Tutorial #2

Control Systems CIE-318

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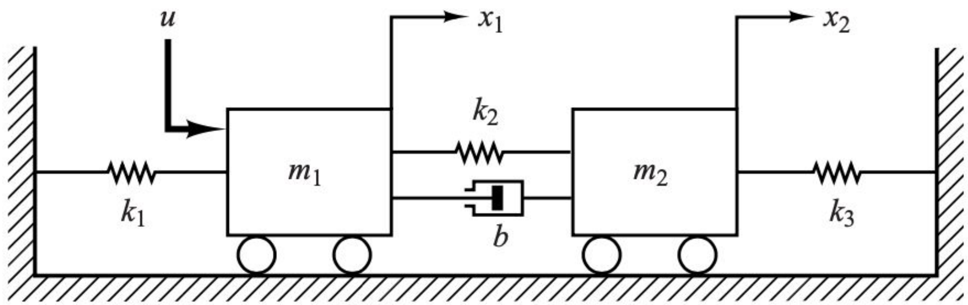
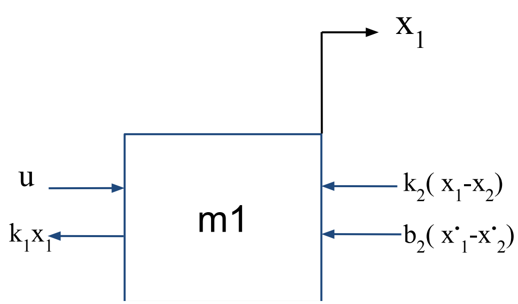
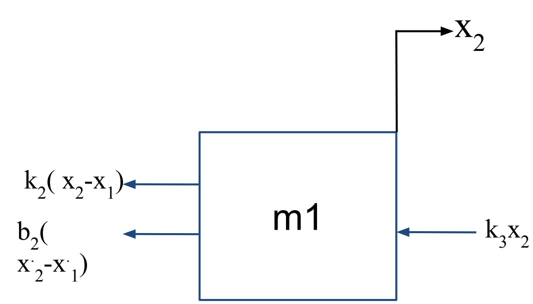
Outline

- More on mathematical modelling of mechanical and electrical systems.
- Laplace transform and inverse laplace transform.
- Methods of Solving ODEs and interpreting the solution.
- Using Matlab & Simulink to plot the system response.

Example 1A: Obtain the mathematical model of the following mass spring damper system.







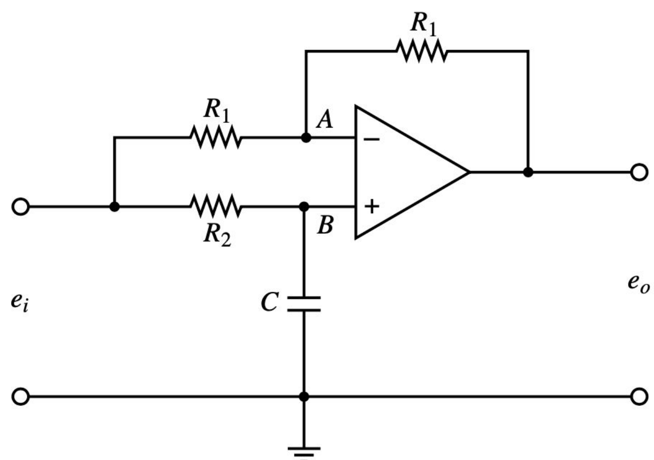
$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u \qquad m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

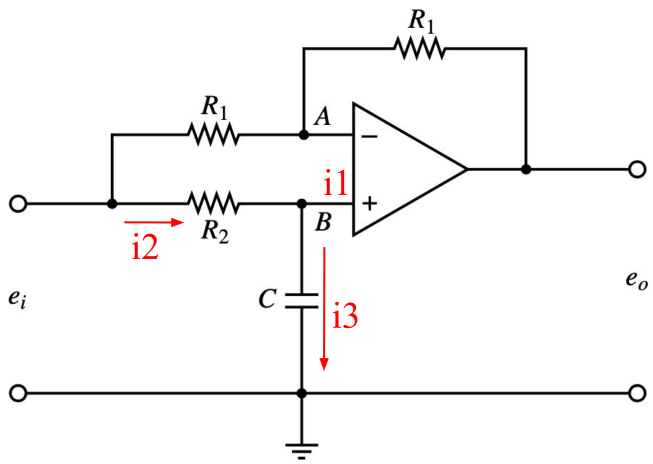
$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Example 1B: Obtain the mathematical model of the following circuit.



