Solutions to Lectures on Symplectic Geometry

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Abstract

I typed this during 2023, symplectic geometry and symplectic topology seminar by Huagui Duan. The reference is [1].

Contents

1 Homework 1: Symplectic Linear Algebra

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal** Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V | \Omega(v, u) = 0, \forall u \in Y\}. \tag{1.1}$$

Problem 1.1. *Show that* $\dim Y + \dim Y^\Omega = \dim V$.

Proof. Consider the linear map below

$$\begin{aligned} V &\to Y^* = \mathrm{Hom}(Y, \mathbb{R}) \\ v &\mapsto \Omega(v, \cdot)|_Y \end{aligned}$$

and we denote it as f, thus naturally $v \in \mathrm{Ker}f$ iff $\Omega(v, \cdot)|_Y = 0$ iff for any $u \in Y$, $\Omega(v, u) = 0$, iff $v \in Y^\Omega$, thus we have $Y^\Omega = \mathrm{Ker}f$.

On the other hand, we will show that f is epimorphism, i.e., surjective. Suppose $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ is a symplectic basis of V, then suppose $\{u_1, \cdots, u_k\}$ is a basis of Y, and $k = \dim Y$, we CLAIM: there exists $\{v_1, \cdots, v_k\} \subset V$ such that $\Omega(u_i, v_j) = \delta_{ij}$, now suppose $u_l = u_l^i e_i + u_l^\alpha f_\alpha$, and let $v_s = v_s^i e_i + v_i^\alpha f_\alpha$, which the coefficients are to be determined. then from $u_1, \cdots, u_k$ are linearly independent, one can always solve the linear equation and find a desired $v_1, \cdots, v_k$, then we for any $g \in Y^*$, we define $v = g(u_i)v_i$, then one can directly check that for any $u \in Y$, $\Omega(v, u) = g(u)$.

So in short, we have $\mathrm{Ker}f = Y^\Omega$ and $\mathrm{Im}f = Y^* \cong Y$, thus we have $\dim Y + \dim Y^* = V$. □

Problem 1.2. *Show that* $(Y^\Omega)^\Omega = Y$.

Proof. On the one hand, from problem 1.1 above, we have $\dim Y = \dim (Y^\Omega)\Omega$, and since for any $u \in Y$, $v \in Y^\Omega$, by definition, we have $\Omega(u, v) = 0$, then $Y \subseteq (Y^\Omega)^\Omega$, thus we have $Y = (Y^\Omega)^\Omega$. □

Problem 1.3. *Show that, if* V *and* W *are subspaces, then* $Y \subseteq W$ *if and only if* $W^\Omega \subseteq Y^\Omega$.

Proof. One the one hand, if $Y \subseteq W$, then for any $w' \in W^\Omega$, since any $y \in Y$ we have naturally $y \in W$, then $\Omega(w', y) = 0$, thus $W^\Omega \subseteq Y^\Omega$. For the other side, since $W^\Omega \subseteq Y^\Omega$, then from problem 1.2, we have $Y = (Y^\Omega)^\Omega \subseteq (W^\Omega)^\Omega = W$, then we finish the proof. □

Problem 1.4. *Show that Y is **symplectic**, i.e., $\Omega|_{Y\times Y}$ is nondegenerate if and only if $Y\cap Y^{\Omega}=\{0\}$ if and only if $V=Y\oplus Y^{\Omega}$.*

Proof. From probelm 1.1, we have $\dim Y+\dim Y^{\Omega}=\dim V$, so it is trivial to see that $Y\cap Y^{\Omega}=\{0\}$ if and only if $V=Y\oplus Y^{\Omega}$, now if $\Omega|_{Y\times Y}$ is nondegenerate, then if $y\in Y\cap Y^{\Omega}$, then for any $y'\in Y$, we have $\Omega(y,y')=0$, which is clearly absurd, then on the other hand, if $Y\cap Y^{\Omega}=\{0\}$, then if $y\in Y$ and for any $y'\in Y$ we have $\Omega(y,y')=0$, then $y\in Y^{\Omega}$, thus $y'=0$, then $\Omega|_{Y\times Y}$ is nondegenerate. Finally we finish the proof. ☐

Remark 1. *The propeties above are all same with the classical orthogonal for Euclidean spaces.*

Problem 1.5. *We call Y is **isotropic** when $Y\subset Y^{\Omega}$, i.e., $\Omega|Y\times Y\equiv 0$, show that if Y is isotropic, then $\dim Y\le\frac{1}{2}\dim V$.*

