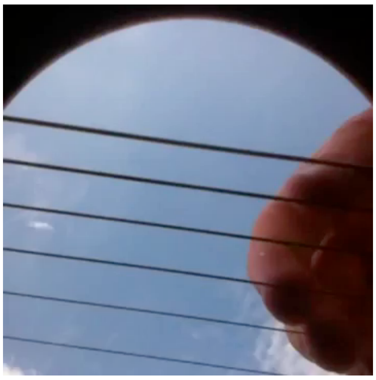
Fourier Series

- Eigen-signals of LSI systems
- Fourier series of continuous-domain periodic signals
- Fourier series of discrete-domain periodic signals
- Filtering

Introduction



Leonhard Euler
1707-1783

Jean B. J. Fourier
1768-1830

Peter G. L. Dirichlet
1805-1859

Eigenvector (بردار ویژه)

- Linear Algebra

$$\mathbf{A} \in \mathbb{C}^{n\times n}$$

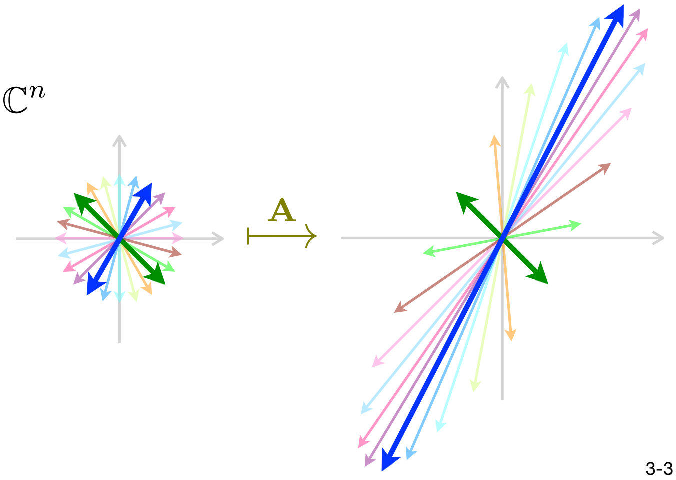
$$\text{If} \quad \lambda \in \mathbb{C},\, \mathbf{v} \in \mathbb{C}^n \quad \& \quad \mathbf{A}\cdot\mathbf{v} = \lambda\,\mathbf{v} \quad \Rightarrow \quad \begin{cases} \lambda & : \text{ eignevalue} \\ \mathbf{v} & : \text{ eignevector} \end{cases}$$

λ : eignevalue (مقدار ویژه)
v : eignevector (بردار ویژه)

- Linear systems theory

$$\mathbf{v} \in \mathbb{C}^n \longrightarrow \boxed{\mathsf{L}} \longrightarrow \mathbf{A}\mathbf{v} \in \mathbb{C}^n$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \quad \Rightarrow$$



Eigenfunction (تابع ویژه)

- Sturm-Liouville equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) + q(t)y(t) = \lambda_y y(t)$$

$$\begin{cases} \alpha_{11}\, y(t_1) & + & \alpha_{12}\, \dot{y}(t_1) & = & 0 \\ \alpha_{21}\, y(t_2) & + & \alpha_{22}\, \dot{y}(t_2) & = & 0 \end{cases}$$

- Linear systems theory

$$y(t) \longrightarrow \boxed{\textsf{Linear Diff. Sys.}} \longrightarrow -\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t)\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) + q(t)y(t)$$

boundary conditions

Eigen-signal (سیگنال ویژه)

- Discrete-domain LSI Systems

$$x_{\text{eig}}[n] \longrightarrow \overset{\text{LSI}}{\boxed{h[n]}} \longrightarrow y_{\text{eig}}[n] = \lambda_{x_{\text{eig}}}\, x_{\text{eig}}[n]$$

$$\lambda_{x_{\text{eig}}}\, x_{\text{eig}}[n] = y_{\text{eig}}[n] = \sum_{m\in\mathbb{Z}} h[m]\, x_{\text{eig}}[n-m]$$

$$\rightsquigarrow \quad \underbrace{\lambda_{x_{\text{eig}}}}_{\text{const.}} = \sum_{m\in\mathbb{Z}} h[m]\, \frac{x_{\text{eig}}[n-m]}{x_{\text{eig}}[n]}$$

$$\rightsquigarrow \quad \frac{x_{\text{eig}}[n-m]}{x_{\text{eig}}[n]} \quad \text{does not depend on } n$$

$$\rightsquigarrow \quad \frac{x_{\text{eig}}[n-m]}{x_{\text{eig}}[n]} = w[m] \qquad \rightsquigarrow \quad \boxed{x_{\text{eig}}[n] = c(\underbrace{w[\text{-}1]}_{z})^n = c\, z^n}$$

Eigen-signal (سیگنال ویژه)

- Discrete-domain LSI Systems

$$x[n] = z^n \longrightarrow \overset{\text{LSI}}{\boxed{h[n]}} \longrightarrow y[n] = H(z)\, z^n$$

$$H(z) = \sum_{m\in\mathbb{Z}} h[m] z^{\text{-}m}$$

$\mathcal{Z}-$transform of $h[\cdot]$

Eigen-signal (سیگنال ویژه)

- Continuous-domain LSI Systems

$$x_{\text{eig}}(t) \longrightarrow \boxed{h(t)} \longrightarrow y_{\text{eig}}(t) = \lambda_{x_{\text{eig}}}\, x_{\text{eig}}(t)$$

LSI

$$\lambda_{x_{\text{eig}}}\, x_{\text{eig}}(t) = y_{\text{eig}}(t) = \int_{\mathbb{R}} h(\tau)\, x_{\text{eig}}(t-\tau)\mathrm{d}\tau$$

$$\rightsquigarrow \quad \underbrace{\lambda_{x_{\text{eig}}}}_{\text{const.}} = \int_{\mathbb{R}} h(\tau)\, \frac{x_{\text{eig}}(t-\tau)}{x_{\text{eig}}(t)}\mathrm{d}\tau$$

$$\rightsquigarrow \quad \frac{x_{\text{eig}}(t-\tau)}{x_{\text{eig}}(t)} \quad \text{does not depend on } t$$

$$\rightsquigarrow \quad \frac{x_{\text{eig}}(t-\tau)}{x_{\text{eig}}(t)} = w(\tau) \qquad \rightsquigarrow \quad \boxed{x_{\text{eig}}(t) = \mathrm{e}^{st}}$$

Eigen-signal (سیگنال ویژه)

- Continuous-domain LSI Systems

LSI

$$x(t) = \mathrm{e}^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(s)\, \mathrm{e}^{st}$$

$$H(s) = \int_{\mathbb{R}} h(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

Laplace transform of $h(\cdot)$

Why Eigen-signals?

