Дз2

воскресенье, 23 марта 2025 г. 17:05

№4

$\theta \in (0, 1)$

$$\xi \sim \frac{\theta}{2}\{(-1,0)\} + \frac{\theta}{2}\{(0,1)\} + \frac{1-\theta}{2}\{0\} + \frac{1-\theta}{2}\{2\}$$

$$\alpha_1 = M[\xi] = \int_{-\infty}^{+\infty} x\, p(x,\theta)\,dx = \int_{-1}^{1} \frac{\theta}{2} x\,dx + \frac{1-\theta}{2}\cdot 0 + \frac{1-\theta}{2}\cdot 2 = 1-\theta$$

$$\alpha_2 = M[\xi^2] = \frac{\theta}{2}\int_{-1}^{1} x^2\,dx + \frac{1-\theta}{2}\cdot 2^2 = 2 - \frac{5}{3}\theta$$

$$\mu_2 = \alpha_2 - \alpha_1^2 = D[\xi] = \ldots = -\theta^2 + \frac{\theta}{3} + 1$$

(a)

I ОММ

$\alpha_1(\theta) = \tilde{\alpha}_1 = \overline{X} = 1-\theta \Rightarrow \boxed{\tilde{\theta}_1 = 1-\overline{X}}$

Несмещ:

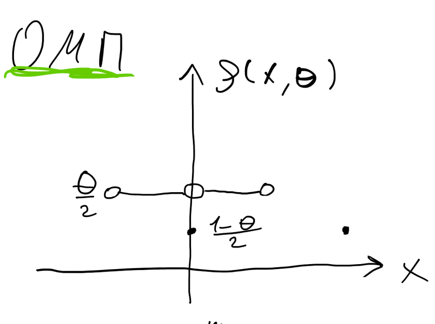
$M[\tilde{\theta}_1] = M[1-\overline{X}] = 1 - M[\xi] = \theta \Rightarrow$ **несмещ**

Состоят:

$D[\tilde{\theta}_1] = D[1-\overline{X}] = \frac{1}{n} D[\xi] \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ сост по дост усл-ю.

II ОМП



$$L = \left(\frac{\theta}{2}\right)^{n-m_0-m_2} \cdot \left(\frac{1-\theta}{2}\right)^{m_0+m_2}$$

$$\ln L = (n-m_0-m_2)\cdot \ln\frac{\theta}{2} + (m_0+m_2)\cdot \ln\left(\frac{1-\theta}{2}\right)$$

$$(\ln L)' = \frac{n-m_0-m_2}{\theta} + \frac{m_0+m_2}{\theta-1} = \frac{n(\theta - 1 + \frac{m_0}{n} + \frac{m_2}{n})}{\theta(\theta-1)}$$

Хотим: $(\ln L)' = 0 \Rightarrow \theta = 1 - \nu_0 - \nu_2$, где $\nu_i = \frac{m_i}{n}$

$$(\ln L)'' = \frac{m_0 - m_2 - n}{\theta^2} - \frac{m_0+m_2}{(\theta-1)^2} = \frac{(m_0+m_2-n)(\theta-1)^2 - (m_0+m_2)\theta^2}{\theta^2(\theta-1)^2} = \frac{(\nu_0+\nu_2)(\nu_0+\nu_2-1)}{\text{„}>0\text{"}} < 0 \text{ т.к. } \sum \nu_i < 1$$

$\Rightarrow \theta = 1 - \nu_0 - \nu_2 \leftarrow$ т max

$\Rightarrow \boxed{\tilde{\theta}_2 = 1 - \nu_1 - \nu_2}$

(б) Несмещ

$M[\tilde{\theta}_2] = 1 - M[\nu_0] - M[\nu_2] = 1 - \frac{1-\theta}{2} - \frac{1-\theta}{2} = \theta \rightarrow$ несмещ

$D[\tilde{\theta}_2] = D[\nu_0 + \nu_2] = \frac{\theta(1-\theta)}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сост

(c) Проверка на регулярность модели:

- $p(x,\theta) \in C^1$ на Θ
- $\frac{\partial}{\partial\theta}\int_A p(x,\theta)\,dx = \int_A \frac{\partial}{\partial\theta} p(x,\theta)\,dx$ на $(0,1)$ $(A = \mathbb{R})$

$$\int_{-\infty}^{+\infty} \frac{\partial p(x,\theta)}{\partial\theta} = \frac{1}{2}\int_{-1}^{1} dx + \left(\frac{1-\theta}{2}\right)'_\theta + \left(\frac{1-\theta}{2}\right)'_\theta = 1 - 2\cdot\frac{1}{2} = 0$$

- $I(\theta) \in C^1$ на Θ и $I(\theta) > 0$ на Θ $(0,1)$

т.к.: $\frac{\partial \ln p(x,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\ln\frac{\theta}{2}\right) = \frac{1}{\theta}$ $\quad \frac{\partial \ln p}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\ln\left(\frac{1-\theta}{2}\right)\right) = \frac{1}{\theta-1}$

