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23i-2623

Day / Date

HOMEWORK #3

(i) $f_x = 2x - 6 - \sqrt{y}$ $\quad$ $f_y = \frac{-x}{2\sqrt{y}} + 1$

$f_{xx} = 2$ $\quad$ $f_{yy} = \frac{x}{4y^{3/2}}$

$f_x = 0$ $\quad$ $f_y = 0$

$\sqrt{y} = 2x - 6$ $\quad$ $0 = \frac{-x}{2\sqrt{y}} + 1$

$x = 4$

$y = 4$ $\quad$ $f_{xy} = \frac{-1}{2\sqrt{y}}$

$P(4,4)$

$f_{xx}(4,4) f_{yy}(4,4) - (f_{xy}(4,4))^2$

$2 \cdot \frac{1}{8} - \frac{1}{16} = \frac{3}{16} > 0$

$f_{xx} = 2 > 0$

Local Minima at $(4,4)$

(ii) $f_x = -4y(x^2+y^2+1)^{-2}(-1)(2x)$ $\quad$ $f_y = \frac{vu' - uv'}{v^2}$

$0 = \frac{8xy}{(x^2+y^2+1)^2}$ $\quad$ $f_y = \frac{(x^2+y^2+1)(-4) - (-4y)(2y)}{(x^2+y^2+1)^2}$

$x = 0$, $y = 0$ $\quad$ $0 = 4x^2 - 4y^2 + 4$

$y = 0$: $4x^2 = -4$ $\quad$ $x = 0$: $-4y^2 = -4$

$x^2 = -1$ $\quad$ $y^2 = 1$

No real solutions $\quad$ $y = \pm 1$

Points $(0,1)$ $(0,-1)$ $\cancel{y}$

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$$f_{xx} = \frac{vu' - uv'}{v^2} = \frac{(x^2+y^2+1)^2(8y) - (8xy)(2)(2x)(x^2+y^2+1)}{(x^2+y^2+1)^4}$$

$$f_{xx} = \frac{(8y)(x^2+y^2+1)^2 - (32x^2y)(x^2+y^2+1)}{(x^2+y^2+1)^4}$$

$$f_{yy} = \frac{vu' - uv'}{v^2} = \frac{(x^2+y^2+1)^2(8y) - (-4x^2+4y^2-4)(x^2+y^2+1)(2)(2y)}{(x^2+y^2+1)^4}$$

@ $(0,1)$ — @ $(0,-1)$

$f_{xx} = 2$ — $f_{xx} = -2$

$f_{yy} = 2$ — $f_{yy} = -2$

$$f_{xy} = \frac{vu' - uv'}{v^2} = \frac{(x^2+y^2+1)^2(8x) - (8xy)(2)(2y)(x^2+y^2+1)}{(x^2+y^2+1)^4}$$

~~@+~~ $f_{xy} = 0$ @ Both $(0,1)$ and $(0,-1)$

$(0,1) \mapsto f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4 > 0$, $f_{xx} > 0$

Local ~~Maxima~~ Minima

$(0,-1) \mapsto f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4 > 0$, $f_{xx} < 0$

Local Maxima

(iii) $f(x,y) = \sin x + \sin y$

$f_x = \cos(x)$ — $f_y = \cos(y)$

$0 = \cos(x)$ — $0 = \cos(y)$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ — $y = \frac{\pi}{2}, \frac{3\pi}{2}$

$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$ $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

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$f_{xx} = -\sin(x)$, $f_{xy} = 0$, $f_{yy} = -\sin(y)$

① @$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $f_{xx} = -1$, $f_{yy} = -1$, $\cancel{f_{xx} \cdot f}$

② @$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$: $f_{xx} = -1$, $f_{yy} = 1$

③ @$\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$: $f_{xx} = 1$, $f_{yy} = -1$

④ @$\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$: $f_{xx} = +1$, $f_{yy} = +1$

