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Contagion

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Each player in an infinite population interacts strategically with a finite subset of that population. Suppose each player's binary choice in each period is a best response to the population choices of the previous period. When can behaviour that is initially played by only a finite set of players spread to the whole population? This paper characterizes when such contagion is possible for *arbitrary* local interaction systems. Maximal contagion occurs when local interaction is sufficiently uniform and there is *low neighbour growth*, *i.e.* the number of players who can be reached in k steps does *not* grow exponentially in k.

1. INTRODUCTION

When large populations interact strategically, players may be more likely to interact with some players than others. A local interaction system describes a set of players and specifies which players interact with which other players. If in addition, each player at each location has a set of available actions and a payoff function from each of his various interactions, we have a *local interaction game*. The strategic problem becomes interesting when it is assumed that players cannot tailor their behaviour for each neighbour, but must choose a constant action for all neighbours.

A recent literature has examined such local interaction games.[1] A key finding of that analysis is that local interaction may allow some forms of behaviour to spread in certain dynamic systems. For example, suppose that players are arranged along a line, and each player interacts with his two neighbours. An action is $\frac{1}{2}$-dominant if it is a best response when a player has at least one neighbour playing that action.[2] Ellison (1993) showed that if an action was $\frac{1}{2}$-dominant at every location and was played at *any* pair of neighbouring locations, then best response dynamics alone would ensure that it would eventually be played everywhere.[3]

A number of papers have explored how robust this type of phenomenon is to the structure of the local interaction. For example, two-dimensional lattices have been much studied (Anderlini and Ianni (1995), Blume (1995), Ellison (2000)). Blume (1995) considered local interaction systems where locations are on an m-dimensional lattice and there is a translation invariant description of the set of neighbours. Unfortunately, it is hard to know what to make of results which rely on a particular geometric structure. It is not

1. The relevant pure game theory literature includes Anderlini and Ianni (1996), Berninghaus and Schwalbe (1996*a*, 1996*b*), Blume (1993, 1995), Ellison (1993, 2000), Goyal (1996), Galeshoot and Goyal (1997), Ianni (1997), Mailath, Samuelson and Shaked (1997*a*) and Young (1998, Chapter VI). See Durlauf (1997) for a survey of the closely related economics literature on local interaction. This paper follows those literatures in taking the local interaction system as exogenous. See Ely (1997) and Mailath, Samuelson and Shaked (1997*b*) for models with endogenous local interaction.

2. If there are only two possible actions in a symmetric two player game, both players choosing the $\frac{1}{2}$-dominant action is risk dominant in the sense of Harsanyi and Selten (1988).

3. Ellison (1993) and much of the literature cited above are concerned with stochastic versions of best response dynamics; as I discuss briefly in Section 7, some of the conclusions are driven by properties of deterministic best response dynamics.

clear that the study of lattices will explain which qualitative features of neighbourhood relations determine strategic behaviour.

The primary purpose of this paper is to develop techniques for analysing *general* local interaction systems. It is useful to focus on one narrow strategic question in order to explore the effect of changes in the local interaction system.[4] In particular, I consider an infinite population of players. Each player interacts with some finite subset of the population and must choose one of two actions (0 and 1) to play against all neighbours. There exists a critical number q between 0 and 1 such that action 1 is a best response for a player if at least proportion q of his neighbours plays 1. Players are assumed to revise their actions according to deterministic best response dynamics. Contagion is said to occur if one action—say, action 1—can spread from a finite set of players to the whole population. In particular, for any given local interaction system, there is a critical *contagion threshold* such that contagion occurs if and only if the payoff parameter q is less than the contagion threshold.

Ellison's argument discussed above shows that the contagion threshold for interaction on a line is $\frac{1}{2}$. In fact, the contagion threshold is at most $\frac{1}{2}$ in *all local interaction systems*. This paper provides a number of characterizations of the contagion threshold. A group of players is said to be *p-cohesive* if every member of that group has at least proportion p of his neighbours within the group. I show that the contagion threshold is the smallest p such that every "large" group (consisting of all but a finite sets of players) contains an infinite, $(1-p)$-cohesive, subgroup. I also show that the contagion threshold is the largest p such that it is possible to label players so that, for any player with a sufficiently high label, proportion at least p of his neighbours has a lower label. These characterizations provide simple techniques for calculating the contagion threshold explicitly in examples.

Contagion is most likely to occur if the contagion threshold is close to its upper bound of $\frac{1}{2}$. I show that the contagion threshold will be close to $\frac{1}{2}$ if two properties hold. First, there is *low neighbour growth*: the number of players who can be reached in k steps grows less than exponentially in k. This will occur if there is a tendency for players' neighbours' neighbours to be their own neighbours. Second, the local interaction system must be sufficiently *uniform*, *i.e.* there is some number α such that for all players a long way from some core group, roughly proportion α of their neighbours are closer to the core group.

While I focus on this one contagion question, the techniques and critical properties described are important in a range of strategic local interaction problems:

- When do there exist equilibria with co-existent conventions, *i.e.* equilibria of the local interaction game where both actions are played? A low contagion threshold implies the existence of such "co-existent equilibria" for a wide range of payoff parameters. In Section 6, I show (under the low neighbour growth assumption) that co-existent equilibria exist whenever the payoff parameter is more than the contagion threshold and less than one minus the contagion threshold. One consequence is that co-existent equilibria *always* exist in the (extreme) case of exactly symmetric payoffs.
- The literature on local interaction games cited above has focused on *stochastic* revision processes. In Section 7, I discuss how stochastic processes built around best response dynamics are related to the deterministic process of this paper, and thus how they depend on the properties of general local interaction systems studied here.

4. Goyal (1996) uses a rich set of examples to examine the effect of changes in the local interaction system.

This paper builds on two literatures. The questions studied and the formal framework used are very close to the earlier literature on local interaction games (see Footnote 1). When applied to lattice examples, the contribution of this paper is to provide a useful language for discussing the structure of local interaction that can be used to generalize arguments already used in that literature (especially, those of Blume (1995)). But much more importantly, this approach allows a discussion of the key qualitative properties of local interaction systems that does not rely on special features of lattices.

The inspiration for this work is an apparently unrelated literature on the role of higher-order beliefs in incomplete information games. It is possible to show a formal equivalence between local interaction games and incomplete information games. The formal techniques in this paper are then analogues of the belief operator techniques, introduced by Monderer and Samet (1989), and used in the higher-order beliefs literature.[5] However, this relationship is explored in detail in a companion piece (Morris (1997*b*)), so in this paper, the ideas are developed independently.

The paper is organized as follows. Local interaction games are introduced, and contagion threshold defined, in Section 2. Some examples are discussed in Section 3; these illustrate the questions studied but also highlight the risks of taking simple interaction systems too seriously. The crucial general properties of interaction systems are introduced and discussed in Section 4. The main results—characterizations of the contagion threshold—are presented in Section 5. Section 6 presents the results on the co-existence of conventions. Various ways of adding random elements to this paper's deterministic dynamic are discussed in Section 7. Section 8 concludes.

2. LOCAL INTERACTION GAMES

A local interaction game consists of a *local interaction system* describing how players interact and their payoffs from those interactions.

A local interaction system consists of an infinite population, each of whom interacts with a finite subset of the population.[6] So fix a countably infinite set of players $\mathscr{X}$ and let $\sim$ be a binary relation on $\mathscr{X}$. If $x' \sim x$, say that "x' is a neighbour of x." The following are assumed, for all $x, x' \in \mathscr{X}$,

1. *Irreflexivity*: $x \nsim x$. No player is his own neighbour.
2. *Symmetry*: $x' \sim x \Rightarrow x \sim x'$. If x' is a neighbour of x, then x is a neighbour of x'.
3. *Bounded Neighbours*: there exists M such that $\#\{y \in \mathscr{X}: y \sim x\} \leq M$. Each player has at most M neighbours.
4. *Connectedness*: there exist $\{x_1, x_2, \ldots, x_K\} \subseteq \mathscr{X}$ such that $x_1 = x'$, $x_K = x$ and $x_k \sim x_{k+1}$ for each $k = 1, \ldots, K-1$. There is some path connecting any pair of players.

A *local interaction system* is a pair $(\mathscr{X}, \sim)$, where $\sim$ satisfies properties [1] through [4].[7] Write $\Gamma(x)$ for the set of neighbours of x, *i.e.* $\Gamma(x) \equiv \{x' : x' \sim x\}$; a *group* of players, X, is

5. The papers of Monderer and Samet (1989, 1996), Morris, Rob and Shin (1995) and Kajii and Morris (1997) are especially relevant.

6. The assumption of an infinite population is for analytic convenience. Analogous results could be proven for large finite populations, although some of the simplicity of the results would be lost.

7. Irreflexivity and Symmetry imply that $(\mathscr{X}, \sim)$ is an (infinite) graph. Bounded Neighbours implies that each player has a small number of neighbours (*i.e.* finite) relative to the whole (infinite) population. Connectedness is assumed for convenience only; if the graph were disconnected, the paper's results could be applied to each connected component. Symmetry is a substantive assumption, necessary for many of the results that follow. It will imply that if player 1 cares about player 2's action, then player 2 must care about player 1's action.

an arbitrary subset of $\mathscr{X}$; the complementary group of X in $\mathscr{X}$ is written as $\bar{X}$, *i.e.* $\bar{X}=\{x\in\mathscr{X}: x\notin X\}$.

