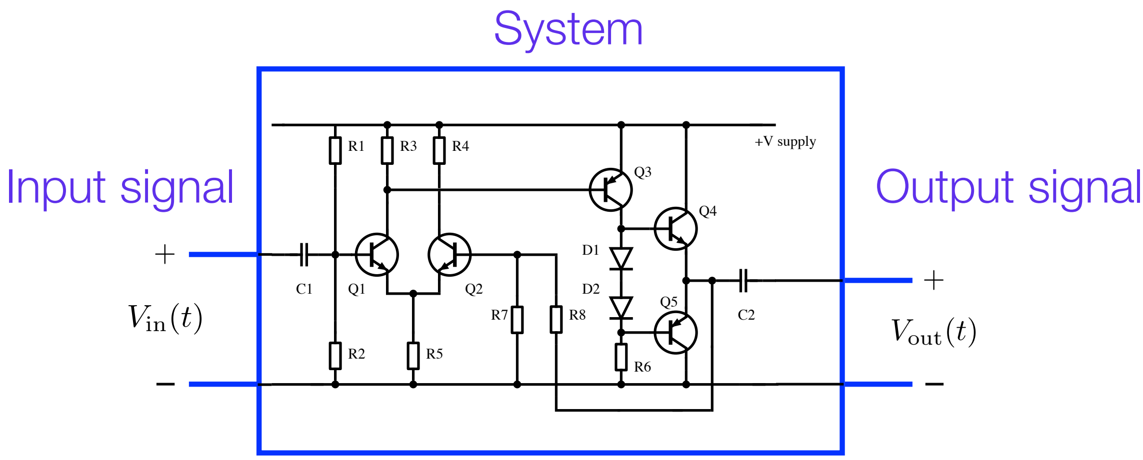
Introduction to Signals and Systems

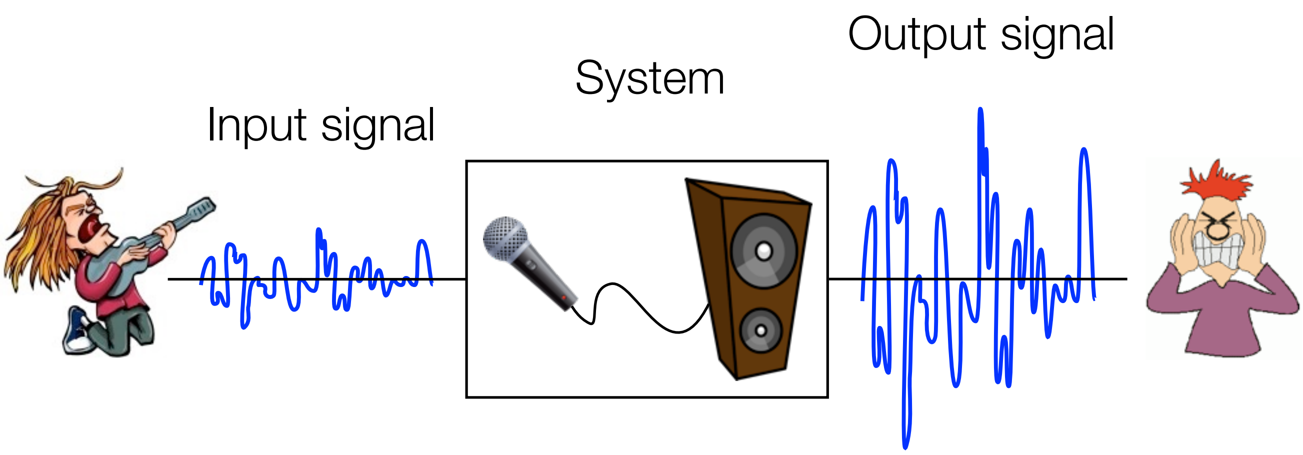
- Basic Concepts
- Signal Properties and Special Signals
- Fundamental System Properties

Signals and Systems

An Electronic Circuit



Sound Amplification System



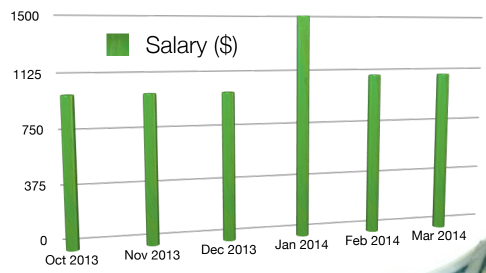
A Camera

Input

System

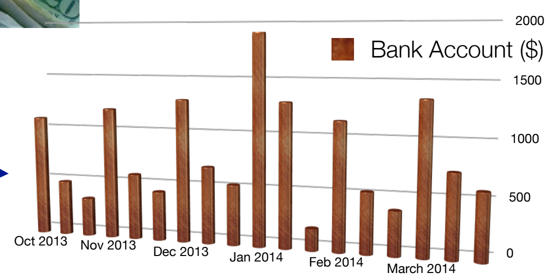
Output

Income and Outlay System



Input signal

System



Output signal

System vs. Function

Input		Output
a number or an n-tuple $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $z \in \mathbb{C}$ $(x, y) \in \mathbb{R}^2$, $\ldots$	$\longrightarrow$ **Function** $\longrightarrow$	a number or an n-tuple $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $z \in \mathbb{C}$ $(x, y) \in \mathbb{R}^2$, $\ldots$
a function $f(t)$, $p(x, y)$, $a[n]$, $\ldots$	$\longrightarrow$ **System** $\longrightarrow$	a function $f(t)$, $p(x, y)$, $a[n]$, $\ldots$

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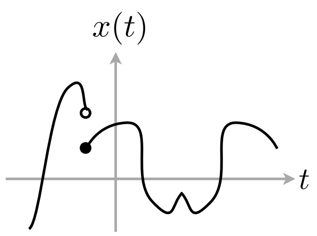
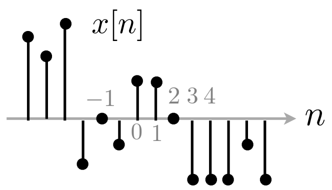
Continuous and Discrete-space signals

