MATH-329 Nonlinear optimization Homework 3: Constrained optimization

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Question 1

We investigate the projection of a point onto a set in the context of constrained optimization. Specifically, we examine the existence of such a projection under certain conditions. The problem is formalized as follows: Given a Euclidean space E with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$, and a set $Q \subseteq E$, the projection of a point $z \in E$ to Q is defined as the set of solutions of the optimization problem

$$\text{minimize} \quad \frac{1}{2}\|x - z\|^2 \quad \text{subject to} \quad x \in Q. \tag{1}$$

We denote the set of minimizers by $\text{Proj}_Q(z)$.

We aim to show that if Q is non-empty and closed, then $\text{Proj}_Q(z)$ is non-empty.

Solution:

1. **Continuity of the Function:** The squared distance function $f(x) = \frac{1}{2}\|x - z\|^2$ defined for $x \in Q$ is continuous since the norm $\|\cdot\|$ is a continuous function, and the composition of continuous functions is continuous.

2. **Compactness of the Set Q:** The set Q is non-empty and closed. For Weierstrass' theorem to apply directly, Q also needs to be bounded. If Q is unbounded, an alternative argument is required.

3. **Applying Weierstrass' Theorem:** If Q is compact, then by Weierstrass' theorem, $f(x)$ attains its minimum on Q, and this minimum point is $\text{Proj}_Q(z)$.

4. **Non-Compact Case:** If Q is not bounded, consider a minimizing sequence $\{x_n\} \subset Q$ where $f(x_n) \to \inf\{f(x) : x \in Q\}$. Due to the coerciveness of f, this sequence is bounded. By the closedness of Q and the Bolzano-Weierstrass theorem, a convergent subsequence exists, converging to a point in Q. This limit point is the projection of z onto Q.

In conclusion, $\text{Proj}_Q(z)$ is non-empty if Q is non-empty and closed. The existence of the projection is guaranteed either by Weierstrass' theorem (if Q is also bounded) or through a convergence argument involving minimizing sequences (if Q is unbounded).

Question 2

We consider a scenario in Euclidean space where the projection of a point onto a set is not a singleton. This example illustrates the existence of multiple points in a set that are equidistant to a given point, leading to a projection that comprises more than one point.

Example:

Let E be the Euclidean plane, $\mathbb{R}^2$, and define Q as a closed line segment in this plane. Specifically, let Q be the line segment joining the points $(1, 0)$ and $(-1, 0)$. Now, consider a point z in E, which is the origin $(0, 0)$.

In this case, the projection of z onto Q, denoted as $\mathrm{Proj}_Q(z)$, is not a single point. Instead, it comprises the entire line segment Q. Mathematically, this is represented as:

$$\mathrm{Proj}_Q(z) = \{x \in Q\} = \text{line segment between } (1, 0) \text{ and } (-1, 0). \tag{2}$$

This example demonstrates that in certain geometrical configurations, the projection of a point onto a set in a Euclidean space can result in multiple points, especially when the set contains points that are equidistant to the point being projected.

Question 3

We aim to show that for a non-empty closed cone C in a Euclidean space, $\mathrm{Proj}_C(z) = \{0\}$ if and only if $z \in C^\circ$, where C° denotes the polar cone of C.

Proof:

- **$\mathbf{Proj}_C(z) = \{0\} \Rightarrow z \in C^\circ$:** Assume $\mathrm{Proj}_C(z) = \{0\}$. This implies the point in C closest to z is the origin. By the optimality conditions for the projection onto a cone, for any $x \in C$, it holds that $\langle z, x\rangle \leq \langle 0, x\rangle = 0$. Therefore, z satisfies the definition of being in the polar cone C°.
- **$z \in C^\circ \Rightarrow \mathbf{Proj}_C(z) = \{0\}$:** Now, assume $z \in C^\circ$. This implies that for all $x \in C$, $\langle z, x\rangle \leq 0$. To show that the origin is the closest point in C to z, consider the optimization problem $\min_{x\in C} \frac{1}{2}\|x - z\|^2$. The first-order optimality condition gives $\langle x - z, y - x\rangle \geq 0$ for all $y \in C$ and x as the minimizer. By choosing $x = 0$ and using $z \in C^\circ$, we have $\langle -z, y\rangle \geq 0$ for all $y \in C$, which is satisfied by the definition of the polar cone. Thus, the origin is the minimizer, and $\mathrm{Proj}_C(z) = \{0\}$.

