[EECS 304]
Control I

Sheet (2)
Transfer Function

Hassan M. F. El-Menier
h.elmenier@eng.cu.edu.eg

Communications and Computer Engineering Department
Credit Hours System, Faculty of Engineering
Cairo University, Egypt

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Outline

Transfer Function

- **Goal**: Develop the mathematical models of physical systems.
- What's meant by mathematical model?
- Why are we concerned with developing models for physical systems?
    1. Predict the system response under certain conditions without testing the physical system.
    2. Illustrate the analysis and design of control systems.

Transfer Function

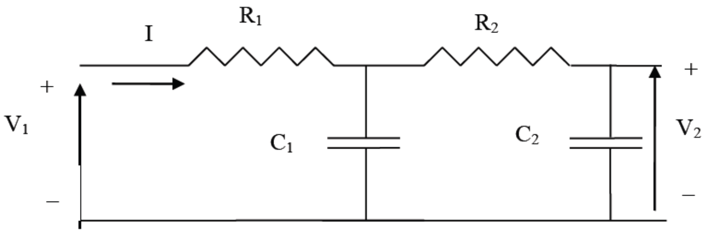
Consider a system with input e(t) and output c(t). If C(s) is the Laplace trnsform of the system output and E(s) is the Laplace transform of the system input, the transfer function G(s) is defined by

$$G(s) = \frac{C(s)}{E(s)}$$

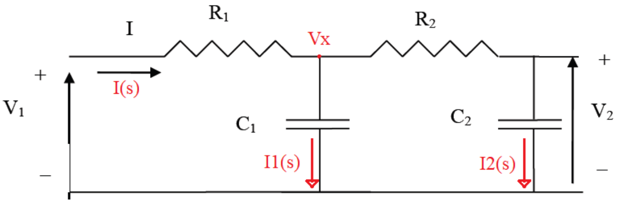
Problem 1

Problem 1

For the circuit shown, obtain the transfer function $V_2(s)/V_1(s)$.



Problem 1



$$\because V_1(s) = I(s)R_1 + V_x(s)$$

$$\because V_x(s) = I_2(s)R_2 + V_2(s)$$

$$\therefore V_1(s) = I(s)R_1 + I_2(s)R_2 + V_2(s)$$

$$\because I(s) = I_1(s) + I_2(s)$$

$$\therefore V_1(s) = [I_1(s) + I_2(s)]R_1 + I_2(s)R_2 + V_2(s)$$

$$\because I_1(s) = V_x(s)sC_1 \because I_2(s) = V_2(s)sC_2$$

$$\therefore V_1(s) = [V_x(s)sC_1 + V_2(s)sC_2]R_1 + [V_2(s)sC_2]R_2 + V_2(s)$$

Problem 1

$$\therefore V_1(s) = V_x(s)sC_1R_1 + V_2(s)sC_2R_1 + V_2(s)sC_2R_2 + V_2(s)$$

$$\therefore V_1(s) = [I_2(s)R_2 + V_2(s)]sC_1R_1 + V_2(s)sC_2R_1 + V_2(s)sC_2R_2 + V_2(s)$$

$$\therefore V_1(s) = I_2(s)R_2sC_1R_1 + V_2(s)sC_1R_1 + V_2(s)sC_2R_1 + V_2(s)sC_2R_2 + V_2(s)$$

$$\therefore V_1(s) = [V_2(s)sC_2]R_2sC_1R_1 + V_2(s)sC_1R_1 + V_2(s)sC_2R_1 + V_2(s)sC_2R_2 + V_2$$

$$\therefore V_1(s) = V_2(s)s^2C_2R_2C_1R_1 + V_2(s)sC_1R_1 + V_2(s)sC_2R_1 + V_2(s)sC_2R_2 + V_2($$

$$\therefore V_1(s) = V_2(s)[s^2R_1R_2C_1C_2 + sR_1C_1 + sR_1C_2 + sR_2C_2 + 1]$$