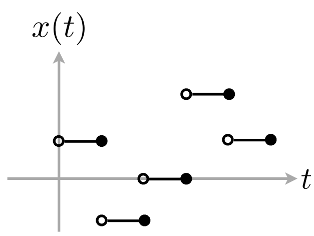
- Continuous-domain (Continuous-time)
  (حوزه پیوسته*



- Discrete-domain (Discrete-time)
  (حوزه گسسته*



- Discrete-valued Continuous-domain

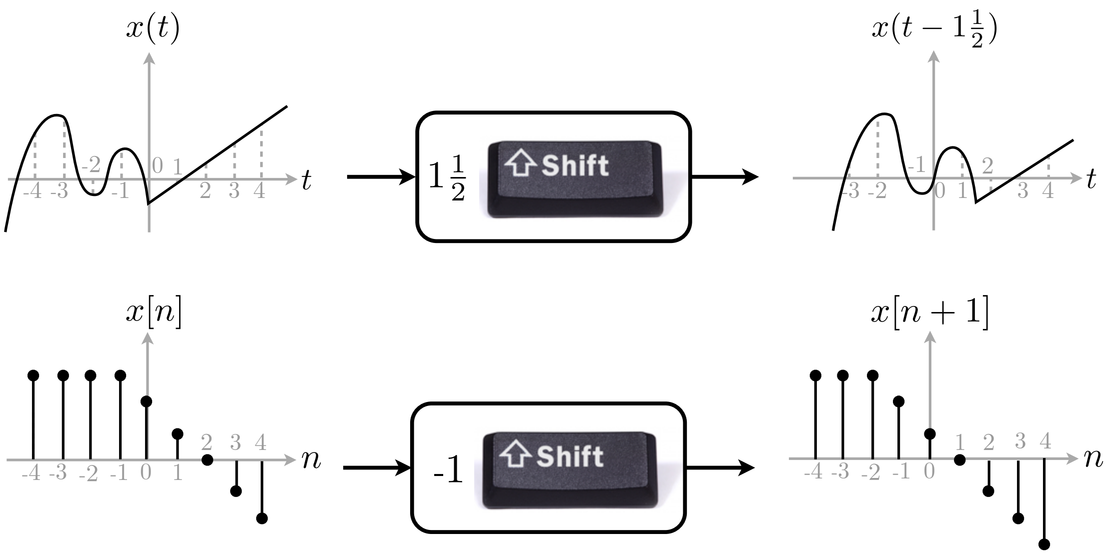


Continuous and Discrete-space signals

- Continuous-domain*حوزه پیوسته(
- Discrete-domain*حوزه گسسته(

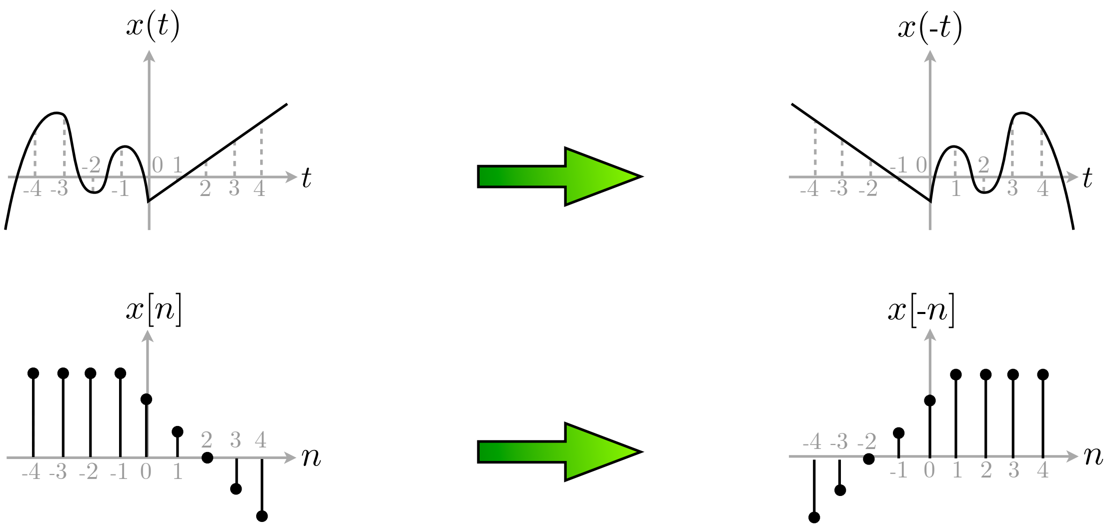
Simple Transformations

- Shift *انتقال(



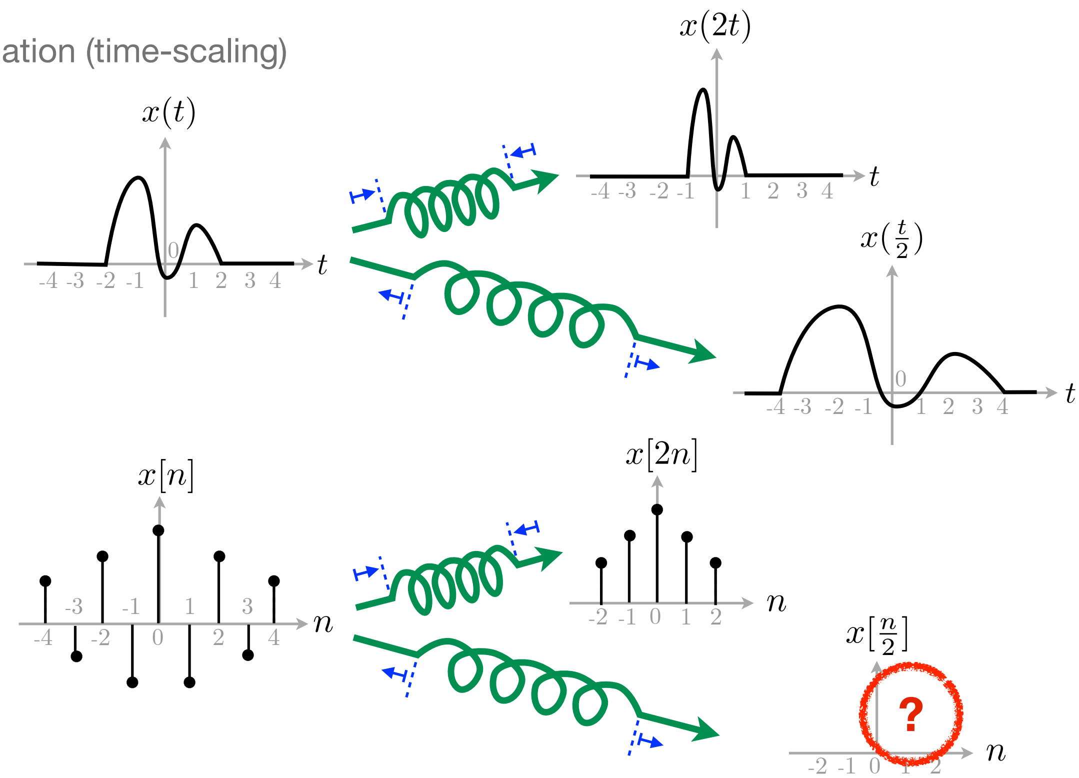
Simple Transformations

- Time-reversal



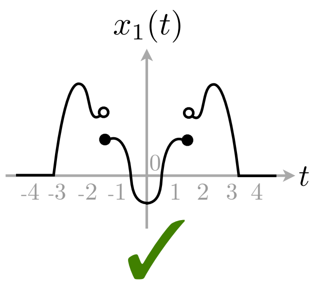
Simple Transformations

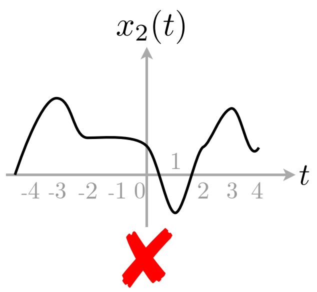
- Dilation (time-scaling)



