Sequences and Series

A sequence is a set of numbers written down in a specific order

1st 2nd 3rd 4th 5th

e.g. 1, 3, 5, 7, 9

−1, −2, −3, −4, −5

9, −11, 1/2, 3, 40

Each number in a sequence is called a term of the sequence.

finite sequence ⟶ 1, 3, 5, 7, 9 — there are 5 terms

2, 4, 6, 8, . . .

an infinite sequence.

Notation:

We [illegible]

We use subscript notation to refer to different terms in a sequence.

$$x \longrightarrow \underset{x_1}{3}, \underset{x_2}{6}, \underset{x_3}{9}, \underset{x_4}{12}, \ldots x_n$$

nth term.

$$t \longrightarrow \underset{t_1}{2}, \underset{t_2}{4}, \underset{t_3}{6}, \underset{t_4}{8}, \ldots, t_n$$

Sometimes, the first term in a sequence is labelled as x_0, 2nd is x_1 and so on.

In computer programming, when considering arrays, first term is labelled a_0, 2nd term a_1 and so on.

Q: The terms of a sequence x are given by

$x_k = 2k + 3$. Write down

x_1, x_4 and x_7.

Soln:

$$x_k = 2k+3$$

$k=1, \quad x_1 = 2(1)+3 = 5$

$k=4, \quad x_4 = 2(4)+3 = 11$

$k=7, \quad x_7 = 2(7)+3 = 17$

Q. Write down the first 4 terms of the sequence given by

$$x_n = 2^n + 3^n \text{ starting from } n=0$$

Solution

$$x_n = 2^n + 3^n$$

$n=0, \quad x_0 = 2^0 + 3^0 = 1+1 = 2$

$n=1, \quad x_1 = 2^1 + 3^1 = 2+3 = 5$

$n=2, \quad x_2 = 2^2 + 3^2 = 4+9 = 13$

$n=3, \qquad x_3 = 2^3 + 3^3 = 8 + 27 = 35$

Arithmetic Progression or Arithmetic Sequence

An arithmetic sequence is one in which each new term in the sequence is obtained by adding a fixed amount to the previous term. The fixed term is called the common difference.

e.g.

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ 2, & 4, & 6, & 8, & 10 \end{array}$$

$$2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \xrightarrow{+2} 10$$

common difference $= +2$

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4$$

Example

Write down the first 6 terms of an arithmetic progression that has first term 5 and common difference -2.

<u>Solution</u>:

$$5,\ 5-2=3,\ 3-2=1,\ 1-2=-1,\ -1-2=-3,$$
$$-3-2=-5$$

$$5,\ 3,\ 1,\ -1,\ -3,\ -5$$

<u>General</u> <u>notation</u> for <u>Arithmetic</u> Pro<u>gression</u>s

Let $\underline{a}$ be the first term and

d be the common difference

An arithmetic progression can be written as

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

Using this, we can find a formula for the nth term of an arithmetic progression

$$\boxed{x_n = a + (n-1)d}$$

nth
term

Q. Find the 10th term of an arithmetic progression with first term 3 and common difference 5.

Solution:

$$x_n = a + (n-1)d$$

$n = 10$

$a = 3$

$d = 5$

$$x_{10} = 3 + (10-1) \times 5$$

$$x_{10} = 3 + 9 \times 5 = \underline{48}$$

$$x_{10} = \underline{48}$$

Q. Find the 20th term of the arithmetic sequence

$1, 3, 5, 7, \ldots$

Soln:

$a = 1$

$d = 3-1 = 5-3 = 2$

$n = 20$

$x_n = a + (n-1)d$

$x_{20} = 1 + (20-1) \times 2$

$= 1 + 19 \times 2 = 39$

$x_{20} = \underline{\underline{39}}$

Geometric Progression (or Geometric Sequence)

We can form a sequence in which each new term

is obtained by multiplying the previous term by a fixed amount.
The fixed amount is called a common ratio and the sequence obtained is

called a geometric sequence.