Each player has two possible actions, 0 and 1. Write $u(a, a')$ for the payoff of a player from a particular interaction if he chooses a and his neighbour chooses a'. This payoff function corresponds to symmetric payoff matrix:

	0	1
0	$u(0,0), u(0,0)$	$u(0,1), u(1,0)$
1	$u(1,0), u(0,1)$	$u(1,1), u(1,1)$

It is assumed that this game has two strict Nash equilibria, so that $u(0,0)>u(1,0)$ and $u(1,1)>u(0,1)$. However, for the analysis of this paper, only the best response correspondence matters. In particular, observe that action 1 is best response for some player exactly if he assigns probability at least

$$q=\frac{u(0,0)-u(1,0)}{(u(0,0)-u(1,0))+(u(1,1)-u(0,1))}$$

to the other player choosing action 1. Thus payoffs are parameterized by the critical probability $q\in(0,1)$. All analysis in the paper is unchanged if we restrict attention to the payoff matrix

	0	1
0	q, q	$0, 0$
1	$0, 0$	$1-q, 1-q$

(2.1)

Now a *local interaction game* is a 3-tuple $(\mathscr{X}, \sim, q)$.

A conventional description of best responses would proceed as follows. A (pure) *configuration* is a function $s: \mathscr{X}\to\{0,1\}$. Given configuration s, player x's best response is to choose an action which maximizes the sum of his payoffs from his interactions with each of his neighbours. Thus action a is a best response to configuration s for player x, *i.e.* $a\in b(s,x)$, if

$$\textstyle\sum_{y\in\Gamma(x)} u(a, s(y))\geqq\sum_{y\in\Gamma(x)} u(1-a, s(y)).$$

Configuration s' is a best response to configuration s if $s'(x)$ is a best response to s for each x, *i.e.* if $s'(x)\in b(s,x)$ for all $x\in\mathscr{X}$.

But notice that action 1 is a best response for a player if at least proportion q of his neighbours choose action 1; and action 0 is a best response if at least proportion $1-q$ of his neighbours choose action 0. So it is convenient to describe configurations of play and best responses as follows. Identify a configuration with the group of players who choose action 1 in that configuration. Thus configuration s is identified with the group $X=\{x: s(x)=1\}$; group X is identified with configuration s where

$$s(x)=\begin{cases}1, & \text{if } x\in X\\ 0, & \text{if } x\notin X.\end{cases}$$

Now let $\pi[X|x]$ be the proportion of x's neighbours who are in group X, *i.e.*

$$\pi[X|x]=\frac{\#(X\cap\Gamma(x))}{\#\Gamma(x)}.$$

Write $\Pi^p(X)$ for the players for whom at least proportion p of their interactions are with players in X, *i.e.*

$$\Pi^p(X)=\{x\in\mathscr{X}:\pi[X|x]\geqq p\}.$$

In this notation, X is a *best response* to Y if $X\subseteq\Pi^q(Y)$ and $\bar{X}\subseteq\Pi^{1-q}(\bar{Y})$. For simplicity, I assume the tie-breaking rule that action 1 is chosen if a player is indifferent between the two actions. Thus $\Pi^q(X)$ will be referred to as *the* best response to X. This tie-breaking rule makes no difference to the contagion results, except for non-generic values of q.[8]

This paper is concerned with the following question. Does there exist a *finite* group of players, such that if that group starts out playing some action (say, without loss of generality, action 1), best response dynamics will ensure that that action is eventually played everywhere? If so, action 1 spreads *contagiously*. The *contagion threshold* ξ is defined to be the largest q for which such contagious dynamics are possible.

Definition 1. The contagion threshold, ξ, of local interaction system $(\mathscr{X},\sim)$ is the largest q such that action 1 spreads by best response dynamics from some finite group to the whole population, *i.e.*[9]

$$\xi=\max\{q:\textstyle\bigcup_{k\geq 1}[\Pi^q]^k(X)=\mathscr{X}\text{ for some finite }X\}.$$

3. EXAMPLES

The examples described in this section provide some intuition for the contagion threshold (as well as fixing the notation described in the previous section). For each example, I state (without proof) the contagion threshold. Techniques described in Section 5 are later used to establish these results. The examples will also be used to illustrate the critical graph theoretic properties of interaction described in the next section. I write $\mathbf{Z}$ for the set of integers.

Example 1: *Interaction on a line.* The population is arranged on a line and each player interacts with the player to his left and the player to his right. See Figure 1. This is represented formally as follows:

- $\mathscr{X}=\mathbf{Z}\,;\,x'\sim x\quad\text{if } x'=x-1\quad\text{or } x'=x+1.$

If $q<\frac{1}{2}$ in the payoff matrix (2.1), action 1 is a best response whenever at least one

FIGURE 1

8. For generic values of q (in particular, as long as $q\neq n/m$ for all integers $n\leqq m\leqq M$), $\pi[X|x]\neq q$ for all $x\in\mathscr{X}$ and $X\subseteq\mathscr{X}$. In this case, $\Pi^q(X)=\Pi^{1-q}(\bar{X})$ for all $X\subseteq\mathscr{X}$ (so there is a unique best response always) and thus the tie-breaking rule does not matter. Section 7.1 describes the precise sense in which the tie-breaking rule does not matter.

9. The maximum can be shown to always exist, using the continuity properties of the operator Π^p described in Lemma 1 in the Appendix.

neighbour chooses action 1. Thus if two neighbours x and $x+1$ initially choose action 1, players $x-1$, x, $x+1$ and $x+2$ must all choose action 1 in the next period, players $x-2$, $x-1$, x, $x+1$, $x+2$ and $x+3$ must all choose action 1 in the period after that, and so on. So action 1 eventually spreads to the whole population. But if $q>\frac{1}{2}$, no player switches to action 1 unless both neighbours are already choosing 1. Thus the contagion threshold is $\frac{1}{2}$.

Example 2: *Nearest neighbour interaction in m dimensions.* More generally, we can imagine the population situated on an m-dimensional lattice. Each player interacts with all players who are immediate neighbours in the lattice, *i.e.* whose coordinates differ in only one dimension. If $m=1$, then we have the interaction on a line of the previous example. See Figure 2 for the case where $m=2$.

- $\mathscr{X}=\mathbf{Z}^m$; $x'\sim x$ if $\sum_{i=1}^{m}|x_i'-x_i|=1$.

In this case, each player has $2m$ neighbours. The contagion threshold is $\frac{1}{2m}$ (*i.e.* contagion occurs if and only if $q\leqq\frac{1}{2m}$). Thus it appears that as interaction becomes "richer" (*i.e.* as the number of dimensions increases) contagion becomes impossible. However, the next example suggests that this conclusion is premature.

Example 3: *n-max distance interaction in m dimensions.* The population is again situated on an infinite m dimensional lattice. Each player interacts with all players who are less than n steps away in each of the m dimensions. See Figure 3 for the case where $m=2$ and $n=1$.

- $\mathscr{X}=\mathbf{Z}^m$; $x'\sim x$ if $1\leqq\max_{i=1,\ldots,m}|x_m'-x_m|\leqq n$.

The contagion threshold is $n(2n+1)^{m-1}/(2n+1)^m-1$. Table 1 gives the values of this expression for different values of m and n.

This example illustrates the lack of robustness of the nearest neighbour analysis. If we simply add players at one diagonal remove (holding $n=1$), then increasing the number of dimensions (m) never lowers the contagion threshold below $\frac{1}{3}$; If we fix the number of dimensions (m) and increase the radius of interaction (n), the contagion threshold tends to $\frac{1}{2}$.

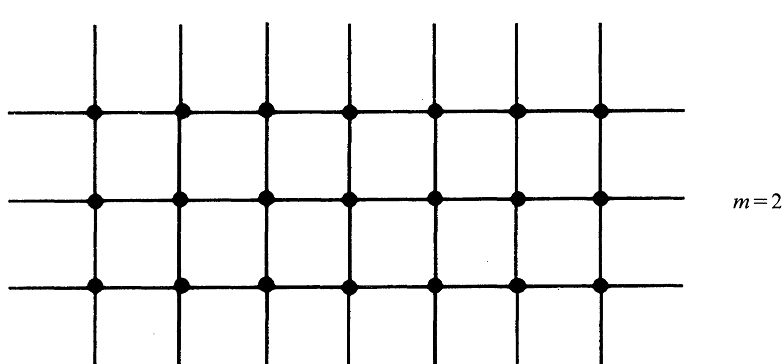

FIGURE 2

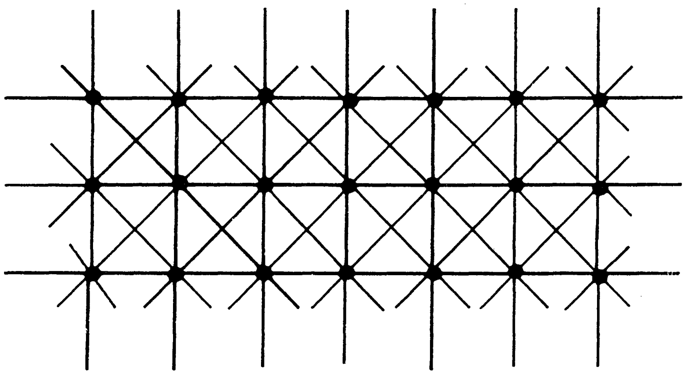

FIGURE 3

	1	2	3	·	n	·	$n\to\infty$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	·	$\frac{1}{2}$	·	$\frac{1}{2}$
2	$\frac{3}{8}$	$\frac{5}{12}$	$\frac{7}{16}$	·	$\frac{n(2n+1)}{(2n+1)^2-1}$	·	$\frac{1}{2}$
3	$\frac{9}{26}$	$\frac{25}{62}$	$\frac{147}{342}$	·	$\frac{n(2n+1)^2}{(2n+1)^3-1}$	·	$\frac{1}{2}$
·	·	·	·	·	·	·	·
m	$\frac{3^{m-1}}{3^m-1}$	$\frac{2\cdot 5^{m-1}}{5^m-1}$	$\frac{3\cdot 7^{m-1}}{7^m-1}$	·	$\frac{n(2n+1)^{m-1}}{(2n+1)^m-1}$	·	$\frac{1}{2}$
·	·	·	·	·	·	·	·
$M\to\infty$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	·	$\frac{n}{2n+1}$	·	$\frac{1}{2}$

TABLE 1

Example 4: *Regions*. The population is divided into an infinite number of "regions" of m players each. Each player in a region interacts with every other player in that region. The regions are arranged in a line and each player also interacts with one player in each neighbouring region. See Figure 4 for the case where $m=3$.

