Course Title: Math 1(a)

Course Code: PME 0101

ALGEBRA

Lecture 1

Partial Fractions

Definition

We know that $\frac{4}{x+1}+\frac{-3}{x+2}=\frac{x+5}{(x+1)(x+2)}$. This is known as reduction of two or more functions. The partial fraction process is the opposite process of this one.

Partial Fractions Techniques

Consider the rational function $f(x)=\frac{P(x)}{Q(x)}$ with the degree of the polynomial $P(x)$ is less than the degree of Q(x). If not we can perform long division to obtain a rational function with degree of numerator less than the degree of denominator.

Analyze the denominator. Its roots are one of the following four cases:

Case 1: If all factors of the denominator are distinct (different) and of 1st degree.

Consider the fraction

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}.$$

We put

$$\frac{P(x)}{(x-x_1)(x-x_2)(x-x_3).....(x-x_n)}=\frac{A_1}{(x-x_1)}+\frac{A_2}{(x-x_2)}+.....+\frac{A_n}{(x-x_n)}$$

Where $A_1, A_2, ...A_n$ are constants.

If we unify denominators in the R.H.S and equate the two numerators we get:

$$P(x)=A_1(x-x_2).....(x-x_n)+A_2(x-x_1)(x-x_3)+.....(x-x_n)$$
$$+......+A_n(x-x_1).......(x-x_{n-1}) \text{ .}$$

We can get the constant $A_1, A_2, ...A_n$ by either comparing the coefficients in both sides or by assuming n different values for x in both sides.

Note:

These constants can be easily obtained from the law:

$$A_i=\lim_{x\to x_i}(x-x_i)\frac{P(x)}{Q(x)}.$$

Example 1

Analyze the fraction $\frac{5x+1}{(x+2)(x-1)}$ to its partial fractions.

Solution

In this example the fraction $\frac{5x+1}{(x+2)(x-1)}$ is proper.

Then we have to put it in the following form:

$$\frac{5x+1}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)},$$

$$A=\lim_{x\to -2}\frac{5x+1}{x-1}=3 \;\&\; B=\lim_{x\to 1}\frac{5x+1}{x+2}=2.$$

$$\therefore \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x-1} + \frac{2}{x+2}.$$

Case 2:If the factors of the denominator are of the 1st degree but some of them are repeated.

Consider the fraction

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a)^k}$$

We put: $$\frac{P(x)}{(x-a)^k} = \frac{A_1}{(x-a)^k} + \frac{A_2}{(x-a)^{k-1}} + \dots + \frac{A_k}{(x-a)}$$

A_1 can be determined directly as in the example. But the other constants are to be determined the other two methods mentioned above.

Example 2

Analyze $\dfrac{2x+5}{(x-3)(x-1)^3}$ to its partial fractions.

Solution

$$\frac{2x+5}{(x-3)(x-1)^3} = \frac{A}{(x-3)} + \frac{b}{(x-1)} + \frac{c}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$A = \lim_{x\to 3}\left[\frac{2x+5}{(x-1)^3}\right] = \frac{11}{8} \quad , \quad D = \lim_{x\to 1}\left[\frac{2x+5}{x-3}\right] = -\frac{7}{2} .$$

Unify the denominator in R.H.S and equate the two numerators in both sides as follows:

$$2x+5 = A(x-1)^3 + B(x-3)(x-1)^2 + C(x-3)(x-1) + D(x-3) .$$

C.O.$x^3: 0 = A + B \Rightarrow B = -A = -\dfrac{11}{8}$.

Put $x = 0 \Rightarrow 5 = -A - 3B + 3C - 3D \Rightarrow C = -\dfrac{11}{4}$.

Thus,

$$\frac{2x+5}{(x-3)(x-1)^3} = \frac{11}{8(x-3)} - \frac{11}{8(x-1)} - \frac{11}{4(x-1)^2} - \frac{7}{2(x-1)^3}$$

Case 3: If the factors of the denominator are of the 2nd degree and not repeated

Consider the fraction $\dfrac{P(x)}{Q(x)} = \dfrac{P(x)}{ax^2 + bx + c}$,

When $Q(x) = ax^2 + bx + c$ cannot be factorized to real factors of the1st degree. We put the fraction $\dfrac{Ax + B}{ax^2 + bx + c}$ corresponds to the factor $ax^2 + bx + c$, where A.B are constants.

Example 3

Analyze the fraction $\dfrac{1}{(x+1)(x^2+1)}$ to its partial fractions.

Solution

At first we notice that the given fraction is proper and the factor $x^2 + 1$ in the denominator cannot be factorized to real So the fraction can be put in the form:

$$\frac{1}{(x+1)(x^2+1)}=\frac{1}{(x+1)}+\frac{Bx+C}{(x^2+1)} \qquad (1)$$

Unify the denominators and equate the numerator in both side of (1) we get:

$$1=A(x^2+1)+(Bx+C)(x+1) \qquad (2)$$

Put x = -1 in (2) we have: $1=2A \,\&\, A=\frac{1}{2}$

Put x = 0 in (2) we have $1=A+C \,\&\, C=\frac{1}{2}$

Put x = 1 in (2) we have $1=2A+2B+2C \,\&\, B=-\frac{1}{2}$

Remark

The constants $A.B$ and C can be calculated as follows:

Comparing the absolute terms in both sides of (2) we have

$1 = A + C$ (3)

Coefficients of x^2 we have

$0=A + B$ (4)

Coefficients of x we have

$0 = B + C$ (5)

Solving the three equations (3),(4) and (5) we have the constants A,B and C.

Case 4: If the factors of the denominator are of the 2nd degree and repeated (or some of them are repeated)

Consider the fraction $\frac{P(x)}{Q(x)}=\frac{P(x)}{(ax^2+bx+c)^k}$ or

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-x_1)(ax^2+bx+c)^k}$$

The factor $(ax^2+bx+c)^k$ generates the K fractions

$$\frac{A_1x+A2}{(ax^2+bx+c)^k}+\frac{A_3x+A2}{(ax^2+bx+c)^{k-1}}+......+\frac{A_{2k-1}x+A2}{(ax^2+bx+c)^k}$$

Example 4

Analyze the fraction $\frac{x^2}{(1-x)(1+x^2)^2}$ to its partial fractions.

Solution

$$\frac{x^2}{(1-x)(1+x^2)^2}+\frac{A}{(1+x)}+\frac{Bx+C}{(1+x^2)^2}+\frac{Dx+E}{(1+x^2)}.$$

Unify the denominators and then equate the numerator in both sides we get:

$$x^2=A(1+x^2)^2+(Bx+c)(1+x)+(Dx+E)(1+x)(1+x^2)$$

Put x = -1 in (1) we have $A=\frac{1}{4}$.

Equate coefficient of x^4 in both sides of (1) we have:

$$0=A+D\Rightarrow D=-\frac{1}{4}.$$

Equate coefficient of x^3 in both sides of (1) we have:

$$0=D+E\Rightarrow E=\frac{1}{4}.$$

Equate the absolute terms in both side of (1) we have:

$$0=A+C+E\Rightarrow C=\frac{1}{2}.$$

Equate coefficient of x in both sides of (2) we have

$$0 = B + C + D + E \Rightarrow B = \frac{1}{2}.$$

Thus the given fraction can be written in the form:

$$\frac{x^2}{(1+x)(1+x^2)} = \frac{1}{4(1+x)} + \frac{x-1}{2(1+x^2)^2} + \frac{x-1}{4(1+x^2)}.$$

Example 5

Analyze the fraction $\dfrac{x^4}{x^3+1}$ to its partial fractions.

Solution

The given fraction is not proper.

Performing the long division we have $\dfrac{x^4}{x^3+1} = x - \dfrac{x}{x^3+1}$

We have to analyze the fraction $\dfrac{x}{x^3+1}$ to its partial fractions

$$\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

Unify the denominators and equate the numerator in both sides of (2) we have:

$$x = A(x^2-x+1) + (Bx+C)(x+1).$$

Equate the absolute terms in both sides:

$$0 = A + C.$$

Equate the coefficients of x^2 in both sides:

$$0 = A + B$$

Equate the coefficients of x in both sides:
1= - A + B + C.
A = - C.
Substitute we get:

1= - 2A + B . $A=-\frac{1}{3}$, $C=\frac{1}{3}$ & $B=\frac{1}{3}$

Thus we have

$$\frac{x^4}{x^3+1}=x+\frac{1}{3(x+1)}-\frac{x+1}{3(x^2-x+1)}.$$

Example 6

Analyze the fraction $\frac{x^3-2}{(x^2+x+2)^2(x^2+x+1)}$ to its partial fractions.

Solution

$$\frac{x^3-2}{(x^2+x+2)^2(x^2+x+1)}=\frac{ax+b}{(x^2+x+2)^2}+\frac{cx+d}{x^2+x+2}+\frac{ex+f}{x^2+x+1}$$

We have:

$$x^3-2=(ax+b)(x^2+x+1)+(cx+d)(x^2+x+2)(x^2+x+1)$$
$$+(ex+f)(x^2+x+2)^2$$

If we want to obtain the two constants e & f , we let
$x^2+x+1=0$
$(ex+f)(1)^2=x(-x-1)-2=-(x^2+x)-2=1-2=-1\Rightarrow$
$e=0$ & $f=-1$.

Similarly, If we want to obtain the two constants e & f , we let

$$x^2+x+2=0$$
$$(ax+b)(-1)=x(-x-2)-2=-x^2-2x-2=x-2x=-x\Rightarrow$$
$$x^2-x-2\Rightarrow a=1\&b=0.$$

For obtaining $c\&d$ compare the coefficients of both $x^5\&x^0$ we get $c=0\&d=1$.

Thus we have:

$$\frac{x^3-2}{(x^2+x+2)^2(x^2+x+1)}=\frac{x}{(x^2+x+2)^2}+\frac{1}{x^2+x+2}-\frac{1}{x^2+x+1}$$

Example 7

Analyze the fraction $\dfrac{x^2-2}{(x^2+2)(x^2+1)(x^2+4)}$ to its partial fractions.

Solution

Put $x^2=y$, then the fraction becomes $\dfrac{y-2}{(y+2)(y+1)(y+4)}$ and we can easy write

$$\frac{y-2}{(y+2)(y+1)(y+4)}=\frac{a}{y+2}+\frac{b}{y+1}+\frac{c}{y+4}\text{, where}$$

$$a=\frac{-2-2}{(-2+1)(-2+4)},\quad b=\frac{-1-2}{(-1+2)(-1+4)}\ \&$$
$$c=\frac{-4-2}{(-4+1)(-4+2)}.$$

Example 8

Expand $f(x)=\dfrac{2x+3}{x^2-4x+3}$ near $x=2$. Find x that make the expansion is true.

Solution

$$\frac{2x+3}{x^2-4x+3}=\frac{2x+3}{(x-3)(x-1)}=\frac{a}{x-3}+\frac{b}{x-1}.$$

$$a=\frac{6+3}{3-1}=\frac{9}{2},\ b=\frac{2+3}{1-3}=-\frac{5}{2}\Rightarrow$$

$$\frac{2x+3}{x^2-4x+3}=\frac{\frac{9}{2}}{x-3}+\frac{-\frac{5}{2}}{x-1}=\frac{-5}{2x(1-\frac{1}{x})}-\frac{3}{2}\frac{1}{1-\frac{x}{3}}.$$

$$=\frac{-5}{2x}(1-\frac{1}{x})^{-1}-\frac{3}{2}(1-\frac{x}{3})^{-1}$$

$$=\frac{-5}{2x}[1+\frac{1}{x}+\frac{1}{x^2}+...]-\frac{3}{2}[1+\frac{x}{3}+\frac{x^2}{9}+...]$$

$$=\frac{-5}{2x}-\frac{5}{2x^2}-\frac{5}{2x^3}-...-\frac{3}{2}-\frac{x}{2}-\frac{x^2}{6}-....$$

Condition for the truth of expansion is

$$\left|\frac{x}{3}\right|<1\ \&\ \left|\frac{1}{x}\right|<1\Rightarrow|x|<3\&|x|>1\Rightarrow$$

$-3 < x < -1 \; or \;\; 1 < x < 3.$

Example 9

Deduce the coefficient of x^n in $f(x) = \dfrac{1}{1-5x+6x^2}$.

Solution

$$f(x) = \frac{1}{1-5x+6x^2} = \frac{2}{2x-1} + \frac{-3}{3x-1}.$$

$$f(x) = 3(1-3x)^{-1} - 2(1-2x)^{-1}$$

$$= 3(\ldots + 3^n x^n + \ldots) - 2(\ldots + 2^n x^n + \ldots).$$

$$C.o.x^n = 3(3)^n - 2(2)^n = 3^{n+1} - 2^{n+1}.$$

Example 10

Deduce the coefficient of x^n in $f(x) = \dfrac{1+x}{(1-x)^3}$.