- The benefit of decomposition

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\mathrm{e}^{st} \longmapsto H(s)\,\mathrm{e}^{st}$$

$$\Rightarrow \quad x(t) = \sum_k a_k \mathrm{e}^{s_k t} \longmapsto y(t) = \sum_k a_k H(s_k)\,\mathrm{e}^{s_k t}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$z^n \longmapsto H(z)\,z^n$$

$$\Rightarrow \quad x[n] = \sum_k a_k z_k^n \longmapsto y[n] = \sum_k a_k H(z_k)\,z_k^n$$

Fourier Series

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- Filtering

Periodic Signals in the span of Eigen-signals

- Eigen-decomposition of periodic functions

$$\left\{\begin{array}{lcl} x(t) & = & \sum_k a_k \mathrm{e}^{s_k t} \\ \& & & \\ x(t) & = & x(t+T),\ \forall t \end{array}\right. \overset{\text{non-trivial}}{\Longrightarrow} \quad \mathrm{e}^{s_k t} = \mathrm{e}^{s_k (t+T)}$$

$$\Rightarrow \quad \mathrm{e}^{s_k T} = 1$$

$$\Rightarrow \quad s_k = \mathrm{j}\frac{2\pi}{T} n_k, \quad n_k \in \mathbb{Z}$$

$$\Rightarrow \quad x(t) = \sum_{k\in\mathbb{Z}} \tilde{a}_k \, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T} k\, t}$$

Periodic Signals in the span of Eigen-signals

- Orthogonal Sets

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\det([\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]) = 6 \neq 0$$

$\Rightarrow$ linear independence

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \quad \Rightarrow \quad a_1, a_2, a_3 = ?$$

$$\begin{cases} \langle \mathbf{v}_1, \mathbf{x}\rangle = a_1 \underbrace{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}_{2} + a_2 \underbrace{\langle \mathbf{v}_1, \mathbf{v}_2\rangle}_{0} + a_3 \underbrace{\langle \mathbf{v}_1, \mathbf{v}_3\rangle}_{0} \\ \langle \mathbf{v}_2, \mathbf{x}\rangle = a_1 \underbrace{\langle \mathbf{v}_2, \mathbf{v}_1\rangle}_{0} + a_2 \underbrace{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}_{3} + a_3 \underbrace{\langle \mathbf{v}_2, \mathbf{v}_3\rangle}_{0} \\ \langle \mathbf{v}_3, \mathbf{x}\rangle = a_1 \underbrace{\langle \mathbf{v}_3, \mathbf{v}_1\rangle}_{0} + a_2 \underbrace{\langle \mathbf{v}_3, \mathbf{v}_2\rangle}_{0} + a_3 \underbrace{\langle \mathbf{v}_3, \mathbf{v}_3\rangle}_{6} \end{cases} \Rightarrow \begin{cases} a_1 = \frac{\langle \mathbf{v}_1, \mathbf{x}\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle} = 1 \\ a_2 = \frac{\langle \mathbf{v}_2, \mathbf{x}\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle} = -\frac{1}{3} \\ a_3 = \frac{\langle \mathbf{v}_3, \mathbf{x}\rangle}{\langle \mathbf{v}_3, \mathbf{v}_3\rangle} = \frac{2}{3} \end{cases}$$

Periodic Signals in the span of Eigen-signals

- Fourier Coefficients

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}}_{\mathbf{v}_k} \quad \Rightarrow \quad a_k = ?$$

- Inner-product for T-periodic signals

$$w(t),\ z(t) = T\text{-periodic} \quad \Rightarrow \quad \langle w(t), z(t)\rangle \triangleq \int_{\substack{\text{one}\\ \text{period}}} \overline{w(\tau)}\, z(\tau)\, \mathrm{d}\tau$$
$$= \int_0^T \overline{w(\tau)}\, z(\tau)\, \mathrm{d}\tau$$

Periodic Signals in the span of Eigen-signals

- Fourier Coefficients

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}}_{\mathbf{v}_k} \quad \Rightarrow \quad a_k = ?$$

- Orthogonal Harmonics

$$k_1, k_2 \in \mathbb{Z}: \quad \Big\langle \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k_1 t}}_{\mathbf{v}_{k_1}}, \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k_2 t}}_{\mathbf{v}_{k_2}} \Big\rangle = \int_0^T \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k_1\tau}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k_2\tau} \mathrm{d}\tau$$

$$= \int_0^T \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}(k_2-k_1)\tau} \mathrm{d}\tau = T\,\delta[k_2 - k_1]$$

$$\Rightarrow \quad \Big\{ \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}}_{\mathbf{v}_k} \Big\}_{k\in\mathbb{Z}} = \text{Orthogonal set}$$

Periodic Signals in the span of Eigen-signals

- Fourier Coefficients

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}}_{\mathbf{v}_k} \quad \Rightarrow \quad a_k = ?$$

$$\Rightarrow \quad a_k = \frac{\langle \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}, x(t)\rangle}{\langle \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\rangle} = \frac{1}{T}\int_0^T x(\tau)\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau$$

Periodic Signals in the span of Eigen-signals

- Fourier series in terms of $\sin$ and $\cos$

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \, e^{j\frac{2\pi}{T}kt} \quad , \quad a_k = \frac{1}{T}\int_0^T x(\tau)\, e^{-j\frac{2\pi}{T}k\tau}\, d\tau$$

$$\begin{aligned} x(t) &= \sum_{k\in\mathbb{Z}} a_k \left(\cos\left(\tfrac{2\pi}{T}kt\right) + j\sin\left(\tfrac{2\pi}{T}kt\right)\right) \\ &= a_0 + \sum_{k=1}^{\infty} \underbrace{(a_k + a_{-k})}_{b_k} \cos\left(\tfrac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} \underbrace{j(a_k - a_{-k})}_{c_k} \sin\left(\tfrac{2\pi}{T}kt\right) \\ &= a_0 + \sum_{k=1}^{\infty} b_k \cos\left(\tfrac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} c_k \sin\left(\tfrac{2\pi}{T}kt\right) \end{aligned}$$

$$a_0 = \frac{1}{T}\int_0^T x(\tau)\, d\tau \qquad b_k = \frac{2}{T}\int_0^T x(\tau)\cos\left(\tfrac{2\pi}{T}k\tau\right) d\tau \qquad c_k = \frac{2}{T}\int_0^T x(\tau)\sin\left(\tfrac{2\pi}{T}k\tau\right) d\tau$$

Periodic Signals in the span of Eigen-signals

- Real and Imaginary parts

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \, e^{j\frac{2\pi}{T}kt} \quad , \quad a_k = \frac{1}{T}\int_0^T x(\tau)\, e^{-j\frac{2\pi}{T}k\tau}\, d\tau$$

$$a_k + a_{-k} \qquad\qquad j(a_k - a_{-k})$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} b_k \cos\left(\tfrac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} c_k \sin\left(\tfrac{2\pi}{T}kt\right)$$

$$\begin{cases} \Re\{x(t)\} = \Re\{a_0\} + \sum_{k=1}^{\infty} \Re\{b_k\}\cos\left(\tfrac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} \Re\{c_k\}\sin\left(\tfrac{2\pi}{T}kt\right) \\ \Im\{x(t)\} = \Im\{a_0\} + \sum_{k=1}^{\infty} \Im\{b_k\}\cos\left(\tfrac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} \Im\{c_k\}\sin\left(\tfrac{2\pi}{T}kt\right) \end{cases}$$

$$x(t) = \text{real-valued} \quad \Rightarrow \quad \Im\{x(t)\} \equiv 0 \quad \Rightarrow \quad \Im\{b_k\} = \Im\{c_k\} = 0 \quad \Rightarrow \quad a_{-k} = \overline{a_k}$$

Fourier Series (سری فوریه)

- Example

$$x(t) = \sin(\omega_0 t) + 2\cos(2\omega_0 t + \tfrac{\pi}{4})$$

$$x\big(t + \underbrace{\tfrac{2\pi}{\omega_0}}_{T_0}\big) = \sin\left(\omega_0(t + \tfrac{2\pi}{\omega_0})\right) + 2\cos\left(2\omega_0(t + \tfrac{2\pi}{\omega_0})\right) = x(t) \quad \Rightarrow \quad T_0\text{-periodic} \checkmark$$

