Numericals

April 8, 2023

Normal Shearing plane

1 Evaluate the normal and shearing stresses on the planes, whose normal has the following direction cosines:

(a) $n_x = n_y = \frac{1}{\sqrt{2}}, n_z = 0$

(b) $n_x = 0, n_y = n_z = \frac{1}{\sqrt{2}}$

(c) $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$

for a point which has 200 MPa, -100 MPa, -100 MPa normal stresses in x,y and z direction and -200 MPa is the value of $\tau_{xy}, \tau_{yz}, \tau_{zx}$.

Solution

(a) $n_x = n_y = \frac{1}{\sqrt{2}}, n_z = 0$

Using Cauchy's equation,

$$T_x^n = n_x\sigma_x + n_y\tau_{yx} + n_z\tau_{zx}$$
$$T_y^n = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{zy}$$
$$T_x^n = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z$$
$$T_x^n = (\frac{1}{\sqrt{2}} \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 200) - (0 \cdot 200)$$
$$T_x^n = 0 \ \ MPa$$
$$T_y^n = -(\frac{1}{\sqrt{2}} \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 100) - (0 \cdot 200)$$
$$T_y^n = -\frac{300}{\sqrt{2}} \ \ MPa$$
$$T_z^n = -(\frac{1}{\sqrt{2}} \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 200) - (0 \cdot 100)$$

$$T_z^n = -\frac{400}{\sqrt{2}} \ \ MPa$$

$$\sigma_n = n_x T_x^n + n_y T_y^n + n_z T_z^n$$

$$\sigma_n = (\frac{1}{(\sqrt{2}} \cdot 0) - (\frac{1}{\sqrt{2}} \cdot \frac{300}{\sqrt{2}}) - (0 \cdot \frac{400}{\sqrt{2}})$$

$$\sigma_n = \frac{-300}{2}$$

$$\sigma_n = -150 \ \ MPa$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$

$$|T^n|^2 = 0 + \frac{300}{2}^2 + \frac{400}{2}^2$$

$$|T^n|^2 = \frac{250000}{2}$$

$$|T^n|^2 = 125000$$

$$\tau_n^2 = 125000 - (-150)^2$$

$$\tau_n^2 = 102500$$

$$\tau_n = 320.16 \ \ MPa$$

(b) $n_x = 0, n_y = n_z = \frac{1}{\sqrt{2}}$
Using Cauchy's equation,

$$T_x^n = n_x \sigma_x + n_y \tau_{yx} + n_z \tau_{zx}$$

$$T_y^n = n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{zy}$$

$$T_x^n = n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z$$

$$T_x^n = (0 \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 200)$$

$$T_x^n = -\frac{400}{\sqrt{2}} \ \ MPa$$

$$T_y^n = -(0 \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 100) - (\frac{1}{\sqrt{2}} \cdot 200)$$

$$T_y^n = -\frac{300}{\sqrt{2}} \ \ MPa$$

$$T_z^n = -(0 \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 200) - (\frac{1}{\sqrt{2}} \cdot 100)$$

$$T_z^n = -\frac{300}{\sqrt{2}} \ \ MPa$$

$$\sigma_n = n_x T_x^n + n_y T_y^n + n_z T_z^n$$

$$\sigma_n = (0 \cdot \frac{400}{\sqrt{2}}) - (\frac{1}{\sqrt{2}} \cdot \frac{300}{\sqrt{2}}) - (\frac{1}{\sqrt{2}} \cdot \frac{300}{\sqrt{2}})$$

$$\sigma_n = \frac{-600}{2}$$

$$\sigma_n = -300 \ \ MPa$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$

$$|T^n|^2 = \frac{400^2}{2} + \frac{300^2}{2} + \frac{300^2}{2}$$

$$|T^n|^2 = \frac{340000}{2}$$

$$|T^n|^2 = 170000$$

$$\tau_n^2 = 170000 - (-350)^2$$

$$\tau_n^2 = 102500$$

$$\tau_n = 282.84 \ \ MPa$$

(c) $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$

Using Cauchy's equation,

$$T_x^n = n_x \sigma_x + n_y \tau_{yx} + n_z \tau_{zx}$$

$$T_y^n = n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{zy}$$

$$T_x^n = n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z$$

$$T_x^n = (\frac{1}{\sqrt{3}} \cdot 200) - (\frac{1}{\sqrt{3}} \cdot 200) - (\frac{1}{\sqrt{3}} \cdot 200)$$

