COLLEGE
Tanta Engineering

إعدادي 2020

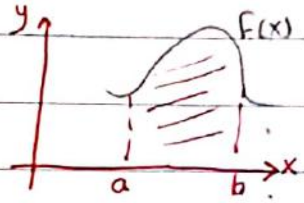
التكامل

طرق التكامل (تجزيء، تعويض)

م. إسراء شوقي

Integration

$$A = \int_a^b F(x)\, dx$$



أنواع التكامل

$\int_a^b$ المحدود

$\int + C$ الغير محدود

$* \int a\, f(x)\, dx = a \int f(x)\, dx$

$* \int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

(المجموعة الأولى)

[1] $\int \frac{F'(x)}{F(x)}\, dx = \ln|F(x)| + C$

[2] $\int (F(x))^n \cdot F'(x)\, dx = \frac{(F(x))^{n+1}}{n+1} + C$

[3] $\int \frac{F'(x)}{\sqrt{F(x)}}\, dx = 2\sqrt{F(x)} + C$

[4] $\int e^{F(x)} F'(x)\, dx = e^{F(x)} + C$

[5] $\int a^{F(x)} F'(x)\, dx = \frac{a^{F(x)}}{\ln a} + C$

$* \int \frac{x}{\sqrt{x+1}}\, dx \rightarrow \int \frac{x+1-1}{\sqrt{x+1}} \rightarrow \int \frac{x+1}{\sqrt{x+1}} - \int \frac{dx}{\sqrt{x+1}}$

$\int (x+1)^{\frac{1}{2}}\, dx - \int (x+1)^{-\frac{1}{2}}\, dx \rightarrow \frac{(x+1)^{\frac{3}{2}}}{3/2} - \frac{(x+1)^{\frac{1}{2}}}{1/2} + C$

[2] $\int \sec^5 x \tan x\, dx$

$\int \sec^4 x \underbrace{\tan x \sec x}_{d(\sec)} = \frac{\sec^5 x}{5} + C$

[3] $\int \sec x\, dx \quad * \frac{\sec x + \tan x}{\sec x + \tan x}$

$I = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx = \ln|\sec x + \tan x| + C$

[4] $\int \tan\, dx$

$= -\int \frac{-\sin x}{\cos x} = -\ln|\cos x| + C$

[5] $\int \cos^4 x\, dx \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$I = \int (\cos^2 x)^2\, dx = \int \left[\frac{1}{2} + \frac{1}{2}\cos 2x\right]^2 dx = \int \left[\frac{1}{4} + \frac{1}{4}\cos^2 2x + \frac{1}{2}\cos 2x\right]$

$I = \int \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) + \frac{1}{2}\cos 2x\, dx$

$I = \int \left[\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x\right] dx = \frac{1}{4}x + \frac{1}{8}x + \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + C$

* Integration By Substitution *

[1] $\int \frac{x^m}{(ax+b)^n}dx$ let $z = ax+b$

[2] $\int \frac{dx}{x^m(ax+b)^n}$ let $z = \frac{ax+b}{x}$

[3] $\int \frac{dx}{x(ax^n+b)}$ let $\frac{1}{z} = x^n$

[4] $\int \frac{\text{البسط}}{\text{المقام من مصغير معين}}$ let $z =$ المتغير

$\tan, \cos, \sin, \tanh, \cosh, \sinh x$

[5] $\int \sqrt{ax+b}\,dx$ $z^2 = ax+b$

[6] $\int \sqrt{a^2-x^2}\,dx$ $x = a\cos\theta$ $\sin^2+\cos^2=1$ $1+\tan^2=\sec^2$

$a^2-a^2\cos^2\theta \to a^2(1-\cos^2\theta) \to \sqrt{a^2\sin^2\theta} = a\sin\theta$

[7] $\int \sqrt{a^2+x^2}\,dx$ let $x = a\tan\theta$

$\sqrt{a^2+a^2\tan^2\theta} \to \sqrt{a^2(1+\tan^2\theta)} \to \sqrt{a^2\sec^2} \to a\sec$

[8] $\int \sqrt{x^2-a^2}\,dx$ let $x = a\sec\theta$

$\sqrt{a^2\sec^2\theta - a^2} \to \sqrt{a^2(\sec^2\theta-1)}$

$\sqrt{a^2\tan^2} = a\tan\theta$

[9] $\int \sqrt{9x+b} + \sqrt[3]{ax+b}\,dx$

$z^6 = ax+b$

[10] $\int \sqrt{9x+b} + \sqrt[4]{ax+b}\,dx$ $z^4 = ax+b$

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[1] $\int \frac{x^3}{(2+3x)^4}dx$ let $z = 2+3x \Rightarrow x = \frac{z-2}{3} \Rightarrow dx = \frac{dz}{3}$

$I = \int \frac{(\frac{z-2}{3})^3}{z^4}\frac{dz}{3} = \frac{1}{81}\int \frac{(z-2)^3}{z^4}dz$

$(a+b)^n = a^n + n a^{n-1}b + \frac{(n)(n-1)}{2!}a^{n-2}b^2 + \dots + b^n$ n موجب ، عدد الحدود $= n+1$

$(z-2)^3 = z^3 + 3z^2(-2) + \frac{3*2}{2!}z(-2)^2 + (-2)^3 \Rightarrow (z-2)^3 = z^3-6z^2+12z-8$

$I = \frac{1}{81}\int \frac{z^3-6z^2+12z-8}{z^4}dx = \frac{1}{81}\left[\int \frac{1}{z} - 6z^{-2} + 12z^{-3} - 8z^{-4}\,dz\right.$

