Mechanics I — Computational Assignment

Group Number: 37

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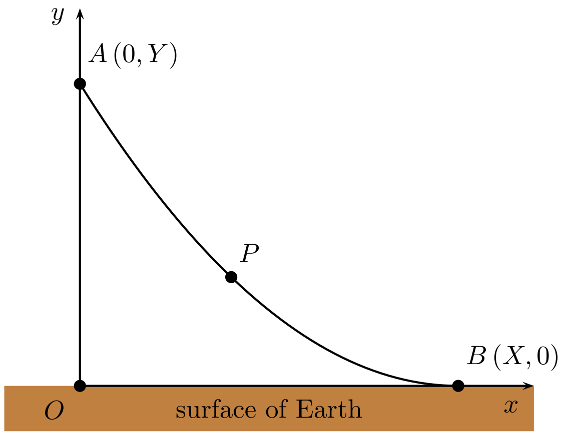
- You must print a hard copy of this assignment and write your answers for each question on your hard copy.
- You must submit your hard copy in person before 16h00 on 25 October 2024. (One hard copy only per group must be submitted.)
- Assignments may be submitted during the following times at room 132 on the first floor of the Mathematical Sciences Building:
  - Wednesday to Friday: 11h00 to 12h30;
  - Monday to Friday: 14h30 to 16h00.
- Assignments that are not submitted in person before the due date will not be accepted.
- The total marks available is 70.

- Consider two points A and B that have coordinates $(0, Y)$ and $(X, 0)$ with respect to the origin O of a coordinate system that is fixed to the surface of Earth as indicated in the diagram given below.
- A particle P is allowed to slide without friction down a thin wire that connects the points A and B.
- It is the purpose of this assignment to determine the shape of the wire which will allow the particle to fall from A to B in the shortest time.



- This problem is called the brachistochrone problem and it was solved by Johann Bernoulii in the late 1690s.
- We will attempt to solve the problem using a simple machine learning algorithm.

- Suppose we want to determine the shape of the wire given that

$$X = Y = 5 \text{ m}.$$

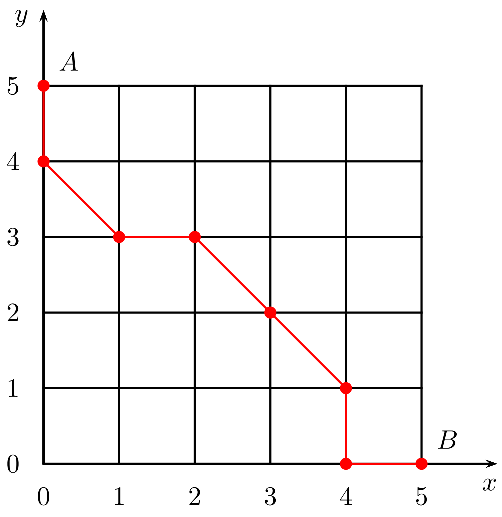
- We begin by constructing a random path from point A to point B using the following stepsizes,

$$\triangle x = \triangle y = 1 \text{ m}.$$

- For example, consider the path that is made up of the following set of points,

$$\text{path } 1 = \{(0,5),(0,4),(1,3),(2,3),(3,2),(4,1),(4,0),(5,0)\},$$

as indicated in the diagram given below.



- If the particle falls from rest from point $(0, Y)$ then, by the conservation of mechanical energy, the particle's speed when it is at the point (x, y) can be calculated from

$$\frac{m\|\underline{v}(x,y)\|^2}{2} + mgy = mgY,$$

that is,

$$\|\underline{v}(x,y)\| = \sqrt{2g(Y-y)}. \tag{1}$$

- If we set $g = 10 \text{ m/s}^2$ then we can use equation (1) to determine the speed of the particle at each point along path 1 as indicated in the table given below.

