e_i,$$

easy to see $e_i \wedge e_j$ coresponds to $E_{ij} - E_{ji}$, hence we can rewirte the connection $1-$form ω as

$$\omega = (\omega_{ij}) = \frac{1}{2} \sum_{i,j=1}^{n} \omega_{ij} \otimes e_i \wedge e_j.$$

And locally, we have

$$\begin{aligned} \nabla_{e_k} X &= (e_k X^i) e_i + X^i \nabla_{e_k} e_i \\ &=: e_k(X) + \sum_{i,j=1}^{n} X^i \omega_{ji}(e_k) e_j \\ &= e_k(X) + \frac{1}{2} \sum_{i,j=1}^{n} \omega_{ij}(e_k) \cdot (e_i \wedge e_j)(X). \end{aligned}$$

Hence bu pullback, and use $\rho_{*,e} : \mathfrak{spin}(n) \to \mathfrak{so}(n)$, we can write ∇^S in basis of $\{e_i\}$, but firstly, recall we have got

$$\boxed{\rho_{*,e}^{-1}\left(\frac{1}{2}\sum_{i<j} A_{ij} e_i \wedge e_j\right) = \frac{1}{4}\sum_{i<j} A_{ij} e_i \cdot e_j,}$$

then we have for any $\psi \in \Gamma(S)|_U$, we have

$$\nabla^S_{e_k} \psi = e_k(\psi) + \frac{1}{4} \sum_{i,j=1}^{n} \omega_{ij}(e_k) e_i \cdot e_j \cdot \psi,$$

then by the linerity, for any $X \in \mathfrak{X}(U)$, we finally have

$$\boxed{\nabla^S_X \psi = e_k(\psi) + \frac{1}{4} \sum_{i,j=1}^{n} \omega_{ij}(X) e_i \cdot e_j \cdot \psi.}$$

Use this local expression, we can prove

Theorem 5.2.4

With the spin connection ∇^S, the spinor bundle (S, γ) is a Dirac bundle over M. More proecisely, ∇^S is compatible with Clifford multiplication.

Proof. Use local expression, for any $X, v \in \mathfrak{X}(U)$, $\psi \in \Gamma(S)$, we have

$$\begin{aligned} \nabla^S_X (v \cdot \psi) &= X(v \cdot \psi) + \frac{1}{4} \sum_{i,j=1}^{n} \omega_{ij}(X) e_i \cdot e_j \cdot v \cdot \psi \\ &= (Xv) \cdot \psi + v \cdot (X\psi) + \frac{1}{4} \sum_{i,j=1}^{n} \omega_{ij}(X) e_i \cdot e_j \cdot v \cdot \psi. \end{aligned}$$

Note that we have

$$\begin{aligned} e_i \cdot e_j \cdot v &= e_i \cdot (-v \cdot e_j - 2g(e_j, v)) \\ &= -e_i \cdot v \cdot e_j - 2g(e_j, v) e_i \\ &= v \cdot e_i \cdot e_j + 2g(e_i, v) e_j - 2g(e_j, v) e_i. \end{aligned}$$

Hence we have

$$\nabla^S_X(v\cdot\psi) = v\cdot(X\psi) + v\cdot\left(\frac{1}{4}\sum_{i,j=1}^n \omega_{ij}(X)e_i\cdot e_j\cdot\psi\right)$$
$$+ (Xv)\cdot\psi + \frac{1}{2}\sum_{i,j=1}^n \omega_{ij}(X)\left(g(e_i,v)e_j - g(e_j,v)e_i\right)\cdot\psi$$
$$= v\cdot\nabla^S_X\psi + \nabla_X v\cdot\psi,$$

then we finally finish the proof. ♣

Summary 2

If M is spin, then the spinor bundle S is a Dirac bundle for M.

Now we can define our favorite object: Dirac operator! On spinor bundle, we have

$$\not{D}:\Gamma(S)\to\Gamma(S),\quad \not{D}:=\sum_{i=1}^n e_i\cdot\nabla^S_{e_i}.$$

¶ Twisted Dirac operators

Easy to see, if M is spin, $E\to M$ is an Hermitian vector bundle with a unitary connection d_A, then consider the Hermitian vector bundle $S\otimes E$, we can also view it as a Dirac bundle:

- Cliffor multiplication: $v\cdot(\psi\otimes\sigma) := (v\cdot\psi)\otimes\sigma$;
- Conisder tensor connection: $\nabla^A := \nabla^S\otimes 1 + 1\otimes \mathrm{d}_A$.

Then we can form the **twisted Dirac opertors** $\not{D}_A:\Gamma(S\otimes E)\to\Gamma(S\otimes E)$, by

$$\not{D}_A := \sum_{i=1}^n e_i\cdot\nabla^A_{e_i},$$

more precisely, we have

$$\not{D}_A(\psi\otimes\sigma) = \sum_{i=1}^n e_i\cdot\nabla^A_{e_i}(\psi\otimes\sigma) = \sum_{i=1}^n e_i\cdot\left(\nabla^S_{e_i}\psi\otimes\sigma + \psi\otimes\mathrm{d}_A\sigma(e_i)\right).$$

Recall for general Dirac bundle, we have Weitzenbock formula

$$\not{D}_A^2 = \Delta^A + \frac{1}{2}\gamma(F_A),$$

recall here $\frac{1}{2}\gamma(F_A) := \sum_{i<j}^n e_i\cdot e_j\circ F_A(e_i,e_j)$. Now for Dirac bundle $S\otimes E$, and the curvature R_A for ∇^A, we have $R_A = R^S\otimes 1 + 1\otimes F_A$, here R^S is the curvature for ∇^S. Hence we have

$$\frac{1}{2}\gamma(R_A) = \sum_{i<j} e_i\cdot e_j\circ\left(R^S(e_i,e_j)\otimes 1 + 1\otimes F_A(e_i,e_j)\right)$$
$$= \left(\sum_{i<j} e_i\cdot e_j\circ R^S(e_i,e_j)\right)\otimes 1 + 1\otimes\frac{1}{2}\gamma(F_A).$$

Now it remians to calculate $\sum_{i<j} e_i \cdot e_j \circ R^S(e_i, e_j) = \frac{1}{2}\sum_{i,j=1}^n e_i \cdot e_j \circ R^S(e_i, e_j)$.

Exercise 3. Prove that

$$R^S(e_i, e_j) = \frac{1}{4}\sum_{k,l=1}^n R_{ijkl} e_k \cdot e_l,$$

here $R_{ijkl} := g\left(F_\nabla(e_i, e_j)e_k, e_l\right)$ is the curvature for Levi-Civita connection ∇.

Hint: use $F_\nabla = \mathrm{d}\omega + \omega \wedge \omega$, and $R^S(e_i, e_j) = \nabla^S_{e_i}\nabla^S_{e_j} - \nabla^S_{e_j}\nabla^S_{e_i} - \nabla^S_{[e_i, e_j]}$. For more details, one can refer the book of Weiping Zhang, proposition 11.5 on page 191.

Now we have

$$\begin{aligned}
\frac{1}{2}\sum_{i,j=1}^n e_i \cdot e_j \circ R^S(e_i, e_j) &= \frac{1}{8}\sum_{i,j,k,l=1}^n R_{ijkl} e_i \cdot e_j \cdot e_k \cdot e_l \\
&= \frac{1}{8}\sum_{l=1}^n \left\{ \frac{1}{3}\sum_{i\neq j\neq k\neq i} \left(\underbrace{R_{ijkl} + R_{kijl} + R_{jkil}}_{\text{by Bianchi identity } = 0} \right) e_i \cdot e_j \cdot e_k \right\} \cdot e_l \\
&\quad + \frac{1}{8}\left(\sum_{i,j=1}^n R_{ijil} e_i \cdot e_j \cdot e_i + \sum_{i,j=1}^n R_{ijjl} e_i \cdot e_j \cdot e_j \right) \cdot e_l \\
&= \frac{1}{4}\sum_{i,j,l=1}^n R_{ijil} e_j \cdot e_l = -\frac{1}{4}\sum_{i,j,l=1}^n R_{ijli} e_j \cdot e_l \\
&= -\frac{1}{4}\sum_{j,l=1}^n \mathrm{Ric}_{jl} e_j \cdot e_l = \frac{1}{4}\sum_{j=1}^n \mathrm{Ric}_{jj} = \frac{1}{4}s_g,
\end{aligned}$$

where s_g denotes the saclar curvature of (M, g), and we use $\mathrm{Ric}_{jl} = \mathrm{Ric}_{lj}$.

Finally, we get

Proposition 5.2.6: Witzenbock formula for twosted Dirac operator

Let M spin with spinor bundle (S, γ), $E \to M$ is Hermitian complex vector bundle with unitary connection d_A, then for the twisted Dirac opertor $\not{D}_A : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$, we have

$$\boxed{\not{D}_A^2 = \Delta^A + \frac{1}{4}s_g + \frac{1}{2}\gamma(F_A).}$$

Remark. Here to be more precise, $\frac{1}{4}s_g = \frac{1}{4}s_g \otimes 1$, and $\frac{1}{2}\gamma(F_A) = 1 \otimes \frac{1}{2}\gamma(F_A)$.

Remark. For E trivial, then we also get

$$\boxed{\not{D}^2 = \Delta^S + \frac{1}{4}s_g}$$

for spin manifold M.

5.2.3 Atiyah-Singer index theorem

What is Atiyah-Singer index theorem? Roughly speaking, it is of the form

$$\text{analytic index} = \text{topological index} = \int_{\text{manifold}} \text{characteristic classes}.$$

In our setting, i.e., the spin manifold, the analytic index we want to study is the index of Dirac opertor. Generally, for a Freholm opertor $T : X \to Y$, its index is defined by

$$\mathrm{Ind}T := \dim \mathrm{Ker}T - \dim \operatorname{coker} T.$$

Since $\not{D}_A$ is elliptic (we have calculate its symobol in the last section), hence it can naturally be viewed as a Fredholm operator after we take Sobolev completion. Howerver, since $\not{D}_A$ is self-adjoint, then we have $\mathrm{Ind}\not{D}_A \equiv 0$, no too much interests.

Luckily, when $n = 4k$ (and from now on we fix this assumptation), we can consider the volume form $\omega := e_1 \cdots e_n$, easy to see $\omega^2 = (-1)^{\frac{n(n+1)}{2}} = 1$, and moreover

$$\omega \cdot e_{i_1} \cdots e_{i_l} = (-1)^{l(n-1)} e_{i_1} \cdots e_{i_l} \cdot \omega,$$

hence $\omega \cdot v = -v \cdot \omega$ for any $v \in V$, i.e., ω anti-commutes with V.

Now turn to the spinor representation, we can spilt $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$, where

$$\Delta_n^\pm = \{\psi : \omega \cdot \psi = \pm\psi\},$$

hence we also have for spinor bundle $S = S^+ \oplus S^-$. Corresponding to **positive/neagive spinor bundles**, which sections are called **positive/negative spinors**.

Proposition 5.2.7

$\not{D}$ anti-commutes with ω.

Proof. Locally at p, choose $(\nabla_{e_i} e_j)_p = 0$, then we have at p,

$$\not{D} \circ \omega \cdot \psi = \sum_{i=1}^n e_i \cdot \nabla^S_{e_i}(\omega \cdot \psi) = \sum_{i=1}^n e_i \cdot \omega \nabla^S_{e_i} \psi = -\omega \cdot \not{D}\psi,$$

then we finish the proof. ♣

From the proposition above, we know that $\not{D} : \Gamma(S^\pm) \to \Gamma(S^\mp)$, more precisely, we have

$$\not{D} = \begin{pmatrix} O & \not{D}^- \\ \not{D}^+ & O \end{pmatrix}, \quad \not{D}^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp).$$

Note that $(\not{D}^\pm)^* = \not{D}^\mp$, since $\not{D}^* = \begin{pmatrix} O & \left(\not{D}^+\right)^* \\ \left(\not{D}^-\right)^* & O \end{pmatrix} = \not{D}$.

Simarly for the twisted Dirac bundle $S\otimes E$ and twisted Dirac operators $\not{D}_A$, we have $S\otimes E = (S^+\otimes E)\oplus(S^-\otimes E)$, and $\not{D}_A = \begin{pmatrix} O & \not{D}_A^- \\ \not{D}_A^+ & O \end{pmatrix}$.

Hence now we can define the really intersting index for Dirac operators:

Definition 5.2.9: Index of Dirac operators

Note that $S\otimes E$ are complex vector bundle, hence we have **index of Dirac operators** $\not{D}_A$ is given by

$$\mathrm{Ind}\not{D}_A^+ := \mathrm{Ind}_{\mathbb{C}}\not{D}_A^+ := \dim\ker\not{D}_A^+ - \dim\operatorname{coker}\not{D}_A^+.$$

Remark. Note that $\left(\not{D}_A^+\right)^* = \not{D}_A^-$, hence we actually have $\mathrm{Ind}\not{D}_A^+ = \dim\ker\not{D}_A^+ - \dim\ker\not{D}_A^- \in \mathbb{Z}$.

Now we can state the famous **Atiyah-Singer index theorem**:

Theorem 5.2.5: Atiyah-Singer index theorem

Let M be a $4k$ dimension spin manifold, (S,γ) is the spinor bundle over M, E is any Hermitian complex vector bundle over M with unitary connection d_A, then we have

$$\boxed{\mathrm{Ind}\not{D}_A^+ = \int_M \widehat{A}(M)\wedge \mathrm{ch}(E).}$$

Remark. Recall we have define the $\hat{A}-$genus and chern character, but we will mainly focus on the four dimensional, so we just take

- $\widehat{A}(M) = 1 - \frac{1}{24}p_1(M) \in H^*(M;\mathbb{R})$, here $p_1(M) = -c_1(TM\otimes\mathbb{C})$;
- $\mathrm{ch}(E) = \mathrm{rank}E + c_1(E) + \frac{1}{2}\left(c_1^2(E) - c_2(E)\right)$,

hence in $M = X^4$, the four dim case, we can rewirte the Atiyah-Singer index theorem as

$$\mathrm{Ind}\not{D}_A^+ = \int_X -\frac{1}{24}p_1(M) + \frac{1}{2}\left(c_1^2(E) - c_2(E)\right).$$

Recall we have

Theorem 5.2.6: Hirzebruch Signature theorem

For X closed oriented four manifold, we have

$$\sigma(X) = -\frac{1}{3}\int_X p_1(X),$$

where $\sigma(X)$ is the signature of X.

Finally, we can get the following easy to calculate formula:

$$\boxed{\mathrm{Ind}\not{D}_A^+ = -\frac{1}{8}\sigma(X)\cdot \mathrm{rank}E + \frac{1}{2}\int_X \left(c_1^2(E) - c_2(E)\right).}$$

Corollary 5.2.1

For X^4 is a spin manifold, then we have $\sigma(X) \equiv 0 \mod 8$.

Proof. Take $E = \mathbb{C}$ the trivial line bundle, then we have $\frac{1}{8}\sigma(X) = -\mathrm{Ind}\not{D}^+ \in \mathbb{Z}$. ♣

Now we see two more interesting applications of Atiyah-Singer index theorem.

¶ Topological obstruction of PSC metric

Definition 5.2.10

Let M^{4k} be spin, with spinor bundle $(S = S^+ \oplus S^-, \gamma)$, for the Dirac operator $\not{D}$, we call the section of ker $\not{D}$ is **harmonic spinors**, and ker $\not{D}^{\pm}$ is **harmonic positive/negative spniors**,

Remark. Easy to see ker $\not{D} = \{0\}$ iff ker $\not{D}^{\pm} = \{0\}$.

Theorem 5.2.7: Licherowicz

If (M^{4k}, g) is closed oriented spin manifold with **positive scalar curvature**, then M has no nontrivial harmonic spinors.

Remark. we call g is **PSC metric** if it has positive scalar curvature.

Proof. Suppose $\psi \in$ ker $\not{D}$, then recall we have Witzenbock formula:

$$\not{D}^2 = \Delta^S + \frac{1}{4}s_g,$$

here $s_g > 0$. Then we have

$$0 = \not{D}^2\psi = \Delta^S\psi + \frac{1}{4}s_g\psi,$$

hence for the Hermitian metric $\langle\ ,\ \rangle$ on S, we have

$$0 = \int_M \left(\langle \Delta^S, \psi\rangle + \frac{1}{4}s_g\langle\psi,\psi\rangle\right)\mathrm{dvol}_X = \int_M \left(|\nabla^S\psi|^2 + \frac{1}{4}s_g|\psi|^2\right)\mathrm{dvol}_X,$$

then we know that $|\psi| \equiv 0$, then $\psi = 0$, i.e., there is no nontrivial harmonic spinors. ♣

Note that ker $\not{D} = \{0\}$ iff ker $\not{D}^{\pm} = \{0\}$, then if M has nontrivial harmonic spinors, then we have

$$\mathrm{Ind}\not{D}^+ = \dim\ker\not{D}^+ - \dim\ker\not{D}^- = 0.$$

Now apply Atiyah singer index theorem, we have:

Theorem 5.2.8: Licherowicz

If X^4 is a closed oriented spin manifold, with PSC metric g, then $\sigma(X) = 0$.

Proof. This is directly from $\operatorname{Ind} \not{D}^+ = -\frac{1}{8}\sigma(X)$. ♣

Remark. In summary, we know that the signature is a topological obstruction for spin four manifold.