$= \frac{1}{81}\left[\ln|z| - \frac{6z^{-1}}{-1} + \frac{12z^{-2}}{-2} - \frac{8z^{-3}}{-3}\right] + c$ # $z = 2+3x$ يا رجاله

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[2] $\int \frac{\sin 2x}{\sqrt{1+\sin x}}dx$ let $z = \sin x \Rightarrow dz = \cos x\,dx \Rightarrow dx = \frac{dz}{\cos x}$

$\sin 2x = 2\sin x\cos x = 2z\cos x$

$I = \int \frac{2z\cos x}{\sqrt{1+z}}\frac{dx}{\cos x} = 2\int \frac{z}{\sqrt{z+1}}dz = 2\int \frac{z+1-1}{\sqrt{z+1}} = 2\left[\int \frac{z+1}{\sqrt{z+1}} - \int \frac{dz}{\sqrt{z+1}}\right]$

$2\int (z+1)^{\frac{1}{2}} - \int (z+1)^{-\frac{1}{2}} = 2\left[\frac{(z+1)^{\frac{3}{2}}}{3/2} - \frac{(z+1)^{\frac{1}{2}}}{1/2}\right]$

آخر خطوة دائماً الجدول

[3] $\int \frac{\cos x}{\sin^2 x + 2\sin x + 1}\,dx$ let $z = \sin x \Rightarrow dz = \cos x\,dx \Rightarrow dx = \frac{dz}{\cos x}$

$I = \int \frac{\cos x}{z^2 + 2z + 1}\,\frac{dz}{\cos x} = \int \frac{dz}{(z+1)^2} = \int (z+1)^{-2}\,dz = \frac{(z+1)^{-1}}{-1} + C$

[4] $\frac{dx}{x^2\sqrt{x^2+9}}$ $1 + \tan^2\theta = \sec^2\theta \rightarrow x = 3\tan\theta \rightarrow dx = 3\sec^2\theta\,d\theta$

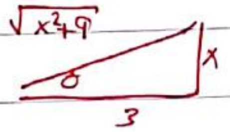
$I = \int \frac{3\sec^2\theta\,d\theta}{9\tan^2\theta\sqrt{9\tan^2\theta + 9}} = \int \frac{3\sec^2\theta\,d\theta}{9\tan^2\theta \cdot 3\sec\theta}$

$I = \int \frac{\sec\theta}{9\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\sin^2\theta/\cos^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \sin^{-2}\theta\cos\theta\,d\theta$

$= \frac{1}{9}\int [\sin\theta]^{-2}\cos\theta\,d\theta = \frac{1}{9}\,\frac{[\sin\theta]^{-1}}{-1} + C$

$\tan\theta = \frac{\text{المقابل}}{\text{المجاور}} = \frac{x}{3}$

$I = \frac{-1}{9\sin\theta} + C \Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+9}}$



$I = -\frac{\sqrt{x^2+9}}{x} + C$

[5] $\int (1-x^2)^{3/2}\,dx$

$I = \int (1-x^2)^1 (1-x^2)^{\frac{1}{2}}\,dx = \int \sqrt{1-x^2}\,(1-x^2)\,dx$ $\boxed{x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta}$

$dx = \cos\theta\,d\theta$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$I = \int \sqrt{1-\sin^2\theta}\,(1-\sin^2\theta)\,\cos\theta\,d\theta = \int \cos\theta\cos^2\theta\cos\theta\,d\theta$

$I = \int \cos^4\theta\,d\theta = \int \left[\frac{1}{2} + \frac{1}{2}\cos 2\theta\right]^2 d\theta = \int \left[\frac{1}{4} + \frac{1}{4}\cos^2 2\theta + \frac{1}{2}\cos 2\theta\right] d\theta$

$= \int \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) + \frac{1}{2}\cos 2\theta]\,d\theta \Rightarrow I = \int \left[\frac{1}{4} + \frac{1}{8} + \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta\right] d\theta$

$I = \frac{1}{4}\theta + \frac{1}{8}\theta + \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + C$ $\theta = \sin^{-1} x$ ...

* Integration By parts *

$\int u\,dv = uv - \int v\,du$

u ↓ سهلة التفاضل $\sin^{-1}x$, $\sinh^{-1}x$, $\ln x$

dv سهلة التكامل e^x كثيرة الحدود

[1] $\int x\cos x\,dx$ $u = x$ $dv = \cos x\,dx$

$du = dx$ $v = \sin x$

$-\int$

$I = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

2) $\int x^5 \ln x^9 \, dx$

$I = \int 9x^5 \ln x \, dx$

$u = \ln x \quad dv = 9x^5$

$du = \frac{dx}{x} \quad v = \frac{9x^6}{6}$

$I = \frac{9x^6}{6} \ln x - \int \frac{9x^5}{6} dx = \frac{9x^6}{6} - \frac{9x^6}{36} + C$

3) $I = \int \sec x \tan^2 x \, dx$

$u = \tan x \quad dV = \sec x \tan x$

$du = \sec^2 x \, dx \quad V = \sec x$

$\tan x \sec x - \int \sec x \cancel{\tan x} \sec^2 x \, dx$

$I = \sec x \tan x - \int \sec x (1 + \tan^2 x) \, dx = \sec x \tan x - \int \sec x \, dx - \int \sec x \tan^2 x \, dx$

$2I = \sec x \tan x = -\ln|\sec x + \tan x|$

$I = [\sec x \tan x - \ln| \quad | \quad] \frac{1}{2} + C$