$(0,5)$ 0 m/s	$(0,4)$ $\sqrt{20}$ m/s	$(1,3)$ $\sqrt{40}$ m/s	$(2,3)$ $\sqrt{40}$ m/s
$(3,2)$ $\sqrt{60}$ m/s	$(4,1)$ $\sqrt{80}$ m/s	$(4,0)$ 10 m/s	$(5,0)$ 10 m/s

- Given that the speed of the particle was 0 m/s at point $(0,5)$ and the speed of the particle was $\sqrt{20}$ m/s at point $(0,4)$ then, using the particle's average speed between these points, the time that it takes the particle to move from point $(0,5)$ to point $(0,4)$ is approximately

$$\triangle t_1 = \frac{\sqrt{(0-0)^2 + (5-4)^2} \text{ m}}{\left[\left(0+\sqrt{20}\right)/2\right] \text{ m/s}} = \frac{2}{\sqrt{20}} \text{ s}.$$

- In the same way we can calculate how long it takes the particle to move between each successive pair of points along the remainder of path 1.

$$(0,4) \to (1,3): \quad \triangle t_2 = \frac{\sqrt{(0-1)^2+(4-3)^2}\ \mathrm{m}}{\left[\left(\sqrt{20}+\sqrt{40}\right)/2\right]\ \mathrm{m/s}} = \frac{2\sqrt{2}}{\sqrt{20}+\sqrt{40}}\ \mathrm{s}$$

$$(1,3) \to (2,3): \quad \triangle t_3 = \frac{\sqrt{(1-2)^2+(3-3)^2}\ \mathrm{m}}{\left[\left(\sqrt{40}+\sqrt{40}\right)/2\right]\ \mathrm{m/s}} = \frac{1}{\sqrt{40}}\ \mathrm{s}$$

$$(2,3) \to (3,2): \quad \triangle t_4 = \frac{\sqrt{(2-3)^2+(3-2)^2}\ \mathrm{m}}{\left[\left(\sqrt{40}+\sqrt{60}\right)/2\right]\ \mathrm{m/s}} = \frac{2\sqrt{2}}{\sqrt{40}+\sqrt{60}}\ \mathrm{s}$$

$$(3,2) \to (4,1): \quad \triangle t_5 = \frac{\sqrt{(3-4)^2+(2-1)^2}\ \mathrm{m}}{\left[\left(\sqrt{60}+\sqrt{800}\right)/2\right]\ \mathrm{m/s}} = \frac{2\sqrt{2}}{\sqrt{60}+\sqrt{80}}\ \mathrm{s}$$

$$(4,1) \to (4,0): \quad \triangle t_6 = \frac{\sqrt{(4-4)^2+(1-0)^2}\ \mathrm{m}}{\left[\left(\sqrt{80}+10\right)/2\right]\ \mathrm{m/s}} = \frac{2}{\sqrt{80}+10}\ \mathrm{s}$$

$$(4,0) \to (5,0): \quad \triangle t_7 = \frac{\sqrt{(4-5)^2+(0-0)^2}\ \mathrm{m}}{\left[(10+10)/2\right]\ \mathrm{m/s}} = \frac{1}{10}\ \mathrm{s}$$

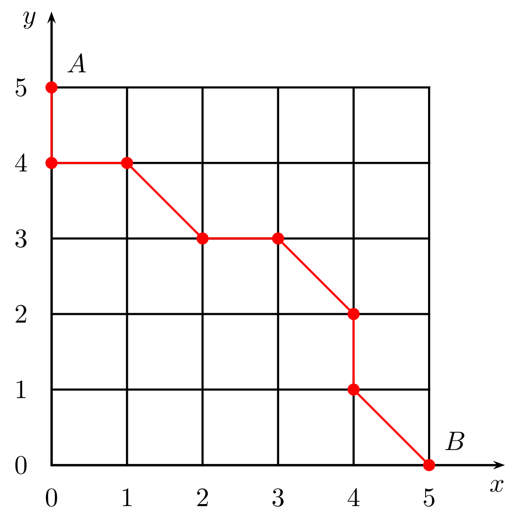
- Thus, the total time that it takes the particle to move from point A to point B along path 1 is

$$\begin{aligned}\triangle t\left(A \xrightarrow{\text{path 1}} B\right) &= \triangle t_1 + \triangle t_2 + \cdots + \triangle t_7 \\ &= 1,443 \text{ s (rounded to three decimal places).}\end{aligned}$$

- Consider a second random path from point A to point B that is made up of the following set of points,

$$\text{path 2} = \{(0,5), (0,4), (1,4), (2,3), (3,3), (4,2), (4,1), (5,0)\},$$

as indicated in the diagram given below.