Solution

$$f(x) = \frac{1+x}{(1-x)^3} = \frac{a=2}{(1-x)^3} + \frac{b=-1}{(1-x)^2} + \frac{c=0}{1-x} \;.$$

$$f(x) = 2(1-x)^{-3} - (1-x)^{-2}$$

$$= 2(\ldots + \frac{3.4.5\ldots(n+2)}{n!} x^n + \ldots) - (\ldots + (n+1)x^n + \ldots).$$

$$C.o.x^n = 2\frac{3.4.5...(n+2)}{n!} - (n+1) = \frac{(n+2)!}{n!} - (n+1)$$
$$= (n+2)(n+1) - (n+1) = (n+1)^2.$$

Mathematical Induction

Mathematical induction is a method used to prove many important laws and Mathematical relations that include certain symbol (say n) takes positive integer values.

This method is preferred to use it to prove the laws and mathematical relations which are true for all positive integers.

Techniques of mathematical induction method

1- Prove the relation when $n = 1$ (the base).

2- Assume that the relation is true for fixed value of n (say $n = k$) and prove it for $n = k + 1$ (the step).

Example 1

Use the mathematical induction method to prove that

$$1^2 + 2^2 + 3^2 ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution

1- Induction base

We try to prove the relation when $n=1$

L.H.S $= 1 = 1$.

R.H.S. $=\frac{1}{6}\times 1\times 2\times 3 = 1$.

Thus the relation is true for $n=1$.

2- Induction step

a- Assume that the relation is true for $n=k$ i.e. let

$$1^2+2^2+3^2 ... + k^2 = \frac{1}{6}k(k+1)(2k+1). \qquad (1)$$

b- We try to prove that the relation is true for $n=k+1$ or

$$1^2+2^2+3^2 ... + k^2 + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3). \qquad (2)$$

Using the relation (1) we have:

$$L.H.S. = 1^2+2^2+3^2 ... + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) = R.H.S.$$

The relation is true for $n=k+1$ assuming that it is true at $n=k$.

And since it is true at $n=1$, then it is true fore every n.

Example 2

Using the mathematical induction method to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!}.$$

Solution

At $n=1$

$$L.H.S.=\frac{1}{2!}=\frac{1}{2},\quad R.H.S.=1-\frac{1}{2}=\frac{1}{2}\Rightarrow L.H.S.=R.H.S.$$

Let

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}\ .$$

The required is to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}.$$

$$L.H.S.=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}+\frac{k+1}{(k+2)!}$$

$$=1-\frac{(n+2)-(n+1)}{(n+2)!}=1-\frac{1}{(n+2)!}=R.H.S.$$

The relation is true for $n=k+1$ assuming that it is true at $n=k$.

And since it is true at $n=1$, then it is true fore every n .

Example 3

Prove that $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^n=\begin{pmatrix}1 & nx\\ 0 & 1\end{pmatrix}$.

Solution

At $n=1$

$$L.H.S.=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=R.H.S..$$

Let

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & kx \\ 0 & 1 \end{pmatrix}.$$

Required to prove $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & (k+1)x \\ 0 & 1 \end{pmatrix}.$

$$L.H.S. = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k . \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & kx \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x+kx \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (k+1)x \\ 0 & 1 \end{pmatrix} = R.H.S.$$

Example 4

Prove that if $y = \dfrac{1}{ax+b}$, then $\dfrac{d^n y}{dx^n} = (-1)^n \dfrac{a^n\, n!}{(ax+b)^{n+1}}$.

Solution

At $n = 1$

$$L.H.S. = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{-a}{(ax+b)^2}.$$

$$R.H.S. = \frac{(-1)^1 a^1\, 1!}{(ax+b)^2} = \frac{-a}{(ax+b)^2}.$$

Hence the law is verified at $n = 1$.

Assume that $\dfrac{d^k y}{dx^k} = (-1)^k \dfrac{a^k\, k!}{(ax+b)^{k+1}}$, and prove that

$$\frac{d^{k+1} y}{dx^{k+1}} = (-1)^{k+1} \frac{a^{k+1}\,(k+1)!}{(ax+b)^{k+2}}.$$

$$L.H.S. = \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right) = \frac{d}{dx}\left[\frac{(-1)^k a^k\, k!}{(ax+b)^{k+1}}\right.$$

$$=(-1)^k a^k k!\frac{(-)(k+1)a}{(ax+b)^{k+2}}=\frac{(-1)^{k+1}a^{k+1}(k+1)!}{(ax+b)^{k+2}}=R.H.S..$$

Example 5

Use the mathematical induction to prove that:

$2^n > n^2, \ \forall n \geq 5.$

Solution

At $n=5$

$L.H.S.=2^5=32, \quad R.H.S.=5^2=25 \Rightarrow L.H.S.>R.H.S.$

Then the relation is true at $n=5$.

Let $\quad 2^k > k^2, \ \forall k \geq 5.$

Prove that:

$2^{k+1} > (k+1)^2, \ \forall k \geq 5.$

$2^{k+1} = 2\,2^k > 2k^2.$

$2^{k+1}-(k+1)^2 > 2k^2-(k+1)^2 = k^2-2k-1.$

But $k^2 > 2k+1$ since $k > 2+\frac{1}{k} \ \forall k \geq 3 \Rightarrow$

$2^{k+1}-(k+1)^2 > 2k^2-(k+1)^2 = k^2-2k-1 > 0.$

Thus $2^{k+1} > (k+1)^2, \ \forall k \geq 5.$

The theorem is true for $n=k+1$ assuming that it is true at $n=k$.

And since it is true at $n=5$, then it is true for every $n \geq 5$.

Example 6

Prove that, for any natural number $n > 1$,

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}.$$

Solution

The base: At $n = 2$

$S_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} = \frac{14}{24} > \frac{13}{24}$. The theorem is true for $n = 2$.

The step:

Assume that

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}.$$

The required to prove is:

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{13}{24}.$$

$$S_{k+1} - S_k = \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} = \frac{1}{2(k+1)(2k+1)}.$$

For any natural number $k > 1$,

$$\frac{1}{2(k+1)(2k+1)} > 0 \Rightarrow S_{k+1} > S_k.$$

But $S_k > \frac{13}{24} \Rightarrow S_{k+1} > \frac{13}{24}$.

The theorem is true for $n = k+1$ assuming that it is true at $n = k$. And since it is true at $n = 5$, then it is true for every $n > 1$.

Lecture 3

Binomial Theorem

Definition

Binomial Theorem is a law which gives the expansion of an algebraic expression of two terms raised to any power without multiplication. We studied before the case, when the power is a positive integer.

Binomial Theorem With n is a positive integer

$$(x+a)^n = x^n + c_1^n x^{n-1}a + c_2^n x^{n-2}a^2 \ldots + c_r^n x^r a^{n-r} + \ldots a^n \,,$$

where $c_r^n = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, $c_r^n = c_{n-r}^n$, $c_0^n = c_n^n = 1$,

Example 1

Expand $(x+2)^5$ in powers of x .

Solution

$$(x+2)^5 = x^5 + 5x^4.2^1 + \frac{(5.4)}{(1.2)}x^3.2^2 + \frac{(5.4.3)}{(1.2.3)}x^2.2^3 + \frac{(5.4.3.2)}{(1.2.3.4)}x.2^4 + 2^5$$

$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32.$$

Binomial Theorem with any Exponent

If n is negative or fraction it can be proved that:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots$$
$$\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}x^r + \ldots$$

Proof

In this proof we make use of Taylor's theorem. This theorem based on the knowledge of the function at only one point (the value of the function and its derivatives at this point) and it states that:

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots$$

$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r+\dots.$$

Now, let $f(x)=(1+x)^n$, $\quad f(0)=1$,

$f'(x)=n(1+x)^{n-1}$, $\quad f'(0)=n$,

$f''(x)=n(n-1)(1+x)^{n-2}$, $\quad f''(0)=n(n-1)$,

$f'''(x)=n(n-1)(n-2)(1+x)^{n-3}$, $\quad f'''(0)=n(n-1)(n-2)$,

$f^{(r)}(x)=n(n-1)(n-2)\dots(n-r+1)(1+x)^{n-r}$,

$f^{(r)}(0)=n(n-1)(n-2)\dots(n-r+1)$.

Substituting, we get the required theorem.

It is true when n is a positive integer in which case the number of terms is $n+1$ (as we know before).

And is also true when n is negative integer, positive or negative fraction but only if $|x|<1$ (condition of convergence of the series or condition for the expansion to be true).

Applying the theorem we can obtain:

$$(1+x)^{-1}=1+(-1)(-x)+\frac{(-1)(-2)}{1.2}(-x)^2+\frac{-1-2-3}{1\,2\,3}(-x)^3+\ldots$$

$$=1+x+x^2+x^3+x^4+\ldots \qquad if\ |x|<1$$

$$(1+x)^{-1}=1+(-1)x+\frac{(-1)(-2)}{1.2}x^2+\frac{(-1)\,(-2)\,(-3)}{1.2.3}x^3+\ldots$$

$$=1-x+x^2-x^3+x^4+\ldots. \qquad if\ |x|<1$$

$$(1-x)^{-2}=1+(-2)(-x)+\frac{(-2)(-3)}{1.2}x^2+\frac{(-2)\,(-3)\,(-4)}{1.2.3}x^3+\ldots$$

$$=1-2x+3x^2+4x^3+\ldots \qquad if\ |x|<1$$

$$(1+x)^{-2}=1+(-2)(x)+\frac{(-2)(-3)}{1.2}x^2+\frac{(-2)\,(-3)\,(-4)}{1.2.3}x^3+\ldots$$

$$=1-2x+3x^2+4x^3+\ldots \qquad if\ |x|<1$$

Example 2

Expand $(1+x)^{-3}$.

Solution

$$(1+x)^{-3}=1+(-3)x+\frac{-3.-4}{1.2}x^2+\frac{-3.-4.-5}{1.2.3}x^3+\ldots$$

$$=1-3x+6x^2-10x^3+\ldots \qquad if\ |x|<1$$

Applications of Binomial Theorem

1) Approximations

When x is small compared to 1 then $x^2, x^3, \ldots$ are very small and can be neglected in the expansion and the 1st two terms of the expansion are sufficient for practical applications, i.e. we can write

$$(1+x)^n = 1+nx$$

$$(1-x)^n = 1-nx$$

$$(1\pm x)^n(1\pm y)^m = 1\pm nx \pm my$$

$$\frac{(1\pm x)^n}{(1\pm y)^m} = 1\pm nx \pm my$$

When x & y are small

Example 4

Find $\sqrt{101}$ using the expansion to four terms and compare with using the approximate rules.

Solution

$$(101)^{1/2} = (100+1)^{1/2} = 10(1+\frac{1}{100})^{1/2}$$

$$(101)^{1/2} = 10(1+.01)^{1/2}$$

$$= 10[1+\frac{1}{2}.01+\frac{(1/2).(-1/2)}{1.2}(.01)^2$$

$$+\frac{(1/2).(-1/2)(-3/2)}{1.2.3}(.01)^3]$$

$$= [10+.005-\frac{1}{8}.0001+\frac{1}{16}.000001]$$

$$= [10+.005-.0000125+.0000000625]$$

$$= 10\ [1.0050000625-.0000125]$$

$$= 10.04987562.$$

Using the approximate law

$$(101)^{1/2} = (100+1)^{1/2} = 10(1+.01)^{1/2}$$

$$= 10[1 + \frac{1}{2}(.01)]$$

$$10 + .05 = 10.05.$$

Example 5

Find an approximate value for $\sqrt{\frac{97}{101}}$ correct to 5 decimals.

Solution

$$(\frac{97}{101})^{1/2} = (\frac{101\text{-}4}{101})^{1/2} = (1 - \frac{4}{101})^{1/2}$$

$$= 1 - \frac{1}{2} * \frac{4}{101} + \frac{1/2.-1/2}{1.2}(\frac{4}{101})^{3}$$

$$= 1 - .0198020 - .0001961 - .0000039$$

$$= .9799980$$

$$= .98000 \qquad \text{correct to 5 decimals.}$$

2) Evaluation the sum of some infinite series

Example 6

Comparing with the binomial expansion, show that

$$S = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots = \sqrt{3} - 1.$$

Solution

$$S = \frac{1}{3} + \frac{(-1)(-3)}{1.2}(\frac{-1}{3})^{2} + \frac{(-1)(-3)(-5)}{1.2.3}(\frac{-1}{3})^{3} + \ldots$$

$$S + 1 = 1 + \frac{1}{3} + \frac{(\frac{-1}{2})(\frac{-3}{2})}{1.2}(\frac{-1}{3})^{2} + \frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{1.2.3}(\frac{-1}{3})^{3} + \ldots$$

$$\therefore S+1=(1+(\frac{-2}{3}))^{\frac{-1}{2}}=\sqrt{3}\Rightarrow\therefore S=\sqrt{3}-1.$$

Example 7

Evaluate the sum of the series

$$S=\frac{1}{5}-\frac{1.4}{5.10}+\frac{1.4.7}{5.10.15}-\ldots\quad.$$

Solution

$$S=-[\frac{-1}{3}(\frac{3}{5})+\frac{(\frac{-1}{3})(\frac{-4}{3})}{1.2}(\frac{3}{5})^2-\frac{(\frac{-1}{3})(\frac{-4}{3})(\frac{-7}{3})}{1.2.3}(\frac{3}{5})^3+\ldots]$$

$$=1-(1+\frac{3}{5})^{\frac{-1}{3}}=1-\sqrt[3]{\frac{5}{8}}.$$

2-3.3 Finding coefficient of x^n in the expansion

Example 8

Find the coefficient of x^n in the expansion of $(\frac{1+x}{1-x})^2$.

