National University of Computer & Emerging Sciences Islamabad

FAST School of Computing | Fall-2024 | Islamabad Campus

MT1004 – Linear Algebra

Homework # 13

1. Let $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$ be an evaluation inner product on $\mathbb{P}_3$ for the vectors $\boldsymbol{p} = p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\boldsymbol{q} = q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ at $x_0 = -3$, $x_1 = -1$, $x_2 = 1$, and $x_3 = 3$.

(a) Find best approximation of $p(x) = x^2$ onto the subspace $\mathbb{P}_1 = \text{Span}\{1, x\}$.
(b) Write an orthogonal basis of $\mathbb{P}_2 = \text{Span}\{1, x, x^2\}$ using the results from (**a**).

2. Construct an orthonormal basis for $M_{2\times2}$ with respect to the inner product
$$\langle A, B \rangle = \text{tr}(A^T B)$$
by applying the Gram-Schmidt Process to the basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right\}$.

3. Find the Fourier approximation to the function $f(x) = x - 1$ on the interval $(-\pi, \pi)$. (Hint: Use $f(x) \approx a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$.)

Questions listed below are for practice only. Do NOT submit.

1. Determine which of the four inner product axioms do not hold and give a specific example in each case.

(a) In $\mathbb{P}_2$, define $\langle p(x), q(x) \rangle = p(1)q(1)$.
(b) In $M_{2\times2}$, define $\langle A, B \rangle = \det(AB)$.

2. Let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1v_1 + 5u_2v_2$ be a weighted Euclidean inner product on $\mathbb{R}^2$ for the vectors $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$. Find a matrix that generates it. (Hint: Express the given inner product as $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T A \boldsymbol{v}$.)