Linear Algebra (MT-1004)
Assignment # 2

Question # 1

Let $T : \mathbb{P}_2 \rightarrow \mathbb{M}_{2\times 2}$, where $\mathbb{P}_2$ is set of polynomial of degree 2 and $\mathbb{M}_{2\times 2}$ is set of all matrices of order 2×2, defined as

$$T(p(x)) = \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}$$

(a) Find the matrix of transformation (w.r.t standard basis of $\mathbb{P}_2$ and $\mathbb{M}_{2\times 2}$).

(b) Is T a linear transformation?

(c) Find $ker(T)$.

(d) Find a basis for $Range(T)$, also write the dimension of $Range(T)$?

(e) Is T an isomorphism? (one-one & onto)

(f) Verify the rank-nullity theorem for T.

Question # 2

Consider a 3D point $P(x, y, z)$ in space, and a camera located at the origin $O(0, 0, 0)$. The camera is oriented such that it projects the point P onto a 2D image plane located at $z = d$ (where $d > 0$) along the z-axis.

Using homogeneous coordinates and the concept of perspective projection:

(a) **Derive the perspective projection matrix** that projects the 3D point $P(x, y, z)$ onto the 2D image plane. Express the 2D coordinates (x', y') of the projected point in terms of the original coordinates (x, y, z) and the distance d of the image plane from the origin.

(b) **Explain** why objects that are farther from the camera appear smaller in the image. Use your derived projection formula to support your explanation.

(c) Given a 3D point $P(4, 3, 8)$ and an image plane located at $z = 4$, **calculate the coordinates** of the projected point P' on the 2D image plane.

Question # 3

(i) Let A be a 3×3 matrix given by:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 6 & 3 & 9 \end{bmatrix}$$

(a) **Matrix Factorization:**
Express A as a product of elementary matrices E_i. Show all steps of the factorization process.

(b) **Inverse via Elementary Matrices:**
Compute the inverse of A by writing A^{-1} as the product of the inverses of the elementary matrices.

(ii) If a 6×6 matrix has $det(A) = \frac{1}{4}$, find $det(2A)$, $det(-A)$, $det(A^2)$ and $det(A^{-1})$.

(iii) A cone with a volume of $26cm^3$ is transformed by the matrix composition AB^2, where

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{bmatrix},$$

moreover it is given that $\det B = 1/13$, calculate the volume of transformed cone.

Question # 4

Consider the A, 3×3 matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(i) **Eigenvalue Problem:**

(a) (a) Find the eigenvalues of A by solving the characteristic equation.

(b) (b) For each eigenvalue, find the corresponding eigenvectors.

(ii) **Basis of** $\lambda - Eigenspace$**:**
Find the basis of each eigen-space.

(iii) **Diagonalization:**

(a) Show that A is diagonalizable by finding a matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix of eigenvalues.

(b) Write out the diagonal matrix D and the orthogonal matrix P.

(iv) **Geometric and Algebraic Multiplicities:**
Discuss the geometric and algebraic multiplicities of the eigenvalues and explain why A is diagonalizable despite having repeated eigenvalues.

(v) **Eigenbasis:**
Is the set of all eigenvectors form the basis of $\mathbb{R}^3$?

(vi) **Matrix of Transformation w.r.t Eigenbasis:**
If $\mathcal{B} = \{v_1, v_2, v_3\}$, and $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$, find the matrix of transformation w.r.t $\mathcal{B}$ basis.

Question # 5

(i) If a 2×2 matrix A has eigenvalues 2 and 3, with eigenvectors $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(a) Find the determinants of A and A^n

(b) If $A \sim B$, i.e, $A = CBC^{-1}$ for some invertible matrix C. What will be the $tr(B)$ and $det(B)$.

(c) Find eigenvector and eigenvalues of $A - cI$, A^{-1} and A^n

(ii) Solve the recurrence relation with the given initial conditions.

(a) $a_0 = 4, a_1 = 1, a_n = a_{n-1} - \frac{a_{n-2}}{4}$, for $n \geq 2$.

(b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, subject to $a_0 = 2, a_1 = 2, a_2 = 4$, for $n \geq 3$.

Question # 6

Consider the following dynamical system

$$y_{k+1} = .8y_k + .3z_k, \quad y_0 = 0$$

$$z_{k+1} = .2y_k + .7z_k, \quad z_0 = 5.$$

(a) Find the limiting values of y_k and z_k, $(k \to \infty)$.

(b) Is the origin an attractor, a repeller, or a saddle point of the dynamical system $x_{k+1} = Ax_k$ where $x_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}$?

(c) Find the directions of greatest attraction and/or repulsion for this dynamical system.

Question # 7

Two species, X and Y, live in a symbiotic relationship i.e., neither species can survive on its own and each depends on the other for its survival. Initially there are 15 of X and 10 of Y. If $X = x(t)$ and $Y = y(t)$ are the sizes of the populations at time t months, the growth rates of the two populations are given by the system

$$x'(t) = -0.8x + 0.4y,$$

$$y'(t) = 0.4x - 0.7y$$

(a) Write the equation $x(t) = Py(t)$ and show the calculation that leads to the uncoupled system $y' = Dy$, specifying P and D.

(b) Determine what happens to these two populations.

(c) Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by $x'(t) = Ax$.

(d) Find the directions of greatest attraction and/or repulsion.

(e) Find e^A [**Hint:** You can take help from Example 4.47 at page 347 of David Poole]

Question # 8

Let the growth rate of the population is governed by the system of differential equations $x'(t) = Ax + b$, where $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and b is a constant vector. Determine what happens to the two populations for the following A and b

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -30 \\ -10 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

[**Hint:** You can take help from Example 4.45 at page 344 of David Poole]

Question # 9

(i) Let W be the subspace spanned by the u_1, u_2. Write y as the sum of a vector in W and a vector orthogonal to $W^{\perp}$.

$$y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \; u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \; u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

(ii) If $\mathbb{V}$ is spanned by vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, find

(a) a basis for the orthogonal complement $\mathbb{V}^{\perp}$

(b) the projection matrix P onto $\mathbb{V}$

(c) the vector in $\mathbb{V}$ closest to the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{V}^{\perp}$

Question # 10

For the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$, use Gram Schmidth process to find the orthogonal basis of column space of the given matrix.