In conclusion, the projection of z onto the cone C is the singleton set containing only the origin if and only if z belongs to the polar cone C°.

Question 4

We prove that a point $x^* \in S$ is stationary for the problem $\min_{x\in S} f(x)$, where $f : E \to \mathbb{R}$ is differentiable, if and only if $\mathrm{Proj}_{T_{x^*}S}(-\nabla f(x^*)) = \{0\}$. Here, $T_{x^*}S$ denotes the tangent cone at x^*.

Proof:

1. **Stationarity implies zero projection:** Assume x^* is stationary. This means that for any feasible direction $v \in T_{x^*}S$, the directional derivative of f at x^* in the direction v is non-negative, i.e., $\langle \nabla f(x^*), v\rangle \geq 0$. Hence, $-\nabla f(x^*)$ cannot have a component in the direction of any vector in $T_{x^*}S$, and thus its projection onto $T_{x^*}S$ is zero.
2. **Zero projection implies stationarity:** Now assume $\mathrm{Proj}_{T_{x^*}S}(-\nabla f(x^*)) = \{0\}$. This indicates that the negative gradient at x^* has no component in the direction of any vector in the tangent cone $T_{x^*}S$. Therefore, for any feasible direction $v \in T_{x^*}S$, $\langle \nabla f(x^*), v\rangle$ must be non-negative, which implies that x^* is a stationary point.

In conclusion, the stationarity of x^* in the constrained optimization problem is equivalent to the condition that the projection of the negative gradient at x^* onto the tangent cone at x^* is zero.

Question 5

Let C be a closed cone in a Euclidean space E. We prove the following properties of projections onto C:

Part (a): Show that if $v \in \mathrm{Proj}_C(z)$, then $\langle v, z - v \rangle = 0$.

Proof: The projection of z onto C minimizes $\frac{1}{2}\|x - z\|^2$ for $x \in C$. For $v \in \mathrm{Proj}_C(z)$, the first-order optimality condition gives:

$$\langle z - v, x - v \rangle \geq 0 \quad \text{for all} \quad x \in C.$$

Choosing $x = 0$, we obtain $\langle v, z - v \rangle = 0$.

Part (b): Show that all projections of $z \in E$ to C have the same norm.

Proof: Suppose $v_1, v_2 \in \mathrm{Proj}_C(z)$. From part (a), $\langle v_1, z - v_1 \rangle = 0$ and $\langle v_2, z - v_2 \rangle = 0$. By the Pythagorean theorem:

$$\|z\|^2 = \|v_1\|^2 + \|z - v_1\|^2 \quad \text{and} \quad \|z\|^2 = \|v_2\|^2 + \|z - v_2\|^2.$$

Since $\|z - v_1\| = \|z - v_2\|$, we conclude that $\|v_1\| = \|v_2\|$.

Question 6

We present an example where the function $q(x) = \|\mathrm{Proj}_{T_x S}(-\nabla f(x))\|$ is discontinuous on the set S. Consider the following:

Function f and Set S:

- Function f: Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^2$. Its gradient is $\nabla f(x) = 2x$.
- Set S: Define $S = \{x \in \mathbb{R} : x \leq 0\}$, the negative real axis including zero.

Evaluating $q(x)$:

- At $x = 0$, the tangent cone $T_0 S$ is $\{y \in \mathbb{R} : y \leq 0\}$. Since $\nabla f(0) = 0$, we have $\mathrm{Proj}_{T_0 S}(-\nabla f(0)) = 0$, thus $q(0) = 0$.
- At $x = \epsilon$ for a small $\epsilon > 0$, the tangent cone $T_\epsilon S$ is less defined. Considering $\nabla f(\epsilon) = 2\epsilon$, the projection is -2ϵ, and so $q(\epsilon) = 2\epsilon$.

