Lecture 2: Propositional Logic

Section 1.1 (Continued)

Implication

Implication

- If p and q are propositions, then $p \to q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example** : If p denotes "I am at home." and q denotes "It is raining." then $p \to q$ denotes "If I am at home then it is raining."
- In $p \to q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

An example

"If you try hard for your exam, then you will succeed".

p = you tried hard for your exam.
q = you succeed

Case 1: "You tried hard for your exam" and "you succeed"

p = True q = True

Compound proposition p -> q is True

Case 2: "You tried hard for your exam" but "you failed"

p = True q = False

Compound proposition p -> q is False.

An example

Case 3: "You haven't tried hard for your exam" and "You succeeded"

p = False q = True

Compound proposition p -> q is True. Why?
because you can make the compound proposition false only when you satisfy the first condition itself i.e. p. If that itself not satisfied then we cannot make compound proposition False. Not False means True.

Case 4: "You haven't tried hard for your exam" and "You failed"

p = False q = False

Compound proposition p -> q is True. (Same reason as above)

Homework Problem:

Determine whether each of these conditional statements is True or False.

1. If $1 + 1 = 3$, then dogs can fly.
2. If $1 + 1 = 2$, then dogs can fly.
3. If monkeys can fly, then $1 + 1 = 3$
4. If $1 + 1 = 2$, then $2 + 2 = 5$
5. If Delhi is the capital of India then Beijing is the capital of China.

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates."
  - "If the moon is made of green cheese then I'm on welfare."
  - "If $1 + 1 = 3$, then you will buy combat boots."

Understanding Implication (cont)

- Another way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \rightarrow q$

- **if** p**, then** q
- **if** p, q
- q **unless** $\neg p$
- q **if** p
- q **whenever** p
- q **follows from** p
- p **implies** q
- p **only if** q
- q **when** p
- q **if** p
- p **is sufficient for** q
- q **is necessary for** p

- **a necessary condition for** p **is** q
- **a sufficient condition for** q **is** p

How "if p then q" and "p only if q" can be same?

Example: "I will stay at home only if I'm sick."

Let p = "I will stay at home" and let q = "I'm sick"

Above statement is of the form p only if q

According to the above statement, becoming sick is the necessary condition that will make you stay at home.

This means "if you're not sick then, you cannot stay at home at any cost."

In order to falsify the above statement, q must be FALSE and p must be TRUE i.e. you are not sick and you still stay at home.

if p then q: "If I'll stay at home then I'm sick"

The only way to falsify the above statement is by making p TRUE and q FALSE. Therefore, p only if q is equivalent to if p then q

Converse, Contrapositive and Inverse

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example : Find the converse, inverse, and contrapositive of "Raining is a sufficient condition for me not going to town."

Solution:

converse : If I do not go to town, then it is raining.

inverse : If it is not raining, then I will go to town.

contrapositive : If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:

  - p **is necessary and sufficient for** q
  - **if** p **then** q **, and conversely**
  - p **iff** q

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- **Example** : Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example : Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with *n* propositional variables?

  Solution : 2^n

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
If the intended meaning is $p \vee (q \rightarrow \neg r)$
then parentheses must be used.