Proof. This is directly from $\dim Y+\dim Y^{\Omega}=\dim V$ and the isotropic definition: $Y\subseteq Y^{\Omega}$. ☐

Remark 2. *Actually, one can trust it intutively, if $\dim Y>\frac{1}{2}\dim V$, then consider the symplectic baisis of V, then it is not hard to find it is absurd.*

Problem 1.6. *We call Y **coisotropic** when $Y^{\Omega}\subseteq Y$, check that every codimension 1 subspace Y is coisotropic.*

Proof. Since Y has codimension 1, then $\dim Y^{\Omega}=1$, suppose $Y^{\Omega}=\{au|a\in\mathbb{R}\}$, then if $u\notin Y$, then u combined with the basis of Y form a basis of V, then since for any $y\in Y$, $\Omega(u,y)=0$ and $\Omega(u,u)=0$, we have $\Omega(u,\cdot)\equiv 0$, which is contracdicted to Ω is nondegenerate. ☐

Remark 3. *For the same reason, if Y has dimension 1, then Y is isotropic.*

Problem 1.7. *An isotropic subspace Y of (V,Ω) is called **Lagrangian** when $\dim Y=\frac{1}{2}\dim V$, check that Y is Lagrangian iff Y is isotropic and coisotropic iff $Y=Y^{\Omega}$.*

Proof. This problem is trivial from definition. ☐

Problem 1.8. *Show that, if Y is a Lagrangian subspace of (V,Ω), then any basis $e_1,\cdots,e_n$ of Y can be extened to a symplectic basis $e_1,\cdots,e_n,f_1,\cdots,f_n$ of (V,Ω).*

Proof. Note that Y is Lagrangian iff $Y=Y^{\Omega}$, so we have $\Omega(e_i,e_j)=0$, since Ω is nondegenerate, so we can find $f_1\in V$ such that $\Omega(e_1,f_1)=1$, then we do the same thing as throrem 1.1 of [1]:

Firstly, we consider $W=\mathrm{span}\{e_1,f_1\}$, then we consider W^{Ω}, then we have $W\cap W^{\Omega}=\{0\}$, then we have $V=W\oplus W^{\Omega}$, and trivially we can see that $V'=\mathrm{span}\{e_2,\cdots,e_n\}$ is the Lagrangian subspace of W^{Ω}, which is because V' is already an isotropic subspace (this comes from the subspace of isotropic space is still isotropic), so we can finish the proof by induction. ☐

Remark 4. *The propoties from problem 1.5 is a little "weired", this is because in Euclidean orthogonal, $\Omega(u,u)=0$ will never happen, thus there will also have $W\cap W^{\perp}=\{0\}$. One should always be careful.*

Problem 1.9. *Show that, if Y is a lagrangian subspace, (V,Ω) is symplectomorphic to the space $(Y\oplus Y^*,\Omega_0)$, where Ω_0 is determined by the formula*

$$\Omega_0(u\oplus,v\oplus\beta)=\beta(u)-\alpha(v). \tag{1.2}$$

In fact, for any vector space E, the direct sum $V=E\oplus E^*$ has a canonical symplectic structure determined by the formula above. If $e_1,\cdots,e_n$ is a basis of E, and $f_1,\cdots,f_n$ is the dual basis, then $e_1\oplus 0,\cdots,e_n\oplus 0,0\oplus f_1,\cdots,0\oplus f_n$ is a symplectic basis for V.

Proof. Suppose $e_1,\cdots,e_n$ is the basis of Y, and from problem 1.8, we have we can extend it to a symplectic basis of V, i.e., $e_1,\cdots,e_n,f_1,\cdots,f_n$, then we define $\varphi:V\to Y\oplus Y^*$, suppose the dual basis of e_i is E_i, then $\varphi(e_i)=e_i$, and $\varphi(f_i)=E_i$, now we check $\varphi^*\Omega_0=\Omega$, note that $\varphi^*\Omega_0(e_i,e_j)=\Omega_0(e_i\oplus 0,e_j\oplus 0)=0$, $\varphi^*\Omega_0(e_i,f_j)=\Omega_0(e_i\oplus 0,0\oplus E_j)=\delta_{ij}$, and $\varphi^*\Omega_0(f_i,f_j)=0$, so we know that the two symplectic spaces are symplectomorphic. ☐

