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DS-B

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Assignment #2

Q1: $T(p(x)) = \begin{bmatrix} p(1)-p(0) & p(2)-p(0) \\ p(-1)-p(0) & p(-2)-p(0) \end{bmatrix}$ $T: \mathbb{P}_2 \to M_{2\times2}$

(a) Matrix of Transformation:

Basis of $\mathbb{P}_2 = \{1, x, x^2\}$

$T(1) = \begin{bmatrix} 1-1 & 1-1 \\ 1-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x) = \begin{bmatrix} 1-0 & 2-0 \\ -1-0 & -2-0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 1-0 & 4-0 \\ 1-0 & 4-0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$

MOT : $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$

(b) Linearity:

Check additivity and ~~m~~ scalar multiplicity.

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1. Additivity:

$$\boxed{\begin{array}{l} \text{Let } p(x) = a_0 + a_1x + a_2x^2 \\ \text{Let: } q(x) = b_0 + b_1x + b_2x^2 \end{array}}$$

$$T(p(x)+q(x)) = \begin{bmatrix} (p+q)(1) - (p+q)(0) & (p+q)(2) - (p+q)(0) \\ (p+q)(-1) - (p+q)(0) & (p+q)(-2) - (p+q)(0) \end{bmatrix}$$

$$= \begin{bmatrix} p(1) + q(1) - p(0) - q(0) & p(2) + q(2) - p(0) - q(0) \\ p(-1) + q(-1) - p(0) - q(0) & p(-2) + q(-2) - p(0) - q(0) \end{bmatrix}$$

$$= \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix} + \begin{bmatrix} q(1) - q(0) & q(2) - q(0) \\ q(-1) - q(0) & q(-2) - q(0) \end{bmatrix}$$

Thus, $T(p(x)+q(x)) = T(p(x)) + T(q(x))$

2. Scalar Multiplicity:

$$T(c.p(x)) = \begin{bmatrix} c.p(1) - c.p(0) & c.p(2) - c.p(0) \\ c.p(-1) - c.p(0) & c.p(-2) - c.p(0) \end{bmatrix}$$

$$= \begin{bmatrix} c(p(1) - p(0)) & c(p(2) - p(0)) \\ c(p(-1) - p(0)) & c(p(-2) - p(0)) \end{bmatrix}$$

$$T(c.p(x)) = c. \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}$$

Thus, $T(c.p(x)) = c.T(p(x))$

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As both additivity and scalar multiplicity hold, T is a linear transformation.

(c) $Ker(T)$: $T(p(x))=0$

$a+bx+cx^2=0$

$p(1)= a+b+c$ $\quad p(-1) = a-b+c$

$p(2)= a+2b+4c$ $\quad p(-2)= a-2b+4c$

$p(0)=a$

$$T(p(x))=\begin{bmatrix}(a+b+c)-(a) & (a+2b+4c)-(a)\\ (a-b+c)-(a) & (a-2b+4c)-(a)\end{bmatrix}$$

$$=\begin{bmatrix} b+c & 2b+4c\\ -b+c & -2b+4c\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$b+c=0 \rightarrow 2b=0 \rightarrow b=0$

$-b+c=0 \rightarrow c=b \leftrightarrow c=0.$

$2b+4c=0$

$-2b+4c=0$

~~For any values of a, and b=c=0, ker T = {0}~~

$Ker\,T = \{p(x)=a \mid a\in\mathbb{R}\}$ as $b=c=0$ but there is no restriction on a. hence $\dim Ker\,T=1$

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(d) Range:

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ R2-2R1 \\ R3+R1 \\ R4+2R1 \end{matrix}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R3-R2 \\ R4-3R2 \end{matrix} \end{array}$$

We can see that x_1 is within the span of x_2 and x_3 as it doesn't have a pivot column.

