# ASM - MST 1

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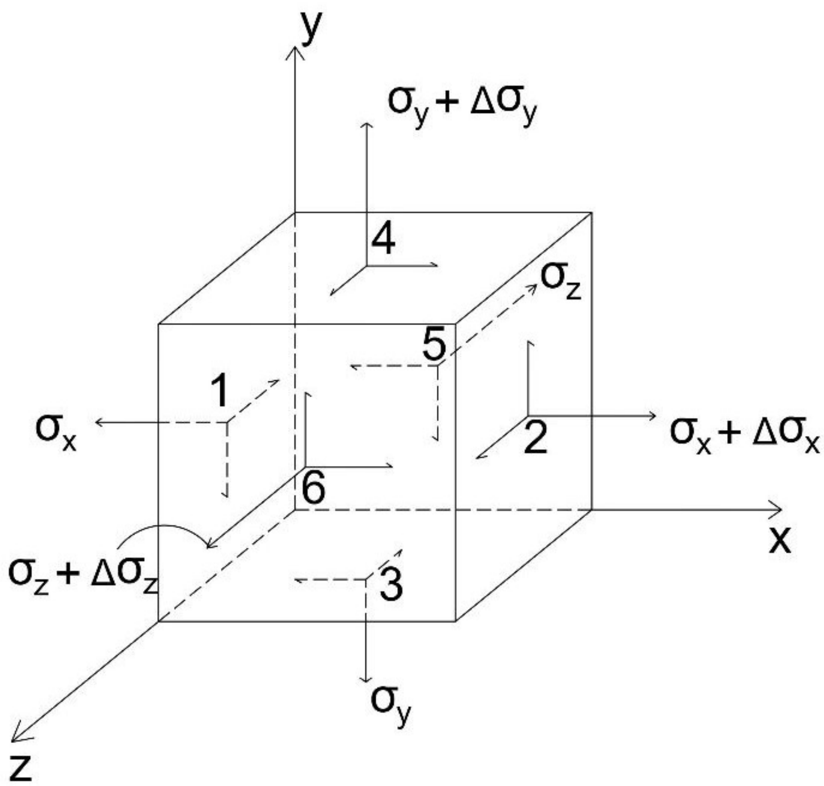
**4th Question**

State following with their role in the analysis of solid mechanics (with respect to cartesian co-ordinates):

1. Equation of equilibrium with body forces
2. Strain-displacement relations
3. Stress-strain relationship
4. Compatibility equations
5. Boundary conditions

**Solution**

1. **Equation of equilibrium with body forces**




Consider a small rectangular element with sides $\Delta x, \Delta y$ and $\Delta z$ isolated

from its parent body. Since in the limit, we are going to make $\Delta x, \Delta y$ and $\Delta z$ tend to zero, we shall deal with average values of the stress components on each face.These stress components are shown in Fig.
The faces are marked as 1, 2, 3 etc. On the left-hand face, i.e. face No. 1, the average stress components are
$\sigma_x, \tau_{xy}$ and $\tau_{xz}$
On the right-hand face, i.e. face No. 2, the average stress components are

$$\sigma_x + \frac{\delta \sigma_x}{\delta_x} \cdot \Delta x$$

$$\tau_{xy} + \frac{\delta \tau_{xy}}{\delta_x} \cdot \Delta x$$

$$\tau_{xz} + \frac{\delta \tau_{xz}}{\delta_x} \cdot \Delta x$$

This is because the right-hand face is $\Delta x$ distance away from the left-hand face.