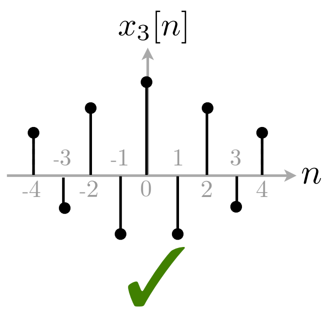
Even and Odd symmetries

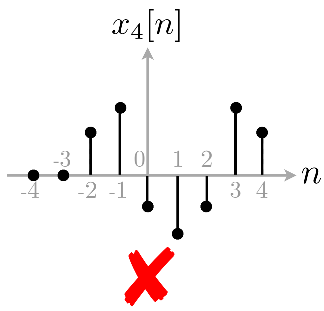
- Even Signals *سیگنالهای زوج(
  - Continuous-domain
    $x(\text{-}t) = x(t)$





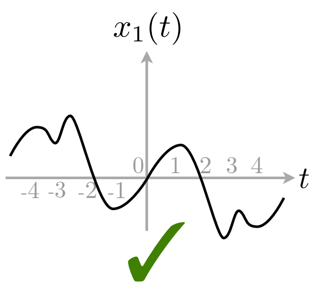
  - Discrete-domain
    $x[\text{-}n] = x[n]$

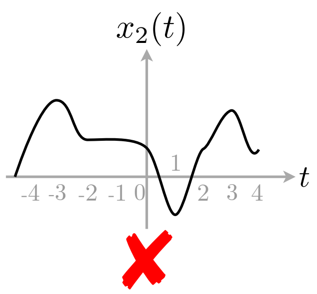




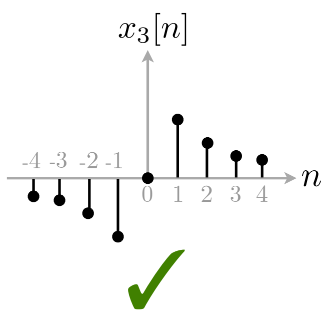
Even and Odd symmetries

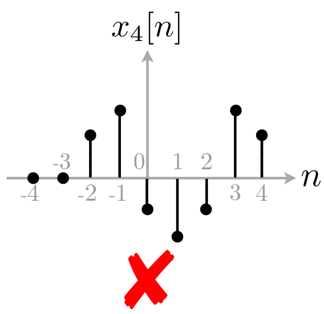
- Odd Signals *سیگنالهای فرد(
  - Continuous-domain
    $x(\text{-}t) = \text{-}x(t)$





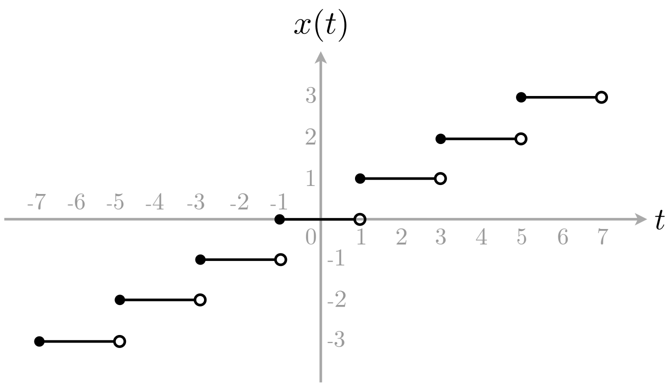
  - Discrete-domain
    $x[\text{-}n] = \text{-}x[n]$





Even and Odd symmetries

Even or odd?