Solution

$$(\frac{1+x}{1-x})^2=(1+x)^2(1-x)^2=(1+2x+x^2)(1+2x+3x^2+\ldots+(n+1)x^n+\ldots)$$

c.o. x^n = (n+1) + 2(n) + 1(n – 1) = 4n, $n\geq 2$,

whereas c.o. x^0, x^1 can be estimated individually such as:

c.o. $x^0=(1)(1)=1,\qquad c.o.x=2+2=4.$

Example 9

Find the coefficient of x^n in the expansion of $\dfrac{1-x}{1+x+x^2}$.

Solution

$$\frac{1-x}{1+x+x^2}=\frac{(1-x)^2}{(1-x)(1+x+x^2)}=\frac{(1-x)^2}{1-x^3}=(1-x)^2(1-x^3)^{-1}$$

$$=(1-2x+x^2)(1+x^3+x^6+\ldots+x^{3r}+\ldots)$$

$$\text{c.o.}x^{3r}=1,\quad \text{c.o.}x^{3r+1}=-2,\quad \text{c.o.}x^{3r+2}=1.$$

Note

$$\binom{-n}{r}=(-1)^r\,\frac{n(n-1)(n-2)\ldots(n+r-1)}{r!}$$

$$=(-1)^r\frac{(n+r-1)!}{(n-r)!\,r!}=\binom{n+r-1}{r}(-1)^r.$$

Lecture 4

Theory of equations

Definition

The function

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0,\ a_0\neq 0, \tag{1}$$

where $n\geq 0$ is an integer , where $a_0,a_1\ldots,a_n$ are constants, is called a polynomial in x with degree n, and the equation

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0=0,\ a_0\neq 0, \tag{2}$$

is called an algebraic equation in x with degree n.

The fundamental Theorem of Algebra

The algebraic equation $f(x)=0$, x belongs to the complex numbers i. e. $x=a+ib,\ i=\sqrt{-1}$ has at least one root.

The basic fundamental Theorem of Algebra

If x belongs to the complex numbers i. e. $x=a+ib,\ i=\sqrt{-1}$. or i is the solution of the equation $x^2+1=0$. Then the algebraic equation (2) has exactly n roots (real or complex).

Remainder theorem and division

If $P(x)$ is divided by the factor $(x-r)$, we obtain the quotient $Q(x)$ which is another polynomial in x with degree $n-1$ and the remainder R is constant, and we write

$P(x)=(x-r)Q(x)+R$.

The number r is called a root of equation (2) if $P(r)=0$, and $(x-r)$ is called a factor of $P(x)$.

$P(x)$ is called a devisor, $Q(x)$ is the quotient and $R(x)$ is the remainder and it is a function of x .

Synthetic division (Horner's Method)

It is a practical and easily applied method to divide a polynomial $P(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$ by (x - r), so write

$$a_0x^n+a_1x^{n-1}+\dots+a_n=(x-r)(b_0x^{n-1}+b_1x^{n-2}+\dots+b_{n-1})$$
$$=b_0x^n+(b_1-rb_0)x^{n-1}+(b_2-rb_1)x^{n-2}+\dots+(R-rb_{n-1}).$$

Comparing the coefficients of different powers gives:

$$b_1 - r\ b_0 = a_1, b_2 - rb_1 = a_2, \ldots, R - rb_{n-1} = a_n,$$

which can be written as:

$$a_0 = b_0, = a_1 + rb_2 = b_1, a_2 + rb_1 = b_2$$

$$a_{n-1} + rb_{n-2} = b_{n-1} \quad , a_n + rb_{n-1} = b_n .$$

This can be seen in the following table which we call Horner's method.

r						
	a_0	a_1	a_2	$\cdots$	a_{n-1}	
		rb_0		$\cdots$		rb_{n-1}
	$b_0 = a_0$	b_1	b_2		b_{n-1}	R

Example 1

Prove that $(x+4)$ is one of the factors of the polynomial $x^3 - 3x^2 - 18x + 40$.

Solution

$P(x) = x^3 - 3x^2 - 18x + 40$. Using the remainder theorem:

$P(-4) = (-4)^3 - 3(-4)^2 - 18(-4) + 40 = 0$.

Therefore $(x-4)$ is a factor of the given polynomial,

where R is the remainder. We notice that $P(r) = R$.

Example 2

Find the result of dividing

$2x^5 - 3x^3 - 42x^2 + 5$ by $(x - 3)$

Solution

2	0	-3	-42	0	
	6		45		27
2	6				$32 = R$

The quotient is:

$Q(x) = 2x^4 + 6x^3 + 15x^2 + 3x + 9$, and the remainder is:

R = 32.

Example 3

Find the result of dividing

$P(x) = x^5 + 2x^4 - 9x^3 - 22x^2 + 8$ by $x^2 - x - 6$.

Solution

$x^2 - x - 6 = (x - 3)(x + 2)$.

Using the synthetic division twice, the first by $(x - 3)$ and the second by $(x + 2)$ as follows:

The first one:

$$
\begin{array}{c|ccccc}
 & 2 & -9 & -22 & 0 & \\
 & & & & -12 & -36 \\
\hline
-2 & & 6 & -4 & -12 & \\
 & -2 & -6 & 0 & & \\
\hline
 & & 0 & -4 & &
\end{array}
$$

The quotient is $Q(x) = x^3 + 3x^2 - 4$.

The remainder is $R = -4(x-3) - 28$.

Solution of equation by remainder theorem

This method can be summarized as follows, we analyze the absolute term to its factors, and then we have to get the root based on the fact that r is a root if the remainder is zero. i.e. $P(r) = 0$

Example 4

Solve the equation $P(x) = x^3 - 10x^2 + 27x - 18 = 0$, where it has integer roots.

Solution

The roots of this equation may be one of the factors of the absolute term -18. These factors are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$.

	-10	27	-18
		-9	
	-9		0
		-18	
6	-6	0	
	6		
	0		

Then the roots are $1, 3, 6$.

Notes

1- The complex of the algebraic equation with real coefficients are conjugates

2- If $a+\sqrt{b}$ is a root of an equation, then $a-\sqrt{b}$ is also a root provided the coefficients are rational numbers.

Example 5

Form the equation which has the roots $(3+i),\ -1,\ -2$.

Solution

Let $F(x)=0$ be the required equation, then its roots are

$(3+i),\ (3-i),\ -1,\ -2$.

And it can be written in terms of its factors of 1st degree as follows:

$$F(x) = (x-(3+i))(x-(3-i))(x+1)(x+2)$$
$$= x^4 - 3x^3 - 6x^2 + 18x + 20 = 0.$$

Example 8

Prove that one of the roots of the equation $2x^5 - 11x^4 + 19x^3 - 10x^2 + 4x - 8 = 0$ is 2 repeated three times and find the other two roots.

Solution

Perform the synthetic division successive three times such that 2 is the root in each times as follows

2	2	−11		−10	4	−8
		4	−14		0	
	2	−7		0	4	0
		4	−6	−2	−4	
	2	−3	−1	−2	0	
		4	2	2		
	2			0		

The remainder R = 0 in every time this means that 2 is a root repeated three times. The given equation can be written in the form:

$$2x^5 - 11x^4 + 19x^3 - 10x^2 + 4x - 8 =$$
$$(x-2)^3(2x^2 + x + 1) = 0.$$

The two other roots are the roots of the equation $2x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1-4(2)(1)}}{4} = \frac{-1}{4} \pm \frac{i\sqrt{7}}{4}.$$

The five roots of the given equation are:

$$2, 2, 2, \frac{-1}{4} + i\frac{\sqrt{7}}{4}, \frac{-1}{4} - i\frac{\sqrt{7}}{4}.$$

Lecture 5

Relation between coefficients of the equation and its roots

Suppose that the n^th degree equation

$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ has the n roots $r_1, r_2, \ldots, r_n$. Then

$(x-r_1), (x-r_2), \ldots (x-r_n)$ are factors i.e.

$$(x-r_1)(x-r_2)\ldots(x-r_n) = x^n + a_1x^{n-1} + x^{n-2} + \ldots + a_n.$$

Comparing the coefficients of equal powers in both sides we get:

$$(\mathrm{r}_1 + \mathrm{r}_2 + \ldots + \mathrm{r}_n) = -\mathrm{r}_1 \Rightarrow \sum_{i=1}^{n} r_i = -a_1$$

$$(\mathrm{r}_1\mathrm{r}_2 + \mathrm{r}_2\mathrm{r}_3 + \mathrm{r}_1\mathrm{r}_3 + \ldots) = a_2 \Rightarrow \sum \sum_{i<j} r_i r_j = a_2$$

$$(-1)^n(\mathrm{r}_1\mathrm{r}_2\ldots\mathrm{r}_n) = a_n \Rightarrow \mathrm{r}_1\mathrm{r}_2\ldots\mathrm{r}_n = (-1)^n a_n.$$

These relations alone are not sufficient to solve the equation but with additional conditions they helping solving the equation.

Example 9

Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$ if it has two equal roots.

Solution

Rewrite the given equation in the form $x^3 - \frac{11}{3}x^2 + \frac{8}{3}x + \frac{4}{3} = 0$.

Suppose that the roots are r_1, r_2, r_3, then we have:

$$r_1 + r_1 + r_2 = -(-\frac{11}{3}) \Rightarrow 2r_1 + r_2 = \frac{11}{3}$$

$$r_1^2 + 2r_1r_2 = \frac{8}{3}.$$

Also, $r_1^2 r_2 = (-1)^3(\frac{4}{3})$.

Solving together we have

$$r_2 = \frac{11}{3} - 2r_1$$

$$r_1^2 + 2r_1(\frac{11}{3} - 2r_1) = \frac{8}{3}$$

$$r_1^2 + \frac{22}{3}r_1 - 4r_1^2 = \frac{8}{3}$$

$$-3r_1^2 + \frac{22}{3}r\text{-}\frac{8}{3} = 0$$

$$9r_1^2 - 22r_1 + 8 = 0$$

$$(9r_1 - 4)(r_1 - 2) = 0$$

$(9r_1 - 4) = 0 \qquad \therefore r_1 = \frac{4}{9}$

or

$(r_1 - 2) = 0 \qquad \therefore r_1 = 2$

If $r_1 = \frac{4}{9} \qquad \therefore r_2 = \frac{25}{9}$

If $r_1 = 2 \qquad \therefore r_2 = \frac{1}{3}$

$r_1 = 2, r_2 = -\frac{1}{3}$.

The roots are $2, 2, -\frac{1}{3}$.

Example 10

If $x^3 + 2x^2 + ax + b = 0$ has the two roots $-3, 5$.

Find the 3rd root then find a, b.

Solution

Suppose the 3rd root is α

$\therefore -3 + 5 + \alpha = -2 \qquad \therefore \alpha = -4$.

Then the three roots are -3,-4,5.

$a = (-3)(-4) + (-4)(5) + (-3)(5)$.

$b = (-3)(-4)(5) = -60$.

Rational roots

If $\frac{b}{c}$ is a rational root of the equation $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0 = 0$.

Then b is one of the factors of a_0, and c is one of the factors of a_n, for example the equation $6x^3 + 5x^2 - 3x - 2 = 0$, for any rational root $\frac{b}{c}$, b is one of the factors of 2 $(\pm 1, \pm 2)$, and c is one of the factors of 6 $(\pm 1, \pm 2, \pm 3, \pm 6)$. Therefore the roots may be between the values $\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}$.

We notice that if the coefficient of x^n equal 1 $(a_n = 1)$, then the roots are integers and factors of the absolute term.

Example 11

Obtain the rational roots of $F(x) = 2x^3 - 7x^2 + 7x - 2 = 0$.

Solution

The roots are from the values $(\pm 1, \pm 2, \pm \frac{1}{2})$

F(1) = 2 – 7 + 2 – 2 = 0 , 1 is a root,

F(-1) = - 2 – 7 – 7 – 2 = - 18 , -1 is not a root,

$F(\frac{1}{2}) = 2(\frac{1}{8}) - 7(\frac{1}{4}) + 7(\frac{1}{2}) - 2 = 0$, $(\frac{1}{2})$ is a root,

$F(-\frac{1}{2})=2(-\frac{1}{2})^3-7(-\frac{1}{2})^2+7(-\frac{1}{2})-2\neq 0$, $-\frac{1}{2}$ is not a root,

$F(2)=2(8)-7(4)+7(2)-2=0$, 2 is a root.

