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DS-B

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Homework #11

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) $A^T A\hat{x} = A^T b$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix} x = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \mapsto \left[\begin{array}{cc|c} 11 & 6 & 5 \\ 6 & 6 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & 6 & 4 \\ 11 & 6 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \textcircled{1} & 1 & 2/3 \\ 11 & 6 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} \textcircled{1} & 1 & 2/3 \\ 0 & -5 & -7/3 \end{array}\right] \sim \left[\begin{array}{cc|c} \textcircled{1} & 0 & 1/5 \\ 0 & \textcircled{1} & 7/15 \end{array}\right]$$

$$x_1 = \frac{1}{5}, \quad x_2 = \frac{7}{15}, \quad \hat{x} = \begin{bmatrix} 1/5 \\ 7/15 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.467 \end{bmatrix}$$

Date: ____/____/____

(b) $A = QR$

$$a_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$u_2 = a_2 - \text{proj}_{u_1} a_2$

$$Q = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{2} \\ 1/\sqrt{11} & -1/\sqrt{2} \\ 1/\sqrt{11} & 0 \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{11} & 5/\sqrt{11} \\ 0 & \sqrt{11}/2 \end{bmatrix}$$

$Rx = Q^T b$

$$\begin{bmatrix} \sqrt{11} & 5/\sqrt{11} \\ 0 & \sqrt{11}/2 \end{bmatrix} x = \begin{bmatrix} 5/\sqrt{11} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\sqrt{11}\,x_1 + (5/\sqrt{11})x_2 = 5/\sqrt{11}$$

$$\sqrt{11}/2\, x_2 = 1/\sqrt{2}$$

$$x_2 = \frac{\sqrt{22}}{11}$$

$x_1 = 0.2$, $x_2 = 0.467$

$$x = \begin{bmatrix} 0.2 \\ 0.467 \end{bmatrix}$$

Date: ___/___/___

(c) $e = b - Ax$

$$e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \underset{3\times 2}{\begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0.2 \\ 0.467 \end{bmatrix}} = \begin{bmatrix} -0.067 \\ 0.333 \\ -0.133 \end{bmatrix}$$

$\|e\| \approx 0.364$

(d) The matrix A has two columns and three rows, so it can have maximum rank 2. Since A has full column rank (2), its columns are linearly independent. This implies that A^TA is invertible because it is a 2×2 matrix with full rank.

(e) $P = A(A^TA)^{-1}A^T$

$$A^TA = \underset{2\times 3}{\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}$$

$3\times 2 \quad 2\times 2 \quad 2\times 3 \quad 3\times 2 \quad 2\times 1$

$$[A^TA]^{-1} = \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} \frac{1}{30} = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.3666 \end{bmatrix}$$

2×2

$$P = \begin{bmatrix} 14 & 6 & 8 \\ 6 & 2 & 4 \\ 8 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0.9666 & 0.1666 & -0.066 \\ 0.1666 & 0.1666 & 0.333 \\ -0.066 & 0.333 & 0.8664 \end{bmatrix}$$

Date: ____|____|____

(f) $A = QR$:

$$A(A^TA)^{-1}A^T = QR(R^T\overbrace{Q^TQ}^{I}R)^{-1}R^TQ^T$$

$$= QR(R^TR)^{-1}R^TQ^T$$

$$= QRR^{-1}Q^T$$

$$= QQ^T$$

(g) $\hat{b} = Pb = \begin{bmatrix} 0.467 & 0.167 & -0.066 \\ 0.167 & 0.167 & 0.333 \\ -0.066 & 0.331 & 0.866 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.068 \\ 0.667 \\ 1.133 \end{bmatrix}$

(h) $A_L^{-1} = (A^TA)^{-1}A^T$

$$= \frac{1}{6}\begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.3666 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$2 \times 2 \qquad 2 \times 3$

$$= \begin{bmatrix} 0.4 & 0 & -0.2 \\ -0.233 & 0.1666 & 0.5332 \end{bmatrix}$$