Example 5.2.9. *Conider Z_d the degree d hypersurface in $\mathbb{C}P^3$, then we have calculated that $c_1(Z_d) = (4-d)i^*\mathrm{PD}.[\mathbb{C}P^1]$, hence $w_2(Z_d) \equiv (4-d) \mod 2$, hence we know Z_d is spin iff d is even.*

Note that $\sigma(Z_d) = \frac{d(4-d^2)}{3}$, hence if Z_d admits PSC metric, then $\sigma(Z_d) = 0$ i.e., $d = 2$. Actually, when $Z_2 \cong S^2 \times S^2$, trivially holds PSC metric.

For $d \neq 2$, we know that Z_d cannot admit PSC metric, the most interesting case is Z_4, the $K3$ surface, it is Calabi-Yau, hence admits a Ricci flat metric, i.e., have zero scalar curvature.

Remark. If we drop the condition of spin, then the result does not hold in general: Consider the non-spin four manifold $\mathbb{C}P^2$, with the following PSC metric, but $\sigma(\mathbb{C}P^2) = 1 \neq 0$.

Exercise 4. Define Remannian metric on $\mathbb{C}^{n+1} - \{0\}$ in the following way: if $Z \in \mathbb{C}^{n+1} - \{0\}$, then for any $V, W \in T_Z\left(\mathbb{C}^{n+1} - \{0\}\right)$, we define

$$g_Z(V, W) := \frac{\operatorname{Re}\langle V, W\rangle}{\langle Z, Z\rangle}.$$

Prove that this metric induces a metric $\widetilde{g}$ on $\mathbb{C}P^n$, and the sectional curvature sec of $\widetilde{g}$ satisfies

$$1 \leq \sec(\sigma) \leq 4, \quad \sigma = \operatorname{span}\{X, Y\}.$$

Hence $s_{\widetilde{g}} > 0$, i.e., $\mathbb{C}P^n$ admits PSC metric.

Hint: this is exercise 8.12 in Do Carmo's book.

¶ The topological manifold has no differential structure

We will firstly prove that

Theorem 5.2.9: Rochlin

The signature of closed oriented spin four manifold X satisfies

$$\sigma(X) \equiv 0 \mod 16.$$

Remark. Note that we have got $\sigma(X) \equiv 0 \mod 8$, this is a refinement.

Example 5.2.10. *For Z_d and d is even, hence Z_d is spin, and note that $\sigma(Z_d) = \dfrac{d(4-d^2)}{3}$. Let $d = 2k$, then we have*

$$\sigma(Z_d) = \frac{2k(4-4k^2)}{3} \equiv 0 \mod 16 \iff k(k-1)(k+1) \equiv 0 \mod 6,$$

which trivially holds.

Proof. After identify $\mathrm{Cl}_4 \cong \mathbb{H}(2)$, and $\Delta_4 = \mathbb{C}^4 = \mathbb{H}^2$, $\mathrm{Spin}(4) = \mathrm{sp}(1) \times \mathrm{sp}(1)$, we have $S \to X$ is quatrnoic bundle of rank 2, and $S^{\pm}$ are quarternoic line bundle.

Note that $\not{D}, \not{D}^{\pm}$ is compatible with quarternoic multiplication, hence we have $\ker \not{D}^{\pm}$ can be viewed as quarternoic space. Hence $\dim_{\mathbb{C}} \ker \not{D}^{\pm}$ are even, then we have

$$-\frac{1}{8}\sigma(X) = \mathrm{Ind}\not{D}^{+} = \dim_{\mathbb{C}} \ker \not{D}^{+} - \dim \ker \not{D}^{-}$$

is even, then we have $\sigma(X) \equiv 0 \mod 16$ as desired. ♣

Example 5.2.11. *Recall for smooth closed oriented simply connected four manifolds, spin is a topological condition which is equivalent to its intersection form is even.*

Hence we have for any smooth and simply connected four manifold X with intersection even, then $\sigma(X) \equiv 0 \mod 16$. For a algebraic topology proof, one can refer this note.

Now we can deduce a striking result: there is a topological four manifold that can NOT be smoothable.

Example 5.2.12. *Firstly, reall we have the unimodular integral bilinear form*

$$E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix},$$

easy to see E_8 is positive-definite, even, and of signature 8.

Recall we have Freedman's amazing result: given any unimodular interal bilinear form Q, there exists a closed simply connected topological manifold X_Q with intersection form Q.

Now consider topological manifold X_{E_8}, the ***E8 manifold****, since its intersection is even, then if X_{E_8} admits smooth structure, from Rochlin's theorem we know that $\sigma(X_{E_8}) = 8 \equiv 0 \mod 16$, which is a contradiction!*

5.3 Spin^c Structures and Spinor Bundles

6 Seiberg-Witten Moduli Spaces

6.1 Seiberg-Witten Equations

Firstly, let's recall our baisc settings and collect the results from the previous that we will need in this chapter:

- (X, g) is a closed oriented smooth Riemanian $4-$manifold, with spin^c structure $\mathfrak{s} = (S, \gamma)$, where $S = S^+ \oplus S^-$ is the corresponding spinor bundles.

  And $\gamma : TX \to \mathrm{Hom}(S^\pm, S^\mp)$ is the Clifford multiplictation.

  - We have characteristic line bundle $L_\mathfrak{s} = \det S^+ = \det S^-$, and then $\det S = L_\mathfrak{s}^{\otimes 2}$. The class of $\mathfrak{s}$ is $c_1(\mathfrak{s}) := c_1(L_\mathfrak{s})$.
  - The spin^c structures over X up to isomorphism is denoted by Spin$^c(X)$, then it is an affine space modelled on $H^2(X; \mathbb{Z})$, which is given by

    $$\mathfrak{s} + \alpha := (S \otimes L_\alpha, \gamma \otimes \mathrm{Id}), \quad \alpha \in \mathrm{H}^2(X; \mathbb{Z}),$$

    and L_α is the unique line bundle over X with $c_1(L_\alpha) = \alpha$.

    Easy to see $c_1(\mathfrak{s}+\alpha) = c_1(S^+ \otimes L_\alpha) = c_1(\mathfrak{s}) + 2\alpha$, hence if $\mathrm{H}^2(X; \mathbb{Z})$ has no $2-$trosion, $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$ is injective.
  - If X is simply connected, then $c_1 : \mathrm{Spin}^c(X) \to \mathrm{Char}(X)$ is isomorphism, where

    $$\mathrm{Char}(X) := \{k \in H^2(X; \mathbb{Z}) : k \cup a \equiv a \cup a \mod 2, \ \forall a \in H^2(X; \mathbb{Z})\}$$

    is the characteristic elements of X.

- For $(X, g, \mathfrak{s})$ as above, there exists spin^c connection A on S, the space of spin^c connections is denoted as $\mathcal{A}_\mathfrak{s}$. A^τ denotes the induced connection on $L_\mathfrak{s}$.

  - A is determined by $A^\tau \in \mathcal{A}(L_\mathfrak{s})$ and ω the Levi-Civita connection of (X, g), more precisely, $A = A^\tau + \omega$ under the identification of the connection of princial bundle.
  - $\mathcal{A}_\mathfrak{s} \cong \mathcal{A}(L_\mathfrak{s})$ is an affine space modelled on $\Omega^1(X; \mathrm{i}\mathbb{R})$. And we have

    $$(A + a)^\tau = A^\tau + 2a, \quad A \in \mathcal{A}_\mathfrak{s}, \ a \in \Omega^1(X; \mathrm{i}\mathbb{R}),$$

    since $\det S = L_\mathfrak{s}^{\otimes 2}$.
  - For spin^c connection A, we have curvature $F_{A^\tau} \in \Omega^2(X; \mathrm{i}\mathbb{R})$, and $F_{A^\tau}^+ \in \Omega^2_+(X; \mathrm{i}\mathbb{R})$.

- We have Dirac operator $\not{D}_A : \Gamma(S) \to \Gamma(S)$, where A is the spin^c connection, and

$$\not{D}_A = \begin{pmatrix} O & \not{D}_A^- \\ \not{D}_A^+ & O \end{pmatrix}, \quad \not{D}_A^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp).$$

  - From Atiyah-Singer index theorem, we have

$$\mathrm{Ind}_{\mathbb{C}} \not{D}_A^+ = \frac{1}{8}\left(c_1^2(\mathfrak{s}) - \sigma(X)\right).$$

To state the Seiberg-Witten equations, we still need some linear algebra:

Proposition 6.1.1

We have isomorphism $\gamma : \bigwedge_+^2 TX \otimes \mathbb{C} \to \mathfrak{su}(S^+)$, here $\gamma(v \wedge w) := \gamma(v) \circ \gamma(w)$, and

$$\mathfrak{su}(S^+) := \{A \in \mathrm{End}(S^+) : A^* = -A,\ \mathrm{tr}A = 0\}.$$

Proof. In local orthognoal basis $\{e_i\}_{i=0}^3$, recall we have

$$\gamma(e_0) = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}, \quad \gamma(e_i) = \begin{pmatrix} O & B_i \\ B_i & O \end{pmatrix}, \quad i = 1, 2, 3.$$

Noet that locally, $\bigwedge_+^2 TX \otimes \mathbb{C}$ is spaned by $E_1 := e_0 \wedge e_1 + e_2 \wedge e_3$, $E_2 := e_0 \wedge e_3 - e_1 \wedge e_4$ and $E_3 := e_0 \wedge e_3 + e_1 \wedge e_2$. Directly calculation, we know that

$$\gamma(E_i) = \begin{pmatrix} 2B_i & O \\ O & O \end{pmatrix}, \quad i = 1, 2, 3.$$

Note that $\mathfrak{su}(2) = \mathrm{span}\{B_1, B_2, B_3\}$, we left the compatibility of this local isomorphsim to the readers, then we finish the proof. ♣

Now for any $\Phi \in \Gamma(S^+)$, the positive spinors, we can define $(\Phi \otimes \Phi^*)_0 \in \sqrt{-1}\mathfrak{su}(S^+)$as follows: for the Hermitian inner product $\langle , \rangle$, we can define

$$\Phi \otimes \Phi^* \in \mathrm{End}(S^+), \quad \psi \mapsto \langle \Phi, \psi \rangle \Phi,$$

and then we take its traceless part by

$$(\Phi \otimes \Phi^*)_0 \in \mathrm{End}(S^+), \quad \psi \mapsto \langle \Phi, \psi \rangle \Phi - \frac{|\Phi^2|}{2}\psi.$$

We can check $(\Phi \otimes \Phi^*)_0 \in \sqrt{-1}\mathfrak{su}(S^+)$ by use local oordinates $\Phi = (a, b)^\mathsf{T}$, then

$$(\Phi \otimes \Phi^*)_0 = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} - \frac{|a|^2 + |b|^2}{2}\mathrm{Id} = \begin{pmatrix} \frac{1}{2}\left(|a|^2 - |b|^2\right) & a\bar{b} \\ \bar{a}b & \frac{1}{2}\left(-|a|^2 + |b|^2\right) \end{pmatrix} \in \sqrt{-1}\mathfrak{su}(2).$$

Definition 6.1.1

We have the **quardic map** $\sigma : \Gamma(S^+) \to \Omega^2_+(X; \mathrm{i}\mathbb{R})$ by

$$\sigma(\Phi) := \gamma^{-1}\left((\Phi \otimes \Phi^*)_0\right),$$

where we used the fact that $(\Phi \otimes \Phi^*)_0 \in \sqrt{-1}\mathfrak{su}(S^+)$.

Exercise 1. One can similary define $(\Phi \otimes \Psi^*)_0 \in \mathfrak{su}(S^+)$, then hence extend the definition of quardic map by

$$\sigma(\Phi, \Psi) := \gamma^{-1}\left((\Phi \otimes \Psi^*)_0\right),$$

please complete this construction in local coordinates.

Now we can state the famous Seiberg-Witten equations:

Definition 6.1.2: Seiberg-Witten equations

For $(A, \Phi) \in \mathcal{A}_{\mathfrak{s}} \times \Gamma(S^+)$, we call

$$\boxed{\begin{cases} \not{D}^+_A \Phi = 0 & \text{Dirac equation,} \\ F^+_{A^\tau} = \sigma(\Phi) & \text{Curvature equation,} \end{cases}}$$

is the **Seiberg-Witten equations**. The solutions (A, Φ) lie in the **configuration space**

$$\mathcal{C}_{\mathfrak{s}} := \mathcal{A}_{\mathfrak{s}} \times \Gamma(S^+).$$

Remark. In physics, they call the above SW equations as **monopole equations**, and the solutions (A, Φ) are called **monopoles**.

We also consider the $\omega-$**perturbued SW equations**,

$$\begin{cases} \not{D}^+_A \Phi = 0 \\ F^+_{A^\tau} = \sigma(\Phi) + \omega \end{cases},$$

where $\omega \in \Omega^2_+(X; \mathrm{i}\mathbb{R})$.

Definition 6.1.3: Parameter Space

The **space of parameters** for the SW equations on X is

$$\mathcal{P} := \{(g, \omega) : g \in \mathrm{Met}(X),\ \omega \in \Omega^2_+(X; \mathrm{i}\mathbb{R})\},$$

here $\mathrm{Met}(X)$ denotes the space of Riemanian metrics on X.

6.2 Moduli Spaces and Compactness

Our goal is to study the solutions of SW equations up to gauge equivalence, to be more precise, study the moduli space of SW equations.

Definition 6.2.1: Gauge Group

The **gauge group** we consider is $\mathcal{G} := \mathrm{Aut}(S, \gamma) = C^\infty(X, S^1)$.

Remark. For the proof of $\mathrm{Aut}(S, \gamma) = C^\infty(X, S^1)$, one can recall the Shur's lemma tells us that for each irreducible Clifford module (S_x, γ_x), we have $\mathrm{Aut}(S_x, \gamma_x) \cong \mathrm{U}(1)$.

Definition 6.2.2

We have gauge group $\mathcal{G}$ acts on the left of configuration space $\mathcal{C}_{\mathfrak{s}}$ by

$$u \cdot (A, \Phi) := ((u^{-1})^* A, u \cdot \Phi), \quad u \in \mathcal{G},\ (A, \Phi) \in \mathcal{C}_{\mathfrak{s}}.$$

Here $u \cdot \Phi$ is just the scalar multiplication.

Recall the pull-back connection is defined by

$$\mathrm{d}_{u^* A} := u^{-1} \circ \mathrm{d}_A \circ u : \Gamma(S) \to \Gamma(S),$$

in the local coordinates, we have

$$\mathrm{d}_{u^* A}(s) = u^{-1} \mathrm{d}_A(us) = u^{-1} \mathrm{d}_A u(s) + u^{-1} \cdot u \mathrm{d}_A(s),$$

note that $\mathrm{d}_A u = \mathrm{d}u + [A, u] = \mathrm{d}u$ since $\mathfrak{u}(1)$ is abealian, hence we have

$$u^* A = A + u^{-1} \mathrm{d}u,$$

then we know that actually,

$$\boxed{u \cdot (A, \Phi) = (A + u \mathrm{d}u^{-1}, u \cdot \Phi).}$$

Definition 6.2.3: Seiberg-Witten Map

We define $\mathrm{SW} : \mathcal{C}_{\mathfrak{s}} \to \Omega^2_+(X; \mathrm{i}\mathbb{R}) \times \Gamma(S^-)$, the **Seiberg-Witten map** as

$$\mathrm{SW}(A, \Phi) := \left(F^+_{A^\tau} - \sigma(\Phi), \not{D}^+_A \Phi \right).$$

Remark. Easy to see $\mathrm{SW}^{-1}(0, 0) \subseteq \mathcal{C}_{\mathfrak{s}}$ is the solutions of SW equations.

Definition 6.2.4: Seiberg-Witten Moduli space

For any $\omega \in \Omega^2_+(X; \mathrm{i}\mathbb{R})$, the $\boldsymbol{\omega}-$**perturbed moduli space** is

$$\mathcal{M}_\omega := \mathrm{SW}^{-1}(\omega, 0)/\mathcal{G}.$$

And $\mathcal{B} := \mathcal{C}_\mathfrak{s}/\mathcal{G}$ is called the **base space**, easy to see $\mathcal{M}_\omega \subseteq \mathcal{B}$.

Generally speaking, to study the moduli space or general quotient space by group actions, we need to first study the stablizer at each point.

For $(A, \Phi) \in \mathcal{C}_\mathfrak{s}$, if $u \in \mathcal{G}$ satisfies that

$$(A, \Phi) = u \cdot (A, \Phi) = (A + u\mathrm{d}u^{-1}, u \cdot \Phi),$$

then $u\mathrm{d}u^{-1} = 0$, hence $\mathrm{d}u = 0$ then u is constant map. Hence we have

- if $\Phi \equiv 0$, then u can be any constant, hence $\mathrm{Stab}_{(A,\Phi)} = \mathrm{U}(1)$;
- if $\Phi \not\equiv 0$, then $u \equiv 1$, hence $\mathrm{Stab}_{(A,\Phi)} = \{1\}$.