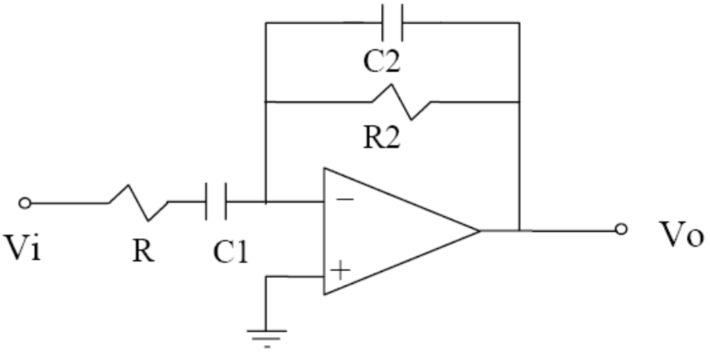
Answer:

$$\therefore \frac{V_2(s)}{V_1(s)} = \frac{1}{s^2R_1R_2C_1C_2 + sR_1C_1 + sR_1C_2 + sR_2C_2 + 1}$$

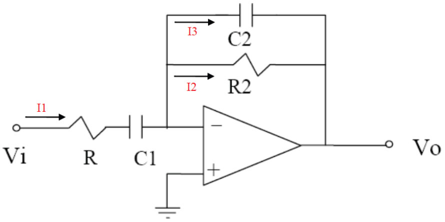
Problem 2

Problem 2

For the ideal OP Amp, obtain the transfer function $V_o(s)/V_i(s)$.



Problem 2



$$\because I_1(s) = I_2(s) + I_3(s)$$

$$\because I_1(s) = \frac{V_{in}(s)}{R_1 + \cfrac{1}{sC_1}}$$

$$\because I_2(s) = \frac{0 - V_o(s)}{R_2} \quad \because I_3(s) = \frac{0 - V_o(s)}{\cfrac{1}{sC_2}}$$

Problem 2

$$\therefore \frac{V_{in}(s)}{R_1 + \dfrac{1}{sC_1}} = \frac{-V_o(s)}{R_2} - V_o(s)sC_2$$

$$\therefore \frac{V_o(s)}{V_{in}(s)} = \frac{-1}{(R_1 + \dfrac{1}{sC_1})(\dfrac{1}{R_2} + sC_2)}$$

$$\therefore \frac{V_o(s)}{V_{in}(s)} = \frac{-1}{sC_2R_1 + \dfrac{1}{sC_1R_2} + \dfrac{C_2}{C_1} + \dfrac{R_1}{R_2}}$$

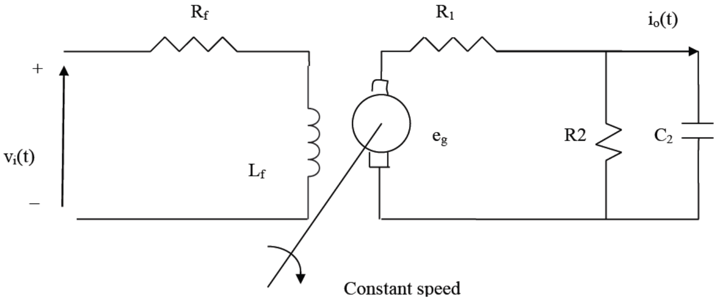
Answer:

$$\therefore \frac{V_o(s)}{V_{in}(s)} = \frac{-sC_1R_2}{s^2C_1C_2R_1R_2 + sC_2R_2 + sC_1R_1 + 1}$$

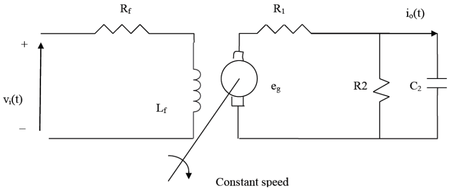
Problem 3

Problem 3

The system shown consists of a DC generator connected to an RC impedence. Find the TF that relates V_i and I_o.



