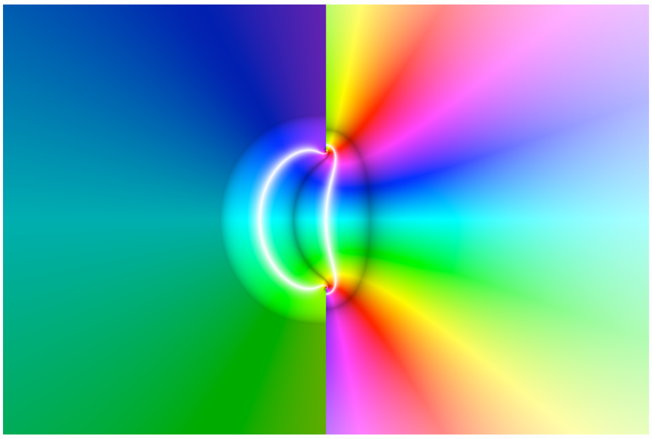


Laplace Transform

- Laplace Transform and Region of Convergence (ROC)
- Properties of Laplace Transform
- Characterization of LSI Systems
- One-sided (unilateral) Laplace Transform

Generalization of Fourier Transform

$$x(t) \longrightarrow \boxed{\text{LSI}} \longrightarrow y(t)$$

Pierre-Simon Laplace
1749-1827

$$\begin{pmatrix}\text{periodic}\\ \text{eigen-signals}\end{pmatrix} \; \mathrm{e}^{\mathrm{j}\omega t} \quad \Rightarrow \quad \begin{pmatrix}\text{Fourier}\\ \text{Transform}\end{pmatrix} \; \widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$\begin{pmatrix}\text{general}\\ \text{eigen-signals}\end{pmatrix} \; \mathrm{e}^{st} \quad \Rightarrow \quad \begin{pmatrix}\text{Laplace}\\ \text{Transform}\end{pmatrix} \; X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

Laplace Transform (تبدیل لاپلاس)

$$\left\{\begin{array}{c} x(t): \text{ complex-valued} \\ s \in \mathbb{C} \\ \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau \text{ exists} \end{array}\right. \quad \Rightarrow \quad X(s) \triangleq \mathcal{L}\{x(t)\}(s) \triangleq \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

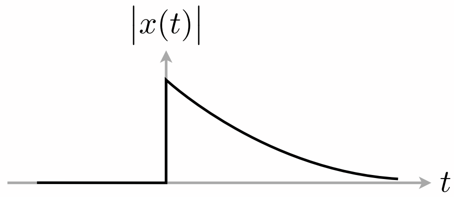
$$\widehat{x}(\omega) = X(s)\big|_{s=\mathrm{j}\omega}$$

$$X(s)\big|_{s=\sigma+\mathrm{j}\omega} = \mathcal{F}\left\{x(t)\mathrm{e}^{-\sigma t}\right\}(\omega)$$

Laplace Transform (تبدیل لاپلاس)

- Example

$$x(t) = \mathrm{e}^{-a\,t}u(t), \quad a \in \mathbb{C}$$



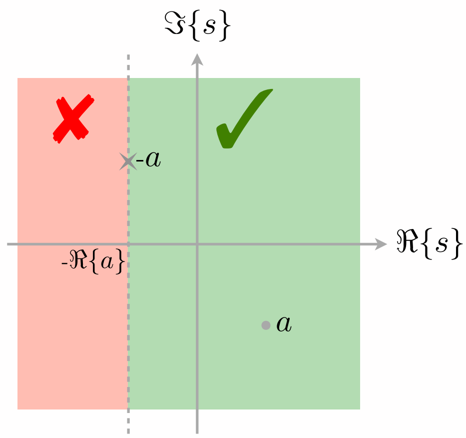
$$X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_0^{\infty} \mathrm{e}^{-a\tau}\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$= -\frac{\mathrm{e}^{-(a+s)\tau}}{a+s}\Bigg|_{\tau=0}^{\infty}$$

convergent $\Leftrightarrow \Re\{a+s\} > 0$

$$\boxed{\overset{\Re\{s\}>-\Re\{a\}}{=} \frac{1}{a+s}}$$

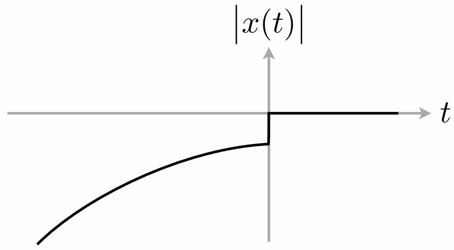
ROC



Laplace Transform (تبدیل لاپلاس)

- Example

$$x(t) = \text{-}\mathrm{e}^{-a\,t}u(\text{-}t), \quad a \in \mathbb{C}$$



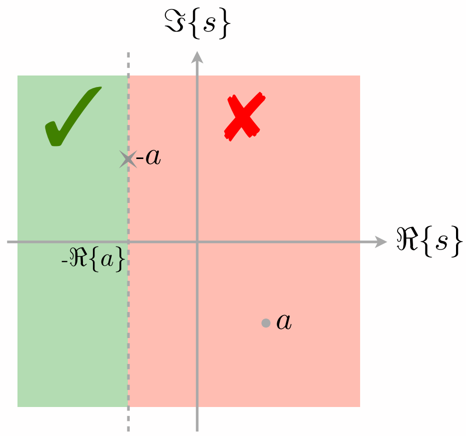
$$X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \text{-}\int_{-\infty}^{0} \mathrm{e}^{-a\tau}\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$= \left.\frac{\mathrm{e}^{-(a+s)\tau}}{a+s}\right|_{\tau=-\infty}^{0}$$

convergent $\Leftrightarrow \Re\{a+s\} < 0$

$$\overset{\Re\{s\}<-\Re\{a\}}{=} \frac{1}{a+s}$$

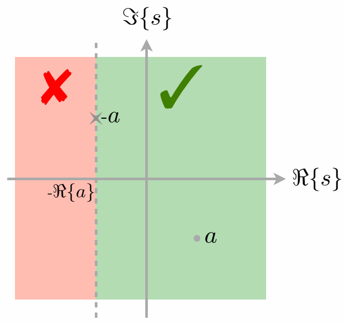
ROC



Laplace Transform (تبدیل لاپلاس)

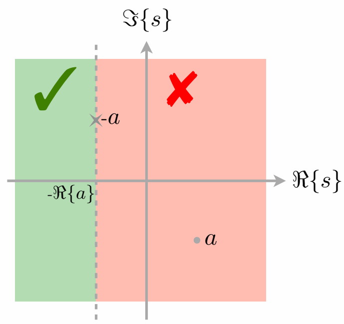
$$\mathrm{e}^{-at}u(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} \frac{1}{a+s}$$

$\Re\{s\} > \text{-}\Re\{a\}$



$$\text{-}\mathrm{e}^{-at}u(\text{-}t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} \frac{1}{a+s}$$

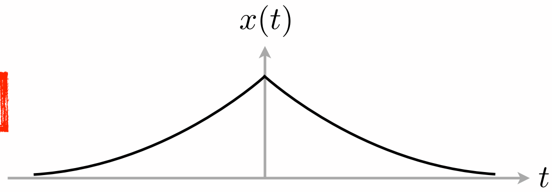
$\Re\{s\} < \text{-}\Re\{a\}$



Laplace Transform (تبدیل لاپلاس)

