Hub Location Network Optimization
OR Nice Organizers

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1 Introduction

The hub location problem (HLP) is one of the novel and thriving research areas in location theory. In order to satisfy a demand, the HLP involves the movement of people, commodities, or information between required origin–destination pairs. Hubs are applied to decrease the number of transportation links between origin and destination nodes. For example, a fully connected network with k nodes and with no hub node has $k(k\text{-}1)$ origin-destination links.

However, if a hub node is selected to connect all other nodes (non-hub nodes, also known as spokes) with each other, there will only be $2(k\text{-}1)$ connections to serve all origin-destination pairs. This idea can be extended to a network with more than one hub node, called a multiple-hub network. Thus, by using fewer resources, demand pairs can be served more efficiently with a hub network than with a fully connected structure.

The costs of a hub network depend on its structure. The total distance of arcs connecting the whole network might be less in a hub network, but the total travel distance may be greater since there is no guarantee that the number of people, merchandise, or information moving on the hub-to-hub connections are greater than those moving between hubs and spokes. Therefore, it is very complicated to determine the location of hub facilities as well as the allocation scheme of clients to them.

It seems that the telecommunication industry is originally one of the oldest users of the hub network concept. However, logistic systems, airline industry and postal companies are one of the main users of hub location network optimization. Today, there are many other areas that can take advantage of hub concept like maritime industry, freight transportation companies, public transit, and message delivery networks.

Figure 1 shows a sample hub location network where all the demand nodes have been connected to least one hub and to at most two hubs. Transferring flow between two spokes consists of three main elements: 1) Spoke-to-Hub or *collection*, 2) Hub-to-Hub or *transfer*, and 3) Hub-to-Spoke or *distribution*. In Figure 1, the path from the origin spoke i to the destination spoke j is: $i \to k \to l \to j$. Figure 2 depicts a real air transportation network under a hub network topology.

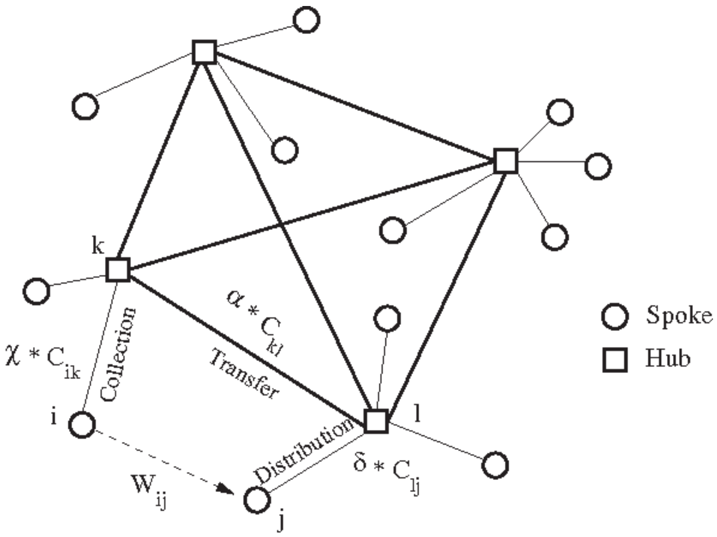


Figure 1: Hub location network.

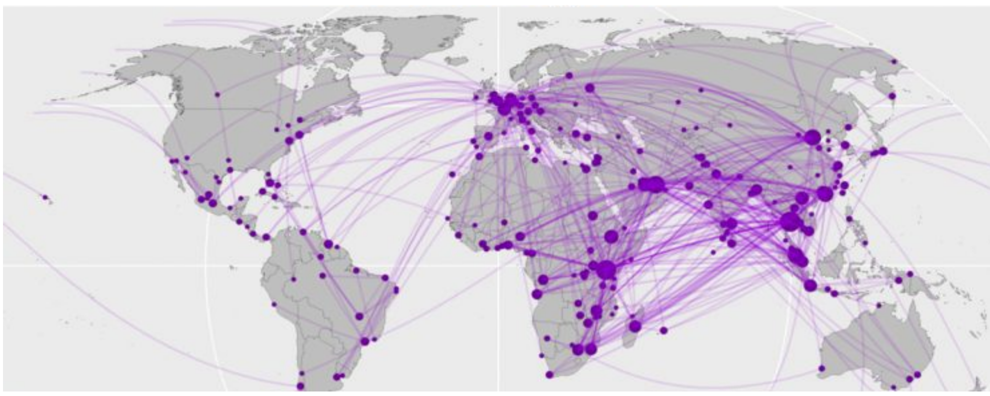
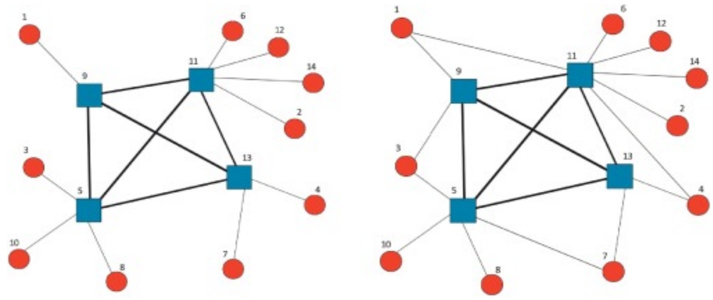


Figure 2: A sample of a hub network in air transportation.