① $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 1 > 0$

$f_{xx} < 0$

Maxima

② $f_{xx} \cdot f_{yy} - (f_{xy})^2 = -1 < 0$

Saddle point

③ $f_{xx} \cdot f_{yy} - (f_{xy})^2 = -1 < 0$

Saddle Point

④ $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 1 > 0$

$f_{xx} \geq 0$

Minima

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2(i) $z^2 = xy - x + 4y + 21$

$D = \sqrt{x^2 + y^2 + z^2}$

$D^2 = x^2 + y^2 + z^2$

$D^2 = x^2 + y^2 + xy - x + 4y + 21$

$D_x = 2x + y - 1 = 0$

$y = 1 - 2x$

$y = -3$

$D_y = 2y + x + 4 = 0$

$= 2(1-2x) + x + 4 = 0$

$2 - 4x + x + 4 = 0$

$6 = 3x$

$x = 2$

$P(2, -3)$

$D_{xx} = 2$, $D_{yy} = 2$, $D_{xy} = 1$

$D_{xx} \cdot D_{yy} - (D_{xy})^2 = 3 > 0$, $D_{xx} > 0$

∴ Minima

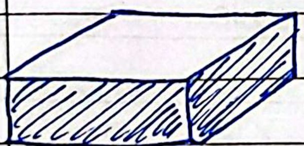
$z^2 = xy - x + 4y + 21$ @ $(2, -3)$

$z = \pm 1$

$(2, -3, 1)$ and $(2, -3, -1)$ are shortest distance from origin.

$D = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

(ii)



$S.A = 48\, ft^2 = 2zy + 2xz + xy$

$V = xyz$

$48 = 2zy + 2xz + xy$

$x = \dfrac{48 - 2zy}{2z + y}$

$V_* = \left(\dfrac{48 - 2zy}{2z + y}\right) yz$

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$$V = \frac{48yz - 2y^2z^2}{2z + y}$$

$$V_y = \frac{vu' - uv'}{v^2} = \frac{(2z+y)(48z - 4yz^2) - (48yz - 2y^2z^2)(1)}{(2z+y)^2}$$

$$V_y = \frac{96z^2 - 8yz^3 - 2y^2z^2}{(2z+y)^2}$$

$$0 = \frac{96z^2 - 8yz^3 - 2y^2z^2}{(2z+y)^2}$$

(1) $48z^2 - 4yz^3 - y^2z^2 = 0$

$$V_z = \frac{vu' - uv'}{v^2} = \frac{(2z+y)(48y - 4y^2z) - (48yz - 2y^2z^2)(2)}{(2z+y)^2}$$

$$V_z = \frac{48y^2 - 4y^2z^2 - 4y^3z}{(2z+y)^2}$$

$$0 = \frac{48y^2 - 4y^2z^2 - 4y^3z}{(2z+y)^2}$$

(2) $12y^2 - y^2z^2 - y^3z = 0$

$48z^2 - 4yz^3 - 12y^2 + y^3z = 0$

$4z^2(12 - yz) - y^2(12 - yz) = 0$

$(4z^2 - y^2)(12 - 4z) = 0$

$4z^2 = y^2$, $yz = 12$

$y = 2z, -2z$, $2z^2 = 12$

$y = 2\sqrt{6}, -2\sqrt{6}$, $z = +\sqrt{6}, -\sqrt{6}$

Points $\mapsto$ $\boxed{(2\sqrt{6}, \sqrt{6}) \quad 39.9}$ $(2\sqrt{6}, -\sqrt{6})$ -693.6 $(-2\sqrt{6}, \sqrt{6})$ -70 $(-2\sqrt{6}, -\sqrt{6})$ -39

Max Volume $\approx 40\ ft^3$ at $(2\sqrt{6}, \sqrt{6})$

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3 (i) $f(x,y,z) = \sqrt{(xy^2 + 6y^2z^2)}$, $P(2,3,-1)$, $v = 2\hat{i} - \hat{k}$

$f_x = \frac{1}{2}(xy^2 + 6y^2z^2)^{-\frac{1}{2}}(y^2) = \frac{y^2}{2\sqrt{xy^2 + 6y^2z^2}}$