- $\mathscr{X}=\mathbf{Z}\times\{1,\dots,m\}$; $x'\sim x$ if either (i) $x_1'=x_1$; or (ii) $|x_1'-x_1|=1$ and $x_2'=x_2$.

The contagion threshold is $1/(m+1)$.

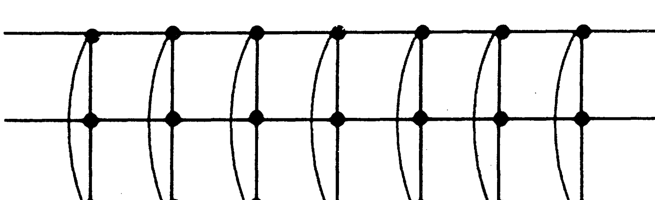

FIGURE 4

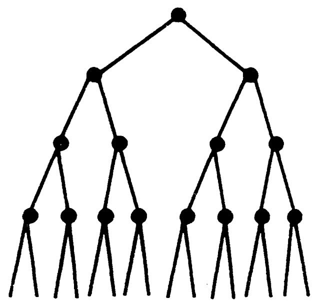

FIGURE 5

Example 5: *Hierarchy*. The population is arranged in a hierarchy. Each player has m subordinates. Each player, except the root player, has a single superior. See Figure 5 for the case where $m=2$.

- $\mathscr{X}=\bigcup_{k=0}^{\infty} X_k$, where $X_0=\{\varnothing\}$ and $X_k=\{1,\ldots,m\}^k$ for all $k\geqq 1$, with $m\geqq 2$;

  $x \sim y$ if and only if $x=(y,n)$ or $y=(x,n)$ for some $n\in\{1,\ldots,m\}$.

  The contagion threshold is again $1/(m+1)$.

4. PROPERTIES OF INTERACTION

Three types of properties important in local interaction systems are described in this section.

4.1. *Cohesion*

One natural measure of the "cohesion" of a social group is the relative frequency of ties among group members compared to non-members.[10] For any given group of players X, do players in the group mostly interact with players within the group or with players outside the group? Let the *cohesion* of group X be the smallest p such that each player in X has proportion p of his interactions within X, *i.e.*

$$c(X)=\min_{x\in X}\pi[X|x]=\max\{p\colon X\subseteq\Pi^p(X)\}.$$

The minimum and maximum exist since, for all players x and groups X, $\pi[X|x]$ is a rational number with denominator less than or equal to M.

Definition 2. Group X is p-cohesive if $c(X)\geqq p$.

We can use the examples of the previous section to illustrate this concept.

- *Example* 1: *Interaction on a line*. Any non-trivial group (non-empty and not equal to the whole population) is at most $\frac{1}{2}$-cohesive, since at least one player must have one neighbour outside the group. For example, consider the group of players

10. Wasserman and Faust (1994, Chapter 7), identify four independent concepts of cohesion in the sociology literature: (i) the relative frequency of ties among group members compared to non-members [*the notion studied here*]; (ii) the mutuality of ties; (iii) the closeness or reachability of members; and (iv) the frequency of ties among members.

$\{1, \ldots, N\}$ for some very large N. Most players have all of their interactions within the group. But the cohesion of the group is only $\frac{1}{2}$, because of the two critical players, 1 and N, who have only $\frac{1}{2}$ their interactions with the group.

- *Example* 2: *Nearest neighbour interaction in m dimensions.* Any group consisting of all players above some horizontal plane (*i.e.* taking the form $\{x \in \mathbf{Z}^m : x_1 \geqq c\}$) will be $(2m-1)/2m$-cohesive.

- *Example* 3: *n-max distance in m dimensions.* Any group consisting of all players above some horizontal plane (*i.e.* taking the form $\{x \in \mathbf{Z}^m : x_1 \geqq c\}$) will be $[(n+1)(2n+1)^{m-1}-1]/[(2n+1)^m-1]$-cohesive. To see why, consider a player x with $x_1 = c$. He has $(2n+1)^{m-1}-1$ neighbours with first coordinate c; he has $n(2n+1)^{m-1}$ neighbours with first co-ordinate greater than c; and he has $n(2n+1)^{m-1}$ neighbours with first co-ordinate less than c. Thus proportion $[(n+1)(2n+1)^{m-1}-1]/[(2n+1)^m-1]$ of his neighbours have first co-ordinate greater than or equal to c.

- *Example* 4: *Regions.* A pair of neighbouring regions (*i.e.* a group of the form $\{x \in \mathbf{Z} \times \{1, \ldots, m\} : x_1 = c \text{ or } c+1\}$) is $m/(m+1)$-cohesive, since each player in that group has m neighbours within that group, and one neighbour outside.
- *Example* 5: *Hierarchy.* Any group consisting of all direct or indirect subordinates of some player $\hat{x} \in \mathscr{X}_n$, *i.e.* taking the form

$$\{x \in \textstyle\bigcup_{n' \geq n} \mathscr{X}_{n'} : x_k = \hat{x}_k \text{ for each } k = 1, \ldots, n\},$$

  will be $m/(m+1)$-cohesive. To see why, note that player $\hat{x}$ has all neighbours except his superior in that group, while all other players in the group have all their neighbours in the group.

4.2. *Weak links, strong links and neighbour growth*

Granovetter (1973) introduced the distinction between "weak" and "strong" social links.[11] Strong social links are often transitive: if A is a close friend of B and B is a close friend of C, then B and C are more likely to be close friends than two randomly chosen individuals. This "neighbour correlation" will be less pronounced the weaker the social link being studied.

In this section, I introduce a natural way of capturing this distinction for the infinite graphs studied in this paper. Fix an individual, and calculate the number of players that can be reached in no more than n steps from that individual. If there is no neighbour correlation in the graph (*i.e.* links are weak), then the numbers of players reached will grow exponentially. But if a significant proportion of players' neighbours' neighbours are their own neighbours, then this will tend to slow down the exponential growth. We will be interested in the case where there is enough neighbour correlation to prevent exponential growth.

The *Erdös distance* between player x and group X is n if it takes at most n steps to reach x from X; *i.e.* writing $\Gamma^n(X)$ for the set of players within Erdös distance n, $\Gamma^0(X) = X$ and $\Gamma^{n+1}(X) = \Gamma^n(X) \cup \{x' : x' \sim x \text{ for some } x \in \Gamma^n(X)\}$.

11. I am grateful to Michael Chwe for bringing this literature to my attention. See Chwe (2000) for more on the strategic implications of different kinds of social links.

Definition 3. Local Interaction System $(\mathscr{X}, \sim)$ satisfies *low neighbour growth* if $\gamma^{-n}\#\Gamma^n(X) \to 0$ as $n \to \infty$, for all finite groups X and $\gamma > 1$.[12]

In the hierarchy example, $\#\Gamma^1(X_0) = 1 + m$, $\#\Gamma^2(X_0) = 1 + m + m^2$, *etc.*, so that

$$\#\Gamma^k(X_0) = (1 + m + \cdots + m^n) = (m^{n+1} - 1)/(m - 1).$$

Thus the low neighbour growth property is not satisfied. In all the other examples considered in Section 3, $\#\Gamma^n(X)$ is a polynomial function of n, and thus low neighbour growth *is* satisfied.

Researchers in the sociology literature have empirically verified that $\#\Gamma^n(X)$ grows slower for graphs describing more important relationships.[13] To get a feel for the growth of $\#\Gamma^n(X)$, consider the experimental finding of Milgram (1967) that the median Erdös distance (derived from the relation "personally acquainted") between two randomly chosen individuals in the U.S. was five. To interpret this finding, consider two extreme cases. The U.S. population at the time was around 200 million and Milgram estimated that each individual had approximately 500 acquaintances. Suppose that one individual has no overlap between his acquaintances, his acquaintances' acquaintances and his acquaintances' acquaintances' acquaintances. Then over half the population would be within Erdös distance 3 of this individual ($500^3 = 125{,}000{,}000$). On the other hand, suppose the population of 200 million was arranged in a circle and each individual knew 250 people on either side, the median Erdös distance would be 200,000.

4.3. *δ-Uniformity*

The last property considered is more technical and requires some additional notation. A *labelling* of players $\mathscr{X}$ is a bijection l: $\mathbf{Z}_{++} \to \mathscr{X}$. Write $\mathscr{L}$ for the set of labellings and $\alpha_l(k)$ for the proportion of neighbours of player $l(k)$ who have a lower label under labelling l, *i.e.*

$$\alpha_l(l) = \frac{\#\{j: l(j) \sim l(k) \text{ and } j < k\}}{\#\{j: l(j) \sim l(k)\}}.$$

Labelling l is an *Erdös labelling* if there exists a finite group X such that $l(i) \in \Gamma^n(X)$ and $l(j) \notin \Gamma^n(X) \Rightarrow i < j$.

Definition 4. Local Interaction System $(\mathscr{X}, \sim)$ satisfies *δ-uniformity* if there exists an Erdös labelling l such that for all sufficiently large K,

$$\max_{k', k \geq K} |\alpha_l(k') - \alpha_l(k)| \leq \delta. \tag{4.1}$$

This seems to be a weak property. There must be *some* way of labelling players, consistent with Erdös distance from *some* finite group X, such that for players sufficiently

12. In fact, requiring the definition to hold "for all" finite X is redundant: if it holds for *any* finite X, it holds for *all* finite X. To see why, suppose that for some finite X and $\gamma > 1$, $\gamma^{-k}\#\Gamma^k(X) > \varepsilon > 0$, for infinitely many k (*i.e.* $\gamma^{-k}\#\Gamma^k(X) \not\to 0$). Fix *any* finite group Y. By connectedness, $X \subseteq \Gamma^n(Y)$ for some n. Now $\Gamma^k(X) \subseteq \Gamma^{n+k}(Y)$, so $\gamma^{-(n+k)}\#\Gamma^{n+k}(Y) > \varepsilon\gamma^{-n} > 0$ for infinitely many k, *i.e.* $\gamma^{-k}\#\Gamma^k(Y) \not\to 0$.

