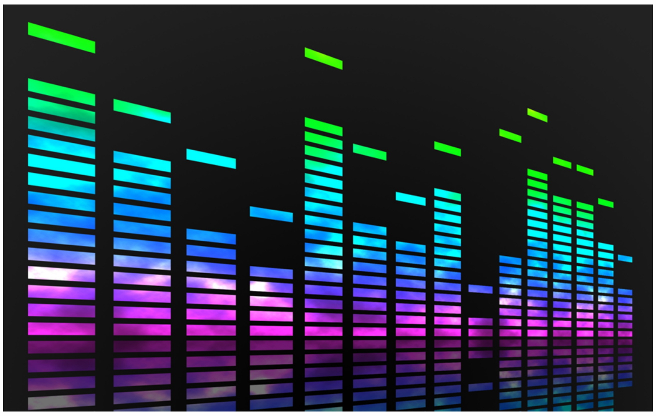


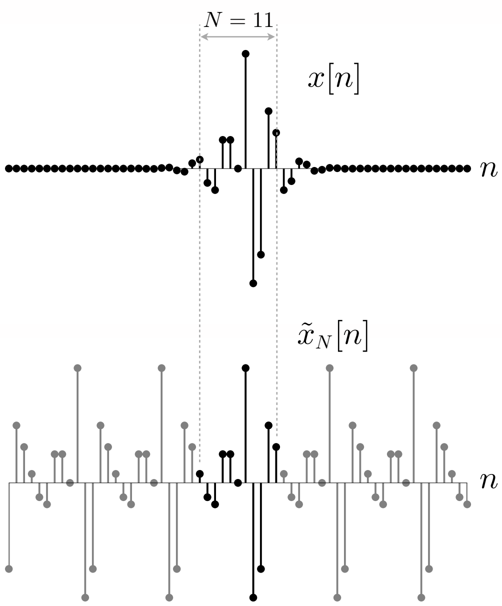
Discrete-domain Fourier Transform

- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of difference equations

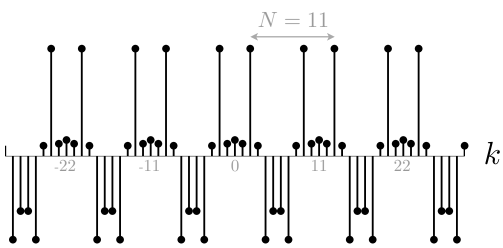
Periodized signals

- Non-periodic discrete-domain signal

$$x[n] = n\, 2^{-|n-1|} \cos\left((2n-1)\tfrac{\pi}{4}\right)$$



$N\Re\{a_k\}$

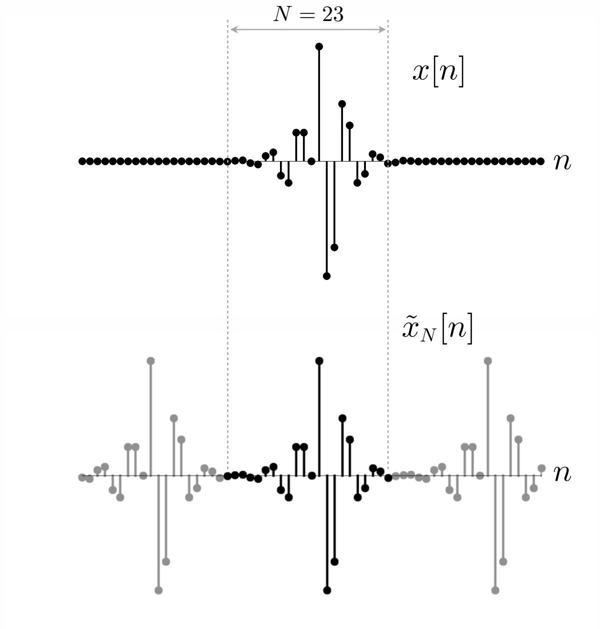


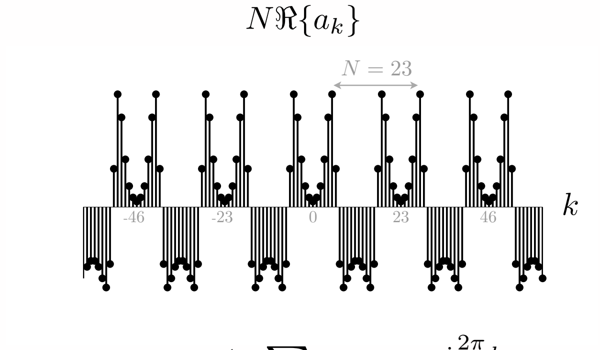
$$a_k = \frac{1}{N} \sum_{m \in \langle N \rangle} \tilde{x}_N[m]\, e^{-j\frac{2\pi}{N} k\, m}$$

Periodized signals

- Non-periodic discrete-domain signal

$$x[n] = n\, 2^{-|n-1|} \cos\left((2n-1)\tfrac{\pi}{4}\right)$$





$$a_k = \frac{1}{N} \sum_{m \in \langle N \rangle} \tilde{x}_N[m]\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N} k\, m}$$

Periodized signals

- Non-periodic discrete-domain signal

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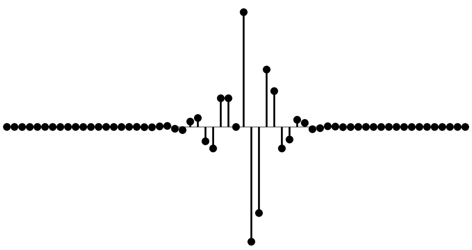
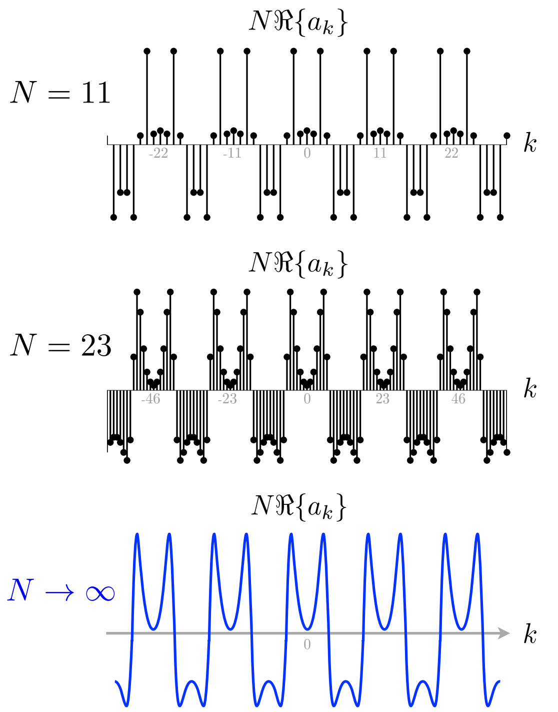




Fourier Transform (تبدیل فوریه)