Following a similar procedure, the stress components on the six faces of the element are as follows :

Face 1 - $\sigma_x, \tau_{xy}$ and $\tau_{xz}$

Face 2 - $\sigma_x + \frac{\delta \sigma_x}{\delta_x} \cdot \Delta x$

$$\tau_{xy} + \frac{\delta \tau_{xy}}{\delta_x} \cdot \Delta x$$

$$\tau_{xz} + \frac{\delta \tau_{xz}}{\delta_x} \cdot \Delta x$$

Face 3 - $\sigma_y, \tau_{yx}$ nd $\tau_{yz}$

Face 4 -

$$\sigma_y + \frac{\delta \sigma_y}{\delta_y} \cdot \Delta y$$

$$\tau_{yx} + \frac{\delta \tau_{yx}}{\delta_y} \cdot \Delta y$$

$$\tau_{yz} + \frac{\delta \tau_{yz}}{\delta_y} \cdot \Delta y$$

Face 5 - $\sigma_z, \tau_{zx}$ and $\tau_{zy}$.

Face 6 -

$$\sigma_z + \frac{\delta\sigma_z}{\delta_z} \cdot \Delta z$$

$$\tau_{zx} + \frac{\delta\tau_{zx}}{\delta_z} \cdot \Delta z$$

$$\tau_{zy} + \frac{\delta\tau_{zy}}{\delta_z} \cdot \Delta z$$

Let the body force components per unit volume in x, y, and z directions be $\gamma_x, \gamma_y, \gamma_z$.

For equilibrium in x direction

$[\sigma_x + \frac{\delta\sigma_x}{\delta x} \cdot \Delta x]\Delta y \Delta z - \sigma_x \Delta y \Delta z + [\tau_{yx} + \frac{\delta\tau_{yx}}{\delta y} \cdot \Delta y]\Delta z \Delta x - \tau_{yx}\Delta z \Delta x + [\tau_{zx} + \frac{\delta\tau_{zx}}{\delta z} \cdot \Delta z]\Delta x \Delta y - \tau_{zx}\Delta x \Delta y + \gamma_x \Delta x \Delta y \Delta z = 0$

Canceling terms , dividing by $\Delta x, \Delta y, \Delta z$ and we going to the limit we get

$$\frac{\delta\sigma_x}{\delta x} + \frac{\tau_{yx}}{\delta y} + \frac{\tau_{zx}}{\delta z} + \gamma_x = 0$$

Similarly, equating forces in the y and z directions respectively to zero, we get two more equations. on the basis of the fact that the cross shears are equal, i.e,

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{xz} = \tau_{zx}$$

we obtain the three differential equations of equilibrium as

$$\frac{\delta\sigma_x}{\delta x} + \frac{\tau_{yx}}{\delta y} + \frac{\tau_{zx}}{\delta z} + \gamma_x = 0$$

$$\frac{\delta\sigma_y}{\delta y} + \frac{\tau_{xy}}{\delta x} + \frac{\tau_{yz}}{\delta z} + \gamma_y = 0$$

$$\frac{\delta\sigma_z}{\delta z} + \frac{\tau_{xz}}{\delta x} + \frac{\tau_{yz}}{\delta y} + \gamma_z = 0$$

2. **Strain - displacement relations**

Consider point P (x,y,z) in a body and it is displaced to a new position $P'(x+u_x, y+u_y, z+u_z)$ where $u_x, u_y, u_z$ are the displacement components. A neighboring point Q with coordinates $(x+\Delta x, y+\Delta y, z+\Delta z)$ gets displaced to Q' with new coordinates $(x+\Delta x+u_x+\Delta u_x, y+\Delta y+u_y+\Delta u_y, z+\Delta z+u_z+\Delta u_z)$.

Hence, it is possible to determine the change in the length of the line element PQ caused by deformation. Let $\Delta s$ be the length of the line element PQ. Its components are

$$\Delta s : (\Delta x, \Delta y, \Delta z)$$

$$\Delta s^2 : (PQ)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Let $\Delta s'$ be the length of P'Q'. Its components are

$$\Delta s' : (\Delta x' = \Delta x + \Delta u_x, \Delta y' = \Delta y + \Delta u_y, \Delta z' = \Delta z + \Delta u_z)$$

$$\Delta s'^2 : (P'Q')^2 = (\Delta x + \Delta u_x)^2 + (\Delta y + \Delta u_y)^2 \ + (\Delta z + \Delta u_z)^2$$