- Example

$x(t) = \mathrm{e}^{-a\,|t|}$, $\boxed{\Re\{a\} > 0}$



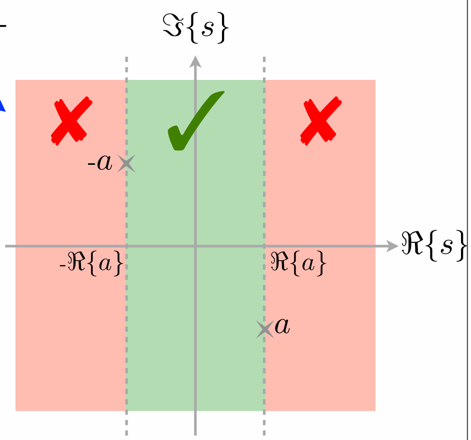
$$X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_{-\infty}^{0} \mathrm{e}^{a\tau}\mathrm{e}^{-s\tau}\mathrm{d}\tau + \int_{0}^{\infty} \mathrm{e}^{-a\tau}\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$= \frac{\mathrm{e}^{(a-s)\tau}}{a-s}\Big|_{\tau=-\infty}^{0} - \frac{\mathrm{e}^{-(a+s)\tau}}{a+s}\Big|_{\tau=0}^{\infty}$$

ROC

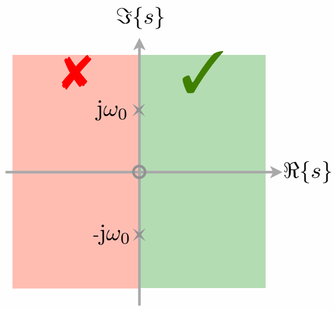
$$\text{convergent} \Leftrightarrow \begin{cases} \Re\{s-a\} < 0 \\ \Re\{s+a\} > 0 \end{cases}$$

$$= \frac{1}{a-s} + \frac{1}{s+a} = \boxed{\frac{2a}{a^2 - s^2}}$$



Laplace Transform (تبدیل لاپلاس)

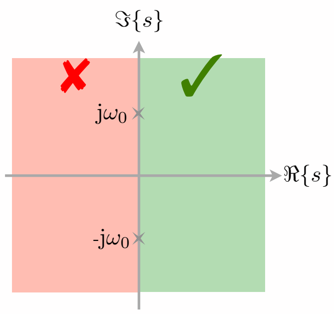
- Example $x_1(t) = \cos(\omega_0 t)u(t), \quad \omega_0 \in \mathbb{R}^+$



$$X_1(s) = \int_0^{\infty} \cos(\omega_0\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_0^{\infty} \frac{\mathrm{e}^{\mathrm{j}\omega_0\tau} + \mathrm{e}^{-\mathrm{j}\omega_0\tau}}{2}\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$\overset{\Re\{s\}>0}{=} \frac{1}{2}\left(\frac{-1}{\mathrm{j}\omega_0 - s} + \frac{1}{\mathrm{j}\omega_0 + s}\right) = \boxed{\frac{s}{\omega_0^2 + s^2}}$$

- Example $x_2(t) = \sin(\omega_0 t)u(t), \quad \omega_0 \in \mathbb{R}^+$



$$X_2(s) = \int_0^{\infty} \sin(\omega_0\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_0^{\infty} \frac{\mathrm{e}^{\mathrm{j}\omega_0\tau} - \mathrm{e}^{-\mathrm{j}\omega_0\tau}}{2\mathrm{j}}\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$\overset{\Re\{s\}>0}{=} \frac{1}{2\mathrm{j}}\left(\frac{-1}{\mathrm{j}\omega_0 - s} - \frac{1}{\mathrm{j}\omega_0 + s}\right) = \boxed{\frac{\omega_0}{\omega_0^2 + s^2}}$$

Laplace Transform (تبدیل لاپلاس)

- Example

$$x(t) = \left(\mathrm{e}^{-2t}\cos(t) + \mathrm{e}^{t}\right)u(t)$$

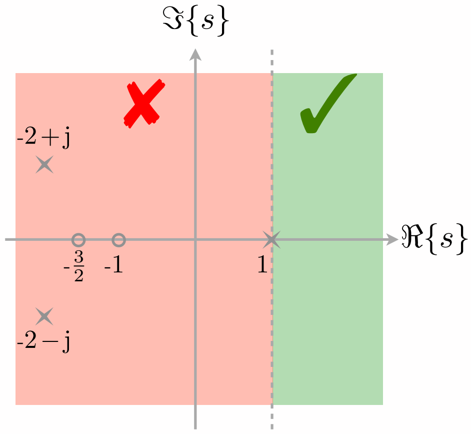
$$X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_0^{\infty}\left(\mathrm{e}^{-2\tau}\frac{\mathrm{e}^{\mathrm{j}\tau}+\mathrm{e}^{-\mathrm{j}\tau}}{2} + \mathrm{e}^{\tau}\right)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$= \left(\frac{-1}{2}\frac{\mathrm{e}^{-(2-\mathrm{j}+s)\tau}}{2-\mathrm{j}+s} - \frac{1}{2}\frac{\mathrm{e}^{-(2+\mathrm{j}+s)\tau}}{2+\mathrm{j}+s} + \frac{\mathrm{e}^{(1-s)\tau}}{1-s}\right)\Bigg|_{\tau=0}^{\infty}$$

convergent $\Leftrightarrow \Re\{s\} > 1$

$$= \frac{\frac{1}{2}}{2-\mathrm{j}+s} + \frac{\frac{1}{2}}{2+\mathrm{j}+s} + \frac{1}{s-1}$$

$$= 2\frac{(s+\frac{3}{2})(s+1)}{(s+2+\mathrm{j})(s+2-\mathrm{j})(s-1)}$$



Poles and Zeros (قطب و صفر)

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{n_N} a_k s^k}{\sum_{k=0}^{n_D} b_k s^k} = \alpha\frac{\prod_{k=0}^{n_N}(s-z_k)}{\prod_{k=0}^{n_D}(s-p_k)}$$

zeros

poles

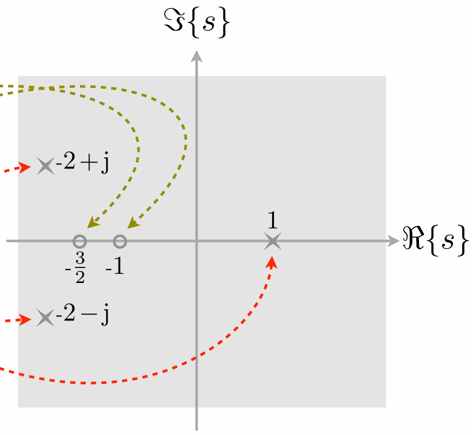
$$a_k, b_k \in \mathbb{C}$$

- Example

zeros

$$X(s) = 2\frac{(s+\frac{3}{2})(s+1)}{(s+2+\mathrm{j})(s+2-\mathrm{j})(s-1)}$$

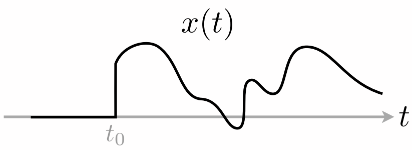
poles

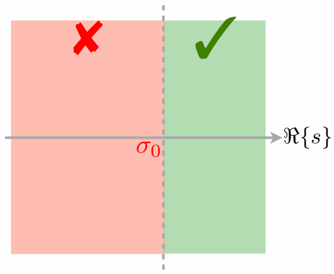


ROC (ناحیه همگرایی)