$f_y = \frac{1}{2}(xy^2 + 6y^2z^2)^{-\frac{1}{2}}(2xy + 12yz^2) = \frac{xy + 6yz^2}{\sqrt{xy^2 + 6y^2z^2}}$

$f_z = \frac{1}{2}(xy^2 + 6y^2z^2)^{-\frac{1}{2}}(12y^2z) = \frac{6y^2z}{\sqrt{xy^2 + 6y^2z^2}}$

@ $(2,3,-1)$

$f_x = \frac{3\sqrt{2}}{8}$, $f_y = 2\sqrt{2}$, $f_z = -\frac{3\sqrt{2}}{4}$

$\nabla f = \frac{3\sqrt{2}}{8}\hat{i} + 2\sqrt{2}\hat{j} - \frac{3\sqrt{2}}{4}\hat{k}$

$\hat{u} = \frac{2\hat{i} - \hat{k}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{k}$

$\nabla f \cdot \hat{u} = D_{\hat{u}} f(2,3,-1) = 3.77$

(ii) $f(x,y,z) = e^x(2\cos y + 3\sin z)$, $P(1, \pi/6, \pi/6)$, $v = 2\hat{i} - \hat{j} + 3\hat{k}$

$f_x = e^x(2\cos y + 3\sin z)\Big|_P = e(\sqrt{3} + 3/2)$

$f_y = -2e^x \sin(y)\Big|_P = -e$

$f_z = 3e^x\cos(z)\Big|_P = \frac{3\sqrt{3}}{2}e$

$\nabla f = e(\sqrt{3} + 3/2)\hat{i} - e\hat{j} + \frac{3\sqrt{3}}{2}e\hat{k}$

$\hat{u} = \frac{2}{\sqrt{14}}\hat{i} - \frac{1}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

$D_{\hat{u}} f(x,y,z) = 11$ Ans.

4 (i) $f(x,y,z) = \ln(x^2 + 2y^2 + 3z^2)$, $P(1,2,-1)$

$f_x = \frac{2x}{x^2+2y^2+3z^2}$, $f_y = \frac{4y}{x^2+2y^2+3z^2}$, $f_z = \frac{6z}{x^2+2y^2+3z^2}$

@ $(1,2,-1)$

$\nabla f = \frac{1}{6}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{2}\hat{k}$ → Max: Vector

Min Vector $= -\nabla f = \frac{-1}{6}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{2}\hat{k}$

Maximum Increase $= \|\nabla f\| = \frac{\sqrt{21}}{6}$

~~Minimum~~

Maximum Decrease $= -\|\nabla f\| = -\frac{\sqrt{21}}{6}$

(ii) $f(x,y,z) = \sqrt{xy}\cos(z)$, $P(4, 1, \pi/4)$

$f_x = \frac{\sqrt{y}\cos z}{2\sqrt{xy}}$, $f_y = \frac{\sqrt{x}\cos z}{2\sqrt{xy}}$, $f_z = -\sqrt{xy}\sin(z)$

@ $(4, 1, \pi/4)$

Maximum Vector $= \nabla f = \frac{\sqrt{2}}{8}\hat{i} + \frac{1}{2\sqrt{2}}\hat{j} - \sqrt{2}\hat{k}$

Minimum Vector $= -\nabla f = -\frac{\sqrt{2}}{8}\hat{i} - \frac{1}{2\sqrt{2}}\hat{j} + \sqrt{2}\hat{k}$

Maximum Increase $= \frac{\sqrt{138}}{8}$

Maximum Decrease $= -\frac{\sqrt{138}}{8}$

5 $f(x,y) = 12 - 3x - 2y$

$f_x = -3$ $f_y = -2$

As first order partials are a constant, no critical points.

Boundary Points:

$f(0,1) = 12 - 3(0) - 2(1) = 10$

$f(2,0) = 6$

$f(1,2) = 5$

Absolute Max at $(0,1) = 10$

Absolute Min at $(1,2) = 5$