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Homework #2

1(a) $\lim_{(x,y)\to(0,0)} \frac{(2x-y)^2}{x-y} = \frac{0}{0}$

Along $x=0$:

$\lim_{y\to 0} \frac{(-y)^2}{-y} = -y \lim_{y\to 0} -y = 0$

Along $y=0$:

$\lim_{x\to 0} \frac{(2x)^2}{x} = \lim_{x\to 0} 4x = 0$

Along $y=x$:

$\lim_{x\to 0} \frac{(2x-x)^2}{x-x} = \lim_{x\to 0} \frac{x^2}{0} = \infty$

As the limit for the paths differ we can say L.D.N.E

1(b)(i) $\lim_{(x,y)\to(0,0)} \frac{\sin(x)(e^y-1)}{xy} = \frac{0}{0}$

$\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{x} \cdot \frac{(e^y-1)}{y}$

$\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{x} \cdot \lim_{(x,y)\to(0,0)} \frac{e^y-1}{y}$

$1 \cdot 1$

$L = 1$

Limit exists.

(ii) $\lim_{(x,y)\to(0,0)} |y|^{x}$

Along $x=0$:

$$\lim_{y\to 0} |y|^{0} = 1$$

Along $y=0$:

$$\lim_{x\to 0} |0|^{x} = 0$$

Limiting value different hence L.D.N.E.

(iii) $\lim_{(x,y)\to(2,1)} \dfrac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$

$x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{r\to 0} \frac{(r\cos\theta-2)(r\sin\theta-1)}{(r\cos\theta-2)^2+(r\sin\theta-1)^2} = \lim_{r\to 0} \frac{r^2(\cos\theta-2)(\sin-1)}{(r\cos\theta-2)^2+(r\sin\theta-1)^2}$$

$$\lim_{r\to 0} r^2 \cdot \lim_{r\to 0} \frac{\cos(\theta-2)(\sin\theta-1)}{r^2-4r\cos\theta+5+r^2-2r\sin\theta}$$

$L = 0$

(iv) $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{x^2+1}{x^2+(y-1)^2}\right)$

$x = r\cos\theta \qquad y = r\sin\theta + 1$

$\lim_{\substack{\cancel{(x,y)\to(0,1)} \\ r\to 0}} \tan^{-1}\left(\frac{r^2\cos^2\theta+1}{r^2\cos^2\theta+(r\sin\theta\,\cancel{1})^2}\right) = \lim_{r\to 0} \tan^{-1}\left(\frac{r^2\cos^2\theta+1}{r^2(1)}\right)$

$\lim_{r\to 0} \tan^{-1}\left(\frac{r^2\cos^2\theta+1}{r^2}\right)$

• Input r

$\tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2}$

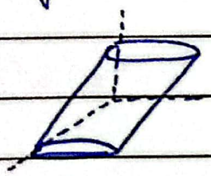
$L = \frac{\pi}{2}$ ← Ans.

c)

$$f(x,y) = \begin{cases} \frac{\sin(x^2-y^2)}{x^2-y^2}, & x^2 \neq y^2 \\ 1, & x^2 = y^2 \end{cases}$$

The function is continous on all points except at $x = y$ where it has a removable/jump/hole discontinuity. By this definition, the 'hole is filled' however a jump discontinuty will remain due to the definition.

2(a)



$(x-h)^2 + (y-k)^2 = r^2$

$x^2 + y^2 = 1, \quad z = y$

Slope in ~~x~~ $+y$: $\left.\frac{\partial z}{\partial y}\right|_{(0,-1,0)} = 1$

Slope in $+x$: $\left.\frac{\partial z}{\partial x}\right|_{(0,-1,0)} = 0$

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b) $f(x,y) = \begin{cases} \dfrac{\cos(x)-\cos(y)}{x-y} & , x \neq y \\ 0 & , x = y \end{cases}$

