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Question #1:-

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

(a) $A = U \Sigma V^T$

Step 1:- find eigen values and vectors (A^TA)

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$\lambda_1 = 2$, $\lambda_2 = 3$

$\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{3}$

$\Sigma \ (3 \times 2) \Rightarrow$

Step 2:- find Σ

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

Step 3:- find V.

$\lambda = 2$

$(A - 2I)v = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 3$

$(A - 3I)v = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigen vectors = $\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ singular vectors

Step 4:-

$U = [u_1 \; u_2 \; u_3]$

$$u_1 = \frac{1}{\delta_1} A v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{\delta_2} A v_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

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$$u_3 = \frac{1}{\sigma_3} A v_3$$

let $[x_1 \; x_2 \; x_3]$ to find v_3
$\perp$ to (v_1, v_2)

$$\frac{1}{\sqrt{2}} x_1 + 0 - \frac{1}{\sqrt{2}} x_3 = 0$$

$$x_3 = x_3$$
$$x_1 = x_3$$
$$2/\sqrt{3}\, x_3 + 1/\sqrt{3}\, x_2 = 0$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \& \quad u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$\& \; A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(B) Part.

(1) ~~col~~ $A = \delta_1 u_1 v_1^T + \delta_2 u_2 v_2^T$

$$A = \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

(C) Part

(1) col A

$$\text{col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

These vectors from orthognal Basis.

(b) Nul A :-

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

(C) Row space :-

$$\text{Row}(A) = \{[1\ 1], [0\ 1]\}$$

(d) Left Null space :-

$$\text{Null } A^T = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

~~Roz~~

(D) Part

$$A^{+} = V \Sigma^{+} U^{T}$$

$$\Sigma^{+} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

(Part e)

$$\hat{x} = (A^{+}A)^{-1} Ab$$

$$\hat{x} = A^{+} b.$$

↳ unknown

Part (f)

$$P = uu^T$$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Question # 2 :-

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

(a) Std Matix :-

$$\mu_1 = 1 + 0 - 1 = 0$$

$$\mu_2 = \frac{1+1+1}{3} = 1$$

So Std Matix is :-

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$$

(b) covariance Matix.

$$C = \frac{1}{n-1} Z^T Z$$

$$C = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) eigen values & vectors of "C" :-

$\lambda = 2$

Eigen vectors :- $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

$x_1 = x_1$

$-2x_2 = 0$

$x_2 = 0$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ vector.

So, eigen value $\lambda = 2$
and vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

thus $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first principal component.

(d) Varience :-

$$= \frac{\lambda_1}{\Sigma \lambda} \times 100$$

$$= \frac{2}{2} \times 100$$

$$= 100\%$$

There is 100% variation in direction of u_1 and all data lies in the direction of u_1

(e) Transformation :-

$$Zu_1 = T$$

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$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(f) $P = 2 u_1 u_1^T$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$$

as full data is in direction of first principal component so all data is fully projected to u_1 so,

$$\boxed{Z = P}$$