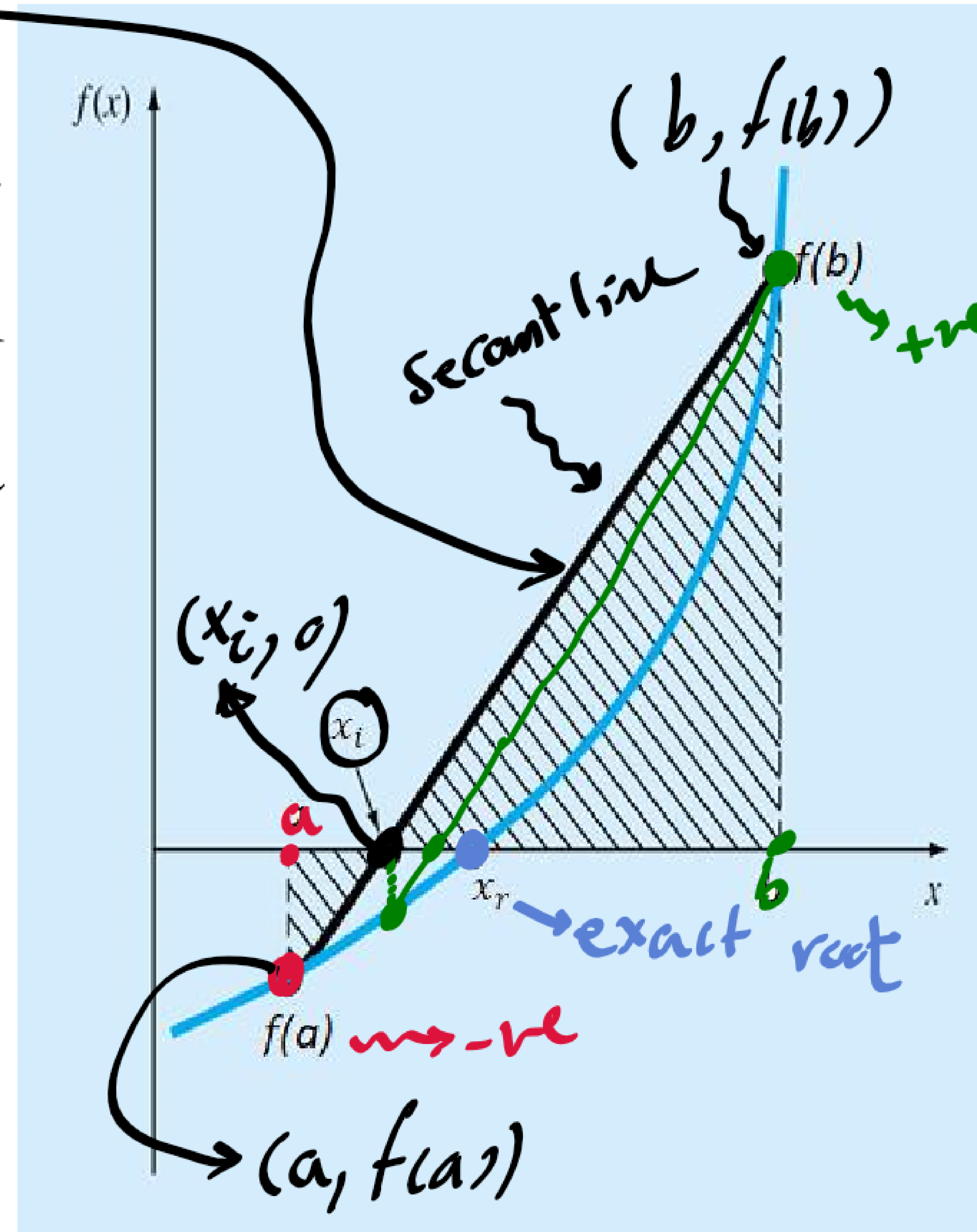
The False-Position Method

$$\frac{-f(b)}{x_i - b} = \frac{f(a)-f(b)}{a-b}$$

$$\frac{\overset{0}{\cancel{y}} - f(b)}{\underset{x_i}{\cancel{x}} - b} = \frac{f(a)-f(b)}{a-b}$$

If a real **root** is bounded by a and b of $f(x) = 0$, then we can estimate this **root** x_r by doing a linear interpolation $L(x)$ between the points $(a, f(a))$ and $(b, f(b))$ to find the x_i value such that $L(x_i) = 0$.