Hence Dim Range(T) = 2

Basis $= \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} \right\}$

~~Dim V = Dim Ker(T) + Dim R(T)~~

~~3 = Dim Ker(T) + 2~~

~~Dim Ker(T) = 1~~

(e) As the ~~f~~ transformation is neither one to one $(Ker\,T \neq 0)$ nor onto (Basis $\neq$ ~~[illegible]~~ $M_{2\times2}$), it is not an isomorphism

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f) Rank Nullity Theorem

$\text{Dim}\, V = \text{Dim}\, \text{Ker}\, T + \text{Rank}\, T$

$\text{Dim}\, V = \text{Dim}(\text{Ker}(T)) + \text{Dim}\, (\text{Range}\, T)$

$3 = 1 + 2$

$3 = 3 \quad \therefore$ Verified

~~Q: $P(x, y, z)$, $O(0,0,0) \rightarrow$ Camera. P and 2D img at $z = d$~~
~~where $d > 0$~~

(a) $x' = \frac{d}{z} x$, $y' = \frac{d}{z} y$

$P = (x, y, z, 1)$

Projection Matrix:

$$M = \begin{bmatrix} d - z_c & 0 & -x_c & x_c d \\ 0 & d - z_c & -y_c & y_c d \\ 0 & 0 & d & -d z_c \\ 0 & 0 & 1 & -z_c \end{bmatrix}$$

Apply M to $(x, y, z, 1)$:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} dx/z \\ dy/z \\ d \end{bmatrix}$$

$x' = \frac{dx}{z}$, $y' = \frac{dy}{z}$

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Q2: Homogenous Coordinates: $P_h = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ Mureeb

$P' = P_o \times P_h$

$$P' = \begin{bmatrix} d-z_c & 0 & x_c & -dx_c \\ 0 & d-z_c & y_c & -dy_c \\ 0 & 0 & d & -dz_c \\ 0 & 0 & 1 & -z_c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

P' = In terms of x', y', z' and w'

$x' = (d-z_c)(x) + (0)(y) + (x_c)z - dx_c$ $\frac{1}{6}$

$x' = (d-z_c)x + x_c z - dx_c$

$y' = (d-z_c)(y) + (0)(x) + (y_c)z - dy_c$

$y' = (d-z_c)y + y_c z - dy_c$

$z' = (0)x + (0)y + dz - dz_c$

$z' = dz - dz_c$

$w' = (0)(x) + (0)(y) + z - z_c$

$w' = z - z_c$

$$P' = \begin{bmatrix} (d-z_c)x + x_c z - dx_c \\ (d-z_c)y + y_c z - dy_c \\ dz - dz_c \\ z - z_c \end{bmatrix}$$

x_o : Normalized Coordinates

$x_o = \frac{x'}{w'} = \frac{(d-z_c)x + x_c z - dx_c}{z - z_c}$

$y_o = \frac{y'}{w'} = \frac{(d-z_c)y + y_c z - dy_c}{z - z_c}$

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(b) Objects that are farther from the camera appear smaller due to the scaling factor $\frac{d}{z}$. As z-increases scale factor decreases so x' and y' decrease and image appears farther away and smaller.

(c) Given $P(4,3,8)$, $z=4$

$$x' = \frac{dx}{z} = \frac{16}{8} = 2 \quad , \quad y' = \frac{dy}{z} = \frac{12}{8} = 1.5$$

So new coordinate $P_o = (2, 3/2)$

Q3: $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 6 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 1/2 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ R2/2

(a) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} \\ R2-2R1 \\ R3-3R1 \end{matrix}$ • Zero row

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = E_3^{-1} E_2^{-1} E_1^{-1} . \tilde{A}$

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(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 6 & 3 & 9 \end{bmatrix}$ $\quad \tilde{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

~~As $\tilde{A}$, has a zero row, ...~~

$\det A = 2\begin{vmatrix} 4 & 7 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 4 \\ 6 & 3 \end{vmatrix} = 0$

As $\det A = 0$, matrix is not invertible.

(ii) $\underset{6\times 6}{\det A} = \frac{1}{4}$, $\det 2A = 2^6 \cdot \frac{1}{4} = 16$, $\det(-A) = (-1^6) \cdot \frac{1}{4} = \frac{1}{4}$, $\det(A^2) = \frac{1^2}{4} = \frac{1}{16}$

$\det A^{-1} = \frac{1}{\det A} = 4$

(iii) ~~$B = \frac{1}{\sqrt{13}}$~~ $\det B = \frac{1}{13}$, $V_{cone} = 26$, $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{bmatrix}$