13. In one classic study, Rapoport and Horvarth (1961) examined levels of friendship among junior high school students. In a graph based on seventh and eighth best friends, $\#\Gamma^n(X)$ grows fast. In a graph based on best and second best friends, $\#\Gamma^n(X)$ grows much more slowly.

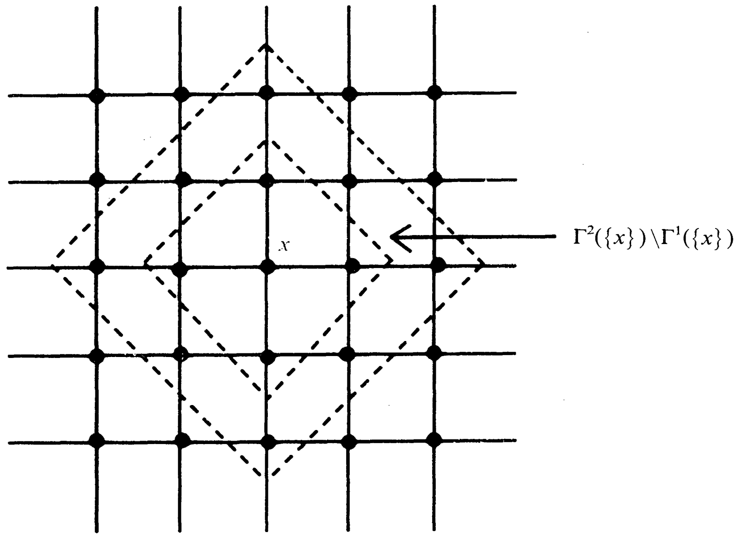

FIGURE 6

far away from X, the proportion of neighbours with lower labels tends (roughly) to something (with no restrictions on what that something is). Two examples illustrate the property.

- *Example* 5: *Hierarchy.* 0-uniformity is satisfied. Consider any Erdös labelling with initial (singleton) group X_0. Now $\Gamma^n(X_0) = \bigcup_{j=0}^{n} X_j$. For any k, player $l(k)$ has exactly one neighbour with a lower label. Thus $\alpha_l(k) = 1/(m+1)$ for all k.
- *Example* 2: *Nearest neighbour interaction in* 2 *dimensions.* For any $\delta < \frac{1}{4}$, δ-uniformity fails. Consider any Erdös labelling l. For any n, there are $4(n+1)$ players who are contained in $\Gamma^{n+1}(\{x\})$ but not $\Gamma^n(\{x\})$ (see Figure 6). Those locations form an empty square with $n+2$ players on each side; now $4n$ of those locations (those *not* on the corners) have $\alpha_l(k) = \frac{1}{2}$. But the four corners have $\alpha_l(k) = \frac{1}{4}$.

The latter example illustrates how δ-uniformity fails because of the lumpiness of the lattice. With sufficiently large neighbourhoods on a lattice, δ-uniformity is satisfied for small δ.

5. CHARACTERIZATIONS OF THE CONTAGION THRESHOLD

Recall that the contagion threshold, for a given local interaction system $(\mathscr{X}, \sim)$, was defined as follows

$$\xi = \max\{q: \textstyle\bigcup_{k\geq 1}[\Pi^q]^k(X) = \mathscr{X} \text{ for some finite } X\}.$$

Note that operator Π^p is non-monotonic: X may contain $\Pi^p(X)$, X may be contained in $\Pi^p(X)$, or neither might be true. It is useful to analyse instead the following monotonic operator

$$\Pi_+^p(X) \equiv X \cup \Pi^p(X).$$

It is straightforward to construct examples where X is finite, $\bigcup_{k\geq 1}[\Pi^p]^k(X)$ is finite, but $\bigcup_{k\geq 1}[\Pi_+^p]^k(X) = \mathscr{X}$. But it turns out that it must then be possible to find another, possibly larger but still finite, group Y with $\bigcup_{k\geq 1}[\Pi^p]^k(Y) = \mathscr{X}$. This means that the same contagion threshold arises if the monotonic operator Π^q_+ is substituted in the definition of the contagion threshold, *i.e.* ξ also equals

$$\max \{q: \bigcup_{k\geq 1} [\Pi^q_+]^k(X) = \mathscr{X} \text{ for some finite } X\}.$$

This equivalence result is key to the characterizations that follow. It and the following two propositions are proved in the Appendix.

Proposition 1. *The contagion threshold is the smallest p such that every co-finite group contains an infinite*, $(1-p)$*-cohesive, subgroup.*[14]

Fix the parameter q in the payoff matrix (2.1). How can contagion be prevented? Suppose that there is a $(1-p)$-cohesive group, with $p<q$, where action 0 is played initially. Since each player in that group has more than proportion $1-q$ of his neighbours playing 0, no player in that group will ever switch away from that group. So a sufficient condition for no contagion is that for any finite initial group X, the complement of X contains a $(1-p)$-cohesive group, for some $p<q$. This sufficient condition turns out to be necessary also, and we get Proposition 1.

Proposition 2. *The contagion threshold is the largest p such that under some labelling* l, $\alpha_l(k)\geqq p$ *for all sufficiently large k. Formally*

$$\xi = \max_{i\in\mathscr{L}}\left(\lim_{K\to\infty}\left(\min_{k\geqq K}\alpha_l(k)\right)\right). \tag{5.1}$$

That is, roughly speaking, the contagion threshold is the largest p such that we can label the players such that at least proportion p of all players' neighbours have a lower label (except for some initial group).

The following immediate corollaries of Propositions 1 and 2 respectively are especially useful in identifying contagion thresholds in practice.

Corollary 1. [*Upper Bound*]. *If every co-finite group contains an infinite*, $(1-p)$*-cohesive, subgroup, then* $\xi\leqq p$.

Corollary 2. [*Lower Bound I*]. *If there exists a labelling* l *with* $\alpha_l(k)\geqq p$ *for all sufficiently large k, then* $\xi\geqq p$.

An even simpler lower bound is a consequence of Corollary 2. Recall that M was an upper bound on the number of possible neighbours.

Corollary 3. [*Lower Bound II*]. *The contagion threshold* ξ *is at least* $1/M$.

These corollaries can be used to establish the contagion threshold in earlier examples.

- *Example* 2: *Nearest neighbour interaction in m dimensions.* Every co-finite group contains an infinite $(2m-1)/2m$-cohesive group of the form $\{x\in\mathbf{Z}^m: x_1\geqq c\}$, so (by Corollary 1) $\xi\leqq 1/2m$. But $\xi\geqq 1/2m$ by Corollary 3.

14. A group X is co-finite if its complement $\bar{X}$ is co-finite.

- *Example* 3: *n-max interaction in m dimensions.* Every co-finite group contains an infinite $((1-n(2n+1)^{m-1})/((2n+1)^m-1))$-cohesive group of the form $\{x\in\mathscr{X}^m: x_1\geqq c\}$, so (by Corollary 1) $\xi\leqq n(2n+1)^{m-1}/((2n+1)^m-1)$. But consider any labelling of players that is increasing in their Euclidean distance from the origin (*i.e.* $l(x')>l(x)\Rightarrow\|x'\|>\|x\|$). For any player x, at most $(2n+1)^{m-1}-1$ players are contained on any $m-1$ dimension plane passing through x. By symmetry, half the remainder are on either side of the plane. Thus at least proportion $n(2n+1)^{m-1}/((2n+1)^m-1)$ are on either side of the plane. Now consider in particular the $m-1$ dimensional plane through x, perpendicular to the line joining x and the origin. If x has a sufficiently high label, all players on the origin's side of that plane will have a lower label than x. Thus we have constructed a labelling with $\alpha_l(k)\geqq n(2n+1)^{m-1}/((2n+1)^m-1)$ for all sufficiently large k, and, by Corollary 2, $\xi\geqq n(2n+1)^{m-1}/((2n+1)^m-1)$.[15]
- *Example* 4: *Regions.* Every co-finite group contains an infinite $m/(m+1)$-cohesive group of the form $\{x\in\mathscr{X}\times\{1,\ldots,m\}: x_1\geqq c\}$, so (by Corollary 1) $\xi\leqq 1/(m+1)$. But $\xi\geqq 1/(m+1)$ by Corollary 3.
- *Example* 5: *Hierarchy.* Every co-finite group contains an infinite $m/(m+1)$-cohesive group of the form $\{x\in\bigcup_{n'\geq n}\mathscr{X}_{n'}: x_k=\hat{x}_k \text{ for each } k=1,\ldots,n\}$, for some $\hat{x}\in\mathscr{X}_n$. So (by Corollary 1) $\xi\leqq 1/(m+1)$. But $\xi\geqq 1/(m+1)$ by Corollary 3.

Proposition 3. *Every local interaction system* $(\mathscr{X},\sim)$ *has a contagion threshold less than or equal to* $\frac{1}{2}$.