The False-Position Method

For the arbitrary equation of one variable, $f(x) = 0$

1. Find a pair of values a and b where $f(a) < 0$ and $f(b) > 0$.
2. Estimate the **root** x_r by the value

$$\boxed{x_i = \frac{af(b) - bf(a)}{f(b) - f(a)}} \qquad (i = 1, 2, \cdots, n).$$

3. If $f(x_i) = 0$ then $x_r = x_i$ will be the exact desired **root**.
4. For $f(x_i) \neq 0$, to start the next iteration, the value of x_i will be used as the new value of a if $f(x_i) < 0$ or it will be used as the new value of b if $f(x_i) > 0$.

The False-Position Method

5. Repeat steps 2, 3, and 4 by using the new interval $[a, b]$.
6. The number of required iterations is n depending on the desired accuracy (ε) where the absolute error $|x_r - x_n| < \varepsilon$ and the root will be $x_r \approx x_n$.

The False-Position Method (Example)

Use the false-position method to obtain the smaller positive root of the algebraic equation $x^3 - 7x + 1 = 0$ correct to three decimal places.

<u>Solution:</u> Let $f(x) = x^3 - 7x + 1$

$$f(0) = 1 \text{ and } f(1) = -5$$

So there is a root inside the interval $[0, 1]$.

$$a = 0 \text{ and } b = 1.$$

$$\xrightarrow{\text{Correct to 3 decimal places}} \varepsilon = 5 \times 10^{-4}$$

The False-Position Method (Example)

i	a (f is +ve)	b (f is -ve)	$x_i=[af(b)-bf(a)]/[f(b)-f(a)]$	$f(x_i)$
0	0	1	0.166666667	-0.162037037
1	0	0.166666667	0.143426295	-0.001033627
2	0	0.143426295	**0.143278199**	-6.08357E-06

The root will be $x_r \approx \mathbf{0.143}$ correct to 3 decimal places.

Bisection
False-Position] → Bracketing Methods

Lecture 2: Roots of Equations
(Open Methods)