$V'_{cone} = \det AB^2 \cdot V_{cone}$

$\det AB^2 = \det A \cdot \det B^2 = 39 \times \frac{1}{169} = \frac{3}{13}$

$V'_{cone} = 6\,cm^3$

Q4: $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

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(i) Eigenvalues :

$$\det(A-\lambda I)=0$$

$$A-\lambda I=\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I)=0=1-\lambda\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix}-1\begin{vmatrix} 1 & 0 \\ 0 & -\lambda \end{vmatrix}+0\begin{vmatrix} 1 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$2\lambda^2-\lambda^3=0$$

$$\lambda^2(2-\lambda)=0$$

$$\lambda^2=0 \quad , \lambda=2$$

$$\lambda_1=\lambda_2=0 \quad \lambda_3=2$$

$\lambda=0$:

$$(A-0I)\vec{v}=0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v_1}+\vec{v_2}=0$ — (1) $\quad \vec{v_2},\vec{v_3}$ free

$\vec{v_3}=0$ — (2)

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$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Eigenvectors} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 2$:

$$(A - 2I)v = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \cancel{0}\,{-2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \cancel{0}\,{-2} \end{bmatrix} \begin{matrix} -R1 \\ \\ \\ \end{matrix} \overset{R2-R1}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\vec{v_1} - \vec{v_2} = 0$ — (1) $\longrightarrow \vec{v_1} = \vec{v_2}$

$-2\vec{v_3} = 0$ — (2)

$$\vec{V} = \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Eigenvector}: \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

two lin. independent vectors and 2 yields one so A.M = G.M. ✓

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ii) Basis for $\lambda = 0 \mapsto \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $\lambda = 2 \mapsto \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

iii) Diagonalization:

(a) $P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$A = PDP^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_1 = 0 \rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \|v_1\| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$= \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 \rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \|v_1\| = \sqrt{2}$$

$$\frac{v}{\|v\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\text{Ortho } P = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ +1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

iv)

	A.M	G.M
0	2	2
2	1	1

$\}\rightarrow$ Diagonalizable

A is diagonalizable because even though 0 is repeated, it yields two lin. independent vectors and 2 yields one so A.M = G.M.

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v) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ spans entire $\mathbb{R}^3$ because 3 Linearly independant vectors from $\mathbb{R}^3$.

vi) Matrix of Transformation :

Eigenbasis: $T(x) = A\vec{x}$

$$T(v_1) = Av_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(v_2) = Av_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(v_3) = Av_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2v_1 .$$

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Q5: $\lambda_1 = 2$, $\lambda_2 = 3$

$v_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(a) $\det A = \lambda_1 . \lambda_2 = 6$, $\det A^n = (\det A)^n = 6^n$

(b) If $A \sim B$, $A = CBC^{-1}$ → Similar

$trace(B) = trace(A) = 5$

$\det(B) = \det(A) = 6$

(c) Eigenvalues of $A - cI$: $\underbrace{\lambda_1 = 2-c \quad , \lambda_2 = 3-c}_{}$ Change of origin

$A^{-1} = \frac{1}{\cancel{6}}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

A^n : $\lambda_1 = 2^n$, $\lambda_2 = 3^n$

$v_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(ii) (a) $a_0 = 4, a = 1$, $a_n = a_{n-1} - \frac{a_{n-2}}{4}$, $x \geq 2$

$a_n - a_{n-1} + \frac{a_{n-2}}{4} = 0$ $\Bigg\{ \lambda_1 = \lambda_2 = \frac{1}{2}$

$\lambda^2 - \lambda + \frac{1}{4} = 0$

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Repeated roots so: $x_n = c_1\lambda_1^n + c_2\lambda_2^n x$

$x_n = (c_1+c_2)\lambda_1^n n$

$x_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{2}\right)^n n$

$n=0: \quad x_0 = c_1$

$n=1: \quad x_1 = \frac{c_1}{2} + \frac{c_2}{2}$

At $c_1 = 4$: $\quad x_1 = \frac{4}{2} + \frac{c_2}{2} = 1$

$\frac{c}{2} = -1$

$c_2 = -2$

$x_n = 4\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right)^n n = \left(\frac{1}{2}\right)^n(4-2n)$