The roots are: $1, \frac{1}{2}, 2$

$$2x^3-7x^2+7x-2=(x-1)(2x-1)(x-2).$$

Complete Synthetic Division

Assume that $P(x)$ is a polynomial of degree 3. If we divide it by the factor $x-r$ three times we obtain

$$P(x)=(x-r)P_1(x)+R_0 \Rightarrow P_1(x)=(x-r)P_2(x)+R_1$$

$$P_2(x)=(x-r)R_3+R_2.$$

Substituting, we get:

$$P(x)=R_3(x-r)^3+R_2(x-r)^2+R_1(x-r)+R_0.$$

Thus, we express $P(x)$ in terms of $x-r$.

If we write $y=x-r$, then the roots of the equation $R_3y^3+R_2y^2+R_1y+R_0=0$ are less than those of the equation $P(x)=0$ by r.

Moreover, if we compare with Taylor expansion of $P(x)$, where $P(x)=(x-r)^3\frac{P'''(r)}{3!}+(x-r)^2\frac{P''(r)}{2!}+(x-r)P'(r)+P(r)$

we obtain the relation between the remainders and the derivatives of $P(x)$ so that:

Notes

Through the use of complete synthetic division we can do the following:

1- We can express any polynomial $P(x)$ of powers of x in terms of $(x-r)$ under dividing $P(x)$ by the number r.

2- We can obtain another equation that has roots less or greeter than those of the equation $P(x)=0$ without obtaining the roots of $P(x)=0$ under dividing $P(x)$ by the number r. or $-r$ respectively.

3- Also, the value of $P(x)$ and its derivatives at a point $x=r$ can be evaluated through the use of the remainders of the complete synthetic division of $P(x)$ by the number r. The following example explains these three useful points.

Example 12

Let $P(x)=0$, where $P(x)=x^4+6x^3-7x^2-36x+37$.

1- Write $P(x)$ in terms of $(x-2)$.

2- Find an equation its roots are less than the given one by 2.

3- Evaluate the value of $P(x)$ and its derivatives at $x=2$.

Solution

As before, it is easy to obtain:

$R_0 = 1,\ R_1 = 40,\ R_2 = 53,\ R_3 = 14,\ R_4 = 1,$

1- $P(x) = (x-2)^4 + 14(x-2)^3 + 53(x-2)^2 + 40(x-2) + 1.$

2- The required equation is:

$y^4 + 14y^3 + 53y^2 + 40y + 1 = 0.$

3- $P(2) = 1, P'(2) = 40, P''(2) = 2!(53) = 106,$
$P'''(2) = 3!(14) = 84, P^{(iv)}(2) = 4!(1) = 24.$

Notes

1- To find an equation whose roots are the reciprocal of those of the equation

$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$ Put $y = \frac{1}{x} \Rightarrow x\ \frac{1}{y}$.

Substituting we get: $a_n (\frac{1}{y})^n + a_{n-1} (\frac{1}{y})^{n-1} + \dots + a_1 (\frac{1}{y}) + a_0 = 0$.

Or

$a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_{n-1} y + a_n = 0$

2- To find an equation whose roots are multiples of those of the equation

$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$ Put $y = mx \Rightarrow x = \frac{y}{m}$.

Substituting we get:

$a_n y^n + a_{n-1} m y^{n-1} + a_{n-2} m^2 y^{n-2} + \dots + a_1 m^{n-1} y + a_0 m^n = 0$

As a special case:

Letting $m=-1$, we obtain an equation whose roots are the same of those of the equation with opposite sins.

3- To find an equation whose roots are the squared of those of the equation $f(x)=a_n x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1 x+a_0=0$ follow the following steps:

a- If $f(x)=(x-r_1)(x-r_2)...(x-r_n)=0$.

b- $\therefore P(x)=(x+r_1)(x+r_2)...(x+r_n)=0$ is the equation whose roots are the same of those of the equation with opposite sins.

c- $\therefore F(x)=f.P=(x^2-r_1^2)(x^2-r_2^2)...(x^2-r_n^2)=0$.

d- Put $x^2=y \Rightarrow g(y)=(y-r_1^2)(y-r_2^2)...(y-r_n^2)=0$.

Example 13

If two of the roots of the equation $x^4+6x^3+hx^2+wx+36=0$ are real and equal twice the value of the other two roots. Solve the equation and find h,w .

Solution

Assume the roots are $a,b,2a,2b$, then we have:

$3a+3b=-6 \Rightarrow a+b=-2$.

Also, $4a^2b^2=(-1)^4(36) \Rightarrow a^2b^2=9 \Rightarrow ab=\pm 3$.

Taking $ab=3 \Rightarrow a+\frac{3}{a}=-2 \Rightarrow a^2+2a+3=0$. This equation has no real roots. So we study the other case $ab=-3$.

It gives

$a-\frac{3}{a}=-2 \Rightarrow a^2+2a-3=0 \Rightarrow a=-3,1 \Rightarrow b=1,-3$.

Consequently the roots are: $1,-3,2,-6$.

We can write the equation in the form:

$(x-1)(x+3)(x-2)(x+6)=0$.

Or

$x^4+6x^3-7x^2-36x+36=0$.

Comparing, we get:

$h=-7, w=-36$.

Example 14

Solve the equation $x^4-16x^3+ax^2+bx+105=0$ if its roots form an arithmetic progression. Find the value of both a and b.

Solution

Assume the roots of the form $c-3d, c-d, c+d, c+3d$.

$\therefore 16=c-3d+c-d+c+d+c+3d=4c \Rightarrow c=4$.

Also, $(-1)^4(105)=(c-3d)(c-d)(c+d)(c+3d)$.

$\therefore (c^2-9d^2)(c^2-d^2)=105 \Rightarrow 9d^4-160d^2+256=105$.

$(9d^2-151)(d^2-1)=0 \Rightarrow$

Case 1

$d^2 - 1 = 0 \Rightarrow d = \pm 1 \Rightarrow roots: 1, 3, 5, 7$.

$a = 86, b = -176$.

Case 2

$(9d^2 - 151) = 0 \Rightarrow d = \pm\frac{\sqrt{151}}{3}$, and the roots are:

$4 - \sqrt{151}, 4 - \frac{\sqrt{151}}{3}, 4 + \frac{\sqrt{151}}{3}, 4 + \sqrt{151}$.

$a = -\frac{650}{9} = 72.222, b = \frac{1136}{9} = 126.222$.

Elimination of Second Term of the Equation:

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0. \qquad (1)$$

To do this, substitute $x = y + k$ in (1). Then we have:

$$a_0 (y + k)^n + a_1 (y + k)^{n-1} + a_2 (y + k)^{n-2} + \ldots + a_{n-1}(y + k) + a_n = 0. \qquad (2)$$

Equating coefficient of y^{n-1} by zero we get:

$$nk\, a_0 + a_1 = 0 \Rightarrow k = -\frac{a_1}{n\, a_0}.$$

Substituting by k in (2) we get the reduced form of equation (1) and it is an equation of order n with the second term is missed. Also equation (2) has roots less than those of (1) by k.

Example 15

Find reduced form of the equation $x^3 - 3x^2 + 2x - 1 = 0$.

Solution

$$k=-\frac{a_1}{n\,a_0}=-\frac{-3}{3(1)}=1.$$

$$\begin{array}{c|ccc} & -3 & 2 & -1 \\ & & -2 & 0 \\ \hline & -2 & 0 & -1 \\ & & -1 & \\ & -1 & -1 & \\ & 0 & & \end{array}$$

The reduced form is the equation $x^3-x-1=0$.

Lecture 6

Numerical Solution of Nonlinear equations

In this lecture we are interested in finding one real root (or the real roots) of the nonlinear equation $f(x)=0$ (either algebraic in the form $x^3-3x+1=0$ or not algebraic such as the equation $\sin x+2x-1=0$). We can obtain the root of the equation (or the roots) by first choosing a starting point x_0 (initial root) which can either be obtained analytically or graphically. Then improve it through the use of one of the following approximated methods:

Numerical Methods

Simple iterative method

This method starts with an initial approximate root x_0 of the equation $f(x)=0$.

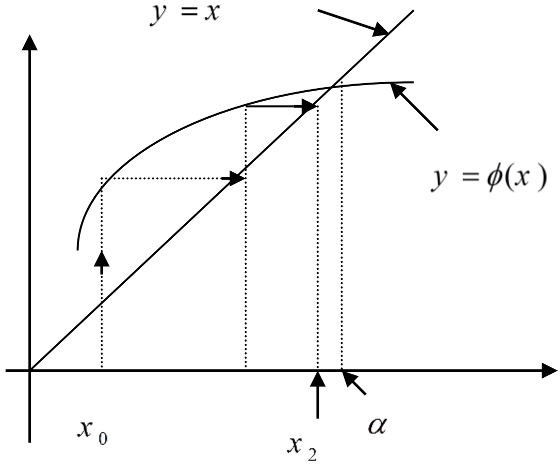
Rewrite the equation $f(x)=0$ in the form $x=\phi(x)$ such that the condition $|\phi'(x)|<1$ is satisfied in the interval containing the required root. In particular, $|\phi'(x_0)|<1$.

This condition is the condition of convergence, i.e. the successive values of x tends to the exact root if this condition is satisfied.

Then improve x_0 using the sequence

$$x_{n+1}=\phi(x_n),\ n=0,1,2,\dots .$$

(1)



$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \phi(x_n) = \alpha$ (the exact root). i.e. $f(\alpha) = 0$.

We will stop the iteration process (1) after one or more of the following conditions is satisfied:

1- Certain number of iterations is achieved.

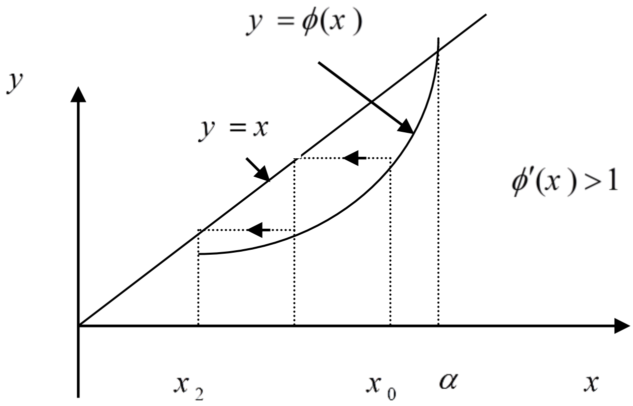
2- If $|x_{n+1} - x_n| \leq E_1$, where E_1 is certain defined accuracy ,say, 10^{-8}.

3- If $|f(x_n)| \leq E_2$, where E_2 is another accuracy, i.e. if we substitute with the obtained root in $f(x)$, its modulus must be less than or equal to E_2 ,say, 10^{-4}.

Theorem 1

If $|\phi'(x)|<1$ in the interval containing the required root, then the iterative process converges, provided that the initial value x_0 is sufficiently close to the required root.

The following figure indicates the case of divergence i.e. $|\phi'(x)|>1$ in the interval containing α.



Example 1

Obtain one real root of the equation $\sin x + 2x - 1 = 0$.

Solution

Rewrite the equation $\sin x + 2x - 1 = 0$ in the form

$$x = \frac{1}{2}(1 - \sin x) = \phi(x).$$

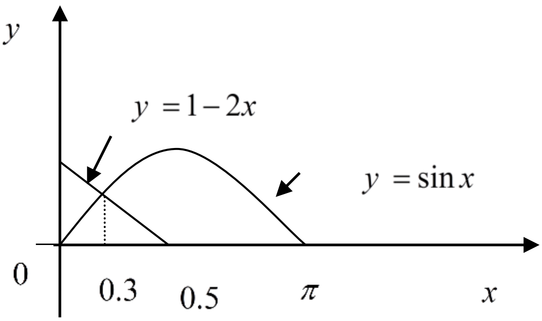
$$\therefore \phi'(x) = -\frac{1}{2}\cos x \ .$$

$\because |\phi'(x)| = \frac{1}{2}|\cos x| \le \frac{1}{2}$, then the solution converges.

Now we find x_0 graphically as follows:

The point $x = x_0$ represents the point of intersection of the two curves:

$y = \sin x$, $y = 1 - 2x$.



$x_0 = 0.3$.

$$\therefore x_{n+1} = \frac{1}{2}(1 - \sin x_n) \Rightarrow$$

$$x_1 = \frac{1}{2}(1 - \sin 0.3) = 0.3522399$$

$$x_2 = \frac{1}{2}(1 - \sin 0.3522399) = 0.3274995$$

$x_3 = 0.3391618$

$x_4 = 0.3336516$

$x_5 = 0.3362523$

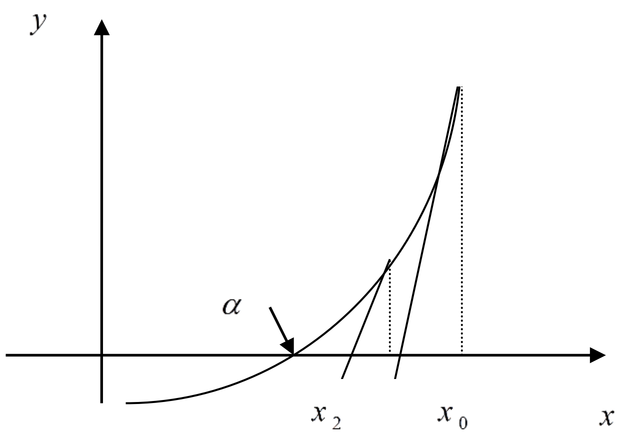
$x_6 = 0.3350242$

$x_7 = 0.335604.$

$x = 0.335604$.