- The total time that it takes the particle to move from point A to point B along path 2 is

$$\begin{aligned}\triangle t\left(A \xrightarrow{\text{path 2}} B\right) &= \triangle t_1 + \triangle t_2 + \cdots + \triangle t_7 \\ &= 1,561 \text{ s (rounded to three decimal places).}\end{aligned}$$

- Consider now a 6×6 matrix W. The elements of the matrix W are random numbers between 1 and 5 except for the elements in the first row of the matrix which are set equal to 0; for example,

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 3 & 5 & 4 & 1 \\ 2 & 5 & 4 & 4 & 4 & 2 \\ 5 & 5 & 2 & 1 & 3 & 1 \\ 4 & 2 & 3 & 4 & 4 & 1 \\ 1 & 5 & 5 & 5 & 2 & 3 \end{bmatrix}.$$

- The elements in the first column of the matrix W are associated with the points $(0, y)$ in descending order from $y = 5$ to $y = 0$; that is,

  - the element $w_{11} = 0$ is associated with the point $(0, 5)$;
  - the element $w_{21} = 5$ is associated with the point $(0, 4)$;
  - the element $w_{31} = 2$ is associated with the point $(0, 3)$ etc;

  the elements in the second column of the matrix W are associated with the points $(1, y)$ in descending order from $y = 5$ to $y = 0$; that is,

  - the element $w_{12} = 0$ is associated with the point $(1, 5)$;
  - the element $w_{22} = 3$ is associated with the point $(1, 4)$;
  - the element $w_{32} = 5$ is associated with the point $(1, 3)$ etc;

  and so forth.

- We will update the elements of the matrix W according to the following rules.

  Rule 1. For each point on the path that gives the minimum time, increase the corresponding elements of W by 1.

  Rule 2. For each point on the path that gives the maximum time, decrease the corresponding elements of W by 1.

- It follows from the above rules that points that are common to both paths remain unchanged.

- Thus, given that

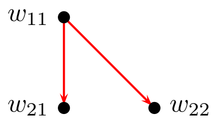
$$\triangle t\left(A \xrightarrow{\text{path 1}} B\right) < \triangle t\left(A \xrightarrow{\text{path 2}} B\right),$$

  the matrix W will update as follows,

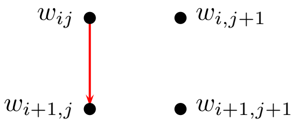
$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 3 & 5 & 4 & 1 \\ 2 & 5 & 4 & 4 & 4 & 2 \\ 5 & 5 & 2 & 1 & 3 & 1 \\ 4 & 2 & 3 & 4 & 4 & 1 \\ 1 & 5 & 5 & 5 & 2 & 3 \end{bmatrix} \quad \rightarrow \quad W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 3 & 5 & 4 & 1 \\ 2 & 6 & 4 & 3 & 4 & 2 \\ 5 & 5 & 2 & 2 & 2 & 1 \\ 4 & 2 & 3 & 4 & 4 & 1 \\ 1 & 5 & 5 & 5 & 3 & 3 \end{bmatrix}.$$

- The updated matrix W will determine a new path from A to B according to the following rules.

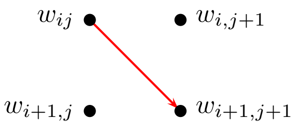
Rule 1. Start at the element w_{11} and move down if $w_{21} > w_{22}$ or sideways if $w_{21} < w_{22}$ (make a random move if $w_{21} = w_{22}$).



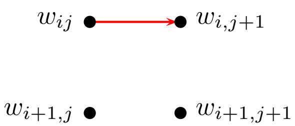
Rule 2: Consider next the element w_{ij} where $i = 2, \ldots, 5$ and $j = 1, \ldots, 5$ and determine the maximum value of $w_{i+1,j}$, $w_{i+1,j+1}$, $w_{i,j+1}$. Move down if the maximum value is $w_{i+1,j}$.