Question 1.10. *From problem 1.8 we know that for lagrangian subspace and any basis $e_1,\cdots,e_n$, we can extend them to a symplectic basis of V, and from Gram-Schmidt orthogonal process, we know that for any basis of a Euclidean subspace, we can extend them to an orthongonal basis, so the question is: for what subspace Y, we can find a basis of Y and then extend them to a symplectic basis?*

Solution. **Claim 1**: if Y is an isotropic subspace, then for any basis of Y, we can extend them to a symplectic basis. Since $Y \subseteq Y^{\Omega}$, if $Y = Y^{\Omega}$, then it is lagrangian space, so it is true from problem 1.8. If $Y \subset Y^{\Omega}$, and the basis is $e_1, \cdots, e_k$, then there exists $e_{k+1} \in Y^{\Omega} \setminus Y$, so consider $Y' = \text{span}\{e_1, \cdots, e_{k+1}\}$, then it is not hard to see that Y' is also an isotropic subspace, so we can continue this procedure until Y' is lagrangian, then from 1.8 we can finish the proof.

Observation 2: consider $\text{span}\{e_1, f_1\}$, then it is not an isotropic subspace, but we can extend this basis to a sympletic basis. $\square$

2 Homework 2: Symplectic Volume

Exercise 2.1. *Let $(\mathscr{U}, x^1, \cdots, x^n)$ be a chart on a smooth manifold X, with associated cotangent coordinates $x^1, \cdots, x^n, \xi_1, \cdots, \xi_n$. Show that on $T^*\mathscr{U}$, $\alpha = \xi_i \mathrm{d}x^i$*

Proof. The tautological 1-form α is defined pointwise at $p = (x, \xi)$ as $\alpha_p = (\mathrm{d}\pi_p)^*\xi$, so if we denote

$$T_xX = \text{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_x, \left.\frac{\partial}{\partial x^2}\right|_x, \cdots, \left.\frac{\partial}{\partial x^n}\right|_x \right\}, \quad T_pM = \text{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p, \left.\frac{\partial}{\partial \xi_1}\right|_p, \cdots, \left.\frac{\partial}{\partial \xi_n}\right|_p \right\} \tag{2.1}$$

then suppose $\alpha_p = a_i \mathrm{d}x^i + b^i \mathrm{d}\xi_i$, then we have $a_i = \alpha_p(\partial_{x^i}|_p) = \xi_x(\mathrm{d}\pi_p(\partial_{x^i}|_p)) = \xi_i$, and since $\mathrm{d}\pi_p(\partial_{\xi_i|_p}) = 0$, thus we have $b_i = 0$, then we have $\alpha = \xi_i \mathrm{d}x^i$. $\square$

Exercise 2.2. *Show that the tautological $1-$form α is uniquely characterized by the property that, for every $\mu \in T^*X$, and $s_\mu : X \to T^*X$ such that $s_\mu(x) = (x, \mu_x)$, then $s_\mu^*\alpha = \mu$.*

Proof. On the one hand, we check $\alpha_{(x,\xi)} = \xi_i \mathrm{d}x^i$ satisfies the property above. At each $x \in X$, we have

$$(s_\mu^*\alpha)_x = s_\mu^*\alpha_{s_\mu(x)} = s_\mu^*\alpha_{(x,\mu_x)} = s_\mu^*\pi^*\mu_x = (\pi \circ s_\mu)^*\mu_x = \mu_x,$$

or we can directly check from the definition of pullback

$$s_\mu^*\alpha_{(x,\mu_x)} = s_\mu^*(\mu_{x,i}\mathrm{d}x^i) = \mu_{x,i} \circ s_\mu(x)\mathrm{d}x^i \circ s_\mu = \mu_{x,i}\mathrm{d}x^i = \mu_x.$$

On the other hand, if for each μ, we have $s_\mu^*\alpha = \mu$, then suppose $\alpha = a_i \mathrm{d}x^i + b^j \mathrm{d}\xi_j$, then we have

$$\mu_x = a_i \circ s_\mu(x)\mathrm{d}x^i \circ s_\mu + b^j \circ s_\mu(x)\mathrm{d}\xi_j \circ s_\mu,$$

then we have $a_i(x, \mu_x) = \mu_{x,i}$ and $b^j(x, \mu_x) = 0$, then we know that $\alpha_{(x,\mu_x)} = \mu_{x,i}\mathrm{d}x^i$, which means that α is the tautological 1-form. $\square$

Exercise 2.3. *Suppose that $f : X \to Y$ is a diffeomorphism, then the map $f_\sharp : M \to N$, where $M = T^*X$ and $N = T^*Y$, given by at point $p = (x, \xi)$ and $q = f_\sharp(p) = (y, \eta)$ is*

$$y = f(x), \quad \xi = f^*\eta, \tag{2.2}$$

here $f^*\eta(X_p) = \eta_{f(p)}(f_{*,p}X_p)$, then check that $f_\sharp$ is a diffeomorphism.