- Region of Convergence

Property 1. $\begin{cases} x(t): & \text{right-sided,} \\ \mathcal{L}\{x\}(s): & \text{absolutely convg. at } s_0 = \sigma_0 + \mathrm{j}\omega_0 \end{cases} \Rightarrow\ X(s) \text{ converges at all } \ \sigma_0 < \Re\{s\}$



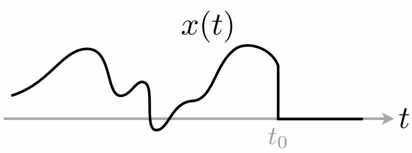


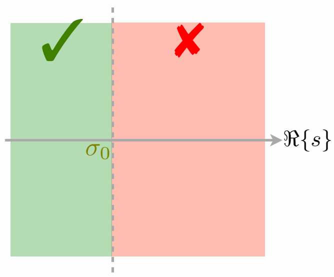
$$|X(s)| = \left|\int_{t_0}^{\infty} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau\right| = \left|\int_{t_0}^{\infty} x(\tau)\mathrm{e}^{-s_0\tau}\mathrm{e}^{-(s-s_0)\tau}\mathrm{d}\tau\right| \underset{\sigma_0 < \Re\{s\}}{\leq} \mathrm{e}^{-\Re\{s-s_0\}t_0} \int_{t_0}^{\infty} |x(\tau)|\mathrm{e}^{-\sigma_0\tau}\mathrm{d}\tau < \infty$$

ROC (ناحیه همگرایی)

- Region of Convergence

Property 2. $\begin{cases} x(t): & \text{left-sided,} \\ \mathcal{L}\{x\}(s): & \text{absolutely convg. at } s_0 = \sigma_0 + \mathrm{j}\omega_0 \end{cases} \Rightarrow\ X(s) \text{ converges at all } \ \Re\{s\} < \sigma_0$



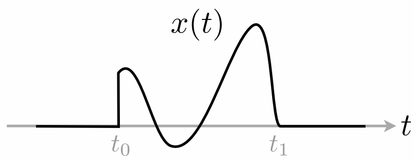


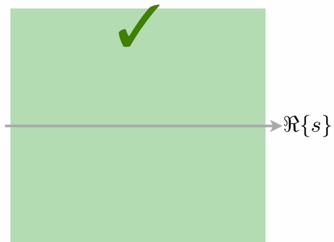
$$|X(s)| = \left|\int_{-\infty}^{t_0} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau\right| = \left|\int_{-\infty}^{t_0} x(\tau)\mathrm{e}^{-s_0\tau}\mathrm{e}^{-(s-s_0)\tau}\mathrm{d}\tau\right| \underset{\Re\{s\} < \sigma_0}{\leq} \mathrm{e}^{-\Re\{s-s_0\}t_0} \int_{-\infty}^{t_0} |x(\tau)|\mathrm{e}^{-\sigma_0\tau}\mathrm{d}\tau < \infty$$

ROC (ناحیه همگرایی)

- Region of Convergence

$$\textbf{Property 3.} \quad \begin{cases} x(t): & \text{compact support,} \\ x(t): & \text{absolutely integrable} \end{cases} \quad \Rightarrow \quad X(s) \text{ converges everywhere}$$





$$|X(s)| = \left| \int_{t_0}^{t_1} x(\tau) \mathrm{e}^{-s\tau} \mathrm{d}\tau \right| \leq \Big(\max_{t_0 \leq \tau \leq t_1} \mathrm{e}^{-\Re\{s\}\tau} \Big) \int_{t_0}^{t_1} |x(\tau)| \mathrm{d}\tau < \infty$$

ROC (ناحیه همگرایی)

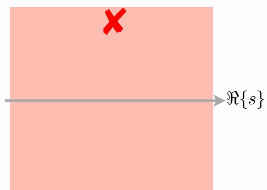
- Region of Convergence

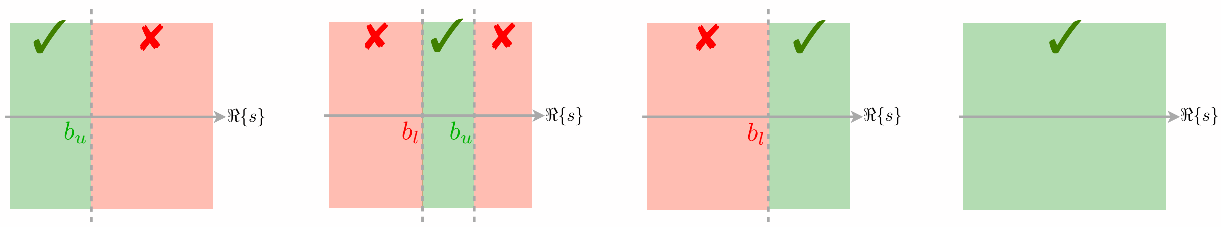
or $\leq$

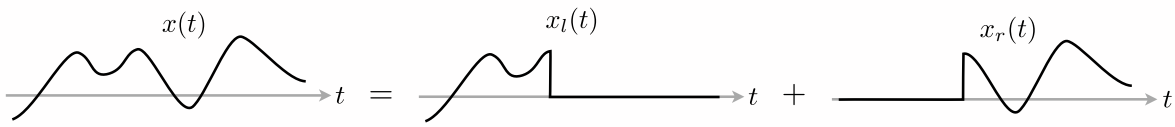
$$\textbf{Property 4.} \quad \text{arbitrary } x(t) \quad \Rightarrow \quad \text{ROC: } \; b_l < \Re\{s\} < b_u$$

$b_u \in \mathbb{R} \cup \{+\infty\}$

$b_l \in \mathbb{R} \cup \{-\infty\}$



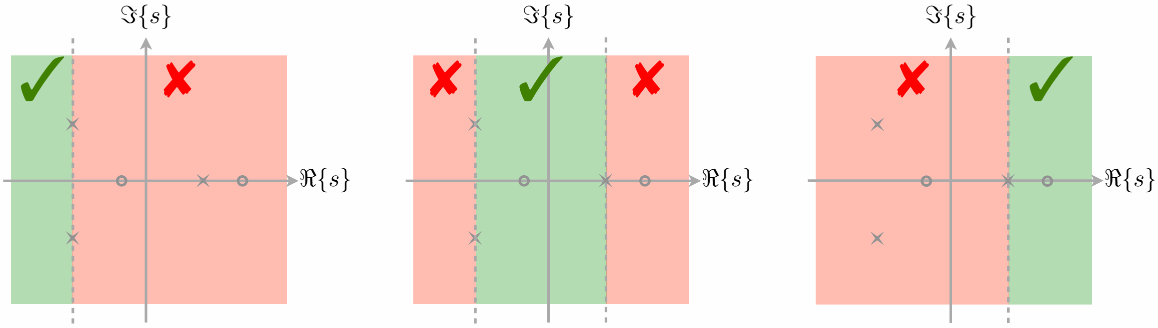




ROC (ناحیه همگرایی)

