Linear Algebra (MT1004)

Final Examination

Course Instructor(s):
Muhammad Ali, Sara Aziz, Sehrish Hassan Shigri, Muhammad Usman Rashid
Sections: AI(A,B), CS(A-G), DS(A-C), SE(A,B)

Total Time (Hrs): 3
Total Marks: 125
Total Questions: 11

Date: Dec 20, 2024

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Roll No **Course Section** **Student Signature**

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Attempt all the questions.

Q 1: *QR-Decomposition*. **[8]**

(a) Factorize the following matrix A as a product of an orthogonal matrix Q and an invertible upper triangular matrix R.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \end{bmatrix}$$

(b) **Bonus Problem:** Use $A = QR$ to show that the projection matrix $A(A^TA)^{-1}A^T$ can be written as QQ^T.

Q 2: *Linear Regression Model.* **[10+3]**

(a) Fit a linear regression model $y = a + bx$ to the given data points $(1,2), (2,3), (3,4)$, and $(4,5)$.

(b) State the criteria for the existence of a left inverse, a right inverse, and a two-sided inverse.

Q 3: *Singular Value Decomposition (SVD).* **[12+2+2]**

(a) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(i) Compute U, Σ, V to find the Singular Value Decomposition (SVD) of $A = U\Sigma V^T$.

(ii) Use the SVD to find the basis of the four fundamental subspaces of A.

(iii) Write A as sum of rank 1 matrices.

(b) **Bonus Problem:** Let A be an $m \times n$ matrix with $m > n$. How many eigenvalues of AA^T must be zero?

Q 4: *Orthogonal Diagonalization.* **[10+2]**

Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Orthogonally diagonalize A.

(b) Find the spectral decomposition of A.

Q 5: ***Principal Component Analysis (PCA).*** **[(2+2+4+2+2+2)+2]**

(a) The two features of three observations are given by the following data matrix:

$$X = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 2 \end{bmatrix},$$

- **(i)** Compute the standardized/centralized matrix.
- **(ii)** Compute the covariance matrix.
- **(iii)** Find the eigenvalues and eigenvectors of the covariance matrix.
- **(iv)** Find the total variance of the data.
- **(v)** Determine how much variance is preserved when projecting the data onto the first principal component.
- **(vi)** Transform the data along the first principal component.

(b) Explain the purpose of principal component analysis.

(c) **Bonus Problem:** Show that $A^T A$ is semi positive definite matrix for any given matrix A.

Q 6: ***Constrained Optimization.*** **[2+5+2]**

Consider a quadratic form

$$Q(x_1, x_2) = {x_1}^2 + 10x_1x_2 + {x_2}^2$$

(a) Classify $Q(x_1, x_2)$ as positive definite, negative definite or indefinite.

(b) Make a change of variable, $\boldsymbol{x} = P\boldsymbol{y}$, that transforms the quadratic form into a quadratic form with no cross-product term. Give P and the new quadratic form.

(c) Find the maximum and minimum values of $Q(x_1, x_2)$ subject to the constraint ${x_1}^2 + {x_2}^2 = 1$. Find the points at which maximum and minimum of $Q(x_1, x_2)$ is attained.

(d) **Bonus Problem:** State the condition(s) under which the given matrix is negative definite:

$$A = \begin{bmatrix} a & 2 \\ 2 & 2 \end{bmatrix}.$$

Q 7: ***Theory of Affine Spaces.*** **[(4+4+2)+2]**

(a) Let

$$\boldsymbol{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 17 \\ 1 \\ 5 \end{bmatrix}.$$

- **(i)** Show that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is affinely independent.
- **(ii)** Write $\boldsymbol{y}$ as affine combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$.
- **(iii)** Find the Barycentric coordinates of $\boldsymbol{y}$ with respect to the affinely independent set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(b) Explain why $x + 2y + 7z = 1$ is a Flat. Furthermore, write the dimension of the Flat.

Q 8: ***Higher Order Singular Value Decomposition (HOSVD).*** **[2+6+6]**

For the Given Tensor: $\mathcal{B} \in \mathbb{R}^{1\times2\times3}$

$$\mathcal{B}(:,:,1) = [1 \quad 0], \qquad \mathcal{B}(:,:,2) = [1 \quad -1], \qquad \mathcal{B}(:,:,1) = [0 \quad 1].$$

(a) Let $A \in \mathbb{R}^{2\times 3}$ be a matrix. Discuss which of the following mode-n products are possible and what will be the dimension of the resulting tensor.

$$\mathcal{B} \times_1 A, \quad \mathcal{B} \times_2 A, \quad \mathcal{B} \times_3 A$$

(b) Compute mode-1 unfolding $B_{(1)}$, mode-2 unfolding $B_{(2)}$, mode-3 unfolding $B_{(3)}$.

(c) For the given factor matrices

$$U^{(1)} = [4], U^{(2)} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \text{and } U^{(3)} = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix},$$

compute the core tensor $S = \mathcal{A} \times_1 U^{(1)} \times_2 U^{(2)} \times_3 U^{(3)}$.

Q 9: ***Orthogonal Projection and Orthogonal Complement.*** **[1+1+2]**

(a) Let W be a subspace spanned by the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

(i) What is the $\dim(W)$.

(ii) What should be $\dim(W^{\perp})$.

(iii) Find the basis for the orthogonal complement $W^{\perp}$ of W.

(b) **Bonus Problem:** Describe how a projection on column space of a matrix A and projection on a row space of a matrix can be constructed using the left and right inverses.

Q 10: ***Inner Product Spaces.*** **[2+5]**

(a) Check if the vectors $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ are orthogonal or not relative to the defined inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T A \boldsymbol{y}$, when $A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$.

(b) Apply the Gram-Schmidt Process to the $\mathcal{B} = \{1, 1 + x\}$ to obtain an orthonormal basis for the subspace of the inner product space $V = \mathbb{P}_2$ relative to the given inner product defined below

$$\langle p(x), q(x) \rangle = p(t_0)q(t_0) + p(t_1)q(t_1), \text{where } t_0 = 1, t_1 = 2 .$$

Q 11: ***3D Transformations.*** **[(6+2)+(2+2+2)]**

(a) A point $(2,3,4)$ is transformed by the following operations in sequence: Translation by $(1, -2, 3)$, rotation of 45° about the z-axis, reflection across the xy-plane and a perspective projection onto the plane $z = 2$, assuming the center of projection is at the origin.

(i) Write the transformation matrices for each operation.

(ii) Compute the final coordinates of the given point after applying the sequence of transformations.

(b) Let $\boldsymbol{u} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and H be the householder matrix given by $H = I - 2\boldsymbol{u}\boldsymbol{u}^T$. Prove that

(i) H is symmetric, **(ii)** H is orthogonal, **(iii)** $H^2 = I$.