Even   Odd

Even and Odd parts

- Even part (قسمت زوج)

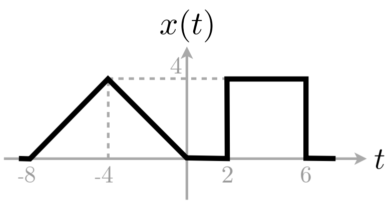
$$\mathcal{E}v\{x(t)\} \triangleq (\mathcal{E}v\ x)(t) \triangleq \frac{x(t)+x(\text{-}t)}{2}$$

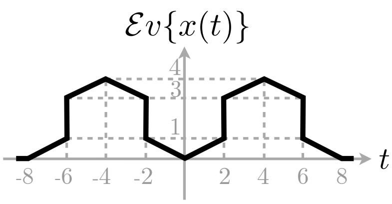
$$\mathcal{E}v\{x[n]\} \triangleq (\mathcal{E}v\ x)[n] \triangleq \frac{x[n]+x[\text{-}n]}{2}$$

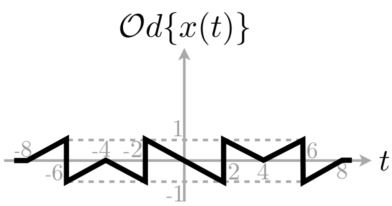
- Odd part (قسمت فرد)

$$\mathcal{O}d\{x(t)\} \triangleq (\mathcal{O}d\ x)(t) \triangleq \frac{x(t)-x(\text{-}t)}{2}$$

$$\mathcal{O}d\{x[n]\} \triangleq (\mathcal{O}d\ x)[n] \triangleq \frac{x[n]-x[\text{-}n]}{2}$$



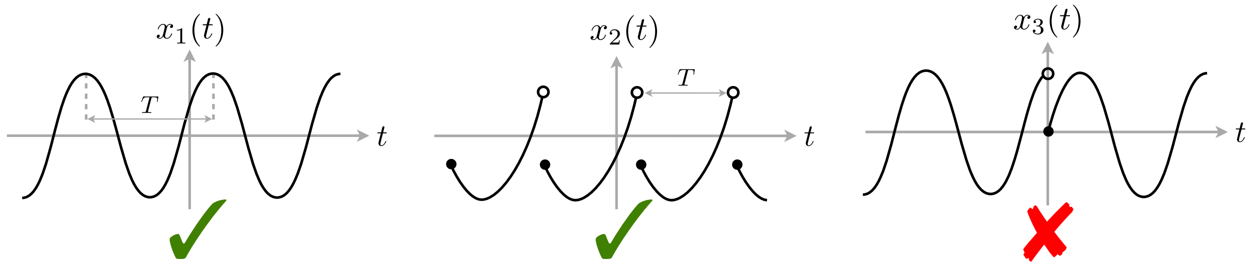




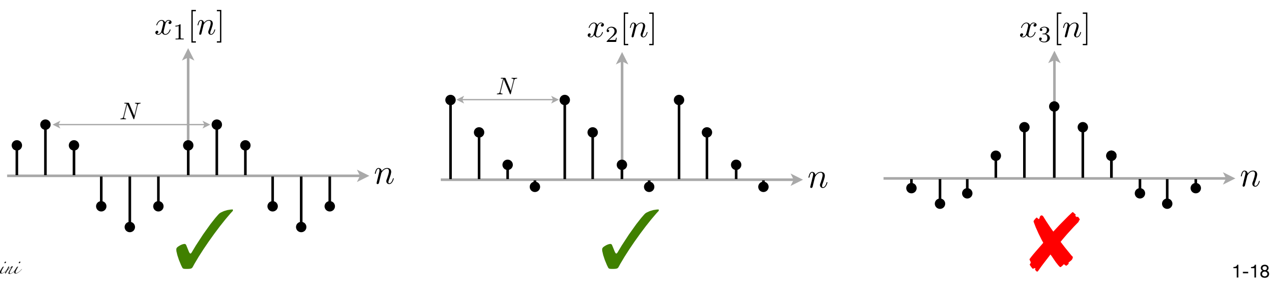
$$\mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\} = x(t)$$

Periodic Signals (سیگنالهای تناوبی)

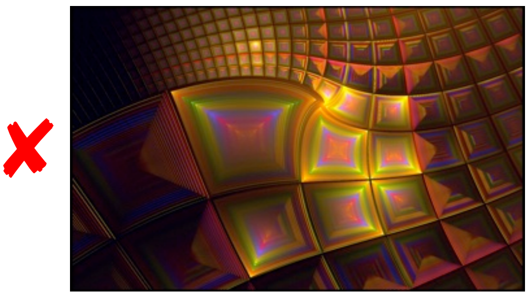
- Continuous-domain $\exists T,\ \forall t: \quad x(t+T) = x(t)$

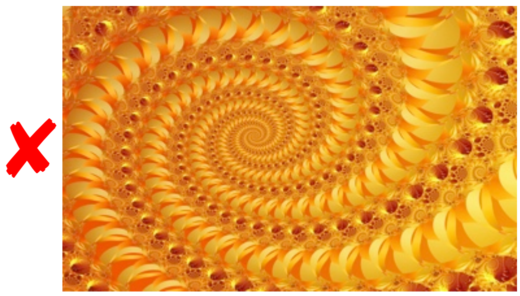


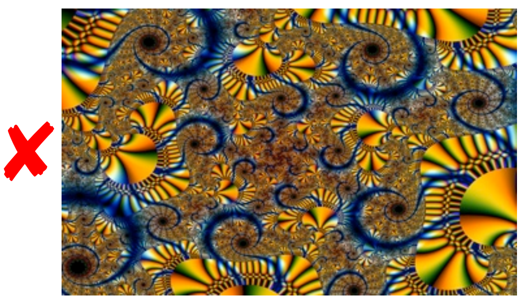
- Discrete-domain $\exists N,\ \forall n: \quad x[n+N] = x[n]$



Periodic Signals (سیگنالهای تناوبی)

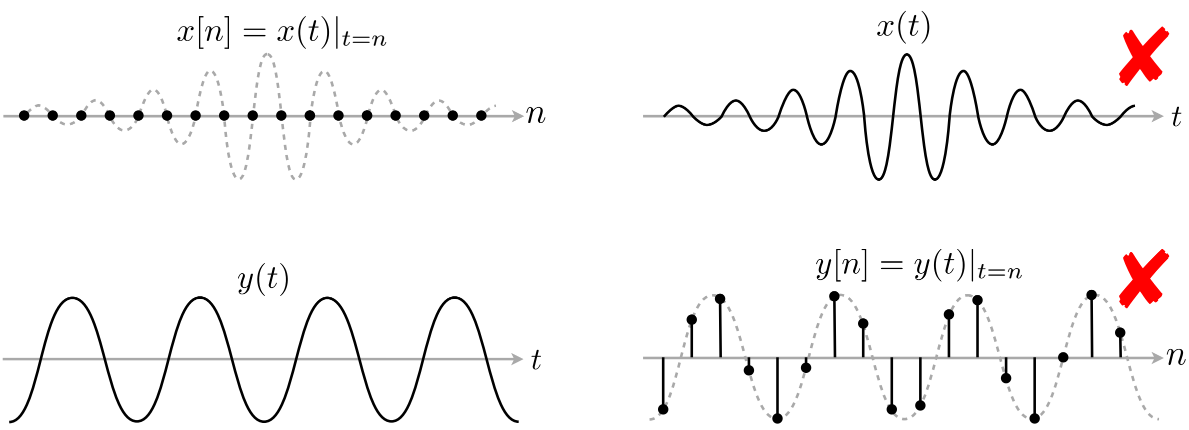






Periodic Signals (سیگنالهای تناوبی)