This can be proved from a result of Kajii and Morris (1997), *via* the incomplete information game/local interaction game equivalence discussed in Morris (1997*b*). However it is possible to give an elementary proof in this simpler setting, exploiting Proposition 2.[16]

Proof. Suppose the contagion threshold were $\xi>\frac{1}{2}$. By Proposition 2, there exists a labelling l and a number K such that $\alpha_l(k)\geqq\xi>\frac{1}{2}$ for all $k\geqq K$. Let $f(k)$ and $g(k)$ be the number of interactions involving player k and players with lower labels and higher labels respectively and let $h(k)$ be the number of interactions consisting of one player with a label greater than or equal to k and another player with a label lower than k, *i.e.*

$$f(k)=\#\{j: l(j)\sim l(k) \text{ and } j<k\},$$

$$g(k)=\#\{j: l(j)\sim l(k) \text{ and } j>k\},$$

$$h(k)=\#\{(i,j): l(i)\sim l(j), i\leqq k-1 \text{ and } j\geqq k\}.$$

By construction, $h(k+1)=h(k)-f(k)+g(k)$. For all $k\geqq K$, $\alpha_l(k)=f(k)/(f(k)+g(k))>\frac{1}{2}$, so $f(k)>g(k)$ and thus $h(k+1)<h(k)$. But since $h(K)$ is finite and $h(\cdot)$ takes integer values, we must have $h(k)$ negative for sufficiently large k, a contradiction. ||

Proposition 4. *If a Local Interaction System satisfies low neighbour growth and* δ-*uniformity, then the contagion threshold* $\xi\geqq\frac{1}{2}-\delta$.

15. For a more formal version of this argument, see Appendix A of the working paper version of this work (Morris 1997*a*). That Appendix describes a general way of identifying the contagion threshold (using Corollaries 1 and 2) for interaction on a lattice.
16. I am grateful to David McAdams of Stanford Business School for suggesting this argument.

Blume (1995) showed that if players interact on a two dimensional lattice in sufficiently large symmetric neighbourhoods, then the contagion threshold is close to $\frac{1}{2}$.[17] Proposition 4 generalizes Blume's result to arbitrary interaction structures.

Proof. Suppose Erdös labelling l satisfies (4.1). Then there exist α and K with

$$\alpha-\delta \leqq \frac{\#\{j: l(j) \sim l(k) \text{ and } j<k\}}{\#\{j: l(j) \sim l(k)\}} \leqq \alpha,$$

for all $k \geqq K$. By Corollary 2, $\xi \geqq \alpha-\delta$. So if $\alpha \geqq \frac{1}{2}$, we are done. Suppose then that $\alpha<\frac{1}{2}$. Now

$$\#\{j: l(j) \sim l(k) \text{ and } j<k\} \leqq\left(\frac{\alpha}{1-\alpha}\right) \#\{j: l(j) \sim l(k) \text{ and } j>k\}, \tag{5.2}$$

for all $k \geqq K$. Since l is an Erdös labelling, there exists a finite group X such that $l(i) \in \Gamma^n(X)$ and $l(j) \notin \Gamma^n(X) \Rightarrow i<j$. Let $X_0=X$ and $X_n=\Gamma^n(X) \cap \overline{\Gamma^{n-1}(X)}$ for $n=1,2, \ldots$. Choose N such that $l(K) \in X_{N-1}$. Now if $n \geqq N$, summing equation (5.2) across all k with $l(k) \in X_n$ implies

$$\left\{\begin{array}{l}\#\{(j, k): l(j) \sim l(k), l(j) \in X_{n-1} \text{ and } l(k) \in X_n\} \\ \qquad\qquad +\#\{(j, k): l(j) \sim l(k),\{l(j), l(k)\} \subseteq X_n \text{ and } j<k\}\end{array}\right\}$$
$$\leqq\left(\frac{\alpha}{1-\alpha}\right)\left\{\begin{array}{l}\#\{(j, k): l(j) \sim l(k), l(j) \in X_{n+1} \text{ and } l(k) \in X_n\} \\ \qquad\qquad +\#\{(j, k): l(j) \sim l(k),\{l(j), l(k)\} \subseteq X_n \text{ and } j>k\}\end{array}\right\}.$$

$$\text{Writing } F^n=\#\{\{x, y\}: x \sim y, x \in X_{n-1} \text{ and } y \in X_n\},$$
$$\text{and } H^n=\#\{\{x, y\}: x \sim y, x \in X_n \text{ and } y \in X_n\},$$

the above expression can be re-written as

$$F^n+H^n \leqq\left(\frac{\alpha}{1-\alpha}\right)(F^{n+1}+H^n).$$

Since $\alpha<\frac{1}{2}$, this implies $F^{n+1} \geqq((1-\alpha) / \alpha) F^n$ for all $n \geqq N$ so $F^n \geqq((1-\alpha) / \alpha)^{n-N} F^N$ for all $n \geqq N$. But $\# X_n \geqq F^n / M \geqq((1-\alpha) / \alpha)^{n-N} F^N / M$. Thus $\gamma^{-n} \# \Gamma^n(X) \rightarrow \infty$ if $\gamma<((1-\alpha) / \alpha)$, contradicting the assumption of low neighbour growth. Thus the hypothesis that $\alpha<\frac{1}{2}$ is false and the lemma is proved. ||

Two examples illustrate why both conditions are required:

- The hierarchy example satisfied 0-uniformity but failed low neighbour growth. The contagion threshold was $1/(m+1)$ and thus not close to $\frac{1}{2}$.
- Nearest neighbour interaction in 2 dimensions satisfied low neighbour growth but failed δ-uniformity, for any $\delta<\frac{1}{4}$. The contagion threshold was $\frac{1}{4}$.

The intuition for Proposition 4 is that behaviour must always spread slowly when contagion occurs: if it spreads fast initially, it must spread to players who do not interact much with each other, and therefore it will not spread further. Given the uniformity condition, low neighbour growth ensures that it spreads slowly.

17. Blume considered a different (stochastic) revision process; but as I show in Section 7, this difference does not influence contagion properties.

The uniformity condition is quite necessary for this result. The following example—where uniformity does not hold—demonstrates this.

Example 6: *Combined weak links and strong links.* Players are situated on a line and each player interacts with all players within r steps. This generates $2r$ neighbours for each player. But in addition, a hierarchy is super-imposed.

- $\mathscr{X} = \mathbf{Z}$; let $\sim_1$ correspond to be r-max distance interaction in 1 dimension (Example 3), *i.e.* $x' \sim_1 x$ if $|x'-x| \leqq r$; let $\sim_2$ correspond to a hierarchy with m subordinates (Example 5); let x' and x be neighbours if they are neighbours under *either* of these relations.

In this example, $\#\Gamma^n(X)$ grows at exponential rate m. But the contagion threshold $\xi \geqq r/(2r+m+1)$, by Corollary 2 (consider the labelling l with $l(k)=\frac{1}{2}k$ if k is even, $l(k) = -\frac{1}{2}(k+1)$ if k is odd). By choosing m large but r larger, it is possible to get arbitrarily large growth of $\#\Gamma^n(X)$ with a contagion threshold arbitrarily close to $\frac{1}{2}$. Thus it is possible to have high neighbour growth (as the evidence of Milgram (1967) suggests for acquaintances in the U.S. population) but still have high contagion if, as in this example, most neighbours are "local" but a few relations generate most of the growth.

6. THE CO-EXISTENCE OF CONVENTIONS

When do there exist equilibria of local interaction games where different players take different actions? How does the answer depend on the structure of interaction? This question has been studied by researchers under the rubric of the co-existence of conventions. Goyal (1996), Galesloot and Goyal (1997), Sugden (1995) and Young (1996) all discuss which properties of the interaction structure make co-existence more or less likely.[18] The contagion threshold can be used to show when co-existence is possible in general interaction structures.

Equilibrium in local interaction game $(\mathscr{X}, \sim, q)$ is defined as follows.

Definition 5. X is an *equilibrium* of $(\mathscr{X}, \sim, q)$ if X is a best response to X, *i.e.* if $X \subseteq \Pi^q(X)$ and $\bar{X} \subseteq \Pi^{1-q}(\bar{X})$.

Thus X is an equilibrium if and only if X is q-cohesive and $\bar{X}$ is $(1-q)$-cohesive.[19] An equilibrium X is said to be a *co-existent equilibrium* if X is an equilibrium and $X \notin \{\varnothing, \mathscr{X}\}$.

Proposition 5. *Suppose local interaction system $(\mathscr{X}, \sim)$ satisfies low neighbour growth and has contagion threshold ξ. Then for all $q \in [\xi, 1-\xi]$, local interaction game $(\mathscr{X}, \sim, q)$ has a co-existent equilibrium.*

Low neighbour growth implies the existence of a non-empty finite $\frac{1}{2}$-cohesive group. For any $q \in [\xi, \frac{1}{2}]$, this finite group is also q-cohesive. By Proposition 1, there exists a disjoint non-empty $(1-q)$-cohesive subgroup. Now there exists an equilibrium with co-existent conventions where action 1 is played by the q-cohesive group and action 0 is

18. See also Shin and Williamson (1996) for an analysis of conventions with a continuum of actions (Morris (1997*b*) shows how their incomplete information results translates to a local interaction setting).

19. For generic q, X is an equilibrium if and only if $X = \Pi^q(X)$, since no one will have exactly proportion q of their neighbours taking action 1; *see* Footnote 8.

played by the $(1-q)$-cohesive group. A symmetric argument shows the existence of a co-existent equilibrium if $q \in [\frac{1}{2}, 1-\xi]$. (The proof of Proposition 5 is in the Appendix.)

Since $\xi \leqq \frac{1}{2}$ (by Proposition 3), the following corollary holds.

Corollary 4. *Local interaction game* $(\mathscr{X}, \sim, \frac{1}{2})$ *has a co-existent equilibrium whenever* $(\mathscr{X}, \sim)$ *satisfies low neighbour growth.*

Thus with low neighbour growth and in the degenerate case of exactly symmetric payoffs, there always exists an equilibrium with co-existent conventions.

7. ADDING RANDOMNESS

Deterministic best response dynamics need not converge to an equilibrium. For example, if players are arranged in a line and odd players choose action 1 and even players choose action 0, then best responses will lead odd players to switch to 0 and even players to switch to action 1. Best response dynamics, then, will lead to a two cycle as every player alternates between actions. Partly to rule out such cycles, a number of researchers have considered adding stochastic elements to the best response dynamics. This section contains a discussion of alternative ways of adding random elements to the dynamic process considered in this paper. We can use this discussion to describe the connection to some of the related literature.