$\tilde{\theta}_1 = 1 - \overline{x} \rightarrow$ несмещ $\oplus$ $D[\tilde{\theta}]$ - огр на $\forall$ компакт. из $(0,1)$

$\Rightarrow \tilde{\theta}_1$ - рег

$\tilde{\theta} = 1 - \nu_0 - \nu_2$ - несмещ, $D[\tilde{\theta}_2]$ - огр на $\forall$ комп из $(0,1)$ $\Rightarrow \tilde{\theta}_2$ - рег

по Нер-ву К-Р:

$\tilde{\theta}_2$: $D[\tilde{\theta}_2] \geq \frac{1}{nI(\theta)} = \frac{\theta(1-\theta)}{n}$

$\frac{(1-\theta)\theta}{n} \geq \frac{\theta(1-\theta)}{n} \Rightarrow$ по дост усл - эфф

$\tilde{\theta}_2 \Leftrightarrow \tilde{\theta}_1$ - не эфф (по Т о ! эф. оц.)

№5

$\xi \sim R(\theta, 2\theta)$

(a1) $p(x,\theta) = \frac{1}{\theta}\{(\theta, 2\theta)\}$

ОММ:

$$\alpha_1 = M[\xi] = \int_{-\infty}^{+\infty} x\,p(x)\,dx = \frac{1}{2\theta-\theta}\int_{\theta}^{2\theta} x\,dx = \frac{3\theta}{2}$$

(б1) $\alpha_1(\theta) = \tilde{\alpha}_1 = \overline{x} \Rightarrow \frac{3\theta}{2} = \overline{x} \Rightarrow \tilde{\theta} = \frac{2}{3}\overline{x}$

Несмещ: $M[\tilde{\theta}] = M[\frac{2}{3}\overline{x}] = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta \rightarrow$ несмещ ✓

Сост: $\alpha_2 = M[\xi^2] = \int_{\theta}^{2\theta} \frac{1}{\theta} x^2\,dx = \frac{7}{3}\theta^2$

$\mu_2 = D[\xi] = \alpha_2 - \alpha_1^2 = \ldots = \frac{\theta^2}{12}$

$D[\tilde{\theta}] = D[\frac{2}{3}\overline{x}] = \frac{4}{9n}\cdot\frac{\theta^2}{12} \xrightarrow[n\to\infty]{} 0$ ✓ сост по дост усл

(a2) (б2) ОМП:

$L = \frac{1}{\theta^n}\{x_i \in [\theta, 2\theta]\}$ т.к. $x_{max} = 2\theta \Rightarrow \tilde{\theta} = \frac{x_{max}}{2}$

$$P(x) = (F(x))^n = \left(\int_0^x \frac{1}{\theta}\,dx\right)^n = \left(\frac{x}{\theta} - 1\right)^n$$

$$M[\tilde{\theta}] = M[x_{max}] = \int_0^{2\theta} \frac{n}{\theta}\left(\frac{x}{\theta}-1\right)^{n-1} x\,dx = \ldots = \frac{2n+1}{n+1}\theta$$

$$\tilde{\theta}^{\bullet} = \frac{n+1}{2n+1}\tilde{\theta} = \frac{n+1}{2n+1}\cdot\frac{x_{max}}{2}$$

$$M[x_{max}^2] = \int_0^{2\theta} x^2 \frac{n}{\theta}\left(\frac{x}{\theta}-1\right)^{n-1} dx = 2\theta^2\,\frac{2n^2+4n+1}{(n+2)(n+1)}$$

$\Rightarrow M[\tilde{\theta}^{\bullet}] = \theta \leftarrow$ несмещ

$$D[\tilde{\theta}^{\bullet}] = D\left[\frac{2(n+1)}{2n+1}\cdot\frac{x_{max}}{2}\right] = \left(\frac{n+1}{2n+1}\right)^2 D[x_{max}] = \ldots = \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

$D[x_{max}] = M[x_{max}^2] - M^2[x_{max}]$

$\tilde{\theta}^{\bullet}$ - сост по дост усл

(c) $D[\tilde{\theta}_1] = \frac{\theta^2}{27n}$ $\geq$ $D[\tilde{\theta}_2^{\bullet}] = \frac{n\theta^2}{(2n+1)^2(n+2)}$

$\Rightarrow \tilde{\theta}_2^{\bullet}$ - эфф-ее

(d+e)

$x_i \in [\theta, 2\theta] \Rightarrow \frac{x_i}{\theta} \in [1; 2]$

$\phi(x_{max}) = (F(x))^n = \left(\int_1^x dx\right)^n = (x-1)^n$

$x = \theta$

$\sqrt[n]{0{,}025} + 1 < x < \sqrt[n]{0{,}975} + 1$

$\sqrt[n]{0{,}025} + 1 < \frac{x_{max}}{\theta} < \sqrt[n]{0{,}975} + 1$

$$\Rightarrow \frac{x_{max}}{\sqrt[n]{0{,}975}+1} < \theta < \frac{x_{max}}{\sqrt[n]{0{,}025}+1}$$