Proof. Trivially, for$\pi_1 : M \to X$ and $\pi_2 : N \to Y$, we have $f \circ \pi_1 = \pi_2 \circ f_\sharp$, then since f is a diffeomorphism, then $f_{*,p}^*$ is a linear isomorphism from T_q^*N to T_p^*M , thus, we can give a coordinate expression of $f_\sharp$ at $p = (x, \xi)$ if the coordinate is $(x^1, \cdots, x^n, \xi_1, \cdots, \xi_n)$, then $f_\sharp(x^1, \cdots, x^n, \xi_1, \cdots, \xi_n) = (f(x^1, \cdots, x^n), (f_{*,p}^*)^{-1}(\xi_1, \cdots, \xi_n))$, in short, we can write it as $f_\sharp = (f, (f_{*,p}^*)^{-1})$, trivially it is smooth, and $f_\sharp^{-1} = (f^{-1}, f_{*,p}^*)$ which is also smooth. $\square$

Problem 2.1. *Given a vector space V, the exterior algebra of its dual space is*

$$\wedge^*(V^*) = \bigoplus_{k=0}^{\dim V} \wedge^k(V^*),$$

*where $\wedge^k(V^*)$ is the set of maps $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ which are linear in each entry, and for any permutation π, $\alpha(v_{\pi_1}, \cdots, v_{\pi_k}) = (\text{sgn}\pi) \cdot \alpha(v_1, \cdots, v_k)$. The elements of $\wedge^k(V^*)$ are known as **skew-symmetric k-linear maps** or **k-forms**.*

1. *Show that any $\Omega \in \wedge^2(V^*)$ is of the form $\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$, where $u_1^*, \cdots, u_k^*, e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*$ is a basis of V^* dual to the standard basis ($k + 2n = \dim V$).*

2. *Show that $\Omega^n = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n}$ does not vanish.*

3. *Deduce that the n-th exterior power ω^n of any symplectic form ω on a $2n$-dimensional manifold M is a volume form. So each symplectic manifold is oriented, then Mobius strip is not a symplectic manifold.*

4. *Conversely, given a $2-$form $\Omega \in \wedge^2(V^*)$, show that, if $\Omega^n \neq 0$, then Ω is symplectic.*

Proof. (1) From a known result, $\wedge^2(V^*)$ is generated by $\{u_i^* \wedge e_j^*, e_k^* \wedge f_l^*, u_s^* \wedge f_t^*\}$, thus Ω can be written as a linear combination of this basis, since we have $\Omega(u_i, v) = 0$ for any $v \in V$, and $\Omega(e_i, v) \neq 0$ iff $v = cf_i$, and $\Omega(e_i, f_i) = 1$, thus we have $\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$.

(2) Suppose $e_1, \cdots, e_n, f_1, \cdots, f_n$ is a symplectic basis of V, then from (1), we have $\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$, if we write $\omega_i = e_i^* \wedge f_i^*$, then we have $\omega_i \wedge \omega_j = (-1)^{2\times 2}\omega_j \wedge \omega_i$, i.e., $\omega_i \wedge \omega_j = \omega_j \wedge \omega_i$, now for $\Omega = \omega_1 + \cdots + \omega_n$,

$$\Omega^n = \sum_{\pi \in S_n} \omega_{\pi_1} \wedge \cdots \wedge \omega_{\pi_n} = \sum_{\pi \in S_n} \omega_1 \wedge \cdots \omega_n = n! \cdot \omega_1 \wedge \cdots \omega_n,$$

from this we know that Ω^n does not vanish.

(3) For each $p \in M$, then ω_p is a symplectic form on T_pM, thus from (2) we know that ω_p^n does not vanish, thus we have ω^n is a volume form.