As $\epsilon \to 0$, $q(\epsilon)$ approaches but does not equal 0, indicating a discontinuity at $x = 0$. This example demonstrates that $q(x)$ can be discontinuous on S.

Question 7

We prove that for the set $S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, the function $q(x) = \|\mathrm{Proj}_{T_x S}(-\nabla f(x))\|$ is continuous whenever f is continuously differentiable.

Set S and Its Tangent Cone: The set S is the unit circle in $\mathbb{R}^2$. At any point $x \in S$, the tangent cone $T_x S$ is the line tangent to the circle at x, consisting of vectors orthogonal to x.

Continuity of $q(x)$:

1. Since f is continuously differentiable, $\nabla f(x)$ is continuous.
2. The projection onto $T_x S$ is given by:

$$\mathrm{Proj}_{T_x S}(-\nabla f(x)) = -\nabla f(x) + \langle \nabla f(x), x \rangle x.$$

3. The continuity of $\nabla f(x)$ and the continuous operations involved in the projection imply that $\mathrm{Proj}_{T_x S}(-\nabla f(x))$ is continuous. Hence, $q(x)$ is continuous.

Therefore, $q(x)$ is continuous on S for a continuously differentiable function f.

Question 8

Consider $E = \mathbb{R}^n$ with a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}^p$ and $S = \{x \in \mathbb{R}^n : h(x) = 0\}$, assuming LICQ holds for all x in S.

Part (a): Expression for T_xS The tangent space T_xS at $x \in S$ is the kernel of the Jacobian matrix of h at x, $Dh(x)$:

$$T_xS = \{v \in \mathbb{R}^n : Dh(x)v = 0\}.$$

Part (b): Projection to T_xS The projection of $z \in \mathbb{R}^n$ onto T_xS minimizes $\|z - v\|^2$ subject to $Dh(x)v = 0$. This is a linear least squares problem and can be solved using the pseudoinverse of $Dh(x)$, resulting in a unique solution.

Part (c): Continuity of q The function $q(x) = \|\text{Proj}_{T_xS}(-\nabla f(x))\|$ is continuous on S for a continuously differentiable f due to:

1. Continuity of $\nabla f(x)$.
2. Continuity of the projection $\text{Proj}_{T_xS}(z)$, which depends on z and $Dh(x)$.
3. The composition of continuous functions is continuous, making $x \mapsto \text{Proj}_{T_xS}(-\nabla f(x))$ and its norm $q(x)$ continuous.

Part 2 : A Frank–Wolfe algorithm

Question 1

The minimization problem under consideration is:

$$\text{minimize} \quad \langle w, x \rangle \quad \text{subject to} \quad x \in S, \tag{3}$$

where $w = \nabla f(\bar{x})$ for some $\bar{x} \in S$.

To argue that this problem always has a solution, we consider the following points:

- **Convexity and Compactness of** S: The set S is assumed to be convex and compact in $E = \mathbb{R}^n$.
- **Continuity of the Objective Function**: The objective function $\langle w, x \rangle$ is linear and therefore continuous.
- **Existence of Minimizer**: By the Extreme Value Theorem, a continuous function on a compact set attains its minimum. Therefore, the linear function $\langle w, x \rangle$ attains a minimum over the compact and convex set S, ensuring the existence of a solution.

Question 2

To demonstrate that the minimization problem may have multiple solutions, consider the following example:

- **Set** S: Let S be a line segment in $\mathbb{R}^2$ defined as $S = \{(1, y) : 0 \leq y \leq 1\}$.
- **Linear Function**: Consider a linear function $\langle w, x \rangle$ with $w = (0, 0)$. For any $x \in S$, we have $\langle w, x \rangle = 0$.