7.1. *Random revision opportunities*

In this paper, all players best responded simultaneously. Consider the more general notion of a best response sequence. Sequence $\{X_k\}_{k=0}^{\infty}$ is a *best response sequence* if

1. $x \in X_{k+1}$ and $x \notin X_k$ for some $k \Rightarrow \pi[X_k | x] \geqq q$.
2. $x \notin X_{k+1}$ and $x \in X_k$ for some $k \Rightarrow \pi[\bar{X}_k | x] \geqq 1-q$.
3. $\pi[X_k | x] > q$ for all $k \geqq K \Rightarrow x \in X_k$ for some $k > K$.
4. $\pi[\bar{X}_k | x] > 1-q$ for all $k \geqq K \Rightarrow x \notin X_k$ for some $k > K$.

Properties (1) and (2) say that if a player switches action, it must be to a best response; properties (3) and (4) say that if an action is always going to be a *unique* best response, it is never abandoned (even if it is played only rarely).

The sequence $\{[\Pi^q]^k(X)\}_{k=0}^{\infty}$ studied in this paper is an example of a best response sequence. Blume (1995) considers a dynamic process where revision opportunities arrive randomly (and only one player revises his action at a time). With probability one, his process will generate a best response sequence. The "noise at the margin" process of Anderlini and Ianni (1996) allows more than one player to switch to a best response at a time. But again, with probability one, a best response sequence is generated. As long as players only switch to best responses and sometimes get opportunities to revise, *the timing of revision opportunities makes no difference to contagion properties*. More specifically, if the contagion threshold is ξ.

- if $q < \xi$, then there exists a finite group X such that *every* best response sequence $\{X_k\}_{k=0}^{\infty}$ with $X_0 = X$ has $\bigcup_{k \geqq 1} X_k = \mathscr{X}$;
- if $q > \xi$, then for *every* finite group X, there exists an infinite group Y such that for *every* best response sequence $\{X_k\}_{k=0}^{\infty}$ with $X_0 = X$, $\bigcup_{k \geqq 1} X_k \subseteq \bar{Y}$.[20]

20. In the non-generic case where $q = \xi$, contagion *is* sensitive to fine details of the best response dynamic.

7.2. *Random initial conditions*

Let the initial actions be chosen randomly, with each player starting out choosing action 1 with (independent) probability $\varepsilon \in (0, 1)$.[21] Let P_ε be the implied probability distribution over initial configurations. Define a modified contagion threshold as follows

$$\xi^* = \max \{q\text{: for all } \varepsilon > 0,\ P_\varepsilon[\{X\text{: } \textstyle\bigcup_{k \geqq 1} [\Pi^q]^k(X) = \mathscr{X}\}] = 1\}.$$

Intuitively, the modified contagion threshold asks whether action 1 will spread from a randomly chosen "small" infinite group of players to the whole population. The contagion threshold asked whether action 1 would spread from a finite group of players.

Say that the local interaction system is *well behaved* if there exist an infinite number of isomorphisms between players that preserve the neighbourhood structure. This property is satisfied by all the examples in this paper except Examples 5 and 6. If the interaction system is well behaved, then there will always exist an infinite number of disjoint groups of fixed finite size from which action 1 will spread, whenever there exists one such group. By the law of large numbers, with probability one, one of those finite groups will start one playing action 1. So we must then have $\xi^* \geqq \xi$ in well-behaved local interaction systems; thus contagion from some finite group is sufficient for contagion from a randomly chosen "small" infinite group.

It is straightforward to provide an upper bound on the modified contagion threshold, in the spirit of Corollary 1[22]

$$\xi^* \leqq \max \left\{ q\text{: } \begin{array}{l} \text{for some } N, \text{ every co-finite group contains an infinite} \\ \text{number of } (1-q)\text{-cohesive groups of size } N \text{ or less.} \end{array} \right\}. \tag{7.1}$$

Thus in the regions example (Example 4), the right-hand side of (7.1) is $1/(m+1)$ and so the modified contagion threshold equals the contagion threshold. But for nearest neighbour interaction (Example 2), the right-hand side of (7.1) is $\frac{1}{2}$.[23] A result of Lee and Valentinyi (2000) suggests that in the case of nearest neighbour interaction in two dimensions, the modified contagion threshold would be $\frac{1}{2}$ (*i.e.* the risk dominant action would always spread), whereas the original contagion threshold was $\frac{1}{4}$.[24] This important result suggests that it might be possible to derive much weaker sufficient conditions for a *modified* contagion threshold close to $\frac{1}{2}$. In particular, a weakening of the δ-uniformity condition must be possible.

7.3. *Random responses*

Blume (1993) and Ellison (1993, 2000) consider (finite population) dynamics in local interaction games where the possibility of mutations implies that players may switch to non-best responses (and thus *all* configurations are played with positive probability). The process is ergodic and in the long run, with small mutation probabilities, the risk dominant action (*i.e.* action 1 if $q < \frac{1}{2}$) will be played by most players most of the time. The risk

21. The models of Blume (1995) and Anderlini and Ianni (1996) both incorporate random initial conditions.

22. For any $\varepsilon > 0$, an infinite number of the finite groups will (with probability one) start out playing action 0 and never switch.

23. With nearest neighbour interaction in m dimensions, there exist non-trivial *infinite* $(2m-1)/(2m)$-cohesive groups; the existence of these groups was used above to show that the contagion threshold is $1/2m$. But non-empty *finite* p-cohesive groups do not exist for any $p > \frac{1}{2}$.

24. The argument of Lee and Valentinyi for large finite lattices could presumably be extended to the infinite lattice of Example 2.

dominant equilibrium is also selected with uniform interaction (Kandori, Mailath and Rob (1993)), but in that case, convergence is very slow. Ellison (1993) used a contagion argument to show that convergence to the risk dominant action would occur very fast if there was interaction on a line. Using the results in this paper, it would be possible to show very fast convergence to action 1, using a contagion argument, in general local interaction systems, if the payoff parameter q were less than the contagion threshold.

In fact, the fast convergence properties with local interaction *do not rely on contagion* (*i.e.* behaviour spreading by best responses alone from small neighbourhoods to much larger neighbourhoods). Young (1998, Chapter VI) provides a set of sufficient conditions on general interaction systems for fast convergence. He requires that all players belong to some small "close-knit" group. These close-knit groups need not even be connected to each other for very fast convergence to occur. Thus there is very fast convergence even when contagion is impossible under any best response dynamic.

8. CONCLUSION

This paper focused on one narrow question: when do we get contagion under deterministic best response dynamics in binary action games? This narrow focus allowed a detailed analysis of the effect of changes in the local interaction system. However, the techniques and some of the results presented here are relevant to a broader range of questions: for example, the existence of equilibria with co-existent conventions and stochastic best response dynamics.

Many of the results extend straightforwardly to more general interaction structures (for example, allowing different interactions to have different weights). A companion paper, Morris (1997*b*), considers a very general class of interaction games and it is straightforward to extend many of the results in this paper.

The contribution of the paper is to characterize contagion in terms of qualitative properties of the interaction system, such as cohesion, neighbour growth and uniformity (rather than in terms of, say, the dimensions of lattices or number of neighbours). But one would like to go one step further and understand how likely these critical properties are to emerge.

APPENDIX

For a sequence of groups X_k, write $X_k \uparrow X$ if $X = \bigcup_{k\geq 1} X_k$ and $X_k \subseteq X_{k+1}$ for each k; and $X_k \downarrow X$ if $X = \bigcup_{k\geq 1} X_k$ and $X_{k+1} \subseteq X_k$ for each k.

Lemma 1. *The following properties hold for all* $X \subseteq \mathscr{X}$.

B1 (*Operator Monotonicity*). $\Pi^p(X) \subseteq \Pi_+^p(X)$.
B2 (*Group Continuity*). *If* $X_k \uparrow X$, *then* $\Pi^p(X) = \bigcup_{k\geq 1} \Pi^p(X_k)$ *and* $\Pi_+^p(X) = \bigcup_{k\geq 1} \Pi_+^p(X_k)$.
B2 *implies*:
B2* (*Group Monotonicity*). *If* $X \subseteq Y$, *then* $\Pi^p(X) \subseteq \Pi^p(Y)$ *and* $\Pi_+^p(X) \subseteq \Pi_+^p(Y)$.
B3 (*Probability Continuity*). *If* $p_k \uparrow p$, *then* $\Pi^{p_k}(X) \downarrow \Pi^p(X)$ *and* $\Pi_+^{p_k}(X) \downarrow \Pi_+^p(X)$.
B3 *implies*:
B3* (*Probability Monotonicity*). *If* $p < r$, *then* $\Pi^r(X) \subseteq \Pi^p(X)$ *and* $\Pi_+^r(X) \subseteq \Pi_+^p(X)$.
B4 (*Inverse Operator*). *If* $p + r > 1$, *then* $\Pi^p(X) \subseteq \overline{\Pi^r(\bar{X})}$.

Proof.

B1: $\Pi^p(X) \subseteq X \cup \Pi^p(X) = \Pi_+^p(X)$.

B2*: If $X \subseteq Y$, then $\pi[X|x] \leq \pi[Y|x]$ for all x; so $\pi[X|x] \geq p$ implies $\pi[Y|x] \geq p$, and thus $\Pi^p(X) \subseteq \Pi^p(Y)$. Now $X \subseteq Y$ and $\Pi^p(X) \subseteq \Pi^p(Y)$ imply that $\Pi_+^p(X) = X \cup \Pi^p(X) \subseteq Y \cup \Pi^p(Y) = \Pi^p(Y)$.