Move sideways if the maximum value is $w_{i+1,j+1}$.



Move across if the maximum value is $w_{i,j+1}$



If the maximum value is not unique then move randomly to one of the elements for which the maximum value is common.

Rule 3: If we reach the element w_{6j} then move across the columns to w_{66}.

Rule 4: If we reach the element w_{i6} then move down the rows to w_{66}.

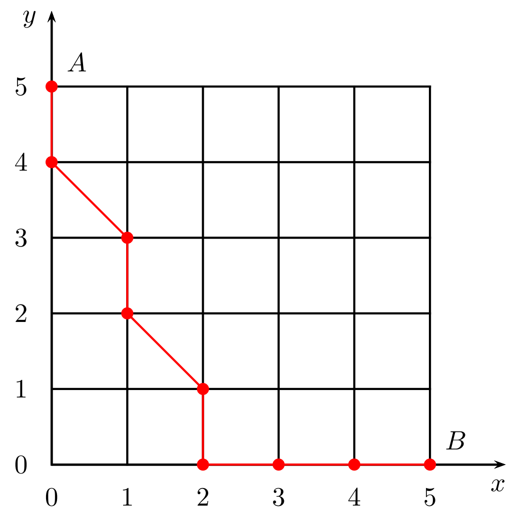
- The elements in the matrix W that are chosen according to these rules are indicated in red in the matrix given below.

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 3 & 5 & 4 & 1 \\ 2 & 6 & 4 & 3 & 4 & 2 \\ 5 & 5 & 2 & 2 & 2 & 1 \\ 4 & 2 & 3 & 4 & 4 & 1 \\ 1 & 5 & 5 & 5 & 3 & 3 \end{bmatrix}.$$

- These elements correspond to a path from point A to point B that is made up of the following set of points,

$$\text{path } 3 = \{(0,5), (0,4), (1,3), (1,2), (2,1), (2,0), (3,0), (4,0), (5,0)\},$$

as indicated in the diagram given below.



- The total time that it takes the particle to move from point A to point B along path 3 is

$$\begin{aligned}\triangle t\left(A \xrightarrow{\text{path 3}} B\right) &= \triangle t_1 + \triangle t_2 + \cdots + \triangle t_8 \\ &= 1,426 \text{ s (rounded to three decimal places).}\end{aligned}$$

- The final step (of the first iteration of the algorithm) is to determine new paths 1 and 2.

- Since

$$\triangle t\left(A \xrightarrow{\text{path 3}} B\right) < \triangle t\left(A \xrightarrow{\text{path 1}} B\right) < \triangle t\left(A \xrightarrow{\text{path 2}} B\right),$$

let

$$\begin{aligned}&\text{new path 1} \\ &= \{(0,5),(0,4),(1,3),(1,2),(2,1),(2,0),(3,0),(4,0),(5,0)\} \\ &= \text{old path 3.}\end{aligned}$$

and

$$\begin{aligned}&\text{new path 2} \\ &= \{(0,5),(0,4),(1,3),(2,3),(3,2),(4,1),(4,0),(5,0)\} \\ &= \text{old path 1.}\end{aligned}$$

- We can now apply the next iteration of the algorithm.

- Use (new) path 1 and (new) path 2 to update the matrix W.

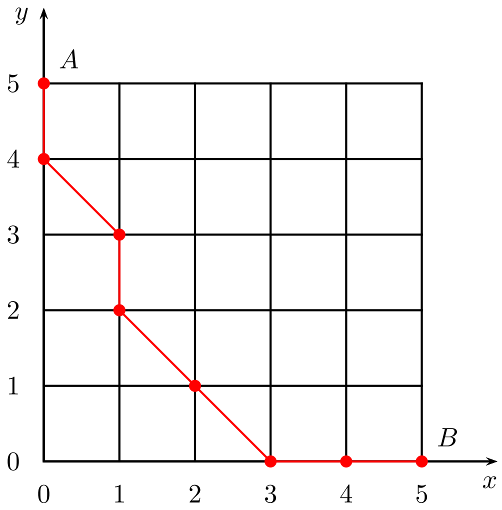
$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 3 & 5 & 4 & 1 \\ 2 & 6 & 4 & 3 & 4 & 2 \\ 5 & 5 & 2 & 2 & 2 & 1 \\ 4 & 2 & 3 & 4 & 4 & 1 \\ 1 & 5 & 5 & 5 & 3 & 3 \end{bmatrix} \quad \rightarrow \quad W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 3 & 5 & 4 & 1 \\ 2 & 6 & 3 & 3 & 4 & 2 \\ 5 & 6 & 2 & 1 & 2 & 1 \\ 4 & 2 & 4 & 4 & 3 & 1 \\ 1 & 5 & 6 & 6 & 3 & 3 \end{bmatrix}$$