This motivates we define the following:

Definition 6.2.5

The **irreducible configuration spaces**

$$\mathcal{C}_\mathfrak{s}^{\mathrm{irr}} := \{(A, \Phi) \in \mathcal{C}_\mathfrak{s} : \Phi \not\equiv 0\}.$$

And the **reducible configuration space**

$$\mathcal{C}_\mathfrak{s}^{\mathrm{red}} := \{(A, 0) : A \in \mathcal{A}_\mathfrak{s}\}.$$

Corresponding to this, we can decompose the modulis space as **irreducible solutions** part and **reducible solutions** part:

$$\mathcal{M}_\omega = \mathcal{M}_\omega^{\mathrm{irr}} \bigsqcup \mathcal{M}_\omega^{\mathrm{red}}.$$

Remark. we will soon prove that for genirc ω, $\mathcal{M}_\omega^{\mathrm{irr}}$ will be a smooth manifold, and $\mathcal{M}_\omega^{\mathrm{red}}$ will be some (actually 0 or 1) signular point(s).

6.2.1 Sobolev completion

As the remark above, we want to prove $\mathcal{M}_\omega^{\mathrm{irr}}$ is generically a smooth manifold, so we need some results in infinite dimensional analysis.Hence we need to use Sobolev completion to make the space we consider is Banach.

From now on , we will need the following assumptions on the quantities that appear in our study of the SW equations:

- positive spinors $\Phi \in \Gamma(S^+)$ lie in $L^2_5(S^+)$;
- sections of $\mathrm{i} \cdot \bigwedge^2_+ T^*X \times S^-$ are elements of $L^2_4\left(\mathrm{i} \cdot \bigwedge^2_+ T^*X \times S^-\right)$;
- spin^c connections $A \in L^2_5(\mathcal{A}_\mathrm{s})$, i.e., fix a smooth spin^c connection A_0, A has the form of $A_0 + a$ where $a \in L^2_5\left(\mathrm{i} \bigwedge^2_+ T^*X\right)$;
- gauge group $\mathcal{G}$ consistis of maps in $L^2_6(X, S^1)$.

We will slightly abuse notation and keeping using the old symbols when we are in actuality referring to corresponding Sobolev spaces. But if not special notes, then we mean it is the sobolev completion.

Regularity of moduli space WEITSEI P231

6.2.2 Compatness of moduli space

Unlike other moduli space, we have $\mathcal{M}_\omega$ is naturally compact , hence doesn't need any effort of compactification. This is called the *Seiberg-Witten miracle*, because we can see in the further chapters that there is no such good moduli spaces again.

Now when talking about compactness of the moduli space, we will always mean sequential compactness, i.e. any sequence has a convergent subsequence.

The precise statement that we will prove is:

Theorem 6.2.1: Compactness of moduli space

Let (A_i, Φ_i) be a sequence of L^2_4 solutions to the SW equations. Then there exists a sequence u_i of L^2_5 gauge transformations such that $u_i \cdot (A_i, \Phi_i)$ is a bounded sequence in L^2_k for all k . Hence the solutions $u_i \cdot (A_i, \Phi_i)$ are C^∞ and there is a subsequence that converges in the C^∞ topology to a C^∞ solution (A, Φ) of the monopole equations. In particular, the moduli space is sequentially compact in the C^∞ topology.

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6.3 Transversality

In this section, our final goal is to prove the following:

Theorem 6.3.1: Transversality theorem

For *generic* $\omega \in \Omega^2_+(X; \mathrm{i}\mathbb{R})$, $\mathcal{M}^{\mathrm{irr}}_\omega$ is either empty or an orientable smooth manifold with dimension $d(\mathfrak{s}, X) := \frac{1}{4}\left(c_1^2(\mathfrak{s}) - 2\chi(X) - 3\sigma(X)\right)$.

Remark. We will explain what *genirc* means in latter paragraph. One may also notice that $d(\mathfrak{s}, X)$ is calculated by indexes of some differential operators.

The proof of transvesality theorem takes a lot of work, firstly, we need to study the structure of gauge group $\mathcal{G}$ and base space $\mathcal{B} = \mathcal{C}_\mathfrak{s}/\mathcal{G}$ to make things easier.

Recall we have fixed the Sobolev completion for the spaces below, but we just abuse of the same notation as the smooth case.

6.3.1 Structure of gauge group

Recall $\mathcal{G} = L^2_6(X, S^1)$, hence from $6 - \frac{4}{2} > 0$ we know that any $u \in \mathcal{G}$ is continuous. Since X is simply connected, we have lift $f : X \to \mathbb{R}$ such that $u = \mathrm{e}^{\mathrm{i}f}$.

Definition 6.3.1

- $\mathcal{G}^h := \{\mathrm{e}^{\mathrm{i}c} : c \in \mathbb{R}\} \cong \mathrm{U}(1)$ is the **harmonic map**;
- $\mathcal{G}^\perp := \left\{\mathrm{e}^{\mathrm{i}f} : f \in L^2_6(X), \int_X f \mathrm{dvol}_g = 0\right\}$.

Remark. Generally we call $u : X \to S^1$ is **harmonic** if $\alpha = u\mathrm{d}u^{-1}$ is harmonic $1-$form, in our case, for $u = \mathrm{e}^{\mathrm{i}f}$, it is equivalent to $\mathrm{d}f$ is harmonic, i.e., $\Delta f = 0$, then we know that f must be constant since X is simply connected.

The reason we consider such definition above is the following

Proposition 6.3.1

There is an isomrophism

$$\mathcal{G}^h \times \mathcal{G}^\perp \to \mathcal{G}, \quad (\mathrm{e}^{\mathrm{i}c}, \mathrm{e}^{\mathrm{i}f}) \mapsto \mathrm{e}^{\mathrm{i}(c+f)}.$$

Proof. We can directly construct the inverse map, for any $u = \mathrm{e}^{\mathrm{i}f}$, then we define

$$\lambda_u := \exp\left(\frac{\mathrm{i}}{\mathrm{Vol}_X}\int_X f \mathrm{dvol}_g\right),$$

one can check this is independent on the choice of the lift f, then the inverse map is given by $n \mapsto (\lambda_u, \lambda_u^{-1} f_u)$. We left the details to the readers. ♣

6.3.2 Structure of base space

Now since $\mathcal{G} = \mathcal{G}^h \times \mathcal{G}^\perp$, then we have

$$\mathcal{B} = \mathcal{C}_\mathfrak{s}/\left(\mathcal{G}^h \times \mathcal{G}^\perp\right) = \left(\mathcal{C}_\mathfrak{s}/\mathcal{G}^\perp\right)/\mathcal{G}^h,$$

luckily we have $\mathcal{C}_\mathfrak{s}/\mathcal{G}^\perp$ is really well-behavioured:

Theorem 6.3.2: Global Slice

The action of $\mathcal{G}^\perp$ on $\mathcal{C}_\mathfrak{s}$ admits a global slice S, i.e., $\mathcal{C}_\mathfrak{s}$ is diffeomorphic to $S \times \mathcal{G}^\perp$. Fix a spin^c connection A_0, then the slice is nothing but the **columb gauge fixing**:

$$S := \left\{(A_0 + a, \Phi) \in \mathcal{C}_\mathfrak{s} : a \in \Omega^1(X; \mathrm{i}\mathbb{R}),\ \mathrm{d}^* a = 0\right\}.$$

Recall $\mathcal{G} \curvearrowright \mathcal{C}_\mathfrak{s}$ by $u \cdot (A, \Phi) = (A + u\mathrm{d}u^{-1}, u \cdot \Phi)$, then we have

$$m : \mathcal{G}^\perp \times S \to \mathcal{C}_\mathfrak{s}, \quad \left(\mathrm{e}^{\mathrm{i}f}, (A_0 + a, \Phi)\right) \mapsto \left(A_0 + a - \mathrm{i}\mathrm{d}f, \mathrm{e}^{\mathrm{i}f} \cdot \Phi\right).$$

To construct the inverse map, we need some prelimanaries about harmonic function. Let $\mathcal{H}(X)$ denotes the hamnoic function on X, hence they are actually constant maps, then $\mathcal{H}(X)^\perp$ is the functions with integral zero.

Note that $\int_X \Delta f \mathrm{dvol}_g = 0$ for any f by divergence theorem, a natural question is the converse, for any $g \in \mathcal{H}(X)^\perp$, does there exists $h \in C^\infty(X)$ such that $\Delta h = g$? This is the classical **Dirichlet problem**, and we can always find such h, hence this motivates us define

Definition 6.3.2: Green Operator

There exists **Green operator** $G : C^\infty(X) \to \mathcal{H}(X)^\perp$ such that

$$\Delta(Gf) := f - \frac{1}{\mathrm{Vol}_X}\int_X f \mathrm{dvol}_g.$$

Now we can directly define $m^{-1} : \mathcal{C}_\mathfrak{s} \to S \times \mathcal{G}^\perp$ as

$$(A_0 + b, \Psi) \mapsto \left(\mathrm{e}^{-G(\mathrm{d}^* b)}, \left(A_0 + b - \mathrm{d}(G(\mathrm{d}^* b)), \mathrm{e}^{G(\mathrm{d}^* b)}\Psi\right)\right).$$

Exercise 1. Check the above m^{-1} is the inverse of m.

Hence we know the existence of the global slice. And we have

$$\mathcal{B} \cong S/\mathcal{G}^h.$$

We also have $S = S^{\mathrm{irr}} \sqcup S^{\mathrm{red}}$, more precisely

$$S^{\mathrm{irr}} = \{(A_0 + a, \Phi) \in \mathcal{C}_{\mathfrak{s}} : \mathrm{d}^* a = 0, \Phi \not\equiv 0\}, \quad S^{\mathrm{red}} = \{(A_0 + a, 0) \in \mathcal{C}_{\mathfrak{s}} : \mathrm{d}^* a = 0\}.$$

Now for $u = \mathrm{e}^{\mathrm{i}c} \in \mathcal{G}^h$, since c is constant, then we know that if $\Phi \not\equiv 0$,

$$u \cdot (A, \Phi) = (A, u \cdot \Phi)$$

hence the action of $\mathcal{G}^h$ on S^{irr} is free, this motivates us that

Theorem 6.3.3

The **irreducible base space** $\mathcal{B}^{\mathrm{irr}} := \mathcal{C}_{\mathfrak{s}}^{\mathrm{irr}}/\mathcal{G} = S^{\mathrm{irr}}/\mathcal{G}^h$ is a smooth Banach manifold with weak homotopy type of $\mathbb{C}P^\infty$.

Proof. We already have $\mathcal{G}^h$ acts on S^{irr} freely, it suffices to prove it is properly discontinuous to ensure $\mathcal{B}^{\mathrm{irr}}$ is Hausdorff. Actually, this is equivalent to show that

$$\Gamma := \left\{((A, \Phi), u \cdot (A, \Phi)) : (A, \Phi) \in \mathcal{C}_{\mathfrak{s}}^{\mathrm{irr}}, u \in \mathcal{G}\right\}$$

is closed in $\mathcal{C}_{\mathfrak{s}} \times \mathcal{C}_{\mathfrak{s}}$. Consider convergent subsequence

$$\Big((A_j, \Phi_j), (A_j - \mathrm{id} f_j, \mathrm{e}^{\mathrm{i} f_j} \Phi_j)\Big) \to ((A, \Phi), (B, \Psi)),$$

note that $\lim_{j\to\infty} (A_j - \mathrm{id} f_j) = B$, hence we have $|\mathrm{d} f_j|_{L^2_5}$ is uniformly bounded, hence $\{f_j\}$ is uniformly bounded in L^2_6, then we have convergent subsequence, $f_{j'} \to f$ such that $B = A - \mathrm{id} f$, then we finish.

Hence $\mathcal{B}^{\mathrm{irr}}$ is a smooth Banach manifold, and we have $\mathrm{U}(1)$ principal bundle $\mathrm{U}(1) \to S^{\mathrm{irr}} \to \mathcal{B}^{\mathrm{irr}}$. Note that S^{irr} is homotopy equivalent to $\Gamma(S^+) \setminus \{0\}$ since $\ker \mathrm{d}^*$ is vector space hence contractible. Since $\Gamma(S^+)$ is infinite vector space hence it minus a point is still contractible.

Then we know that $\mathcal{B}^{\mathrm{irr}}$ is weakly homotopy equivalent to the classifying space $B\mathrm{U}(1) = \mathbb{C}P^\infty$, then we finish the proof. ♣

Remark. we have $H^*(\mathcal{B}^{\mathrm{irr}}; \mathbb{Z}) \cong \mathbb{Z}[e]$, where $e = c_1(S^{\mathrm{irr}})$ of this $\mathrm{U}(1)-$bundle.

Remark. Note that $\mathcal{M}_\omega^{\mathrm{irr}} \subseteq \mathcal{B}^{\mathrm{irr}}$, hence it is also Hausdorff.

6.3.3 Proof of transversality theorem

In this section, we prove our main result— —the transvarsality theorem. Firstly, let's recall the baisc results we need from infinite dimensional analysis:

Theorem 6.3.4: Implicit function theorem

Let $f : M \to N$ be a smooth map between two C^l Banach manifolds, and $q \in N$ is a regular value of f, i.e., for any $p \in f^{-1}(q)$, we have $f_{*,p} : T_pM \to T_qN$ is surjective. Then we have $f^{-1}(q) \subseteq M$ is a C^l Banach manifold, and furthermore if f is Fredholm, we have $f^{-1}(q)$ is a manifold with dimension $\mathrm{Ind}(f)$.

Just like Sard's theorem in finite dimension, we have

Theorem 6.3.5: Sard-Smale

Let $f : M \to N$ be a C^l Fredholm map between two seperable Banach manifolds and $l > \max\{0, \mathrm{Ind}(f)\}$, then the set of regular values of f is *generic* in N, i.e., residual or equivalently is the intersection of countbly many open dense subsets.

Remark. Easy to see, from Baire's categroy theorem, we have the set of regular vaulues is dense.

Remark. The detailed proof of these two theorems can be found in the Appendix B of Samalon's "dictionary" about spin geometry and Seiberg-Witten theory.

Now to prove $\mathcal{M}_\omega^{\text{irr}}$ is a smooth manifold for genric ω, the classical way is to firstly consider this paramter as perturbtion, then use the map projected to the parameter space is Fredholm to show that for generic paramter we have the space is smooth manifold.

Consider SW map with gauge fixing, i.e.,

$$\begin{aligned}\widetilde{\mathrm{SW}} : \mathcal{C}_{\mathfrak{s}} \times \Omega^2_+(X; i\mathbb{R}) &\to \Omega^2_+(X; i\mathbb{R}) \times \Gamma(S^-) \times \widetilde{\Omega}^0(X; i\mathbb{R}) \\ ((A, \Phi), \omega) &\mapsto \left(F^+_{A^\tau} - \sigma(\Phi) - \omega, \not{D}^+_A \Phi, \mathrm{d}^*(A - A_0)\right),\end{aligned}$$

where $\widetilde{\Omega}^0(X) := \mathrm{d}^*\Omega^1(X) \cong \Omega^0(X)/\mathbb{R}$, since $f = \mathrm{d}^*a$ iif f has integration over X zero.

We denote $\mathcal{N} := \widetilde{\mathrm{SW}}^{-1}(0, 0, 0) \subseteq S \times \Omega^2_+(X; i\mathbb{R})$ as the solution space of $\widetilde{\mathrm{SW}}$, hence we also have

$$\mathcal{N} = \mathcal{N}^{\text{irr}} \sqcup \mathcal{N}^{\text{red}},$$

then our first goal is to prove that $\mathcal{N}^{\text{irr}}$ is a smooth Banach manifold.

Then consider $\mathrm{pr}_2 : \mathcal{C}_{\mathfrak{s}} \times \Omega^2_+(X; i\mathbb{R}) \to \Omega^2_+(X; i\mathbb{R})$ the natural projection map, it restricts to $\mathcal{N}^{\text{irr}} \subseteq \mathcal{C}_{\mathfrak{s}} \times \Omega^2_+(X; i\mathbb{R})$, we have

$$\pi := \mathrm{pr}_2|_{\mathcal{N}^{\text{irr}}} : \mathcal{N}^{\text{irr}} \to \Omega^2_+(X; i\mathbb{R}),$$

the next goal we want to prove is $\pi : \mathcal{N}^{\mathrm{irr}} \to \Omega^2_+(X; i\mathbb{R})$ is Fredholm with index $d(\mathfrak{s}, X) + 1$.

Hence for geneirc $\omega \in \Omega^2_+(X; i\mathbb{R})$, by Sard-Smale theorem above, we have

$$\mathcal{N}^{\mathrm{irr}}_\omega := \pi^{-1}(\omega) \subseteq \mathcal{N}^{\mathrm{irr}}$$

is a smooth manifold with dimension $d(\mathfrak{s}, X) + 1$.

It may be abstract for you to imagine what $\mathcal{N}^{\mathrm{irr}}_\omega$ is, explicitly,

$$\mathcal{N}^{\mathrm{irr}}_\omega \cong \left\{(A, \Phi) \in \mathcal{C}^{\mathrm{irr}}_{\mathfrak{s}} : F^+_{A^\tau} - \sigma(\Phi) = \omega, \not{D}^+_A \Phi = 0, \mathrm{d}^*(A - A_0) = 0\right\} \subseteq S^{\mathrm{irr}},$$

then under this identification, it is easy to see

$$\mathcal{M}^{\mathrm{irr}}_\omega = \mathcal{N}^{\mathrm{irr}}_\omega / \mathcal{G}^h,$$

note that $\mathcal{G}^h$ acts freely on S^{irr} hence freely on $\mathcal{N}^{\mathrm{irr}}_\omega$, then we know that $\mathcal{M}^{\mathrm{irr}}_\omega$ is a smooth manifold with dimeniso $d(\mathfrak{s}, X)$, finish the proof of the transversality theorem!