$$\Delta x' = (1 + \frac{\delta u_x}{\delta x})\Delta x + \frac{\delta u_x}{\delta y})\Delta y + \frac{\delta u_x}{\delta z}\Delta z$$

$$\Delta y' = \frac{\delta u_y}{\delta x}\Delta x + (1 + \frac{\delta u_y}{\delta y})\Delta y + \frac{\delta u_y}{\delta z}\Delta z$$

$$\Delta z' = \frac{\delta u_z}{\delta x}\Delta x + \frac{\delta u_z}{\delta y}\Delta y + (1 + \frac{\delta u_z}{\delta z})\Delta z$$

We take the difference between $\Delta s'^2 and \Delta s^2$

$$(P'Q')^2 - (PQ)^2 = \Delta s'^2 - \Delta s^2$$

$= (\ \Delta x'^2 + \Delta y'^2 + \Delta z'^2) - (\Delta x^2 + \Delta y^2 + \Delta z^2)$

=2 ( E$_{xx}\Delta x^2 + E_{yy}\Delta y^2 + E_{zz}\Delta z^2 + E_{xy}\Delta x\Delta y + E_{yz}\Delta y\Delta z + E_{xz}\Delta x\Delta z)(1)$

where

$$E_{xx} = \frac{\delta u_x}{\delta x} + \frac{1}{2}[(\frac{\delta u_x}{\delta x})^2 + (\frac{\delta u_y}{\delta x})^2 + (\frac{\delta u_z}{\delta x})^2]$$

$$E_{yy} = \frac{\delta u_y}{\delta y} + \frac{1}{2}[(\frac{\delta u_x}{\delta y})^2 + (\frac{\delta u_y}{\delta y})^2 + (\frac{\delta u_z}{\delta y})^2]$$

$$E_{zz} = \frac{\delta u_z}{\delta z} + \frac{1}{2}[(\frac{\delta u_x}{\delta z})^2 + (\frac{\delta u_y}{\delta z})^2 + (\frac{\delta u_z}{\delta z})^2]$$

$$E_{xy} = \frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x} + \frac{\delta u_x}{\delta x}\frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x}\frac{\delta u_y}{\delta y} + \frac{\delta u_z}{\delta x}\frac{\delta u_z}{\delta y}$$

$$E_{yz} = \frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y} + \frac{\delta u_x}{\delta y}\frac{\delta u_x}{\delta z} + \frac{\delta u_y}{\delta y}\frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y}\frac{\delta u_z}{\delta z}$$

$$E_{xz} = \frac{\delta u_x}{\delta z} + \frac{\delta u_z}{\delta x} + \frac{\delta u_x}{\delta x}\frac{\delta u_x}{\delta z} + \frac{\delta u_y}{\delta x}\frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta x}\frac{\delta u_z}{\delta z}$$

It is observed that

$$E_{xy} = E_{yx}, E_{yz} = E_{zy}, E_{xz} = E_{zx}$$

If the deformation imposed on the body is small, the quantities like $\frac{\delta u_x}{\delta x}, \frac{\delta u_y}{\delta y},$, etc. are extremely small so that their squares and products can be neglected. Retaining only linear terms, the following equations are obtained

$$\epsilon_{xx} = \frac{\delta u_x}{\delta x}$$

$$\epsilon_{yy} = \frac{\delta u_y}{\delta y}$$

$$\epsilon_{zz} = \frac{\delta u_z}{\delta z}$$

$$\gamma_{xy} = \frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x}$$

$$\gamma_{yz} = \frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y}$$

$$\gamma_{xz} = \frac{\delta u_x}{\delta z} + \frac{\delta u_z}{\delta x}$$

$$E_{PQ} = \epsilon_{PQ} = \epsilon_{xx}n_x^2 + \epsilon_{yy}n_y^2 + \epsilon_{zz}n_z^2 + \epsilon_{xy}n_xn_y + \epsilon_{yz}n_yn_z + \epsilon_{zx}n_zn_x \quad (2)$$

The equation (2 ) directly gives the linear strain at point P in the direction PQ with direction cosines $n_x, n_y, n_z$.