B2: Suppose $X_k \uparrow X$. First, $x \in \bigcup_{k \geq 1} \Pi^p(X_k) \Rightarrow x \in \Pi^p(X_k)$ for some $X_k \Rightarrow x \in \Pi^p(X)$ (by B2*); so $\bigcup_{k \geq 1} \Pi^p(X_k) \subseteq \Pi^p(X)$. But for any x, there exists k such that $\Gamma(x) \cap X \subseteq X_k$ (by finiteness of $\Gamma(x)$). So $x \in \Pi^p(X) \Rightarrow x \in \Pi^p(X_k)$ for some $k \Rightarrow x \in \bigcup_{k \geq 1} \Pi^p(X_k)$; therefore, $\Pi^p(X) = \bigcup_{k \geq 1} \Pi^p(X_k)$. Now $\Pi^p_+(X) = X \cup \Pi^p(X) = [\bigcup_{k \geq 1} X_k] \cup [\bigcup_{k \geq 1} \Pi^p(X_k)] = \bigcup_{k \geq 1} [X_k \cup \Pi^p(X_k)] = \bigcup_{k \geq 1} \Pi^p_+(X_k)$.

B3*: If $\pi[X|x] \geq r$ and $r > p$, then $\pi[X|x] \geq p$. Therefore, $r > p$ implies $\Pi^r(X) \subseteq \Pi^p(X)$. Now $\Pi^r_+(X) = X \cup \Pi^r(X) \subseteq X \cup \Pi^p(X) = \Pi^p_+(X)$.

B3: Suppose $p_k \uparrow p$. By B3*, $\Pi^{p_k}(X)$ is a decreasing sequence of sets and $\Pi^p(X) \subseteq \Pi^{p_k}(X)$ for all k. But now if $x \in \bigcap_{k \geq 1} \Pi^{p_k}(X)$, $\pi[X|x] \geq p_k$ for all k, so $\pi[X|x] \geq p$, so $x \in \Pi^p(X)$. Thus $\Pi^{p_k}(X) \downarrow \Pi^p(X)$. Now $\Pi^{p_k}_+(X) = [X \cup \Pi^{p_k}(X)] \downarrow [X \cup \Pi^p(X)] = \Pi^p_+(X)$.

B4: Suppose $p + r > 1$; $x \in \Pi^p(X) \Rightarrow \pi[X|x] \geq p \Rightarrow \pi(\bar{X}|x) \leq 1 - p < r \Rightarrow x \notin \Pi^r(\bar{X}) \Rightarrow x \in \overline{\Pi^r(\bar{X})}$. ||

Lemma 2. *If ξ be the contagion threshold of local interaction system $(\mathscr{X}, \sim)$, the following properties are equivalent*:

[0] $p \leq \xi$;

[1] $\bigcup_{k \geq 1} [\Pi^p]^k(X)$ *is co-finite, for some finite* X;

[2] $\bigcup_{k \geq 1} [\Pi^p_+]^k(X)$ *is co-finite, for some finite* X;

[3] $[\Pi^p_+]^k(X) \uparrow \mathscr{X}$, *for some finite* X;

[4] $[\Pi^p]^k(X) \uparrow \mathscr{X}$, *for some finite* X.

Proof. By B3* and the definition of $\xi, p \leq \xi$ if and only if $\bigcup_{k \geq 1} [\Pi^p]^k(X) = \mathscr{X}$, for some finite X. Thus [4] $\Rightarrow$ [0] $\Rightarrow$ [1]. Thus it is sufficient to show the equivalence of [1], [2], [3] and [4].

If $X \subseteq Y$ and X is co-finite, then Y is co-finite. With property B1, this gives [1] $\Rightarrow$ [2].

To show [2] $\Rightarrow$ [3], suppose $\bigcup_{k \geq 1} [\Pi^p_+]^k(X)$ is co-finite, for some finite X; let $Y = X \cup (\overline{\bigcup_{k \geq 1} [\Pi^p_+]^k(X)})$; Y is the union of finite sets, and thus finite. But by property B2*, $\bigcup_{k \geq 1} [\Pi^p_+]^k(X) \subseteq \bigcup_{k \geq 1} [\Pi^p_+]^k(Y)$; and

$$\overline{\bigcup_{k \geq 1} [\Pi^p_+]^k(X)} \subseteq Y \subseteq \bigcup_{k \geq 1} [\Pi^p_+]^k(Y),$$

so $\bigcup_{k \geq 1} [\Pi^p_+]^k(Y) = \mathscr{X}$. But $[\Pi^p_+]^k(Y)$ is increasing by construction, so $(\Pi^p_+]^k(X) \uparrow \mathscr{X}$.

To show [3] $\Rightarrow$ [4], I first show by induction that for all groups X and $k \leq 1$,

$$[\Pi^p_+]^k(X) = X \cup \Pi^p([\Pi^p_+]^{k-1}(X)). \tag{8.1}$$

This is true by definition for $k = 1$. Suppose it is true for arbitrary k. Now

$$\begin{aligned}
[\Pi^p_+]^{k+1}(X) &= \Pi^p_+([\Pi^p_+]^k(X)) \\
&= [\Pi^p_+]^k(X) \cup \Pi^p([\Pi^p_+]^k(X)), \qquad \text{by definition of } \Pi^p_+ \\
&= X \cup \Pi^p([\Pi^p_+]^{k-1}(X)) \cup \Pi^p([\Pi^p_+]^k(X)), \qquad \text{by inductive hypothesis} \\
&= X \cup \Pi^p([\Pi^p_+]^k(X)), \qquad \text{by B2*, since } [\Pi^p_+]^{k-1}(X) \subseteq [\Pi^p_+]^k(X).
\end{aligned}$$

Now suppose that X is finite and $\bigcup_{k \geq 1} [\Pi^p_+]^k(X) = \mathscr{X}$. Let $Y = X \cup \{x: \Gamma(x) \cap X \neq \varnothing\}$; since X is finite, Y is finite, and we can choose K such that $Y \subseteq [\Pi^p_+]^K(X)$ and therefore, $Y \subseteq [\Pi^p_+]^k(X)$ for all $k \geq K$. Since $x \in X \Rightarrow \Gamma(x) \subseteq Y \Rightarrow \Gamma(x) \subseteq [\Pi^p_+]^k(X)$ for all $k \geq K \Rightarrow x \in \Pi^p([\Pi^p_+]^k(X))$ for all $k \geq K$. Thus $X \subseteq \Pi^p([\Pi^p_+]^k(X))$ for all $k \geq K$. Now by (8.1), $[\Pi^p_+]^{k+1}(X) = X \cup \Pi^p([\Pi^p_+]^k(X)) = \Pi^p([\Pi^p_+]^k(X))$ for all $k \geq K$. So $[\Pi^p]^k([\Pi^p_+]^K(X)) = [\Pi^p_+]^{K+k}(X)$ for all $k \geq 0$. Thus $[\Pi^p]^k([\Pi^p_+]^K(X))$ is increasing and $\bigcup_{k \geq 1} [\Pi^p]^k([\Pi^p_+]^K(X)) = \mathscr{X}$. Thus $[\Pi^p_+]^K(X)$ is a finite group satisfying property [4].

Finally, since $\mathscr{X}$ is co-finite, [4] $\Rightarrow$ [1]. ||

Lemma 3. *For any local interaction system $(\mathscr{X}, \sim)$ and probability $p \in (0, 1)$, there exists $\varepsilon > 0$ such that $\bigcup_{k \geq 1} [\Pi^r_+]^k(X)$ is $(1-p)$-cohesive for all $X \subseteq \mathscr{X}$ and $r \leq p + \varepsilon$.*

Proof. Consider the following *finite* subset of [0, 1]:

$$F(M) = \left\{ \alpha \in (0, 1): \alpha = \frac{n}{m}, \quad \begin{array}{l} \text{for some integers } m, n \\ \text{with } 0 < m \leq M \text{ and } 0 \leq n \leq m \end{array} \right\}.$$

Given p, choose $\varepsilon > 0$ such that $F(M) \cap (p, p + \varepsilon)$ is empty. Since $(\#(X \cap \Gamma(x)) / (\#\Gamma(x)) \in F(M)$ for all $x \in \mathscr{X}$ and $X \subseteq \mathscr{X}$, we have $\Pi^r(X) = \Pi^{r'}(X)$ for all $X \subseteq \mathscr{X}$ and $r, r' \in (p, p + \varepsilon)$. So for any $X \subseteq \mathscr{X}$, there exists a group Y with

$$Y = \overline{\bigcup_{k \geq 1} [\Pi^r_+]^k(X)} \quad \text{for all } r \in (p, p + \varepsilon).$$

Now for all $x \in Y$ and $r \in (p, p + \varepsilon)$, $\pi(Y|x) > 1 - r$, so $\pi(Y|x) \geq 1 - p$ for all $x \in Y$. ||

Proof of Proposition 1.

The proposition can be re-stated as: "every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup if and only if $\xi \leqq p$." Suppose every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup. Let X by any finite group. Let Y be any infinite, $(1-p)$-cohesive, subgroup of co-finite group $\bar{X}$. Fix $r > p$. We will show by induction that $Y \subseteq \overline{[\Pi_+^r]^k(X)}$. True for $k=0$ (since $Y \subseteq \bar{X}$). Suppose true for k. Now

$$\begin{aligned} Y &\subseteq \Pi^{1-p}(Y), \quad \text{since } Y \text{ is } (1-p)\text{-cohesive} \\ &\subseteq \Pi^{1-p}(\overline{[\Pi_+^r]^k(X)}), \quad \text{buy inductive hypothesis and B2*} \\ &\subseteq \overline{\Pi^r([\Pi_+^r]^k(X))}, \quad \text{by B4.} \\ \text{Thus } Y &\subseteq \overline{[\Pi_+^r]^k(X)} \cap \overline{\Pi^r([\Pi_+^r]^k(X))} \\ &= \overline{[\Pi_+^r]^k(X) \cup \Pi^r([\Pi_+^r]^k(X))} \\ &= \overline{[\Pi_+^r]^{k+1}(X)}. \end{aligned}$$

So $\bigcup_{k\geq 1}[\Pi_+^r]^k(X)$ is not co-finite for all $X \subseteq \mathscr{X}$ and thus, by Lemma 2, $\xi < r$. But $\xi < r$ for all $r > p$ implies $\xi \leqq p$.