$$T_x^n = -\frac{200}{\sqrt{3}} \;\; MPa$$

$$T_y^n = -(\frac{1}{\sqrt{3}} \cdot 200) - (\frac{1}{\sqrt{3}} \cdot 100) - (\frac{1}{\sqrt{3}} \cdot 200)$$

$$T_y^n = -\frac{500}{\sqrt{3}} \;\; MPa$$

$$T_z^n = -(\frac{1}{\sqrt{3}} \cdot 200) - (\frac{1}{\sqrt{3}} \cdot 200) - (\frac{1}{\sqrt{3}} \cdot 100)$$

$$T_z^n = -\frac{500}{\sqrt{3}} \;\; MPa$$

$$\sigma_n = n_x T_x^n + n_y T_y^n + n_z T_z^n$$

$$\sigma_n = -(\frac{1}{\sqrt{3}} \cdot \frac{200}{\sqrt{3}}) - (\frac{1}{\sqrt{3}} \cdot \frac{500}{\sqrt{3}}) - (\frac{1}{\sqrt{3}} \cdot \frac{500}{\sqrt{2}})$$

$$\sigma_n = \frac{-1200}{3}$$

$$\sigma_n = -400 \;\; MPa$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$

$$|T^n|^2 = \frac{200^2}{2} + \frac{500^2}{2} + \frac{500^2}{2}$$

$$|T^n|^2 = \frac{540000}{3}$$

$$|T^n|^2 = 180000$$

$$\tau_n^2 = 180000 - (-400)^2$$

$$\tau_n^2 = 20000$$

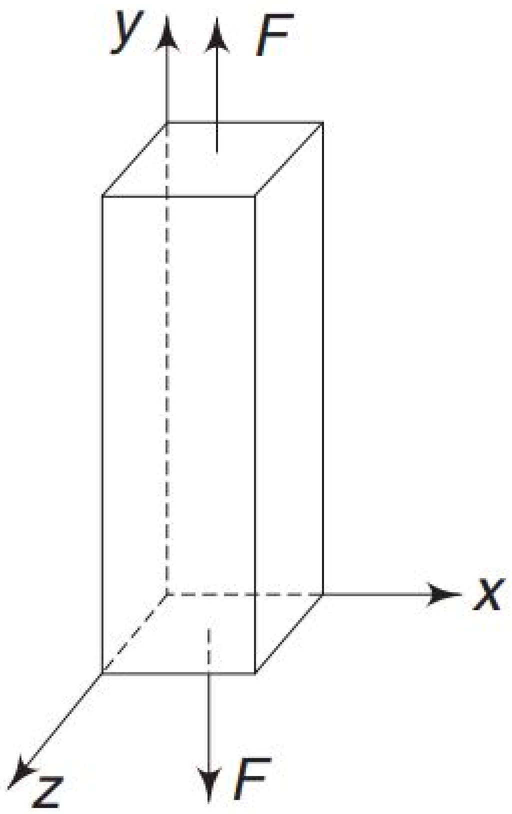
$$\tau_n = 141.42 \;\; MPa$$

2 A rectangular steel bar having a cross-section 2 cm X 3 cm is subjected to a tensile force of 6000 N (612.2 kgf). If the axes are chosen as shown in Fig., determine the normal and shear stresses on a plane whose normal has the following direction cosines:

(a) $n_x = n_y = \frac{1}{\sqrt{2}}, n_z = 0$

(b) $n_x = 0, n_y = n_z = \frac{1}{\sqrt{2}}$

(c) $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$



Solution

Cross-sectional area of bar = 20 X 30 mm

$$= 600mm^2$$

Normal stress in y-direction = $\frac{Force}{Area}$

$$= 6000/600$$

$$= 10 \ MPa$$

$\sigma_x = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$

(a) $n_x = n_y = \frac{1}{\sqrt{2}}, n_z = 0$
Using Cauchy's equation,

$$T_x^n = n_x\sigma_x + n_y\tau_{yx} + n_z\tau_{zx}$$
$$T_y^n = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{zy}$$
$$T_x^n = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z$$
$$T_x^n = (\frac{1}{\sqrt{2}} \cdot 0) + (\frac{1}{\sqrt{2}} \cdot 0) + (0)$$
$$T_x^n = 0 \ \ MPa$$
$$T_y^n = (\frac{1}{\sqrt{2}} \cdot 0) + (\frac{1}{\sqrt{2}} \cdot 10) + (0)$$
$$T_y^n = \frac{10}{\sqrt{2}} \ \ MPa$$
$$T_z^n = (\frac{1}{\sqrt{2}} \cdot 0)(\frac{1}{\sqrt{2}} \cdot 0) + (0)$$
$$T_z^n = 0 \ \ MPa$$
$$\sigma_n = n_xT_x^n + n_yT_y^n + n_zT_z^n$$
$$\sigma_n = (\frac{1}{(\sqrt{2}} \cdot 0) + (\frac{1}{\sqrt{2}} \cdot \frac{10}{\sqrt{2}}) + (0)$$
$$\sigma_n = \frac{10}{2}$$
$$\sigma_n = 5 \ \ MPa$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$
$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$
$$|T^n|^2 = 0 + \frac{10^2}{2} + 0$$
$$|T^n|^2 = \frac{100}{2}$$
$$|T^n|^2 = 50$$
$$\tau_n^2 = 50 - (5)^2$$
$$\tau_n^2 = 25$$
$$\tau_n = 5 \ \ MPa$$

(b) $n_x = 0, n_y = n_z = \frac{1}{\sqrt{2}}$
Using Cauchy's equation,

$$T_x^n = n_x\sigma_x + n_y\tau_{yx} + n_z\tau_{zx}$$
$$T_y^n = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{zy}$$
$$T_x^n = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z$$
$$T_x^n = (0) - (\frac{1}{\sqrt{2}} \cdot 0) - (\frac{1}{\sqrt{2}} \cdot 0)$$
$$T_x^n = 0 \ \ MPa$$
$$T_y^n = (0) + (\frac{1}{\sqrt{2}} \cdot 10) + (\frac{1}{\sqrt{2}} \cdot 0)$$
$$T_y^n = \frac{10}{\sqrt{2}} \ \ MPa$$
$$T_z^n = (0) + (\frac{1}{\sqrt{2}} \cdot 0) + (\frac{1}{\sqrt{2}} \cdot 0)$$
$$T_z^n = 0 \ \ MPa$$
$$\sigma_n = n_xT_x^n + n_yT_y^n + n_zT_z^n$$
$$\sigma_n = 0 + (\frac{1}{\sqrt{2}} \cdot \frac{10}{\sqrt{2}}) + 0$$
$$\sigma_n = \frac{10}{2}$$
$$\sigma_n = 5 \ \ MPa$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$
$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$
$$|T^n|^2 = 0 + \frac{10^2}{2} + 0$$
$$|T^n|^2 = \frac{100}{2}$$
$$|T^n|^2 = 50$$
$$\tau_n^2 = 50 - (5)^2$$
$$\tau_n^2 = 25$$
$$\tau_n = 5 \ \ MPa$$

(c) $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$
Using Cauchy's equation,

$$T_x^n = n_x\sigma_x + n_y\tau_{yx} + n_z\tau_{zx}$$
$$T_y^n = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{zy}$$
$$T_z^n = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z$$
$$T_x^n = (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot 0)$$
$$T_x^n = 0 \;\; MPa$$
$$T_y^n = (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot 10) + (\frac{1}{\sqrt{3}} \cdot 0)$$
$$T_y^n = \frac{10}{\sqrt{3}} \;\; MPa$$
$$T_z^n = (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot 0)$$
$$T_z^n = 0 \;\; MPa$$
$$\sigma_n = n_x T_x^n + n_y T_y^n + n_z T_z^n$$
$$\sigma_n = (\frac{1}{\sqrt{3}} \cdot 0) + (\frac{1}{\sqrt{3}} \cdot \frac{10}{\sqrt{3}}) + (\frac{1}{\sqrt{3}} \cdot 0)$$
$$\sigma_n = \frac{10}{3}$$
$$\sigma_n = 3.34 \;\; MPa$$

Normal and shear components of resultant stress vector are:
$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$
$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$
The components of resultant stress vector are:
$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$
$$|T^n|^2 = 0 + \frac{10^2}{3} + 0$$
$$|T^n|^2 = \frac{100}{3}$$
$$|T^n|^2 = 33.34$$
$$\tau_n^2 = 33.34 - (3.34)^2$$
$$\tau_n^2 = 22.18$$
$$\tau_n = 4.71 \;\; MPa$$

3 At a point P in a body, σ_x = 10,000 N/cm^2 (1020 kgf/cm^2), σ_y = –5,000 N/cm^2 (–510 kgf/cm^2), σ_z = –5,000 N/cm^2, τ_{xy} =τ_{yz} =τ_{xz}= $10,000$cm^2.