Newton's method

This method starts with an initial (approximate) root x_0 of the equation $f(x)=0$. This initial root can be improved to the value x_1 using the tangent line at the point $(x_0, f(x_0))$, and so on until we obtain the required root with the required accuracy as follows:



Remember that the real root α of the equation $f(x)=0$ is the point of intersection of the curve $y=f(x)$ with the $x-$axis. The equation of the tangent line to this curve at the point p and whose slope is $f'(x_0)$, is given by $y-f(x_0)=f'(x_0)(x-x_0)$.

At $y=0$ we have $x=x_1$ then we get:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \qquad (2)$$

x_1 is called the first approximation to the root x_0.

In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0,1,2,\dots \;. \qquad (3)$$

Formula (3) is called Newton-Raphson's method (formula).

Comparing with the iterative method we have:

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = 1 - \frac{f'f' - f\,f''}{f'^2} = \frac{f(x) f''(x)}{f'^2(x)} \to 0 \, as \; x \to \alpha.$$

This is the condition of convergence of this method.

The speed of convergence is given by:

$$\phi'(\alpha) = \lim_{n\to\infty} \frac{x_{n+1} - \alpha}{x_n - \alpha}.$$

Note that, if the equation $f(x) = 0$ is algebraic), then equation (1) can be written as:

$$x_1 = x_0 - \frac{R_0}{R_1}.$$

Example 2

Use Newton's method to obtain all roots of the equation:

$x^3 - 3x + 1 = 0$.

Solution

We will use the algebraic method for finding the initial root as follows:

$f(x) = x^3 - 3x + 1 = 0 \Rightarrow f'(x) = 3x^2 - 3$.

Substitute x with the integer values $0,1,2,\ldots$ until the sign of $f(x)$ changes. If (say) $f(a)>0$ and $f(b)<0$ then, from the continuity of f , there exists at least one root of the equation between a,b .

$$f(0)=1, f(1)=-1 \Rightarrow x_0=\frac{0+1}{2}=\frac{1}{2}.$$

$$f(\frac{1}{2})=\frac{1}{8}-\frac{3}{2}+1=-\frac{3}{8},$$

$$f'(\frac{1}{2})=\frac{3}{4}-3=-\frac{9}{4}.$$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=\frac{1}{2}-\frac{-3/8}{-9/4}=\frac{1}{3}=0.333333.$$

Repeating this step we get:

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{1}{3}-\frac{(\frac{1}{3})^3-3(\frac{1}{3})+1}{3(\frac{1}{3})^2-3}=0.347222.$$

Continue to obtain the required accuracy if it mentioned in the problem.

If we apply the synthetic division, we have:

0.5	0	-3	
	0.5		-1.375
	0.5	-2.75	$-0.375 = R_0$
	0.5	0.5	

$$\therefore\ x_1 = x_0 - \frac{R_0}{R_1} = 0.5 - \frac{-0.375}{-2.25} = 0.333333.$$

Repeating this step we get:

0.333333	0	-3	
	0.333333	.1111089	
	0.333333	2.8888911	$0.0370459 = R_0$
	0.333333	.2222178	

$$x_2 = x_1 - \frac{R_0}{R_1} = .333333 - \frac{.0370459}{-2.6666733} = 0.3472222.$$

Repeating this step we obtain

0.3472222	0	−3	
	0.3472222		−0.9998044
	0.3472222	−2.8794367	$0.0001956 = R_0$

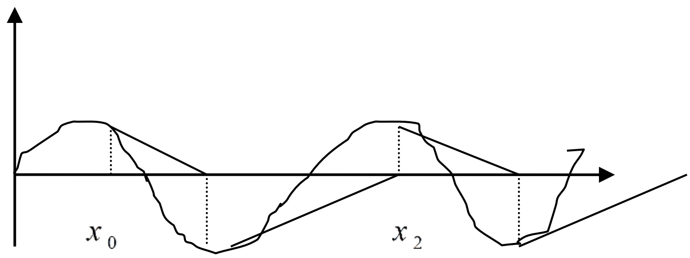
$R_0 = 0.0001956$, which is a small value.

Then the first root is $x = 0.3472222$.

Solving the second degree equation $x^2 + 0.3472222x - 2.8794367 = 0$, we get the other two roots and they are:

$x = 1.5321373, -1.87935974$.

Note that the following graph shows how the starting point can effect on the computations.



Example 3

Find the smallest positive root of the equation $x^3 - 7x + 4 = 0$ correct to four decimal places using:

1- Simple iterative method. 2- Newton's method.

Solution

$f(x) = x^3 - 7x + 4 = 0 \Rightarrow f(0) = 4,\ f(1) = -2 \Rightarrow x_0 = 0$.

Note that we take the left end of the interval as a starting point. One can take the right end or the midpoint of the interval in which the required root exists.

1- Simple iterative method

The given equation can be put in the form:

$$x = \frac{x^3 + 4}{7} = \phi(x) \Rightarrow \phi'(x) = \frac{3x^3}{7}.$$

If $|\phi'(x)| < 1 \Rightarrow \left|\frac{3x^2}{7}\right| < 1 \Rightarrow |x| < \sqrt{\frac{7}{3}} = 1.5275251.$

Hence the interval $(0,1)$ containing the required root satisfies the condition of convergence. Thus we have:

$$x_{n+1} = \frac{x_n^3 + 4}{7},\ n = 0,1,2,\dots.$$

$$x_1 = \frac{0+4}{7} = 0.5714287$$

$$x_2 = \frac{[(0.5714287)^3 + 4]}{7} = 0.59808414$$

$x_3 = 0.60199114, \quad x_4 = 0.602594,$

$x_5 = 0.60268771, \quad x_6 = 0.60270280,$

$x_7 = 0.60270459 \quad \Rightarrow x = 0.60270459.$

2- Newton's method

$f'(x) = 3x^2 - 7.$

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n + 4}{3x_n^2 - 7}, \; n = 0, 1, 2, \ldots.$$

$x_1 = 0.57142857, \quad x_2 = 0.60242128,$

$x_3 = 0.6027463, \quad x_4 = 0.60270503,$

$\therefore x = 0.60270503.$

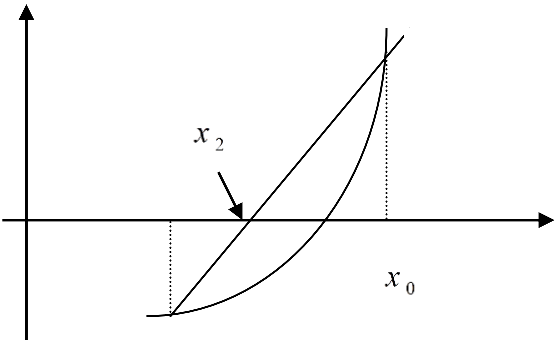
Secant method

This method is essentially a modification of the conventional Newton's method with the derivative replaced by a difference expression. This is advantageous if the function is difficult to differentiate, and is also convenient to program in the sense that it is only necessary to supply a function subprogram to the method rather subprogram for the function and its derivatives.

From the blow figure we have;

The slope of the secant $AB = m = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

Starting with the two points x_0, x_1 , at which the signs of f are different in the begging, we can obtain x_2 as follows:



Equation of the line AB is given by:

$y - f(x_0) = m(x - x_0)$.

$y = 0$ at $x = x_2$.

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{m}$.

In general, we have:

$$x_{n+1} = x_n - \frac{f(x_n)}{m}, \; n = 1, 2, \ldots. \qquad (4)$$

Example 6

Evaluate a root for the equation $f(x) = x^3 - 8x + 1 = 0$.

Solution

$f(0) = 1, \; f(1) = -6$.

$x_0 = 1,\ x_1 = 0 \Rightarrow f(x_0) = -6,\ f(x_1) = 1.$

$\therefore x_2 = 0 - (1)\dfrac{0-1}{1+6} = 0.14285714.$

$f(0.14285714) = -0.1399417.$

$\therefore x_3 = .14285714 + (.1399417)\dfrac{.14285714 - 0}{-.1399417 - 1} = .1253197.$

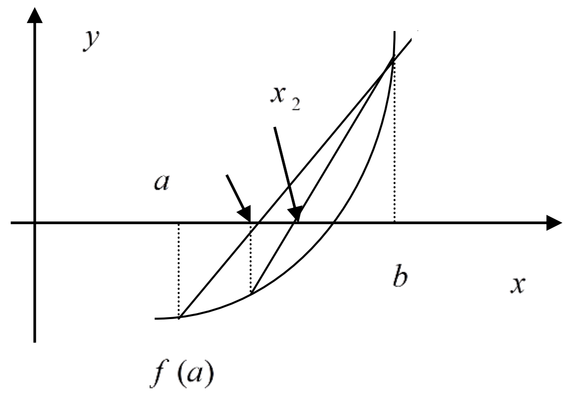
$f(0.1253197) = -0.0005895.$

$x_4 = .1253197 + (.0005895)\dfrac{.1253197 - .14285714}{-.0005895 + .1399417} = .1252455.$ 5-2.5

$f(.1252455) = .0000006.$

$\therefore x_5 = .1252455 - (.0000006)\dfrac{.1252455 - .1253197}{.0000006 + .0005895} = .1224557.$

The false position method



This method starts with two points $a, b \ni f(a) \cdot f(b) < 0$.

As above, we can obtain x_1 from the relation:

$$x_1 = a - \frac{f(a)}{m},\quad m = \frac{f(a) - f(b)}{a - b}.$$

Example 7

Find a root for the equation $x^3 - x - 1 = 0$ between $1.3, 1.4$ approximated to three decimal places.

Solution

x			
	-0.103	–	–
	0.344	0.224	-0.077
	-0.00731	0.219	0.0016
1.3246	-0.000503	0.219	
1.32471			

Lecture 7

Matrices

Introduction

A matrix is a rectangular array of scalars (real or complex) enclosed by a pair of brackets such as:

$$A = \begin{bmatrix} 3 & 7 & 2 \\ 1 & -5 & 4 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \\ 6 & -1 & 2 \\ 0 & 3 & 7 \end{bmatrix}.$$

The matrix A could be considered as the coefficient matrix of the system of linear equations:

$3x+7y+2z=0$ & $x-5y+4z=0$, and the matrix B express the coefficient matrix of the system of linear equations: $2x+3y+4z=8,\ \ 6x-y+2z=10$ & $3y+7z=-1$.

In the matrix

$$A_{mn}=\begin{pmatrix} a_{11} & a_{12} & a_{13}\dots a_{1n} \\ a_{21} & a_{22} & a_{23}\dots a_{2n} \\ a_{m1} & a_{m2} & a_{m3}\dots a_{mn} \end{pmatrix} \qquad (1)$$

The numbers or functions a_{ij} are called elements in the double subscript notation, the first subscript indicates the row number and the second subscript indicates the column number.

A matrix of m rows and n columns are said to be of order $m\times n$. When $m=n$, the matrix is square. Otherwise it is rectangular matrix.

In a square matrix, the elements $a_{11},a_{22},\dots,a_{nn}$ are called its main diagonal elements. The sum of the main diagonal elements of a square matrix A is called the trace of A and its notation is trA.

$$\therefore trA=\sum_{i=}^{n}a_{ii}\ .$$

Two matrices $A=[a_{ij}],\ B=[b_{ij}]$ are said to be equal or $A=B$ if and only if they have the same size and each element of A is equal to the corresponding element in B i.e. $a_{ij}=b_{ij}$.

Common Types of Matrices

In most physical applications we frequently meet some special forms of matrix like the following:

1- Diagonal matrix D

This is a square matrix whose off-diagonal elements are zeros, i.e. $a_{ij}=0\ \forall i\neq j$.

2- Unit matrix I (the identity matrix)

It is a diagonal matrix where $a_{ii}=1,\ \forall i$.

$I_{1\times1}=(1),\ I_{2\times2}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ I_{3\times3}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. It behaves as the number one in algebra.

3- Zero or null matrix 0

All of its elements are zero, i.e. $a_{ij}=0\ \forall i,j$.

$0_{1\times1}=(0),\ 0_{2\times2}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ 0_{3\times3}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$. It behaves as the number zero in algebra.