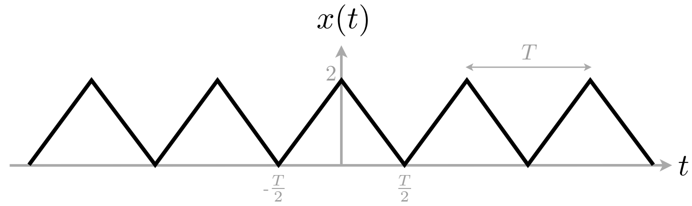
$$x(t) = \overbrace{\underbrace{\tfrac{1}{2\mathrm{j}}}_{a_1} \, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T_0}t} + \underbrace{\tfrac{-1}{2\mathrm{j}}}_{a_{-1}} \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_0}t}}^{\sin(\omega_0 t)} + \overbrace{\underbrace{\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}}_{a_2} \, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T_0}2t} + \underbrace{\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}}_{a_{-2}} \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_0}2t}}^{2\cos\left(2\omega_0 t + \frac{\pi}{4}\right)} \checkmark$$

$$a_1 = \tfrac{1}{2\mathrm{j}}, \quad a_{-1} = \tfrac{-1}{2\mathrm{j}}, \quad a_2 = \mathrm{e}^{\mathrm{j}\frac{\pi}{4}}, \quad a_{-2} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}$$

$$a_n = 0, \quad n \notin \{\pm 1, \pm 2\}$$

Fourier Series (سری فوریه)

- Example



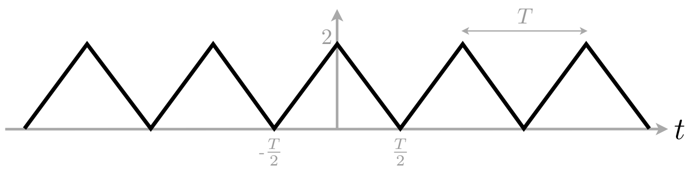
$$x(t) = T\text{-periodic} \checkmark \qquad x(t) \overset{?}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

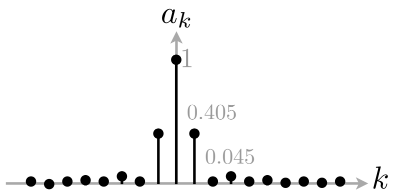
$$a_k \triangleq \frac{1}{T}\int_{\substack{\text{one}\\\text{period}}} x(\tau)\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(1 - \tfrac{2|\tau|}{T}\right) \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau$$

$$= \frac{4}{T}\int_{0}^{\frac{T}{2}} \left(1 - \tfrac{2\tau}{T}\right) \cos\left(\tfrac{2\tau}{T}\pi k\right) \mathrm{d}\tau = \left(\frac{\sin\left(\frac{\pi k}{2}\right)}{\frac{\pi k}{2}}\right)^2$$

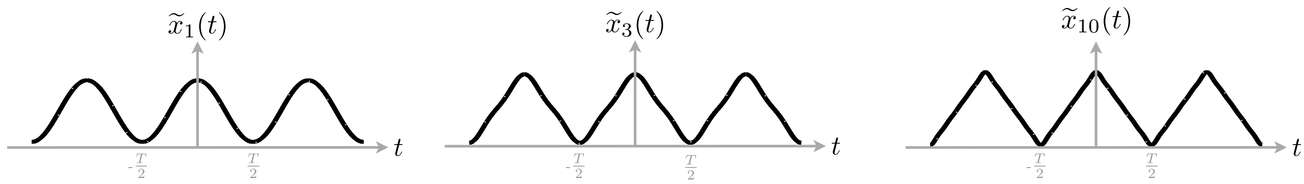
Fourier Series (سری فوریه)

- Example





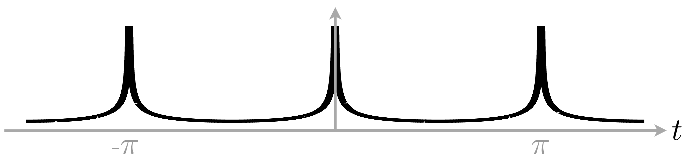
$$\widetilde{x}_N(t) \triangleq \sum_{k=-N}^{N} a_k\, e^{j\frac{2\pi}{T}kt}$$



Fourier Series (سری فوریه)

- Example

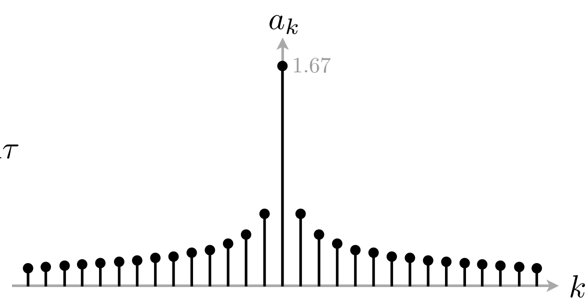
$$x(t) = \frac{1}{\sqrt{|\sin(t)|}}$$



$x(t) = \pi$-periodic ✓

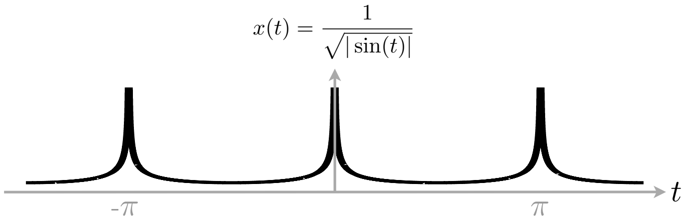
$$x(t) \stackrel{?}{=} \sum_{k\in\mathbb{Z}} a_k e^{j\frac{2\pi}{T}kt}$$

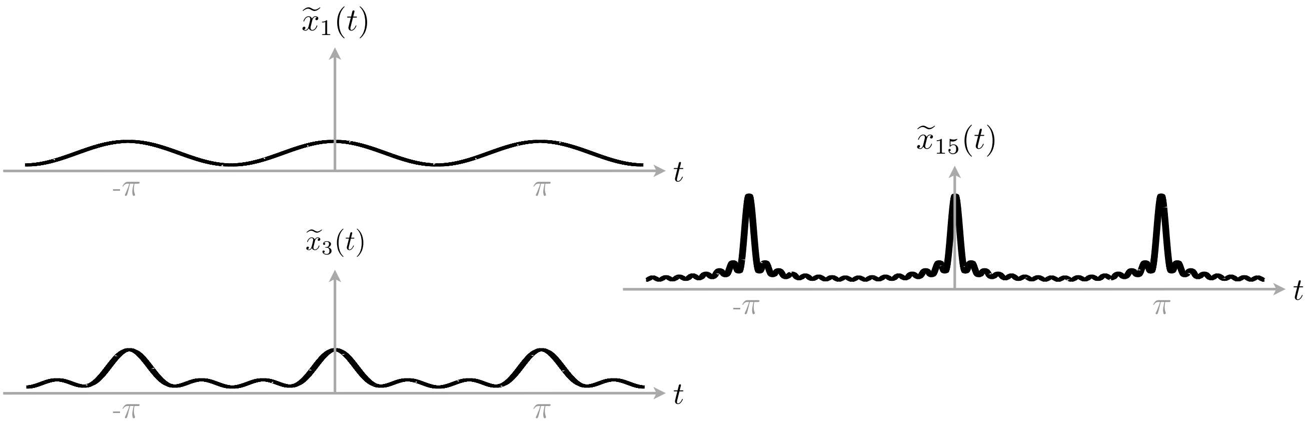
$$a_k \triangleq \frac{1}{T}\int_{\substack{\text{one}\\\text{period}}} x(\tau)\, e^{-j\frac{2\pi}{T}k\tau}\, d\tau$$



Fourier Series (سری فوریه)