- Region of Convergence

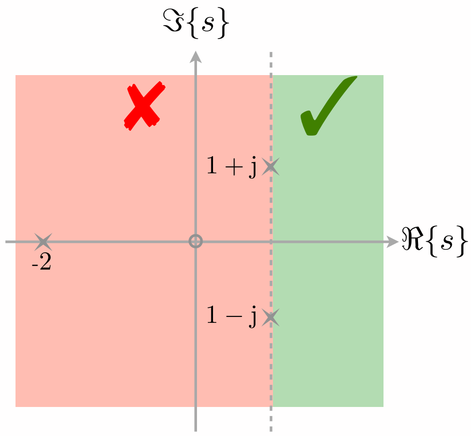
Property 5. $X(s) = \dfrac{N(s)}{D(s)} = \alpha \dfrac{\prod_{k=0}^{n_N}(s - z_k)}{\prod_{k=0}^{n_D}(s - p_k)} \Rightarrow \begin{cases} p_k \notin \text{ROC} \\ \text{there is at least one pole on each finite boundary of ROC} \end{cases}$



ROC (ناحیه همگرایی)

- Example

$$\begin{cases} x(t): & \text{right-sided}, \\ X(s) = & \frac{-s}{(s+2)(s-1-j)(s-1+j)} \end{cases}$$



Inverse Laplace Transform (تبدیل عکس لاپلاس)

$$x(t) \xmapsto{\mathcal{L}.\text{T.}} X(s)$$

$$X(s) \xmapsto{?} x(t)$$

$$x_1(t) = \mathrm{e}^{-at}u(t) \xmapsto{\mathcal{L}.\text{T.}} X_1(s) = \frac{1}{a+s}$$

$$x_2(t) = -\mathrm{e}^{-at}u(-t) \xmapsto{\mathcal{L}.\text{T.}} X_2(s) = \frac{1}{a+s}$$

$$X_1(s) \cancel{\overset{?!?!}{=}} X_2(s)$$

$$\text{ROC}_1 \neq \text{ROC}_2$$

$$X(s),\ \text{ROC} \xmapsto{\mathcal{L}^{-1}.\text{T.}} x(t)$$

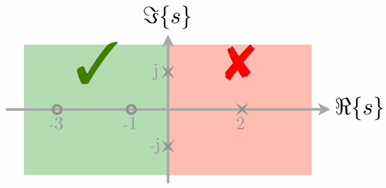
$$\mathcal{L}^{-1}.\text{T.}: \begin{cases} x(t) \xmapsto{\mathcal{L}.\text{T.}} X(s) \\ \left\{ s \mid \Re\{s\} = \sigma_0 \right\} \subset \text{ROC} \end{cases} \Rightarrow x(t) = \mathrm{e}^{\sigma_0 t}\,\mathcal{F}_\omega^{-1}\{X(\sigma_0 + \mathrm{j}\omega)\}(t)$$

$$= \boxed{\frac{1}{2\pi}\int_{\Re\{s\}=\sigma_0} X(s)\mathrm{e}^{st}\mathrm{d}s}$$

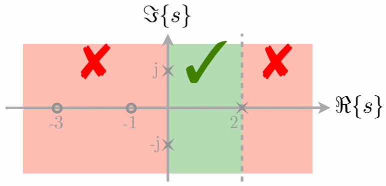
Inverse Laplace Transform (تبدیل عکس لاپلاس)

- Example

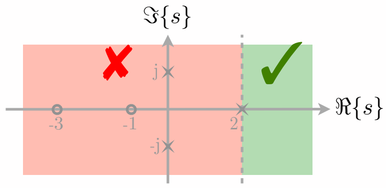
$$X(s) = \frac{(s+1)(s+3)}{(s^2+1)(s-2)} = \frac{3}{s-2} - \frac{2s}{s^2+1}$$



$$\Rightarrow x(t) = -3\mathrm{e}^{2t}u(-t) + 2\cos(t)u(-t)$$



$$\Rightarrow x(t) = -3\mathrm{e}^{2t}u(-t) - 2\cos(t)u(t)$$



$$\Rightarrow x(t) = 3\mathrm{e}^{2t}u(t) - 2\cos(t)u(t)$$

Geometrical Perspective of Fourier Transform

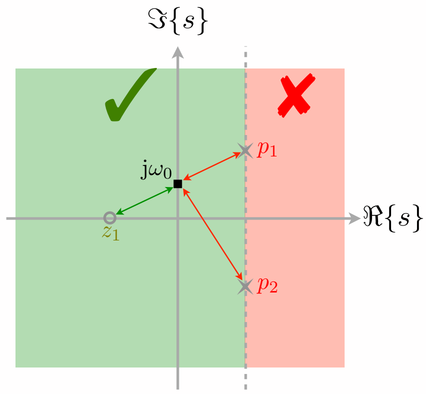
Fourier Transform $\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$

$\rightarrow \quad \mathcal{F}\mathrm{T\ exists} \;\Leftrightarrow\; \{\mathrm{j}\omega \text{ axis}\} \subseteq \mathrm{ROC}$

Laplace Transform $X(s) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$

$$X(s) = \frac{N(s)}{D(s)} = \alpha\,\frac{\prod_{k=0}^{n_N}(s - z_k)}{\prod_{k=0}^{n_D}(s - p_k)}$$

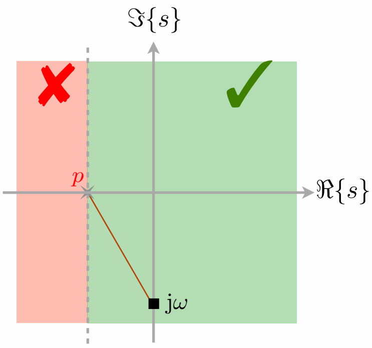
$$\Rightarrow \quad \widehat{x}(\omega) = \alpha\,\frac{\prod_{k=0}^{n_N}(\mathrm{j}\omega - z_k)}{\prod_{k=0}^{n_D}(\mathrm{j}\omega - p_k)}$$

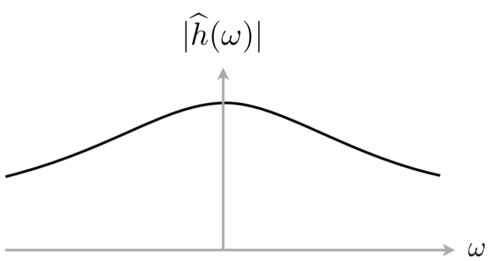


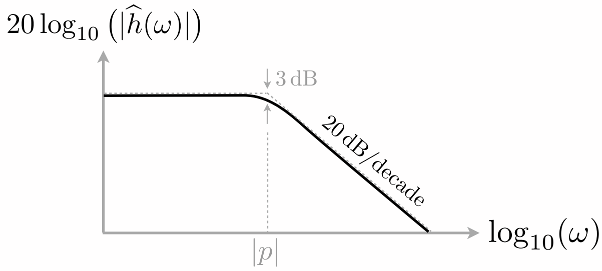
Geometrical Perspective of Fourier Transform

- 1st-order Systems

$$h(t) = \mathrm{e}^{pt}u(t), \quad p \in \mathbb{R}^- \quad \Rightarrow \quad H(s) = \frac{1}{s-p}$$