Example 1B: Solution



$$e_A = \left(\frac{R1}{R1 + R1}\right) \cdot (e_i - e_o) + e_o$$

$$e_A = e_B = 0.5 \cdot (e_i + e_o) \qquad 1$$

$$i_3 = C\left(\frac{de_B}{dt}\right) \qquad 2$$

$$i_2 = \frac{e_i - e_B}{R_2} \qquad 3$$

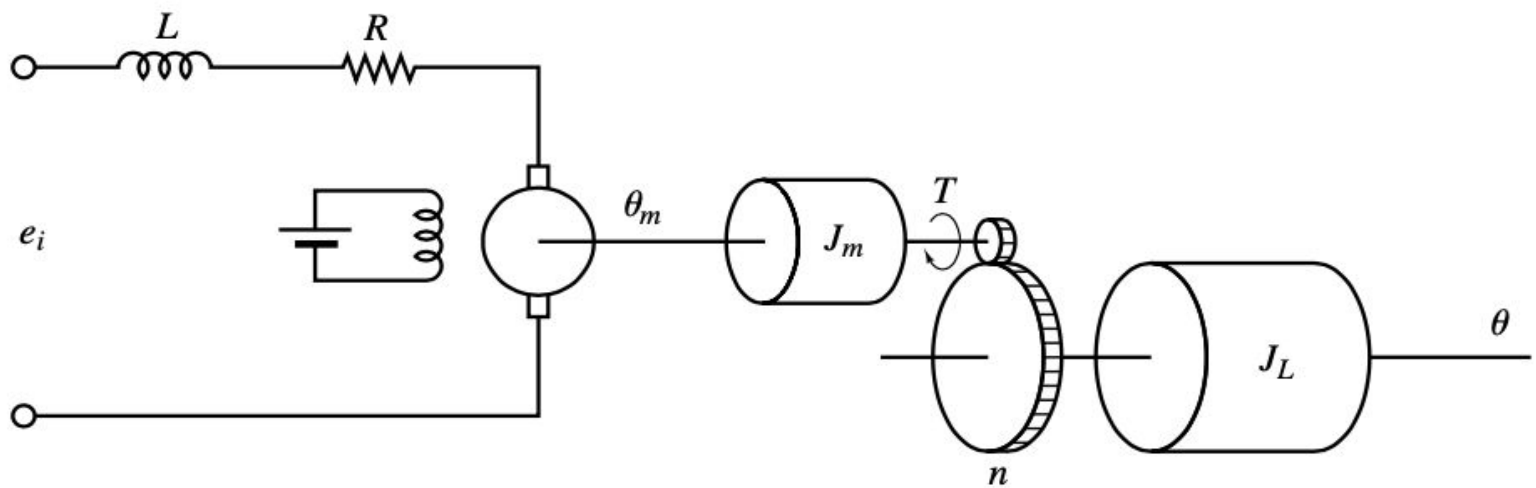
i1=0

∴ i2=i3 equating 2 & 3 and substituting e_B using 1

$$\frac{1}{R_2} \cdot (e_i - 0.5\,(e_i + e_o)) = 0.5C \cdot ({e_i}' + {e_o}')$$

$$C R_2 {e_i}' + C R_2 {e_o}' + e_o - e_i = 0$$

Example 1C: Obtain the mathematical model of the following armature-controlled dc servomotor that drives a load consisting of the moment of inertia JL . The torque developed by the motor is T. The moment of inertia of the motor rotor is Jm . The angular displacements of the motor rotor and the load element are um and u, respectively. The gear ratio is n = Θ/Θm.



Laplace Transform (LT)

Why do we need LT?

- To solve differential equations.
- It transforms the DEs to simple algebraic equations.
- Simplifies cascading many functions together as it becomes simple algebraic equations instead of complex convolution (review the lecture).
- Makes it simple to analyse the transient and frequency response of LTI systems as we will discover through the course.

Why do we need LT?

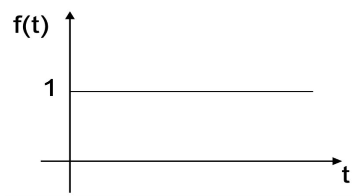
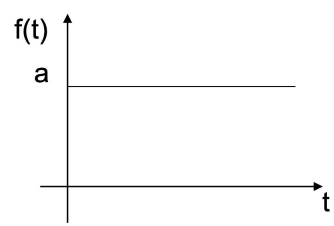
In Summary

- Laplace Transform is a tool to use to analyze and design a control system. Start with the dynamic equation of a system (usually low order or reduced to low order via linearization) which is made of differentials, integrals, and etc.

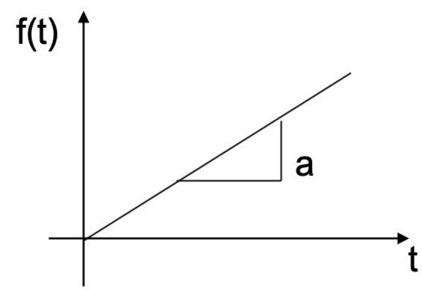
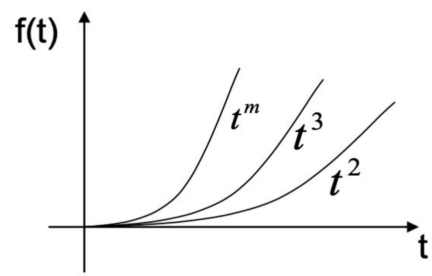
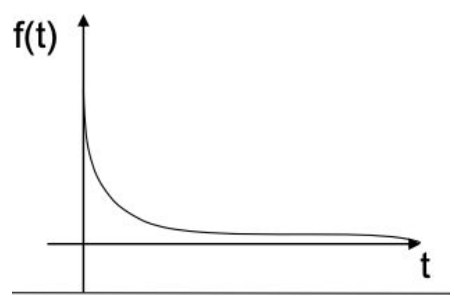
Laplace Transform (LT)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$shorthand : L\{f(t)\} = F(S)$$

Function	Shape	F(S)
Unit step f(t) = 1		$F(s) = \frac{1}{s}$
Step f(t) = at		$F(s) = \frac{a}{s}$

Laplace Transform (LT)

Function	Shape	F(S)
Ramp $f(t) = a^* t$		$F(s) = \frac{a}{s^2}$
Parabolic or higher $f(t) = a^* t^m$		$F(s) = \frac{a * m!}{s^{m+1}}$
Exponential $f(t) = e^{-at}$		$F(s) = \frac{1}{s-(-a)} = \frac{1}{s+a}$

Example 2: Find the Laplace transforms of the given functions.