4- Echelon form of matrix

This form has the following properties:

a- The first non-zero element in each row is one.

b- In each row after the first, the number of zeros preceding the first non-zero element exceeds the number of zeros preceding the first non-zero element in the previous row.

c- When the first non-zero element in the i^{th} row lies in the j^{th} column, all other elements in the j^{th} column are zero.

d- The first row has a non-zero element.

$$A=\begin{pmatrix}1&0&0&0&0\\0&0&1&2&0\\0&0&0&0&1\end{pmatrix}$$ is in the Echelon form.

Conjugate and Transpose of a Matrix

The conjugate of a matrix $A=(a_{ij})$ is denoted by $\bar{A}$ is obtained by taking the conjugate of all elements of the matrix, i.e. $\bar{A}=(\bar{a}_{ij})$. If A is real, then $\bar{A}=A$.

The transpose of $A=(a_{ij})$ is denoted by A' and equal to $A'=(a_{ji})$.

If both operations are simultaneously applied on A we obtain the so called conjugate transpose of A denoted by A^*.

Theorem 1

If A and B are two matrices and k is a scalar we have:

$$1-\ (A')'=A, \quad (KA)'=KA'$$

$$2-\ (A+B)'=A'+B'$$

$$3-\ (AB)'=B'.A'$$

Example 1

$$\text{If } A=\begin{pmatrix} 3 & 1+i & 2i \\ 2-3i & i & 0 \\ 0 & 1 & 1+i \end{pmatrix} \Rightarrow \bar{A}=\begin{pmatrix} 3 & 1+i & 2i \\ 2+3i & -i & 0 \\ 0 & 1 & 1+i \end{pmatrix},$$

$$A'=\begin{pmatrix} 3 & 2-3i & 0 \\ 1+i & i & 1 \\ 2i & 0 & 1+i \end{pmatrix}, A^*=\begin{pmatrix} 3 & 2+3i & 0 \\ 1-i & -i & 1 \\ -2i & 0 & 1-i \end{pmatrix}.$$

Operations with Matrices

Addition and subtraction

If $A=[a_{ij}], B=[b_{ij}]$ are two $m\times n$ matrices, then their sum (difference) is defined as the $m\times n$ matrix C , where

$C = [c_{ij}]$, and each element of C is the sum (difference) of the corresponding elements of A , B . In other word:

$c_{ij} = a_{ij} \pm b_{ij}$, $\forall i, j$.

Example 2

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{pmatrix} \Rightarrow$$

$$A + B = \begin{pmatrix} 1+2 & 2+3 & 3+0 \\ 5+(-1) & 1+2 & 4+5 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{pmatrix}.$$

$$A - B = \begin{pmatrix} 1-2 & 2-3 & 3\text{-}0 \\ 5-(-1) & 1-2 & 4\text{-}5 \end{pmatrix} = \begin{pmatrix} -1 & \text{-}1 & 3 \\ 1 & -1 & \text{-}1 \end{pmatrix}.$$

Notes

1- Two matrices of different orders can not be added or subtracted.

2- Two matrices of the same order are said to be conformable for addition or subtraction.

Assuming that the matrices A, B, C are conformable for addition, we state:

a- $A+B=B+A$. (commutative law)

b- $A+(B+C)=(A+B)+C$. (associative law)

c- $k(A+B)=kA+kB=(A+B)k$, k is a scalar.

d- There exists a matrix D such that $A+D=B$.

Multiplication

If A is multiplied by a scalar k, then the matrix $kA=(ka_{ij})$. i.e. all elements of A multiplied by k .

If A is multiplied by another matrix B, then $C=AB, c_{ij}=\sum_{k=1}^{n} a_{ik}\, b_{kj}$. Hence the condition for post-multiplying A by B is that the number of columns of A must be equal to the number of rows of B .

In general, by the product AB in that order of the $m\times p$ matrix $A=[a_{ij}]$ and the $p\times n$ matrix $B=[b_{ij}]$ is meant the $m\times n$ matrix $C=[c_{ij}]$ where

$$cij=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{i}b_{j}=\sum_{k=1}^{p} a_{ik}b_{kj} \;,$$

where $\mathrm{i}=1,2,\ldots,\mathrm{m}\;,\;\mathrm{j}=1,2,\ldots\mathrm{p}$.

The product AB is defined or A is conformable to B for multiplication only when the number of columns of A is equal to the number of rows of B .

If A is conformable to B for multiplication (AB is defined), B is not necessarily conformable to A for multiplication i.e. BA may or may not be defined.

Assuming that A,B,C are conformable for indicated sums and products we have:

a – $A(B + C) = AB + AC$

b- $(A + B)C = AC + BC$

c- $A(BC) = (AB)C$

d- $AB \neq BA$

e- AB = 0 does not necessarily imply A = 0 or B = 0,

f- AB = AC does not necessarily imply B = C,

$k_1(A+B) = k_1A + k_1B, (k_1 + k_2)A = k_1A + k_2A, k_1, k_2$.

Example 3

Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 3 & -1 \end{bmatrix}.$$

Find A+B, A - B , 3A , - A.

Solution

$$A+B=\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix}+\begin{bmatrix}3 & -4 & 1 & 2\\ 1 & 5 & 0 & 3\\ 2 & -2 & 3 & -1\end{bmatrix}.$$

$$\text{A-B}=\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix}+\begin{bmatrix}3 & -4 & 1 & 2\\ 1 & 5 & 0 & 3\\ 2 & -2 & 3 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1-3 & 2-(-4) & -1-1 & 0-2\\ 4-1 & 0-5 & 2-0 & 1-3\\ 2-2 & -5-(-2) & 1-3 & 2-(-1)\end{bmatrix}=\begin{bmatrix}-2 & 6 & -2 & -2\\ 3 & -5 & 2 & -2\\ 0 & -3 & -2 & 3\end{bmatrix}.$$

$$3A=3\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix}=\begin{bmatrix}3 & 6 & -3 & 0\\ 12 & 0 & 6 & 3\\ 6 & -15 & 3 & 6\end{bmatrix}.$$

$$-A=-\begin{bmatrix}1 & 2 & -1 & 0\\ 4 & 0 & 2 & 1\\ 2 & -5 & 1 & 2\end{bmatrix}=\begin{bmatrix}-1 & -2 & 1 & 0\\ -4 & 0 & -2 & -1\\ -2 & 5 & -1 & -2\end{bmatrix}.$$

Example 4

If

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}, B=\begin{bmatrix}-3 & -2\\ 1 & -5\\ 4 & 3\end{bmatrix}.$$

Find $D=\begin{bmatrix}p & q\\ r & s\\ t & u\end{bmatrix}$ such that $A+B-D=0$.

Solution

If $A+B-D=0$

$$\therefore \begin{bmatrix} 1\text{ - }3\text{-p} & 2\text{-}2\text{-q} \\ 3+1\text{-r} & 4\text{-}5\text{-s} \\ 5+4\text{-t} & 6+3\text{-u} \end{bmatrix} = \begin{bmatrix} \text{-}2\text{-p} & \text{-q} \\ 4\text{-r} & \text{-}1\text{-s} \\ 9\text{-t} & 9\text{-u} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\therefore$ -2-P=0 P=-2

4-r=0 r=4

9-t= 0 t=9

-9=0 9=0

-1-s=0 s=-1

9-u=0 u=9

$$D = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}.$$

Products by Partitioning

Let $A=[a_{ij}]$ is $m \times p$ matrix and $B=[b_{ij}]$ be of order $p \times n$. In forming the product AB, the matrix A is in effect partitioned into m matrices of order $i \times p$ and B into n matrices of order $p \times i$. Other partitions may be used. For example, let A and B partitioned as follows

$$A=\begin{pmatrix}(m_1xp_1) & (m_1xp_2) & (m_1xp_3)\\ (m_2xp_1) & (m_2xp_2) & (m_2xp_3)\end{pmatrix}, B=\begin{bmatrix}(p_1xn_1) & (p_1xn_2)\\ (p_2xn_1) & (p_2xn_3)\\ (p_3xn_1) & (p_3xn_3)\end{bmatrix}$$

$$A=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\end{pmatrix}, B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\\ B_{31} & B_{32}\end{bmatrix}.$$

In any such partitioning it is necessary that the columns of A and B the rows of B partitioned in exactly the same way, hover m_1, m_2, n_1, n_2 may be any non-negative

integers such that $m_1+m_2=m, n_1+n_2=n$, then

$$AB=\begin{pmatrix}A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31}+A_{11}B_{12}+A_{12}B_{22}+ & A_{13}B_{32}\\ A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31}+A_{21}B_{12}+A_{22}B_{22}+ & a_{23}B_{32}\end{pmatrix}$$

$$=\begin{pmatrix}c_{11}c\\ c_{21}c_{22}\end{pmatrix}=c.$$

Example 5

Find AB where $A=\begin{pmatrix}2\ 1\ 0\\ 3\ 2\ 0\\ 1\ 0\ 1\end{pmatrix}$ and B= $\begin{pmatrix}1 & 1 & 1 & 0\\ 2 & 1 & 1 & 0\\ 2 & 3 & 1 & 2\end{pmatrix}$

Solution

We write A, B as:

$$A=\begin{pmatrix}2 & 1 & 0\\ 3 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}.$$

And

$$B=\begin{pmatrix}1&1&1&0\\1&1&1&0\\2&3&1&2\end{pmatrix}=\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}$$

$$AB=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}$$

$$=\begin{pmatrix}A_{11}B_{11}+A_{12}B_{21}&A_{11}B_{12}+A_{12}B_{22}\\A_{21}B_{11}+A_{22}B_{21}&A_{21}B_{12}+A_{22}B_{22}\end{pmatrix}$$

$$=\begin{pmatrix}\begin{bmatrix}2&1\\3&2\end{bmatrix}\begin{bmatrix}1&1&1\\2&1&1\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}2&3&1\end{bmatrix}&\begin{bmatrix}2&1\\3&2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}2\end{bmatrix}\\\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}1&1&1\\2&1&1\end{bmatrix}+\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}2&3&1\end{bmatrix}&\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}2\end{bmatrix}\end{pmatrix}$$

$$\therefore AB=\begin{pmatrix}\begin{bmatrix}4&3&3\\7&5&5\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}&\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\\\begin{bmatrix}1&1&1\end{bmatrix}+\begin{bmatrix}2&3&1\end{bmatrix}&\begin{bmatrix}0\end{bmatrix}+\begin{bmatrix}2\end{bmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}\begin{bmatrix}4&3&3\\7&5&5\end{bmatrix}&\begin{bmatrix}0\\0\end{bmatrix}\\\begin{bmatrix}3&4&2\end{bmatrix}&\begin{bmatrix}2\end{bmatrix}\end{pmatrix}=\begin{bmatrix}4&3&3&0\\7&5&5&0\\3&4&2&2\end{bmatrix}$$

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}&\dots&a_{1n}\\0&a_{22}&a_{23}&\dots&a_{2n}\\0&0&a_{33}&\dots&a_{3n}\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&a_{nn}\end{pmatrix}\text{is upper triangular}$$

and $\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$ is lower triangular

The matrix $\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$ which is both

Upper and lower triangular is called diagonal matrix.

A matrix A for which $A^{x+1} = A$ where k is a positive integer is called periodic A is said to be period k if k is the least positive integer for which $A^{x+1} = A$

When $k = 1 \rightarrow A^2 = A$ then A is called idempotent.

A matrix A for which $A^p = 0$, p is positive integer for which the condition $A^p = 0$ is satisfied, is called nilpotent. When A is said to be nilpotent of index p .

Example 6

Show that $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent

Solution

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$= A.$

$\therefore$ A is idempotent.

Example 7

Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of order 3.

Solution

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}.$$

Also,

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ A is nilpotent of order 3.

Note

If $AB = A$ and $BA = B$, then A and B are idempotent.

Proof

ABA = (AB)A = AA = A^2.

$\therefore$ A is idempotent .

Similarly

BAB = (BA) B = BB = B^2.

B is idempotent.

5- Symmetric matrices

A square matrix A which satisfies that $A' = A$ is called symmetric, thus a square matrix $A = [a_{ij}]$ is symmetric if and only if $a_{ij} = a_{ji} \; \forall i, j$.

Theorem 2

If A is an n - square matrix, then $A + A'$ is symmetric.

A square matrix A such that A = -A is called skew-symmetric. Thus a square matrix A is skew – symmetric provided $a_{ij} = -a_{ji} \; \forall i, j$.

Clearly the diagonal elements are zeros. For example the matrix $A = \begin{bmatrix} 0 & -3 & 4 \\ 2 & 0 & 6 \\ 3 & -6 & 0 \end{bmatrix}$ is skew-symmetric and so also is kA for any scalar k .

Theorem 3

If A is any n-square matrix. Then the matrix A - A' is skew-symmetric. From theorems 2 and 3 we have the following theorem:

Theorem 4

Every square matrix A can be written as the sum of a symmetric matrix $B = \frac{1}{2}(A + A')$ and as skew-symmetric matrix $C = \frac{1}{2}(A + A')$.

Theorem 5

If A and B are n-square symmetric matrices then AB is symmetric if and only if A and B commute

Proof

Suppose A and B commute so that $AB = BA$.