$X(\mathrm{e}^{\mathrm{j}\omega})$

$X_{2\pi}(\omega)$

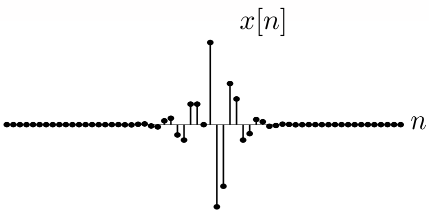
$$\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \triangleq \mathrm{DTFT}\{x[n]\}(\omega) \triangleq \sum_{m\in\mathbb{Z}} x[m]\,\mathrm{e}^{-\mathrm{j}\omega m} \triangleq \lim_{N\to\infty} \sum_{m=-N}^{N} x[m]\,\mathrm{e}^{-\mathrm{j}\omega m}$$

$x[n] \in \ell_1$,

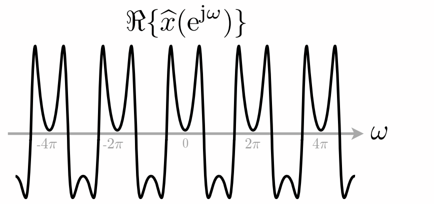
$\tilde{x}_N[n]$: N-periodized version of $x[n]$,

a_k : $\mathcal{F}$.S. of $\tilde{x}_N[n]$

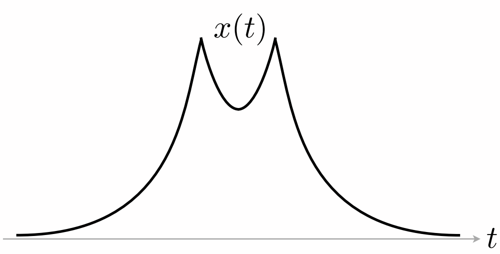
$$\Rightarrow \quad N\,a_k = \sum_{m=1+\lfloor\frac{-N}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor} x[m]\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}mk} \overset{N\gg 1}{\approx} \sum_{m=-\infty}^{\infty} x[m]\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}km} = \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\Big|_{\omega=2\pi\frac{k}{N}}$$



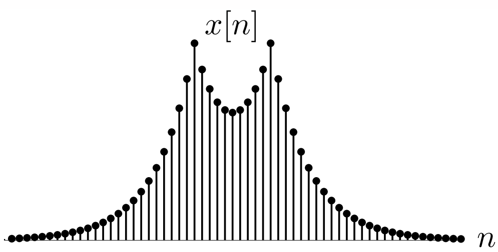
$\xrightarrow{\text{DTFT}}$



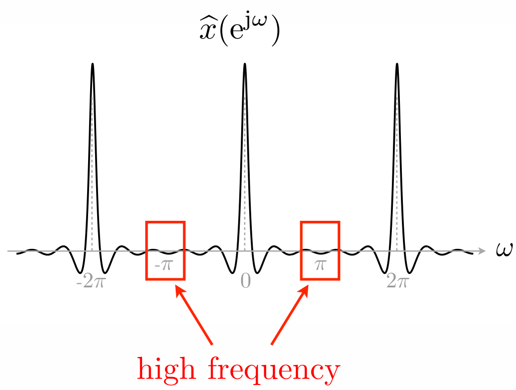
DTFT vs. CTFT



$\xrightarrow{\mathcal{F}.\text{T.}}$



$\xrightarrow{\text{DTFT}}$

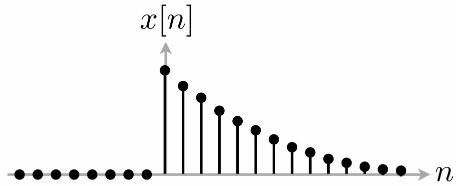


$\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = 2\pi$-periodic

Fourier Transform (تبدیل فوریه)

- Example

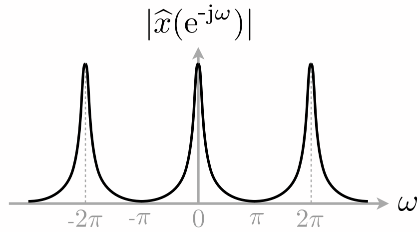
$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$



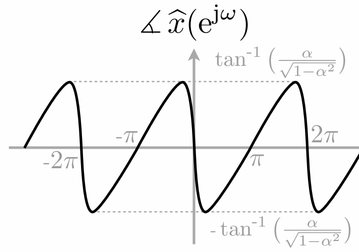
$$\widehat{x}(e^{j\omega}) = \sum_{m\in\mathbb{Z}} x[m]e^{-j\omega m} = \sum_{m=0}^{\infty} \alpha^m e^{-j\omega m}$$

$$= \frac{1}{1-\alpha\, e^{-j\omega}}$$

$$= \frac{1}{\sqrt{1+|\alpha|^2-2\Re\{\alpha e^{-j\omega}\}}} \measuredangle \tan^{-1}\left(\frac{-\Im\{\alpha e^{-j\omega}\}}{1-\Re\{\alpha e^{-j\omega}\}}\right)$$



$\alpha \in \mathbb{R}^+$

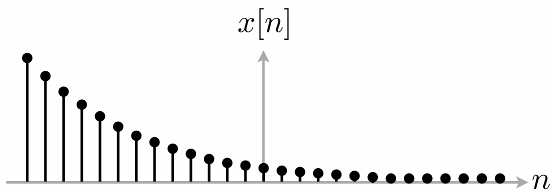


Amini

Fourier Transform (تبدیل فوریه)

- Example

$$x[n] = \alpha^n, \quad |\alpha| < 1$$

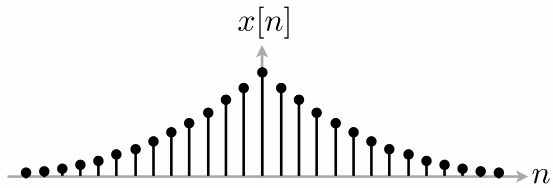


$$\sum_{m\in\mathbb{Z}} x[m]e^{-j\omega m} = \sum_{m\in\mathbb{Z}} \alpha^m e^{-j\omega m} = \underbrace{\sum_{m=-\infty}^{-1} \alpha^m e^{-j\omega m}}_{\text{infinite}} + \underbrace{\sum_{m=0}^{\infty} \alpha^m e^{-j\omega m}}_{\text{finite}} \Rightarrow$$

no Fourier transform ✘

- Example

$$x[n] = \alpha^{|n|}, \quad |\alpha| < 1$$



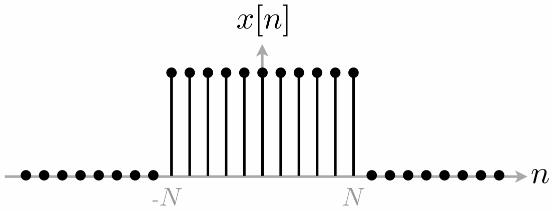
$$x(e^{j\omega}) = \sum_{m=-\infty}^{-1} \alpha^{-m} e^{-j\omega m} + \sum_{m=0}^{\infty} \alpha^m e^{-j\omega m} = \frac{\alpha e^{j\omega}}{1-\alpha e^{j\omega}} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(\omega)}$$