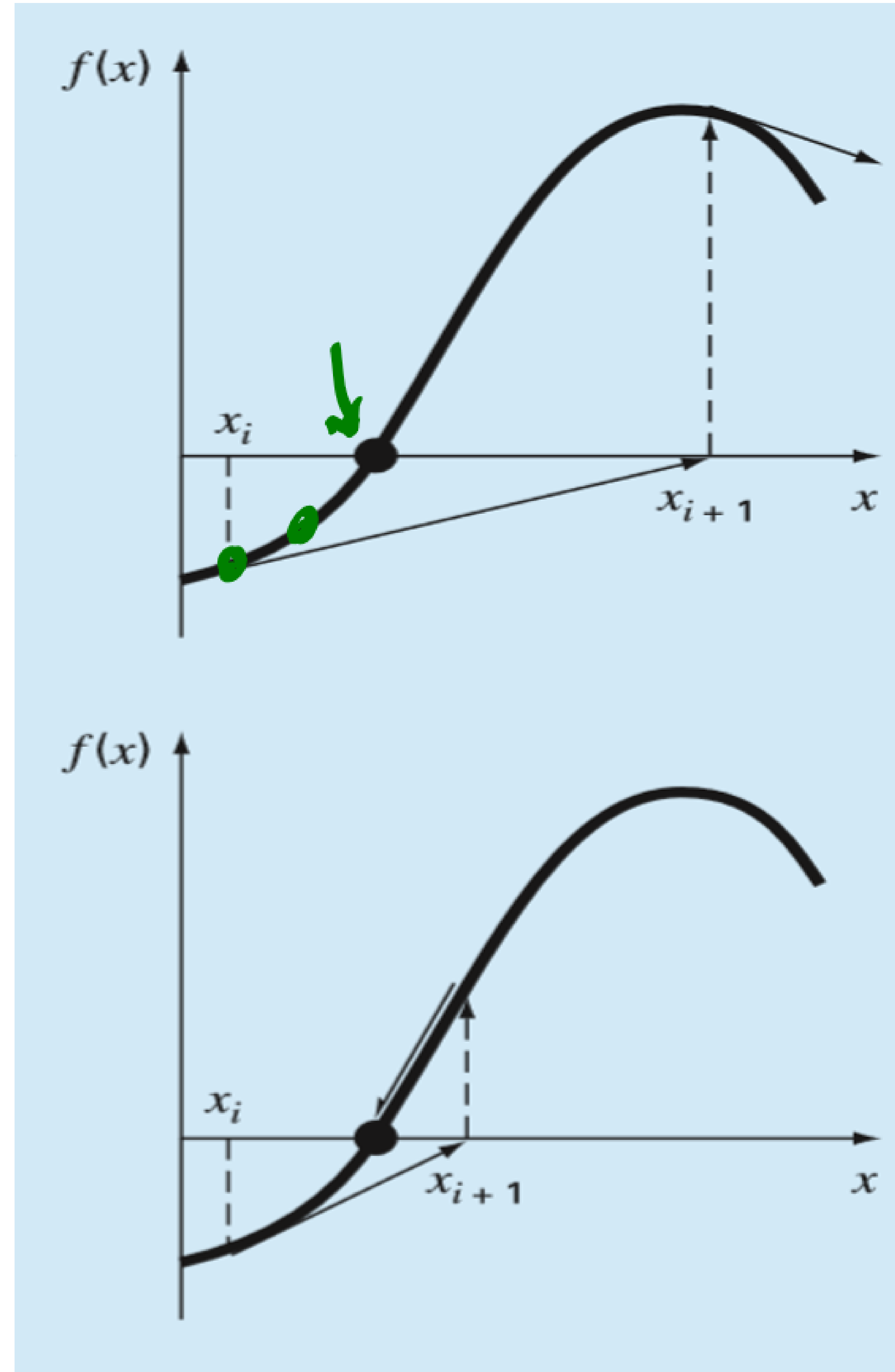
By
Dr. Khaled M. Abdelgaber

Open Methods

- **Open methods** are based on formulas that require only a single starting value of x or two starting values that do **not** necessarily **bracket** the root.

- **They** sometimes move away (diverge) from the true root as the computation progresses.

- However, when the **open methods** converge, they usually do so much more quickly than the **bracketing** methods.



Simple Fixed-point Iteration

$7x = x^3 + 1$

$\boxed{x = \frac{x^3+1}{7}}$

For the arbitrary equation of one variable, $\boxed{f(x) = 0}$

1. Rearrange the function so that x is on the left side of the equation:

   $x(x^2-7) = -1 \rightarrow \boxed{x = \frac{-1}{x^2-7}}$

   $f(x) = x^3 - 7x + 1 = 0$

   $$\boxed{f(x) = 0} \rightarrow \boxed{x = g(x)}$$

   $x^3 = 7x - 1$

   $\boxed{x = \sqrt[3]{7x-1}}$

2. Estimate the **root** x_r by the value

   $x_0 = 2$

   $$x_i = g(x_{i-1}), \qquad i = 1, 2, \cdots, n$$

   where the iteration process is started by assuming an initial guess x_0.

Simple Fixed-point Iteration

Fixed-point methods may sometime "**diverge**", depending on the

x_0

stating point (initial guess, x_0) and how the function $g(x)$ looks like.