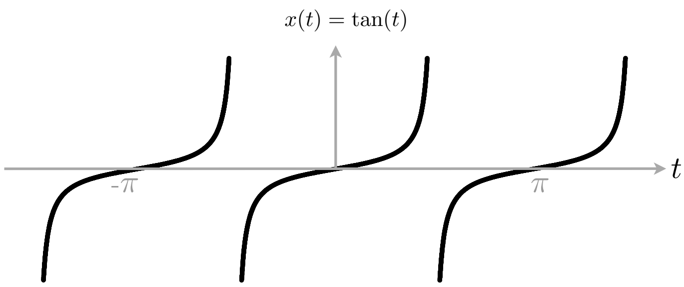
- Example





Fourier Series (سری فوریه)

- Example



$x(t) = \pi\text{-periodic}$

$x(t) \stackrel{?}{=} \sum_{k\in\mathbb{Z}} a_k e^{j\frac{2\pi}{T}kt}$

$$a_k \stackrel{?}{=} \frac{1}{T}\int_{\text{one period}} x(\tau)\, e^{-j\frac{2\pi}{T}k\tau}\, d\tau$$

$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(\tau)d\tau = \frac{-1}{\pi}\log\big(\cos(\tau)\big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$\rightarrow$ not well-defined

$k \neq 0,\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(\tau)\sin(2k\tau)d\tau = (-1)^k$

$\Rightarrow$ coefficients do not decay

Convergence of Fourier Series (همگرایی سری فوریه)

$x(t) = T\text{-periodic}$ $\qquad a_k \overset{?}{=} \frac{1}{T}\int_{\substack{\text{one}\\ \text{period}}} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau$

$$\Rightarrow \lim_{N\to\infty}\sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \overset{?}{=} x(t)$$

Peter G. L. Dirichlet
1805-1859

$$\begin{cases} \textbf{Cond. 1} & x(t)\in L_1([0,T[) \\ \textbf{Cond. 2} & x(t) \text{ has finitely many extrema in } [0,T[\\ \textbf{Cond. 3} & x(t) \text{ has finitely many discontinuities} \\ & \text{in } [0,T[\text{ and each discontinuity is finite} \end{cases}$$

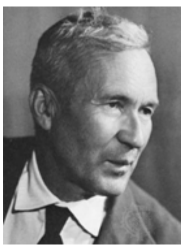
$$\Rightarrow \lim_{N\to\infty}\sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} = \frac{x(t^+)+x(t^-)}{2}$$

Convergence of Fourier Series (همگرایی سری فوریه)

$x(t) = T\text{-periodic}$ $\qquad a_k \overset{?}{=} \frac{1}{T}\int_{\substack{\text{one}\\ \text{period}}} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau$

$$\Rightarrow \lim_{N\to\infty}\sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \overset{?}{=} x(t)$$

$$x(t)\in L_1([0,T[) \;\Rightarrow\; a_k = \frac{1}{T}\int_0^T x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau \overset{\checkmark}{=} \text{finite and well-defined}$$



Andrey Kolmogorov
1903-1987

$$\exists\, x(t)\in L_1([0,T[), \qquad \sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \;\text{ diverges at all } t \text{ as } N\to\infty$$

Convergence of Fourier Series (همگرایی سری فوریه)

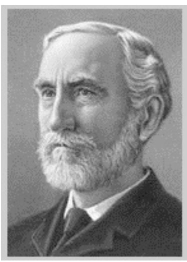
$$x(t) = T\text{-periodic} \qquad a_k \overset{?}{=} \frac{1}{T}\int_{\substack{\text{one}\\\text{period}}} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau$$

$$\Rightarrow \quad \lim_{N\to\infty}\sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \overset{?}{=} x(t)$$

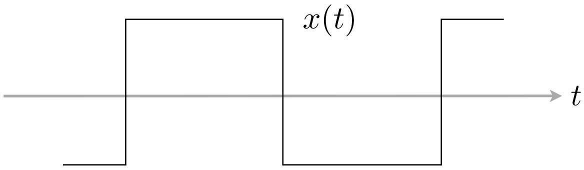
Lennart Carleson
1928-now

$$x(t) \in L_2([0,T[) \quad \Rightarrow \quad \lim_{N\to\infty}\sum_{k=-N}^{N} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \overset{\text{a.e.}}{=} x(t)$$

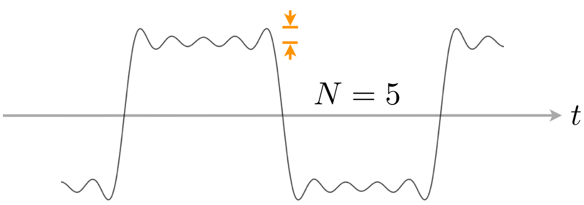
Gibbs Phenomena (اثر یا پدیده گیبس)

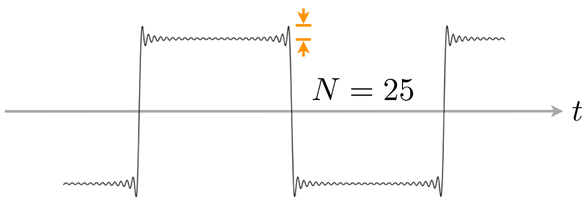


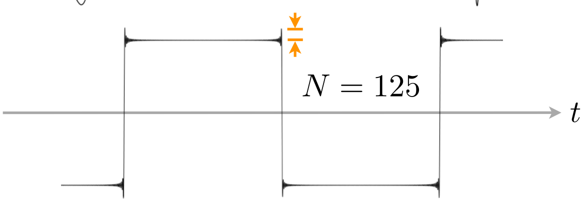
Josiah W. Gibbs
1839-1903



$$\tilde{x}_N(t) \triangleq \sum_{k=-N}^{N} a_k\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

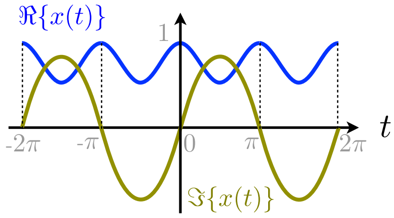






Fourier Series (سری فوریه)

- Example $\quad x(t) = \mathrm{e}^{\mathrm{j}\sin(t)}$



$$x(t) = 2\pi\text{-periodic} \; ✓ \qquad \int_0^{2\pi} |x(\tau)| \mathrm{d}\tau = 2\pi < \infty \quad \Rightarrow \quad x(t) \in L_1([0, 2\pi[) \; ✓$$

$$\begin{cases} x(t) \;\; \text{is continuous} \\ \Re\{x(t)\} : 4 \text{ extrema in } [0, 2\pi[\\ \Im\{x(t)\} : 2 \text{ extrema in } [0, 2\pi[\end{cases} \; ✓$$

$$a_k = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} x(\tau) \mathrm{e}^{-\mathrm{j}k\tau} \mathrm{d}\tau = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{j}\left(\sin(\tau) - k\tau\right)} \mathrm{d}\tau = J_k(1) \quad \Rightarrow \quad x(t) = \sum_{k\in\mathbb{Z}} J_k(1) \mathrm{e}^{\mathrm{j}kt}$$

Bessel function
(1st kind)

Properties of Continuous-domain Fourier Series

- Linearity

$$x(t),\, w(t) \; : \; T\text{-periodic}, \; \begin{cases} x(t) \quad \xrightarrow{\mathcal{F}.\mathrm{S}.} \quad \{a_k\}_{k\in\mathbb{Z}} \\ w(t) \quad \xrightarrow{\mathcal{F}.\mathrm{S}.} \quad \{b_k\}_{k\in\mathbb{Z}} \end{cases}$$

$$\Rightarrow \quad \forall \alpha, \beta \in \mathbb{C}, \; \begin{cases} \alpha\, x(t) + \beta\, w(t) \quad = \quad T\text{-periodic}, \\ \alpha\, x(t) + \beta\, w(t) \quad \xrightarrow{\mathcal{F}.\mathrm{S}.} \quad \{\alpha\, a_k + \beta\, b_k\}_{k\in\mathbb{Z}} \end{cases}$$