A. $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

B. $f(t) = t^2 + e^{-2t}\sin(3t)$

C. $h(t) = 3\sinh(2t)$

D. $g(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

A. $f(t) = 6e^{-5t} + e^{3t} + 5t^{3} - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$
$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^{4}} - \frac{9}{s}$$

B. $f(t) = t^2 + e^{-2t}\sin(3t)$

$$\begin{aligned} f(t) &= t^2 + e^{-2t}\sin 3t \\ F(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{e^{-2t}\sin 3t\} \\ &= \frac{2!}{s^3} + \frac{3}{(s+2)^2+9} \\ &= \frac{2}{s^3} + \frac{3}{(s+2)^2+9}. \end{aligned}$$

C. h (t) = 3 sinh(2t)

$$
\begin{aligned}
h(t) &= 3\sinh t \\
&= 3\,\frac{e^{t}-e^{-t}}{2} \\
H(s) = \mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\frac{e^{t}}{2}\right\} - \mathcal{L}\left\{\frac{e^{-t}}{2}\right\} \\
&= \frac{3}{2}\left(\frac{1}{s-1}\right) - \frac{3}{2}\left(\frac{1}{s+1}\right) \\
&= \frac{3}{s^2-1}.
\end{aligned}
$$

D. $g(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

$$G(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Inverse Laplace Transform

Example 3A: Find the time function corresponding to the following Laplace Transform:

$$\frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$$

performmpartial fractionexpansion :

$$F(s)=\frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}=\frac{c_1}{(s+2)}+\frac{c_2 s+c_3}{(s^2+5s+11)}$$

$$\therefore\frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}=\frac{c_1(s^2+5s+11)+(c_2 s+c_3)(s+2)}{(s+2)(s^2+5s+11)}$$

$\Rightarrow$ *equatingnumerators*:

$$3s^2+9s+12=c_1(s^2+5s+11)+(c_2 s+c_3)(s+2)$$

$$3s^2+9s+12=(c_1+c_2)s^2+(5c_1+2c_2+c_3)s+11c_1+2c_3$$

Example 3A: Solution

Solution (cont'd):

$$3s^2 + 9s + 12 = (c_1 + c_2)s^2 + (5c_1 + 2c_2 + c_3)s + 11c_1 + 2c_3$$

$$\Rightarrow \text{equating coeff. of } s^2 : \; c_1 + c_2 = 3$$

$$\Rightarrow \text{equating coeff. } of\ s : \; 5c_1 + 2c_2 + c_3 = 9$$

$$\Rightarrow \text{equating coeff. of } s^0 : \; 11c_1 + 2c_3 = 12$$

$$Solve three equations in three unkowns:$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 5 & 2 & 1 \\ 11 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 2 & 1 \\ 11 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.8 \\ -0.6 \end{bmatrix}$$

Example 3A: Solution

$$F(s)=\frac{c_1}{(s+2)}+\frac{c_2s+c_3}{(s^2+5s+11)}=\frac{1.2}{(s+2)}+\frac{1.8s-0.6}{(s^2+5s+11)}$$

$$\Rightarrow \frac{1.2}{(s+2)}\xrightarrow{L^{-1}}1.2e^{-2t}$$

$$\Rightarrow \frac{1.8s-0.6}{(s^2+5s+11)}=\frac{1.8s-0.6}{(s+2.5)^2+\left(\frac{19}{4}\right)}=\frac{1.8(s+2.5)-5.1}{(s+2.5)^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$=1.8\frac{(s+2.5)}{(s+2.5)^2+\left(\frac{\sqrt{19}}{2}\right)^2}-\frac{5.1}{\frac{\sqrt{19}}{2}}\frac{\frac{\sqrt{19}}{2}}{(s+2.5)^2+\left(\frac{\sqrt{19}}{2}\right)^2}\xrightarrow{L^{-1}}$$

$$1.8\cos\left(\frac{\sqrt{19}}{2}t\right)e^{-2.5t}-\frac{5.1}{\frac{\sqrt{19}}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)e^{-2.5t}$$

$$\therefore f(t)=1.2e^{-2t}+1.8\cos\left(\frac{\sqrt{19}}{2}t\right)e^{-2.5t}-\frac{5.1}{\frac{\sqrt{19}}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)e^{-2.5t}$$

Solving ODEs Using LT

Important Property:

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0) \mapsto \text{initial condition}$$

