Lecture 3: Propositional Equivalence

Section 1.3

Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition which is always true.
  - Example: $\boldsymbol{p \lor \neg p}$
- A *contradiction* is a proposition which is always false.
  - Example: $\boldsymbol{p \land \neg p}$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as $\boldsymbol{p}$

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logical Equivalence: Examples

- Two compound propositions p and q are *logically equivalent* if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table shows that $\neg p \lor q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

Example: De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Example: Distributive Laws

$$(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$$

$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

This truth table shows that distributive law of disjunction over conjunction holds.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**

Predicates and Quantifiers

Section 1.4

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x)$, $M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain <u>variables</u> and a <u>predicate</u>, e.g., $P(x)$
  - Variables can be replaced by elements from their ***domain*** .

Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:
  - P(-3) is false.
  - P(0) is false.
  - P(3) is true.
- Often the domain is denoted by U. So in this example U is the integers.

Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers.
  Find these truth values:
  R(2,-1,5) **Solution: F**
  R(3,4,7) **Solution: T**
  R(x, 3, z) **Solution: Not a Proposition**

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:
  Q(2,-1,3) **Solution: T**
  Q(3,4,7) **Solution: F**
  Q(x, 3, z) **Solution: Not a Proposition**

Compound Expressions

- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution** : T

  $P(3) \wedge P(-1)$ **Solution** : F

  $P(3) \rightarrow P(-1)$ **Solution** : F

  $P(3) \rightarrow \neg P(-1)$ **Solution** : T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, **"**For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ means $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ means $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

  Examples :

  - If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
  - If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
  - If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples :

  - If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
  - If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
  - If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- When the domain is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
  - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples** :
  - If U is the positive integers and $P(x)$ is the statement "$x < 2$", then **$\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false** .
  - If U is the negative integers and $P(x)$ is the statement "$x < 2$", then **both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true** .
  - If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then **both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true** .
  - But if $P(x)$ is the statement "$x < 2$", then **both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false** .

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, \mathbf{(}P(x) \vee Q(x)\mathbf{)}$.

Translating from English to Logic

Example 1 : Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution :

First decide on the domain U.

Solution 1 : If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2 : But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2 : Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution :

First decide on the domain U.

Solution 1 : If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2 : But if U is all people, then translate as

$\exists x\, (S(x) \land J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example** : $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "*It is not the case that* every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

Translation from English to Logic

Examples :

1. "Some student in this class has visited Mexico."

   Solution : Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x \ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution : Add $C(x)$ denoting "x has visited Canada."

   $$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

Some Translating from English into Logical Expressions

- U = {fast, slow, turning}

  $F(x)$: x is fast

  $S(x)$: x is slow

  $T(x)$: x is turning

  Translate "Everything is fast."

  Solution: $\forall x\, F(x)$

Translation (cont)

- U = {fast, slow, turning}

  $F(x)$: x is fast

  $S(x)$: x is slow

  $T(x)$: x is turning

  "Nothing is slow."

Solution: $\neg \exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x \neg S(x)$

Translation (cont)

- U = {fast, slow, turning}

  F(x): *x* is fast

  S(x): *x* is slow

  T(x): x is turning

 "All fast things are slow."

Solution: $\forall x\ (F(x) \rightarrow S(x))$

Equivalent to: For all things, fast implies slow.

Translation (cont)

- U = {fast, slow, turning}

  F(x): *x* is fast

  S(x): *x* is slow

  T(x): x is turning

"Some fast things are turning."

Solution: $\exists x\, (F(x) \land T(x))$

Equivalent to: Some things are fast and turning

Translation (cont)

- U = {fast, slow, turning}

  $F(x)$: x is fast

  $S(x)$: x is slow

  $T(x)$: x is turning

"No slow thing is a turning."

Solution: $\neg \exists x\, (S(x) \wedge T(x))$ What is this equivalent to?

Solution: $\forall x\, (\neg S(x) \vee \neg T(x))$

Translation (cont)

- U = {fast, slow, turning}

  F(x): *x* is fast

  S(x): *x* is slow

  T(x): x is turning

"If anything fast is a slow then it is also a turning."

Solution: $\forall x\ ((F(x) \wedge S(x)) \rightarrow T(x))$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

- The first two are called *premises* and the third is called the *conclusion*.
  - "All lions are fierce."
  - "Some lions do not drink coffee."
  - "Some fierce creatures do not drink coffee."
- Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.
  - $\forall x\,(P(x) \rightarrow Q(x))$
  - $\exists x\,(P(x) \land \neg R(x))$
  - $\exists x\,(Q(x) \land \neg R(x))$
- Later we will see how to prove that the conclusion follows from the premises.