Amini

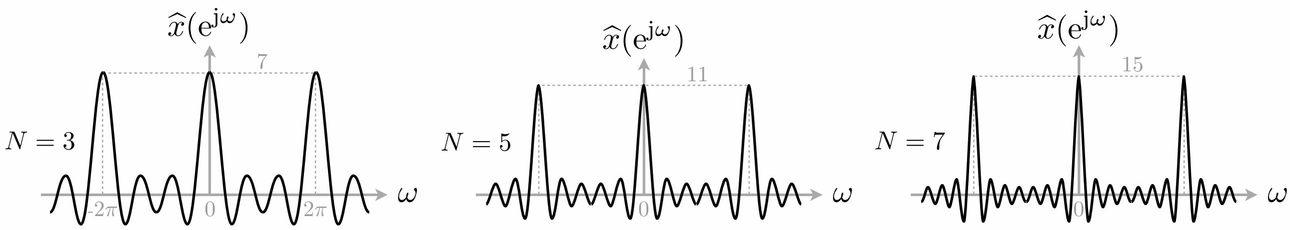
Fourier Transform (تبدیل فوریه)

- Example

$$x[n] = \begin{cases} 1 & |n| \le N, \\ 0 & \text{otherwise.} \end{cases}$$



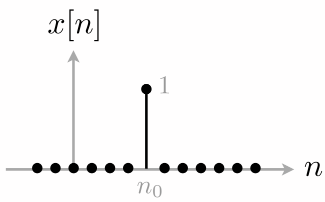
$$\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{-\mathrm{j}\omega m} = \sum_{m=-N}^{N} \mathrm{e}^{-\mathrm{j}\omega m} = \frac{\mathrm{e}^{\mathrm{j}\omega N} - \mathrm{e}^{-\mathrm{j}\omega(N+1)}}{1-\mathrm{e}^{-\mathrm{j}\omega}} = \frac{\sin\left((2N+1)\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$



Fourier Transform (تبدیل فوریه)

- Example

$$x[n] = \delta[n-n_0], \quad n_0 \in \mathbb{Z}$$



$$\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{-\mathrm{j}\omega m} = \sum_{m\in\mathbb{Z}} \delta[m-n_0]\mathrm{e}^{-\mathrm{j}\omega m} = \mathrm{e}^{-\mathrm{j}\omega n_0}$$

no need for adjoint operator!!!

$$\boxed{n_0 = 0: \quad \delta[n] \xrightarrow{\text{DTFT}} \widehat{\delta}(\mathrm{e}^{\mathrm{j}\omega}) \equiv 1}$$

Convergence of Fourier Transform

$$x[n] \in \ell_1 \quad \Rightarrow \quad \begin{cases} \widehat{x}(e^{j\omega}) \text{ exists,} \\ \widehat{x}(e^{j\omega}) \text{ is continuous,} \\ \widehat{x}(e^{j\omega}) \in L_\infty([0, 2\pi[). \end{cases}$$

$$x[n] \in \ell_2 \quad \Rightarrow \quad \begin{cases} \widehat{x}(e^{j\omega}) \text{ exists,} \\ \widehat{x}(e^{j\omega}) \in L_2([0, 2\pi[). \end{cases}$$

Inverse Fourier Transform (تبدیل عکس فوریه)

$$\tilde{x}_N[n] = N\text{-periodic} \quad \Rightarrow \quad \tilde{x}_N[n] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N} \sum_{m=\langle N\rangle} \tilde{x}_N[m]\, e^{-j\frac{2\pi}{N}km} \underset{N\gg 1}{\overset{x\in\ell_1}{\approx}} \frac{1}{N}\widehat{x}(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

$$\Rightarrow \quad \tilde{x}_N[n] \underset{N\gg 1}{\overset{x\in\ell_1}{\approx}} \frac{1}{N} \sum_{k=\langle N\rangle} e^{j\frac{2\pi}{N}kn}\, \widehat{x}(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \underset{N\gg 1}{\overset{x\in\ell_1}{\approx}} \frac{1}{2\pi}\int_{2\pi} \widehat{x}(e^{j\omega})e^{j\omega n}d\omega$$

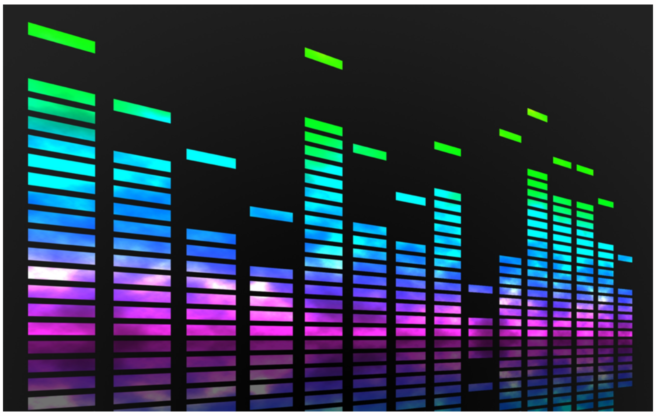
$$|n| < \frac{N}{2} \quad \Rightarrow \quad x[n] = \tilde{x}_N[n] \quad \Rightarrow \quad \forall n,\ x[n] = \lim_{N\to\infty} \tilde{x}_N[n]$$

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} \widehat{x}(e^{j\omega})e^{j\omega n}d\omega}$$

Inverse Fourier Transform (تبدیل عکس فوریه)

$$x[n] \stackrel{?}{=} \frac{1}{2\pi}\int_{2\pi} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}\mathrm{d}\omega$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{2\pi} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}\mathrm{d}\omega &= \frac{1}{2\pi}\int_{2\pi}\Big(\sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{\text{-}\mathrm{j}\omega m}\Big)\mathrm{e}^{\mathrm{j}\omega n}\mathrm{d}\omega \\
&\stackrel{x\in\ell_1}{=} \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} x[m] \underbrace{\int_{2\pi} \mathrm{e}^{\text{-}\mathrm{j}\omega m}\mathrm{e}^{\mathrm{j}\omega n}\mathrm{d}\omega}_{2\pi\,\delta[n-m]} \\
&= \sum_{m\in\mathbb{Z}} x[m]\,\delta[n-m] = x[n] \quad \checkmark
\end{aligned}$$

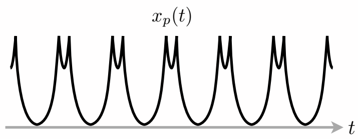


Discrete-domain Fourier Transform

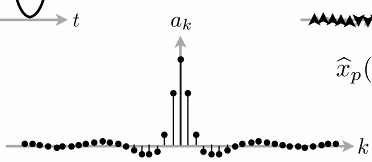
- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of difference equations

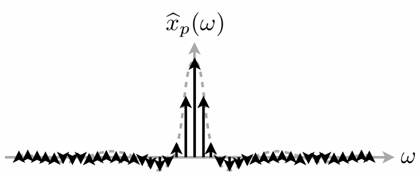
Periodic Signals

Continuous-Domain



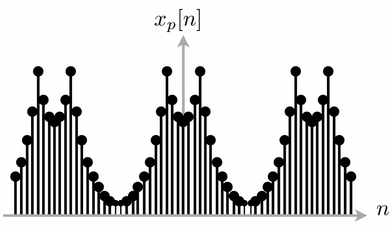
$$x_p(t) = \sum_{k\in\mathbb{Z}} a_k \, e^{j\frac{2\pi}{T}kt}$$