~~Q6: $y_{k+1} = 0.8y_k + 0.3$~~

(b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$

$a_0 = 2, a_1 = 2, a_2 = 4, \quad n \geq 3$

$a_n - 2a_{n-1} + a_{n-2} - 2a_{n-3} = 0$

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\lambda_1 = -1$

$\lambda_2 = 1$

$\lambda_3 = 2$

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$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n + c_3 d_3^n$$

$$x_n = c_1(-1)^n + c_2(1)^n + c_3(2)^n$$

$$x_0 = c_1(-1)^0 + c_2(1)^0 + c_3(2)^0 = c_1 + c_2 + c_3 \quad -(1)$$

$$x_1 = -c_1 + c_2 + 2c_3 \quad -(2)$$

$$x_2 = c_1 + c_2 + 4c_3 \quad -(3)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & -3 & -2 \end{array}\right]$$

$$-3c_3 = -2 \qquad c_3 = \frac{2}{3}$$

$$2c_2 + 3\left(\frac{2}{3}\right) = 4$$

$$2c_2 + 2 = 4$$

$$c_2 = 1$$

$$c_1 + c_2 + c_3 = 2$$

$$c_1 = \frac{1}{3}$$

$$\therefore x_n = \frac{1}{3}(-1)^n + 1(1)^n + \frac{2}{3}(2)^n$$

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Q6: $y_{k+1} = 0.8y_k + 0.3z_k$, $y_0 = 0$

$z_{k+1} = 0.2y_k + 0.7z_k$, $z_0 = 5$

As $z_{k+1} = A\vec{x}_k$

$x_{k+1} = \begin{bmatrix} y_{k+1} \\ z_{k+1} \end{bmatrix}$ $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$x_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}$

$\therefore x_k = c_1 \lambda_1^k \vec{v_1} + c_2 \lambda_2^k \vec{v_2}$

$\lambda^2 - tr(A) + det(A) = 0$

$\lambda^2 - 1.5\lambda + 0.5 = 0$

$\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$

$\lambda = 1$:

$(A - I)\vec{x} = 0$

$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$

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$-0.2x_1 + 0.3x_2 = 0$

$+0.2x_1 = 0.3x_2$

$x_1 = \frac{3}{2}x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$\lambda = 1/2$

$(A - 1/2 I)\vec{v} = 0$

$= \left[\begin{array}{cc|c} 0.3 & 0.3 & 0 \\ 0.2 & 0.2 & 0 \end{array}\right]$

$0.3x_1 + 0.3x_2 = 0$

$0.3(x_1 + x_2) = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

or

$x_2 = -x_1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cancel{-x_2} \\ \cancel{x_2} \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x_k = c_1(1)^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$k=0: \; x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & -1 & 5 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right]$

$c_1 = 1$

$c_2 = -3$

$x_k = (1)(1)^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-3)\left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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(b) $\lambda_1 = 1$ $\quad$ $\lambda_1 = 1$ and $\lambda_2 < 1$

$\lambda_2 = \frac{1}{2}$ $\quad$ $k \to \infty$, $\vec{x}_k = 1$ eigenspace

Corresponding to eigenvalues:

Origin is saddle point.

(c) For $\lambda_2 = \frac{1}{2} < 1$, greatest attractor is along line spanned by $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_1 = 1$, direction is neutral in direction of $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Q7: $x'(t) = -0.8x + 0.4y$

$y'(t) = 0.4x - 0.7y$

a) $x(t)$ as $Py(t)$

$x(t) = Py(t)$

$y'(t) = 0.4Py - 0.7y$ $\quad$ $x'(t) = -0.8Py + 0.4y$

$y'(t) = (0.4P - 0.7)y$ $\quad$ $x'(t) = (-0.8P + 0.4)y$

$y' = Dy$

$y' = (0.4P - 0.7)y$

$D = 0.4P - 0.7$

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To find P, analyze system

$$A = \begin{pmatrix} -0.8 & 0.4 \\ 0.4 & -0.7 \end{pmatrix}$$

b) Population Check:

Stability of populations can be found by finding eigenvalues of matrix A.

$$\det(A - \lambda I) = 0$$

$$= \begin{bmatrix} -0.8 & 0.4 \\ 0.4 & -0.7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.8 - \lambda & 0.4 \\ 0.4 & 0.7 - \lambda \end{bmatrix}$$

$$\det = \lambda^2 + 1.5\lambda + 0.40 = 0$$

$$\lambda = \frac{-1.5 \pm 0.1}{2}, \quad \lambda_1 = -0.7, \quad \lambda_2 = -0.4$$

Both $\lambda < 0$ so $x(t)$ and $y(t)$ will approach equilibrium point at origin, showing both population tend to stabilize.