Determine the normal and shearing stresses on a plane that is equally inclined to all the three axes.

Solution

A plane that is equally inclined to all the three axes will have

$$n_x = n_y = n_z = \frac{1}{\sqrt{3}}$$

since,

$$n_x^2 + n_y^2 + n_z^2 = 1$$

Using Cauchy's equation,

$$T_x^n = n_x\sigma_x + n_y\tau_{yx} + n_z\tau_{zx}$$
$$T_y^n = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{zy}$$
$$T_x^n = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z$$

$$T_x^n = (\frac{1}{\sqrt{3}} \cdot 10000) + (\frac{1}{\sqrt{3}} \cdot 10000) + (\frac{1}{\sqrt{3}} \cdot 10000)$$

$$T_x^n = \frac{30000}{\sqrt{3}} \ \frac{N}{cm^2}$$

$$T_y^n = (\frac{1}{\sqrt{3}} \cdot 10000) - (\frac{1}{\sqrt{3}} \cdot 5000) + (\frac{1}{\sqrt{3}} \cdot 10000)$$

$$T_y^n = \frac{15000}{\sqrt{3}} \ \frac{N}{cm^2}$$

$$T_z^n = (\frac{1}{\sqrt{3}} \cdot 10000) + (\frac{1}{\sqrt{3}} \cdot 10000) - (\frac{1}{\sqrt{3}} \cdot 5000)$$

$$T_z^n = 15000 \ \frac{N}{cm^2}$$

$$\sigma_n = n_xT_x^n + n_yT_y^n + n_zT_z^n$$

$$\sigma_n = (\frac{1}{\sqrt{3}} \cdot \frac{30000}{\sqrt{3}}) + (\frac{1}{\sqrt{3}} \cdot \frac{15000}{\sqrt{3}}) + (\frac{1}{\sqrt{3}} \cdot \frac{15000}{3})$$

$$\sigma_n = \frac{60000}{3}$$

$$\sigma_n = 20000 \ \frac{N}{cm^2}$$

Normal and shear components of resultant stress vector are:

$$|T^n|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n^2 = |T^n|^2 - \sigma_n^2$$

The components of resultant stress vector are:

$$|T^n|^2 = T_x^{n2} + T_y^{n2} + T_z^{n2}$$

$$|T^n|^2 = \frac{30000}{3}^2 + \frac{15000}{3}^2 + \frac{15000}{3}^2$$

$$|T^n|^2 = \frac{1350 \cdot 10^6}{3}$$

$$|T^n|^2 = 450 \cdot 10^6$$

$$\tau_n^2 = (450 \cdot 10^6) - (20000)^2$$

$$\tau_n^2 = 50 \cdot 10^6$$

$$\tau_n = 7071.1 \ \frac{N}{cm^2}$$

Principal Stresses

4 At a point P, the rectangular stress components are σ_x = 1, σ_y = -2, σ_z = 4, τ_{xy} = 2, τ_{yz} = -3 and τ_{xz} = 1 all in units of kPa. Find the principal stresses.

Soultion

Stresses can be written in form of a matrix as:

$$[\tau_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$
$$[\tau_{ij}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 1 & -3 & 4 \end{bmatrix}$$

The first stress invariant is given as

$$l_1 = \sigma_x + \sigma_y + \sigma_z$$
$$l_1 = 1 - 2 + 4$$
$$l_1 = 3$$

The second stress invariant is given as:

$$\text{l}_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix}$$
$$l_2 = \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

l$_2$ = (-2 - 4) + (-8 - 9) + (4 - 1)
l_2 = -20

The third stress invariant is given as:

$$l_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$
$$l_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 1 & -3 & 4 \end{vmatrix}$$
$$l_3 = 1(-8 - 9) - 2(8 + 3) + 1(-6 + 2)$$

$l_3 = -43$

$$f(\sigma) = \sigma^3 - l_1 \cdot \sigma^2 + l_2 \cdot \sigma - l_3 = 0 \tag{1}$$

Putting values of l_1, l_2 and l_3 in eq. (1).