Periodic $\Longrightarrow$ Periodic ?

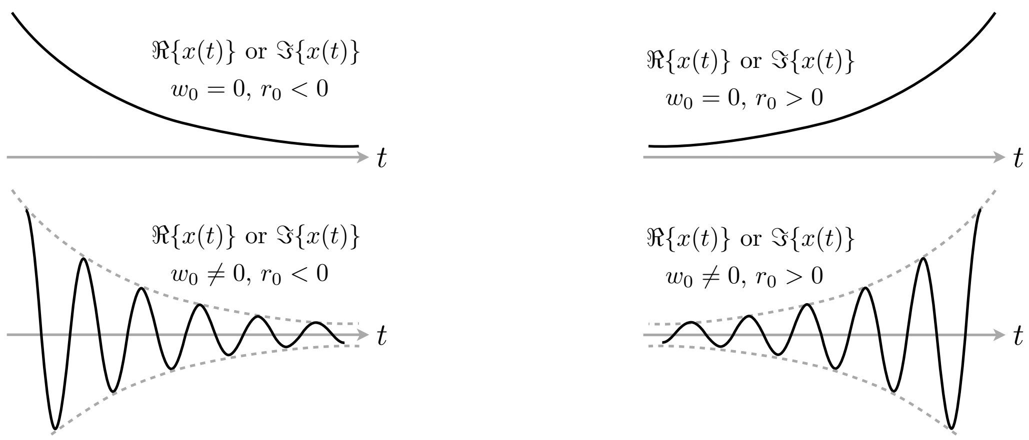


Exponential Signals (سیگنالهای نمایی)

- Continuous-domain

$$x(t) = c\,\mathrm{e}^{z_0 t} \qquad c = |c|\measuredangle\theta_c \qquad z_0 = r_0 + \mathrm{j}\omega_0$$

$$\Re\{x(t)\} = |c|\,\mathrm{e}^{r_0 t}\cos(\omega_0 t + \theta_c) \qquad \Im\{x(t)\} = |c|\,\mathrm{e}^{r_0 t}\sin(\omega_0 t + \theta_c)$$

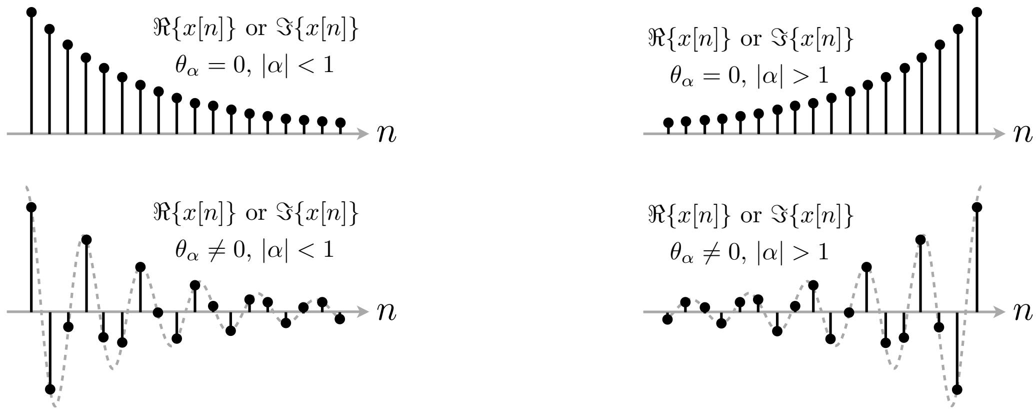


Exponential Signals (سیگنالهای نمایی)

- Discrete-domain

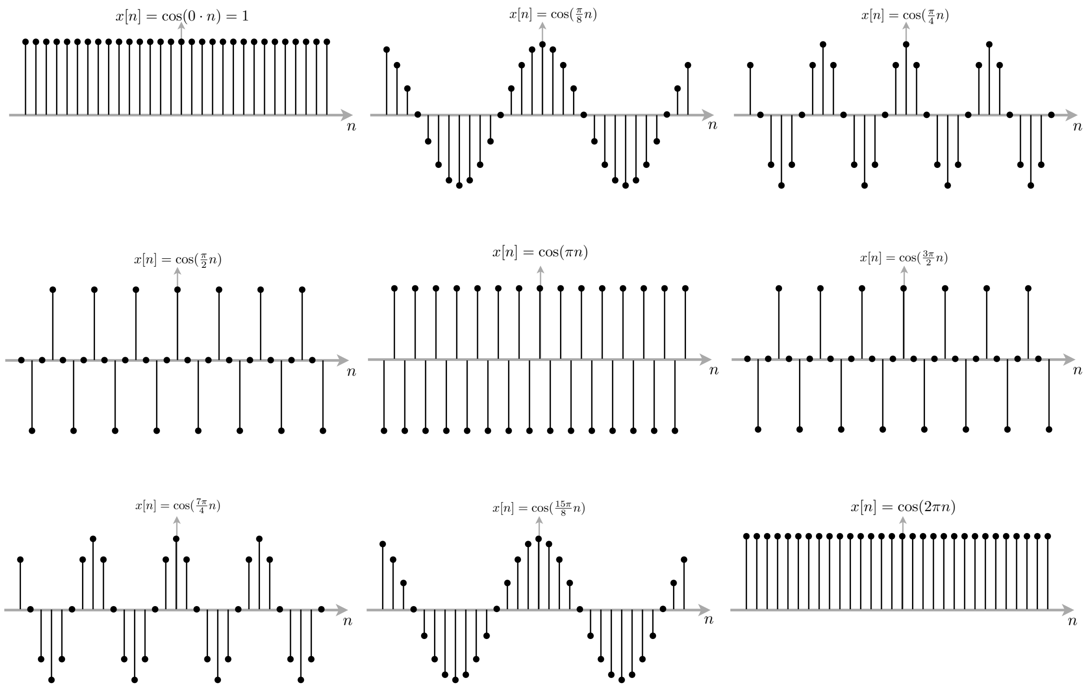
$$x[n] = c\,\alpha^n \qquad c = |c|\measuredangle\theta_c \qquad \alpha = |\alpha|\measuredangle\theta_\alpha$$