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c) Nature of origin: Since both are negative, origin is attractor.

d) Direction of greatest attraction or repulsion:

For $\lambda_1 = -0.7$:

$(A + 0.7I)\vec{v} = 0$

$$\begin{pmatrix} -0.1 & 0.4 \\ 0.4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -0.1 & 0.4 & 0 \\ 0.4 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ 4 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 16 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

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$Nul(A+0.7I) = x_1 - 4x_2 = 0$

$x_1 = 4x_2$

$Nul(A+0.7I) = x_2\begin{pmatrix}4\\1\end{pmatrix}$ $\quad v_1 = \begin{pmatrix}4\\1\end{pmatrix}$

$\lambda = -0.4$

$(A+0.4I)\vec{v} = 0$

$\left[\begin{array}{cc|c} -0.4 & 0.4 & 0 \\ 0.4 & -0.3 & 0 \end{array}\right]$ $\quad -0.4v_1 + 0.4v_2 = 0$

$+0.4v_1 = 0.4v_2$

$v_1 = v_2$

$Nul(A+0.4I)\vec{v} = v_2\begin{pmatrix}1\\1\end{pmatrix}$

Direction of greatest attraction correspond to these eigenvectors.

c) $e^A = ?$

$A = PDP^{-1}$

$D = \begin{pmatrix}-0.7 & 0\\0 & -0.4\end{pmatrix}$

$e^D = \begin{pmatrix}e^{-0.7} & 0\\0 & e^{-0.4}\end{pmatrix}$

$e^D = \begin{pmatrix}0.4966 & 0\\0 & 0.6703\end{pmatrix}$

$P = \begin{pmatrix}4 & 1\\1 & 1\end{pmatrix}$

$P^{-1} = \begin{pmatrix}\frac{1}{3} & \frac{-1}{3}\\ \frac{-1}{3} & \frac{4}{3}\end{pmatrix}$

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$$e^{A} = Pe^{0}P^{-1}$$

$$= \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4966 & 0 \\ 0 & 0.6703 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4386 & -1.5557 \\ -0.0579 & -1.0592 \end{bmatrix}$$

QB: $x'(t) = Ax + B$

$b =$ Constant vector

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -30 \\ -10 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$dx = Ax + B$$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \cancel{1-\lambda}\ \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda_1 = 1 + i \qquad \lambda_2 = 1 - i$$

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b) For $\lambda_1 = 1+i$:

$(A-(1+i)I)\vec{v} = 0 = B$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - 1+i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix}$$

$$\begin{bmatrix} 1-1+(-i) & 1-0 \\ -1-0 & 1-1-i \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Nul } B = \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$\cancel{-iv_1 + v_2 = 0}$ $\quad v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\cancel{= v_1} =$

for $\lambda = 1-i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$v2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$x(t) = c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$A x_p + b = 0 \Rightarrow A x_p = -b$

$\begin{bmatrix} x_p \\ y_p \end{bmatrix}$ and $b = \begin{bmatrix} -30 \\ -10 \end{bmatrix}$

$x_p + y_p = 30$ $\quad 2y_p = 40$

$-x_p + y_p = 10$ $\quad y_p = 20$ $\quad x_p = 10$

$x_p = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

for $x(0) = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$

$x(0) = c1 \begin{pmatrix} 1 \\ i \end{pmatrix} + c2 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$

$c1 + c2 + 10 = 20$

$c1 + c2 = 10$

$i(c1 - c2) = 30 - 20$

$c_1 - c2 = -10$

$c1 - (10 - c1) = -10$

$c1 = 5 - 5i$

$c2 = 10 - 5 - 5i = 5 + 5i$

$\begin{bmatrix} u2 \\ u3 \end{bmatrix} \begin{bmatrix} u3 \\ u3 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$

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$$x(t) = (5-5i)e^{(1+i)t}\begin{bmatrix}1\\i\end{bmatrix} + (5+5i)e^{(1-i)t}\begin{bmatrix}1\\-i\end{bmatrix} + \begin{bmatrix}10\\20\end{bmatrix}$$

