National University of Computer & Emerging Sciences

Homework # 14

1. Let $\mathbf{u^T} = [\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}, -\mathbf{\frac{1}{2}}, -\mathbf{\frac{1}{2}}]$
   (a) Find the projection matrix P the projects each vector on $\mathbf{u}$ (**Hint**: $P = A(AA^T)^{-1}A^T = \mathbf{u}(\mathbf{u^T u})^{-1}\mathbf{u} = \dfrac{\mathbf{uu^T}}{\mathbf{u^T u}}$ as $\mathbf{u^T u = 1}$. Hence $\boxed{P = \mathbf{uu^T}}$)
   (b) Find $Q = I - 2\mathbf{uu^T}(= I - 2P)$ and show that Q is a symmetric orthogonal matrix. (known as a Householder transformation or reflection about the subspace orthogonal to $\mathbf{u}$)
   (c) Show that $P^2 = P$ (i.e., P is idempotent) and $Q^2 = I$ (i.e., Q has order 2)
   (d) What will be the eigenvalues and eigenvectors of P and Q.

2. Orthogonally diagonalize the following symmetric matrix

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Practice Questions (Not to be Submitted)

3. Find the matrix of transformation that reflects each vector about the plane $x + 2y + z = 0$(**Hint**: First find the unit vector $\mathbf{u}$ orthogonal to given plane.)

4. Find a third column so that the matrix

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \\ \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} & \end{bmatrix}$$

   is orthogonal

5. Find the projection of e^x in $C[0, 1]$ with inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$$

   onto subspace of polynomials of degree 1 or less i.e., $\mathbb{P}_1(x)$.
   (Hint: Use Gram Schmidth to find the orthogonal basis of $\mathbb{P}_1(x)$)

6. Consider the subspace $W = span\{1, \cos(nt), \sin(nt)\}$ where $n \in \mathbb{N}$ and the inner product

$$\langle f(t), g(t)\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

   - Show that W is an orthogonal set
   - Find $\|1\|^2$, $\|\cos(nt)\|^2$, and $\|\sin(nt)\|^2$
   - Let $f(t) \in C[-\pi, \pi]$ such that $f(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$. Use orthogonality of W to find values of a_0, a_n and b_n
   - Take $f(t) = t$ and then the values of a_0, a_n and b_n