2 Hub network properties

Each hub network is caracterized by particular structural properties. In the following, some of the main features are provided.

- **Single Allocation vs. Multiple Allocation**. In single allocation, each spoke must be allocated to exactly a single hub, while in multiple allocation, each spoke can be allocated to more than one hub. Figure 3 shows the difference between these two features.



(a) single allocation (b) multiple allocation

Figure 3: Single allocation vs. multiple allocation

- **Complete vs. Incomplete Hub Network**. In a complete hub network, the graph of hubs is complete and all the hubs are directly connected. In an incomplete hub network, the graph of hubs is incomplete but there is at least one indirect path between each pair of hubs. Figures 4a to 4c illustrates three different types of incomplete hub networks: a tree, a ring and a general hub network. A tree structure is a graph where no cycle can be found. An incomplete hub network that does not have a ring or tree structure is called general network. Figure 4d depicts a complete hub network.
- **Capacitated vs. Uncapacitated Hub Network**. In a capacitated hub network, a limited number/volume of flows can enter the hub while in the uncapacitated network, the hubs can process infinite number/volume of flows.

3 Problem statement

Consider a transportation/telecommunication network that contains a set of origin/destination nodes and the need of transferring a flow between each

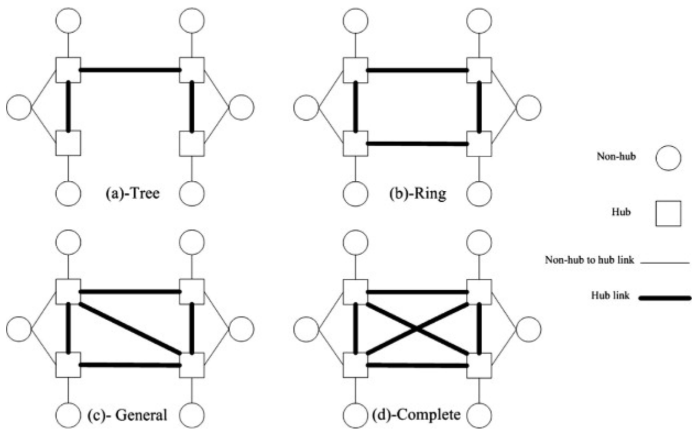


Figure 4: Various hub graphs

pair of origin-destination (O-D) nodes. Each node can be an origin or a destination at the same time. The aim is to **locate** a set of nodes as hubs and to **allocate** the remaining nodes (spokes) to the located hub to minimize the network's total cost (i.e., fixed and variable costs).

3.1 Assumptions

You are provided with the following information/assumptions:

- The location of the nodes is fixed,
- Each spoke should be allocated to only one hub (single allocation)
- All the spokes should be allocated to a hub
- There is a fixed cost for locating a hub
- There is a variable cost, per the quantity flows, transferred through the network
- Transferring flow between hubs takes advantage of a discount factor (transferring the flow in big volumes and lower costs)
- Each hub has finite capacity.

3.2 Problem representation

The problem can be represented as a graph $G = (N, E)$ whose set of nodes, $N = \{1, ..., n\}$, represents the set of origins and destinations of a certain flow (e.g., information, cargo, passenger) that is routed through G via some transshipment nodes that are called hubs. In addition, E is the set of links between each pair of nodes. In order to make the problem simpler, the following assumptions are provided:

- The graph G is a complete graph wherein all vertices are connected. In a complete graph with $|N|$ number of vertices, there are in total $(|N| \times (|N| - 1))/2$ arcs (i.e., $|E| = \frac{|N|\times(|N|-1)}{2}$). Figure 5 depicts a complete graph with 4 vertices ($|N| = 4$) and 6 arcs ($|E| = 6$).

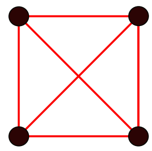


Figure 5: A complete graph.