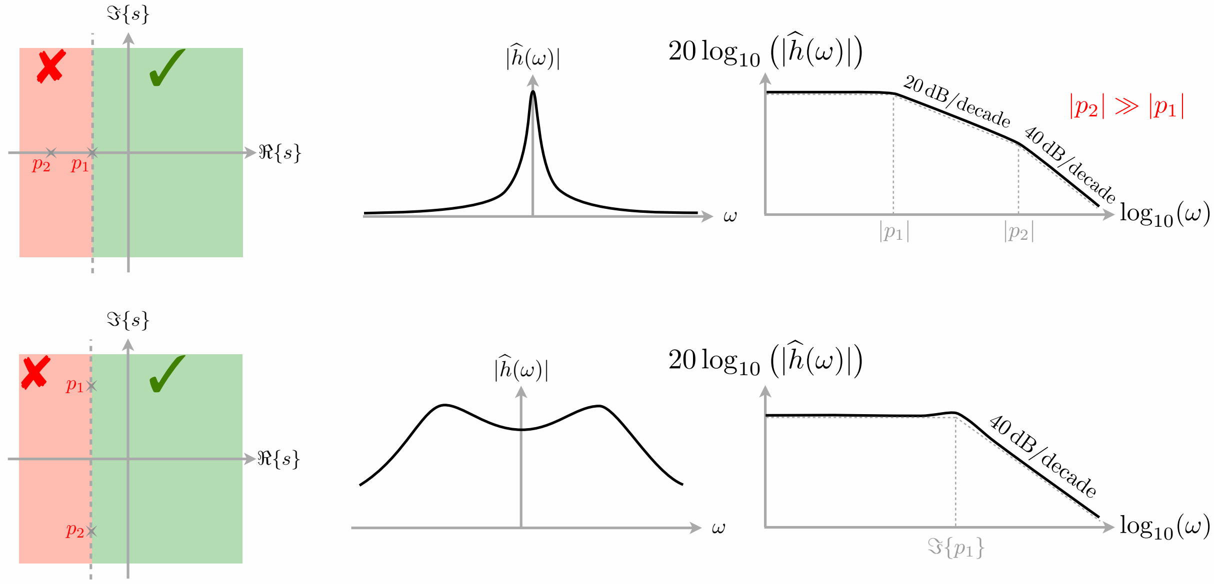




Geometrical Perspective of Fourier Transform

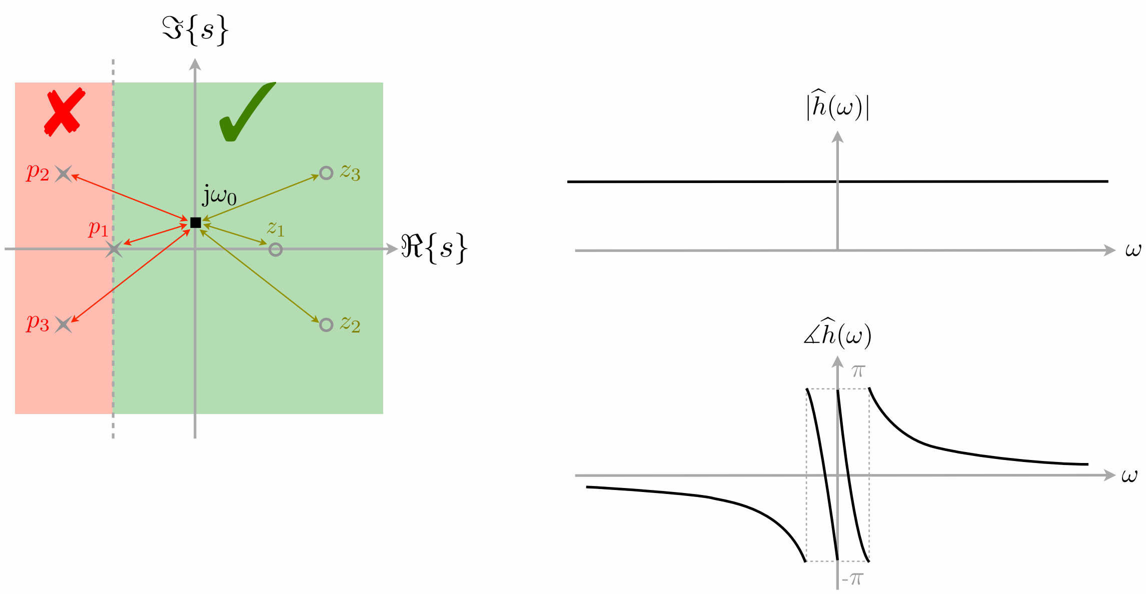
- 2nd-order Systems

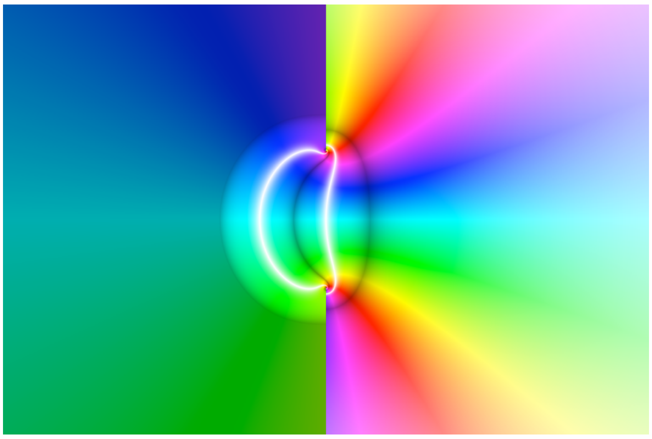
$$h(t) = \left(e^{p_1 t} - e^{p_2 t}\right)u(t),\ \Re\{p_i\} \in \mathbb{R}^- \quad \Rightarrow \quad H(s) = \frac{p_2 - p_1}{(s-p_1)(s-p_2)}$$



Geometrical Perspective of Fourier Transform

- All-pass Systems





Laplace Transform

- Laplace Transform and Region of Convergence (ROC)
- Properties of Laplace Transform
- Characterization of LSI Systems
- One-sided (unilateral) Laplace Transform

Properties of Laplace Transform

- Linearity

$$\begin{cases} x(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} X(s), & s \in \mathrm{ROC}_x \\ w(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} W(s), & s \in \mathrm{ROC}_w \end{cases}$$

$$\Rightarrow \quad \forall \alpha, \beta \in \mathbb{C}, \quad \alpha\, x(t) + \beta\, w(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} \alpha\, X(s) + \beta\, W(s), \quad s \in \mathrm{ROC}_x \cap \mathrm{ROC}_w \text{ or larger}$$

- Shift in Time-domain

$$x(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} X(s), \quad s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad x(t - t_0) \xrightarrow{\mathcal{L}.\mathrm{T}.} \mathrm{e}^{-st_0}\, X(s), \quad s \in \mathrm{ROC}_x$$

$$\int_{\mathbb{R}} x(\tau - t_0) \mathrm{e}^{-s\tau} \mathrm{d}\tau = \int_{\mathbb{R}} x(\tilde{\tau}) \mathrm{e}^{-s(\tilde{\tau} + t_0)} \mathrm{d}\tilde{\tau} = \mathrm{e}^{-st_0}\, X(s)$$

Properties of Laplace Transform

- Shift in s-domain

$$x(t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x$$

$$\overset{s_0\in\mathbb{C}}{\Rightarrow} \quad \mathrm{e}^{s_0 t}\, x(t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(s-s_0),\ s \in \mathrm{ROC}_x + \Re\{s_0\}$$

$$\int_{\mathbb{R}} \mathrm{e}^{s_0\tau} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-(s-s_0)\tau}\mathrm{d}\tau = X(s-s_0)$$

- Dilation (Time-scaling)

$$x(t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad \underset{\alpha\in\mathbb{R}}{x(\alpha\, t)} \xmapsto{\mathcal{L}.\mathrm{T}.} \tfrac{1}{|\alpha|} X\big(\tfrac{s}{\alpha}\big),\ s \in \alpha\, \mathrm{ROC}_x$$

$$\int_{\mathbb{R}} x(\underbrace{\alpha\tau}_{\tilde{\tau}})\mathrm{e}^{-s\tau}\mathrm{d}\tau = \tfrac{1}{|\alpha|}\int_{\mathbb{R}} x(\tilde{\tau})\mathrm{e}^{-\frac{s}{\alpha}\tilde{\tau}}\mathrm{d}\tilde{\tau}$$