$\therefore\ (AB)' = B'A' = BA = AB$

AB *is symmetric*.

Now assume that AB is symmetric.

$\therefore\ (AB)' = AB$

since $(AB)' = B'A' = BA$

$AB = BA$.

Hence, A and A commute.

Lecture 8

Inverse of a Matrix

If A and B are square matrices such that $AB = BA = I$ then B is called the inverse of A and denoted by $B = A^{-1}$.

Example 8

If $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}$, $B=\begin{bmatrix}6 & -2 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$

is $B=A^{-1}$?

Solution

We find AB

$$AB=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}6 & -2 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

$AB=I \qquad B=A^{-1}$

and also $A=B^{-1}$.

Not every square matrix has an inverse, however if A has an inverse that inverse is unique.

If A and B are square matrix of the same order with inverse A^{-1} and B^{-1} respectively then

$(AB)^{-1}=B^{-1}A^{-1}$.

To get the inverse of the matrix $A=(a_{ij})$ in the beginning let us defined the cofactor of the element a_{ij} .

Let A be the n -square matrix whose determinate $|A|$.

When the elements of ith row and jth column are removed the determinate of the remaining $n-1-$ square matrix is called a first minor of A and donated by $|M_{ij}|$ or it is called the minor of a_{ij} and the amount $(-1)^{i+j}|M_{ij}|$ is called the cofactor of a_{ij} and is donated by C_{ij} .

Example 9

Find the cofactors of a_{11}, a_{21}, a_{23} if the matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Solution

$$\text{Cofactor of } a_{11} = \alpha_{11} = (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{23} - a_{23}a_{32}$$

$$\text{Cofactor of } a_{21} = \alpha_{21} = (-1)^3 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -(a_{12}a_{23} - a_{13}a_{32})$$

$$\text{Cofactor of } a_{23} = \alpha_{23} = (-1)^5 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}a_{32} - a_{12}a_{31})$$

Theorem 8

An n-square matrix A has an inverse if and only if it is non-singular.

i.e. A has an inverse if and only if $|A| \neq 0$.

Now we define the adjoint of the square matrix $A = (a_{ij})$.

If α_{ij} be the cofactor of a_{ij} , then by definition

$$\text{adjioint A=adj A=}\begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & & \alpha_{nn} \end{bmatrix}.$$

Note that the cofactors of the elements of i^{th} row (column) of A are the elements of the i^{th} column (row) of $adj\,(A)$.

Hence we can get the inverse of a matrix A from the relation:

$$A^{-1} = \frac{adjA}{|A|}.$$

Example 10

Find A^{-1} when $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & 4 & 5 \end{bmatrix}$.

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & 4 & 5 \end{vmatrix} = 15.$$

$$\left[\alpha_{ij}\right]=\begin{bmatrix} -9 & -15 & 12 \\ 8 & 5 & -4 \\ 2 & 5 & -1 \end{bmatrix}.$$

$$adj\,(A)=\begin{bmatrix} -9 & 8 & 2 \\ -15 & 5 & 5 \\ 12 & -4 & -1 \end{bmatrix}.$$

Hence

$$A^{-1}=\frac{1}{15}\begin{bmatrix} -9 & 8 & 2 \\ -15 & 5 & 5 \\ 12 & -4 & -1 \end{bmatrix}.$$

Vector Norms and Related Matrix Norms

The notation of vector norms is introduced as a scalar function to consider the length of a vector $x \in \mathbb{R}^n$ and is denoted by $\|x\|$, is a non negative real scalar which satisfies the following relations:

$1-\ \|x\|>0, \quad x \neq 0,$

$2-\|\alpha x\|\|\alpha\|\|x\|, \quad \alpha \in \mathbb{R}^n,$

$3-\ \|x+y\| \leq \|x\|+\|y\|, \ \forall x, y \in \mathbb{R}^n.$

The most popular and widely used norms are:

1- $\|x\|_1=\sum_{i=1}^{n}|x_i|$

2- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$, usually referred to as Euclidian norm.

3- $\|x\|_\infty = \max_i |x_i|$.

The matrix norm can given by:

$$\|A\|_p = \max_x \frac{\|Ax\|_p}{\|x\|_p},$$

$$1- \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|,$$

$$2- \|A\|_2 = \max_j \sum_{i=1}^{n} |a_{ij}|^2,$$

$$3- \|A\|_\infty = \max_x \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

Condition Number of A Matrix

The condition number of a square matrix A denoted by $\gamma(A)$, is a non-negative real scalar defined as:

$$\gamma(A) = \max_{u,v} \frac{\|Au\|_p}{\|Av\|_p}, \quad \|u\|_p = \|v\|_p = 1.$$

This number has a simple interpretation. Let the surface $\|x\| = 1$ be mapped by the transformation $y = Ax$ onto the surface S. Then this number represents the ratio between the smallest and the largest distances from the origin to points on S. Thus $\gamma(A) \geq 1$.

To compute $\gamma(A)$ write:

$$\gamma(A) = \max_{u,v} \frac{\|Au\|}{\|Av\|} = \frac{\max_{u} \frac{\|Au\|}{\|u\|}}{\min_{v} \frac{\|Av\|}{\|v\|}}$$

$$= \frac{\|A\|}{\min_{v} \frac{\|Av\|}{\|v\|}}.$$

But
$$\|v\| = \|A^{-1}Av\| \le \|A^{-1}\|.\|Av\| \Rightarrow$$
$$\frac{\|Av\|}{\|v\|} \ge \|A^{-1}\|^{-1} \Rightarrow \min_{v} \frac{\|Av\|}{\|v\|} = \|A^{-1}\|^{-1}.$$

And we finally obtain:

$$\gamma(A) = \|A^{-1}\|\|A\|.$$

If $\gamma(A)$ is large, A is called ill-conditioned. Usually if the value $|A|$ is small then we find that $\gamma(A)$ is large. This situation is not welcomed when solving system of linear equations, since small change in the coefficients of the equations causes a large displacement in the solution.

To see this, consider the following example.

Example 11

Consider the following system:

.835x + .667y = .168,
.333x + .266y = .067,

for which the exact solution is:

x = 1 and y = −1.

If b2 = .067 is only slightly perturbed to become ˆb2 = .066, then the exact solution changes dramatically to become

ˆx = −666 and ˆy = 834.

This is due to, in two dimensions, the angle between the two lines is very small.

Ill- conditioning may be regarded as an approach towards singularity. It gives inaccurate solutions due to the loss of significant figures during the computation.

Theory of Linear Equations

Linear dependence and independence of vectors

Definition

1- A set of vectors $u_1, u_2, \ldots, u_n$ are called linearly independent if there exists a set of scalars $c_1, c_2, \ldots, c_n$ such that the relation $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0$ implies that $c_1 = c_2 = \ldots = c_n = 0$. But if one or more of the scalars $c_1, c_2, \ldots, c_n$ can be nonzero, then the vectors are linearly dependent.

Notes

1. Any set of vectors containing the zero vector is linearly dependent.

2- Any subset of linearly independent set of vectors is also a set of linearly independent vectors.

3- If the vectors $u_1, u_2, \ldots, u_n$ are linearly dependent, at least one of them can be written as a linear combination of the others.

4- If the vector x can be written as a linear combination of the vectors $u_1, u_2, \ldots, u_n$ the set of vectors $x, u_1, u_2, \ldots, u_n$ form a linearly dependent set.

5- The rank of the set of vectors $u_1, u_2, \ldots, u_n$ is the maximum number of linearly independent vectors in the set.

6- The rank of the matrix A $\rho(A)$ is the maximum number of linearly independent vectors (rows or columns) in A .

Orthogonality of vectors

Definitions

1- The dot (inner) product of two vectors x, y is a scalar quantity denoted by $\langle x, y \rangle$ and is given by:

$$\langle x, y \rangle = x^* y \Rightarrow \langle x, x \rangle = x^* x = \|x\|^2 \Rightarrow \|x\| = \sqrt{\langle x, x \rangle} .$$

2- A set of vectors $u_1, u_2, \ldots, u_n$ are called orthogonal if their dot (inner) product noted by $\langle u_i, u_j \rangle = 0, \quad \forall i \neq j$.

3- A set of vectors $u_1, u_2, \ldots, u_n$ are called orthonormal if

$$\langle u_i, u_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

4- A matrix A is called orthogonal if $A'A = I \Rightarrow A^{-1} = A'$.

5- A complex matrix A is called a unitary matrix if $A^*A = I \Rightarrow A^{-1} = A^* = A^{-1}$.

Example 12

Let $u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}, y = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$. Obtain c_1, c_2, c_3 such that $y = c_1 u_1 + c_2 u_2 + c_3 u_3$.

Solution

The set u_1, u_2, u_3 are orthogonal.

Taking the inner product with u_1 we get: $c_1 = \dfrac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} = \dfrac{-3}{5}$.

Similarly, $c_2 = \dfrac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle} = \dfrac{-5}{3}$, and $c_3 = \dfrac{\langle y, u_3 \rangle}{\langle u_3, u_3 \rangle} = \dfrac{-7}{15}$.

Theorem 11

Non-zero orthogonal vectors are linearly independent.

Solution of Linear Equations Using Matrices

Consider a system of n linear equations in the n unknowns

$x_1, x_2, \ldots x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
\qquad (*)
$$

In matrix notation the system of linear equations (*) may be written as:

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.
$$

Or

$$Ax = B \quad , \qquad (**)$$

Where A is the coefficients matrix, x is $n \times 1$of variable matrix, c is the constant matrix.

The solution of this system depends on the matrix A as follows:

Case 1

A is nonsingular matrix(i.e. $|A| \neq 0$: the system $Ax = B$ has a unique solution.

1- If $B = 0$, the system $Ax = B$ has trivial solution $x = 0$.

If $B \neq 0$, the unique solution of the system may be obtained by many methods:

Inverse matrix method

If the matrix A is non- singular matrix then equation (**) has only one(a unique) solution, then solution can be obtained as follows:

Since $Ax = B$. Then multiply both sides in A^{-1} from left

$$A^{-1}AX = A^{1}B$$
$$\therefore I\ X = A^{-1}B$$
$$\therefore X = A^{-1}B.$$

Example 14

Solve the following linear equation

$x_1 + 2x = 1$, $3x_1 - x_2 + x_3 = 2, 4x_2 + 5x_3 = -1.$

Solution

These equations can be written in matrix from as:

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & -1 \\ 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

i.e. $Ax = B$.

$\therefore x = A^{-1}B,$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & -1 \\ 0 & 4 & 5 \end{bmatrix}.$$

From the above example, we have: $A^{-1}=\frac{1}{15}\begin{bmatrix}-9 & 8 & 2\\ -15 & 5 & 5\\ 12 & -4 & -1\end{bmatrix}$.

$$\therefore x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{15}\begin{bmatrix}-9 & 8 & 2\\ -15 & 5 & 5\\ 12 & -4 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}=\frac{1}{15}\begin{bmatrix}-9+16-2\\ -15+10-5\\ 12-4+1\end{bmatrix}$$

$$x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{15}\begin{bmatrix}5\\ -10\\ 5\end{bmatrix}=\begin{bmatrix}\frac{1}{3}\\ \frac{-2}{3}\\ \frac{1}{3}\end{bmatrix}$$

$$\therefore x=\frac{1}{3},\quad x_2=\frac{-2}{3}\quad ,x_3=\frac{1}{3}.$$

Gauss elimination method

Consider the system of n equations (*) in the unknowns can be written in the system from (**) and we assume that the matrix of coefficients A is nonsingular matrix then we can use Gauss Elimination method to solve this system. This method is very simple where less product operations will be needed. The main idea of this method is elimination and substitution where we define certain variable for one equation and try to eliminate it from the other equations using rows operations only, i.e. multiply the row in certain number then it is added to the following row and soon. To detect this method see the following example.