## 3. Stress-strain relationship

A small rectangular parallelepiped or a box which remains rectangular after strain. The normals to the faces of this box were called the principal axes of strain. Since in an isotropic material, a small rectangular box the faces of which are subjected to pure normal stresses, will remain rectangular after deformation (no asymmetrical deformation), the normal to these faces coincide with the principal strain axes. Hence, for an isotropic material, principal stresses $\sigma_x$, $\sigma_y$, $\sigma_z$ can relate with the three principal strains $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ through suitable elastic constants. Let the axes x, y and z coincide with the principal stress and principal strain directions. For the principal stress , the equation becomes

$$\sigma_1 = a\epsilon_1 + b\epsilon_2 + c\epsilon_3$$

where a, b and c are constants. b and c should be equal since the effect of $\sigma_1$ in the directions of $\epsilon_3$ and $\epsilon_3$ which are both at right angles to $\sigma_1$. The effect of $\sigma_1$, in any direction transverse to it is the same in an isotropic material. Hence, for $\sigma_1$, the equation becomes:

$$\sigma_1 = a\epsilon_1 + b(\epsilon_2 + \epsilon_3)$$

adding and subtracting $b\epsilon_1$

$$\sigma_1 = (a - b)\epsilon_1 + b(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

$\epsilon_1 + \epsilon_2 + \epsilon_3$ is the first invariant of strain $J_1$ or the cubical dilatation $\Delta$. Denoting 'b' by $\lambda$ and (a - b) by $2\mu$, the equation for $\sigma_1$ becomes:

$$\sigma_1 = \lambda\Delta + 2\mu\epsilon_1$$

Similarly,

$$\sigma_2 = \lambda\Delta + 2\mu\epsilon_2$$

$$\sigma_3 = \lambda\Delta + 2\mu\epsilon_3$$

The constants $\lambda$ and $\mu$ are called Lame's coefficients. Thus, there are only two elastic constants involved in the relations between the principal stresses and principal strains for an isotropic material.

## 4. Compatibility equations

It was observed that the displacement of a point in a solid body can be represented by a displacement vector u, which has components,

$$u_x, u_y, u_z$$

along the three axes x, y and z respectively. The deformation at a point is specified by the six strain components,

$$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{zx}$$

The three displacement components and the six rectangular strain components are related by the six strain displacement relations of Cauchy, given by

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

The determination of the six strain components from the three displacement functions is straightforward since it involves only differentiation.

However, the reverse operation, i.e. determination of the three displacement functions from the six strain components is more complicated since it involves integrating six equations to obtain three functions.

The total number of these relations are six and they fall into two groups.

**FIRST GROUP**

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Differentiate the first two of the above equations as follows:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \cdot \frac{\partial u_x}{\partial y}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y} \cdot \frac{\partial u_y}{\partial x}$$

Adding these two, we get

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial^2 \gamma xy}{\partial x \partial y}$$

i.e.

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma xy}{\partial x \partial y}$$

Similarly, by considering $\epsilon_{yy}, \epsilon_{zz}$ and $\gamma_{yz}$ and $\epsilon_{zz}, \epsilon_{xx}$ and $\gamma_{zx}$ we get two more conditions.

This leads us to the first group of conditions.