- Shift

$$x(t) \; : \; T\text{-periodic}, \; x(t) \xrightarrow{\mathcal{F}.\mathrm{S}.} \{a_k\}_{k\in\mathbb{Z}}$$

$$\Rightarrow \quad x(t - t_0) = T\text{-periodic}, \quad x(t - t_0) \xrightarrow{\mathcal{F}.\mathrm{S}.} \left\{ \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T} k t_0} a_k \right\}_{k\in\mathbb{Z}}$$

$$x(t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \quad \Rightarrow \quad x(t - t_0) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} \left(a_k \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}kt_0} \right) \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

Properties of Continuous-domain Fourier Series

- Time-reversal

$$x(t) \;:\; T\text{-periodic},\; x(t) \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_k\}_{k\in\mathbb{Z}}$$

$$\Rightarrow \quad x(\text{-}t) = T\text{-periodic}, \quad x(\text{-}t) \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_{\text{-}k}\}_{k\in\mathbb{Z}}$$

$$x(t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \quad \Rightarrow \quad x(\text{-}t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}(\text{-}k)t} = \sum_{k\in\mathbb{Z}} a_{\text{-}k}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

- Conjugation

$$x(t) \;:\; T\text{-periodic},\; x(t) \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_k\}_{k\in\mathbb{Z}}$$

$$\Rightarrow \quad \overline{x(t)} = T\text{-periodic}, \quad \overline{x(t)} \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{\overline{a_{\text{-}k}}\}_{k\in\mathbb{Z}}$$

$x(t) = \substack{\text{real-valued} \\ \text{\& even}} \Rightarrow \{a_k\} = \substack{\text{real-valued} \\ \text{\& even}}$ $\qquad$ $x(t) = \substack{\text{real-valued} \\ \text{\& odd}} \Rightarrow \{a_k\} = \substack{\text{purely imaginary} \\ \text{\& odd}}$

Properties of Continuous-domain Fourier Series

- Dilation (Time-scaling)

$$x(t) \;:\; T\text{-periodic},\; x(t) \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_k\}_{k\in\mathbb{Z}}$$

$$\Rightarrow \quad x(\alpha t) = \left(\tfrac{T}{\alpha}\right)\text{-periodic}, \quad x(\alpha t) \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_k\}_{k\in\mathbb{Z}}$$

$$x(t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \quad \Rightarrow \quad x(\alpha t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k\alpha t} = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T/\alpha}kt}$$

- Multiplication

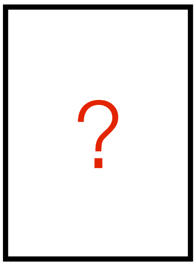
$$x(t),\, w(t) \;:\; T\text{-periodic}, \begin{cases} x(t) & \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \quad \{a_k\}_{k\in\mathbb{Z}} \\ w(t) & \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \quad \{b_k\}_{k\in\mathbb{Z}} \end{cases}$$

$$\Rightarrow \quad \begin{cases} x(t)\, w(t) & = \quad T\text{-periodic}, \\ x(t)\, w(t) & \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \quad \left\{(a * b)[k] = \sum_{m\in\mathbb{Z}} a_m b_{k-m}\right\}_{k\in\mathbb{Z}} \end{cases}$$

Properties of Continuous-domain Fourier Series

- Parseval's Theorem

$$x(t)\ :\ T\text{-periodic},\ x(t) \xmapsto{\mathcal{F}.\mathrm{S}.} \{a_k\}_{k\in\mathbb{Z}}$$



Mark-Antoine Parseval
1755-1836

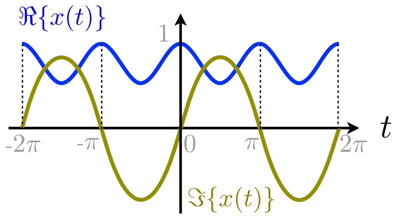
$$\begin{cases} x(t)\ :\ T\text{-periodic}, \\ \underset{0\leq t<T}{x(t)} \in L_2([0,T[) \\ x(t) \xmapsto{\mathcal{F}.\mathrm{S}.} \{a_k\}_{k\in\mathbb{Z}} \end{cases} \quad \Rightarrow \quad x(t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

$$\Rightarrow \quad \frac{1}{T}\int_0^T |x(\tau)|^2 \mathrm{d}\tau \quad = \quad \sum_{k\in\mathbb{Z}} |a_k|^2$$

$$\frac{1}{T}\int_0^T \overline{\Big(\sum_{k_1\in\mathbb{Z}} a_{k_1}\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k_1\tau}\Big)}\Big(\sum_{k_2\in\mathbb{Z}} a_{k_2}\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k_2\tau}\Big)\mathrm{d}\tau = \frac{1}{T}\sum_{k_1,k_2\in\mathbb{Z}} \overline{a_{k_1}}a_{k_2}\int_0^T \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}(k_2-k_1)\tau}\mathrm{d}\tau = \sum_{k_1,k_2\in\mathbb{Z}} \overline{a_{k_1}}a_{k_2}\delta[k_2-k_1]$$

Parseval's Theorem

- Example $\qquad x(t) = \mathrm{e}^{\mathrm{j}\sin(t)}$



$x(t) = 2\pi$-periodic ✓ $\qquad x(t) = \sum_{k\in\mathbb{Z}} J_k(1)\mathrm{e}^{\mathrm{j}kt}$ ✓

$$\frac{1}{T}\int_0^T |x(\tau)|^2 \mathrm{d}\tau = \tfrac{1}{2\pi}\int_0^{2\pi} \left|\mathrm{e}^{\mathrm{j}\sin(\tau)}\right|^2 \mathrm{d}\tau = 1$$

$$\Rightarrow \quad \sum_{k\in\mathbb{Z}} \left|J_k(1)\right|^2 = 1$$

Fourier Series using Properties

- Example

$$x(t) \equiv 1 \;\Rightarrow\; T\text{-periodic}, \;\forall T$$

$$x(t),\; T\text{-periodic} \;\Rightarrow\; x(t) = \sum_{k\in\mathbb{Z}} a_k^{(T)}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

$$x(t),\; \tfrac{T}{m}\text{-periodic} \;\Rightarrow\; x(t) = \sum_{k\in\mathbb{Z}} a_k^{(T/m)}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}(mk)t}$$

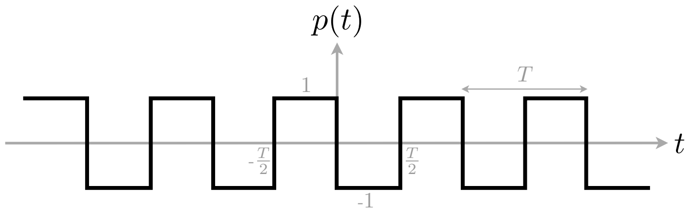
$$\Rightarrow\; a_k^{(T)} = \begin{cases} a_l^{(T/m)} & k = ml, \\ 0 & \text{otherwise.}\end{cases}$$

$$m \in \mathbb{N}^+, \text{ arbitrary} \;\to\; a_k^{(T)} \neq 0 \iff k = 0$$

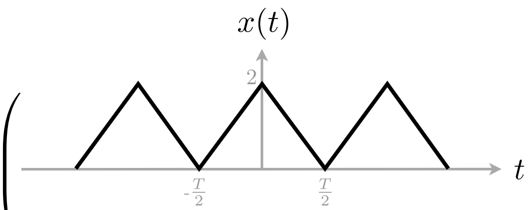
$$x(t) \equiv 1 \;\xmapsto{\mathcal{F}.\mathrm{S}.}\; a_k = \delta[k]$$