$$\Re\{x[n]\} = |c|\,|\alpha|^n\cos(\theta_\alpha n + \theta_c) \qquad \Im\{x[n]\} = |c|\,|\alpha|^n\sin(\theta_\alpha n + \theta_c)$$



Exponential Signals (سیگنالهای نمایی)

- Discrete-domain

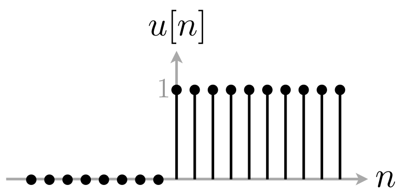


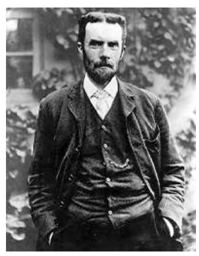
Exponential Signals (سیگنالهای نمایی)

$x_\omega(t) = \mathrm{e}^{\mathrm{j}\omega t}$	$x_\omega[n] = \mathrm{e}^{\mathrm{j}\omega n}$
❁ Uniqueness $x_{\omega_2}(t) \equiv x_{\omega_1}(t) \;\Leftrightarrow\; \omega_2 = \omega_1$	2π-uniqueness $x_{\omega_2}[n] \equiv x_{\omega_1}[n] \;\Leftrightarrow\; \omega_2 = \omega_1 + 2k\pi,\; k \in \mathbb{Z}$
❁ Periodicity $x_\omega(t) = $ periodic, $\forall \omega$	Periodicity $x_\omega[n] = $ periodic $\;\Leftrightarrow\; \omega = 2\frac{m}{N}\pi,\; m \in \mathbb{Z},\; N \in \mathbb{N}$
❁ Fundamental period No T if $\omega = 0$ $T = \frac{2\pi}{\omega}$ if $\omega \neq 0$	Fundamental period $N = 1$ if $\omega = 0$ $N = m\frac{2\pi}{\omega}$ if $\omega \neq 0$

Heaviside's Step function (تابع پله)

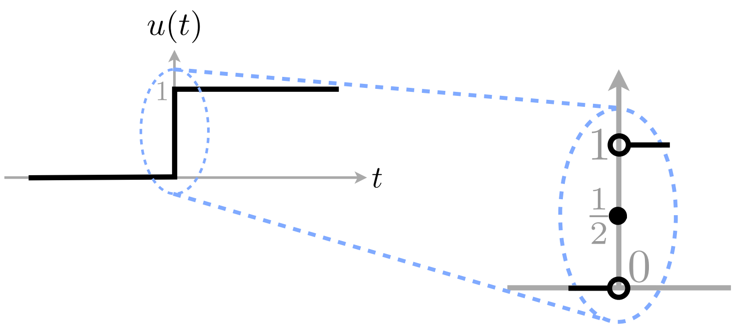
- Discrete-domain





Oliver Heaviside
1850-1925

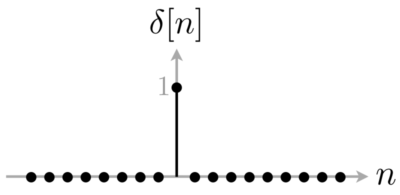
- Continuous-domain

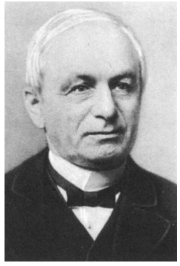


Kronecker delta function (تابع ضربه گسسته)

- Discrete-domain

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$





Leopold Kronecker
1823-1891

$$\sum_{m=-\infty}^{\infty} \delta[m] = 1$$

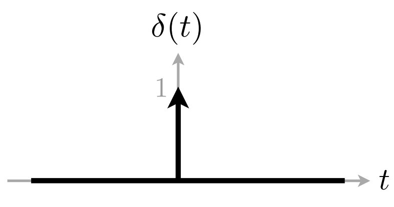
$$\langle \delta, x \rangle = \sum_{m=-\infty}^{\infty} x[m]\delta[m] = x[0]$$

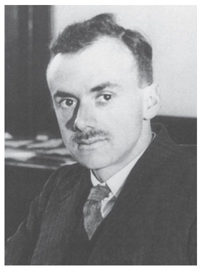
$$\sum_{m=-\infty}^{n} \delta[m] = u[n]$$

Dirac delta functional (impulse) (تابعک ضربه پیوسته)

- Continuous-domain

$$\delta(t) = \begin{cases} ?!?! & t = 0 \\ 0 & t \neq 0 \end{cases}$$





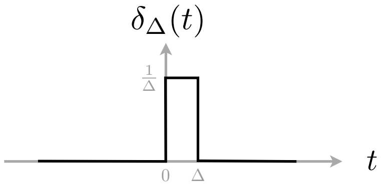
Paul Dirac
1902-1984

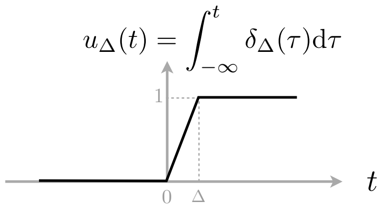
$$\sum_{m=-\infty}^{\infty} \delta[m] = 1 \quad \Rightarrow \quad \int_{\mathbb{R}} \delta(\tau)\mathrm{d}\tau = 1$$

$$\sum_{m=-\infty}^{n} \delta[m] = u[n] \quad \Rightarrow \quad \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau = u(t)$$