Remark. Here more precisely,

$$\mathcal{N}^{\mathrm{irr}}_\omega = \left\{((A, \Phi), \omega) \in \mathcal{C}^{\mathrm{irr}}_{\mathfrak{s}} \times \Omega^2_+(X; i\mathbb{R}) : F^+_{A^\tau} - \sigma(\Phi) = \omega, \not{D}^+_A \Phi = 0, \mathrm{d}^*(A - A_0) = 0\right\},$$

but we just omit the superfluous ω.

Remark. More generaly, we also have

$$\mathcal{M}_\omega = \mathcal{N}_\omega / \mathcal{G}^h, \quad \mathcal{M}^{\mathrm{red}}_\omega = \mathcal{N}^{\mathrm{red}}_\omega / \mathcal{G}^h,$$

but note that $\mathcal{G}^h$ acts on $\mathcal{N}^{\mathrm{red}}_\omega$ nolonger freely.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Now it suffices to prove:

- $\mathcal{N}^{\mathrm{irr}}$ is a smooth Banach manifold;
- $\pi : \mathcal{N}^{\mathrm{irr}} \to \Omega^2_+(X; i\mathbb{R})$ is Fredholm with index $d(\mathfrak{s}, X) + 1$.

¶ $\mathcal{N}^{\mathrm{irr}}$ is a smooth Banach manifold

We prove this result through the following proposition:

Proposition 6.3.2

Consider the following map

$$\begin{aligned} F : S^{\mathrm{irr}} \times \Omega^2_+(X; i\mathbb{R}) &\to \Omega^2_+(X; i\mathbb{R}) \times \Gamma(S^-) \\ (A, \Phi, \omega) &\mapsto \left(F^+_{A^\tau} - \sigma(\Phi) - \omega, \not{D}^+_A \Phi\right), \end{aligned}$$

then $(0,0)$ is the regular value of F, hence $F^{-1}(0,0)$ is a Banach manifold.

Remark. Note that $\mathcal{N}^{\text{irr}} = F^{-1}(0,0)$, hence is a smooth Banach manifold.

Proof. Our goal is to show that if $F(A,\Phi,\omega)=0$, then we have the differential

$$F_{*,(A,\Phi,\omega)} : T_{(A,\Phi)}S^{\text{irr}} \times T_\omega\Omega^2_+(X;\mathrm{i}\mathbb{R}) \to T_0\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times T_0\Gamma(S^-)$$

is surjective. So firstly, let's calculate this differential:

Step 1 : Besides S^{irr}, the other spaces we consider are vector spaces hence thier tangent spaces are isomorphic to themselves, i.e.,

$$T_\omega\Omega^2_+(X;\mathrm{i}\mathbb{R}) = \{\beta : \beta \in \Omega^2_+(X;\mathrm{i}\mathbb{R})\}$$
$$T_0\Omega^2_+(X;\mathrm{i}\mathbb{R}) = \{\eta : \eta \in \Omega^2_+(X;\mathrm{i}\mathbb{R})\}$$
$$T_0\Gamma(S^-) = \{\psi : \psi \in \Gamma(S^-)\}.$$

Now recall

$$S^{\text{irr}} = \{(A_0+a,\Phi) \in \mathcal{C}_\mathfrak{s} : \mathrm{d}^*a = 0,\ \Phi \not\equiv 0\} \cong \ker\left(\mathrm{d}^*|_{\Omega^1(X;\mathrm{i}\mathbb{R})\to\Omega^0(X;\mathrm{i}\mathbb{R})}\right) \times \Gamma(S^+)\setminus\{0\},$$

then we have

$$T_{(A,\Phi)}S^{\text{irr}} = \{(a,\phi) : a \in \Omega^1(X;\mathrm{i}\mathbb{R}),\ \mathrm{d}^*a = 0,\ \phi \in \Gamma(S^+)\}.$$

Now as the notations above, we calculate $F_{*,(A,\Phi,\omega)}(a,\phi,\beta)$, easy to see

$$\begin{aligned} F_{*,(A,\Phi,\omega)}(a,0,0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A+ta,\Phi,\omega) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(F^+_{(A+ta)^\tau} - \sigma(\Phi) - \omega, \not{D}^+_{A+ta}\Phi\right) \\ &= \left(2\mathrm{d}^+a, \gamma(a)\Phi\right), \end{aligned}$$

where we used $F^+_{(A+ta)^\tau} = F^+_{A^\tau+2ta} = F^+_{A^\tau} + t\cdot 2\mathrm{d}^+a + o(t^2)$, and $\not{D}^+_{A+ta}\Phi = \not{D}^+_A\Phi + \gamma(ta)\Phi$.

$$\begin{aligned} F_{*,(A,\Phi,\omega)}(0,\phi,0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A,\Phi+t\phi,\omega) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(F^+_{A^\tau} - \sigma(\Phi+t\phi) - \omega, \not{D}^+_A(\Phi+t\phi)\right) \\ &= \left(-\sigma(\Phi,\phi) - \sigma(\phi,\Phi), \not{D}^+_A\phi\right), \end{aligned}$$

and we also have $F_{*,(A,\Phi,\omega)}(0,0,\beta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(A,\Phi,\omega+t\beta) = (\beta,0)$, then finally, we get

$$\boxed{F_{*,(A,\Phi,\omega)}\begin{pmatrix} a \\ \phi \\ \beta \end{pmatrix} = \begin{pmatrix} 2\mathrm{d}^+a - \sigma(\Phi,\phi) - \sigma(\phi,\Phi) + \beta \\ \not{D}^+_A\phi + \gamma(a)\Phi \end{pmatrix}}.$$

Step 2 : Now we prove that $F_{*,(A,\Phi,\omega)}$ is surjective. Since after Sobolev completion, and we all choose L_k^2 completion, hence the spaces we consider are naturally Hilbert, with the inner product given by either Riemannian metric g or the Hermitian metric $\langle , \rangle$ on S.

Hence it suffices to prove that $\left(\mathrm{Im}F_{*,(A,\Phi,\omega)}\right)^{\perp} = \{(0,0)\}$. More precisely, choose $(\eta,\psi) \in \Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \Gamma(S^-)$ lies in $\left(\mathrm{Im}F_{*,(A,\Phi,\omega)}\right)^{\perp} = \{(0,0)\}$, which means that, for any a,ϕ,β, the following holds:

$$\int_X \left\{ g(2\mathrm{d}^+ a - \sigma(\Phi,\phi) - \sigma(\phi,\Phi) + \beta, \eta) + \langle \not{D}_A^+ \phi + \gamma(a)\Phi, \psi\rangle \right\} \mathrm{dvol}_g = 0. \tag{6.1}$$

Firstly let $a = 0$ and $\phi = 0$, then we just get for any β,

$$\int_X g(\beta,\eta)\mathrm{dvol}_g = 0,$$

then it is easy to see $\boxed{\eta = 0}$. Hence we reduce (6.1) to

$$\int_X \langle \not{D}_A^+ \phi + \gamma(a)\Phi, \psi\rangle \mathrm{dvol}_g = 0,$$

take $a = 0$ again, we get

$$\int_X \langle \not{D}_A^+ \phi, \psi\rangle \mathrm{dvol}_g = 0, \quad \Rightarrow \quad \int_X \langle \gamma(a)\Phi, \psi\rangle \mathrm{dvol}_g = 0 \quad \forall a.$$

Note that $0 = \int_X \langle \not{D}_A^+ \phi, \psi\rangle \mathrm{dvol}_g = \int_X \langle \phi, \not{D}_A^- \psi\rangle \mathrm{dvol}_g$, holds for all ϕ, hence we have $\not{D}_A^- \psi = 0$. Since $\Phi \not\equiv 0$, there exists $p \in X$ such that $\Phi(p) \neq 0$, then we take a such that $\gamma(a_p)\Phi(p) = \psi(p)$ then in a neighborhood U of p, we can ensure $\langle \gamma(a)\Phi, \psi\rangle|_U \geq 0$, then take the cutoff function to ensure a is support on U, then we know that the integration is non-negative.

Hence we know that if $\Phi(p) \neq 0$, then $\psi(p) = 0$, then by the continuity of p, we know that ψ vanishes on an open subset of X. Note that we also have $\not{D}_A^- \psi = 0$, then by the *unique continuation theorem*, we have $\boxed{\psi = 0}$.

Fnially, we show that $F_{*,(A,\Phi,\omega)}$ is surjective for any $(A,\Phi,\omega) \in F^{-1}(0,0)$, then $(0,0)$ is the regular value. Hence by implicit function theorem, $F^{-1}(0,0)$ is a Banach manifold. ♣

Remark. The *unique continuation theorem* says that if the solution of the Dirac equation vanishes on an open subset, then it is identically zero. For the detailed proof, one could refer Samalon's "dictionary", theorem E.8 on P549.

¶ $\pi : \mathcal{N}^{\text{irr}} \to \Omega^2_+(X;\mathbb{R})$ **is Fredholm with index** $d(\mathfrak{s}, X) + 1$

We have proved $\mathcal{N}^{\text{irr}}$ is a Banach manifold, now we furthermore prove the following:

Proposition 6.3.3

$\pi : \mathcal{N}^{\text{irr}} \to \Omega^2_+(X;\mathbb{R})$ is Fredholm with index $d(\mathfrak{s}, X) + 1$.

Remark. From this proposition, then by Sard-Smale's theorem, we know that for generic $\omega \in \Omega^2_+(X; i\mathbb{R})$, $\mathcal{N}^{\text{irr}}_\omega = \pi^{-1}(\omega)$ is either empty or a smooth manifold of dimension $d(\mathfrak{s}, X) + 1$.

Proof. Recall

$$\mathcal{N}^{\text{irr}} = \{(A, \Phi, \omega) : F^+_{A^\tau} - \sigma(\Phi) - \omega = 0, \not{D}^+_A \Phi = 0, \mathrm{d}^*(A - A_0) = 0, \Phi \not\equiv 0\},$$

and more precisely, $\mathcal{N}^{\text{irr}} = F^{-1}(0, 0)$, hence for any $(A, \Phi, \omega) \in \mathcal{N}^{\text{irr}}$, we have

$$\begin{aligned} T_{(A,\Phi,\omega)}\mathcal{N}^{\text{irr}} &= \ker\left(F_{*,(A,\Phi,\omega)} : T_{(A,\Phi)}\mathcal{S}^{\text{irr}} \times T_\omega \Omega^2_+(X; i\mathbb{R}) \to T_0\Omega^2_+(X; i\mathbb{R}) \times T_0\Gamma(S^-)\right) \\ &= \left\{(a, \phi, \beta) : 2\mathrm{d}^+ a - \sigma(\Phi, \phi) - \sigma(\phi, \Phi) = -\beta, \not{D}^+_A \phi + \gamma(a)\Phi = 0, \mathrm{d}^* a = 0, \right\}. \end{aligned}$$

Note that $\pi_{*,(A,\Phi,\omega)} : T_{(A,\Phi,\omega)}\mathcal{N}^{\text{irr}} \to T_\omega \Omega^2_+(X; i\mathbb{R})$ is given by

$$\pi_{*,(A,\Phi,\omega)}(a, \phi, \beta) = \beta,$$

our goal is to show that for any $(A, \Phi, \omega) \in \mathcal{N}^{\text{irr}}$, $\ker \pi_{*,(A,\Phi,\omega)}$ and $\operatorname{coker} \pi_{*,(A,\Phi,\omega)}$ has finite dimension.

From now on, we fix (A, Φ, ω).

Step 1 : Easy to see

$$\ker \pi_{*,(A,\Phi,\omega)} = \left\{(a, \phi, \mathbf{0}) : 2\mathrm{d}^+ a - \sigma(\Phi, \phi) - \sigma(\phi, \Phi) = \mathbf{0}, \not{D}^+_A \phi + \gamma(a)\Phi = 0, \mathrm{d}^* a = 0, \right\},$$

where $a \in \Omega^1(X; i\mathbb{R})$ and $\phi \in \Gamma(S^-)$. This motivates us to define the following operator:

$$\begin{aligned} L : \Omega^1(X; i\mathbb{R}) \times \Gamma(S^+) &\to \Omega^2_+(X; i\mathbb{R}) \times \Gamma(S^-) \times \widetilde{\Omega}^0(X; i\mathbb{R}) \\ (a, \phi) &\mapsto \left(2\mathrm{d}^+ a - \sigma(\Phi, \phi) - \sigma(\phi, \Phi), \not{D}^+_A \phi + \gamma(a)\Phi, \mathrm{d}^* a\right), \end{aligned}$$

then it is clear that $\boxed{\ker \pi_{*,(A,\Phi,\omega)} \cong \ker L}$.

So it is natural to hope $\operatorname{coker} \pi_{*,(A,\Phi,\omega)} \cong \operatorname{coker} L$, note that

$$\begin{aligned} \operatorname{Im}\pi_{*,(A,\Phi,\omega)} &= \left\{\beta \in \Omega^2_+(X; i\mathbb{R}) : \exists a, \phi \text{ s.t. } (a, \phi, \omega) \in T_{(A,\Phi,\omega)}\mathcal{N}^{\text{irr}}\right\} \\ &= \left\{\beta \in \Omega^2_+(X; i\mathbb{R}) : \exists a, \phi \text{ s.t. } (-\beta, 0, 0) = L(a, \phi)\right\} \\ &\cong \operatorname{Im}L \cap \left(\Omega^2_+(X; i\mathbb{R}) \times \{0\} \times \{0\}\right) \end{aligned}$$

this is almost nonsense by carefully checking the above definitions.

Now to prove $\operatorname{coker}\pi_{*,(A,\Phi,\omega)} \cong \operatorname{coker} L$, it suffices to prove their orthogonal complement are isomorphic, this needs take care because they don't live in the same space, so we use notation $Y^{\perp}|_X$ to denote the complement of Y in X, hence it suffices to prove

$$(\mathrm{Im}\pi_{*,(A,\Phi,\omega)})^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})} \cong (\mathrm{Im}L)^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})\times\Gamma(S^-)\times\widetilde{\Omega}^0(X;\mathrm{i}\mathbb{R})},$$

note that $\mathrm{Im}\pi_{*,(A,\Phi,\omega)} \cong \mathrm{Im}L \cap \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)$ by the natural inclusion $\Omega^2_+(X;\mathrm{i}\mathbb{R}) \hookrightarrow \Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}$, then we have

$$\begin{aligned}
&(\mathrm{Im}\pi_{*,(A,\Phi,\omega)})^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})}\\
\cong\ & \left(\mathrm{Im}L \cap \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)\right)^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})\times\{0\}\times\{0\}}\\
=\ & \left(\mathrm{Im}L \cap \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)\right)^{\perp} \cap \ \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)\\
=\ & \Big((\mathrm{Im}L)^{\perp} \cup \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)^{\perp}\Big) \cap \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)\\
=\ & (\mathrm{Im}L)^{\perp} \cap \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right),
\end{aligned}$$

where in above notation we mean

$$\begin{aligned}
(\mathrm{Im}L)^{\perp} &= (\mathrm{Im}L)^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})\times\Gamma(S^-)\times\widetilde{\Omega}^0(X;\mathrm{i}\mathbb{R})}\\
\left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)^{\perp} &= \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)^{\perp}|_{\Omega^2_+(X;\mathrm{i}\mathbb{R})\times\Gamma(S^-)\times\widetilde{\Omega}^0(X;\mathrm{i}\mathbb{R})}
\end{aligned}$$

Hence it suffices to prove that $(\mathrm{Im}L)^{\perp} \subseteq \left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)$, or equivalently,

$$(\mathrm{Im}L)^{\perp} \ \cap \ \Big(\{0\} \times \Gamma(S^-) \times \widetilde{\Omega}^0(X;\mathrm{i}\mathbb{R})\Big) = \{(0,0,0)\}, \tag{6.2}$$

because it is trivial to see that $\left(\Omega^2_+(X;\mathrm{i}\mathbb{R}) \times \{0\} \times \{0\}\right)^{\perp} = \{0\} \times \Gamma(S^-) \times \widetilde{\Omega}^0(X;\mathrm{i}\mathbb{R})$.

Now we focus on proving (6.2), suppose we have $(0,\sigma,f)$ lies in the intersection, then we know that for any $a \in \Omega^1(X;\mathrm{i}\mathbb{R})$ and $\phi \in \Gamma(S^+)$, we have $(0,\sigma,f) \perp L(a,\phi)$, more precisely, we have

$$\int_X \Big(\langle \not{D}^+_A\phi + \gamma(a)\Phi, \sigma\rangle + g(f, \mathrm{d}^*a)\Big)\, \mathrm{dvol}_g = 0.$$

Again take $a = 0$, we have $\displaystyle\int_X \langle \not{D}^+_A\phi, \sigma\rangle \mathrm{dvol}_g = 0$ for all ϕ hence $\not{D}^-_A\sigma = 0$, and also have

$$\int_X \left(\langle\gamma(a)\Phi, \sigma\rangle + g(f, \mathrm{d}^*a)\right) \mathrm{dvol}_g = 0.$$

Recall when we prove the propoerty of self-adjoint of Dirac operator, we actually prove $\langle \not{D}_A\sigma_1, \sigma_2\rangle - \langle\sigma_1, \not{D}_A\sigma_2\rangle = \mathrm{div}X$, where X is determined by $g(X,Y) = \langle Y\cdot\sigma_1, \sigma_2\rangle$. This motivates us to define $b \in \Omega^1(X;\mathrm{i}\mathbb{R})$ such that $g(a,b) = \langle\gamma(a)\Phi, \sigma\rangle$, then we have

$$\mathrm{d}^*b = \mathrm{div}b^{\sharp} = \langle \not{D}_A\Phi, \sigma\rangle - \langle\Phi, \not{D}_A\sigma\rangle = 0,$$

since $\not{D}_A\Phi = 0$ and $\not{D}_A\sigma = 0$.