- Use the updated matrix to determine a new path from A to B.

$$W = \begin{bmatrix} \color{red}{0} & 0 & 0 & 0 & 0 & 0 \\ \color{red}{5} & 2 & 3 & 5 & 4 & 1 \\ 2 & \color{red}{6} & 3 & 3 & 4 & 2 \\ 5 & \color{red}{6} & 2 & 1 & 2 & 1 \\ 4 & 2 & \color{red}{4} & 4 & 3 & 1 \\ 1 & 5 & 6 & \color{red}{6} & \color{red}{3} & \color{red}{3} \end{bmatrix}$$

- These elements correspond to a path from point A to point B that is made up of the following set of points,

$$\text{path 3} = \{(0,5),(0,4),(1,3),(1,2),(2,1),(3,0),(4,0),(5,0)\}.$$



- The total time that it takes the particle to move from point A to point B along path 3 is

$$\begin{aligned}\triangle t\left(A \xrightarrow{\text{path 3}} B\right) &= \triangle t_1 + \triangle t_2 + \cdots + \triangle t_7 \\ &= 1,370 \text{ s (rounded to three decimal places).}\end{aligned}$$

- Thus, after two iterations of the algorithm, the path that will allow the particle to fall from point A to point B in the shortest time is made up of the following set of points,

$$\{(0,5),(0,4),(1,3),(1,2),(2,1),(3,0),(4,0),(5,0)\},$$

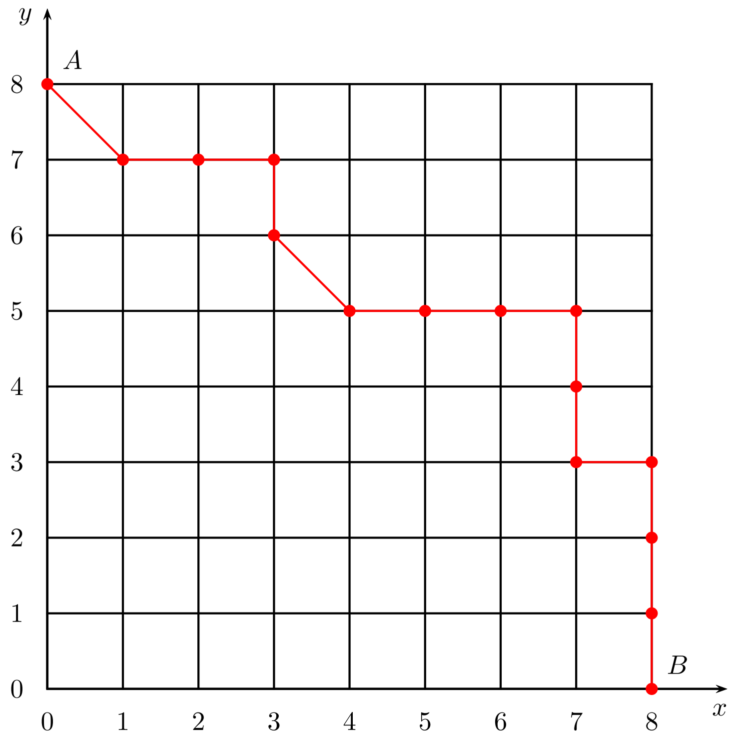
and the shortest time is $1,370$ s (rounded to three decimal places).