Example 13

Solve the system of linear equations using Gauss elimination method.

$$2x_1 + x_2 + x_3 = 1,$$
$$4x_1 + x_2 = -2,$$
$$-2x_1 + 2x_2 + x_3 = 7.$$

Solution

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

The augmented matrix is given by:

$$\begin{bmatrix} 2 & 1 & 1 & \vdots & 1 \\ 4 & 1 & 0 & \vdots & -2 \\ -2 & 2 & 1 & \vdots & 7 \end{bmatrix}.$$

The matrix A must be transformed to an upper triangle matrix using elimination. In the last we find that the third equation has only one variable hence we can obtain the value of this variable similarly the second equation will have only two variables one of them is known so we can get the second and so on. Now we will apply the method to solve this example. Multiply the first row by -2 and adding to the second row , and adding the first to the third we have:

$$\begin{bmatrix} 2 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & -4 \\ 0 & 3 & 2 & \vdots & 8 \end{bmatrix}.$$

The number 2 in the first equation is called (pivot) and in general we can divide the first row by 2 before multiplication operations where the pivot becomes 1 hence the calculation be easier to complete: we neglect the first equation and deal with the second equation as before, then

$$\begin{bmatrix} 2 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -2 & \vdots & -4 \\ 0 & 0 & -4 & \vdots & -4 \end{bmatrix}.$$

Now the third equation is

$$-4x_3 = -4 \Rightarrow \qquad x_3 = 1.$$

The second equation is

$$-x_2 - 2x_3 = -4$$
$$\therefore x_2 = -2(1) + 4 = 2,$$

and finally the first equation is:

$$2x_1 + x_2 + x_3 = 1,$$
$$x_1 = \frac{1}{2}[1 - x_2 - x_3] = \frac{1}{2}[1 - 2 - 1] = -1,$$
$$x_1 = -1, x_2 = 2, x_3 = 1.$$

Gauss-Jordan method

Operate on the augmented matrix by series of row elementary operation to transform the matrix A into the unit matrix I , the vector B transformed to the solution.

$$\tilde{A} = \begin{array}{|c|c|} \hline \quad A \quad & \quad B \quad \\ \hline \downarrow & \downarrow \\ I & x \end{array}$$

Example 13

Use Gauss-Jordan' method to solve the system of linear equations $Ax + 2B = C$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 3 \\ 3 & 4 & -4 \end{pmatrix}, B = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}, C = \begin{pmatrix} 7 \\ 10 \\ 2 \end{pmatrix}.$$

Solution

The above system equivalent to the system:

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 3 \\ 3 & 4 & -4 \end{pmatrix} x = \begin{pmatrix} 11 \\ 8 \\ -10 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 4 & 11 \\ 2 & 5 & 3 & 8 \\ 3 & 4 & -4 & -10 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 4 & 11 \\ 0 & -1 & -5 & -14 \\ 0 & -5 & -16 & -43 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -11 & -31 \\ 0 & 1 & 5 & 14 \\ 0 & 0 & 9 & 27 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Note that we can obtain the inverse of A through the use of Gauss-Jordan method by solving the system $Ax = I$.

$$\tilde{A} = \begin{array}{|c|c|} \hline \quad A \quad & \quad I \quad \\ \hline \downarrow & \downarrow \\ I & A^{-1} \end{array}$$

Example 14

Use Gauss-Jordan method to find the inverse of

$$A = \begin{pmatrix} -1 & 2 & 3 & -3 \\ 2 & -4 & -3 & 5 \\ 1 & -3 & 2 & 2 \\ -2 & 5 & 11 & -8 \end{pmatrix}.$$

Solution

Operate by row elementary operations on the matrix

$$\left(\begin{array}{cccc} -1 & 2 & 3 & -3 \\ 2 & -4 & -3 & 5 \\ 1 & -3 & 2 & 2 \\ -2 & 5 & 11 & -8 \end{array} \vdots \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ until arriving the form

$$\left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 21 & 11 & -4 & -2 \\ 0 & 1 & 0 & 0 & -13 & -5 & 1 & 2 \\ 0 & 0 & 1 & 0 & -7 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & -23 & -10 & 3 & 3 \end{array}\right).$$

$$\therefore A^{-1} = \begin{pmatrix} 21 & 11 & -4 & -2 \\ -13 & -5 & 1 & 2 \\ -7 & -3 & 1 & 1 \\ -23 & -10 & 3 & 3 \end{pmatrix}.$$

Lecture 9

Case2

A is singular or rectangular with $m \le n$.

The system $Ax = B$ has either no solution if $\rho(A) \neq \rho(\tilde{A})$ or infinite number of solutions if $\rho(A) = \rho(\tilde{A})$ that can be obtained as follows:

Operate on the above system of equations by a series of elementary row operations until the matrix A is transformed into an echelon form like the following:

$$\begin{bmatrix} \text{Echelon form} \\ 0_{m-n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1' \\ b_\rho' \\ b_{\rho+1}' \\ b_n' \end{bmatrix}.$$

$$\begin{bmatrix} I_{\rho,\rho} & Q_{\rho,n-\rho} \\ 0_{m-n,n} & \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_\rho \\ x'_{\rho+1} \\ x'_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_\rho \\ \vdots \\ 0 \end{bmatrix}.$$

And the general solution of the system is given by:

$$\begin{bmatrix} x'_1 \\ x'_\rho \\ x'_{\rho+1} \\ x'_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_\rho \\ \vdots \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} q_1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} q_2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \ldots + k_{n-\rho} \begin{bmatrix} q_{n-\rho} \\ 0 \\ \vdots \\ -1 \end{bmatrix},$$

where $k_1, k_2, \ldots, k_{n-\rho}$ are any arbitrary constants.

Example 15

Solve the system

$$\left[\begin{array}{cccccc} 1 & 2 & -3 & -4 & \vdots & 6 \\ 1 & 3 & 1 & -2 & \vdots & 4 \\ 2 & 5 & -2 & -5 & \vdots & 10 \end{array}\right].$$

Solution

$$[A:B] = \left[\begin{array}{cccccc} 1 & 2 & -3 & -4 & \vdots & 6 \\ 1 & 3 & 1 & -2 & \vdots & 4 \\ 2 & 5 & -2 & -5 & \vdots & 10 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & -4 & -8 & 10 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 0 & 10 \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Replacing the 4^{th} and the third one we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -4 \\ 4 \\ 0 \\ -1 \end{bmatrix}.$$

Note that: in some problems we find that there is a pivot has a value zero and hence we can not complete the problem.

In this case to complete the solution we exchange this row(which contains a zero pivot) by its following row or the second following row then we complete the method to get the solution.

The Matrix Eigenvalue Problem

If A is n-square matrix then the square matrix $A-\lambda I$ (where I is the unit matrix) is called the characteristic matrix of A while the determinate $|A-\lambda I|$ is called characteristic polynomial of A and the equation $|A-\lambda I|=0$ is called characteristic equation of A. Its roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called characteristic roots or (eigenvalues) of A. For every eigen-value there exists a vector called the eigenvector which is the solution of the system of equations:

$$(A-\lambda I)x = 0, \quad x \neq 0.$$

Example 16

Find the eigenvalues and eigenvectors of

$$A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}.$$

Solution

The characteristic equation of A is $|A-\lambda I|=0$.

$$\left|\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=\left|\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}-\begin{pmatrix}1 & \lambda\\ \lambda & 1\end{pmatrix}\right|$$

$$=\left|\begin{pmatrix}1-\lambda & 0\\ 0 & 2\text{-}\lambda\end{pmatrix}\right|=(1-\lambda)(2-\lambda)-0=(1-\lambda)(2-\lambda)=0.$$

The characteristic roots of A are $\lambda=1$, $\lambda=2$.

For $\lambda=1$, we have $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}x_1^{'}=0\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}x_1=0$

$$\therefore x_1=\begin{bmatrix}1\\ -1\end{bmatrix}.$$

For $\lambda=2\Rightarrow\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}x_2=0\Rightarrow x_2=\begin{bmatrix}1\\ -1\end{bmatrix}.$

Example 17

Determine the eigenvalues and eigenvectors of

$$A=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix},$$

the characteristic equation of A is $|A-\lambda I|=0$.

$$\therefore \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = -(\lambda^3 - 7\lambda^2 + 11\lambda - 5) = 0$$

$$\therefore \lambda^3 - \lambda\lambda^2 + 11\lambda - 5 = 0$$

The characteristic roots are $\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 1.$

When $\lambda = 5,$ we have:

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} x_1^{'} = 0 \Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{pmatrix} x_1^{'} = 0 \Rightarrow$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} x_1^{'} = 0 \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} x_1^{'} = 0 \Rightarrow$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{At } \lambda = 1 \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} x_2^{'} = 0 \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_{2,3} = 0 \Rightarrow$$

$$x_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \& \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Notes

1- If $Ax = \lambda x$. Then, multiplying by A from left, we get:

$$AAx = A\lambda x = \lambda Ax = \lambda^2 x \ .$$

This means that the eigenvalues of A^2 equal to λ^2 and has same eigenvector x .

2- Multiplying by A^{-1}, we get

$$Ax = \lambda x \Rightarrow A^{-1}Ax = \lambda A^{-1}x \Rightarrow x = \lambda A^{-1}x$$

$$A^{-1}x = \lambda^{-1}x = \frac{1}{\lambda}x \quad .$$

Lecture 10

Cayley – Hamilton Theorem

A matrix satisfies its own characteristic equation.

Example 21

Verify the Cayley–Hamilton theorem for the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}.$$

Solution

The characteristic equation is:

$$\begin{vmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 14 = 0.$$

$$A^2 = \begin{pmatrix} 7 & 14 \\ -7 & 14 \end{pmatrix}, \ -7A = \begin{pmatrix} -21 & -14 \\ 7 & -28 \end{pmatrix}.$$

$$A^2 - 7A + 14I = \begin{pmatrix} 7 & 14 \\ -7 & 14 \end{pmatrix} + \begin{pmatrix} -21 & -14 \\ 7 & -28 \end{pmatrix} + \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = 0.$$

Then A satisfies its own characteristic equation.

Example 19

Prove that $A=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$ satisfies Cayley- Hamilton theorem.

Solution

$$A=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$$

$$A^2=AA=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}=\begin{bmatrix}7 & 12 & 6\\ 6 & 13 & 6\\ 6 & 12 & 7\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}7 & 12 & 6\\ 6 & 13 & 6\\ 6 & 12 & 7\end{bmatrix}\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}=\begin{bmatrix}32 & 62 & 31\\ 31 & 63 & 31\\ 31 & 62 & 32\end{bmatrix}.$$

The characteristic equation of A is $|A-\lambda I|=0$

$$\therefore \lambda^3-7\lambda^2+11\lambda-5=0$$

$$\therefore \begin{bmatrix}32 & 62 & 31\\ 31 & 63 & 31\\ 31 & 62 & 32\end{bmatrix}\begin{bmatrix}7 & 12 & 6\\ 6 & 13 & 6\\ 6 & 12 & 7\end{bmatrix}+11\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$$

$$-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=0$$

$\therefore$ A satisfies

$$A^3-7A^2+11A-5I=0$$

$\therefore A$ satisfies Cayley – Hamilton' Theorem.

Example 20

If $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$.

Find

1- Characteristic matrix of A .

2- Characteristic equation of A .

3- Characteristic roots.

4- Use Cayley-Hamilton theorem to obtain the inverse matrix of A .

Solution

1- characteristic matrix of A is $(A-\lambda I) = \begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}$

2- the characteristic equation of A is $|A-\lambda I| = 0$.

$\therefore \lambda^3 - 2\lambda^2 + 5\lambda + 6 = 0$

3- characteristic roots of A are

$\lambda_1 = 1 \quad , \lambda_2 = 2 \quad , \lambda_3 = 3$.

4- from Cayley- Hamilton theorem every square matrix satisfies its characteristic equation

$\therefore A^3 - 2A^2 - 5A + 6I = 0$. (1)

Multiply (1) in A^{-1}

$$\therefore\ A^3 - 2A - 5I + 6A^{-1} = 0$$

$$\therefore A^{-1} = \frac{-1}{6}[A^2 - 2A - 5I]$$

$$A^2 = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 3 \\ 4 & -2 & 7 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I].$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -5 & -3 & -1 \\ -4 & -2 & -3 \\ -4 & 2 & -7 \end{bmatrix} + \begin{pmatrix} 4 & -4 & 6 \\ 2 & 2 & 2 \\ 2 & 6 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -7 & 5 \\ 2 & 5 & -1 \\ -2 & 8 & 10 \end{bmatrix}.$$

Theorems on eigenvalues and eigenvectors

Theorem 13

1- The eigenvalues of a Hermitian matrix (or symmetric real matrix) are real.

2- The eigenvalues of a skew-Hermitian matrix (or skew-symmetric real matrix) are zero or pure imaginary.

3- The eigenvalues of a unitary matrix (or real orthogonal matrix) all have absolute value equal to 1.

Theorem 14

The eigenvectors belonging to different eigenvalues of a Hermitian matrix (or symmetric real matrix) are orthogonal.

Theorem 15

Reduction of matrix to diagonal form

If a nonsingular matrix A has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with corresponding eigenvectors written as columns in the matrix

$B = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$, then

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = D_\lambda .$$

The matrix $B^{-1}AB$, called the transform of A by B, is a diagonal matrix containing the eigenvalues of A in the main diagonal and zeros elsewhere. We say that A is transformed or reduced to diagonal form.

Note

$A = BD_\lambda B^{-1} \Rightarrow A^2 = (BD_\lambda B^{-1})(BD_\lambda B^{-1}) = BD_\lambda^2 B^{-1}$

$A^k = BD_\lambda^k B^{-1}$.

Example 24

Transform $A = \begin{pmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{pmatrix}$ to a diagonal form.

Solution

The characteristic equation is given by:

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow \lambda = 1, 2, 3.$$

The corresponding eigenvectors are:

$$x_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \; x_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \; x_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\therefore B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} -1 & -3 & 2 \\ 2 & 5 & -3 \\ -1 & -1 & 1 \end{pmatrix}.$$

$$\therefore B^{-1}AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$