Now suppose $\xi \leqq p$. Let X be any co-infinite group. By Lemma 3, there exists $\varepsilon > 0$ such that $\bigcup_{k\geq 1}[\Pi_+^{p+\varepsilon}]^k(\bar{X})$ is $(1-p)$-cohesive. Since $p+\varepsilon > \xi$, we have by Lemma 2 that $\bigcup_{k\geq 1}[\Pi_+^{p+\varepsilon}]^k(\bar{X})$ is infinite. Since $\bigcup_{k\geq 1}[\Pi_+^{p+\varepsilon}]^k(\bar{X}) \subseteq X$, every co-finite group contains an infinite $(1-p)$-cohesive subgroup. ||

Proof of Proposition 2.

The proposition can be re-stated as: "There exists a labelling l with $\alpha_l(k) \geqq p$ for all sufficiently large k if and only if $\xi \geqq p$." Suppose $\alpha_l(k) \geqq p$ for all $k > K$. Now let X be the finite group $\{l(j): j \leqq K\}$. Therefore, by induction $\{l(j): j \leqq K+k\} \subseteq [\Pi_+^p]^k(X)$, so $\bigcup_{k\geq 1}[\Pi_+^p]^k(X) = \mathscr{X} \Rightarrow p \leqq \xi$ (by Lemma 2).

Conversely suppose $p \leqq \xi$. By Lemma 2, there exists finite group X_0 such that $\bigcup_{n\geq 1}[\Pi_+^p]^n(X_0) = \mathscr{X}$. Let $X_n = [\Pi_+^p]^n(X_0) \cap \overline{[\Pi_+^p]^{n-1}(X_0)}$ for $n = 1, 2 \ldots$. Consider any labelling with $j > k$ whenever $l(j) \in X_m$, $l(k) \in X_n$ and $m > n$. Now $\alpha_l(k) \geqq p$ for all $k > \#X_0$. ||

Four additional lemmas are required to prove Proposition 5.

Lemma 4. *Suppose X is finite and $p > \frac{1}{2}$. Then $\#(\bigcup_{k\geq 1}[\Pi_+^p]^k(X)) \leqq (M+1)\#X$.*

Proof. Let $X_n = [\Pi_+^p]^n(X)$, for each $n = 0, 1 \ldots$ Take any labelling l with $j > k$ and $l(j) \in X_n \Rightarrow l(k) \in X_n$. Let $X = \{l(1), \ldots, l(K)\}$. As in the proof of Proposition 3, let

$$\begin{aligned} f(k) &= \#\{j: l(j) \sim l(k) \text{ and } j < k\} \\ g(k) &= \#\{j: l(j) \sim l(k) \text{ and } j > k\} \\ h(k) &= \#\{(i,j): l(i) \sim l(j), i \leqq k-1 \text{ and } j \geqq k\}. \end{aligned}$$

By construction, we have $h(k+1) = h(k) - f(k) + g(k)$; but $\alpha_l(k) = f(k)/(f(k)+g(k)) \geqq p > \frac{1}{2}$ if $k > K$ and $l(k) \in \bigcup_{k\geq 1}[\Pi_+^p]^k(X)$. So $h(k+1) < h(k)$ for all $k > K$. But since $h(K) \leqq MK$, $\#(\bigcup_{k\geq 1}[\Pi_+^p]^k(X)) \leqq (M+1)K$. ||

Lemma 5. *Suppose $(\mathscr{X}, \sim)$ satisfies low neighbour growth. Then for all $\varepsilon > 0$ and all finite groups $X \subseteq \mathscr{X}$, there exists n such that $[\#(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)}/[\#\Gamma^{n+1}(X)] < \varepsilon$.*

Proof. Suppose $[\#(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)})]/[\#\Gamma^{n+1}(X)] \geqq \varepsilon$ for all n. Then, for all n, $\#(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)}) \geqq (\varepsilon/(1-\varepsilon))\#\Gamma^n(X)$, so $\#\Gamma^{n+1}(X) \geqq (1+\varepsilon/(1-\varepsilon))\#\Gamma^n(X) = (1/(1-\varepsilon))\#\Gamma^n(X)$ and $\#\Gamma^n(X) \geqq (1/(1-\varepsilon))^n \#X$. This contradicts low neighbour growth. ||

Lemma 6. *$(\mathscr{X}, \sim, q)$ has a co-existent equilibrium if and only if there exist disjoint non-empty q-cohesive and $(1-q)$-cohesive groups in $\mathscr{X}$.*

Proof. [only if] follows from the definition of equilibrium. For [if], let X_0 and Y_0 be disjoint non-empty q-cohesive and $(1-q)$-cohesive groups in $\mathscr{X}$. Define X_k inductively as follows: $X_{k+1} = \Pi_+^q(X_k) \cap \bar{Y}_0$. Let $X_* = \bigcup_{k\geq 1} X_k$. Now suppose $x \in X_*$. If $x \in X_0$, then $x \in \Pi^q(X_0) \subseteq \Pi^q(X_*)$. If $x \notin X_0$, then $x \in X_{k+1} \backslash X_k$ for some

$k \geqq 0$, so (by definition of X_{k+1}), $x \in \Pi^q(X_k) \subseteq \Pi^q(X_*)$. Thus X_* is q-cohesive. But

$$\begin{aligned} X_* &= \textstyle\bigcup_{k \geq 1} X_k \\ &= \textstyle\bigcup_{k \geq 1} X_{k+1}, \qquad \text{since } X_1 \subseteq X_2 \\ &= \textstyle\bigcup_{k \geq 1} ((X_k \cup \Pi^q(X_k)) \cap \bar{Y}_0) \\ &= ((\textstyle\bigcup_{k \geq 1} X_k) \cup (\textstyle\bigcup_{k \geq 1} \Pi^q(X_k))) \cap \bar{Y}_0 \\ &= (X_* \cup \Pi^q(X_*)) \cap \bar{Y}_0, \qquad \text{by B2.} \\ &= \Pi^q(X_*) \cap \bar{Y}_0. \end{aligned} \tag{8.2}$$

Now suppose $x \in \bar{X}_*$. If $x \in Y_0$, then $x \in \Pi^{1-q}(Y_0) \subseteq \Pi^{1-q}(\bar{X}_*)$, since $Y_0 \subseteq \bar{X}_*$. If $x \notin Y_0$, then by (8.2) $x \notin \Pi^q(X_*)$, so $x \in \Pi^{1-q}(\bar{X}_*)$. Thus $\bar{X}_*$ is $(1-q)$-cohesive. So X_* is an equilibrium. ||

Lemma 7. *If $(\mathscr{X}, \sim)$ satisfies low neighbour growth, then there exists a non-empty, finite, $\frac{1}{2}$-cohesive, group.*

Proof. By Lemma 3, there exists $\varepsilon > 0$ such that $\bigcup_{k \geq 1} [\Pi_+^{(1/2+\varepsilon)}]^k(X)$ is $\frac{1}{2}$-cohesive for all $X \subseteq \mathscr{X}$. Fix any finite group Y. Let

$$Z_n = \Gamma^{n+1}(Y) \cap \overline{\textstyle\bigcup_{k \geq 1} [\Pi_+^{(1/2+\varepsilon)}]^k(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})} \qquad \text{for all } n = 0, 1 \ldots.$$

If $x \in Z_n$, then $x \in \Gamma^{n+1}(Y)$. But $x \in \overline{\bigcup_{k \geq 1} [\Pi_+^{(1/2+\varepsilon)}]^k(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})}$ implies $x \notin \Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}$. Thus $x \in \Gamma^n(Y)$. By construction of $\Gamma^{n+1}(Y)$, this implies that $\pi(\Gamma^{n+1}(Y)|x) = 1$. Now since $\overline{\bigcup_{k \geq 0} [\Pi_+^{(1/2+\varepsilon)}]^k(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})}$ is $\frac{1}{2}$-cohesive, we have that Z_n is $\frac{1}{2}$-cohesive. Also, Z_n is finite since $\Gamma^{n+1}(Y)$ is finite. Now observe that by Lemma 4,

$$\#([\Pi_+^{1/2+\varepsilon}]^k(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})) \leq (M+1)\#(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}), \qquad \text{for all } k \geq 1.$$

$$\begin{aligned} \text{Thus } \#Z_n &\geq \#\Gamma^{n+1}(Y) - \#(\textstyle\bigcup_{k \geq 1} [\Pi_+^{(1/2+\varepsilon)}]^k(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})) \\ &\geq \#\Gamma^{n+1}(Y) - (M+1)\#(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}) \\ &= \#\Gamma^{n+1}(Y)\left(1 - (M+1)\frac{\#(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})}{\#\Gamma^{n+1}(Y)}\right). \end{aligned}$$

By Lemma 5, $[\#(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)})]/[\#\Gamma^{n+1}(Y)] < 1/(M+1)$ for *some* n. Thus $\#Z_n > 0$ and thus Z_n is non-empty for that n. Thus Z_n is a non-empty, finite $\frac{1}{2}$-cohesive group. ||

Proof of Proposition 5.

Suppose that $\xi \leq q \leq \frac{1}{2}$ (a symmetric argument applies if $\frac{1}{2} \leq q \leq 1 - \xi$). By Lemma 7, there exists a non-empty finite, $\frac{1}{2}$-cohesive group X. Thus X is q-cohesive. By Proposition 1, $q \geq \xi \Rightarrow \bar{X}$ contains an infinite (and thus non-empty) $(1-q)$-cohesive subgroup. Thus by Lemma 6, $(\mathscr{X}, \sim, q)$ has a co-existent equilibrium. ||

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