

# الجبر

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## Mathematical Induction principle (MIP)

\* Sum. Notation:  $\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$

$$\Rightarrow \sum_{i=1}^n K = K + K + K + \dots = nK$$

$$\Rightarrow \sum_{i=m_0}^{n_0} a_i = 0 \quad ; \quad m_0 > n_0$$

A method to prove some theorems contains an integer say (n) as "n" is positive

Consider  $C(n) \quad n \in \mathbb{N}$  is  $\Rightarrow$

①  $C(1)$  is true (base)

(Step) ② If  $C(K)$ ,  $K \in \mathbb{N}$  is true implies  
 $C(K+1)$  is also true

then  $C(n)$  is true  $\forall n \in \mathbb{N}$

4-17

Ex: Use MIP to prove the following

$$1 - \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

At  $n=1$ ; L.H.S =  $1^2 = 1$

R.H.S =  $\frac{1 \times 2 \times 3}{6} = 1$

$\therefore C(1)$  is true

Assume  $C(K)$ ;  $1^2 + 2^2 + \dots + K^2 = \frac{K \times (K+1) \times (2K+1)}{6}$

R.T.P:  $C(K+1) = 1^2 + 2^2 + \dots + (K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}$

L.H.S =  $1^2 + 2^2 + \dots + K^2 + (K+1)^2 = \frac{(K)(K+1)(2K+1)}{6} + (K+1)^2$

$$= \frac{(K+1)(2K^2 + K + 6K + 6)}{6}$$

$$= \text{R.H.S}$$

$$C(n) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}^n = \begin{bmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{bmatrix}$$

$$C(1) : \text{L.H.S.} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \text{R.H.S.} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

$$\text{Assume } C(k) : \begin{bmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{bmatrix}^k = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}$$

$$C(k+1) : \begin{bmatrix} \sin(k+1)x & -\sin(k+1)x \\ \sin(k+1)x & \cos(k+1)x \end{bmatrix}$$

$$\text{L.H.S.} = \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix}^k \cdot \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}^1$$

$$= \begin{pmatrix} \cos kx & -\sin kx \\ \sin kx & \cos kx \end{pmatrix} \cdot \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

$$= \begin{pmatrix} \cos kx \cos x + \sin kx \sin x & -\sin x \cos kx - \cos kx \sin x \\ \sin kx \cos x + \cos kx \sin x & -\sin x \sin kx + \cos x \cos kx \end{pmatrix}$$

$$= \begin{pmatrix} \cos(k+1)x & -\sin(k+1)x \\ \sin(k+1)x & \cos(k+1)x \end{pmatrix} = \text{R.H.S.}$$



Prove that  $2^n > n^2 \quad \forall n \geq 5$

At  $n=5$  :  ~~$(5)$~~

$$L.H.S = 2^5 = 32$$

$$R.H.S = 5^2 = 25$$

$\therefore C(5)$  is true

Assume that  ~~$2^k$~~   $2^k > k^2 \Rightarrow k \in \mathbb{N}$

R.T.P  $2^{(k+1)} > (k+1)^2$

$$L.H.S = 2^k \cdot 2 > 2k^2$$

$$2^{k+1} - (k+1)^2 > 2k^2 - (k+1)^2 = k^2 - 2k - 1$$

$$k = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2} \quad 2^{k+1} - (k+1)^2 > [k-1-\sqrt{2}][k-1+\sqrt{2}] > 0$$

$$\therefore 2^{k+1} > (k+1)^2$$

Prove that  $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24} \quad : n > 1$

At  $n=2$

$$L.H.S = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \times \frac{2}{2} = \frac{14}{24} > \frac{13}{24}$$

At  $n=k$

Assume that  $S_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}, \quad k > 1$

At  $n=k+1$

R.T.P:  $\frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24}$

$$S_{k+1} - S_k = \frac{1}{2k+2} + \frac{1}{2k+1} - \frac{1}{k+1} = \frac{2k+1 + 2k+2 - 2(2k+2)}{(2k+2)(2k+1)} = \frac{1}{(2k+2)(2k+1)}$$

$$\therefore S_{k+1} > S_k$$

$$\therefore S_{k+1} > \frac{13}{24}$$