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DS-B

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HOMEWORK #13

1. $\langle p,q\rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$

$p = p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
$q = q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$
@ $\cancel{x_1 = 1}$ $x_0 = -3$, $x_1 = -1$, $x_2 = 1$, and $x_3 = 3$

(a) $p(x) = x^2 \mapsto \mathbb{P}_1 = \text{Span}\{1, x\}$

Let $f(x) = a + bx$ in $\mathbb{P}_1$
$f(x) = 1 + x \longmapsto f_1 = 1,\ f_2 = x$

$$\text{proj}_{\mathbb{P}_1} p = \frac{\langle p, f_1\rangle}{\langle f_1, f_1\rangle} f_1 + \frac{\langle p, f_2\rangle}{\langle f_2, f_2\rangle} f_2$$

$$\frac{\langle p, f_1\rangle}{\langle f_1, f_1\rangle} f_1 = \frac{p(x_0)f_1(x_0) + p(x_1)f(x_1) + p(x_2)f_1(x_2) + p(x_3)f_1(x_3)}{f_1(x_0)^2 + f_1(x_1)^2 + f_1(x_2)^2 + f(x_3)^2} f_1$$

$$= \cancel{(9)(1)}\ \frac{9 + 1 + 1 + 9}{1 + 1 + 1 + 1}(1) = \frac{20}{4}(1) = 5$$

$$\frac{\langle p, f_2\rangle}{\langle f_2, f_2\rangle} f_2 = \frac{p(x_0)f_2(x_0) + p(x_1)f_2(x_1) + p(x_2)f_2(x_2) + p(x_3)f(x_3)}{f_2(x_0)^2 + f_2(x_1)^2 + f_2(x_2)^2 + f_2(x_3)^2} f_2$$

$$= \frac{(9)(-3) + (1)(-1) + (1)(1) + (9)(3)}{9 + 1 + 1 + 9} x = 0$$

$\text{proj}_{\mathbb{P}_1} p = 5 + 0 = 5$ Ans

$p_{approx}(x) = 5$

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(b) $\mathbb{P}_2 = \text{Span}\{1, x, x^2\}$

$\cancel{f_1(x)}\ f_1(x) = 1 \quad , \quad f_2(x) = x \qquad f_3(x) = x^2$

$U_1 = f_1 \mapsto$ orthogonal basis $= \{U_1, U_2, U_3\}$

$U_2 = f_2 - \underbrace{\text{proj}_{U_1} f_2}_{\text{known from (a)}} = x - \cancel{\frac{\langle U_1, f_2\rangle}{\langle f}} \frac{\langle f_2, U_1\rangle}{\langle U_1, U_1\rangle} U_1$

$= x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}(1) = x - 0 = x$

$U_3 = f_3 - \text{proj}_{U_1} f_3 - \text{proj}_{U_2} f_3$

$U_3 = x^2 - \underbrace{\frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}(1)}_{\text{(a)}} - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x$

$U_3 = x^2 - 5 - 0 = x^2 - 5$

Orthogonal Basis $= \{1, x, x^2 - 5\}$

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2. $\langle A,B\rangle = tr(A^T B)$, $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$ Manual
$\qquad\qquad\qquad M_1 \qquad\quad M_2 \qquad\quad M_3 \qquad\quad M_4$

Orthogonal $\beta = \{u_1, u_2, u_3, u_4\}$

Orthonormal $\beta = \{e_1, e_2, e_3, e_4\}$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e_2 = M_2 - proj_{e_1} M_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - 1\langle A_2, e_1\rangle e_1$$

$$e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tr(A_2^T e_1) = \frac{1}{\sqrt{2}} tr \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$e_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - proj_{e_1} M_3 - proj_{e_2} M_3$$

$$u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \langle M_3, e_1\rangle e_1 - \langle M_3, e_2\rangle e_2$$

$$u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - tr(A_3^T e_1)\, e_1 - tr(A_3^T e_2)\, e_2$$

$$u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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1 $U_4 = M_4 - \text{proj}_{e_1} M_4 - \text{proj}_{e_2} M_4 - \text{proj}_{e_3} M_4$

$U_4 = M_4 - \langle M_4, e_1 \rangle e_1 - \langle M_4, e_2 \rangle e_2 - \langle M_4, e_3 \rangle e_3$

$$U_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - [0]\, 4 - [0] - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$E = \left\{ \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \right\}$$

3. $f(x) = x - 1 \quad , [-\pi, \pi]$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x - 1 \, dx = \frac{1}{2\pi} \left| \frac{x^2}{2} - x \right|_{-\pi}^{\pi} = -1$$

$$\cancel{a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-1) \cos kx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \, dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \, dx}$$

~~$a_k = 0$ as odd function~~

~~Odd functio so change limits~~

$$\cancel{a_1 = \frac{2}{\pi} \left[\int_0^{\pi} x \cos x \, dx - \int_0^{\pi} \cos x \, dx \right]}$$

$$\cancel{a_k = -\frac{4}{\pi}}$$

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$$\cancel{b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-1)\sin x \, dx}$$

$$\cancel{= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} x\sin x - \int_{-\pi}^{\pi} \sin x \, dx\right)}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-1)\cos(kx)\, dx$$

$$a_k = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos kx \, dx - \int_{-\pi}^{\pi} \cos kx \, dx\right]$$

$$\cancel{= \frac{2}{\pi}\left[\int_{0}^{\pi} x\cos kx \, dx - \int_{0}^{\pi} \cos kx \, dx\right]}$$

$$a_k = \frac{-2\sin(\pi k)}{\pi k}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-1)\sin kx \, dx$$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin kx \, dx - \int_{-\pi}^{\pi} \sin kx \, dx\right]$$

$$b_k = \frac{2\sin(\pi k) - 2\pi k\cos(\pi k)}{\pi k^2}$$

Fourier Approximation: $\frac{2\sin(\pi k) - 2\pi k\cos(\pi k)}{\pi k^2} - \frac{2\sin(\pi k)}{\pi k} - 1$