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma xy}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma yz}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma zx}{\partial z \partial x}$$

**SECOND GROUP**

This group establishes the conditions among the shear strains

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

Diffrentiating

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u_x}{\partial z \partial y} + \frac{\partial^2 u_y}{\partial z \partial x}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 u_y}{\partial x \partial z} + \frac{\partial^2 u_x}{\partial y \partial z}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 u_z}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial y \partial z}$$

Adding the last two equations and subtracting the first

$$\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} = 2\frac{\partial^2 u_z}{\partial x \partial y} \tag{3}$$

Also,

$$\frac{\partial^3 u_z}{\partial x \partial y \partial z} = \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y}$$

Differentiating equation (1) once more with respect to z.

$$\frac{\partial}{\partial z}(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}) = 2\frac{\partial^3 u_z}{\partial x \partial y \partial z} = 2\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y}$$

This is one of the required relations of the second group. By a cyclic change of the letters we get the other two equations.

Collecting all equations, the six strain compatibility relations are

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma xy}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma yz}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma zx}{\partial z \partial x}$$

$$\frac{\partial}{\partial z}(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}) = 2\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}) = 2\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z}$$

$$\frac{\partial}{\partial y}(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}) = 2\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z}$$

The above six equations are called Saint-Venant's equations of compatibility

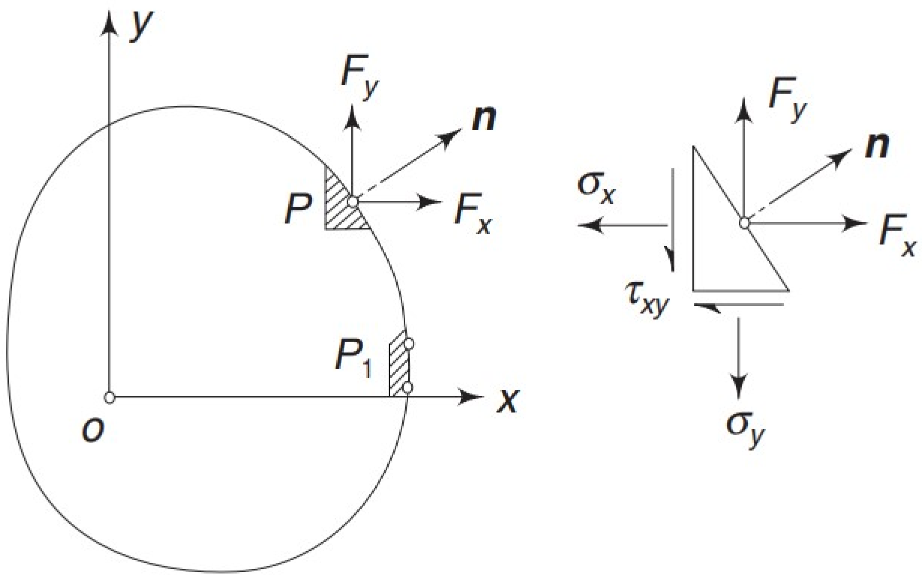
5. **Boundary Condition**

Differential Equation of Equilibrium i.e.

$$\frac{\partial \sigma_x}{\partial_x} + \frac{\partial \tau_x y}{\partial_y} = 0$$

$$\frac{\partial \sigma_y}{\partial_y} + \frac{\partial \tau_x y}{\partial_x} = 0$$

these above equation are satisfied throughout the volume of body. When the stresses vary over the plate (i.e. the body having the plane stress state), the stress components $\sigma_x$, $\sigma_y$ and $\tau_x y$ must be consistent with the externally applied forces at a boundary point.




Consider the two-dimensional body show in below fig. At a boundary point P, the outward drawn normal is **n**. Let $F_x$ and $F_y$ be the components of

the surface forces per unit area at this point.
$F_x$ and $F_y$ must be continuations of stresses $\sigma_x$,$\sigma_y$ and $\tau_x y$ at the boundary. Hence, Cauchy's equations

$$T_x^n = F_x = \sigma_x n_x + \tau_x y n_y$$
$$T_y^n = F_y = \sigma_y n_y + \tau_x y n_x$$

If the boundary of the plate happens to be parallel to y axis, as at point P1, the boundary conditions become

$$F_x = \sigma_x$$

and

$$F_y = \tau_x y$$