(i) $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{\frac{\cosh - \cos 0}{h-0} - 0}{h} = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$\lim\limits_{h \to 0} \dfrac{\cos(h)}{h} = 0$

~~(ii)~~ $f_y(x,y) = \cancel{\lim\limits_{h\to 0}} \dfrac{f(x,y+h) - f(x,y)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{\cos x - (\cos(y+h))}{x-y+h} - \frac{\cos x - \cos y}{x-y}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\frac{\cos(x) - \cos(y)\cosh - \sin y \sinh}{x-y+h} - \frac{\cos x - \cos y}{x-y}}{h}$

(ii) $f_y(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{\cos(x)-\cos(y)}{x-y}\right)\Bigg|_{(x,y)}$

$= \dfrac{u'v - v'u}{v^2}$

$= \dfrac{\sin(y)(x-y) - (-1)(\cos(x) - \cos(y))}{(x-y)^2} = \dfrac{\sin(y)(x-y) + \cos(x) - \cos(y)}{(x-y)^2}$

(c) $f(x,y) = x^2 + y^3$, $(-1,1,2)$

$f_x = 2x$, $f_y = 3y^2$

(i) $z = (-1)^2 + y^3 = y^3 + 1$

$\dfrac{\partial z}{\partial y} = 3y^2$

$\dfrac{\partial z}{\partial y}\Bigg|_{(-1,1,2)} = 3$

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ii) $z = x^2 + 1^3 = x^2 + 1$

$$\frac{\partial z}{\partial x} = 2x$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,1,2)} = -2 \quad \text{An.}$$

(d) No, a function $f(x,y)$ with continuous first partial derivatives throughout an open region R does not necessarily have to be continuous on R.

Example:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

At $(0,0)$:

$$\lim_{r \to 0} \frac{\cancel{r^2} \cos\theta \sin\theta}{\cancel{r^2}} = \cos\theta \sin\theta$$

Limit does not exist hence not continuous.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \qquad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

These exist for all (x,y)

The partial derivatives of f are continuous for all (x,y) however the function itself is discontinuous at $(0,0)$

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c) $f(x,y) = \begin{cases} xy \cdot \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(i) $f_x = \dfrac{u'v - v'u}{v^2} = \dfrac{(3x^2y - y^3)(x^2+y^2) - (2x)(x^3y - xy^3)}{(x^2+y^2)^2}$

$f_x(x,0) = 0$

$f_y = \dfrac{u'v - v'u}{v^2} = \dfrac{(x^3 - 3xy^2)(x^2+y^2) - (2y)(x^3y - xy^3)}{(x^2+y^2)^2}$

$f_y(0,y) = 0$

(ii) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$

$f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{x^4y - x^3y^3 - y^3x^2 - y^5}{(x^2+y^2)^2}\right) = \dfrac{u'v - v'u}{v^2}$

$f_{xy} = \dfrac{(x^4 - 3x^3y^2 - 3y^2x^2 - 5y^4)(x^2+y^2)^2 - (2(x^2+y^2)(2y))(x^4y - x^3y^3 - y^3x^2 - y^5)}{(x^2+y^2)^4}$

$f_{xy}(0,0) = 0$

$f_{yx} = \dfrac{\partial}{\partial x}(f_y)$

$f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{(x^2+y^2)^2}\right)$

$f_{yx} = \dfrac{u'v - v'u}{v^2} = \cancel{5x^4 + 3x^2y^2 - 9x^2y^2 - 3y^4 - 6x^2y^2 + 2y^4}$

$- 3xy^4 - 2x^3y^2 + 2xy^4)$

$= \dfrac{(5x^4 + 3x^2y^2 - 9x^2y^2 - 3y^4 - 6x^2y^2 + 2y^4)(x^2+y^2)^2 - (2(x^2+y^2)(2y))(x^5 + x^3y^2 - 3x^3y^2}{(x^2+y^2)^4}$

$f_{yx} = \dfrac{0}{0}$

$\dfrac{0}{0} \neq 0 \rightarrow$ Shown.