In General: $L\{f^m(t)\} = s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \cdots\cdots\cdots - f^{(m-1)}(0)$

where $f^m(t)$ denotes the mth derivative of $f(t)$ with respect to time

$$L\left\{\frac{d^2}{dt^2} f(t)\right\} = s^2 F(s) - sf(0) - \dot{f}(0)$$

Ex. $\dfrac{d^2 i(t)}{dt^2} \xrightarrow{L} s^2 I(s) - si(0) - \dot{i}(0)$

Solving ODEs Using LT

Example 4A: Solve the following ODE using LT

$$\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$$

Solution:

$$s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)}$$

Example 4A: Solution

Using PFE
$$
\begin{aligned}
Y(s) &= \frac{s^3+3s^2+s+4}{s(s+1)(s^2+1)} \\
&= \frac{C_1}{s}+\frac{C_2}{s+1}+\frac{C_3 s+C_4}{s^2+1}
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \frac{s^3+3s^2+s+4}{(s+1)(s^2+1)}\Big|_{s=0} = 4 \\
C_2 &= \frac{s^3+3s^2+s+4}{s(s^2+1)}\Big|_{s=-1} = -\frac{5}{2}
\end{aligned}
$$

Equate the numerators and match coefficients of like power s to get C_3 & C_4

$$
\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}+\frac{C_3 s+C_4}{s^2+1} = \frac{s^3+3s^2+s+4}{s(s+1)(s^2+1)}
$$

Example 4A: Solution

$$\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}+\frac{C_3 s+C_4}{s^2+1} = \frac{s^3+3s^2+s+4}{s\,(s+1)\,(s^2+1)}$$

$$s^3\left(\frac{3}{2}+C_3\right)+s^2\,(4+C_3+C_4)+s\left(\frac{3}{2}+C_4\right)+4 = s^3+3s^2+s+4.$$

$$\begin{aligned} C_4+\frac{3}{2} &= 1 \qquad \Longrightarrow C_4=-\frac{1}{2} \\ C_3+\frac{3}{2} &= 1 \qquad \Longrightarrow C_3=-\frac{1}{2} \end{aligned}$$

$$\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}+\frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1}=\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}-\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}.$$

Finally $$y\,(t)=4-\frac{5}{2}e^{-t}-\frac{1}{2}\cos t-\frac{1}{2}\sin t.$$

Example 5: Obtain the TF of Examples 1A & 1B & 1C

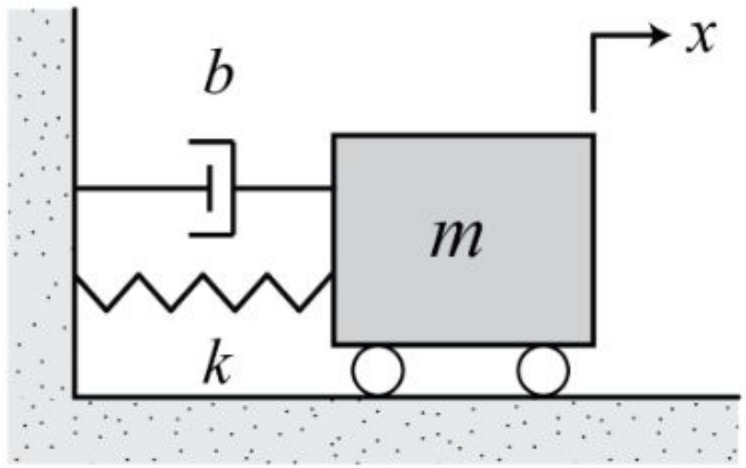
Example 1A: We need to obtain the transfer functions **X1**(s)/**U**(s) and **X2**(s)/**U**(s) of the mechanical system .

Example 1B: Obtain **Eo**(s)/**Ei**(s)

Example 1C: Obtain ***ϴ***(s)/**Ei**(s)

Introduction to Simulink

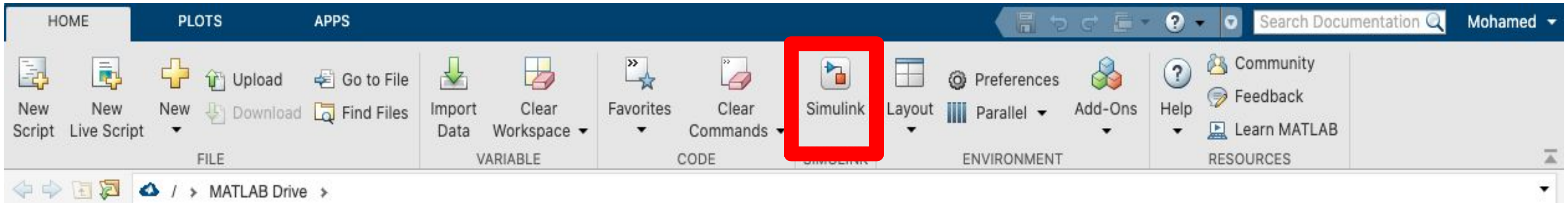
Example 6: Plot the response of the following system using simulink. where $\dot{x}(0) = 3$ and $x(0)=0$