Properties of Laplace Transform

- Time-reversal

$$x(t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad x(-t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(-s),\ s \in -\mathrm{ROC}_x$$

Dilation with $\alpha = -1$

- Conjugation

$$x(t) \xmapsto{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad \overline{x(t)} \xmapsto{\mathcal{L}.\mathrm{T}.} \overline{X(\overline{s})},\ s \in \mathrm{ROC}_x$$

$x(t) =$ real-valued $\Rightarrow$ $X(\overline{s}) = \overline{X(s)}$ $\Rightarrow$ conjugate-symmetric poles and zeros

Properties of Laplace Transform

- Convolution

$$x(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} X(s),\ s \in \mathrm{ROC}_x$$
$$z(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} Z(s),\ s \in \mathrm{ROC}_z$$
$$(x * z)(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} ?$$

$$y(t) \triangleq (x * z)(t) = \int_{\mathbb{R}} x(t-\tau) z(\tau)\, \mathrm{d}\tau \quad \Rightarrow \quad Y(s) = \iint_{\mathbb{R}^2} x(t-\tau) z(\tau) \mathrm{e}^{-st} \mathrm{d}\tau\, \mathrm{d}t$$

$$= \int_{\mathbb{R}} z(\tau) \Big(\underbrace{\int_{\mathbb{R}} x(t-\tau) \mathrm{e}^{-st} \mathrm{d}t}_{\mathcal{L}\{x(\cdot - \tau)\}(s)} \Big) \mathrm{d}\tau = X(s) \int_{\mathbb{R}} z(\tau)\, \mathrm{e}^{-s\tau} \mathrm{d}\tau = X(s) Z(s)$$

$$\Rightarrow \quad (x * z)(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} X(s)\, Z(s),\ s \in \mathrm{ROC}_x \cap \mathrm{ROC}_z$$

or larger

Properties of Laplace Transform

- Differentiation in Time-domain

$$x(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} sX(s),\ s \in \mathrm{ROC}_x$$

or larger

$$x(t) = \tfrac{1}{2\pi} \int_{\Re\{s\}=\sigma_0} X(s) \mathrm{e}^{st} \mathrm{d}s \quad \Rightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = \tfrac{1}{2\pi} \int_{\Re\{s\}=\sigma_0} X(s)\, \underbrace{\tfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{st}}_{s e^{st}}\, \mathrm{d}s$$

- Differentiation in s-domain

$$x(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad -tx(t) \overset{\mathcal{L}.\mathrm{T}.}{\longmapsto} \tfrac{\mathrm{d}}{\mathrm{d}s} X(s),\ s \in \mathrm{ROC}_x$$

$$\tfrac{\mathrm{d}}{\mathrm{d}s} X(s) = \int_{\mathbb{R}} x(\tau) \tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{-s\tau} \mathrm{d}\tau = -\int_{\mathbb{R}} \tau x(\tau) \mathrm{e}^{-s\tau} \mathrm{d}\tau$$

Properties of Laplace Transform

- Integration in Time-domain

$$x(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x$$

$$\Rightarrow \quad \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \xrightarrow{\mathcal{L}.\mathrm{T}.} \tfrac{1}{s}X(s),\ s \in \mathrm{ROC}_x \cap \{s \,|\, \Re\{s\} > 0\}$$

or larger

- Initial/Final-value Theorem

$$\begin{cases} x(t) = 0, \text{ for } t < 0 \\ x(t) : \text{non-singular at } t = 0 \\ x(t) \xrightarrow{\mathcal{L}.\mathrm{T}.} X(s),\ s \in \mathrm{ROC}_x \end{cases}$$

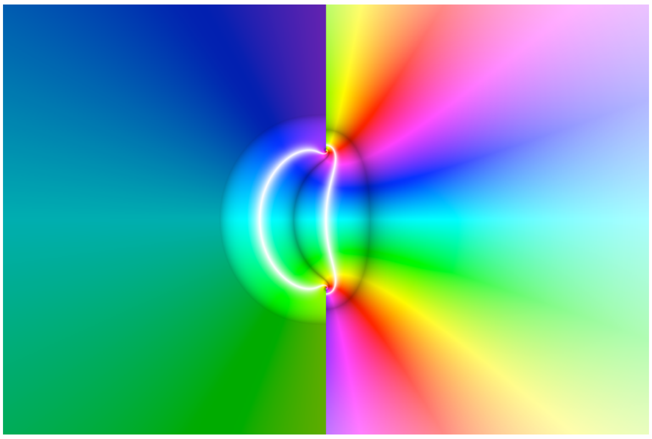
$$\Rightarrow \quad x(0^+) = \lim_{\Re\{s\} \to \infty} sX(s) \quad , \quad x(\infty) = \lim_{s \to 0} sX(s)$$

Properties of Laplace Transform

- Example

$$x(t) = (t - t_0)^n \mathrm{e}^{-a(t-t_0)} u(t - t_0) \quad \Rightarrow \quad X(s) = ?$$

$$\begin{aligned} X(s) &= \mathcal{L}\left\{(t-t_0)^n \mathrm{e}^{-a(t-t_0)} u(t-t_0)\right\}(s) \\ &= \mathrm{e}^{-st_0} \mathcal{L}\left\{t^n \mathrm{e}^{-at} u(t)\right\}(s) \\ &= \mathrm{e}^{-st_0} \mathcal{L}\left\{t^n u(t)\right\}(s+a) \\ &= \mathrm{e}^{-st_0} \mathcal{L}\left\{n \int_{-\infty}^{t} \tau^{n-1} u(\tau)\mathrm{d}\tau\right\}(s+a) \\ &= \mathrm{e}^{-st_0} \tfrac{n}{s+a} \mathcal{L}\left\{t^{n-1} u(t)\right\}(s+a) \\ &= \ldots \\ &= \mathrm{e}^{-st_0} \tfrac{n!}{(s+a)^{n+1}} \mathcal{L}\left\{\delta(t)\right\}(s+a) \\ &= \frac{n!\,\mathrm{e}^{-st_0}}{(s+a)^{n+1}} \end{aligned}$$



Laplace Transform

- Laplace Transform and Region of Convergence (ROC)
- Properties of Laplace Transform
- Characterization of LSI Systems
- One-sided (unilateral) Laplace Transform

Transfer Function (تابع تبدیل)

- LSI Systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = \big(x * h\big)(t)$$

$$\Rightarrow \quad Y(s) = \underbrace{H(s)}_{\text{Transfer Function}} X(s)$$

$$\underset{s \in \mathrm{ROC}_h}{x(t) = \mathrm{e}^{st}} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)\mathrm{e}^{st}$$

LSI System Properties

- Causality

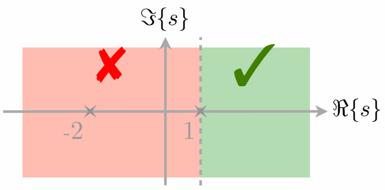
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\text{Causality} \iff h(t) = 0,\ \forall\, t < 0 \qquad \Rightarrow\ h(t) = \text{right-sided}$$

$$\Rightarrow\ \boxed{\mathrm{ROC}_h = \text{right-sided}}$$

- Example

$$\text{Causal System},\ \ H(s) = \frac{\mathrm{e}^{-s}}{(s+2)(s-1)} \quad \Rightarrow \quad h(t) = ?$$



$$H(s) = \frac{\mathrm{e}^{-s}}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)$$

$$\Rightarrow \quad h(t) = \frac{1}{3}\left(\mathrm{e}^{(t-1)} - \mathrm{e}^{-2(t-1)}\right)u(t-1)$$