- Building complete hub networks may unnecessarily increase the total investment cost in designing hub networks. Furthermore, in reality, many less-than-truckload and telecommunication networks do not operate on a complete hub network structure. Therefore, we require the hubs to be connected by means of a (non-directed) tree. In a tree graph, no cycle is allowed in the network. Figure 6 shows tree hub networks in telecommunication (Figure in the left) and transportation (Figure in the right) applications wherein the path between each pair of O-D nodes is unique. For every tree, we have $|E| = |N| - 1$. In Figures 6a and 6b, the intersection between at least two links is considered as a hub.

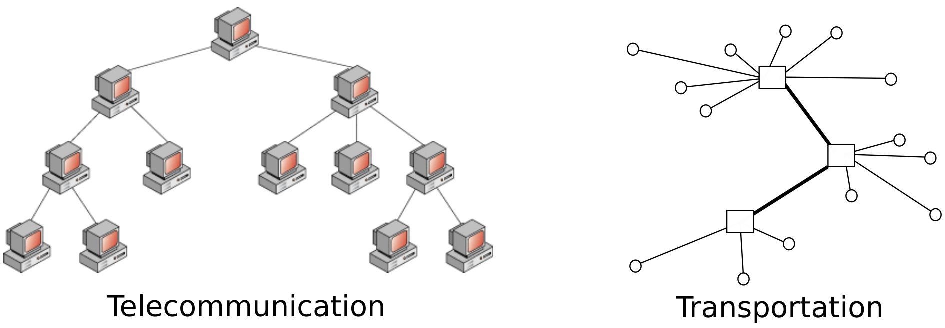


Figure 6: Tree hub networks.

Accordingly, the aim in this problem is to construct a hub network that has a tree structure. For this goal, we need to select $|N|-1$ links from the whole possible $\frac{|N|\times(|N|-1)}{2}$ number of links in the network. These links MUST form a tree and provide the optimal structure for the hub network.

3.3 Sets and parameters

Table 1 presents the sets and parameters of the problem.

Table 1: List of sets and parameters.

Sets:	
N	Set of vertices
$i, j \in N$	Indices of nodes ($i, j \in \{1, 2, \cdots, \lvert N\rvert\}$)
$k, l \in N$	Indices of hubs ($k, l \in \{1, 2, \cdots, \lvert N\rvert\}$)
Parameters:	
w_{ij}	Amount of flow from node i to node j
f_k	Fixed cost of locating a hub at node k
c_{ij}	Variable transfer cost of flow through the link from node i to node j
α	Hub-to-hub discount factor ($0 < \alpha < 1$)
C_k^{max}	Capacity of hub k
$O_i = \sum_{j=1}^{N} w_{ij}$	Total flow originating from node i
$D_i = \sum_{j=1}^{N} w_{ji}$	Total flow destination in node i

3.4 Decision variables

The main decisions in this problem are to first locate a number of hubs among the N possible number of nodes and then to allocate the remaining nodes to the located hubs. Therefore, the decision variables are defined as follows.

$Y_{kl} = 1$ if arc (k, l) links two hubs; 0 otherwise.

The decision variable Y_{kl} helps to design a tree hub-to-hub network.

$Z_{ik} = 1$ if the spoke i is allocated to the hub k; 0 otherwise.

When $i = k$, variable Z_{ik} represents whether or not a hub is located at node k. The Z variables will be referred to as location/allocation variables.

X_{ikl} = the amount of flow with origin in $i \in N$ traversing arc (k, l).

The X variables are independent flow variables and their value is calculated based on the structural Y and Z variables.

4 Problem formulation

In this section, you are expected to formulate the problem and provide a mathematical model. In this regard, you are guided through a set of consecutive steps to build the model.

Step 1: Objective function development

In this step, you develop different terms of the objective function. In this problem, we aim at minimizing the *Total Cost* of the network. The *Total Cost* is the sum of the *Total Fixed Cost* and the *Total Variable Cost*. Hence, we have:

Objective Function Z = MINIMIZE Total Cost

Total Cost = Total Fixed Cost + Total Variable Cost

The *Total Fixed Cost* is the sum of locating the hubs in the network and the *Total Variable Cost* is the cost of transferring the flow in the network. Based on the provided parameters and the decision variables, formulate each part of the objective function.

Term (Total...)		**Formulation**
Total Fixed Cost	=	$\sum_{k \in N} f_k Z_{kk}$
Total Variable Cost	=	$\sum_{i \in N} C_{ik} Oi Z_{ik} + \sum_{i \in N} C_{ik} Di Z_{ik} + \sum_{k \in N} \sum_{l \in N} \alpha C_{kl} Z_{kl}$

Finally, the objective function Z is:

$$\text{Total Cost} = \sum_{k \in N} f_k Z_{kk} + \sum_{i \in N} C_{ik} Oi Z_{ik} + \sum_{i \in N} C_{ik} Di Z_{ik} + \sum_{k \in N} \sum_{l \in N} \alpha C_{kl} Z_{kl}$$

Step 2: Constraints

In this section, you are expected to develop the mathematical formulation of the constraints. Each constraint is explained bellow. you just need to provide the mathematical formulation.