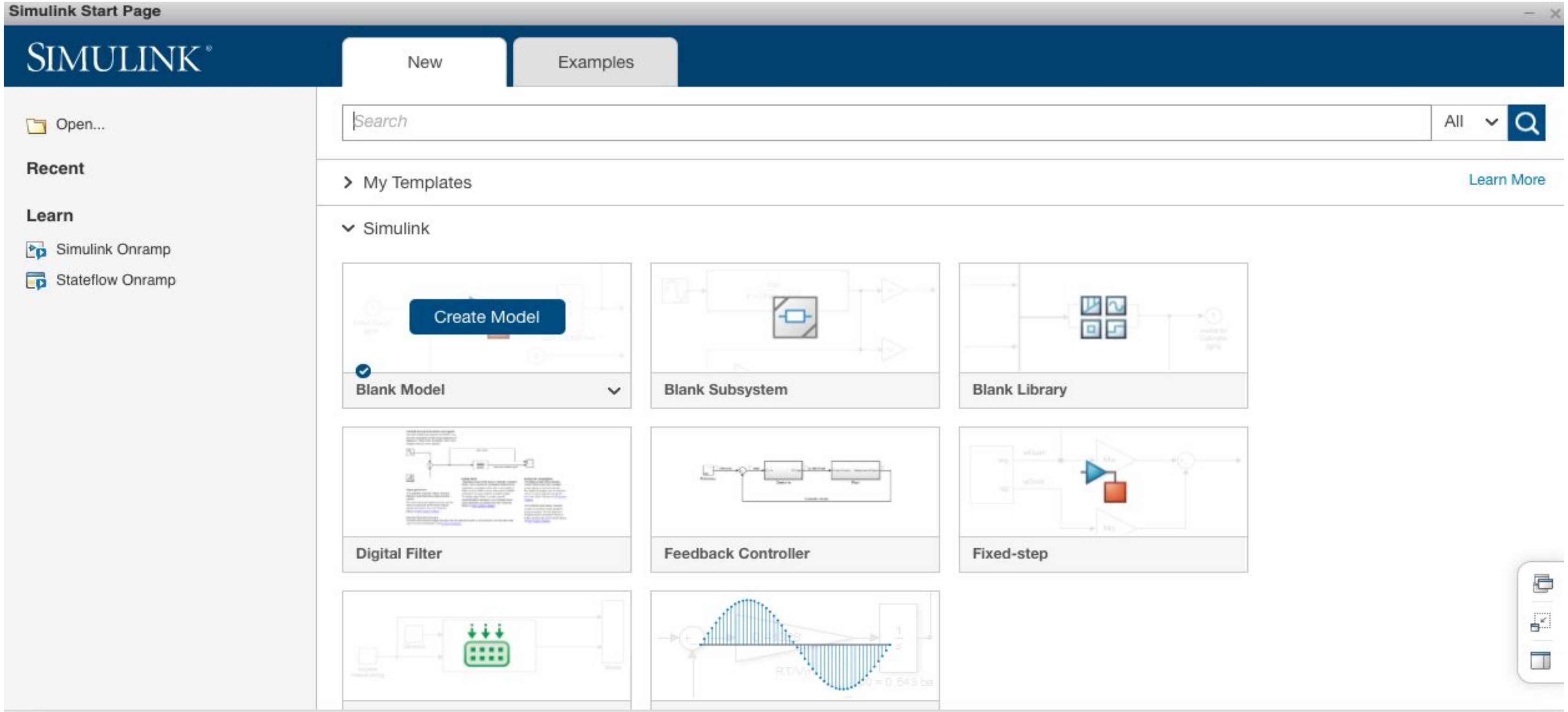
m=1
k=2
b=2

$$m\ddot{x} = -k\,x - b\,\dot{x}$$

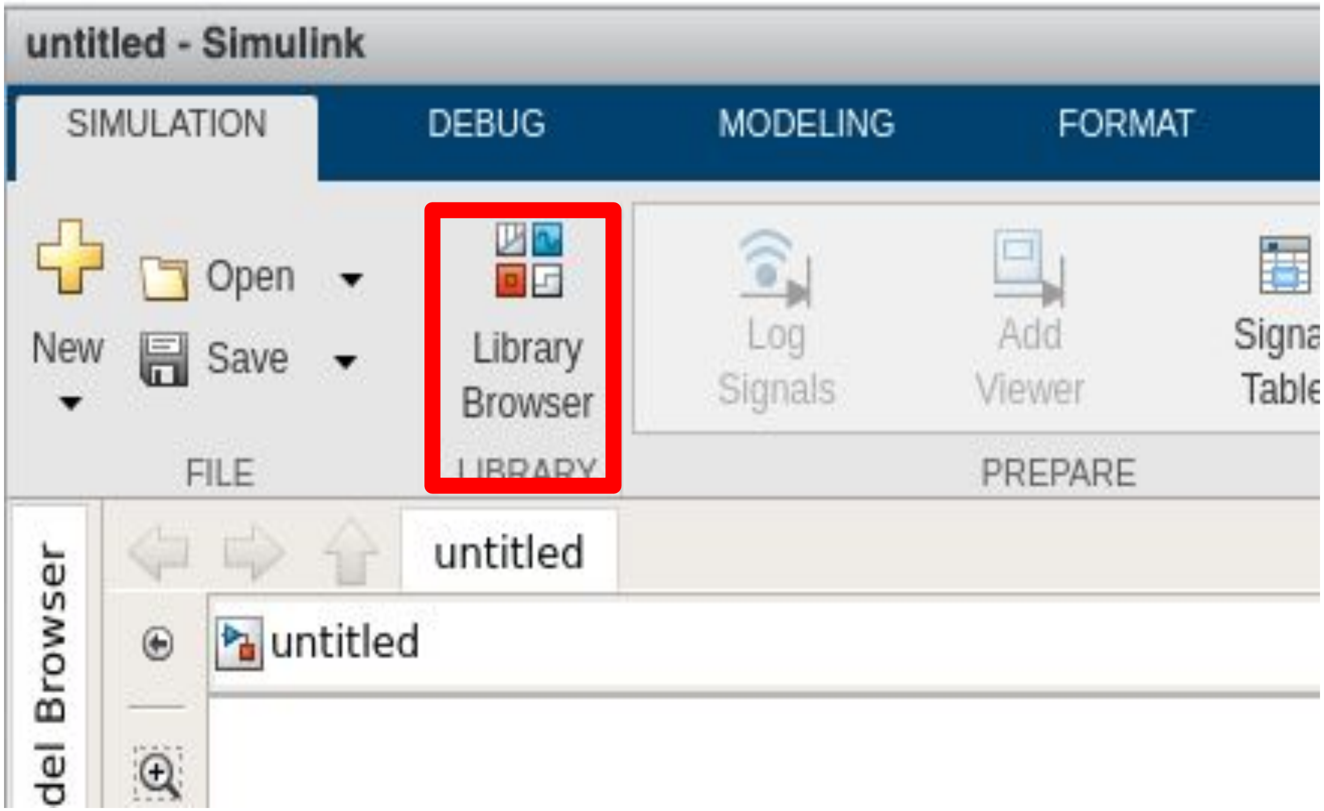
Example 6:
Step1: open matlab and click on simulink button