Now take $a = b$, we get

$$0 = \int_X (\langle\gamma(b)\Phi, \sigma\rangle + g(f, \mathrm{d}^*b))\,\mathrm{dvol}_g = \int_X \langle\gamma(b)\Phi, \sigma\rangle \mathrm{dvol}_g = \int_X g(b, b)\mathrm{dvol}_g,$$

then we know that $b = 0$, hence for all a, we have

$$\langle\gamma(a)\Phi, \sigma\rangle = 0,$$

then similarly as before, we can show that if $\Phi|_U \neq 0$, then $\sigma|_U = 0$, hence by the unique continuation theorem, we know that $\boxed{\sigma \equiv 0}$. Then for any a, we have

$$\int_X g(f, \mathrm{d}^*a)\mathrm{dvol}_g = 0,$$

we immediately know that $\boxed{f = 0}$, then we prove (6.2), hence $\boxed{\operatorname{coker} \pi_{*,(A,\Phi,\omega)} \cong \operatorname{coker} L}$.

In summary, we have proved

$$\ker \pi_{*,(A,\Phi,\omega)} \cong \ker L, \quad \operatorname{coker} \pi_{*,(A,\Phi,\omega)} \cong \operatorname{coker} L.$$

Step 2 : Now let

$$\begin{aligned} L_0 : \Omega^1(X; i\mathbb{R}) \times \Gamma(S^+) &\to \Omega^2_+(X; i\mathbb{R}) \times \Gamma(S^-) \times \widetilde{\Omega}^0(X; i\mathbb{R}) \\ (a, \phi) &\mapsto \left(2\mathrm{d}^+a, \not{D}^+_A\phi, \mathrm{d}^*a\right), \end{aligned}$$

then $L = L_0 + K$ where K is zero order terms then is compact operator, since X is closed.

Easy to see L and L_0 is elliptic hence Fredholm after we take Sobolev completion, then we know that $\ker L$ and $\operatorname{coker} L$ has finite dimension, hence we know $\pi_{*,(A,\Phi,\omega)}$ is also Fredholm.

Now its suffices to calculate the index of L. Since K compact, then we have $\mathrm{Ind}(L) = \mathrm{Ind}(L_0)$, note that $L_0 = \not{D}^+_A \oplus (\mathrm{d}^+ \oplus \mathrm{d}^*)$, hence

$$\mathrm{Ind} L_0 = \mathrm{Ind}_{\mathbb{R}}\left(\not{D}^+_A\right) + \mathrm{Ind}(\mathrm{d}^+ + \mathrm{d}^*).$$

By Atiyah-Singer index theorem, we have

$$\mathrm{Ind}_{\mathbb{R}}\left(\not{D}^+_A\right) = 2 \times \frac{1}{8}\left(c_1(\mathfrak{s})^2 - \sigma(X)\right).$$

By Hodge decomposition, we easily have $\ker(\mathrm{d}^+ + \mathrm{d}^*) \cong \mathcal{H}^1(X)$, and $\operatorname{coker}(\mathrm{d}^+ + \mathrm{d}^*) \cong \mathcal{H}^2_+(X)$, hence

$$\mathrm{Ind}(\mathrm{d}^+ + \mathrm{d}^*) = \dim \mathcal{H}^1(X) - \dim \mathcal{H}^2_+(X) = -b_2^+(X),$$

since X is simply connected.

Then we finally have

$$\mathrm{Ind}\pi_{*,(A,\Phi,\omega)} = \mathrm{Ind} L = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - \sigma(X)\right) - b_2^+(X) = d(\mathfrak{s}, X) + 1,$$

then we finally finish the proof, i.e., we show that π is Fredholm with index $d(\mathfrak{s}, X) + 1$. ♣

¶ Orientation of moduli space

We have shown that for generic ω, $\mathcal{N}_\omega^{\text{irr}}$ is a smooth manifold of dimension $d(\mathfrak{s}, X)+1$. Now we want to prove that it is orienatble.

This is equivalently to show that the budnle

$$\bigwedge^{d(\mathfrak{s},X)+1}\left(T\mathcal{N}_\omega^{\text{irr}}\right) = \det\left(T\mathcal{N}_\omega^{\text{irr}}\right)$$

admits a nonwhere zero section, i.e., is the trivial bundle.

Note that for each $(A,\Phi)\in\mathcal{N}_\omega^{\text{irr}}$, we have

$$T_{(A,\Phi)}\mathcal{N}_\omega^{\text{irr}} \cong \ker \pi_{*,(A,\Phi,\omega)} \cong \ker L_{A,\Phi},$$

where $L_{A,\Phi}$ is just the L we defined in the previuos section, here we just mention that it depends on (A,Φ).

Note that we actually have a **continuous family of Fredholm operators**:

$$\begin{aligned} L: \mathcal{N}_\omega^{\text{irr}} &\to \text{Fred}\left(\Omega^1(X;\text{i}\mathbb{R})\times\Gamma(S^+), \Omega^2_+(X;\text{i}\mathbb{R})\times\Gamma(S^-)\times\widetilde{\Omega}^0(X;\text{i}\mathbb{R})\right)\\ (A,\Phi)&\mapsto L_{A,\Phi}, \end{aligned}$$

note that we take Sobolev completion of these spaces.

It is well known that given such continuous family of Fredholm operators, we can well define the **determinant line bundle** $\det L$ of L over $\mathcal{N}_\omega^{\text{irr}}$, and if $\{L_t\}$ is homotopic, then we have $\det L_t$ are all isomorphic.

$\boxed{\textit{Remark.}}$ For the details, one can refer Mcduff, Salamon's *J-holomorphic curves and symplectic topology*, Appendix A.

Now let

$$(L_t)_{A,\Phi}(a,\phi) := \left(2\text{d}^+a - t\sigma(\Phi,\phi) - t\sigma(\phi,\Phi), \not{D}_A^+\phi + t\gamma(a)\Phi, \text{d}^*a\right),$$

then easy to see L is homotopic to $L_0 = \not{D}_A^+ \oplus (\text{d}^*+\text{d}^+)$, hence we have

$$\det\left(T\mathcal{N}_\omega^{\text{irr}}\right) \cong \det L \cong \det L_0 \cong \det \not{D}_A^+ \otimes \det\left(\text{d}^*\oplus\text{d}^+\right).$$

Since $\ker \not{D}_A^+$ and $\operatorname{coker} \not{D}_A^+$ are complex, then they are oriented, hence $\det \not{D}_A^+$ is trivial, and also since X is orientable, then $\det(\text{d}^*\oplus\text{d}^+)$ will also have orientation induced from X.

Finally we know that $\det\left(T\mathcal{N}_\omega^{\text{irr}}\right)$ is trivial, hence $\mathcal{N}_\omega^{\text{irr}}$ is orientable.

Since $\text{e}^{\text{i}\theta}(A,\Phi) = (A, \text{e}^{\text{i}\theta}\Phi)$, then we know that $\mathcal{G}^h \curvearrowright \mathcal{N}_\omega^{\text{irr}}$ is orientation preserving, hence $\mathcal{M}_\omega^{\text{irr}}$ is also orientable.

6.4 Reducible Solutions and the Parameter Space

In the previous section, we have seen that for generic ω, $\mathcal{M}_\omega^{\text{irr}}$ is a smooth manifold, so to fully understand $\mathcal{M}_\omega$, ite suffices to study the reducible part $\mathcal{M}_\omega^{\text{red}}$.

For the reducible solutions $(A, 0)$ of $\omega-$perturbed SW equations, we have the Dirac equation becomes trivial, and we just need to solve the curvature equation $F_{A^\tau}^+ = \omega$.

Now the question is: for what $\omega \in \Omega_+^2(X; \mathrm{i}\mathbb{R})$, we can find a spin^c connection A solves the curvature equation?

Definition 6.4.1: Wall and Chambers

For a spin^c structure $\mathfrak{s}$ on X, we define the **wall** as the subset of $\mathcal{P}$,

$$\mathcal{W}_\mathfrak{s} := \left\{(g, \omega) \in \mathcal{P} : \mathrm{pr}_{\mathcal{H}_+^2}(\omega + 2\pi \mathrm{i} c_1(\mathfrak{s})) = 0\right\},$$

where $\mathrm{pr}_{\mathcal{H}_+^2} : \Omega^2(X) \to \mathcal{H}_+^2(X)$ is the orthogonal projection according to the Hodge decomposition. We call the complement of the wall $\mathcal{P} \setminus \mathcal{W}_\mathfrak{s}$ is the **chambers**.

$\boxed{\textit{Remark.}}$ Easy to see, the wall is an infinite dimensional affine space with codimension $b_2^+(X)$.

Theorem 6.4.1

$\omega-$perturbed SW equations have a reducible solution if and only if the parameter lies in the wall, i.e., $(g, \omega) \in \mathcal{W}_\mathfrak{s}$.

Proof. Fix a spin^c connection A_0, then the curvature equation is equvaliently to find $a \in \Omega^1(X; \mathrm{i}\mathbb{R})$ such that

$$\omega = F_{(A_0+a)^\tau}^+ = F_{A_0}^\tau + \mathrm{d}^+ a,$$

where we use in $\mathfrak{u}(1)$, $a \wedge a = 0$. Hence it suffices to find a such that

$$\mathrm{d}^+ a = \omega - F_{A_0^\tau}^+.$$

Since we have Hodge decomposition $\Omega_+^2(X) = \mathcal{H}_+^2(X) \oplus \mathrm{d}^+\left(\Omega^1(X)\right)$, hence we know that it ie equivalently

$$\mathrm{pr}_{\mathcal{H}_+^2}\left(\omega - F_{A_0^\tau}^+\right) = 0,$$

note that $\mathrm{pr}_{\mathcal{H}_+^2}\left(F_{A_0^\tau}^-\right) = \mathrm{pr}_{\mathcal{H}_+^2}(\mathrm{d}\alpha) = 0$, and $\left[F_{A_0^\tau}\right] = -2\pi \mathrm{i} c_1(\mathfrak{s})$ by Chern-Weil theorem, hence we have we can solve the curvature equation iff

$$\mathrm{pr}_{\mathcal{H}_+^2}(\omega + 2\pi \mathrm{i} c_1(\mathfrak{s})) = 0, \quad \text{i.e., } (g, \omega) \in \mathcal{W}_\mathfrak{s},$$

then we finish the proof. ♣

Now we can study $\mathcal{M}_\omega^{\mathrm{red}}$, the discussion depends on $b_+^2(X)$, i.e., the codimension of the wall $\mathcal{W}_{\mathfrak{s}}$. Rougly speaking, we don't want the reducible solutions, and if $b_+^2(X)$ is bigger, the codimenison is bigger, then it is easier to avoid the wall hence find (g, ω) such that there is no reducible solution for $\omega-$perturbed SW equations.

More precisely, we have

- if $b_2^+(X) = 0$, then for all $(g, \omega) \in \mathcal{P}$, $\mathcal{M}_\omega^{\mathrm{red}}$ is non-empty and has only one point, i.e., $\mathcal{M}_\omega^{\mathrm{red}} = \{[A, 0]\}$. The proof is as follows:

  Given $(g, \omega) \in \mathcal{P}$, since $b_2^+(X) = 0$, then we always have $\mathrm{pr}_{\mathcal{H}_+^2}(\omega + 2\pi i c_1(\mathfrak{s})) = 0$, i.e., $\mathcal{W}_{\mathfrak{s}} = \mathcal{P}$. So it suffices to prove the uniqueness up to gauge equivalence. Suppose a_1, a_2 solves $\mathrm{d}^+ a_i = \omega - F_{A_0^\tau}^+$, then we have $\mathrm{d}^+(a_1 - a_2) = 0$, hence $\mathrm{d}(a_1 - a_2) = 0$ (recall d^+ closed iff d^- closed iff d closed).

  Then if X is simply connected, we have $a_1 - a_2 = \mathrm{d}f$, hence we can take gauge transformation $u = \mathrm{e}^f$ such that $A_1 = A_0 + a_1$ and $A_2 = A_0 + a_2$ is gauge equvalient.

  $\boxed{\textit{Remark.}}$ We will study the modulis space of this case more detailedly in the next section.

- if $b_2^+(X) > 0$, now since codim $\mathcal{W}_{\mathfrak{s}} > 0$, for generic $(g, \omega) \in \mathcal{W}_{\mathfrak{s}}$, $\mathcal{M}_\omega^{\mathrm{red}} = \varnothing$, hence

  $$\mathcal{M}_\omega = \mathcal{M}_\omega^{\mathrm{irr}}$$

  is either empty or a smooth manifold of dimension $d(\mathfrak{s}, X)$.

  - if $b_2^+(X) = 1$, then the chamber $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ has two path connected components

    $$\begin{aligned} (\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}})^+ &:= \left\{(g, \omega) : \mathrm{pr}_{\mathcal{H}_+^2}(\omega + 2\pi i c_1(\mathfrak{s})) > 0\right\} \\ (\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}})^- &:= \left\{(g, \omega) : \mathrm{pr}_{\mathcal{H}_+^2}(\omega + 2\pi i c_1(\mathfrak{s})) < 0\right\} \end{aligned},$$

    parameters (g_1, ω_1) and (g_2, ω_2) can only be connected by a path avoiding the wall if they are in the same conneted componet of the chamber.

  - if $b_2^+(X) > 1$, paths transverse to the wall are actually disjoint from the wall, hence parameters in the complement of the wall can always be connected by a curve avoiding the wall, i.e. for any (g_1, ω_1) and (g_2, ω_2) lie in the chamber $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$, there exists a smooth family $(g_t, \omega_t) \in \mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ connecting (g_1, ω_1) and (g_2, ω_2) .

6.5 Donaldson's Diagonalizibility Theorem

In this section, we give a Seiberg-Witten way to prove the following celebarting theorem:

Theorem 6.5.1: Donaldson's Diagonalizibility Theorem, 1982

If X is a closed, connected, oriented smooth manifold with definite intersection form Q_X, then Q_X is diagonalizable over $\mathbb{Z}$.

The proof we give here is due to Peter Kronhiemier, with a finishing touch from N.Elkies. To prove this theorem, we can make some assumptations to make this problem easier:

WLOG, we assume Q_X is negative definite, i.e., $b_2^+(X) = 0$.

If Q_X is positive definite, we can consider $\overline{X}$, the manifold with inverse orientation, then $Q_{\overline{X}} = -Q_X$ is negative definite, since for a matrix A, it is trivial to see A is diagonalizable over $\mathbb{Z}$ iff $-A$ is diagonalizable over $\mathbb{Z}$, then we know that there is nothing loss.

WLOG, we assume X is simply connected, also $b_1(X) = 0$.

Note that we establish the SW moduli space for simply connected four manifolds, so this step is essential for us to further consider the SW moduli space over X.

We do surgery! Assume $\pi_1(X)$ is generated by $\gamma_1, \cdots, \gamma_k$, and homotopy them to make them disjoint. Hence consider their tubular neighborhoods $\nu(\gamma_i) \cong S^1 \times \mathbb{D}^3$. And $\nu(\gamma_i)$ are also disjoint, then we consider

$$Y = \left(X \setminus \bigsqcup_{i=1}^{k} \nu(\gamma_i)\right) \bigcup_{\sqcup_{i=1}^{k} S^1 \times S^2} \left(\bigsqcup_{i=1}^{k} \mathbb{D}^2 \times S^2\right),$$

where one should note that $\partial \nu(\gamma_i) \cong S^1 \times S^2$.

It is easy to see $\pi_1\left(X \setminus \bigsqcup_{i=1}^{k} \nu(\gamma_i)\right) \cong \pi_1(X)$, since we can see this inductively, note that $X = (X \setminus \nu(\gamma)) \cup_{S^1 \times S^2} (S^1 \times \mathbb{D}^3)$, then by SVK, note that $i_*\pi_1(S^1 \times S^2) = \pi_1(S^1 \times \mathbb{D}^3)$, hence we know $\pi_1(X \setminus \nu(\gamma)) \cong \pi_1(X)$.

Note that by SVK again,

$$\pi_1(Y) \cong \pi_1(X)/\langle \gamma_1, \cdots, \gamma_k \rangle = 0,$$

where note that the genertor of $S^1 \times S^2$ corresponds to the γ_i is just γ_i itself.

Now it suffices to prove that $Q_X \cong_{\mathbb{Z}} Q_Y$, inductively, for $X' = (X \setminus \nu(\gamma)) \cup_{S^1 \times S^2} (\mathbb{D}^2 \times S^2)$, we show that X' and X have isomorphic intersection form.