Fourier Series using Properties

- Example



$$p(t) = \frac{T}{4}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\; x(t) \;\Big)$$



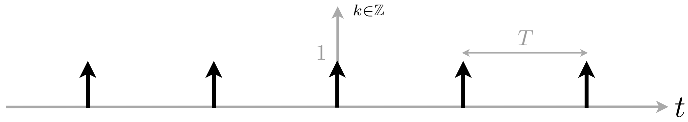
$$= \frac{T}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\sin(\pi k/2)}{\pi k/2}\right)^2 \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\right) = \sum_{k\in\mathbb{Z}} \underbrace{\mathrm{j}\frac{1-\cos(\pi k)}{\pi k}}_{\mathcal{F}.\mathrm{S}.\text{ of } p(t)} \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

Fourier Series using Properties

- Example (Dirac Comb)

  (قطار ضربه)

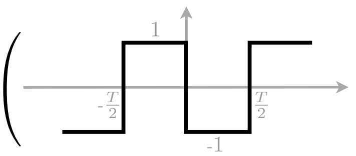
$$\text{Ш}_T(t) = \sum_{k\in\mathbb{Z}} \delta(t-kT)$$



$\text{Ш}_T(t) =$ a T-periodic functional!

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \text{Ш}(\tau)\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(\tau)\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau = \frac{1}{T}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p(t) \right) = 2\Big(\text{Ш}_T(t-\tfrac{T}{2}) - \text{Ш}_T(t)\Big)$$



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k\in\mathbb{Z}} \mathrm{j}\frac{1-\cos(\pi k)}{\pi k}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\right) = 2\Big(\sum_{k\in\mathbb{Z}} \tfrac{1}{T}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}k(t-T/2)} - \sum_{k\in\mathbb{Z}} \tfrac{1}{T}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\Big)$$

Fourier Series

- Eigen-signals of LSI systems
- Fourier series of continuous-domain periodic signals
- Fourier series of discrete-domain periodic signals
- Filtering

Periodic Eigen-signals

- Eigen-decomposition of periodic functions

$$\left\{\begin{array}{lcl} x_{\mathrm{eig}}[n] & = & c\, z^n \\ \& & & \\ x_{\mathrm{eig}}[n] & = & N\text{-periodic} \end{array}\right. \Rightarrow\ c\, z^n = c\, z^{n+N}$$

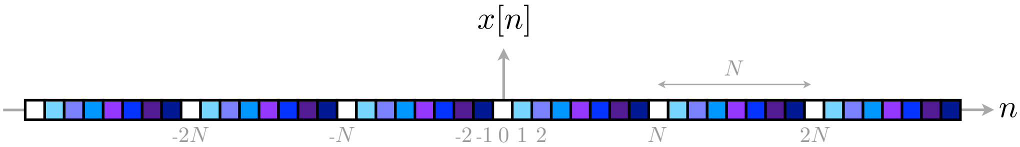
$$\Rightarrow z^N = 1$$

$$\Rightarrow z_k = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}, \quad 0 \leq k \leq N-1$$

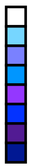
$$\Rightarrow \quad x_{\mathrm{eig},k}[n] = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}, \quad \underbrace{0 \leq k \leq N-1}_{N\ \text{eigen-signals!}}$$

$$\forall\, \{a_k\}, \quad \sum_{k=0}^{N-1} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} = N\text{-periodic} \qquad x[n] = N\text{-periodic} \overset{?}{\Rightarrow} x[n] = \sum_{k=0}^{N-1} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

Periodic Signals as vectors



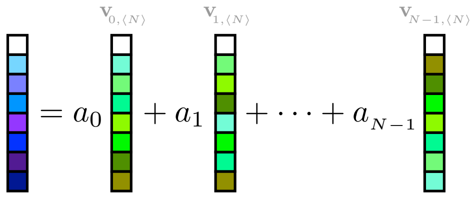
$$\mathbf{x}_{\langle N\rangle} \triangleq \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} =$$



$$v_k[n] \triangleq k\text{th } N\text{-periodic eigen-signal}, \quad \mathbf{v}_{k,\langle N\rangle} = \begin{bmatrix} v_k[0] \\ v_k[1] \\ \vdots \\ v_k[N-1] \end{bmatrix}$$

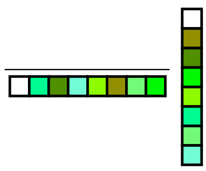
$$= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

$$x[n] = \sum_{k=0}^{N-1} a_k v_k[n] \quad \Leftrightarrow \quad \mathbf{x}_{\langle N\rangle} = \sum_{k=1}^{N} a_k \mathbf{v}_{k,\langle N\rangle} \quad \Leftrightarrow$$



Fourier Series (سری فوریه)

$$\left\langle \mathbf{v}_{k_1,\langle N\rangle}, \mathbf{v}_{k_2,\langle N\rangle} \right\rangle = \left(\mathbf{v}_{k_1,\langle N\rangle}\right)^H \mathbf{v}_{k_2,\langle N\rangle} =$$



$$= \sum_{m=0}^{N-1} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k_1 m} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_2 m} = \sum_{m=0}^{N-1} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(k_2-k_1)m} = N\delta[k_2-k_1]$$

$$\Rightarrow \quad \left\{\mathbf{v}_{k,\langle N\rangle}\right\}_{k=0}^{N-1} = \text{Orthogonal set}$$

$$\Rightarrow \quad \mathbf{x}_{\langle N\rangle} \overset{\checkmark}{=} \sum_{k=0}^{N-1} a_k\, \mathbf{v}_{k,\langle N\rangle}, \qquad \underset{0\le k<N}{a_k} = \frac{\left\langle \mathbf{v}_{k,\langle N\rangle}, \mathbf{x}_{\langle N\rangle}\right\rangle}{\left\langle \mathbf{v}_{k,\langle N\rangle}, \mathbf{v}_{k,\langle N\rangle}\right\rangle}$$

$$\Updownarrow \qquad\qquad = \frac{1}{N}\sum_{m=0}^{N-1} x[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km}$$

$$x[n] = \sum_{k=0}^{N-1} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

Fourier Series (periodic extension)

$$x[n] = N\text{-periodic} \quad \Rightarrow \quad x[n] = \sum_{k=0}^{N-1} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}, \quad a_k = \frac{1}{N}\sum_{m=0}^{N-1} x[m]\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km}$$

$$\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} \equiv \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(k+N)n} \quad \Rightarrow \quad x[n] = \sum_{k=0}^{N-1} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} = \sum_{k=1}^{N} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

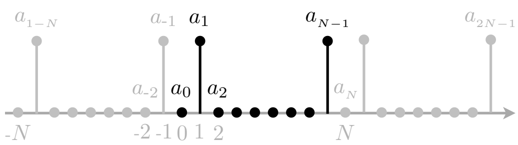
$$= \sum_{k=\langle N\rangle} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

one period

- Example $\quad x[n] = \cos(\frac{2\pi}{N}n) \quad \Rightarrow \quad N\text{-periodic}$