For example: $\boxed{f(x) = x^2 - x - 2, \quad x > 0}$

$$f(x) = 0 \rightarrow x = x^2 - 2 \rightarrow g(x) = x^2 - 2$$

$$f(x) = 0 \rightarrow x^2 = x + 2 \rightarrow x = \sqrt{x+2} \rightarrow g(x) = \sqrt{x+2}$$

$$f(x) = 0 \rightarrow x^2 = x + 2 \xrightarrow{\div x} x = 1 + \frac{2}{x} \rightarrow g(x) = 1 + \frac{2}{x}$$

What is the suitable $g(x)$ to converge to solution?

Simple Fixed-point Iteration

Fixed-point iteration converges if

$$|g'(x_i)| < 1, \qquad \text{for all } x_i, \qquad i = 0, 1, 2, \cdots$$

For example:

$$\boxed{f(x) = x^2 - x - 2, \quad x > 0}$$

$$g(x) = x^2 - 2 \rightarrow g'(x) = 2x \rightarrow |g'(x)| = |2x| < 1 \text{ if } -\frac{1}{2} < x < \frac{1}{2}$$

$\rightarrow$ **Not suitable for all cases?!**

$$g(x) = \sqrt{x+2} \rightarrow g'(x) = \frac{1}{2\sqrt{x+2}} \rightarrow |g'(x)| = \frac{1}{2\sqrt{x+2}} < 1$$

if $x > -\frac{7}{4}$ $\rightarrow$ **Suitable for all cases?!**

$$g(x) = 1 + \frac{2}{x} \rightarrow g'(x) = -\frac{2}{x^2} \rightarrow |g'(x)| = \frac{2}{x^2} < 1$$

if $x > \sqrt{2}$ and $x < -\sqrt{2}$ $\rightarrow$ **Suitable for some cases?!**

Simple Fixed-point Iteration (Example)

Use the simple fixed-point iteration to obtain the smaller positive root of the algebraic equation $x^3 - 7x + 1 = 0$ correct to three decimal places.

<u>Solution:</u> $$f(x) = x^3 - 7x + 1$$

$$x = \frac{x^3 + 1}{7} \rightarrow \boxed{g(x) = \frac{1}{7}(x^3 + 1)}$$

Select the initial guess to be $\boxed{x_0 = 0}$

$$x_i = \frac{1}{7}\left(x_{i-1}^3 + 1\right), \qquad i = 1, 2, \cdots, n$$

Simple Fixed-point Iteration (Example)

i	x_{i-1}	$x_i = \frac{1}{7}\left(x_{i-1}^3 + 1\right)$	$\lvert x_i - x_{i-1}\rvert$
1	0	**0.142857143**	0.142857143
2	**0.142857143**	**0.143273636**	**0.000416493**

The root will be $\boldsymbol{x_r \approx 0.143}$ correct to 3 decimal places.

$$\frac{-f(b)}{x_i - b} = \frac{f(a) - f(b)}{a - b}$$

$$f(b)\,[b-a] = \left[f(a) - f(b)\right]\left[x_i - b\right]$$

$$\frac{f(b)\,(b-a)}{f(a) - f(b)} = x_i - b$$

$$x_i = \frac{b}{1} + \frac{f(b)\,(b-a)}{f(a) - f(b)}$$

$$x_i = \frac{b f(a) - \cancel{b f(b)} + f(b)(\cancel{b} - a)}{f(a) - f(b)}$$

false position

$$\boxed{x_i = \frac{b\,f(a) - a\,f(b)}{f(a) - f(b)}}$$

value close to x_r

Use the false-position method to obtain the smaller positive root of the algebraic equation $x^3 - 7x + 1 = 0$ correct to three decimal places.

Sol $f(x) = x^3 - 7x + 1$

$x=0$ $\quad f(0) = 0 - 0 + 1 = 1$ (+ve)

$x=1$ $\quad f(1) = 1 - 7 + 1 = -5$ (-ve)

$\therefore x_r \in [0, 1]$

c

$x_i = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

a (+ve)	b (-ve)	$x_i = \frac{a f(b) - b f(a)}{f(b) - f(a)}$
0	1	0.16666666---- (-ve)
0	0.166666	0.1434262948 (-ve)
0	0.1434262948	0.1432781986

Fixed pt

$f(x)=0 \longrightarrow x = g(x) \rightarrow$!!!