By MV-sequence, one can easy to show that $H_2(X') \cong H_2(X \setminus \nu(\gamma)) \cong H_2(X)$, and hence the elements in $H_2(X)$ and $H_2(X')$ can all be generated by the $[\Sigma]$, where $\Sigma \hookrightarrow X \setminus \nu(\gamma)$ is embedded surface, hence the intersection form $[\Sigma_1] \cdot [\Sigma_2]$ are naturally isomorphic.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Now recall if $Q : V \times V \to \mathbb{Z}$ be a unimordular bilinear form, the characteristc element is defined by

$$\mathrm{Char}(Q) := \{\xi \in V : \forall v \in V,\ Q(\xi, v) \equiv Q(v, v) \mod 2\}.$$

And also recall we have proved $\mathrm{Char}(X) = \mathrm{Char}(Q_X) \cong \mathrm{Spin}^c(X)$, and the isomorphism is given by c_1.

Thanks to Elikes, we have the following easier characterization for diagonalizibility:

Theorem 6.5.2: Elkies

If $Q : V \times V \to \mathbb{Z}$ is unimordular and negative definite integral bilinear form, then Q is diagonalizable over $\mathbb{Z}$ iff for all $\xi \in \mathrm{Char}(Q)$,

$$Q(\xi, \xi) + \mathrm{rank} Q \leq 0.$$

Use this amazing theorem, we can reduce the problem to the following:

Theorem 6.5.3: Donaldson's thm, reformulated

If X is simply connected, closed, oriented smooth four manifold, if Q_X is negative definite, then for all $\mathfrak{s} \in \mathrm{Spin}^c(X) \cong \mathrm{Char}(X)$, we have

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 0.$$

Remark. the number $c_1(\mathfrak{s})^2 + b_2(X)$ is strongly related to the virtual-dimension $d(\mathfrak{s}, X)$ of SW moduli space, more precisely, note that in this case $b_2 = b_2^-$, $\sigma(X) = -b_2^-$, hence

$$\mathrm{d}(\mathfrak{s}, X) = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)\right) = \frac{c_1(\mathfrak{s})^2 + b_2(X)}{4} - 1.$$

Remark. Based on the reformulated version, we can also use Heegaard Floer to give a proof.

Note that $c_1(\mathfrak{s})^2 + b_2(X) = c_1(\mathfrak{s})^2 - \sigma(X) = 8\mathrm{Ind}\not{D}_A^+$, and recall we have proved: *if* $b_2^+(X) = 0$ *and* $\mathrm{Ind}\not{D}_A^+ > 0$, *then for generic* ω, $\mathcal{N}_\omega$ *is a oriented smooth manifold with dimension* $2 \cdot \mathrm{Ind}\not{D}_A^+$. Note that in this case $\mathcal{N}_\omega$ is not empty since it always contain a reducible solution.

Now we can start the proof of reformulated version of Donadlson's theorem:

Proof. Suppose there exists $\mathfrak{s}$ such that $c_1(\mathfrak{s})^2 + b_2(X) > 0$, equivalently, $k := \mathrm{Ind}\not{D}_A^+ > 0$, hence for generic ω, we have $\mathcal{N}_\omega$ is a smooth oriented manifold with dimension $2 \cdot \mathrm{Ind}\not{D}_A^+ = 2k$.

Note that $\mathcal{G}^h = \mathrm{U}(1)$ acts on $\mathcal{N}_\omega$, it has unique fixed point $[(A, 0)]$, i.e., the unique reducible solution. And $\mathrm{U}(1)$ acts on $\mathcal{N}_\omega \setminus [(A, 0)] = \mathcal{N}_\omega^{\mathrm{irr}}$ freely, more precisely, if $\Phi \neq 0$, then we have $\mathrm{e}^{\mathrm{i}\theta}[(A, \Phi)] = [(A, \mathrm{e}^{\mathrm{i}\theta}\Phi)]$, i.e., action by scalar multiplication.

Hence it is not hard to see that around $[(A,0)]$, the neighborhood V of this point in the moduli space $\mathcal{M}_\omega$ is homeomorphic to

$$\mathbb{C}^k/S^1 \cong C(\mathbb{C}P^{k-1}),$$

where $C(M)$ denotes the cone of M.

If $k=1$, then $\mathbb{C}P^0$ is just a point, then we have $\mathcal{M}_\omega \backslash V$ is a closed $1-$manifold with boundary $\mathbb{C}P^0$, only one point. But this is impossible, since we know that the number of boundaries of $1-$manifold is even, then we get a contracdiction.

If $k \geq 2$, then we have $\partial(\mathcal{M}_\omega \setminus V) = \mathbb{C}P^{k-1}$, and the restriction of $c_1(\mathcal{N}_\omega)$ on $\mathbb{C}P^{k-1}$ is the generator of its cohomology group, which can be seen directly from the construction, hence we have

$$0 \neq \int_{\mathbb{C}P^{k-1}} c_1(\mathcal{N}_\omega)^{k-1} = \int_{\partial(\mathcal{M}_\omega \backslash V)} c_1(\mathcal{N}_\omega)^{k-1} = \int_{\mathcal{M}_\omega \backslash V} \mathrm{d}\left(c_1(\mathcal{N}_\omega)^{k-1}\right) = 0,$$

which is a contracdiction, where we used the fact that Chern class is cloed.

Then we finish the proof. ♣

When paired with

- Serre's classification of indefinite forms;
- Freedman's classification of topological simply connected $4-$manifolds;
- Donaldson's diagonalizibility theorem,

they together yield the following remarkable corollary:

Corollary 6.5.1: Topological classification of simply connected smooth 4 manifold

Let X be a compact, oriented, simply connected smooth 4-manifold. Then X is homeomorphic to one of the following:

$$S^4, \quad m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}, \quad a\cdot(S^2\times S^2)\#b\cdot X_{E_8}.$$

Remark. We cannot strength the homeomorphic to diffeomorphic, because we can construct many examples of homeomorphic but not diffeomorphic 4-manifolds in the next chapter.

6.6 Exotic smooth structures on $\mathbb{R}^4$

7 Seiberg-Witten Invariants

7.1 Basic Settings and Definitions

Let (X, g) be a closed, oriented and simply connected smooth Riemannian $4-$manifold, with $b_2^+(X) > 0$, and $\mathfrak{s} \in \mathrm{Spin}^c(X)$ is a spin^c structure, fix a spin^c connection A_0.

$\boxed{\textit{Remark.}}$ We have seen in the last chapter, if $b_2^+(X) = 0$, then we have Donaldson's diagonalizibility theorem, hence such manifolds homeomophic to the connected sum of $\overline{\mathbb{C}P^2}$, then has less interests. So we focus on the $b_2^+(X) > 0$ case.

Now we want to use SW moduli spaces to define some numeric invariants, recall for

$$\begin{aligned} \mathrm{SW}_{\mathfrak{s}} : \mathcal{A}_{\mathfrak{s}} \times \Gamma(S^+) &\to \Omega^2_+(X; \mathrm{i}\mathbb{R}) \times \Gamma(S^-) \times \Omega^1(X; \mathrm{i}\mathbb{R}) \\ (A, \Phi) &\mapsto \left(F^+_{A^\tau} - \sigma(\Phi), \not{D}^+_A \Phi, \mathrm{d}^*(A - A_0)\right). \end{aligned}$$

Then for the $\mathcal{M}_{\mathfrak{s},\omega} := \mathrm{SW}_{\mathfrak{s}}^{-1}(\omega, 0, 0)/\mathcal{G}^h$, where $\mathcal{G}^h \cong \{\mathrm{e}^{\mathrm{i}c} : c \in \mathbb{R}\} \cong \mathrm{U}(1)$, we have proved:

Theorem 7.1.1

If $b_2^+(X) > 0$, then for generic $\omega \in \Omega^2_+(X; \mathrm{i}\mathbb{R})$, there is no reducible solution for $\omega-$perturbed SW equations, hence

$$\mathcal{M}_{\mathfrak{s},\omega} = \mathcal{M}^{\mathrm{irr}}_{\mathfrak{s},\omega}$$

is an orientable smooth manifold with dimension

$$d(\mathfrak{s}, X) = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)\right).$$

Actually, we have $\mathrm{SW}_{\mathfrak{s}}^{-1}(\omega, 0, 0) =: \mathcal{N}_{\mathfrak{s},\omega} = \mathcal{N}^{\mathrm{irr}}_{\mathfrak{s},\omega}$, which gives a $\mathrm{U}(1)-$principal bundle:

$$\mathrm{U}(1) \to \mathcal{N}_{\mathfrak{s},\omega} \to \mathcal{M}_{\mathfrak{s},\omega}.$$

Then we can use this structure to define the **SW invariants**.

¶ **If** $b_2^+(X) > 1$

Definition 7.1.1: Seiberg-Witten invariants

Let X be a closed oriented simply connected smooth four manifold with $b_2^+(X) > 1$, then we can define the **SW invariants**:

$$\mathrm{SW}_X(\mathfrak{s}) := \begin{cases} \left\langle c_1\left(\mathcal{N}_{\mathfrak{s},\omega}\right)^{\frac{d(\mathfrak{s},X)}{2}}, [\mathcal{M}_{\mathfrak{s},\omega}] \right\rangle & \text{if } d(\mathfrak{s},X) \equiv 0 \mod 2 \\ 0 & \text{if } d(\mathfrak{s},X) \equiv 1 \mod 2 \end{cases}.$$

Remark. Easy to see, suppose $c_1(\mathfrak{s})^2 - \sigma(X) = 8k$ by Atiyah-Singer index theorem, then we have $d(\mathfrak{s}, X) = 2k - 1 - b_2^+(X)$, hence we know that $d(\mathfrak{s}, X)$ is even iff $b_2^+(X)$ is odd.

Remark. Easy to see again, if $d(\mathfrak{s}, X) = 0$, i.e., $c_1(\mathfrak{s})^2 = 2\chi(X) + 3\sigma(X)$, then we directly have $\mathrm{SW}_X(\mathfrak{s}) = \#\mathcal{M}_{\mathfrak{s},\omega}$, the signed number of the points in $\mathcal{M}_{\mathfrak{s},\omega}$.

Definition 7.1.2: Simple type

X is called **simple type**, if for any $\mathfrak{s}$ such that $d(\mathfrak{s}, X) > 0$, $\mathrm{SW}_X(\mathfrak{s}) = 0$.

Remark. There is a **simple type conjecture** by Witten: *all four manifolds are simple type*. It has been proved that symplectict manifolds are simple type.

Now we have to prove that the defined invariants are well defined, i.e., it does NOT depend on the choice of paramters $(g, \omega) \in \mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$.

Theorem 7.1.2

If $b_2^+(X) > 1$, then $\mathrm{SW}_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$ is well-defined.

Proof. Let (g_0, ω_0) and $(g_1, \omega_1) \in \mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$, now recall $\mathcal{W}_{\mathfrak{s}}$ has codimension 2, hence there is a path $\{(g_t, \omega_t)\} \in \mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ connects (g_0, ω_0) and (g_1, ω_1).

Roughly speaking, one can use same methods as we proved $\mathcal{M}_\omega^{\mathrm{irr}}$ is manifold, we can show that

$$\mathcal{M}_{\mathfrak{s}} := \bigcup_{t\in[0,1]} \mathcal{M}_{\mathfrak{s},(g_t,\omega_t)}$$

is again a smooth manifold with dimension $d(\mathfrak{s}, X) + 1$.

If we denote $i_t : \mathcal{M}_{\mathfrak{s},(g_t,\omega_t)} \hookrightarrow \mathcal{M}_{\mathfrak{s}}$ as the inclusion, then since $\mathcal{M}_{\mathfrak{s}}$ actually gives a cobordism between $\mathcal{M}_{\mathfrak{s},(g_0,\omega_0)}$ and $\mathcal{M}_{\mathfrak{s},(g_1,\omega_1)}$. Then we have

$$i_{0,*}\left[\mathcal{M}_{\mathfrak{s},(g_0,\omega_0)}\right] = i_{1,*}\left[\mathcal{M}_{\mathfrak{s},(g_1,\omega_1)}\right] \in H^*\left(\mathcal{M}_{\mathfrak{s}}; \mathbb{Z}\right).$$

And also collect the U(1) bundele $\mathrm{U}(1) \to \mathcal{N}_{\mathfrak{s},(g_t,\omega_t)} \to \mathcal{M}_{\mathfrak{s}.(g_t,\omega_t)}$, we have

$$\mathrm{U}(1) \to \mathcal{N}_{\mathfrak{s}} \to \mathcal{M}_{\mathfrak{s}},$$

hence we have

$$\begin{aligned}
\mathrm{SW}_X(\mathfrak{s},(g_0,\omega_0)) &= \left\langle c_1\left(\mathcal{N}_{\mathfrak{s},(g_0,\omega_0)}\right)^{\frac{d(\mathfrak{s},X)}{2}}, \left[\mathcal{M}_{\mathfrak{s},(g_0,\omega_0)}\right]\right\rangle \\
&= \left\langle c_1\left(\mathcal{N}_{\mathfrak{s}}\right)^{\frac{d(\mathfrak{s},X)}{2}}, i_{0,*}\left[\mathcal{M}_{\mathfrak{s},(g_0,\omega_0)}\right]\right\rangle \\
&= \left\langle c_1\left(\mathcal{N}_{\mathfrak{s}}\right)^{\frac{d(\mathfrak{s},X)}{2}}, i_{1,*}\left[\mathcal{M}_{\mathfrak{s},(g_1,\omega_1)}\right]\right\rangle \\
&= \left\langle c_1\left(\mathcal{N}_{\mathfrak{s},(g_1,\omega_1)}\right)^{\frac{d(\mathfrak{s},X)}{2}}, \left[\mathcal{M}_{\mathfrak{s},(g_1,\omega_1)}\right]\right\rangle = \mathrm{SW}_X(\mathfrak{s},(g_1,\omega_1)),
\end{aligned}$$

finally, we finsih the proof. ♣

Exercise 1. Suppose $\mathfrak{s} = (S,\gamma)$, then we can define $\bar{\mathfrak{s}} := (\overline{S},\gamma)$, by noticing that $\mathrm{End}(S) \cong \mathrm{End}(\overline{S})$. Prove that the SW solutions for $\mathfrak{s}$ and $\bar{\mathfrak{s}}$ are one to one correspondence, hence $\mathrm{SW}_X(\mathfrak{s}) = \pm\mathrm{SW}_X(\bar{\mathfrak{s}})$.

¶ **If** $b_2^+(X) = 1$

In this case, since $\mathcal{W}_{\mathfrak{s}}$ has codimension 1, i.e., $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ has two chambers, we cannot always find a path in $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ connects any two parameters.

And actually we have

Definition 7.1.3: Seiberg-Witten invariants

Let X be a closed oriented simply connected smooth four manifold with $b_2^+(X) = 1$, then we can define the **SW invariants**:

$$\mathrm{SW}_X^+(\mathfrak{s}) := \begin{cases} \left\langle c_1\left(\mathcal{N}_{\mathfrak{s},(g,\omega)}\right)^{\frac{d(\mathfrak{s},X)}{2}}, \left[\mathcal{M}_{\mathfrak{s},(g,\omega)}\right]\right\rangle & \text{if } d(\mathfrak{s},X) \equiv 0 \mod 2 \\ 0 & \text{if } d(\mathfrak{s},X) \equiv 1 \mod 2 \end{cases},$$

where $(g,\omega) \in (\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}})^+$, similarly, we can define $\mathrm{SW}_X^-(\mathfrak{s})$.

There is a hard fact about these two invariants:

Theorem 7.1.3: Wall Crossing Formula

If $b_2^+(X) = 1$, then we have $\mathrm{SW}_X^+(\mathfrak{s}) - \mathrm{SW}_X^-(\mathfrak{s}) = \pm 1$.

Remark. For the detailed proof, one can refer Salamon's dictinoary, chapter 9.

7.2 Basic Properties and Baisc Applications

Throught in this section, X is a closed, oriented, simply connected $4-$manifold with $b_2^+(X) \geq 2$. Hence we have the well defined **SW invariants**,

$$\mathrm{SW}_X : \mathrm{Spin}^c(X) \cong \mathrm{Char}(X) \to \mathbb{Z},$$

where $\mathrm{Char}(x) = \{\xi \in H^2(X;\mathbb{Z}) : \xi \cup a \equiv a \cup a \mod 2, \forall a \in H^2(X;\mathbb{Z})\}$, and the isomorphism is given by c_1.

7.2.1 Vanishing and non-vanishing theorems

Definition 7.2.1: Basic calsses

A characteristic element or equivalently a spin^c structure $\mathfrak{s}$ is called **basic** if $\mathrm{SW}_X(\mathfrak{s}) \neq 0$.

A quick result is

Proposition 7.2.1: SW_X has finite support

The basic classe of X are finite, i.e., there exists at most finite $\mathfrak{s}$ such that $\mathrm{SW}_X(\mathfrak{s}) \neq 0$.

Proof. If $\mathrm{SW}_X(\mathfrak{s}) \neq 0$, then we know that for any generic ω, $\mathcal{M}_{\mathfrak{s},\omega} \neq \varnothing$, hence the $\omega-$perturbed SW equations have solutions. Actually since generic ω is dense, we can ensure that for any $||\omega||_{L^2} \leq 1$, we also have solutions.