Problem 3



- The current i_f flowing through the coil will produce a magnetic flux $\phi \rightarrow \phi \propto i_f$
- The generator plate rotating with an angular speed $\dot{\theta}$ through the flux ϕ will generate emf that depends on $\dot{\theta}$ and ϕ

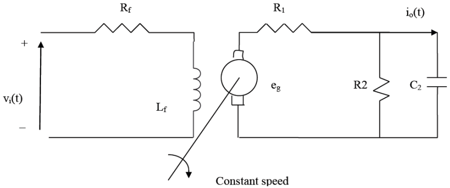
$$e_g \propto \dot{\theta}\phi \rightarrow \because \phi \propto i_f$$

$$\therefore e_g = k\dot{\theta}i_f$$

$\because$ the plate rotates with a constant speed

$\therefore e_g = Ki_f$ where K is constant

Problem 3



$$v_i(t) = i_f(t)R_f + L_f\frac{di_f}{dt}$$

$$e_g(t) = Ki_f(t)$$

$$e_g(t) = R_1[i_o(t) + \frac{v_1(t)}{R_2}] + v_1(t)$$

$$i_o(t) = C_2\frac{dv_1(t)}{dt}$$

$$V_i(s) = (R_f + L_f s)I_f(s)$$

$$E_g(s) = KI_f(s)$$

$$E_g(s) = R_1[I_o(s) + \frac{V_1(s)}{R_2}] + V_1(s)$$

$$I_o(s) = C_2 sV_1(s)$$

Problem 3

$$\because V_i(s) = (R_f + L_f s)I_f(s) \rightarrow I_f(s) = V_i(s)\frac{1}{R_f + L_f s}$$

$$\because E_g(s) = KI_f(s) \rightarrow E_g(s) = V_i(s)\frac{K}{R_f + L_f s}$$

$$\because E_g(s) = V_1(s)[\frac{R_1}{R_2} + 1] + R_1 I_o(s)$$

$$\because I_o(s) = C_2 s V_1(s) \rightarrow V_1(s) = I_o(s)\frac{1}{sC_2}$$

$$\therefore E_g(s) = I_o(s)\frac{1}{sC_2}[\frac{R_1}{R_2} + 1] + R_1 I_o(s)$$

$$\therefore I_o(s) = E_g(s)\frac{1}{R_1 + \frac{1}{sC_2}[\frac{R_1}{R_2} + 1]}$$

Problem 3

$$\therefore I_o(s) = V_i(s)\frac{K}{(R_f + sL_f)(R_1 + \frac{1}{sC_2}[\frac{R_1}{R_2} + 1])}$$

$$\therefore \frac{I_o(s)}{V_i(s)} = \frac{K}{(R_f + sL_f)(R_1 + \frac{1}{sC_2}[\frac{R_1}{R_2} + 1])}$$

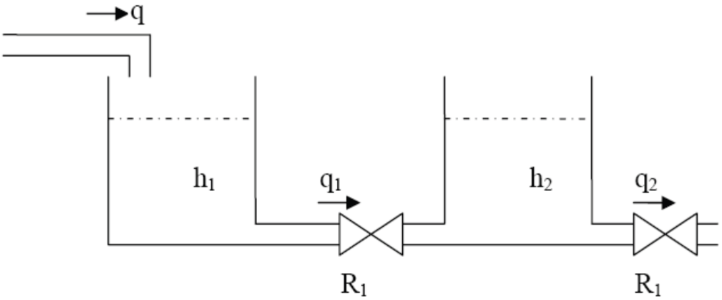
Answer:

$$\therefore \frac{I_o(s)}{V_i(s)} = \frac{sC_2K}{(R_f + sL_f)(sC_2R_1 + \frac{R_1}{R_2} + 1)}$$

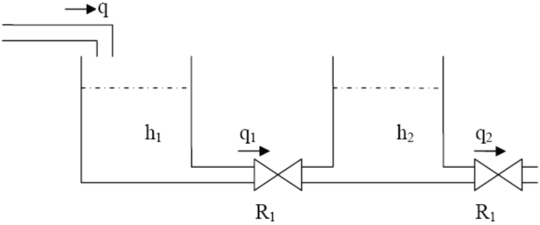
Problem 4

Problem 4

The figure shows a process plant containing of two tanks of areas A_1 and A_2. Derive the transfer function that relates q_2 to q.