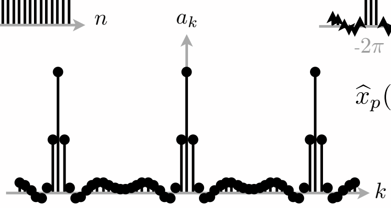


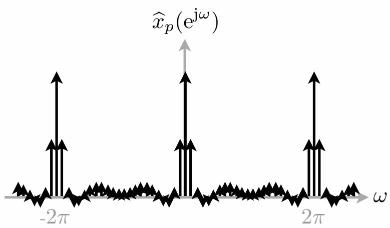
$$\widehat{x}_p(\omega) = \sum_{k\in\mathbb{Z}} (2\pi a_k)\, \delta(\omega - \tfrac{2\pi}{T}k)$$

Discrete-Domain



$$x_p[n] = \sum_{k=\langle N\rangle} a_k \, e^{j\frac{2\pi}{N}kn}$$



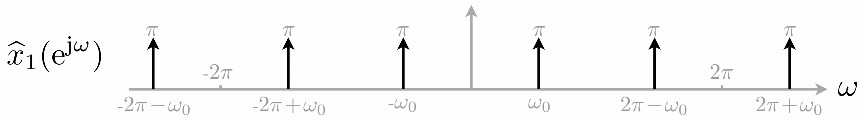


$$\widehat{x}_p(e^{j\omega}) = \sum_{m\in\mathbb{Z}} \sum_{k=\langle N\rangle} (2\pi a_k)\, \delta(\omega - \tfrac{2\pi}{N}k - 2\pi m)$$

Fourier Transform of DDPS

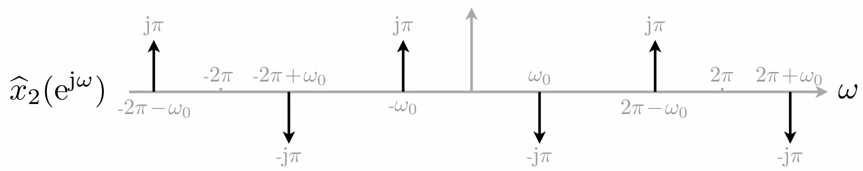
- Example $\quad x_1[n] = \cos\big(\underbrace{\tfrac{p}{q}2\pi\, n}_{\omega_0}\big), \quad p < q \in \mathbb{N}^+$

$$x_1[n] = \tfrac{1}{2}\mathrm{e}^{\mathrm{j}\omega_0 n} + \tfrac{1}{2}\mathrm{e}^{\text{-}\mathrm{j}\omega_0 n} \quad \Rightarrow \quad \widehat{x}_1(\mathrm{e}^{\mathrm{j}\omega}) = \pi \sum_{m\in\mathbb{Z}} \delta(\omega - \omega_0 - 2\pi m) + \pi \sum_{m\in\mathbb{Z}} \delta(\omega + \omega_0 - 2\pi m)$$



- Example $\quad x_2[n] = \sin\big(\underbrace{\tfrac{p}{q}2\pi\, n}_{\omega_0}\big), \quad p < q \in \mathbb{N}^+$

$$x_2[n] = \tfrac{1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\omega_0 n} - \tfrac{1}{2\mathrm{j}}\mathrm{e}^{\text{-}\mathrm{j}\omega_0 n} \quad \Rightarrow \quad \widehat{x}_2(\mathrm{e}^{\mathrm{j}\omega}) = \text{-}\mathrm{j}\pi \sum_{m\in\mathbb{Z}} \delta(\omega - \omega_0 - 2\pi m) + \mathrm{j}\pi \sum_{m\in\mathbb{Z}} \delta(\omega + \omega_0 - 2\pi m)$$

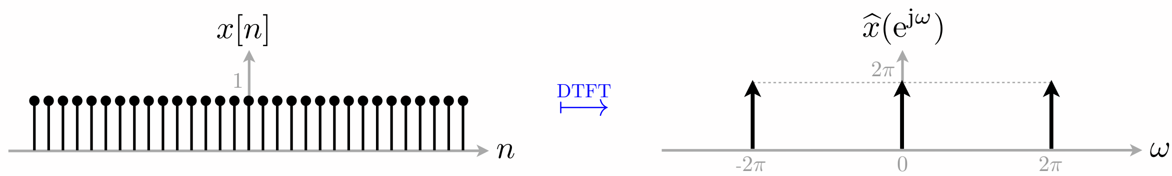


Fourier Transform of DDPS

- Example $\quad x[n] \equiv 1$

$$x[n] = \mathrm{e}^{\mathrm{j}\omega_0 n}\Big|_{\omega_0 = 0}$$

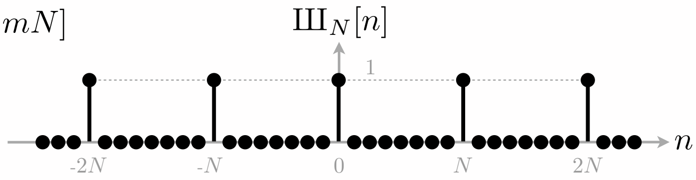
$$\Rightarrow \quad \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = 2\pi \sum_{m\in\mathbb{Z}} \delta(\omega - \omega_0 - 2\pi m)\Big|_{\omega_0=0} = \boxed{2\pi \sum_{m\in\mathbb{Z}} \delta(\omega - 2\pi\, m)}$$



Fourier Transform of DDPS

- Example (قطار ضربه)

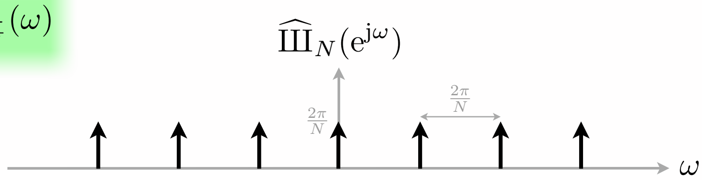
$$Ш_N[n] = \sum_{m\in\mathbb{Z}} \delta[n - mN]$$

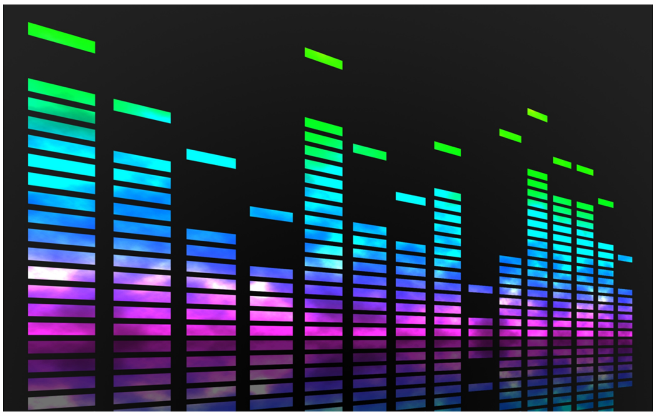


$$Ш_N[n] = N\text{-periodic}, \quad a_k = \tfrac{1}{N}, \ \forall k \in \mathbb{Z} \quad \Rightarrow \quad Ш_N[n] = \sum_{k=\langle N\rangle} \tfrac{1}{N} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}kn}$$

$$\Rightarrow \quad \widehat{Ш}_N(\mathrm{e}^{\mathrm{j}\omega}) = \tfrac{2\pi}{N} \sum_{m\in\mathbb{Z}} \sum_{k=\langle N\rangle} \delta(\omega - \tfrac{2\pi}{N}k - 2\pi m) = \tfrac{2\pi}{N} \sum_{k\in\mathbb{Z}} \delta(\omega - \tfrac{2\pi}{N}k)$$

$$= \tfrac{2\pi}{N} Ш_{\frac{2\pi}{N}}(\omega)$$





Discrete-domain Fourier Transform

- Fourier series to Fourier transform
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- Systems of difference equations