LSI System Properties

- Stability

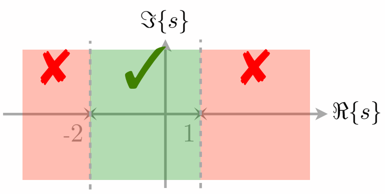
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\text{Stability} \iff h(t) \in L_1 \qquad \Leftrightarrow\ \widehat{h}(\omega) \text{ exists (simple definition)}$$

$$\Leftrightarrow\ \boxed{\mathrm{j}\mathbb{R} \subseteq \mathrm{ROC}_h}$$

- Example

$$\text{Stable System},\ \ H(s) = \frac{\mathrm{e}^{-s}}{(s+2)(s-1)} \quad \Rightarrow \quad h(t) = ?$$



$$H(s) = \frac{\mathrm{e}^{-s}}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)$$

$$\Rightarrow \quad h(t) = -\frac{\mathrm{e}^{(t-1)}u(1-t) + \mathrm{e}^{-2(t-1)}u(t-1)}{3}$$

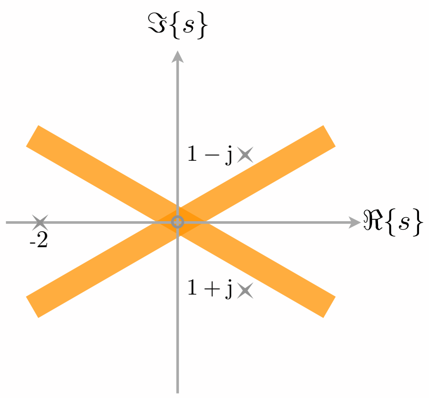
LSI System Properties

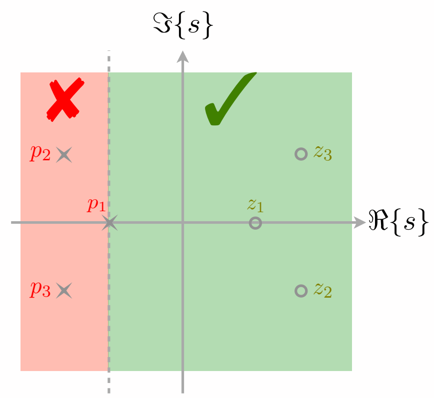
- Causal and Stable

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\text{Causal+Stable} \quad \Rightarrow \quad \{s \,|\, \Re\{s\} \geq 0\} \subseteq \text{ROC}_h$$

$$\Rightarrow \quad \text{all poles in the left half-plane}$$





Differential Systems

- Linear constant-coefficient differential systems

$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t) = \sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t)} \longrightarrow y(t)$$

$$\text{proper boundary conditions} \quad \Rightarrow \quad \text{LSI} \quad \Rightarrow \quad Y(s) = \underbrace{H(s)}_{?}\, X(s)$$

$$\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t) = \sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t) \quad \Rightarrow \quad \mathcal{L}\left\{\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t)\right\}(s) = \mathcal{L}\left\{\sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t)\right\}(s)$$

$$\Rightarrow \quad \sum_{k=0}^{N_y} a_k s^k\, Y(s) = \sum_{k=0}^{N_x} b_k s^k\, X(s) \quad \Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{N_x} b_k s^k}{\sum_{k=0}^{N_y} a_k s^k}$$

System Identification

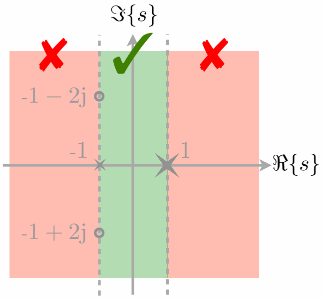
- Example

$$x(t) = \mathrm{e}^{-t}\cos(2t)u(t) \longrightarrow \boxed{\text{LSI}} \longrightarrow y(t) = t\,\mathrm{e}^{t}u(-t)$$

$$\Rightarrow \quad h(t) = ?$$

$$X(s) = \mathcal{L}\{\mathrm{e}^{-t}\cos(2t)u(t)\}(s) = \frac{s+1}{\big(s+(1+2\mathrm{j})\big)\big(s+(1-2\mathrm{j})\big)} \qquad \Re\{s\} > -1$$

$$Y(s) = \mathcal{L}\{t\mathrm{e}^{t}u(-t)\}(s) = \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{-\mathrm{e}^{t}u(-t)\}(s) = \tfrac{\mathrm{d}}{\mathrm{d}s}\tfrac{1}{s-1} = \tfrac{-1}{(s-1)^2} \qquad \Re\{s\} < 1$$



$$H(s) = \frac{Y(s)}{X(s)} = \frac{-(s^2+2s+5)}{(s-1)^2(s+1)}$$

$$= -\frac{4}{(s-1)^2} - \frac{1}{s+1}$$

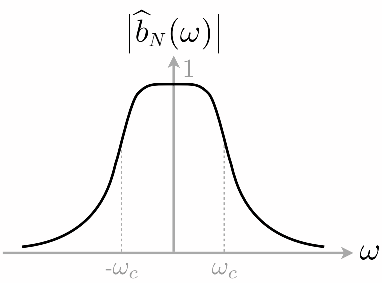
$$\Rightarrow \quad h(t) = 4t\mathrm{e}^{t}u(-t) - \mathrm{e}^{-t}u(t)$$

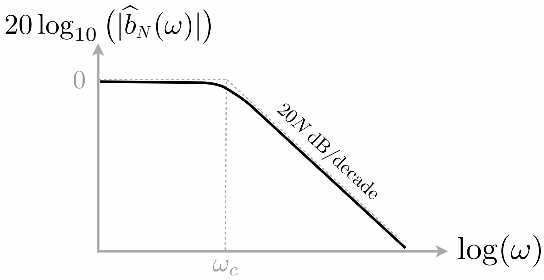
Amini

Filter Design

- Butterworth Filter of order *N*

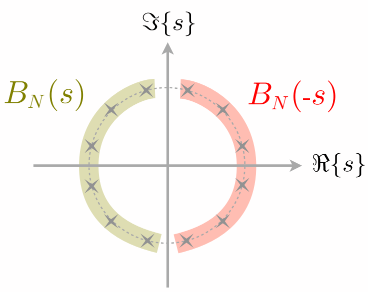
$$\left|\widehat{b}_N(\omega)\right|^2 = \frac{1}{1+(\frac{\omega}{\omega_c})^{2N}}$$





$$\Rightarrow \quad B_N(s)B_N(-s) = \frac{1}{1+\left(\frac{s}{\mathrm{j}\omega_c}\right)^{2N}}$$

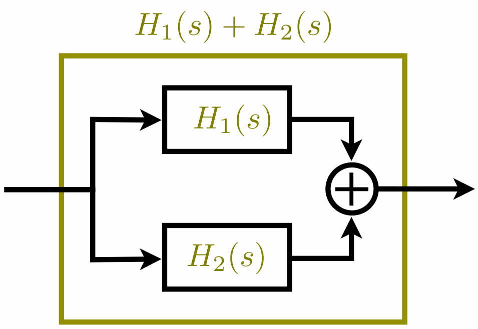
$$\Rightarrow \quad \text{poles:} \quad \underset{1\le k\le 2N}{p_k} = \mathrm{j}\omega_c\,\mathrm{e}^{\mathrm{j}\frac{\pi}{2N}(2k+1)}$$