Step2: create a blank model

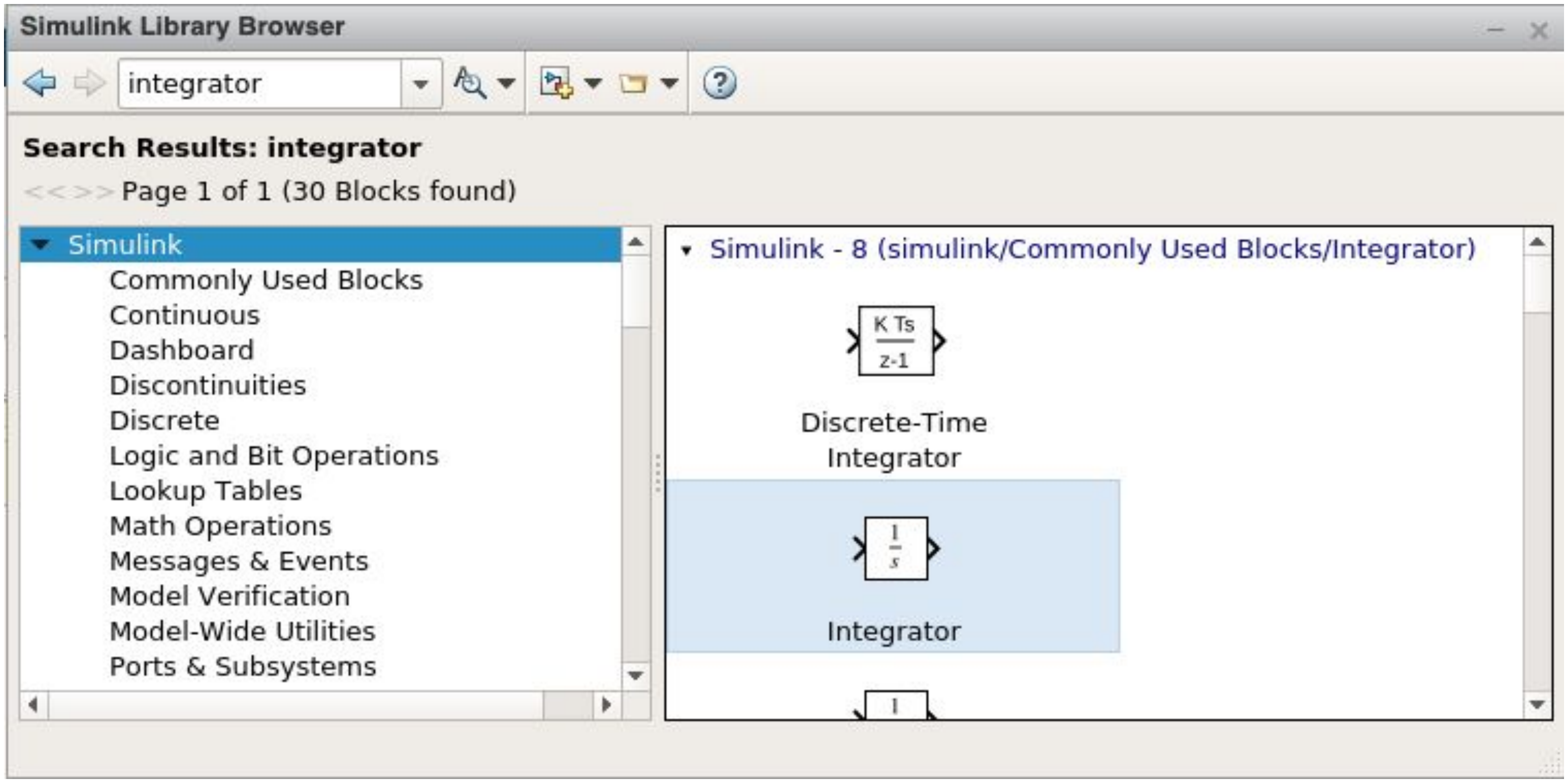


Example 6:
Step3: click on library browser



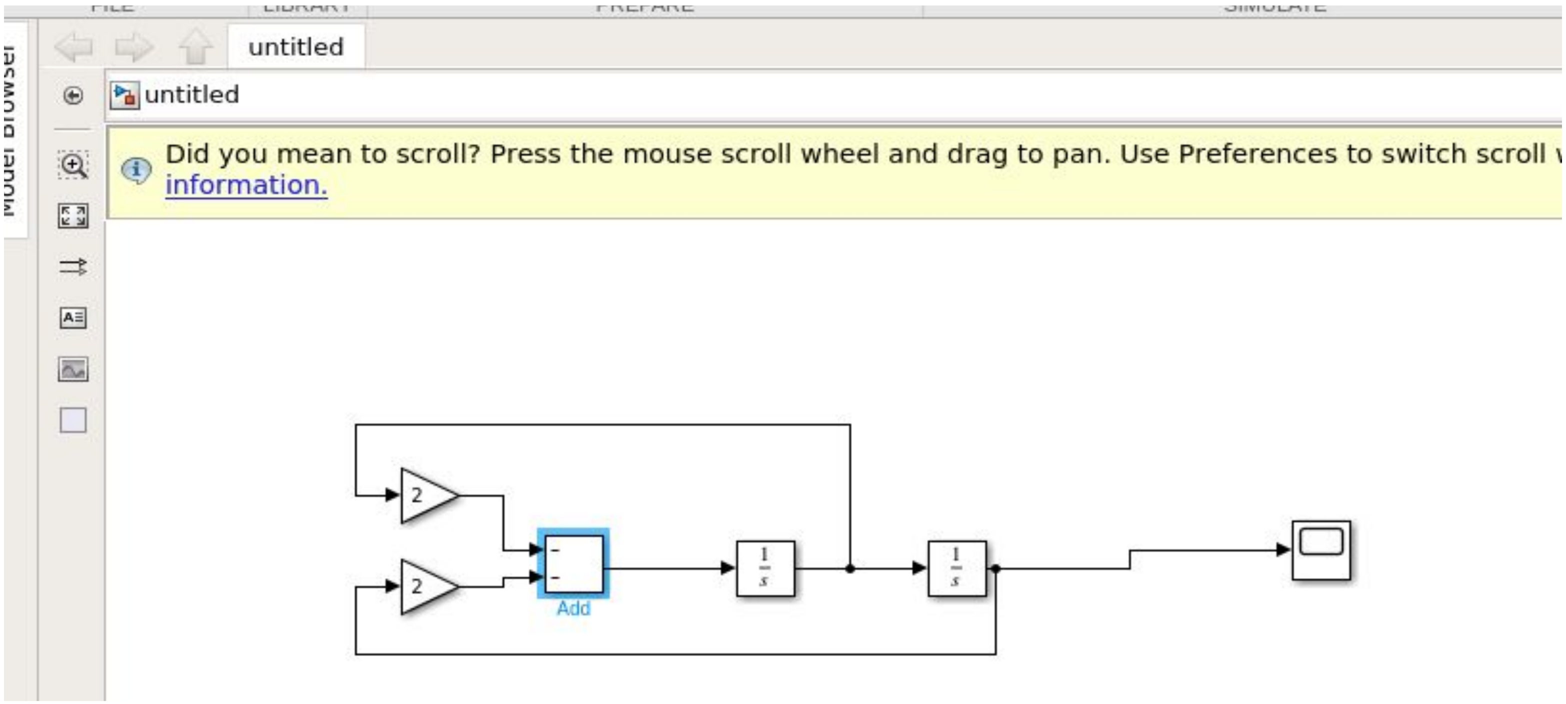
Example 6:

Step4: search for integrator and double click to add to your simulation



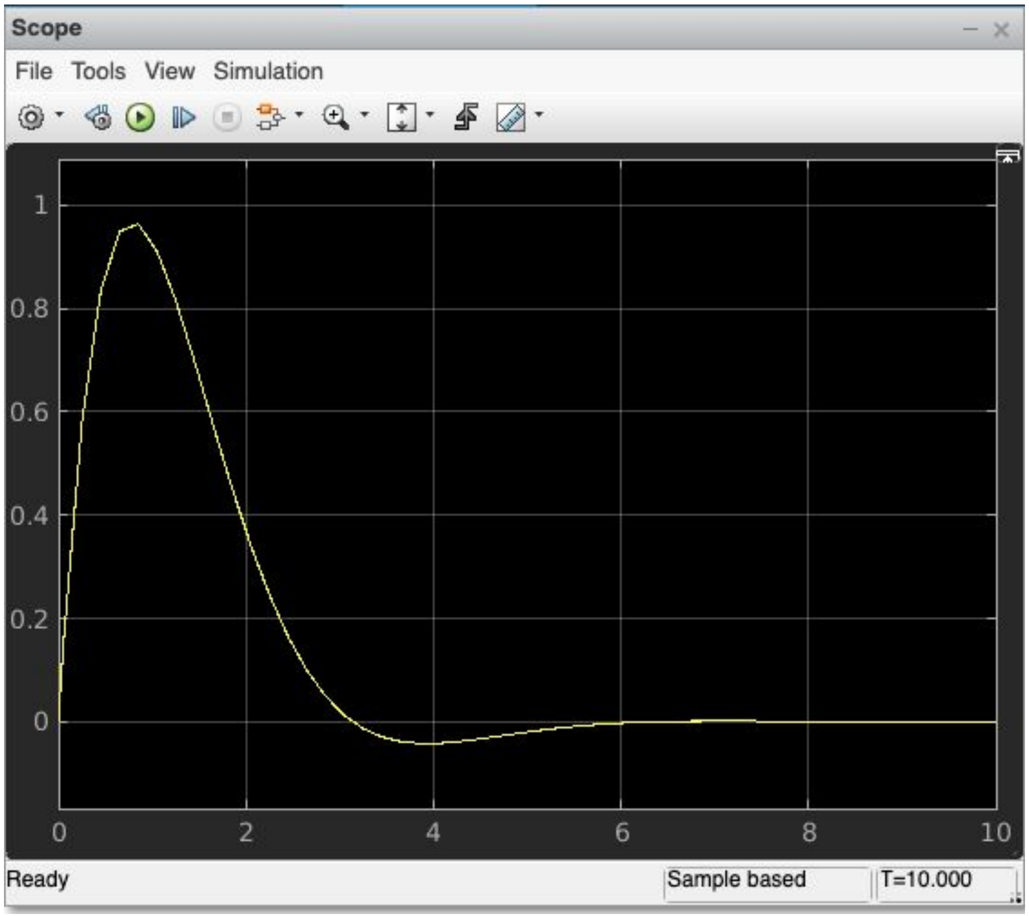
Example 6:

Step 5: repeat step 4 to add “Gain”, “Add” and “Scope” and connect them as shown below



Example 6:

Step 6: click run and observe the output

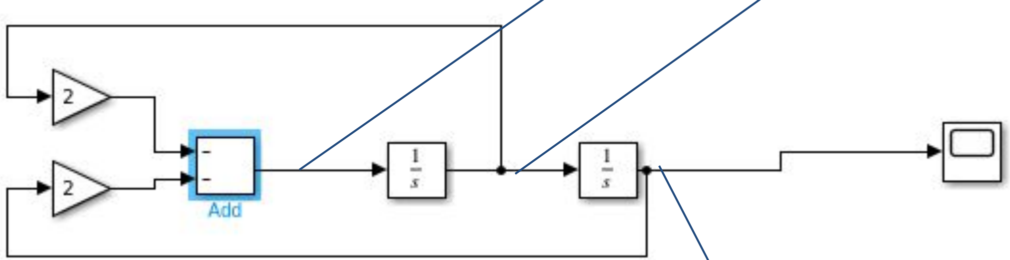


Example 6: Some explanations

here is the acceleration

$m\ddot{x} = -k x - b\dot{x}$

here is the velocity $\dot{x}$



here is the position x

Lab Task :

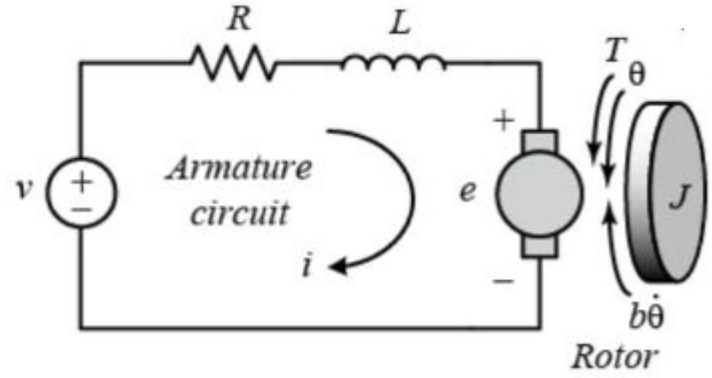
Plot the speed response of a step input of 1v the following motor model using Simulink.

Hint: you will need to get two differential equations

Hint2: Kb=Ki=K

Hint3: to simulate the step in voltage add the step block or add a signal builder block and modify it to step.

- **Physical setup**
  - V: voltage source
  - R:electric resistance
  - L:electric inductance
  - e: back EMF
  - T: torque Generated by the motor
  - $\dot{\theta}$: angular velocity
  - J:moment of inertia of the rotor



Lab Task :

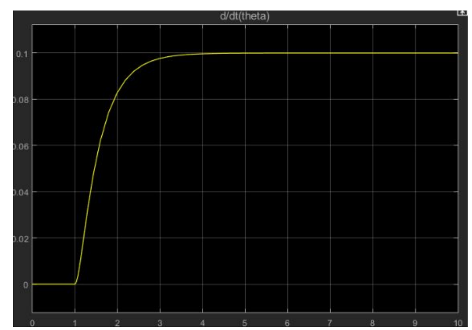
```
J=0.01;
b=0.1;
Ke=0.01;
Kt=0.01;
K=0.01;
R=1;
L=0.5;
```

Hint: you can use this equation you should solve the equation in terms of the highest order variable then start from there and add integrators. It should be then easy to solve like example 6

$$\frac{d^2\theta}{dt^2} = \frac{1}{J}\text{(expression)}$$

$$\frac{di}{dt} = \frac{1}{L}(\text{ expression } \quad . \quad)$$

Expected Output:



Assignment Task 2:

- Will be posted today on classroom
- Deadline next Wednesday 24/03