Properties of Discrete-domain FT

- Periodicity

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(e^{j\omega})$$

$$\Rightarrow \quad \widehat{x}\left(e^{j\omega}\right) = 2\pi\text{-periodic}$$

$$\widehat{x}\left(e^{j(\omega+2\pi)}\right) = \widehat{x}(e^{j\omega})$$

- Linearity

$$x[n],\, w[n] \in \ell_1, \quad \begin{cases} x[n] & \overset{\text{DTFT}}{\longmapsto} & \widehat{x}(e^{j\omega}) \\ w[n] & \overset{\text{DTFT}}{\longmapsto} & \widehat{w}(e^{j\omega}) \end{cases}$$

$$\Rightarrow \quad \forall\, \alpha, \beta \in \mathbb{C}, \quad \alpha\, x[n] + \beta\, w[n] \overset{\text{DTFT}}{\longmapsto} \alpha\, \widehat{x}(e^{j\omega}) + \beta\, \widehat{w}(e^{j\omega})$$

Properties of Discrete-domain FT

- Time-Shift

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\Rightarrow \quad x[n-n_0] \overset{\text{DTFT}}{\longmapsto} \mathrm{e}^{-\mathrm{j}\omega n_0}\,\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\sum_{m\in\mathbb{Z}} x[m-n_0]\mathrm{e}^{-\mathrm{j}\omega m} = \sum_{\tilde{m}\in\mathbb{Z}} x[\tilde{m}]\mathrm{e}^{-\mathrm{j}\omega(\tilde{m}+n_0)} = \mathrm{e}^{-\mathrm{j}\omega n_0}\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

- Frequency-Shift

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

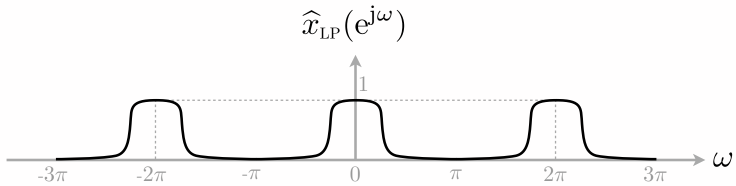
$$\Rightarrow \quad \mathrm{e}^{\mathrm{j}\omega_0 n}\,x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}\big(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)}\big)$$

$$\sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{\mathrm{j}\omega_0 m}\mathrm{e}^{-\mathrm{j}\omega m} = \sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{-\mathrm{j}(\omega-\omega_0)m} = \widehat{x}\big(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)}\big)$$

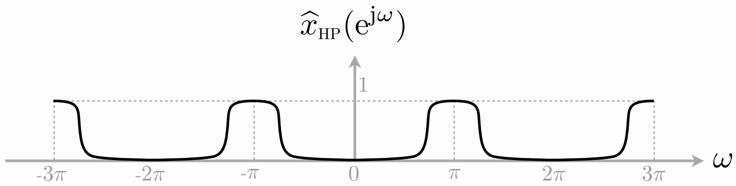
Properties of Discrete-domain FT

- Example

$x_{\text{LP}}[n]$,



$\widehat{x}_{\text{HP}}(\mathrm{e}^{\mathrm{j}\omega}) \triangleq \widehat{x}_{\text{LP}}(\mathrm{e}^{\mathrm{j}(\omega-\pi)})$



$$\Rightarrow \quad x_{\text{HP}}[n] = \mathrm{e}^{\mathrm{j}\pi n}x_{\text{LP}}[n] = (\text{-}1)^n x_{\text{LP}}[n]$$

Properties of Discrete-domain FT

- Time-reversal

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(e^{j\omega}) \qquad \Rightarrow \quad x[-n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(e^{-j\omega})$$

$$\sum_{m\in\mathbb{Z}} x[-m]e^{-j\omega m} = \sum_{\tilde{m}\in\mathbb{Z}} x[\tilde{m}]e^{j\omega\tilde{m}} = \widehat{x}(e^{-j\omega})$$

$$x[n] = \text{even} \;\Rightarrow\; \widehat{x}(e^{j\omega}) = \text{even w.r.t. } \omega \qquad\qquad x[n] = \text{odd} \;\Rightarrow\; \widehat{x}(e^{j\omega}) = \text{odd w.r.t. } \omega$$

- Conjugation

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(e^{j\omega}) \qquad \Rightarrow \quad \overline{x[n]} \overset{\text{DTFT}}{\longmapsto} \overline{\widehat{x}(e^{-j\omega})}$$

$$\sum_{m\in\mathbb{Z}} \overline{x[m]}\, e^{-j\omega m} = \overline{\sum_{m\in\mathbb{Z}} x[m]\, e^{j\omega m}} = \overline{\widehat{x}(e^{-j\omega})}$$

$$x[n] = \text{real-valued} \;\Rightarrow\; \widehat{x}(e^{-j\omega}) = \overline{\widehat{x}(e^{j\omega})} \qquad\qquad x[n] = \begin{array}{c}\text{real-valued}\\ \&\\ \text{even}\end{array} \;\Rightarrow\; \widehat{x}(e^{j\omega}) = \begin{array}{c}\text{real-valued}\\ \&\\ \text{even}\end{array}$$

Properties of Discrete-domain FT

$$x[n] = \text{real-valued}, \quad x[n] = \underbrace{\mathcal{E}v\{x[n]\}}_{x_e[n]} + \underbrace{\mathcal{O}d\{x[n]\}}_{x_o[n]}$$

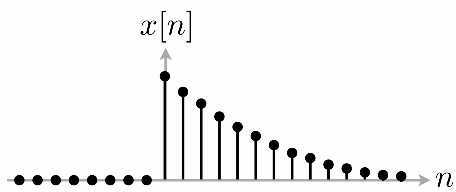
$$\Rightarrow \quad \widehat{x}(e^{j\omega}) = \widehat{x_e}(e^{j\omega}) + \widehat{x_o}(e^{j\omega})$$

real-valued

purely imaginary

- Example

$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$
$$\alpha \in \mathbb{R}$$



$$\mathcal{E}v\{x[n]\} = \tfrac{1}{2}(\delta[n] + \alpha^{|n|}) \overset{\text{DTFT}}{\longmapsto} \Re\{\widehat{x}(e^{j\omega})\} = \Re\{\frac{1}{1-\alpha e^{-j\omega}}\} = \frac{1-\alpha\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}$$

