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DS-B

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Homework #8

Q1: $M = \begin{bmatrix} d-z_c & 0 & -x_c & x_c d \\ 0 & d-z_c & -y_c & y_c d \\ 0 & 0 & d & -d z_c \\ 0 & 0 & 1 & -z_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$A(0,0,0)$: $\text{Img}(A) = MA = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 1 \end{bmatrix}$

$B(1,0,0)$: $\text{Img}(B) = MB = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -2 \\ 1 \end{bmatrix}$

$C(0,1,0)$: $\text{Img}(C) = MC = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}$

$D(1,1,1)$: $\text{Img}(D) = MD = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

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Q2: (i) $T: \mathbb{P}_2 \to \mathbb{R}^3$ by $T(a+bx+cx^2) = \begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix}$

$$T = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\ker(T) = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R1/2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 3/2 & -3 & 0 \\ 0 & -1/2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R2-R1 \\ R3+R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \begin{array}{l} 2R1 \\ 2R2 \\ 2R3 \end{array} \sim \begin{array}{l} \\ R2/3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \\ R3-R1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R3/2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R1-R2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R2+2R1 \\ \\ \end{array}$$

$a=0,\ b=0,\ c=0 \Rightarrow a=b=c$

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As the solution is trivial, $\ker(T)=\{0\}$ Mursel

As the vectors are linearly independent, $\operatorname{Im}(T)=\mathbb{R}^3$

- T is one-to-one ($\ker(T)=\{0\}$), onto ($\operatorname{Im}(T)=\mathbb{R}^3$) and hence is isomorphic.

(ii) $T: M_{22} \to M_{22}$ by $T(A) = AB$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(E_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$T(E_2) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $T(E_3) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, $T(E_4) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$

$\ker(T)$ will consist of matrices that lie in the nullspace of the transformation since B has rank 1 and is non-invertible. $\ker(T) \neq 0$ hence not one to one.

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$\ker(T) = A\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore a=b, \ c=d$$

$$A = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

$\ker(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \middle| \ a, c \in \mathbb{R} \right\}$

$\ker(T) = \operatorname{Span}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right)$

$\cancel{\operatorname{Im}\{T\}}(A) = \cancel{T(A)} = \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} x & -x \\ y & -y \end{bmatrix}$

$\operatorname{Im}(T) = \left\{ \begin{bmatrix} x & -x \\ y & -y \end{bmatrix} \middle| \ x, y \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right)$

T is not one to one ($\ker(T) \neq \{0\}$) nor on to ($\operatorname{Im}(T) \in M_{22}$) and is hence not an isomorphism.

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(iii) $T:\mathbb{R}^3 \to W$ by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c & b-2c \\ b-2c & a-c \end{bmatrix}$ Mmed 2

W is vector space of all symmetric 2×2 matrices.

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c & b-2c \\ b-2c & a-c \end{bmatrix}$$

$$\ker(T) = \begin{bmatrix} a+b+c & b-2c \\ b-2c & a-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} a+b+c &= 0 \\ b-2c &= 0 \\ a-c &= 0 \end{aligned}\right\} \quad 4c = 0 \to c = 0$$

As $a=b=c$, $\ker(T) = \{0\}$

~~a+b+c~~

$$\underbrace{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{X} + \underbrace{b\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{Y} + \underbrace{c\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}}_{Z}$$

Since X, Y and Z are linearly independent,

$Im(T) = W$

As $\ker(T) = \{0\}$ and $Im(T) = W$, T is isomorphic.

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Q3: $T : \mathbb{R}^2 \to \mathbb{P}_2$ $\quad T\begin{bmatrix}1\\1\end{bmatrix} = 1-2x$, $T\begin{bmatrix}3\\-1\end{bmatrix} = x+x^2$

$T\begin{bmatrix}\cancel{-7}\\\cancel{9}\end{bmatrix}$ $\begin{bmatrix}a\\b\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}3\\-1\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 3 & a\\1 & -1 & b\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & 3 & a\\0 & 1 & (-b+a)/4\end{array}\right] -R2/4$$

$$\sim\left[\begin{array}{cc|c}1 & 3 & a\\0 & -4 & b-a\end{array}\right] R2-R1 \qquad \left[\begin{array}{cc|c}1 & 0 & a-\left(\frac{3}{4}\right)(-b+a)\\0 & 1 & (-b+a)/4\end{array}\right] R1-3R2$$

$\cancel{-4\alpha = b-a}$ $\quad \cancel{a+3}$ $\quad \alpha = a - \frac{3}{4}(-b+a)$

$\cancel{-4\beta = b-a}$ $\quad \cancel{\alpha+3\beta = a}$

$\cancel{-4\beta = b-\alpha-3\beta}$ $\quad \beta = \frac{-b+a}{4}$

$\cancel{\beta = b-\alpha}$

T $\quad \alpha = \frac{a+3b}{4}$, $\beta = \frac{a-b}{4}$

(i) $T\begin{bmatrix}-7\\9\end{bmatrix} = \alpha T\begin{bmatrix}1\\1\end{bmatrix} + \beta T\begin{bmatrix}3\\-1\end{bmatrix}$

$= \left(\frac{-7+3(9)}{4}\right)(1-2x) + \left(\frac{-7-9}{4}\right)(x+x^2)$

$= (5)(1-2x) + (-4)(x+x^2)$

$= 5 - 14x - 4x^2$ ← Ans

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(ii) General form for $T\begin{bmatrix} a \\ b \end{bmatrix}$:

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$$T\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a+3b}{4} - \frac{(a+3b)}{4}x + \frac{(a-b)}{4}x + \frac{(a-b)}{4}x^2$$