ASM - MST 1

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1st Question

Derive the expression for strain components, assuming small deformation.

Solution

Consider a point P (x,y,z) in a body and it is displaced to a new position $P'(x + u_x, y + u_y, z + u_z)$ where u_x, u_y, u_z are the displacement components. A neighboring point Q with coordinates $(x+\Delta x, y+\Delta y, z+\Delta z)$ gets displaced to Q' with new coordinates $(x+\Delta x+u_x+\Delta u_x, y+\Delta y+u_y+\Delta u_y, z+\Delta z+u_z+\Delta u_z)$.

Hence, it is possible to determine the change in the length of the line element PQ caused by deformation. Let Δs be the length of the line element PQ. Its components are

$$\Delta s : (\Delta x, \Delta y, \Delta z)$$
$$\Delta s^2 : (PQ)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Let $\Delta s'$ be the length of P'Q'. Its components are

$$\Delta s' : (\Delta x' = \Delta x + \Delta u_x, \Delta y' = \Delta y + \Delta u_y, \Delta z' = \Delta z + \Delta u_z)$$
$$\Delta s'^2 : (P'Q')^2 = (\Delta x + \Delta u_x)^2 + (\Delta y + \Delta u_y)^2 \ + (\Delta z + \Delta u_z)^2$$

$$\Delta x' = (1 + \frac{\delta u_x}{\delta x})\Delta x + \frac{\delta u_x}{\delta y})\Delta y + \frac{\delta u_x}{\delta z}\Delta z$$

$$\Delta y' = \frac{\delta u_y}{\delta x}\Delta x + (1 + \frac{\delta u_y}{\delta y})\Delta y + \frac{\delta u_y}{\delta z}\Delta z$$

$$\Delta z' = \frac{\delta u_z}{\delta x}\Delta x + \frac{\delta u_z}{\delta y}\Delta y + (1 + \frac{\delta u_z}{\delta z})\Delta z$$

We take the difference between $\Delta s'^2 and \Delta s^2$

$$(P'Q')^2 - (PQ)^2 = \Delta s'^2 - \Delta s^2$$

$= (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) - (\Delta x^2 + \Delta y^2 + \Delta z^2)$
$= 2(E_{xx}\Delta x^2 + E_{yy}\Delta y^2 + E_{zz}\Delta z^2 + E_{xy}\Delta x\Delta y + E_{yz}\Delta y\Delta z + E_{xz}\Delta x\Delta z)(1)$
where

$$E_{xx} = \frac{\delta u_x}{\delta x} + \frac{1}{2}[(\frac{\delta u_x}{\delta x})^2 + (\frac{\delta u_y}{\delta x})^2 + (\frac{\delta u_z}{\delta x})^2]$$

$$E_{yy} = \frac{\delta u_y}{\delta y} + \frac{1}{2}[(\frac{\delta u_x}{\delta y})^2 + (\frac{\delta u_y}{\delta y})^2 + (\frac{\delta u_z}{\delta y})^2]$$

$$E_{zz} = \frac{\delta u_z}{\delta z} + \frac{1}{2}[(\frac{\delta u_x}{\delta z})^2 + (\frac{\delta u_y}{\delta z})^2 + (\frac{\delta u_z}{\delta z})^2]$$

$$E_{xy} = \frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x} + \frac{\delta u_x}{\delta x}\frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x}\frac{\delta u_y}{\delta y} + \frac{\delta u_z}{\delta x}\frac{\delta u_z}{\delta y}$$

$$E_{yz} = \frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y} + \frac{\delta u_x}{\delta y}\frac{\delta u_x}{\delta z} + \frac{\delta u_y}{\delta y}\frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y}\frac{\delta u_z}{\delta z}$$

$$E_{xz} = \frac{\delta u_x}{\delta z} + \frac{\delta u_z}{\delta x} + \frac{\delta u_x}{\delta x}\frac{\delta u_x}{\delta z} + \frac{\delta u_y}{\delta x}\frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta x}\frac{\delta u_z}{\delta z}$$

It is observed that

$$E_{xy} = E_{yx}, E_{yz} = E_{zy}, E_{xz} = E_{zx}$$

If the deformation imposed on the body is small, the quantities like $\frac{\delta u_x}{\delta x}, \frac{\delta u_y}{\delta y}$, etc. are extremely small so that their squares and products can be neglected. Retaining only linear terms, the following equations are obtained:

$$\epsilon_{xx} = \frac{\delta u_x}{\delta x}$$

$$\epsilon_{yy} = \frac{\delta u_y}{\delta y}$$

$$\epsilon_{zz} = \frac{\delta u_z}{\delta z}$$

$$\gamma_{xy} = \frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x}$$

$$\gamma_{yz} = \frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y}$$

$$\gamma_{xz} = \frac{\delta u_x}{\delta z} + \frac{\delta u_z}{\delta x}$$

$$E_{PQ} = \epsilon_{PQ} = \epsilon_{xx} n_x^2 + \epsilon_{yy} n_y^2 + \epsilon_{zz} n_z^2 + \epsilon_{xy} n_x n_y + \epsilon_{yz} n_y n_z + \epsilon_{zx} n_z n_x \qquad (2)$$

The equation (2) directly gives the linear strain at point P in the direction PQ with direction cosines n_x, n_y, n_z.