(4) If Ω is not symplectic, then form (1), we have a standard basis $u_1^*, \cdots, u_k^*, e_1^*, e_l^*, f_1^*, \cdots, f_l^*$, where $l < n$ and $k + 2l = 2n$, then we have $\Omega = e_1^* \wedge f_1^* + \cdots + e_l^* \wedge f_l^*$, since $l < n$, then in each term of Ω^n, there must have a $e_i^* \wedge f_i^*$ appears twice, thus this term vanishes, so we have $\Omega^n = 0$, which is a contracdiction. $\square$

Problem 2.2. *Let (M, ω) be a $2n-$dimensinoal symplectic manifold, and let ω^n be the volume form obtained by wedging ω with itself n times.*

1. *Show that, if M is compact, the de Rham cohomology class $[\omega^n] \in H^{2n}(M, \mathbb{R})$ is non-zero.*

2. *conclude that $[\omega]$ itself is non-zero, i.e., ω is not exact.*

3. *Show that if $n > 1$, then $\mathbb{S}^{2n}$ cannot be a symplectic manifold.*

Proof. (1) On the one hand $\mathrm{d}\omega = 0$, thus we have $\mathrm{d}\omega^n = 0$, then ω^n is closed, then if ω^n is closed, and suppose $\omega^n = \mathrm{d}\eta$, where $\eta \in H^{2n-1}(M)$, then since M is compact and I think it means that it is compact and without boundary, thus from Stokes' theorem

$$\int_M \omega^n = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = 0,$$

however, from Darboux theorem, there is an atlas, such that $M = \cup U_i$, and local chart (U_i, φ_i) such that

$$(\varphi_i^{-1})^* \omega = \sum_{k=1}^n \mathrm{d}x^k \wedge \mathrm{d}y^k,$$

then we have

$$(\varphi_i^{-1})^* \omega^n = n! \cdot \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n,$$

thus consider $\{\rho_i\}$ is the partion of unitiy subcordinate to $\{U_i, \varphi_i\}$, then we have

$$\int_M \omega^n = \sum_i \int_{\phi_i(U_i)} \rho_i (\varphi_i^{-1})^* \omega^n = n! \cdot \int_M \mathrm{d}V = n! \cdot \mathrm{Vol}(M) > 0,$$

which is a contradiction.

(2) If not, then suppose $\omega = \mathrm{d}\alpha$, then we have $\omega^n = \mathrm{d}(\alpha \wedge \mathrm{d}\alpha \wedge \cdots \wedge \mathrm{d}\alpha)$, which is a contradiction.

(3) This is directly from $H^2(\mathbb{S}^{2n}) = 0$. $\square$

3 Homework 3: Tautological Form and Symplectomorphisms

Exercise 3.1. *Show that a map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.*

Proof. $\square$

Exercise 3.2. *Check that, if X is compact(and not just oone point) and $f \in C^\infty(X)$, then $\#(X_{\mathrm{d}f} \cap X_0) \geq 2$.*

Proof. Since X is compact, then we have for $f \in C^\infty(X)$, there must exisit minimal and maximal points, then at these two points, $\mathrm{d}f = 0$, so we have $\#(X_{\mathrm{d}f} \cap X_0) \geq 2$. $\square$

Exercise 3.3. *The conormal bundle N^*S is an $n-$dimensional submanifold of T^*X.*

Proof. As a set $N^*S = \{(x, \xi) \in T^*X | x \in S, \xi \in N_x^*S\}$, so we inherit it subspace topology, and then naturally, it is Hausdorff and second-countable, so now it sufficies to give it a smooth chart. Fortunately, since S is a submanifold of X, then there exists an open cover $\{U_i\}$ of X, and with coordinate map φ_i such that on each $S \cap U_i$, $\varphi_i(S \cap U_i) = \varphi_i(U_i) \cap \{x^{k+1} = \cdots = x^n = 0\}$, so we further have for $(x, \xi) \in N^*S \cap T^*U_i$, we have $\xi = a_{k+1}\mathrm{d}x^{k+1} + \cdots + a_n \mathrm{d}x^n$, thus we can find a local chart $N^*S \cap T^*U_i$, and $\psi_i : N^*S \cap T^*U_i \to \mathbb{R}^n$, and $\psi_i(x, \xi) = (x^1, \cdots, x^k, a_{k+1}, \cdots, a_n)$, not hard to check that these give a smooth atlas of N^*S, then we finish the proof. $\square$

Problem 3.1. *Let (M, ω) be a symplectic manifold, and be a $1-$form such that*

$$\omega = -\mathrm{d}\alpha.$$

Show that there exists a unique vector field v such that its interior product with ω is α, i.e., $i(v)\omega = \alpha$.

Proof. $\square$

References

[1] A. C. da Silva, *Lectures on Symplectic Geometry.* Springer, 2008, vol. 1764.