e.g. $2, 4, 8, 16, 32$

$\times 2 \quad \times 2 \quad \times 2 \quad \times 2$

Common ratio $= 2$

A geometric progression can be written as

$$\underset{x_1}{a}, \underset{x_2}{ar}, \underset{x_3}{ar^2}, \underset{x_4}{ar^3}, \ldots$$

where a — first term

r — common ratio

$\Rightarrow$ The nth term of a geometric progression is given by

$$\boxed{x_n = ar^{n-1}}$$

Q. Find the 7th term of the geometric progression

$x_1 \quad x_2 \quad x_3 \quad x_4$

$\frac{1}{2}, 1, 2, 4, 8, \ldots$

Soln:

$$x_n = ar^{n-1}$$

$$a = \frac{1}{2}$$

$$r = \frac{x_2}{x_1} = \frac{x_3}{x_2} = \frac{x_4}{x_3} = \cdots$$

$$r = \frac{1}{\frac{1}{2}} = 1 \times \frac{2}{1} = \underline{\underline{2}}$$

$n = 7$

$$x_7 = \frac{(1/2)\, 2^{7-1}}{}$$

$$= 2^{-1} \times 2^{6}$$

$$x_7 = 2^{-1+6} = 2^{5} = \underline{\underline{32}}$$

Series and Sigma notation

If the terms of a sequence are added, the result is known as a series

e.g. 1, 2, 3, 4, 5 ← sequence
we obtain the series

1+2+3+4+5 ← series

A series is just the sum of terms in a sequence

- If the series contains a finite number of terms we are able to add them up.

- If it contains an infinite number of terms, we may have a finite sum, in which case we say the series <u>converges</u>.

- If it does not have a finite sum, then we say it <u>diverges</u>

<u>Convergent sequence</u>

Consider the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots$$

$$x_k = \frac{1}{k}, \quad k = 1, 2, 3, \ldots$$

as k gets larger and larger, x_k gets smaller and smaller and closer to zero.

e.g $k = 1000$, $x_{1000} = \frac{1}{1000}$,

$$= 0.001$$

$$k = 10000, \quad x_{10000} = \frac{1}{10000}$$
$$= 0.0001$$

$$k = 100000, \quad x_{100000} = \frac{1}{100000} = 0.00001$$

We say x_k tends to zero as k tends to infinity.

Alternatively we say as k tends to infinity the limit of the sequence tends to zero.

$$\boxed{\lim_{k \to \infty} \left(\frac{1}{k}\right) = 0}$$

Eg Consider the sequence

$$3, 5, 7, 9, \ldots$$

$$x_k = 2k+1, \quad k = 1, 2, 3, \ldots$$

as k get larger and larger, so do the terms of the sequence.

The sequence is said to <u>diverge</u>

$$\boxed{\lim_{k \to \infty} (2k+1) = \infty}$$

<u>Q</u>. a) Write down the first four terms of the sequence

$$x_k = 3 + \frac{1}{k^2}, \quad k = 1, 2, 3, \ldots$$

b) Find, if possible, the limit of this sequence as k tends to infinity.

<u>Soln:</u>

a) $x_k = 3 + \frac{1}{k^2}$

$k=1,\ x_1 = 3 + \frac{1}{1^2} = 4$

$k=2,\ x_2 = 3 + \frac{1}{2^2} = 3 + \frac{1}{4} = 3\frac{1}{4}$

$k=3,\ x_3 = 3 + \frac{1}{3^2} = 3 + \frac{1}{9} = 3\frac{1}{9}$

$k=4,\ x_4 = 3 + \frac{1}{4^2} = 3 + \frac{1}{16} = 3\frac{1}{16}$

b) Observe that as k gets larger and larger, $\frac{1}{k^2}$ gets closer and closer to zero.

$$\lim_{k \to \infty} \left(3 + \frac{1}{k^2}\right) = 3 + 0 = \underline{\underline{3}}.$$

Sigma Notation

Sigma notation: $\sum$ $\leftarrow$ this provides a

Σ ← Sigma — concise and convenient way of writing long sums.

e.g. the sum $1, 2, 3, 4, \dots, 10, 11, 12$

$$\underline{1+2+3+\dots+10+11+12}$$

$x_k = k, \quad k = 1, \dots, 12$

can be written as

max. value of k →
$$\boxed{\sum_{k=1}^{k=12} k} = 1+2+3+\dots+12$$
min value of k →

$$\Downarrow$$

$$\sum_{k=1}^{12} k$$

4. Write explicitly what is meant by

$$\sum_{k=1}^{5} k^3$$

Soln:

$$\sum_{k=1}^{\rightarrow 5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

$$= 1 + 8 + 27 + 64 + 125$$

Q. Express $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ concisely using sigma notation

Soln:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\sum_{k=1}^{4} 1/k$$

Q Write out fully what is meant by

$$\sum_{k=1}^{4} (-1)^k 2^k$$

Soln:

$$\sum_{k=1}^{4} (-1)^k 2^k = (-1)^1 2^1 + (-1)^2 2^2 + (-1)^3 2^3 + (-1)^4 2^4$$

$$= -2 + 4 - 8 + 16$$

Q: Express the following series in sigma notation with correct limits

a) $4^2 + 5^2 + 6^2 + \cdots + 84^2$

b) $(2\times3)^2 + (3\times4)^2 + \cdots + (22\times23)^2$

b) $(2\times 3)+(3\times 4)+\cdots+(22\times 23)$

Solution

form

a) —, $k=1$ to 5

a) $4^2+5^2+6^2+\cdots+84^2$

$$\sum_{k=1}^{81}(k+3)^2 = 4^2+5^2+6^2+\cdots+84^2$$

$$\boxed{\sum_{k=4}^{84} k^2}$$

b) $(2\times 3)^2+(3\times 4)^2+(4\times 5)^2+\cdots+(22\times 23)^2$

$(2\times 3)^2 = 2^2\times 3^2$

$$\boxed{\sum_{k=1}^{21}\left[(k+1)(k+2)\right]^2}$$ ✓

$$\boxed{\sum_{k=0}^{20}\left[(k+2)(k+3)\right]^2}$$ ✓

Arithmetic series

Example of arithmetic series

$2+4+6+8+10+\cdots$

$3+6+9+12+\cdots$

The sum of the first n terms of an arithmetic series with first term a and common difference d is denoted by S_n and is given by

$$\boxed{S_n = \frac{n}{2}(a+l)},\quad l - \text{last term}$$

$$\boxed{l = a+(n-1)d}$$

$\rightarrow$ $2+4+6+ \cdots + 12$

$a = 2,\ d = 4-2 = 2$

$l = 12$

we need to find n.

$l = a+(n-1)d$

$12 = 2+(n-1)\times 2$

$$12 - 2 = 2n - 2$$

$$10 = 2n - 2$$

$$2n - 2 = 10$$

$$2n = 10 + 2$$

$$2n = 12$$

$$n = \frac{12}{2} = 6$$

$$S_6 = \frac{6}{2}(2 + 12)$$

$$S_6 = 3(14) = 42$$

$$\boxed{S_n = \frac{n}{2}\left(2a + (n-1)d\right)}$$

Q. Find the sum of the first 10 terms of the series

$$3+7+11+15+\cdots$$

Soln:

$$3+7+11+15+\cdots$$

$$a=3, \quad d=7-3=4$$

$$n=10$$

$$S_{10}=\frac{10}{2}\left(2(3)+(10-1)\times 4\right)$$

$$=5\left(6+36\right)$$

$$S_{10}=5(42)=\underline{210}$$

Geometric Series

The sum of the first n terms of a geometric series with first term a and common ratio r is denoted by S_n and given by

$$\longrightarrow S_n = \frac{a(1-r^n)}{1-r}$$

Provided $r \neq 1$
(otherwise denominator will be zero)

Q. Find the sum of the first 5 terms of the geometric series with first term 2 and common ratio 3.

Solution: $n=5,\ a=2,\ r=3$

$$S_5 = \frac{2(1-3^5)}{1-3}$$

$$= \frac{\cancel{2}^{1}(1-3^5)}{-\cancel{2}_{1}}$$

$$= -(1-3^5)$$

$$= -(1-243)$$

$$S_5 = -(-242) = \underline{\underline{242}}$$

Infinite geometric series

The sum of an infinite number of terms of a geometric series is denoted S_∞ and given by

$$S_\infty = \frac{a}{1-r}, \quad \text{Provided } |r| < 1$$

$\leftarrow$ modulus

$$\Rightarrow -1 < r < 1$$

Note: If $r > 1$ or $r < -1$, the series does not converge and the sum cannot be found.

Q. Find the sum ...

4 Find the sum of the infinite geometric series

$$4+2+1+\cdots$$

Solution:

$$4+2+1+\cdots$$

$$a=4,\quad r=\frac{1}{2}$$

$$S_\infty = \frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 4\times\frac{2}{1} = 8$$

$r=\frac{1}{2}$ $\quad \boxed{\left|\frac{1}{2}\right|<1}$