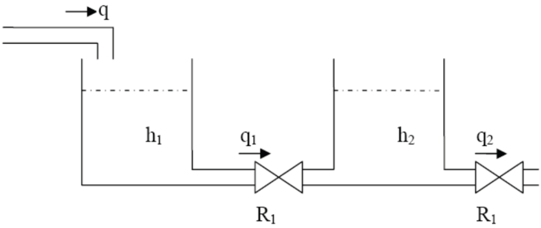


Problem 4



- q_{in}: flow rate input to tank (m^3/sec)
- q_{out}: flow rate output from tank (m^3/sec)
- A: cross section area of the tank (m^2)
- h: liquid level (m)
- h_{next}: liquid level of the next tank (m)
- R_{in}: resistance of the valve controlling the input liquid (ohm)

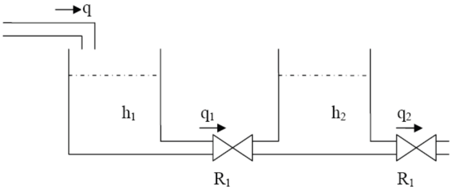
Problem 4



For each tank, we should apply:

1. $$q_{in}(t) - q_{out}(t) = A\frac{dh}{dt}$$
2. $$\frac{h(t) - h_{next}(t)}{R_{out}} = q_{out}(t)$$

Problem 4



$$q - q_1 = A_1 \frac{dh_1}{dt}$$

$$q_1 - q_2 = A_2 \frac{dh_2}{dt}$$

$$\frac{h_1(t) - h_2(t)}{R_1} = q_1(t)$$

$$\frac{h_2(t) - 0}{R_1} = q_2(t)$$

$$Q(s) - Q_1(s) = A_1 s H_1(s)$$

$$Q_1(s) - Q_2(s) = A_2 s H_2(s)$$

$$\frac{H_1(s) - H_2(s)}{R_1} = Q_1(s)$$

$$H_2(s) = R_1 Q_2(s)$$

Problem 4

$$\because Q(s) - Q_1(s) = A_1 s H_1(s) \rightarrow Q_1(s) = Q(s) - A_1 s H_1(s)$$

$$\because Q_1(s) - Q_2(s) = A_2 s H_2(s) \rightarrow Q_2(s) = Q_1(s) - A_2 s H_2(s)$$

$$\therefore Q_2(s) = Q(s) - A_1 s H_1(s) - A_2 s H_2(s)$$

$$Q_2(s) = Q(s) - A_1 s[Q_1(s)R_1 + H_2(s)] - A_2 s[R_1 Q_2(s)]$$

$$Q_2(s) = Q(s) - A_1 s[[Q_2(s) + A_2 s H_2(s)]R_1 + R_1 Q_2(s)] - A_2 s R_1 Q_2(s)$$

$$Q_2(s) = Q(s) - A_1 s Q_2(s) R_1 - A_1 A_2 s^2 H_2(s) R_1 - A_1 s R_1 Q_2(s) - A_2 s R_1 Q_2(s)$$

$$Q_2(s) = Q(s) - A_1 s Q_2(s) R_1 - A_1 A_2 s^2 [R_1 Q_2(s)] R_1 - A_1 s R_1 Q_2(s) - A_2 s R_1 Q_2(s)$$

$$Q_2(s)[1 + A_1 s R_1 + A_1 A_2 s^2 {R_1}^2 + A_1 s R_1 + A_2 s R_1] = Q(s)$$

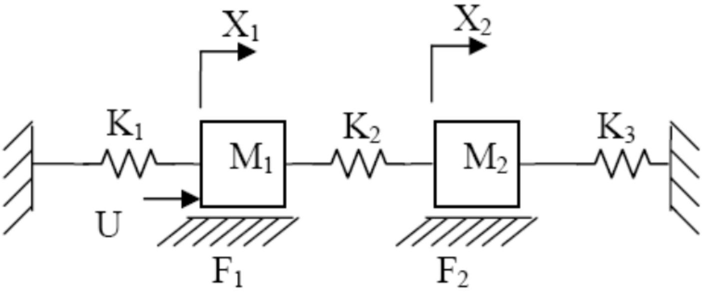
Answer:

$$\therefore \frac{Q_2(s)}{Q(s)} = \frac{1}{A_1 A_2 {R_1}^2 s^2 + (2A_1 R_1 + A_2 R_1)s + 1}$$