Constraint 1: Single Allocation

As mentioned in the assumptions, each spoke must be allocated to exactly one hub.

Single Allocation Each spoke must be allocated to exactly one hub:

$$\sum_{k \in N} Z_{ik} = 1 \quad \forall i \in N \tag{1}$$

Constraint 2: Hub logical structure

These constraints ensure that the a sample node can either be allocated to a hub or itself be a hub; but not both cases at the same time. If it is allocated to a hub, so that hub must be already allocated; but if it becomes a hub, there must be at least a hub-to-hub connection link (i.e., variable Y) from it to another already located hub. These constraints also guarantee that if $Y_{kl} = 1$, both nodes k and l must be already located as hubs. In addition, we conclude that two variables Z_{kl} and Y_{kl} cannot be equal to 1 at the same time.

$$Z_{ik} \le Z_{kk} \quad \forall i, k \in N \quad \text{Spoke } i \text{ needs } k \text{ to be a hub.} \tag{2}$$
$$Y_{kl} \le Z_{kk} \quad \forall k, l \in N \quad \text{Link } k\text{-to-}l \text{ requires } k \text{ as hub.} \tag{3}$$
$$Y_{kl} \le Z_{ll} \quad \forall k, l \in N \quad \text{Link } k\text{-to-}l \text{ requires } l \text{ as hub.} \tag{4}$$
$$Z_{kl} + Y_{kl} \le 1 \quad \forall k, l \in N \quad k \text{ can't be spoke and hub to } l. \tag{5}$$

Constraint 3: Transit of flow through the tree

We need a constraint to determine the X variables and to assure that the flow between hubs will only move through the small tree.

Flow only moves if k-to-l is a hub link:

$$X_{ikl} \le MY_{kl} \quad \forall i, k, l \in N \tag{6}$$

Constraint 4: Flow conservation at each hub

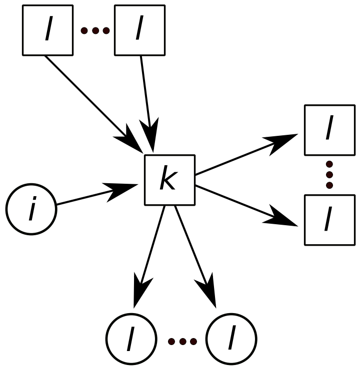


Figure 7: Flow conservation.

We need a constraint that guarantees the conservation of the flow with origin i in node k. Figure 6 shows the entering and exiting flow from node k. These flows should be equal.

Flow into hub k equals flow out for origin i:

$$\sum_{l \in N} X_{ikl} - \sum_{l \in N} X_{ilk} = w_{ik}.Z_{ik} \quad \forall i, k \in N \tag{7}$$

Constraint 5: Capacity constraint

In the hub network, limited volume of flow can enter the hub. Write a constraint in which the total volume of flow entering a hub is less than or equal to the capacity of that hub.

Hub k's flow can't exceed its capacity :

$$\sum_{i \in N} X_{ikl} \leq C_k^{\max} \quad \forall k \in N \tag{8}$$

Constraint 6: Tree structure

We need a constraint that guarantees the tree structure of the hub-to-hub network. In this constraint, the number of created arcs must be equal to the number of located hubs minus one.

Hub links form a tree with one less arc than hubs:

$$\sum_{k \in N} \sum_{l \in N} Y_{kl} = \sum_{k \in N} Z_{kk} - 1 \tag{9}$$

Constraint 7: Domain of variables

These constraints provide the domain for the variable if they are binary, integer, real etc.

$$Y_{kl} \in \{0, 1\} \quad \forall k, l \in N \quad Y_{kl} = 1 \text{ if } k\text{-to-}l \text{ is a hub link, else } 0. \tag{10}$$

$$Z_{ik} \in \{0, 1\} \quad \forall i, k \in N \quad Z_{ik} = 1 \text{ if } i \text{ assigned to hub } k\text{, else } 0. \tag{11}$$

$$X_{ikl} \geq 0 \quad \forall i, k, l \in N \quad \text{Flow } X_{ikl} \text{ is non-negative.} \tag{12}$$

5 Numerical experiment: Small-sized instance

In this section, you are expected to solve a small-sized instance and to find the optimal solution. You are expected to write the proposed model in PuLP, a free open source software written in Python used to describe optimization problems as mathematical models.

You are provided with the data for a small-sized telecommunication network with eight nodes. These data can be downloaded from the website of your course.

After solving the problem using PuLP, please report the objective function value and draw the network below .