Condition $\quad |g'(x)| < 1 \quad$ for all x

Ex $\quad f(x) = x^2 - x - 2 = 0$

$x = x^2 - 2$

$\boxed{g(x) = x^2 - 2}$

$g' = 2x$

$|g'| = |2x|$

$\boxed{|g'| < 1} \rightsquigarrow |2x| < 1$

$-1 < 2x < 1$

$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

α

$x^2 = x + 2$

$x = \sqrt{x+2}$

$\boxed{g(x) = \sqrt{x+2}}$

$g' = \frac{1}{2\sqrt{x+2}}$

$|g'| < 1 \rightsquigarrow \frac{1}{2\sqrt{x+2}} < 1$

$\rightsquigarrow 2\sqrt{x+2} > 1$

$\sqrt{x+2} > \frac{1}{2} \rightsquigarrow x + 2 > \frac{1}{4}$

$\boxed{\cancel{x > -\frac{7}{4}}}$ ✓

$x^2 = x + 2$

$\div x \rightsquigarrow x = 1 + \frac{2}{x}$

$\boxed{g(x) = 1 + \frac{2}{x}}$

$g' = -\frac{2}{x^2} \qquad |g'| = \frac{2}{x^2}$

$\boxed{|g'| < 1} \rightsquigarrow \frac{2}{x^2} < 1$

$\frac{x^2}{2} > 1 \rightsquigarrow x^2 > 2$

$\boxed{x > \sqrt{2} \quad \text{or} \quad x < -\sqrt{2}}$ ✓✓✓

$x(x-1) = 2 \rightarrow x = \frac{2}{x-1}$

$\boxed{g(x) = \frac{2}{x-1}}$

$g' = \frac{-2}{(x-1)^2}$

$\cancel{|g'| = \frac{2}{(x-1)^2} \quad \frac{2}{(x-1)^2} < 1 \quad (x-1)^2 > 2 \quad x-1 > \sqrt{2} \quad x - 1 < -\sqrt{2}}$

Use the simple fixed-point iteration to obtain the smaller positive root of the algebraic equation $x^3 - 7x + 1 = 0$ correct to three decimal places.

Sol ① $x = \frac{x^3+1}{7} \longrightarrow g(x) = \frac{x^3+1}{7} \to g' = \frac{3}{7}x^2 \to \frac{3}{7}x^2 < 1 \rightsquigarrow x^2 < \frac{7}{3}$

$x \in \left] -\frac{\sqrt{7}}{\sqrt{3}}, \frac{\sqrt{7}}{\sqrt{3}} \right[$ الفترة

② $x^3 = 7x - 1 \rightsquigarrow x = \sqrt[3]{7x-1} \longrightarrow g(x) = \sqrt[3]{7x-1} \rightsquigarrow g' = \frac{1}{3}(7x-1)^{-2/3}(7)$

$g' = \frac{7}{3\sqrt[3]{(7x-1)^2}} \xrightarrow{<1} \frac{7}{3\sqrt[3]{(7x-1)^2}} < 1 \rightsquigarrow \frac{3\sqrt[3]{(7x-1)^2}}{7} > 1$

$\sqrt[3]{(7x-1)^2} > \frac{7}{3} \rightsquigarrow (7x-1)^2 > \frac{7^3}{27}$

$7x - 1 > \sqrt{\frac{7^3}{27}}$ or $7x - 1 < -\sqrt{\frac{7^3}{27}}$

Use the simple fixed-point iteration to obtain the smaller positive root of the algebraic equation $x^3 - 7x + 1 = 0$ correct to three decimal places.

$$x = \frac{x^3+1}{7}$$

$$x_{i+1} = g(x_i)$$

$\boxed{x_0 = 0}$

x_i (ans)	$x_{i+1} = \frac{x_i^3+1}{7}$ (ans) (ans)
0	0.1428571
	0.143273636
	0.14327772894