Now from Chern-Weil theorem, we have

$$c_1(\mathfrak{s}) = c_1(L_{\mathfrak{s}}) = \frac{\mathrm{i}}{2\pi} F_{A^\tau},$$

hence we can easily have

$$\langle c_1(\mathfrak{s})^2, [X] \rangle = \frac{1}{4\pi^2}\left(||F_{A^\tau}^+||_{L^2}^2 - ||F_{A^\tau}^-||_{L^2}^2\right) \leq \frac{1}{4\pi^2}||F_{A^\tau}^+||_{L^2}^2.$$

Recall we have deduced the L^2-bound for $F_{A^\tau}^+$ in previous chapters:

$$\begin{aligned} ||F_{A^\tau}^+||_{L^2} &\leq \frac{1}{2\sqrt{2}}|| - s_g + 2\sqrt{2}|\omega| \, ||_{L^2} + ||\omega||_{L^2} \\ &\leq \frac{1}{2\sqrt{2}}||s_g||_{L^2} + 2||\omega||_{L^2} \\ &\leq \frac{1}{2\sqrt{2}}||s_g||_{L^2} + 2 =: C(g), \end{aligned}$$

i.e., if $\mathrm{SW}_X(\mathfrak{s}) \neq 0$, then there exists a finite constant $C'(g)$ depends only on g such that

$$\langle c_1(\mathfrak{s})^2, [X] \rangle \leq C'(g).$$

Now note that $c_1(\mathfrak{s}) \in H^2(X;\mathbb{Z}) \cong \mathbb{Z}^{b_2(X)}$, then we have such $c_1(\mathfrak{s})$ is finite, hence we know that the desired $\mathfrak{s}$ is finite. ♣

¶ Vanishing theorems

Besides the signature for spin four manifolds, we can show that SW invariants also give obstructions for the existence of PSC metric:

Theorem 7.2.1: Witten,PSC matric with vanishing SW

If X has $b_2^+(X) \geq 2$ and admits a positive scalar curvature g, i.e., $s_g > 0$, then $\forall \mathfrak{s}$, $\mathrm{SW}_X(\mathfrak{s}) = 0$.

Proof. Since X is compact, then we know that $\min\limits_{p\in X} s_g(p) > \delta > 0$. If there exists $\mathfrak{s}$ such that $\mathrm{SW}_X(\mathfrak{s}) \neq 0$, then we know that for some $\max\limits_{p\in X} |\omega(p)| < \frac{\delta}{4}$, there exists solution (A, Φ) of $\omega-$pertirbed SW equations. Note that $\Phi \neq 0$, since we can ensure that there are no reducible solutions from $b_2^+(X) \geq 2$.

However, note that for any p, we have $s_g(p) - \frac{1}{2\sqrt{2}}|\omega(p)| > 0$, then by the C^0-estimation of Φ, we have

$$|\Phi|^2 \leq \min_{p\in X}\left\{0, s_g(p) - \frac{1}{2\sqrt{2}}|\omega(p)|\right\} = 0,$$

hence $\Phi = 0$, which is a contradiction. ♣

There is also an amazing results about vanishing of SW invariants:

Theorem 7.2.2: Connected sum with vanishing SW

Suppose $X = X_1 \# X_2$, where $b_2^+(X_1), b_2^+(X_2) \geq 1$, then we have $\mathrm{SW}_X \equiv 0$.

Remark. Easy to see $b_2^+(X) \geq b_2^+(X_1) + b_2^+(X_2) \geq 2$, hence SW_X is well-defined.

This theorem is related to the **gluing problem**: suppose $X = X_1 \cup_Y X_2$, how to compute SW_X in terms of SW_{X_1} and SW_{X_2}? – the method is **monopole Floer homology**!

Sketch. For simplicity, we assume X is of simple type. $X_1 \# X_2$ can be viewed as

$$X_1 \cup \left(S^3 \times [-R, R]\right) \cup X_2,$$

since $S^3 \times [-R, R]$ admits PSC metric, hence somehow, we can *strech the neck*, i.e., sends $R \to +\infty$. When stretching, the geometry of $X_1 \# X_2$ becomes on average dominated by the connecting neck $S^3 \times [-R, R]$. This implies that all Seiberg-Witten solutions must vanish on this neck. (We will explain this more details in proving adjunction inequalities).

Therefore any solution on $X_1 \# X_2$ must come from a solution on X_1 and a solution on X_2. In other words, for every spin^c structure on X , we have

$$\mathcal{N}_{\mathfrak{s},\omega}(X) \cong \mathcal{N}_{\mathfrak{s}_1,\omega_1}(X_1) \times \mathcal{N}_{\mathfrak{s}_2,\omega_2}(X_2),$$

where $\mathfrak{s}_i$ and ω_i are the restrictions of $\mathfrak{s}$ and ω on X_i.

Note that $c_1(\mathfrak{s}_1) = c_1(\mathfrak{s}_2) = c_1(\mathfrak{s})$, $\chi(X) = \chi(X_1) + \chi(X_2) - 2$ and $\sigma(X) = \sigma(X_1) + \sigma(X_2)$, hence we have

$$d(\mathfrak{s}, X) = d(\mathfrak{s}_1, X_1) + d(\mathfrak{s}_2, X_2) + 1.$$

Since we have assumed X is of simple type, hence $d(\mathfrak{s}, X) = 0$ is only posiible case interesting, which means that there exists $i \in \{1, 2\}$ such that $d(\mathfrak{s}_i, X_i) < 0$, which means that $\mathcal{N}_{\mathfrak{s}_i,\omega_i}(X_i)$ is empty! (since $b_2^+(X_i) \geq 1$, then $\mathcal{N}_{\mathfrak{s}_i,\omega_i}(X_i)$ is either empty or a msnifold of dimension $d(\mathfrak{s}_i, X_i)$)

Hence $\mathcal{N}_{\mathfrak{s},\omega}(X)$ is empty, then we have $\mathrm{SW}_X(\mathfrak{s}) = 0$. ##

There is another result about connected sum:

Theorem 7.2.3: Blow up formula

Suppose X is simply connected, $b_2^+(X) \geq 2$, simple type, and $\{\kappa_i\}$ is the basic classes of X, then the topological blow up $X\#\overline{\mathbb{C}P^2}$ has basic classes $\{\kappa_i \pm E\}$, where $E = \mathrm{PD}.[\overline{\mathbb{C}P^1}]$, then we have

$$\mathrm{SW}_{X\#\overline{\mathbb{C}P^2}}(\kappa_i \pm E) = \pm \mathrm{SW}_X(\kappa_i).$$

Sketch. Similarly to the strategy used for the vanishing theo rem for connected sums, we start by stretching the connecting neck $S^3 \times [-R, R]$ between X and $\overline{\mathbb{C}P^2}$, and thus we also have

$$\mathcal{N}\left(X\#\overline{\mathbb{C}P^2}\right) \cong \mathcal{N}(X) \times \mathcal{N}\left(\overline{\mathbb{C}P^2}\right).$$

Note that $b_2^+\left(\overline{\mathbb{C}P^2}\right) = 0$, and the Fubini-Study metric is a PSC metric on $\overline{\mathbb{C}P^2}$, hence generically, $\mathcal{N}\left(\overline{\mathbb{C}P^2}\right)$ is a single point, i.e., the reducible solution.

Hence we have $\mathcal{N}\left(X\#\overline{\mathbb{C}P^2}\right) \cong \mathcal{N}(X)$, then $\mathcal{M}\left(X\#\overline{\mathbb{C}P^2}\right) \cong \mathcal{M}(X)$, then by the dimension formula, we know that $d(\mathfrak{s}, \overline{\mathbb{C}P^2}) = -1$, then easy to see $c_1(\mathfrak{s}) = \pm E$ is the only possibility. ##

Remark. I hope I could give a *detailed proof* about above theorems in the part of *Monopole Floer Homology*.

Remark. The detailed proof of connected sum vanishing and blow up formula can be found in *Notes on Seiberg-Witten Theory*, chapter 4.

¶ Non-vanishing theorems

The most improtant results about SW invariants is the following:

Theorem 7.2.4: Taubes

Let X be a simply connected $b_2^+(X) \geq 2$ symplectic manifold with symplectic form ω. Let J be a almost complex structure compatible with ω. Then $c_1(TX, J)$ is a basic class, and

$$\text{SW}_X\left(\pm c_1(TX, J)\right) = \pm 1.$$

This is a huge result proven by Taubes using hard analysis. The result is called ”SW = GW”, in which it was shown that the Gromov Witten invariants and Seiberg-Witten invariants are equal (when they are both defined). While SW counts solutions to the SW equations, GW counts the number of J-holomorphic curves.

We may (?) outline a proof in the later chapters, but we will do give a explicit proof of the following corollary: note that Kähler surfaces are special symplectic four manifolds,

Corollary 7.2.1

Let X be a simply connected $b_2^+(X) \geq 2$ Kähler surface, then $c_1(TX)$ is a basic class, and

$$\text{SW}\left(\pm c_1(TX)\right) = \pm 1.$$

We will give it a proof in the next section.

7.2.2 Basic applications

In this section, let's see some quick but very interesting (and also amazing!) results deduced from SW invariants:

¶ Interesting results in symplectic topology

From Taube's big theorem, we know that symplectict manifolds always admit nonvanishing SW invariants. Now suppose X is a simply connected symplectic four manifold, with $b_2^+(X) \geq 2$, hence we have:

- **symplectic manifold X does NOT admit PSC metric** .

  This is immediately from Witten's vanishing theorem about PSC metric.

  Remark. This result in nolonger true if we drop the assumptions of $b_2^+(X) \geq 2$, because $\mathbb{C}P^2$ with Fubini-Study metric is an counter-example.

  Remark. People also conjecture that symplectic manifold does NOT admit hyperbolic structure, but there is only few known examples to support this conjecture. For examples, Ian Agol and Francesco Lin's paper, they constructed a closed arithmetic hyperbolic 4-manifolds with vanishing Seiberg-Witten invariants.

- **symplectict manifold X is irreducible** .

  A four manifold is called **irreducible** if it can be spilted into $X_1\#X_2$ with $b_2^+(X_1), b_2^+(X_2) \geq 1$. Then this is also immediately from the vanishing theorem about connected sums.

  Remark. This result is also false if we drop the assumptions of $b_2^+(X) \geq 2$, because

  $$\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$$

  are counter-eamples if $q \geq 1$, since they are **symplectict blow up** of $\mathbb{C}P^2$, then we know that they are also symplectic.

- **$\{$symplectic 4-manifolds$\} \subsetneq \{$almost complex 4-manifolds$\}$**

  It is well-known that any symplectic manifolds admit an almost complex structure, but the converse is false, i.e., there exists simply connected $4-$manifold admit almost complex structure but NOT symplectic.

  Recall we have proved in the part 1, *a simply connected $4-$manifold X admits almost complex structure iff $b_2^+(X)$ is odd*. Now consider

  $$X_{p,q} := p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2},$$

then easy to see when $p \geq 3$ is odd, $X_{p,q}$ is almost complex since $b_2^+(X) = p$ is odd, but $p > 2$ and $X_{p,q}$ is connected sum of $\mathbb{C}P^2$ and $(p-1)\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, both have $b_2^+ \geq 1$, hence $X_{p,q}$ is irreducible, i,e,, not symplectic.

$\boxed{\textit{Remark.}}$ There is also a "geometric proof" of $X_{p,q}$ is not symplectic if $p \geq 2$: Schoen and Yau proved that *if M_1, M_2 are manifolds of dimension n at least 3, and M_1 and M_2 each admit metrics with positive scalar curvature, then so does the connected sum.* Hence from the symplectic manifolds does not admit PSC metric, we can also get there.

Exercise 1. Try to find elementray proof of $\mathbb{C}P^2\#\mathbb{C}P^2$ is not symplectic.

¶ Homeomorphic but NOT diffeomorphic manifolds

Since SW inavriant is defined for smooth manifolds, hence it can play a role as diffeomorphic invariant, which helps us find a of homeomorphic but NOT diffeomorphic manifolds:

The basic ingredients are complex projective surfaces Z_d, recall we have proved in part 1, that for $d \geq 2$,

- If d is odd, then we have Z_d has intersection form

$$\left(\frac{1}{3}d^3 - 2d^2 + \frac{11}{3}d - 1\right) \cdot (1) \oplus \left(\frac{2}{3}d^3 - 2d^2 + \frac{7}{3}d - 1\right) \cdot (-1).$$

- If d is even, then we have Z_d has intersection form

$$\left(\frac{1}{3}d^3 - 2d^2 + \frac{11}{3}d - 1\right) \cdot H \oplus \left(\frac{d(4-d^2)}{24}\right) \cdot E_8.$$

And for smooth simply connected 4 manifolds X_1, X_2, if $Q_{X_1} \cong_{\mathbb{Z}} Q_{X_2}$, then they are homeomorphic by Freedman's celebarating classification theorem.

Example 7.2.1. *Consider Z_5, then direct calculation tells us that*

$$Q_{Z_5} \cong 9(1) \oplus 44(-1),$$

hence consider $X_{9,44} = 9\mathbb{C}P^2\#44\overline{\mathbb{C}P^2}$, then they are homeomorphic.

But note that since Z_5 is Kähler surface, hence SW_{Z_5} not identically zero. But $X_{9,44}$ is reducible hence has zero SW invariants. Then we know that Z_5 and $X_{9,44}$ are not diffeomorphic.

Exercise 2. Prove that $K3\#\overline{\mathbb{C}P^2}$ and $X_{3,20} = 3\mathbb{C}P^2\#20\overline{\mathbb{C}P^2}$ are homeomorphic but not diffeomorphic.

$\boxed{\text{Hint:}}$ $K3$ is topologically just Z_4.

7.3 Seiberg-Witten Invariants on Kähler Surfaces

7.4 Computations for Special Manifolds

7.4.1 $K3$ and T^4

7.4.2 Eienstein Manifolds

8 SW Theory and Low Dimensional Topology

8.1 The Thom Conjecture

In this section, we give the very first application of SW invariants——the proof of (classical) Thom conjecture:

Theorem 8.1.1: Classical Thom Conjecture. Kronheimer, Mrowka, 1994

Let Σ be an oriented closed surface smoothly embedded in $\mathbb{C}P^2$ such that $[\Sigma] = d[\mathbb{C}P^1]$, where $d \in \mathbb{N}$, then we have the genus of Σ, $g(\Sigma)$ satisfies

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}.$$

One may wonder why $\frac{(d-1)(d-2)}{2}$? It is nothing but the genus of the complex representative of the homology class $d[\mathbb{C}P^1]$:

Proposition 8.1.1

Consider $C_d \subset \mathbb{C}P^2$, the degree d algebraic curve in $\mathbb{C}P^2$, i.e., the zero locus of a generic homogeneous polynomial of degree d. Then we have $[C_d] = d[\mathbb{C}P^1]$, and $g(C_d) = \frac{(d-1)(d-2)}{2}$.

Proof. By algebra fundamental theorem and the natural orientation of complex submanifold, we have $[C_d] \cdot [\mathbb{C}P^1] = d$, i.e., intersect positively at d points, then we know that $[C_d] = d[\mathbb{C}P^1]$.

Now from $T\mathbb{C}P^2 = TC_d \oplus NC_d$, and $c_1(NC_d)[C_d] = [C_d] \cdot [C_d]$, $c_1(T\mathbb{C}P^2) = -3[\mathbb{C}P^1]$, then we have

$$2 - 2g(C_d) = d^2 - 3d, \quad \Rightarrow \quad g(C_d) = \frac{(d-1)(d-2)}{2},$$

which is as desired. ♣

From the above proposition, we can also directly prove the Thom conjecture for $d = 1, 2$, since in both cases $\frac{(d-1)(d-2)}{2} = 0$, which always holds.

Firstly, We will give a detailed proof of $d = 3$ case by using Rohlin's theorem, one can also refer the page 46 of *4-Manifolds and Kirby Calculus*.

¶ **Proof of $d = 3$ case**

The proof relies on the improtant techniques we introduced in the prlimanaries: the blow-up process and proper transform. Recall :

- Blow-up of X is just $X\#\overline{\mathbb{C}P^2}$, and conversely for the blow-down:
  - *If there exists exceptional sphere Σ_- in smooth $4-$manifold X', i.e., with $\Sigma_- \cdot \Sigma_- = -1$, then there exists smooth $4-$manifold X such that X' is diffeomorphic to $X\#\overline{\mathbb{C}P^2}$.*
  - *If there exists co-exceptional sphere Σ_+ in smooth $4-$manifold X', i.e., with $\Sigma_+ \cdot \Sigma_+ = 1$, then there exists smooth $4-$manifold X such that X' is diffeomorphic to $X\#\mathbb{C}P^2$.*
- Proper transform of embedded surface Σ of X, just blow up X at the point $p \in \Sigma$, then we get $\widetilde{\Sigma} \subset X\#\overline{\mathbb{C}P^2}$, which is $\Sigma\#L$, where L denotes the "line" of Σ across p.