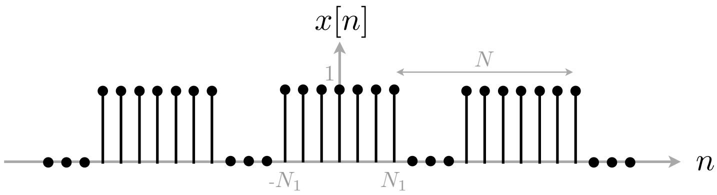
$$x[n] = \tfrac{1}{2}\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n} + \tfrac{1}{2}\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}n}$$

$$= \underbrace{\frac{1}{2}}_{a_1}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}n} + \underbrace{\frac{1}{2}}_{a_{N-1}}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(N-1)n}$$

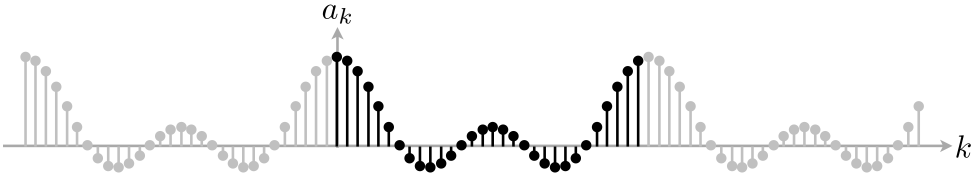


Fourier Series (سری فوریه)

- Example



$$a_k = \frac{1}{N}\sum_{m=\langle N\rangle} x[m]\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km} = \frac{1}{N}\sum_{m=-N_1}^{N_1} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km} = \begin{cases} \dfrac{\sin\left(\frac{\pi}{N}k(2N_1+1)\right)}{N\,\sin\left(\frac{\pi}{N}k\right)}, & k \notin N\,\mathbb{Z} \\[2ex] \dfrac{2N_1+1}{N}, & k = 0, \pm N, \dots \end{cases}$$



$N_1 = 2, \quad N = 30$

Properties of Discrete-domain Fourier Series

$$x[n],\, w[n] \;:\; N\text{-periodic}, \quad \begin{cases} x[n] & \xrightarrow{\mathcal{F}.\text{S.}} \quad \{a_k\} \\ w[n] & \xrightarrow{\mathcal{F}.\text{S.}} \quad \{b_k\} \end{cases}$$

- Linearity $$\forall\, \alpha, \beta \in \mathbb{C}, \quad \begin{cases} \alpha\, x[n] + \beta\, w[n] & = \quad N\text{-periodic}, \\ \alpha\, x[n] + \beta\, w[n] & \xrightarrow{\mathcal{F}.\text{S.}} \quad \{\alpha\, a_k + \beta\, b_k\} \end{cases}$$

- Shift $$x[n - n_0] = N\text{-periodic}, \quad x[n - n_0] \xrightarrow{\mathcal{F}.\text{S.}} \left\{ \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N} k n_0}\, a_k \right\}$$

- Time-reversal $$x[\text{-}n] = N\text{-periodic}, \quad x[\text{-}n] \xrightarrow{\mathcal{F}.\text{S.}} \{a_{\text{-}k}\}$$

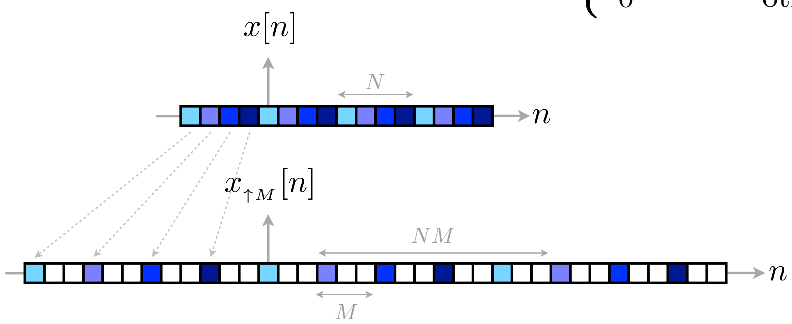
- Conjugation $$\overline{x[n]} = N\text{-periodic}, \quad \overline{x[n]} \xrightarrow{\mathcal{F}.\text{S.}} \{\overline{a_{\text{-}k}}\}$$

Properties of Discrete-domain Fourier Series

- Dilation (Time-scaling)

$$x[n] \;:\; N\text{-periodic}, \; x[n] \xrightarrow{\mathcal{F}.\text{S.}} \{a_k\}$$

$$M \in \mathbb{N}^+, \quad x_{\uparrow M}[n] \triangleq \begin{cases} x\big[n/M\big] & n \in M\,\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$



$$\Rightarrow \quad x_{\uparrow M}[n] \;=\; (NM)\text{-periodic}$$

$(NM\text{-periodic})$

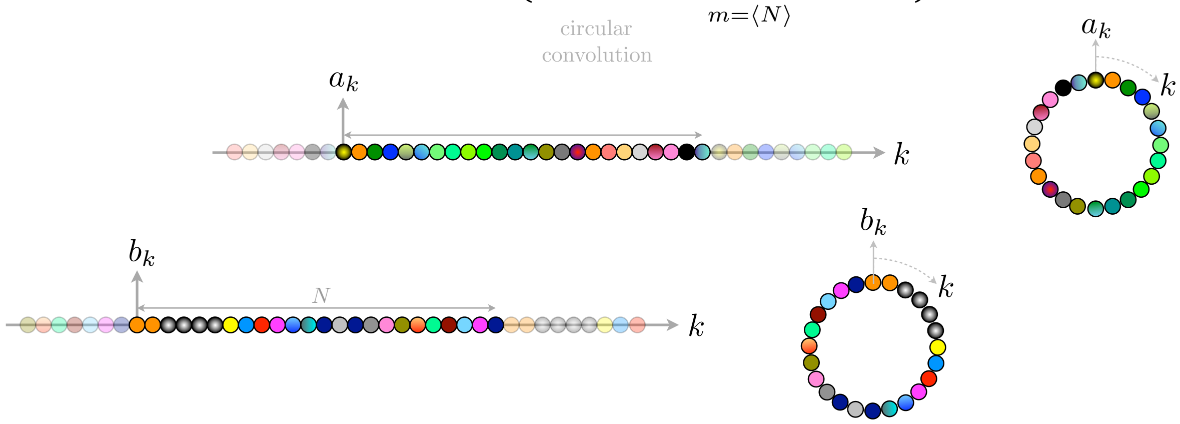
$$\big(\mathcal{F}.\text{S. of } x_{\uparrow M}[\cdot]\big)_k \;=\; \frac{1}{NM} \sum_{m=\langle NM\rangle} x_{\uparrow M}[m]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{NM}mk} \;=\; \frac{1}{NM} \sum_{\widetilde{m}=\langle N\rangle} x[\widetilde{m}]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}\widetilde{m}k} = \frac{a_k}{M}$$

Properties of Discrete-domain Fourier Series

- Multiplication

$$x[n],\, w[n] \;:\; N\text{-periodic}, \quad \begin{cases} x[n] & \overset{\mathcal{F}.\text{S.}}{\longmapsto} \quad \{a_k\} \\ w[n] & \overset{\mathcal{F}.\text{S.}}{\longmapsto} \quad \{b_k\} \end{cases}$$

$$x[n]\, w[n] \overset{\mathcal{F}.\text{S.}}{\longmapsto} \Big\{ \underbrace{(a \overset{N}{\circledast} b)[k]}_{\text{circular convolution}} = \sum_{m=\langle N \rangle} a_m b_{k-m} \Big\}$$