Properties of Discrete-domain FT

- Differencing

$$x[n] \xrightarrow{\text{DTFT}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\Rightarrow \underbrace{x[n]-x[n-1]}_{\text{difference}} \xrightarrow{\text{DTFT}} (1-\mathrm{e}^{-\mathrm{j}\omega})\,\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

equivalent of $\mathrm{j}\omega$

- Accumulation

$$x[n] \xrightarrow{\text{DTFT}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\Rightarrow \sum_{m=-\infty}^{n} x[m] \xrightarrow{\text{DTFT}} \left(\frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)\right) \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

inverse of differencing

DC compensation

Properties of Discrete-domain FT

- Accumulation property

$$\forall\,|\alpha|<1,\quad z_\alpha[n] \triangleq x[n] - \tfrac{1-\alpha}{1+\alpha}\,\widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\alpha^{|n|} \quad\Rightarrow\quad \widehat{z}_\alpha(\mathrm{e}^{\mathrm{j}\omega}) = \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) - \widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\tfrac{(1-\alpha)^2}{1+\alpha^2-2\alpha\cos(\omega)}$$

$$\sum_{-\infty}^{n} z_\alpha[m] - \sum_{-\infty}^{n-1} z_\alpha[m] = z_\alpha[n] \;\Rightarrow\; (1-\mathrm{e}^{-\mathrm{j}\omega})\,\mathrm{DTFT}\left\{\sum_{-\infty}^{n} z_\alpha[m]\right\}(\mathrm{e}^{\mathrm{j}\omega}) = \underbrace{\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) - \widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\tfrac{(1-\alpha)^2}{1+\alpha^2-2\alpha\cos(\omega)}}_{\text{at } \omega=0 \text{ equals } 0}$$

$$\Rightarrow\; \mathrm{DTFT}\left\{\sum_{-\infty}^{n} z_\alpha[m]\right\}(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) - \widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\frac{(1-\alpha)^2}{1+\alpha^2-2\alpha\cos(\omega)}}{1-\mathrm{e}^{-\mathrm{j}\omega}}$$

$$\sum_{-\infty}^{n} z_\alpha[m] = \sum_{-\infty}^{n} x[m] - \tfrac{\widehat{x}(\mathrm{e}^{\mathrm{j}0})}{1+\alpha} - \widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\tfrac{\mathrm{sign}(n)\,\alpha^{u[n]}}{1+\alpha}\,(1-\alpha^{|n|})$$

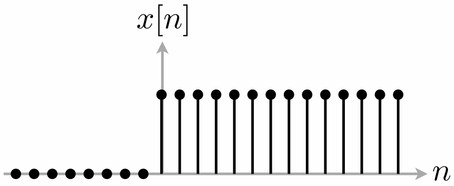
$$\Rightarrow\; \mathrm{DTFT}\left\{\sum_{-\infty}^{n} x[m]\right\} = \frac{\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) - \widehat{x}(\mathrm{e}^{\mathrm{j}0})\frac{(1-\alpha)^2}{1+\alpha^2-2\alpha\cos(\omega)}}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \tfrac{2\pi}{1+\alpha}\,\widehat{x}(\mathrm{e}^{\mathrm{j}0})\,\mathrm{III}_{2\pi}(\omega)$$

$$+\;\tfrac{\widehat{x}(\mathrm{e}^{\mathrm{j}0})}{1+\alpha}\,\mathrm{DTFT}\left\{\tfrac{\mathrm{sign}(n)\,\alpha^{u[n]}}{1+\alpha}(1-\alpha^{|n|})\right\}(\mathrm{e}^{\mathrm{j}\omega}) \qquad \alpha \to 1 \;\Rightarrow\; ✓$$

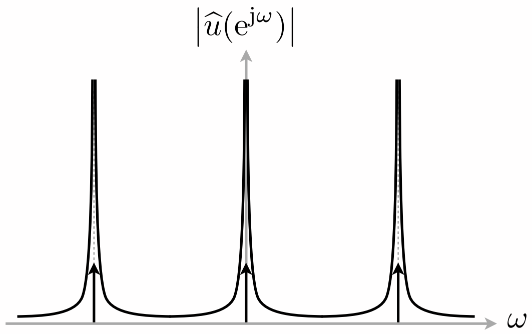
Properties of Discrete-domain FT

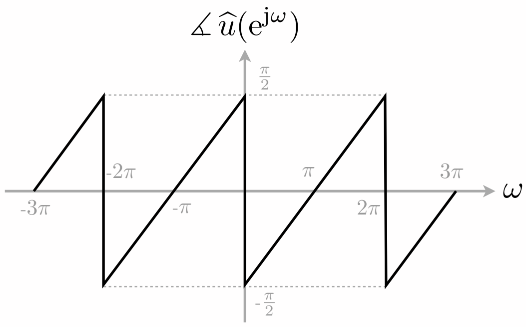
- Example

$$x[n] = u[n] \quad \Rightarrow \quad \widehat{x}(e^{j\omega}) = ?$$



$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \quad \Rightarrow \quad \widehat{u}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$





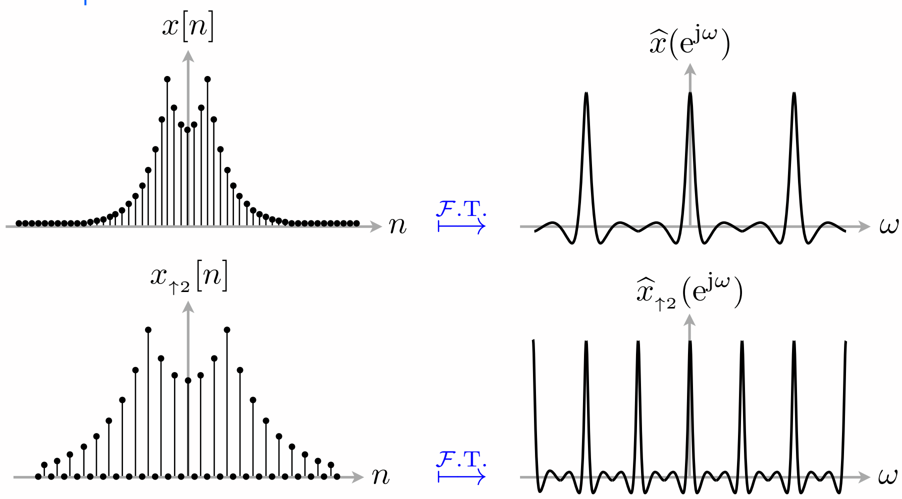
Properties of Discrete-domain FT

- Dilation (Time-Scaling)

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(e^{j\omega})$$

$$M \in \mathbb{N}^+, \quad x_{\uparrow M}[n] \triangleq \begin{cases} x[n/M] & n \in M\,\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \quad \Rightarrow \quad x_{\uparrow M}[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}_{\uparrow M}(e^{j\omega}) = \widehat{x}(e^{j\omega M})$$

up-sample



Properties of Discrete-domain FT

- Frequency Differentiation

$$x[n] \xmapsto{\text{DTFT}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\Rightarrow \boxed{\text{-j}n\, x[n] \xmapsto{\text{DTFT}} \tfrac{\mathrm{d}}{\mathrm{d}\omega}\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}\omega}\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = \tfrac{\mathrm{d}}{\mathrm{d}\omega}\sum_{m\in\mathbb{Z}} x[m]\mathrm{e}^{\text{-j}\omega m} = \sum_{m\in\mathbb{Z}} (\text{-j}\, m\, x[m])\mathrm{e}^{\text{-j}\omega m}$$