Computational Assignment

- You are required to apply the algorthim outlined in the preceding discussion to determine the shape of the wire that will allow the particle to fall from point A to point B in the shortest time if $g = 10$ m/s^2 given that

$$X = Y = 8 \text{ m} \quad \text{and} \quad \triangle x = \triangle y = 1 \text{ m}.$$

- Consider the first random path that is illustrated on the grid given below.

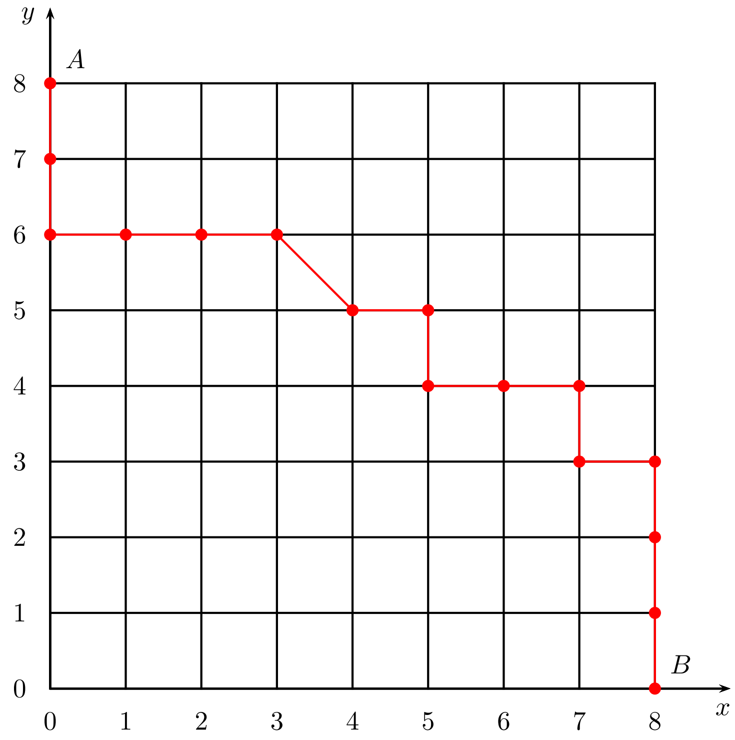


$$\triangle t\left(A \xrightarrow{\text{path 1}} B\right) = \underline{\hspace{5cm}}$$

(in seconds rounded to three decimal places)

[5 MARKS]

- Consider the second random path that is illustrated on the grid given below.



$$\triangle t\left(A \xrightarrow{\text{path 2}} B\right) = \underline{\hspace{5cm}}$$

(in seconds rounded to three decimal places)

[5 MARKS]

- Given the matrix W.

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 3 & 3 & 2 & 2 & 3 & 1 & 4 \\ 3 & 2 & 3 & 3 & 3 & 5 & 2 & 4 & 1 \\ 2 & 2 & 5 & 3 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 5 & 4 & 4 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 & 4 & 4 & 2 & 5 & 4 \\ 4 & 1 & 4 & 3 & 1 & 3 & 1 & 2 & 2 \\ 1 & 5 & 5 & 1 & 4 & 1 & 2 & 5 & 1 \\ 4 & 2 & 4 & 1 & 5 & 3 & 5 & 2 & 5 \end{bmatrix}$$

- Update the matrix W.

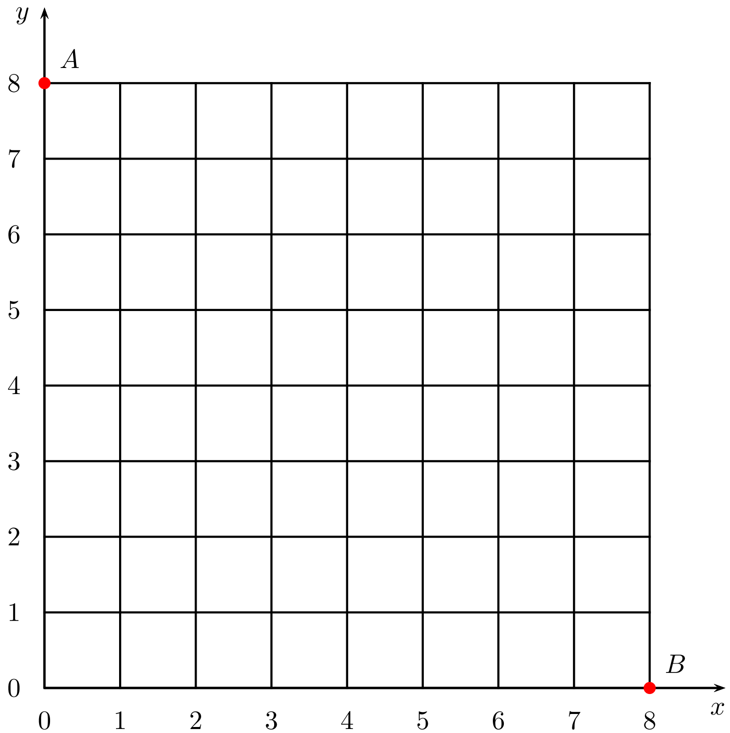
[10 MARKS]