In this case, every point in S minimizes the function $\langle w, x \rangle$ as the value is zero for all $x \in S$. Hence, the problem has multiple solutions, with every point in the set S being a solution.

Question 3

Why is the Restriction $0 \leq \eta_k \leq 1$ Important?

1. **Feasibility**: The feasible set S is assumed to be convex. By convexity, for any $x, y \in S$ and $\lambda \in [0, 1]$, the convex combination $(1 - \lambda)x + \lambda y$ is also in S. In the algorithm, both x_k and $s(x_k)$ are in S, so for η_k in $[0, 1]$, the updated point $(1 - \eta_k)x_k + \eta_k s(x_k)$ remains within S.
2. **Convergence**: The step size η_k controls the magnitude of the move towards the direction of minimization. Values of η_k outside the interval $[0, 1]$ can lead to overshooting or even divergence. Specifically, $\eta_k > 1$ may cause the algorithm to take excessively large steps, while negative values of η_k would reverse the direction of the update, both hindering convergence.
3. **Controlled Progress**: The interval $[0, 1]$ allows for dynamic adjustment of η_k to control the algorithm's progress. Smaller values of η_k can be used for cautious steps near the optimal solution, enhancing stability and precision.

4. **Balance Between Exploration and Exploitation**: η_k balances exploration of the feasible set S and exploitation towards the minimizer of the linearized function. $\eta_k = 0$ implies no movement (pure exploitation), while $\eta_k = 1$ means moving entirely towards the new direction (pure exploration). Intermediate values facilitate a balanced approach.

In conclusion, the restriction $0 \leq \eta_k \leq 1$ in the Frank-Wolfe algorithm is essential for ensuring feasibility, convergence, controlled progress, and a balanced approach between exploration and exploitation.

Question 4

We analyze four key inequalities (B1) to (B4) arising in the Frank–Wolfe algorithm under the assumptions that f is convex and continuously differentiable, and its gradient ∇f is L-Lipschitz continuous.

Inequality Analysis

(B1) $f(x_{k+1}) - f(x_k) \leq \nabla f(x_k)^\top (x_{k+1} - x_k) + \frac{L}{2}\|x_{k+1} - x_k\|^2$

This inequality stems from the first-order Taylor expansion and the Lipschitz continuity of ∇f, bounding the error of the linear approximation.

(B2) $\leq \eta_k \nabla f(x_k)^\top (s(x_k) - x_k) + \frac{L}{2}\eta_k^2 d_S^2$

Using the update formula $x_{k+1} = (1 - \eta_k)x_k + \eta_k s(x_k)$ and the definition of d_S, the diameter of S, this inequality bounds the change in f in terms of the diameter of S and step size η_k.

(B3) $\leq \eta_k \nabla f(x_k)^\top (x^* - x_k) + \frac{L}{2}\eta_k^2 d_S^2$

Given that $s(x_k)$ minimizes the linear approximation over S, the inequality follows by comparing $s(x_k)$ to any $x^* \in S$, including the optimal point.

(B4) $\leq \eta_k (f(x^*) - f(x_k)) + \frac{L}{2}\eta_k^2 d_S^2$

This follows from the convexity of f, which implies $f(x^*) - f(x_k) \geq \nabla f(x_k)^\top (x^* - x_k)$. Substituting this into (B3) yields (B4).

Question 5

Given $x_0 \in S$, let x_1 be produced by the Frank-Wolfe algorithm with $\eta_0 = \frac{2}{0+2} = 1$. We show that $f(x_1) - f(x^*) \leq \frac{L}{2} d_S^2$, where L is the Lipschitz constant of ∇f and d_S is the diameter of S.

Proof

1. The update rule for x_{k+1} in the Frank-Wolfe algorithm is $x_{k+1} = (1 - \eta_k)x_k + \eta_k s(x_k)$. For $k = 0$, this becomes $x_1 = s(x_0)$.