ОММ:

$$\sqrt{n}\,\frac{(g(\tilde{\alpha}) - g(\alpha))}{\sigma(\alpha)} \rightsquigarrow N(0,1)$$

$$\sigma(\alpha) = \sqrt{\nabla^T g\, K\, \nabla g} = \frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}$$

$$\sqrt{n}\,\frac{\frac{2}{3}\tilde{\alpha} - \frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}} = \underbrace{\sqrt{n}\cdot\frac{\tilde{\theta}-\theta}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}}_{\text{Т лем. слуцкого } \rightsquigarrow N(0,1)} \cdot \underbrace{\frac{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}}_{\to 1} \rightsquigarrow N(0,1)$$

$$\Rightarrow -1{,}96 < \sqrt{n}\,\frac{(\tilde{\theta}-\theta)}{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}} < 1{,}96$$

$$-\frac{1{,}96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2} + \tilde{\theta} < \theta < \frac{1{,}96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2} + \tilde{\theta}$$

№6

$$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1$$

(a) $L(x,\theta) = \frac{(\theta-1)^n}{\prod_{i=1}^n x_i^\theta}$

$\ln L = n\cdot\ln(\theta-1) - \theta\sum_i \ln(x_i)$

$\frac{\partial L}{\partial\theta} = \frac{n}{\theta-1} - \sum_i \ln(x_i) = 0$

$\frac{n}{\theta-1} = \sum_i \ln(x_i)$

$\rightarrow \tilde{\theta} = 1 + \frac{n}{\sum_i \ln x_i}$

$\frac{\partial^2 \ln L}{\partial\theta^2} = -\frac{n}{(\theta-1)^2} < 0 \Rightarrow \theta = 1 + \frac{1}{\sum \ln x_i}$ - т max

(б) дов. инт для med:

$$\left.\begin{aligned} &\frac{\partial}{\partial\theta}\int \frac{\theta-1}{x^\theta}\,dx = x^{1-\theta}\ln x \\ &\qquad\qquad \| \\ &\int \frac{\partial}{\partial\theta}\left(\frac{\theta-1}{x^\theta}\right)dx = x^{1-\theta}\ln x \end{aligned}\right\} \Rightarrow \text{модель - сильнорегулярн}$$

$$\int_1^{\hat{x}} \frac{\theta-1}{x^\theta}\,dx = -\frac{1}{\hat{x}^{\theta-1}} + 1 = \frac{1}{2}, \text{ где } \hat{x} = 2^{\frac{1}{\theta-1}}$$

$$g(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$$

$$\sqrt{n}\cdot\frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\theta)} \rightsquigarrow N(0,1), \quad \sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta}), \quad \sigma(\tilde{\theta}) = \sqrt{\nabla^T g(\tilde{\theta})\cdot I^{-1}(\tilde{\theta})\,\nabla g(\tilde{\theta})}$$

$$I(\tilde{\theta}) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = M\left[\left(\frac{\partial(\ln(\theta-1) - \theta\ln(x))}{\partial\theta}\right)^2\right] = M\left[\left(\frac{1}{\theta-1} - \ln x\right)^2\right] = \int_1^{+\infty}\left(\frac{1}{\theta-1} - \ln x\right)^2 p(x,\theta)\,dx = \frac{1}{(\theta-1)^2} \rightarrow \in C \text{ при } \theta > 1$$

$p(x,\theta) = \frac{\theta-1}{x^\theta}$

$$\nabla g(\tilde{\theta}) = -\frac{\ln(2)\cdot 2^{\frac{1}{\tilde{\theta}-1}}}{(\tilde{\theta}-1)^2}, \quad \sigma(\tilde{\theta}) = -\frac{\ln(2)\cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\tilde{\theta}-1}$$

$$\sqrt{n}\cdot\frac{(g(\tilde{\theta}) - g(\theta))}{\sigma(\tilde{\theta})} \rightsquigarrow N(0,1)$$

$$\frac{1{,}96\cdot\sigma(\tilde{\theta})}{\sqrt{n}} + g(\tilde{\theta}) < g(\theta) < -\frac{1{,}96\cdot\sigma(\tilde{\theta})}{\sqrt{n}} + g(\tilde{\theta})$$

(в)

$$\sqrt{n}\,\frac{\tilde{\theta}-\theta}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$\{\sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta})\}$

$$\sigma(\tilde{\theta}) = \theta - 1 \Rightarrow \sqrt{n}\,\frac{\tilde{\theta}-\theta}{\theta-1} \rightsquigarrow N(0,1)$$

$$\frac{-1{,}96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_i \ln x_i} < \theta < \frac{1{,}96(\theta-1)}{\sqrt{n}} + 1 + \frac{n}{\sum_i \ln x_i}$$