Dirac delta functional (impulse) (تابعک ضربه پیوسته)

- Heuristic definition





$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t)$$

- Rigorous definition

$$x(t): \text{ continuous at } t = 0 \quad \Rightarrow \quad \langle \delta, x \rangle = \int_{\mathbb{R}} \delta(\tau) x(\tau) \mathrm{d}\tau = x(0)$$

$$\xcancel{\int_{\mathbb{R}} u(\tau)\delta(\tau)\mathrm{d}\tau}$$

ℓ_p Spaces

- Definition

Frigyes Riesz
1880-1956

$$0 < p < \infty: \quad \ell_p \triangleq \left\{ x[\cdot] \;\middle|\; \sum_{m=-\infty}^{\infty} |x[m]|^p < \infty \right\}$$

$$p = \infty: \quad \ell_\infty \triangleq \left\{ x[\cdot] \;\middle|\; \sup_n |x[n]| < \infty \right\}$$

$$p = 0: \quad \ell_0 \triangleq \left\{ x[\cdot] \;\middle|\; x[n] = 0 \text{ if } |n| \text{ is large enough} \right\}$$

$$x, y \in \ell_p \quad \Rightarrow \quad c_x x + c_y y \in \ell_p, \; \forall\, c_x, c_y \in \mathbb{C}$$

- Examples

$$\begin{cases} u[n] \notin \ell_p, & 0 < p < \infty \\ u[n] \in \ell_\infty \end{cases} \qquad x[n] = \frac{1}{\sqrt{1+n^2}} \Rightarrow \begin{cases} x \notin \ell_p, & 0 < p \le 1 \\ x \in \ell_p, & 1 < p \le \infty \end{cases}$$

L_p Spaces

- Definition $\quad I \subseteq \mathbb{R}$ (e.g., $I = [0, 2\pi[\ ,\ I = \mathbb{R}\ ,\ \ldots$)

$$0 < p < \infty : \quad L_p(I) \triangleq \left\{ x(\cdot) \;\middle|\; \int_I |x(\tau)|^p \mathrm{d}\tau < \infty \right\}$$

$$p = \infty : \quad L_\infty(I) \triangleq \left\{ x(\cdot) \;\middle|\; \sup_{\tau \in I} |x(\tau)| < \infty \right\}$$

$$p = 0 : \quad L_0(I) \triangleq \left\{ x(\cdot) \;\middle|\; x(\tau) = 0 \text{ if } |\tau| \text{ is large enough} \right\}$$

Henri Lebesgue
1875-1941

$$x, y \in L_p(I) \quad \Rightarrow \quad c_x x + c_y y \in L_p(I),\ \forall\, c_x, c_y \in \mathbb{C}$$

- Examples

$$x_1(t) = \sin(t) \Rightarrow \begin{cases} x \notin L_p(\mathbb{R}), & 0 < p < \infty \\ x \in L_\infty(\mathbb{R}) & \end{cases}$$

$$x_2(t) = \frac{1}{\sqrt{t}} \Rightarrow \begin{cases} x \in L_p(]0,1]), & 0 < p < 2 \\ x \notin L_p(]0,1]), & 2 \le p \le \infty \end{cases}$$

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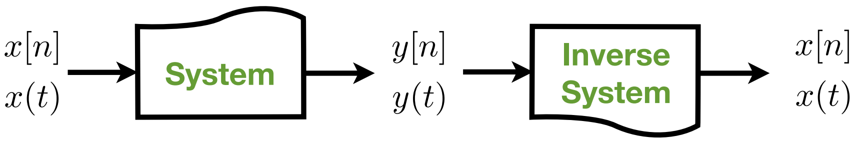
Invertible Systems (سیستمهای وارون پذیر*



- Examples

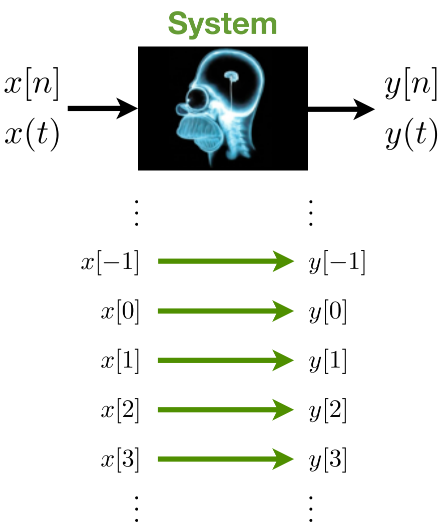
✓ $x(t) \rightarrow \boxed{y(t) = \mathrm{e}^{t} \tan\left(\frac{x(t)}{2}\right)} \rightarrow y(t) \rightarrow \boxed{w(t) = 2\tan^{-1}\left(\mathrm{e}^{-t} y(t)\right)} \rightarrow w(t) = x(t)$

✓ $x[n] \rightarrow \boxed{y[n] = \sum_{m=-\infty}^{n} x[m]\mathrm{e}^{m-n}} \rightarrow y[n] \rightarrow \boxed{w[n] = y[n] - \mathrm{e}^{-1} y[n-1]} \rightarrow w[n] = x[n]$

✗ $x[n] \rightarrow \boxed{y[n] = \frac{x[n] + x[n-1]}{2}} \rightarrow y[n]$

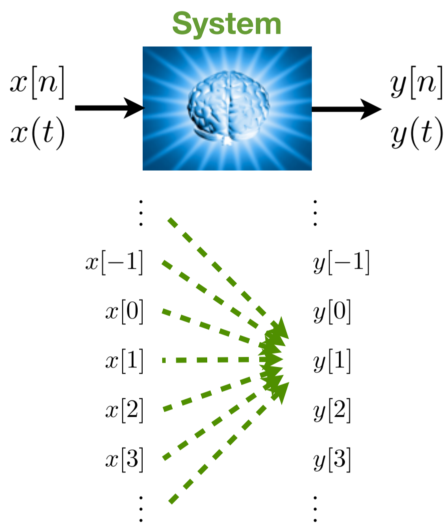
Systems with[out] memory (سیستمهای حافظه دار و بی حافظه*