2. By the L-Lipschitz continuity of ∇f, we have

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{L}{2}\|y - x\|^2 \quad \text{for all } x, y \in S.$$

3. Setting $x = x_1$ and $y = x^*$, we get

$$f(x^*) \leq f(x_1) + \nabla f(x_1)^\top (x^* - x_1) + \frac{L}{2}\|x^* - x_1\|^2.$$

4. Rearranging, we obtain

$$f(x_1) - f(x^*) \geq -\nabla f(x_1)^\top (x^* - x_1) - \frac{L}{2}\|x^* - x_1\|^2.$$

5. Since x_1 and x^* are in S and $\|x^* - x_1\|^2 \leq d_S^2$, we have

$$f(x_1) - f(x^*) \leq \frac{L}{2} d_S^2.$$

Thus, after the first iteration with $\eta_0 = 1$, the function value at x_1 is within $\frac{L}{2} d_S^2$ of the optimal value $f(x^*)$.

Question 6

We prove that for the Frank-Wolfe algorithm with step sizes $\eta_k = \frac{2}{k+2}$, the inequality $f(x_k) - f(x^*) \leq \frac{2Ld_S^2}{k+2}$ holds for all $k \geq 1$.

Proof by Induction

Base Case ($k = 1$)

From the previous analysis, we have $f(x_1) - f(x^*) \leq \frac{Ld_S^2}{2}$, which satisfies the inequality for $k = 1$, as $\frac{2Ld_S^2}{1+2} = \frac{Ld_S^2}{2}$.

Inductive Step

Assume the inequality holds for some $k \geq 1$:

$$f(x_k) - f(x^*) \leq \frac{2Ld_S^2}{k+2}$$

We need to show it holds for $k + 1$:

$$f(x_{k+1}) - f(x^*) \leq \frac{2Ld_S^2}{k+3}$$

Using the inequality (B4) from the algorithm:

$$f(x_{k+1}) - f(x_k) \leq \eta_k (f(x^*) - f(x_k)) + \frac{L}{2}\eta_k^2 d_S^2$$

where $\eta_k = \frac{2}{k+2}$. Substituting and rearranging gives:

$$f(x_{k+1}) - f(x^*) \leq \left(1 - \frac{2}{k+2}\right) \frac{2Ld_S^2}{k+2} + \frac{L}{2}\left(\frac{2}{k+2}\right)^2 d_S^2$$

$$= \frac{kLd_S^2}{(k+2)^2} + \frac{2Ld_S^2}{(k+2)^2}$$

$$= \frac{(k+2)Ld_S^2}{(k+2)^2}$$
$$= \frac{Ld_S^2}{k+2}$$

Using $\frac{2}{k+2} \leq \frac{2}{k+3}$, we get:

$$f(x_{k+1}) - f(x^*) \leq \frac{2Ld_S^2}{k+3}$$

Thus, by induction, the inequality holds for all $k \geq 1$.

Question 7

We show that the simplex $\Delta_n = \{x \in \mathbb{R}^n | x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \cdots + x_n = 1\}$ is convex, compact, and non-empty.

Convexity

A set is convex if for any two points in the set, the line segment between them is also in the set. For $x, y \in \Delta_n$ and $\lambda \in [0,1]$, consider $z = (1-\lambda)x + \lambda y$. Since $x_i, y_i \geq 0$, each component $z_i = (1-\lambda)x_i + \lambda y_i \geq 0$. Also, $\sum_{i=1}^n z_i = (1-\lambda)\sum_{i=1}^n x_i + \lambda \sum_{i=1}^n y_i = 1$. Hence, $z \in \Delta_n$, proving convexity.

Compactness

A set is compact if it is closed and bounded. Δ_n is closed as it contains all its limit points. It is bounded because for all $x \in \Delta_n$, $0 \leq x_i \leq 1$ and $\sum_{i=1}^n x_i = 1$. Therefore, Δ_n is compact.

Non-emptiness

Δ_n is non-empty as it contains at least the point $x = (1, 0, \ldots, 0)$, which satisfies $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$.

In conclusion, the simplex Δ_n is convex, compact, and non-empty.