- Both will grow exponentially.

$$\begin{vmatrix}u2\\u3\end{vmatrix} \quad \begin{bmatrix}u3\\u3\end{bmatrix} \quad \begin{bmatrix}1\end{bmatrix}$$

u1=

u2

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Q9: $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $u1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $u2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

$W = \text{Span}\{\vec{u1}, \vec{u2}\}$

$\text{proj}_W y = \text{proj}_{u1} y + \text{proj}_{u2} y$

$= \frac{y.u1}{u1.u1} \vec{u1} + \frac{y.u2}{u2.u2} \vec{u2}$

$= \frac{(1)(1)+(3)(3)+(5)(-2)}{1^2+3^2+(-2)^2} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{(1)(5)+(3)(1)+(5)(4)}{5^2+1^2+4^2} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

$= \frac{-6}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} -3/7 \\ -9/7 \\ 6/7 \end{bmatrix} + \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 61/21 \\ -13/21 \\ -38/21 \end{bmatrix} = \text{proj}_W \vec{y} \times 21 = \begin{bmatrix} 61 \\ -13 \\ -38 \end{bmatrix}$

$y_{w^\perp} = y - y_w = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 61 \\ -13 \\ -38 \end{bmatrix} = \begin{bmatrix} -60 \\ 16 \\ 43 \end{bmatrix}$

$y = \begin{bmatrix} 61 \\ -13 \\ -38 \end{bmatrix} + \begin{bmatrix} -60 \\ 16 \\ 43 \end{bmatrix}$ Ans.

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(ii) $V = \text{Span}\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$

(a) Basis for $V^\perp$

$V^\perp$ = Null Space of V^T

$$V^\perp = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \end{array}$$

$v_1 + v_2 + v_4 = 0$ —① $\quad v_3 = 0$ —② $\quad v_4, v_2$ free

~~$v_2 + v_4 =$~~ $v_1 = -v_2 - v_4$

$$V^\perp = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -v_2 - v_4 \\ v_2 \\ 0 \\ v_4 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$V^\perp = \text{Span}\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$$

(b) Projection Matrix P

$$P = W(W^TW)^{-1}W^T = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\left(\underbrace{\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}}_{1\times4}\overbrace{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}^{4\times1}\right)^{-1}\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}$$

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$$= \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\left(\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\left((1)(1)+(1)(1)+(0)(0)+(1)(1)\right)^{-1}\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\left(\frac{1}{3}\right)\begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}$$

$$= \left[(1)\left(\tfrac{1}{3}\right)+(1)\left(\tfrac{1}{3}\right)+(0)(0)+\left(\tfrac{1}{3}\right)(1)\right]$$

$$= \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = [1]$$

$P = [1]$ ← Ans

(c) $\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} \in V^{\perp}$ Closest to V is orthogonal vector (proj)

$\rightarrow \vec{x}$

$$\text{proj}_{V^{\perp}}\vec{x} = \frac{\vec{x}\cdot v_1^{\perp}}{v_1^{\perp}\cdot v_1^{\perp}}\,v_1^{\perp} + \frac{\vec{x}\cdot v_2^{\perp}}{v_2^{\perp}\cdot v_2^{\perp}}\,v_2^{\perp}$$

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$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\text{proj}_{V^\perp} \vec{x} = \frac{(1)}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

$$\text{Closest vector} = \text{proj}_{V^\perp} \vec{x} - \vec{x} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ -3/2 \end{bmatrix}$$

Q10:
$$\begin{array}{c} \begin{matrix} \vec{v1} & \vec{v2} & \vec{v3} \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \end{array}$$

$\vec{u1} = \vec{v1}$

$$\text{proj}_{u1} v2 = \frac{v2 \cdot u1}{u1 \cdot u1} \vec{u1} = \frac{2+8+18}{1+4+9} u1 = \frac{28}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\text{proj}_{u1} v3 = \frac{v3 \cdot u1}{u1 \cdot u1} \vec{u1} = \frac{1+6+12}{1+4+9} u1 = \frac{19}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19/14 \\ 19/7 \\ 57/14 \end{bmatrix}$$

$$\text{Orthogonal Basis} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 19/14 \\ 19/7 \\ 57/14 \end{pmatrix} \right\}$$ Ans.