Assignment 6

Problem 1. For continuous-domain signals $x(t)$, $y(t) \in L_1$,

(a) show that the Fourier transform of $z(t) = x(t)y(t)$ is

$$\widehat{z}(\omega) = \tfrac{1}{2\pi}\big(\widehat{x} * \widehat{y}\big)(\omega)$$

where $\widehat{x}(\omega)$ and $\widehat{y}(\omega)$ are the Fourier transforms of $x(t)$ and $y(t)$, respectively.

(b) Using the result of the previous part, find the Fourier transform of the following signals

(i) $x_1(t) = t\mathrm{e}^{-\alpha|t|}\cos(\beta t), \quad \alpha > 0$

(ii) $x_2(t) = \frac{\sin(\pi t)}{\pi t}\,\frac{\sin\big(2\pi(t-1)\big)}{\pi(t-1)}$

(c) Prove the following relation:

$$\int_{-\infty}^{+\infty} x(t)\, y^*(t)\,\mathrm{d}t = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{x}(\omega)\,\widehat{y}^*(\omega)\,\mathrm{d}\omega$$

This relation is known as the generalized Parseval's theorem.

(d) Using the result of the previous part, evaluate the following integrals:

(i) $I_1 = \int_0^{+\infty} \frac{1}{(a^2+x^2)^2}\mathrm{d}x$

(ii) $I_2 = \int_0^{+\infty} \frac{\sin^4(t)}{t^4}\mathrm{d}x$

Problem 2. Hilbert transform of a continuous-domain signal $x(t)$ is defined as

$$x_H(t) = \tfrac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau}\mathrm{d}\tau$$

(a) Find the Fourier transform of $x_H(t)$ in terms of the Fourier transform of the original signal, $\widehat{x}(\omega)$.

(b) Find the Hilbert transform of the following signals, using the result of the previous part

(i) $x_1(t) = \cos(\omega t + \phi)$

(ii) $x_2(t) = \frac{1}{t^2+1}$

(iii) $x_3(t) = \frac{\sin(t)}{t}$

(c) Show that the Hilbert transform of $x'(t)$ is equal to $\frac{\mathrm{d}}{\mathrm{d}t}x_H(t)$.

(d) Show that $\big(x_H\big)_H(t) = -x(t)$.

Problem 3. For a fast decaying signal $x(t)$ and an arbitrary T_0, let $x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$.

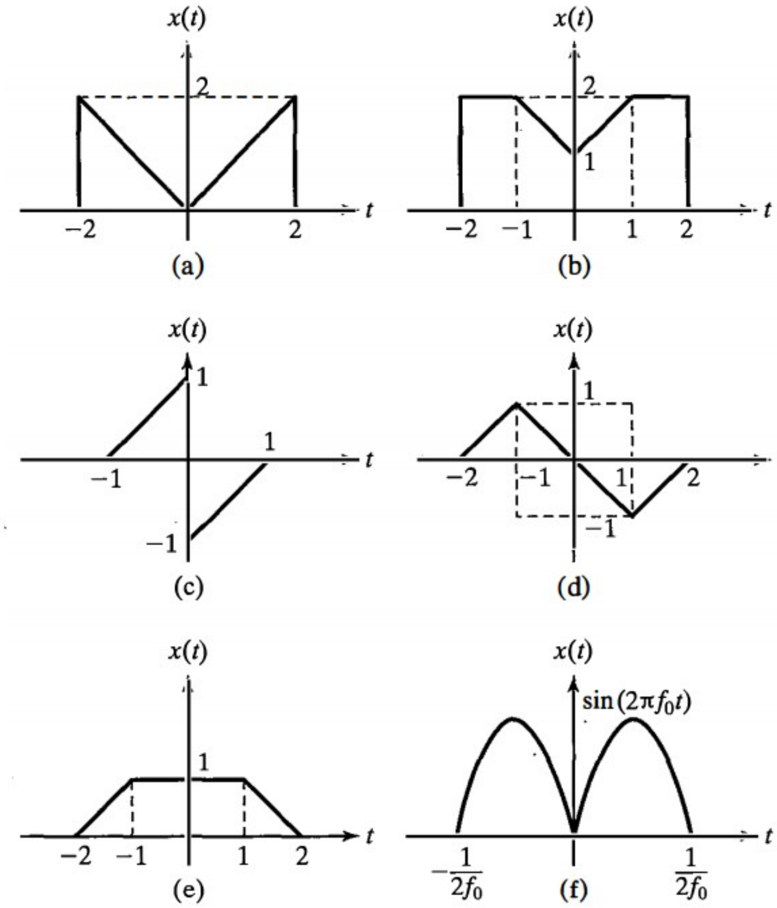

Figure 1: Signals for which the Fourier transform shall be determined.