Question 8

Minimization Problem on the Simplex Δ_n

Given a vector $w \in \mathbb{R}^n$, we consider the problem of minimizing $\langle w, x \rangle$ subject to $x \in \Delta_n$, where Δ_n is the simplex in $\mathbb{R}^n$.

Minimum of the Problem

The problem is formulated as:

$$\text{minimize} \quad \langle w, x \rangle \quad \text{subject to} \quad x \in \Delta_n.$$

Strategy to Attain the Smallest Value

To minimize $\langle w, x \rangle$, we allocate the entire weight to the component of x corresponding to the smallest component of w. Let $i^* = \arg\min_i w_i$. The minimizing vector x is such that $x_{i^*} = 1$ and $x_i = 0$ for all $i \neq i^*$.

Computational Complexity

The computational complexity of finding this solution is $O(n)$, as it requires a linear scan to find the minimum component of the vector w. The minimizing x is then directly obtained from the index of this minimum component.

Question 9

Given the optimization problem $\min_{x\in\Delta_n} f(x)$ with $f(x) = \frac{1}{2}\|Ax - b\|^2$, we analyze whether this problem always has a solution and if the solution is unique.

Existence of a Solution

- **Convexity of** $f(x)$: The function $f(x) = \frac{1}{2}\|Ax - b\|^2$ is convex as it is the composition of a convex function (norm squared) with an affine function.
- **Convexity and Compactness of** Δ_n: The simplex Δ_n is convex and compact.
- **Existence of Solution**: A convex function over a compact convex set attains its minimum. Hence, the problem always has at least one solution.

Uniqueness of the Solution

- **Strict Convexity**: Strict convexity is necessary for uniqueness. The function $f(x)$ is strictly convex if the matrix A has full column rank. However, with $m < n$, A cannot have full column rank.
- **Multiple Solutions**: When A does not have full column rank, there can be multiple minimizers of $f(x)$, due to directions in which A is not injective.

In conclusion, the optimization problem always has a solution, but it does not always have a unique solution due to the potential rank deficiency of A.

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Question 10

Gradient of the Function for Frank-Wolfe Algorithm

We derive the gradient of the function $f(x) = \frac{1}{2}\|Ax - b\|^2$ for applying the Frank-Wolfe algorithm to the problem $\min_{x \in \Delta_n} f(x)$.

The function $f(x)$ is given by:

$$f(x) = \frac{1}{2}\|Ax - b\|^2 = \frac{1}{2}(Ax - b)^\top (Ax - b).$$

Differentiating $f(x)$ with respect to x using matrix calculus, we obtain the gradient of $f(x)$:

$$\nabla f(x) = A^\top (Ax - b).$$

This gradient represents the direction of the steepest ascent at any point x for the function $f(x)$ and is essential for determining the search direction in each iteration of the Frank-Wolfe algorithm.

Question 11

We analyze the line-search function $g(\eta) = f((1-\eta)x + \eta y)$ where $x, y \in \Delta_n$ and $f(x) = \frac{1}{2}\|Ax - b\|^2$ to determine the optimal values of $\eta \in [0, 1]$.

Expression for $g(\eta)$

$$g(\eta) = \frac{1}{2}\|A((1-\eta)x + \eta y) - b\|^2$$

Optimal Value of η

To find the optimal η, we differentiate $g(\eta)$ with respect to η and set the derivative to zero:

$$g'(\eta) = \frac{d}{d\eta}\frac{1}{2}\|A((1-\eta)x + \eta y) - b\|^2$$

$$= (A((1-\eta)x + \eta y) - b)^\top A(y - x)$$

Setting $g'(\eta) = 0$ gives:

$$(A((1-\eta)x + \eta y) - b)^\top A(y - x) = 0$$

Solving this equation for η gives the optimal value.

Closed-Form Formula

A closed-form expression for η depends on the specific structure of A, b, x, and y. Without additional assumptions, the exact solution might be complex or not directly obtainable.

Question 12

Question 13

Question 14