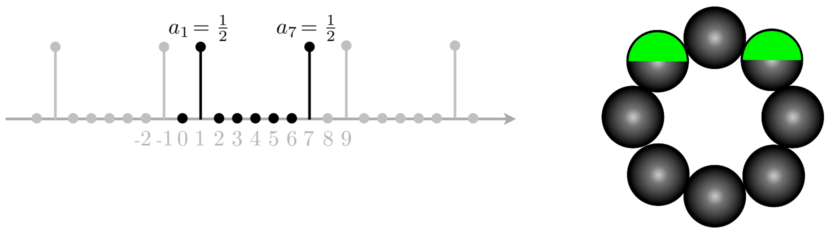


Properties of Discrete-domain Fourier Series

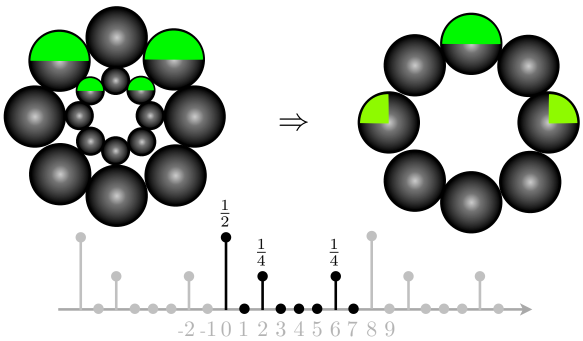
- Example

$$x[n] = \cos\left(\tfrac{2\pi}{8} n\right), \qquad w[n] = \cos^2\left(\tfrac{2\pi}{8} n\right) = x[n] \times x[n]$$

$x[n] \overset{\mathcal{F}.\text{S.}}{\longmapsto}$



$w[n] = x[n] \times x[n] \overset{\mathcal{F}.\text{S.}}{\longmapsto}$



Properties of Discrete-domain Fourier Series

- Parseval's Theorem

$$x[n]\ :\ N\text{-periodic},\ x[n] \overset{\mathcal{F}.\mathrm{S}.}{\longmapsto} \{a_k\}$$

$$\Rightarrow \boxed{\frac{1}{N}\sum_{m=\langle N\rangle} |x[m]|^2 = \sum_{k=\langle N\rangle} |a_k|^2}$$

$$\sum_{m=\langle N\rangle} |x[m]|^2 = \sum_{m=\langle N\rangle}\Big(\sum_{k_1=\langle N\rangle} a_{k_1}\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_1 m}\Big)\overline{\Big(\sum_{k_2=\langle N\rangle} a_{k_2}\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_2 m}\Big)}$$

$$= \sum_{k_1,k_2=\langle N\rangle} a_{k_1}\overline{a_{k_2}}\underbrace{\sum_{m=\langle N\rangle}\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(k_1-k_2)m}}_{N\,\delta[k_1-k_2]} = N\sum_{k_1=\langle N\rangle}|a_{k_1}|^2$$

Parseval's Theorem

- Example

$$x[n] = \cos^2\left(\tfrac{2\pi}{8}n\right)$$

$x[n] = 4\text{-periodic}$ ✓

$$x[n] = \begin{cases} 1, & n = 0 \pmod 4,\\ \frac{1}{2}, & n = 1 \pmod 4,\\ 0, & n = 2 \pmod 4,\\ \frac{1}{2}, & n = 3 \pmod 4.\end{cases}$$

$$x[n] = \frac{1+\cos\left(\frac{4\pi}{8}n\right)}{2} = \frac{1}{2} + \frac{1}{4}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}n} + \frac{1}{4}\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}n} \quad\Rightarrow\quad a_k = \begin{cases} \frac{1}{2}, & k = 0 \pmod 4,\\ \frac{1}{4}, & k = 1 \pmod 4,\\ 0, & k = 2 \pmod 4,\\ \frac{1}{4}, & k = 3 \pmod 4.\end{cases}$$

$$\Rightarrow \frac{1}{4}\sum_{m=0}^{3}|x[m]|^2 = \frac{3}{8} = \sum_{k=0}^{3}|a_k|^2$$

Fourier Series

- Eigen-signals of LSI systems
- Fourier series of continuous-domain periodic signals
- Fourier series of discrete-domain periodic signals
- Filtering

Periodic Signals through LSI systems

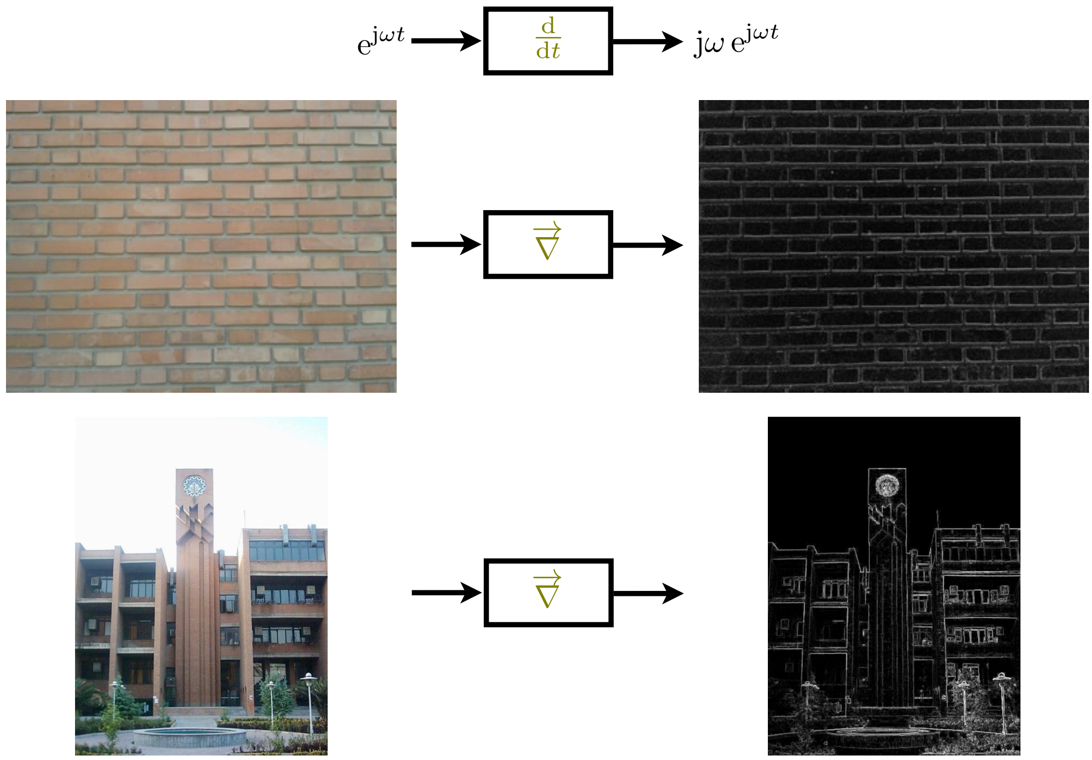
$$x(t) = T\text{-periodic} \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad \Rightarrow \begin{cases} y(t) & = \quad T\text{-periodic} \\ \{b_k\}_k & = \quad \mathcal{F}\text{.S. of } y(t) \end{cases}$$

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \longmapsto y(t) = \sum_{k\in\mathbb{Z}} \underbrace{a_k H(\mathrm{j}\tfrac{2\pi}{T}k)}_{b_k} \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

$$x[n] = N\text{-periodic} \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad \Rightarrow \begin{cases} y[n] & = \quad N\text{-periodic} \\ \{b_k\}_k & = \quad \mathcal{F}\text{.S. of } y[n] \end{cases}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn} \longmapsto y[n] = \sum_{k=\langle N\rangle} \underbrace{a_k H\left(\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k}\right)}_{b_k} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

LSI systems: Frequency-Shaping filters



LSI systems: Frequency-Selective filters

- Power quality