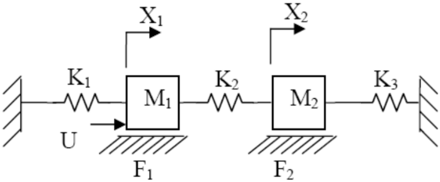
Problem 5

Problem 5

Derive the differential equations that represent the mechanical system shown, where U is a force that affects the mass M1 and hence derive the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$.
If the force U has a value of 1 N find the steady state values of X_1 and X_2.



Problem 5



- K: spring constant (N/m or Kg/s)
- F or β: viscous function coefficient ($N/m/s$ or Kg/s)
- X: displacement due to U or f(t) $\rightarrow$ Tension Force

The mass will oscillate under an acceleration $a = \ddot{x}$, so the equilibrium modelling equation of the system is

$$f(t) - F_1\dot{x} - Kx = M\ddot{x}$$

$$\sum Force = Ma = M\ddot{x}$$

Problem 5

Applying $\sum Force = M\ddot{x}$ at each M:

At M_1:

$$u - K_1 x_1 - F_1 \dot{x}_1 - K_2(x_1 - x_2) = M_1 \ddot{x}_1$$

$$\therefore M_1 \ddot{x}_1 + F_1 \dot{x}_1 + K_1 x_1 + K_2(x_1 - x_2) - u = 0$$

At M_2:

$$-K_2(x_2 - x_1) - K_3 x_2 - F_2 \dot{x}_2 = M_2 \ddot{x}_2$$

$$\therefore M_2 \ddot{x}_2 + F_2 \dot{x}_2 + K_3 x_2 + K_2(x_2 - x_1) = 0$$

Taking Laplace Transform for both equations:

$$\therefore X_1(s)[M_1 s^2 + F_1 s + K_1 + K_2] = U(s) + K_2 X_2(s)$$

$$\therefore X_2(s)[M_2 s^2 + F_2 s + K_3 + K_2] = K_2 X_1(s)$$

Problem 5

$$\therefore X_1(s)[M_1s^2 + F_1s + K_1 + K_2] = U(s) + K_2[\frac{K_2X_1(s)}{M_2s^2 + F_2s + K_3 + K_2}]$$

$$\therefore X_1(s)[M_1s^2 + F_1s + K_1 + K_2 - \frac{K_2^2}{M_2s^2 + F_2s + K_3 + K_2}] = U(s)$$

$$\therefore \frac{X_1(s)}{U(s)} = \frac{M_2s^2 + F_2s + K_3 + K_2}{(M_1s^2 + F_1s + K_1 + K_2)(M_2s^2 + F_2s + K_3 + K_2) - K_2^2}$$

Note:

Steady state of $X_1 \rightarrow \lim_{s\rightarrow 0} sX_1(s)$

$$\therefore X_1|_{ss} = \frac{K_3 + K_2}{(K_1 + K_2)(K_3 + K_2) - K_2^2}$$

Problem 5

$$\therefore \frac{X_2(s)}{U(s)} = \frac{K_2}{(M_1 s^2 + F_1 s + K_1 + K_2)(M_2 s^2 + F_2 s + K_3 + K_2) - K_2^2}$$

$$\therefore X_2|_{ss} = \frac{K_2}{(K_1 + K_2)(K_3 + K_2) - K_2^2}$$

Thank you!

Questions?

`h.elmenier@eng.cu.edu.eg`