- Memoryless



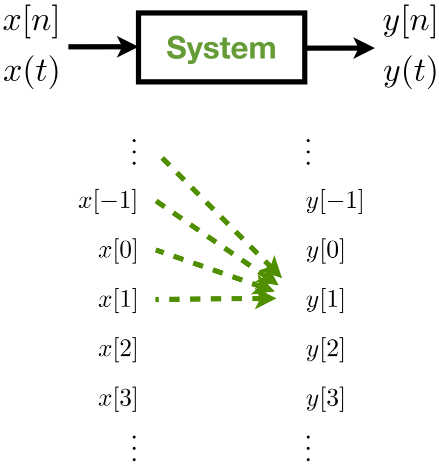
$y(t) = te^{x(t)}$ $\qquad$ $y[n] = \dfrac{x[n]}{1+(x[n])^4}$

- With Memory



$y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) - x(0)$ $\qquad$ $y[n] = \dfrac{1}{7}\sum_{m=n-3}^{n+3} x[m]$

Causal Systems (سیستمهای عِلّی*



- Examples

✓ Memoryless Systems

✓ Weather Forecast

✓ $y[n] = x[n] - x[n-1]$

✗ $y[n] = x[n+1] - x[n]$

✓ $y(t) = \int_{-\infty}^{t} \mathrm{e}^{-|x(\tau)|}\mathrm{d}\tau$

✗ $y(t) = x(2t)$

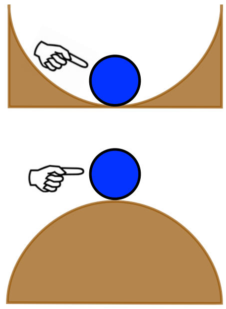
✗ $y(t) = \dfrac{x(t) - x(0)}{t}$

Stable Systems (سیستمهای پایدار*

- BIBO Stability

$$x[\cdot] \in \ell_\infty \quad \Rightarrow \quad y[\cdot] \in \ell_\infty$$

$$x(\cdot) \in L_\infty \quad \Rightarrow \quad y(\cdot) \in L_\infty$$



- Examples

✓ $y(t) = x^2(t) + 3x(t)$

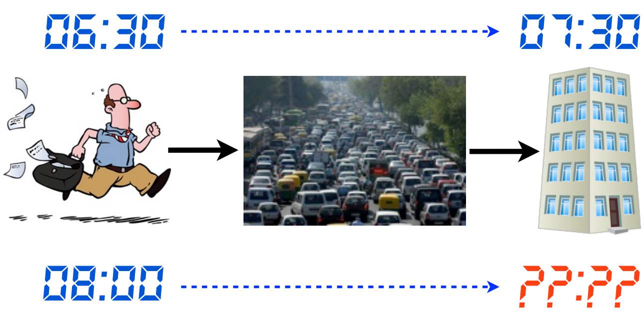
✓ $y(t) = \int_{\mathbb{R}} \frac{\mathrm{e}^{x(t-\tau)}}{\cosh(\tau)} \mathrm{d}\tau$

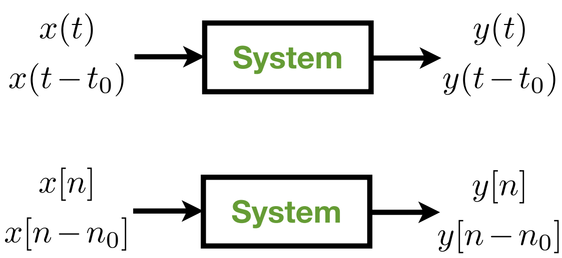
✗ $y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$

✗ $y[n] = n \log \left(|x[n]| + 1 \right)$

✓ $y[n] = \frac{1}{3} \sum_{m=n-1}^{n+1} \sin(m\, x[m])$

Shift [time]-Invariant Systems (سیستمهای تغییر ناپذیر با زمان*





- Examples

✓ $y[n] = \sin(x[n])$

✓ $y(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(t) + 2x(t)$

✗ $y[n] = n\, x[n]$

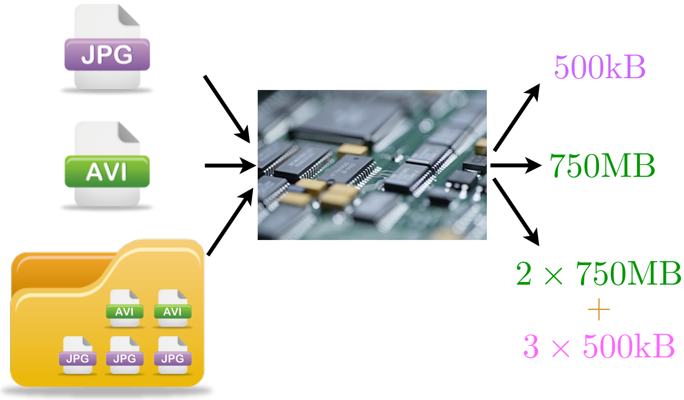
✓ $y[n] = x[n+1] - x[n]$

✓ $y(t) = \int_{-\infty}^{t} \mathrm{e}^{-|x(\tau)|} \mathrm{d}\tau$

✗ $y(t) = x(2t)$

✗ $y(t) = \frac{x(t) - x(0)}{t}$

Linear Systems (سیستمهای خطی)



$$\left\{\begin{array}{l} x_1 \mapsto y_1 \\ x_2 \mapsto y_2 \end{array}\right. \Rightarrow \quad a\,x_1 + b\,x_2 \mapsto a\,y_1 + b\,y_2$$

- Examples

✗ $y[n] = \sin(x[n])$

✓ $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) + 2x(t)$

✓ $y[n] = n^2\,x[n]$

✓ $y[n] = x[n+1] - x[n]$

✗ $y(t) = \int_{-\infty}^{t} \mathrm{e}^{-|x(\tau)|}\mathrm{d}\tau$

✓ $y(t) = x(2t)$

✓ $y(t) = \frac{x(t) - x(0)}{t}$