$$\sigma^3 - 3 \cdot \sigma^2 - 20 \cdot \sigma + 43 = 0$$

Three roots of this equation are: 5.25 kPa, -4.2 kPa and 1.95 kPa

$$\sigma_1 > \sigma_2 > \sigma_3$$

Hence,
σ_1 = 5.25 kPa
σ_2 = 1.95 kPa
σ_3 = -4.2 kPa

5 The following state of stress exists. Determine the principal stresses and their associated directions.

$[\tau_{ij}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution
The first stress invariant is given as:

$$l_1 = \sigma_x + \sigma_y + \sigma_z$$

$$l_1 = 1 + 1 + 1$$

$$l_1 = 3$$

The second stress invariant is given as:
l$_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix}$
$l_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$

l_2 = (1 - 4) + (1 - 1) + (1 - 1)
l_2 = -3

The third stress invariant is given as:

$l_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$

$l_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$l_3 = 1(1-1) - 2(2-1) + 1(2-1)$
$l_3 = -1$

$$f(\sigma) = \sigma^3 - l_1 \cdot \sigma^2 + l_2 \cdot \sigma - l_3 = 0 \qquad (2)$$

Putting values of l_1, l_2 and l_3 in eq. (2).

$$\sigma^3 - 3 \cdot \sigma^2 - 3 \cdot \sigma + 1 = 0$$

Three roots of this equation are: -1 kPa, 0.27 kPa and 3.73 kPa

$$\sigma_1 > \sigma_2 > \sigma_3$$

Hence,
σ_1 = 3.73 kPa
σ_2 = 0.27 kPa
σ_3 = -1 kPa

$$n_x(\sigma_x - \sigma) + n_y\tau_{yx} + n_z\tau_{zx} = 0$$
$$n_x\tau_{xy} + n_y(\sigma_y - \sigma) + n_z\tau_{zy} = 0$$
$$n_x\tau_{xz} + n_y\tau_{yz} + n_z(\sigma_z - \sigma) = 0$$

Putting the roots or values of σ_1, σ_2 and σ_3 one by one in above three equations.
For $\sigma = 3.73$

$$(1 - 3.73)n_x + 2n_y + 1n_z = 0 \qquad (3)$$
$$2n_x + (1 - 3.73)n_y + 1n_z = 0 \qquad (4)$$
$$1n_x + 1n_y + (1 - 3.73)n_z = 0 \qquad (5)$$

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{6}$$

Subtracting eq (4) from eq (3)

$$-4.73n_x + 4.73n_y = 0$$
$$n_x = n_y$$

Put in eq (5)

$$-2.73n_x + 2n_x + 1n_z = 0$$
$$n_z = 0.73n_x$$

Put in eq(6)

$$n_x^2 + n_x^2 + (0.73)^2 \cdot n_x^2 = 1$$
$$2.53n_x^2 = 1$$
$$n_x = 0.63$$
$$n_y = 0.63$$
$$n_z = 0.46$$

For $\sigma = 0.27$

$$(1 - 0.27)n_x + 2n_y + 1n_z = 0 \tag{7}$$
$$2n_x + (1 - 0.27)n_y + 1n_z = 0 \tag{8}$$

Subtracting eq (8) from eq (7)

$$-1.27n_x + 1.27n_y = 0$$
$$n_x = n_y$$

Put in eq (5)

$$0.73n_x + 2n_x + 1n_z = 0$$
$$n_z = -2.73n_x$$

Put in eq(6)

$$n_x^2 + n_x^2 + (-2.73)^2 \cdot n_x^2 = 1$$
$$9.45n_x^2 = 1$$

$$n_x = 0.33$$
$$n_y = 0.33$$
$$n_z = -0.9$$

For $\sigma = -1$

$$(1-(-1))n_x + 2n_y + 1n_z = 0 \tag{9}$$
$$2n_x + (1-(-1))n_y + 1n_z = 0 \tag{10}$$
$$1n_x + 1n_y + (1-(-1)n_z = 0 \tag{11}$$

Multiplying eq (11) with 2 and subtracting it from eq (10)

$$0 \cdot n_x + 0_y - 3n_z = 0$$
$$n_z = 0$$

Put in eq (10)

$$2n_x + 2n_y = 0$$
$$n_x = -n_y$$

Put in eq(6)

$$(-n_y)^2 + n_y^2 = 1$$
$$2n_y^2 = 1$$
$$n_x = -0.71$$
$$n_y = 0.71$$
$$n_z = 0$$