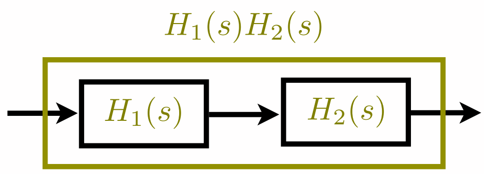


Amini

Transfer Function Algebra

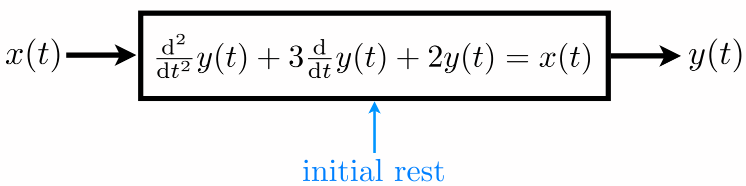
- Block diagram representation



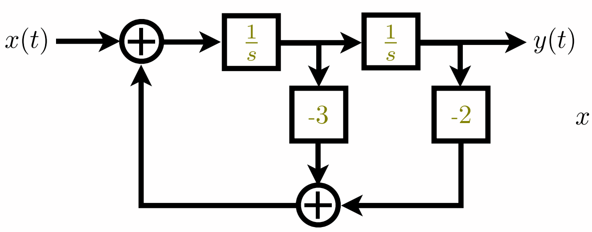


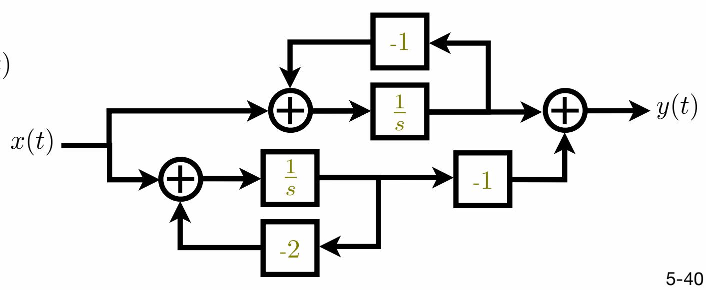
Transfer Function Algebra

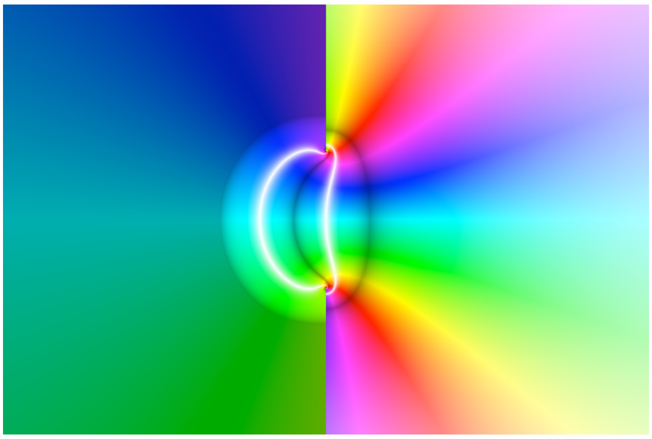
- Example



$$H(s) = \frac{1}{s^2 + 3s + 2} = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right) = \frac{1}{s+1} - \frac{1}{s+2}$$







Laplace Transform

- Laplace Transform and Region of Convergence (ROC)
- Properties of Laplace Transform
- Characterization of LSI Systems
- One-sided (unilateral) Laplace Transform

Unilateral Laplace Transform

$$\underline{X}(s) = \mathcal{UL}\{x(t)\}(s) = \int_{0^-}^{\infty} x(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

$$\Rightarrow \quad \text{ROC} = \text{right-sided}$$

$$\underline{X}(s) = \mathcal{L}\{x(t)u(t-0^-)\}(s)$$

- Example $\quad x(t) = \mathrm{e}^{-a|t|}$

$$X(s) = \mathcal{L}\{x(t)\}(s) = \frac{2a}{a^2 - s^2}$$

$$\underline{X}(s) = \mathcal{UL}\{x(t)\}(s) = \mathcal{L}\{\mathrm{e}^{-at}u(t)\}(s) = \frac{1}{s+a}$$

Properties

	Property	Time-domain	s-domain
	Linearity	$\alpha x(t) + \beta w(t)$	$\alpha\, \underline{X}\,(s) + \beta\, \underline{W}\,(s)$
	s-domain shift	$\mathrm{e}^{s_0 t} x(t)$	$\underline{X}\,(s - s_0)$
	Dilation	$x(\alpha t),\ \alpha > 0$	$\frac{1}{\alpha}\, \underline{X}\,\left(\frac{s}{\alpha}\right)$
	Conjugation	$\overline{x(t)}$	$\overline{\underline{X}\,(\overline{s})}$
x, z right-sided $\longrightarrow$	Convolution	$(x * z)(t)$	$\underline{X}\,(s)\ \underline{Z}\,(s)$
	Time-domain diff.	$\frac{\mathrm{d}}{\mathrm{d}t} x(t)$	$s\, \underline{X}\,(s) - x(0^-)$
	s-domain diff.	$-t\, x(t)$	$\frac{\mathrm{d}}{\mathrm{d}s}\, \underline{X}\,(s)$
	Time-domain Integration	$\int_{0^-}^{t} x(\tau)\mathrm{d}\tau$	$\frac{1}{s}\, \underline{X}\,(s)$
$x(t)$ continuous at $t = 0^+$ $\longrightarrow$	Initial-value Theorem	$x(0) = \lim_{\Re\{s\}\to\infty} s\, \underline{X}\,(s)$	
	Final-value Theorem	$\lim_{t\to\infty} x(t) = \lim_{\Re\{s\}\to 0} s\, \underline{X}\,(s)$	

Application

- Example

$$x(t) \longrightarrow \boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) + 2\frac{\mathrm{d}}{\mathrm{d}t} y(t) + 2y(t) = x(t)} \longrightarrow y(t)$$

$$y(0^-) = \beta,\ \dot{y}(0^-) = \gamma$$

$$\begin{cases} \mathcal{UL}\{\frac{\mathrm{d}}{\mathrm{d}t} y(t)\}(s) &= s\, \underline{Y}\,(s) - \underbrace{y(0^-)}_{\beta} \\ \mathcal{UL}\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t)\}(s) &= s\mathcal{UL}\{\frac{\mathrm{d}}{\mathrm{d}t} y(t)\}(s) - \underbrace{\dot{y}(0^-)}_{\gamma} = s\big(s\, \underline{Y}\,(s) - \underbrace{y(0^-)}_{\beta}\big) - \underbrace{\dot{y}(0^-)}_{\gamma} \end{cases}$$

$$\Rightarrow \quad \underline{Y}\,(s) = \underbrace{\frac{\underline{X}\,(s)}{s^2 + 2s + 2}}_{\text{zero-state response}} + \underbrace{\frac{\beta s + 2\beta + \gamma}{s^2 + 2s + 2}}_{\text{zero-input response}}$$

$$\hookrightarrow \Big(\beta\cos(t) + (\beta + \gamma)\sin(t)\Big)\mathrm{e}^{-t} u(t)$$