  If we take $E = [\overline{\mathbb{C}P^1}] \in H_2(\overline{\mathbb{C}P^2}; \mathbb{Z})$, then easy to see $[\widetilde{\Sigma}] = [\Sigma] - E$, becase $[L] = -E$. But note that L s actually a sphere hence $\widetilde{\Sigma}$ and Σ has same genus.
  - *proper tansform helps us elimanate the positive intersections, suppose Σ_1 and Σ_2 positively intersect transversely at p, then if we blow up at p, we get $\widetilde{\Sigma_1}$ and $\widetilde{\Sigma_2}$, in this case*

$$[\widetilde{\Sigma_1}] \cdot [\widetilde{\Sigma_2}] = ([\Sigma_1] - E) \cdot ([\Sigma_2] - E) = [\Sigma_1] \cdot [\Sigma_2] - 1.$$

Now we can use the above techniques to give the proof:

Proof. Suppose there exists sphere Σ represents $3[\mathbb{C}P^1]$, then easy to see $\Sigma \cdot \Sigma = 9$, hence it at least contains 9 positve self intersections. Blow up at any eight of the positive intersections, then we get $\widetilde{\Sigma}$ in $X\#8\overline{\mathbb{C}P^2}$.

If we deonte the exceptional spheres as $E_1, \cdots, E_8$, then we have

$$[\widetilde{\Sigma}] = 3[\mathbb{C}P^2] - E_1 - E_2 - \cdots - E_8,$$

in this case $\widetilde{\Sigma} \cdot \widetilde{\Sigma} = 1$, and $\widetilde{\Sigma}$ is still a sphere. More precisely, $\widetilde{\Sigma}$ is the co-exceptinal sphere in $X\#8\overline{\mathbb{C}P^2}$. Hence we can co-blow down $\widetilde{\Sigma}$, i.e., there exists a smooth four manifold W such that $W\#\mathbb{C}P^2$ diffeomorhic to $X\#8\overline{\mathbb{C}P^2}$.

Note that $H_2(W\#\mathbb{C}P^2) = H_2(W) \oplus H_2(\mathbb{C}P^2)$, and the later is generated by $[\widetilde{\Sigma}] = 3[\mathbb{C}P^2] - E_1 - E_2 - \cdots - E_8$. We claim that $H_2(W)$ is generated by

$$\left\{E_1 - E_2, E_2 - E_3, \cdots, E_7 - E_8, E_6 + E_7 + E_8 - [\mathbb{C}P^1]\right\},$$

one can verify this by firstly each of them is orthognoal to $[\widetilde{\Sigma}]$, and then show that the intersection form of them is unimodular (to ensure they are lineraly independent).

However, more interstingly, we can directly calculate that the intersection form of the basis aove is given by matrix $-E_8$! (here E_8 may be little confusing, it is the more well-known one)., i.e., $Q_W = -E_8$ and W is smooth. But by Rohlin's theorem $\sigma(W) = -8$ must be divisble by 16, which is a contracdiction! ♣

Remark. We will use this proper tansform techniques frequently when we dealing with the positive intersections.

Exercise 1. Prove that $3[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2;\mathbb{Z}) \subset H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2};\mathbb{Z})$ cannot be represented by an embedded sphere in $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$.

Hint: Again consider blow up at eight positive self intersections, and get a smooth manifold $W\#\mathbb{C}P^2 \cong \mathbb{C}P^2\#(n+8)\overline{\mathbb{C}P^2}$, and show that $Q_W \cong -E_8 \oplus n(-1)$, which cannot be diagnoalized, hence contradicts to Donadlson's theorem.

Remark. To prove $Q_1 := E_8 \oplus n(1) \ncong_{\mathbb{Z}} (n+8)(1) =: Q_2$, although they have same rank, signature and parity, but we can count the number of $v \in \mathbb{Z}^{n+8}$ such that $Q_i(v,v) = 1$. Note that for Q_2, the number is $2(n+8)$. But for Q_1, easy to see $Q_1(v,v) = 2k + \sum_{i=9}^{n+8} v_i^2$, hence only have $2n$ such v, where $2k$ from E_8 is even.

¶ Proof of $d > 3$ case

So from now on, we follow Kronheimer and Mrowka's original proof, and focus on the $d > 3$ case. This assumption will be essential in the proof, we will see this later.

Remark. Roughly speaking, the resriction of 3 comes from $c_1(K_{\mathbb{C}P^2}) = 3$.

¶ Symplectict Thom conjecture

sketch

¶ Immersed Thom conjecture

sketch

8.2 More about the Minimum Genus of Embedded Surfaces

As we have seen about the Thom conjecture, the following genus function is one of the intersting objects we concered in low dimensional topology:

Definition 8.2.1: Genus Fnuction

Suppose X is a closed oriented simply connected $4-$manifold, then the **genus function** G_X is defined on $H_2(X;\mathbb{Z})$ as follows: For $\alpha \in H_2(X;\mathbb{Z})$,consider

$$\mathsf{G}_X(\alpha) := \min\{g(\Sigma) : \Sigma \hookrightarrow X, \quad [\Sigma] = \alpha\},$$

where Σ ranges over closed, connected, oriented surfaces smoothly embedded in X.

Remark. One may wonder, why we don't condier the maximal genus? Because we can add genus as much as we want! For any embedded surface Σ, take a small ball B^4 away from Σ and consider a tours $T^2 \subset B^4$ which also bounds a solid torus, hence $[T^2] = 0$, then consider $\Sigma \# T^2$, easy to see $[\Sigma \# T^2] = [\Sigma] + [T^2] = [\Sigma]$, but with genus plus 1.

Easy to see, we have $\mathsf{G}_X(\alpha) \geq 0$, $\mathsf{G}_X(\alpha) = \mathsf{G}_X(-\alpha)$ and $\mathsf{G}(0) = 0$. But one should note that $\mathsf{G}_X(\alpha) + \mathsf{G}_X(\beta)$ may not be greater then $\mathsf{G}_X(\alpha+\beta)$, so G_X will not be a semi-norm.

Remark. The reason I methion seminorm here because there is a so called **Thurston norm** in $3-$manifolds, which plays a similar role as G_X in $4-$dimension, and for Thurston "norm", it is indeed a semi-norm, i.e., satisfies the triangle inequality. For the details, one can ref *Wild world of $4-$manifolds*, section 11.3.

Example 8.2.1. *We have the following known examples:*

- $\mathsf{G}_{S^4} \equiv 0$, *since* $H_2(S^4;\mathbb{Z}) = 0$. *LOL...*
- *From the thom conjecture, we know that*

$$\mathsf{G}_{\mathbb{C}P^2}\left(d[\mathbb{C}P^1]\right) = \frac{(|d|-1)(|d|-2)}{2},$$

then we can see that $\mathsf{G}_{\mathbb{C}P^2}(6[\mathbb{C}P^1]) = 10 > 1{+}1$, *where* $\mathsf{G}_{\mathbb{C}P^2}(3[\mathbb{C}P^1]) = 1$, *is a counter-example of the previous discussion.*

proof of Ruberman's thm of $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, article. (adjunction formula)

8.3 Slice Genus and the Milnor Conjecture

Ref 283notes 4.6,4.7 and jianfeng lin lec 15 loacl Thom conjecture and Milnor conjecture and further on knots.

8.4 The Fruta's $\frac{10}{8}$−theorem

Fruta theorem and $11/8$ conj (Ref 283notes 4.10 and jianfeng lin)

8.5 The Fintushel-Stern Surgery

knot surgery ref: six lectures on SW, wild word chap 12.

$$\mathrm{SW}_{X_K} = \mathrm{SW}_X \cdot \Delta_K \left(\mathrm{e}^{2[T]}\right),$$

X_K and X are homeomorhic but not diffeomorphic.

conj: $\mathrm{K3}_K$ is diffeomorphic to $\mathrm{K3}_{K'}$ iff $\pi_1(S^3 - K) \cong \pi_1(S^3 - K')$.

8.6 Rational Blow-downs

Wild chap12 notes

9 SW Theory and Symplectic Topology

Ref An introduction to the Seiberg-Witten equations on symplectic manifolds Michael Hutchings and Clifford Henry Taubes

Part III

Instanton Floer Homology

10 ASD Connections

10.1 Basic Settings

Throughout this chapter, we will consider the $\mathrm{SU}(2)-$bundle P over compact oriented Riemanian four manifold X. For the curvature $F_A \in \Omega^2(\mathrm{ad}P)$, locally, $F_A|_U \in \Omega^2(U) \otimes \mathfrak{su}(2)$, hence we can define the inner product $\langle , \rangle$ on $\Omega^2(\mathrm{ad}P)$ by

$$\langle \alpha, \beta \rangle \mathrm{dvol}_X := \mathrm{tr}(\alpha \wedge *\beta^{\mathsf{H}}) = -\mathrm{tr}(\alpha \wedge *\beta),$$

which is combined the inner products $\omega \wedge *\eta$ on $\Omega^2(U)$ and $\mathrm{tr}(AB^{\mathsf{H}})$ on $\mathfrak{su}(2)$.

Hence we can define the very first interesting functional

Definition 10.1.1: Yang-Mills Functional

For the connection $A \in \mathcal{A}(P)$, we define the **Yang-Mills functional** of A as

$$\mathcal{YM}(A) := \int_X |F_A|^2 \mathrm{dvol}_X := \int_X \langle F_A, F_A \rangle \mathrm{dvol}_X.$$

Now we can calculate the critial points of $\mathcal{YM}$ on $\mathcal{A}(P)$. Note that $\mathcal{A}(P)$ is affine space modelled on $\Omega^1(\mathrm{ad}P)$, then if A is the critical point, then for any $a \in \Omega^1(\mathrm{ad}P)$, we have $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{YM}(A + ta) = 0$.

Proposition 10.1.1

For any $a \in \Omega^1(\mathrm{ad}P)$, $F_{A+a} = F_A + \mathrm{d}_A a + \frac{1}{2}[a \wedge a]$, here $\mathrm{d}_A : \Omega^*(\mathrm{ad}P) \to \Omega^{*+1}(\mathrm{ad}P)$ is defined in proposition 1.3.2.

Proof. Still thanks to the isomorphism of π^*, we note that

$$\widehat{F_{A+a}} = \mathrm{d}A + \mathrm{d}a + \frac{1}{2}[A \wedge A] + [A \wedge a] + \frac{1}{2}[a \wedge a],$$

note that $\pi^* \mathrm{d}_A a = \mathrm{d}a + A \cdot a = \mathrm{d}A + \mathrm{ad}_{*,e}(A) \wedge a = \mathrm{d}a + [A \wedge a]$, hence we have $\widehat{F_{A+a}} = \widehat{F_A} + \pi^* \mathrm{d}_A a + \frac{1}{2}[a \wedge a]$, then we know the proof. ♣

Now we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathcal{YM}(A+ta) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_X |F_{A+ta}|^2\mathrm{dvol}_X \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_X \left(|F_A|^2 + 2t\langle F_A, \mathrm{d}_A a\rangle + o(t^2)\right)\mathrm{dvol}_X \\ &= 2\int_X \langle F_A, \mathrm{d}_A a\rangle \mathrm{dvol}_X = 2\int_X \langle \mathrm{d}_A^* F_A, a\rangle \mathrm{dvol}_X,\end{aligned}$$

where d_A^* is the dual operator of d_A.

Hence if A is the critical point, then we have $\boxed{\mathrm{d}_A^* F_A = 0}$, which is the **Yang-Mills equations**.

Remark. Actually we usually denote the above $\mathrm{d}_A^* F_A = 0$ and the Bianchi identity $\mathrm{d}_A F_A = 0$ together as Yang-Mills equations. It somehow can be viewed as an non-linear analogue of harmonic forms, i.e., $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$.

In general, this equation is not so easy to study, luckily, we can use a little calculation to find the minimum of Yang-Mills functional.

To do this, we need the decomposition of $\Omega^2(X)$ through the Hodge star $*$, reacll $*^2 = (-1)^{p(n-p)}$, hence $* : \Omega^2(X) \to \Omega^2(X)$ satisfies $*^2 = 1$, then we have $\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$ by the eigenvalue of $*$.

This decomposition is local, so we can get the similar result as $\Omega^2(\mathrm{ad}P) = \Omega^2_+(\mathrm{ad}P) \oplus \Omega^2_-(\mathrm{ad}P)$, hence $F_A =: F_A^+ + F_A^-$, where $*F_A^+ = F_A^+$ and $*F_A^- = -F_A^-$, then we have

$$\begin{aligned} c_2(P) &= \frac{1}{8\pi^2}\mathrm{tr}\left(F_A^+ \wedge F_A^+ + F_A^+ \wedge F_A^- + F_A^- \wedge F_A^+ + F_A^- \wedge F_A^-\right) \\ &= \frac{1}{8\pi^2}\mathrm{tr}\left(-F_A^+ \wedge *(F_A^+)^{\mathsf{H}} + F_A^+ * (\wedge F_A^-) - F_A^- \wedge *(F_A^+)^{\mathsf{H}} + F_A^- \wedge *(F_A^-)^{\mathsf{H}}\right) \\ &= \frac{1}{8\pi^2}\left\{-|F_A^+|^2 + |F_A^-|^2 - \langle F_A^+, F_A^-\rangle + \langle F_A^-, F_A^+\rangle\right\}\mathrm{dvol}_X \\ &= \frac{1}{8\pi^2}\left(|F_A^-|^2 - |F_A^+|^2\right)\mathrm{dvol}_X.\end{aligned}$$

Note that $\langle F_A^+, F_A^-\rangle = 0$, we have

$$\mathcal{YM}(A) = \int_X |F_A|^2\mathrm{dvol}_X = \int_X \left(|F_A^+|^2 + |F_A^-|^2\right)\mathrm{dvol}_X,$$

hence we know that

$$\begin{aligned}\mathcal{YM}(A) &= 8\pi^2\int_X c_2(P) + 2\int_X |F_A^+|^2\mathrm{dvol}_X \\ &= -8\pi^2\int_X c_2(P) + 2\int_X |F_A^-|^2\mathrm{dvol}_X,\end{aligned}$$

then we know that,

$$\mathcal{YM}(A) \geq \left|8\pi^2\langle c_2(P), [X]\rangle\right| =: |\kappa(P)|,$$

where $\kappa(P)$ is a topological invariant independent on the choice of connection.

Hence we know that the Yang-Mills functional has the minmum $|\kappa(P)|$, which we also called the **instanton number** of P. And the equality holds if and only if $F_A^+ = 0$ when $\kappa(P) \geq 0$, or $F_A^- = 0$ when $\kappa(P) < 0$.

Definition 10.1.2

For $A \in \mathcal{A}(P)$, we said A is

- **Yang-Mills connctions**, if $\mathrm{d}_A^* F_A = 0$;
- **ASD connetions**, if $F_A^+ = 0$.
- **SD connetions**, if $F_A^- = 0$.

Remark. If the connection A admits the mimunm of Yang-Mills functional, we sometimes call it is the **instanton solution** or just **instanton**.

From the discussion above we know that if A is ASD then $\mathcal{YM}(A)$ attains the minmum, hence A is also the critial point, i.e., A is Yang-Mills. Then we have

$$\{\text{ASD conncetions}\} \subseteq \{\text{Yang-Mills Connections}\}, \quad F_A^+ = 0 \Rightarrow \mathrm{d}_A^* F_A = 0.$$

Remark. One can try to prove $F_A^+ = 0 \Rightarrow \mathrm{d}_A^* F_A = 0$ directly. Only suffices to note that

$$\mathrm{d}_A^* F_A = \pm *_g \mathrm{d}_A *_g F_A = \mp *_g \mathrm{d}_A F_A,$$

then it is immediate from Bianchi identity.

In the following, we want to study the modulis spaces of ASD/SD connections, i.e the connection spaces modulo the gauge transformations. After some suitable (Uhlenbeck) compactification, this moduli space will plays a central role in the study of four dimensional topology.

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Beofore we go further, it is necessary to give some examples of ASD connections. $\mathrm{SU}(2)$ case is little tricky, we look for a example in the next section, we consider the $\mathrm{U}(1)$ case here, i.e., ASD connections for the line bundle.

Suppose $P \to X$ is a $\mathrm{U}(1)-$bundle, A is the connection on P, then we have $c_1(P) = \left[\frac{\mathrm{i}}{2\pi} F_A\right] \in H^2(X;\mathbb{R})$. Hence we know that $\mathrm{d}F_A = 0$, i.e., the curvature form $F_A \in \Omega^2(X;\mathrm{i}\mathbb{R})$ is closed.

Proposition 10.1.2

Given $\mathrm{U}(1)-$bundle $P \to X$, there exists connection A such that F_A is harmonic.

Proof. Fix connection $A_0 \in \mathcal{A}_P$, the connections space of P, then we know that $A = A_0 + a$ for some $a \in \Omega^1(X; \mathrm{i}\mathbb{R})$, hence we have

$$F_A = F_{A_0+a} = F_{A_0} + \mathrm{d}a + \frac{1}{2}[a \wedge a] = F_{A_0} + \mathrm{d}a.$$

♣

10.2 Instantons on S^4: ADHM Constructions

10.3 Gauge Transformations and ASD Moduli Space

10.4 Uhlenbeck's Compactness Theorem

10.5 Taubes' Gluing Theorem

10.6 Reducible Solutions

10.7 The Donaldson Polynomial Invariant

11 Chern Simons Functional

11.1 Basic Definitions

11.2 Critical Points:Flat Connection

Part IV

Monopole Floer Homology

Part V

Heegaard Floer Homology