- Parseval's Theorem

$$x[n] \xmapsto{\text{DTFT}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

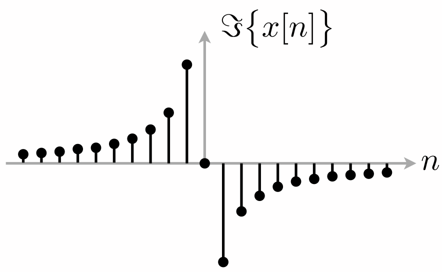
$$\Rightarrow \boxed{\sum_{n\in\mathbb{Z}} \big|x[n]\big|^2 = \tfrac{1}{2\pi}\int_{2\pi} \big|\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\big|^2 \mathrm{d}\omega}$$

$$\int_{2\pi} \big|\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\big|^2 \mathrm{d}\omega = \sum_{m,n\in\mathbb{Z}} x[n]\,\overline{x[m]}\underbrace{\int_{2\pi} \mathrm{e}^{\mathrm{j}\omega(m-n)}\mathrm{d}\omega}_{2\pi\delta[n-m]} = 2\pi\sum_{n\in\mathbb{Z}}\big|x[n]\big|^2$$

Properties of Discrete-domain FT

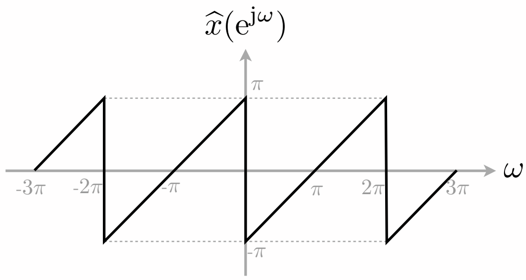
- Example

$$x[n] = \frac{u\big[|n|-1\big]}{\mathrm{j}n} \quad\Rightarrow\quad \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = ?$$



$$\text{-j}n\, x[n] = \text{-}\,u\big[|n|-1\big] = \delta[n]-1 \quad\Rightarrow\quad \tfrac{\mathrm{d}}{\mathrm{d}\omega}\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = 1 - 2\pi\sum_{k\in\mathbb{Z}}\delta(\omega-2\pi k)$$

$\xRightarrow[\text{is odd}]{x[\cdot]}$



$$\xRightarrow{\text{Parseval}} \quad 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \tfrac{1}{2\pi}\int_0^{2\pi}(\omega-\pi)^2\mathrm{d}\omega = \tfrac{\pi^2}{3}$$

Properties of Discrete-domain FT

- Convolution

$$x[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$z[n] \overset{\text{DTFT}}{\longmapsto} \widehat{z}(\mathrm{e}^{\mathrm{j}\omega})$$

$$(x * z)[n] \overset{\text{DTFT}}{\longmapsto} \; ?$$

$$y[n] \triangleq (x * z)[n] = \sum_{m\in\mathbb{Z}} x[n-m]z[m] \quad \Rightarrow \quad \widehat{y}(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{m,n\in\mathbb{Z}} x[n-m]z[m]\mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \sum_{m\in\mathbb{Z}} z[m] \underbrace{\Big(\sum_{n\in\mathbb{Z}} x[n-m]\mathrm{e}^{-\mathrm{j}\omega n}\Big)}_{\mathrm{e}^{-\mathrm{j}\omega m}\,\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})} = \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \sum_{m\in\mathbb{Z}} z[m]\,\mathrm{e}^{-\mathrm{j}\omega m} = \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\,\widehat{z}(\mathrm{e}^{\mathrm{j}\omega})$$

$$(x * z)[n] \overset{\text{DTFT}}{\longmapsto} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega})\,\widehat{z}(\mathrm{e}^{\mathrm{j}\omega})$$

Properties of Discrete-domain FT

- Example

frequency response پاسخ فرکانسی

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \quad \Rightarrow \quad \widehat{y}(\mathrm{e}^{\mathrm{j}\omega}) = \widehat{h}(\mathrm{e}^{\mathrm{j}\omega})\,\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})$$

$$\longrightarrow \boxed{\text{Diff.}} \longrightarrow$$

$$\widehat{h}(\omega) = 1 - \mathrm{e}^{-\mathrm{j}\omega}$$

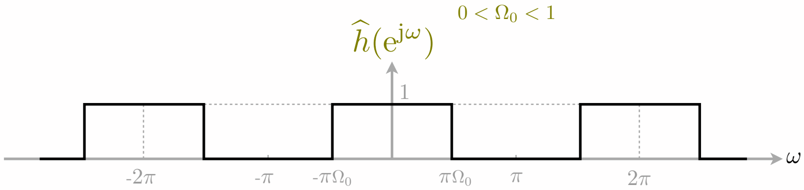
$$\longrightarrow \boxed{\sum_{-\infty}^{n}} \longrightarrow$$

$$\widehat{h}(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \pi \sum_{k\in\mathbb{Z}} \delta(\omega - 2\pi k)$$

ideal lowpass filter

$$\longrightarrow \boxed{h[n] = \Omega_0\,\mathrm{sinc}(\Omega_0 n)} \longrightarrow$$

$0 < \Omega_0 < 1$



Properties of Discrete-domain FT

- Multiplication

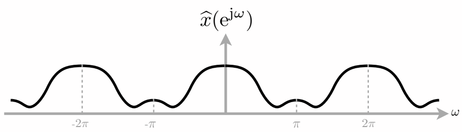
circular convolution

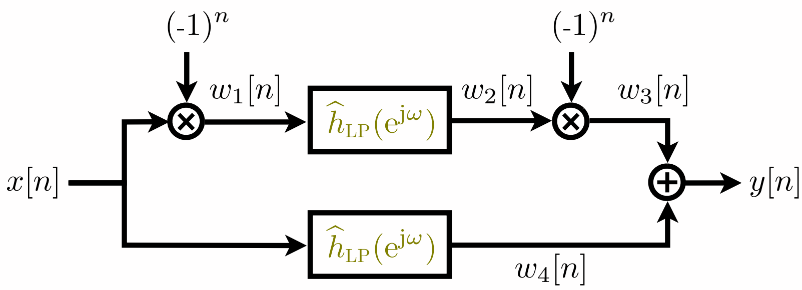
$$\left\{\begin{array}{lcl} x[n] & \stackrel{\text{DTFT}}{\longmapsto} & \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \\ z[n] & \stackrel{\text{DTFT}}{\longmapsto} & \widehat{z}(\mathrm{e}^{\mathrm{j}\omega}) \end{array}\right. \Rightarrow x[n]\,z[n] \stackrel{\text{DTFT}}{\longmapsto} \frac{1}{2\pi}\big(\widehat{x} \overset{2\pi}{\circledast} \widehat{z}\big)(\mathrm{e}^{\mathrm{j}\omega}) \triangleq \frac{1}{2\pi}\int_{2\pi} \widehat{x}(\mathrm{e}^{\mathrm{j}(\omega-\theta)})\,\widehat{z}(\mathrm{e}^{\mathrm{j}\theta})\,\mathrm{d}\theta$$