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

- Use the elements of the updated matrix W to determine the points that make up a third path from point A to B and clearly illustrate the path on the grid given below.

path 3 $= \{(0,8)\,,$

$,(8,0)\}$



$\triangle t\left(A \xrightarrow{\text{path 3}} B\right) =$ ____________________

(in seconds rounded to three decimal places)

[15 MARKS]

- List the points that make up the new path 1.

new path 1 $= \{(0,8)\,,$

$,(8,0)\}$

- List the points that make up the new path 2.

new path 2 $= \{(0,8)\,,$

$,(8,0)\}$

[5 MARKS]

- Use new path 1 and new path 2 to update the matrix W.

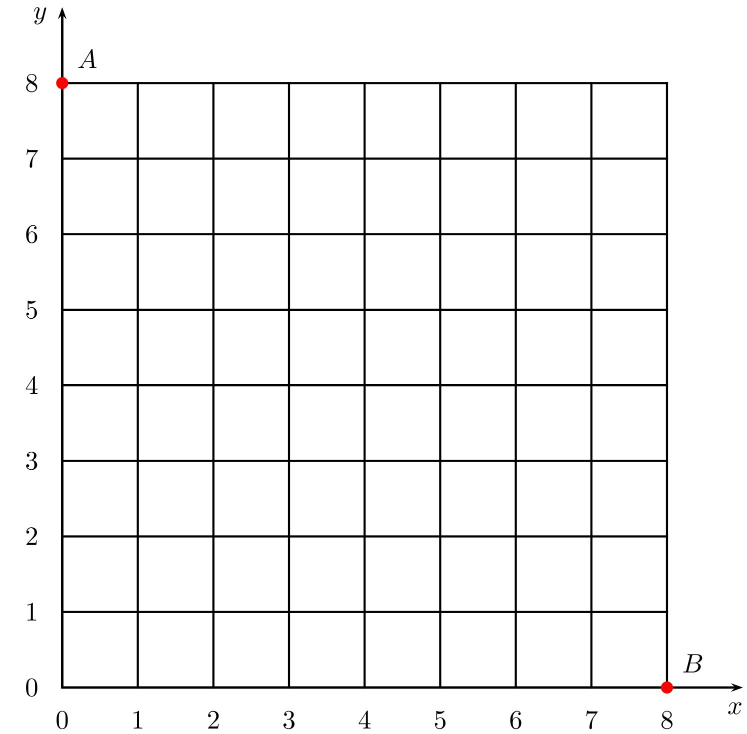
[10 MARKS]

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

- Use the elements of the updated matrix W to determine the points that make up a third path from point A to B and clearly illustrate the path on the grid given below.

path 3 $= \{(0,8)\,,$

$,(8,0)\}$



$$\triangle t \left(A \xrightarrow{\text{path 3}} B \right) = \underline{\qquad\qquad\qquad\qquad}$$

(in seconds rounded to three decimal places)

[15 MARKS]

- Thus, after two iterations of the algorithm, the path that will allow the particle to fall from point A to point B in the shortest time is made up of the following set of points,

$\{(0, 8)\,,$

$, (8, 0)\}\,,$

and the shortest time is __________ s (rounded to three decimal places).

[5 MARKS]