(a) Express $x_p(t)$ in terms of $x(t)$ and $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$.

(b) Find the Fourier transform of $x_p(t)$ in terms of the Fourier transform of $x(t)$.

Problem 4. Determine the Fourier transform of the signals shown in Figure 1.

Problem 5. Let $\widehat{x}(\omega)$ denote the Fourier transform of the signal shown in Figure 2. Determine the value of the following expressions without calculating $\widehat{x}(\omega)$.

- $\angle\widehat{x}(\omega)$
- $\int_{-\infty}^{\infty} \widehat{x}(\omega) \frac{2\sin(\omega)}{\omega} e^{j2\omega} d\omega$
- $\int_{-\infty}^{\infty} |\widehat{x}(\omega)|^2 d\omega$
- $\int_{-\infty}^{\infty} \widehat{x}(\omega) d\omega$
- $\widehat{x}(0)$
- $\int_{-\infty}^{\infty} \omega\, \widehat{x}(\omega) e^{j\omega} d\omega$
- Sketch the inverse Fourier transform of $Re\widehat{x}(\omega)$.

Problem 6. The system defined by the input-output relation

$$y(t) = x(t) \cos(2\pi f_0 t)$$

where f_0 is a constant, is called a modulator.

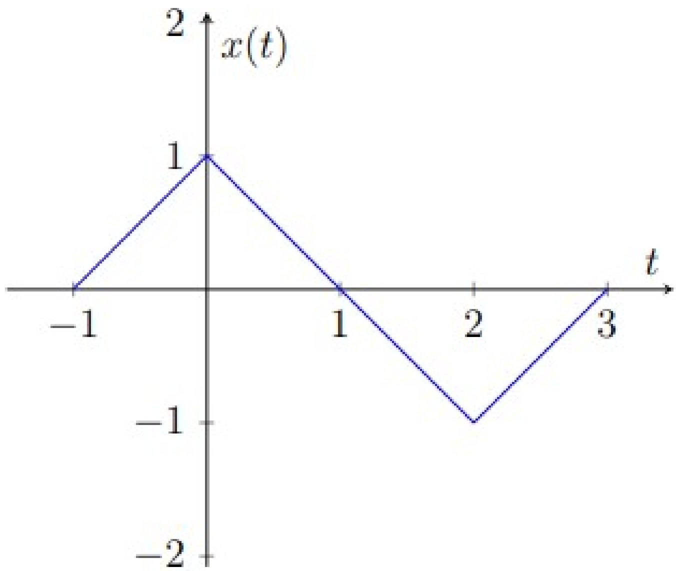

Figure 2: A continuous-domain signal.

(a) Find the Fourier transform of the output signal in terms of the Fourier transform of the input signal and f_0.

(b) If $x(t) = \frac{\sin(\pi t)}{\pi t}$ and $f_0 = 2$, sketch the Fourier transform of the input and output signals.

(c) Assuming the above input signal and f_0, sketch the Fourier transform of the below signal

$$z(t) = x(t)\cos(2\pi f_0 t) - x_H(t)\sin(2\pi f_0 t)$$

where $x_H(t)$ is the Hilbert transform of $x(t)$.

Do you see a difference between the Fourier transforms of the outputs?

Problem 7. Assume a LSI system with impulse response

$$h(t) = \frac{\sin\big(3\pi(t-2)\big)}{\pi(t-2)}$$

Find the response of the system to the following inputs:

- $x_1(t) = \sum_{k=0}^{\infty} (\frac{1}{3})^k \sin(2kt)$
- $x_2(t) = \left(\frac{\sin(2\pi t)}{\pi t}\right)^2$

Problem 8. Solve the following problems from the reference book.

- "Advanced Problems" of Chapter 4, Problem 4.38
- "Basic Problems" of Chapter 4, Problem 4.25