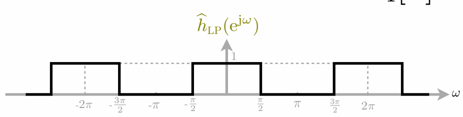
$$\begin{aligned}\int_{2\pi} \widehat{x}(\mathrm{e}^{\mathrm{j}(\omega-\theta)})\,\widehat{z}(\mathrm{e}^{\mathrm{j}\theta})\,\mathrm{d}\theta &= \int_{2\pi} \Big(\sum_{n\in\mathbb{Z}} x[n]\mathrm{e}^{-\mathrm{j}(\omega-\theta)n}\Big)\Big(\sum_{m\in\mathbb{Z}} z[m]\mathrm{e}^{-\mathrm{j}\theta m}\Big)\,\mathrm{d}\theta \\ &= \sum_{n,m\in\mathbb{Z}} x[n]\,z[m]\,\mathrm{e}^{-\mathrm{j}\omega n}\Big(\underbrace{\int_{2\pi} \mathrm{e}^{\mathrm{j}(n-m)\theta}\mathrm{d}\theta}_{2\pi\delta[n-m]}\Big) \\ &= 2\pi\sum_{n\in\mathbb{Z}} x[n]\,z[n]\,\mathrm{e}^{-\mathrm{j}\omega n}\end{aligned}$$

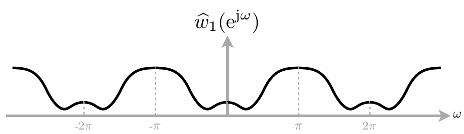
Properties of Discrete-domain FT

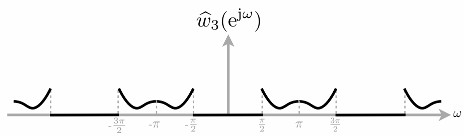
- Example

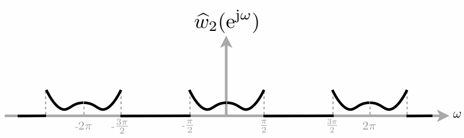


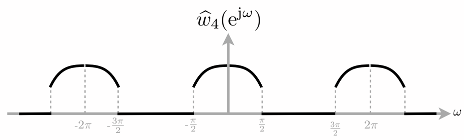












Properties of Discrete-domain FT

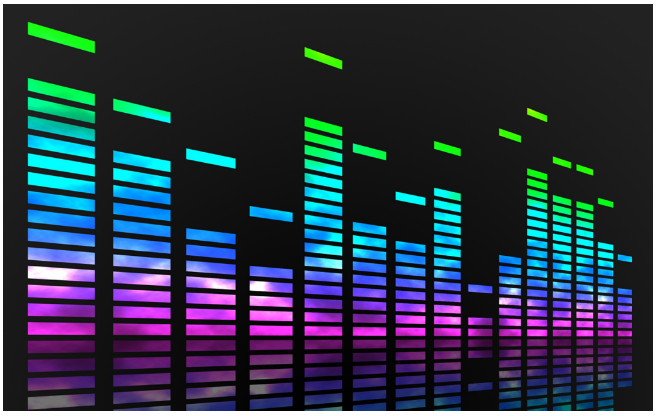
- Duality

$$x(t) \xrightarrow{\mathcal{F}.\text{T.}} \widehat{x}(\omega) \xrightarrow{\mathcal{F}.\text{T.}} 2\pi\, x(\text{-}t)$$

$$x[n] \xrightarrow{\text{DTFT}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \cancel{\xrightarrow{\text{DTFT}} 2\pi\, x[\text{-}n]}$$

$$x[n] \xrightarrow{\text{DTFT}} \underbrace{\widehat{x}(\mathrm{e}^{\mathrm{j}\omega})}_{2\pi\text{-periodic}} \xrightarrow{\mathcal{F}.\text{S.}} x[\text{-}n]$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \mathrm{e}^{\mathrm{j}\omega n} \mathrm{d}\omega \qquad a_k = \frac{1}{2\pi} \int_{\substack{\text{one}\\ \text{period}}} \widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) \mathrm{e}^{\text{-}\mathrm{j}\omega k} \mathrm{d}\omega$$



Discrete-domain Fourier Transform

- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of difference equations

LSI Difference Systems

- Linear constant-coefficient difference systems

$$x[n] \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k y[n-k] = \sum_{k=0}^{N_x} b_k x[n-k]} \longrightarrow y[n]$$

$$\text{proper boundary conditions} \;\Rightarrow\; \text{LSI} \;\Rightarrow\; \widehat{y}(e^{j\omega}) = \underbrace{\widehat{h}(e^{j\omega})}_{?}\, \widehat{x}(e^{j\omega})$$

$$\sum_{k=0}^{N_y} a_k y[n-k] = \sum_{k=0}^{N_x} b_k x[n-k] \quad \overset{\text{DTFT}}{\longmapsto} \quad \sum_{k=0}^{N_y} a_k e^{-j\omega k}\, \widehat{y}(e^{j\omega}) = \sum_{k=0}^{N_x} b_k e^{-j\omega k}\, \widehat{x}(e^{j\omega})$$

$$\Rightarrow \quad \widehat{h}(e^{j\omega}) = \frac{\widehat{y}(e^{j\omega})}{\widehat{x}(e^{j\omega})} = \frac{\sum_{k=0}^{N_x} b_k e^{-j\omega k}}{\sum_{k=0}^{N_y} a_k e^{-j\omega k}}$$

LSI Difference Systems

- Example

$$x[n] \longrightarrow \boxed{6y[n] + y[n-1] - y[n-2] = x[n]} \longrightarrow y[n]$$

initial rest

frequency response

$$\Rightarrow \quad \widehat{h}(e^{j\omega}) = \frac{\widehat{y}(e^{j\omega})}{\widehat{x}(e^{j\omega})} = \frac{1}{6 + e^{-j\omega} - e^{-j2\omega}}$$

$$= \frac{\frac{1}{10}}{1 + \frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{15}}{1 - \frac{1}{3}e^{-j\omega}}$$

impulse response

$$\Rightarrow \quad h[n] = \left(\tfrac{1}{10}(\tfrac{-1}{2})^n + \tfrac{1}{15}(\tfrac{1}{3})^n\right)u[n]$$

$$= \tfrac{1}{5}\left((\tfrac{1}{3})^{n+1} - (\tfrac{-1}{2})^{n+1}\right)u[n]$$

step response

$$x[n] = u[n] \quad \Rightarrow \quad \widehat{y}(e^{j\omega}) = \widehat{h}(e^{j\omega})\widehat{x}(e^{j\omega}) = \tfrac{1}{6+e^{-j\omega}-e^{-j2\omega}}\left(\tfrac{1}{1-e^{-j\omega}} + \pi\textstyle\sum_{k\in\mathbb{Z}}\delta(\omega - 2\pi k)\right)$$

$$= \tfrac{1}{30}\left(\tfrac{1}{1+\frac{1}{2}e^{-j\omega}} - \tfrac{1}{1-\frac{1}{3}e^{-j\omega}} + \tfrac{5}{1-e^{-j\omega}}\right) + \tfrac{\pi}{6}\textstyle\sum_{k\in\mathbb{Z}}\delta(\omega - 2\pi k)$$

$$\Rightarrow \quad y[n] = \tfrac{1}{30}\left((\tfrac{-1}{2})^n u[n] - (\tfrac{1}{3})^n u[n]\right) + \tfrac{1}{6}u[n]$$

$$= \tfrac{1}{30}\left((\tfrac{-1}{2})^n - (\tfrac{1}{3})^n + 5\right)u[n]$$