Sampling

- Sampling Theorem
- Aliasing
- Digital Processing

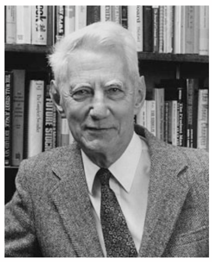
Theory of Sampling

Edmund T. Whittaker
1873-1956

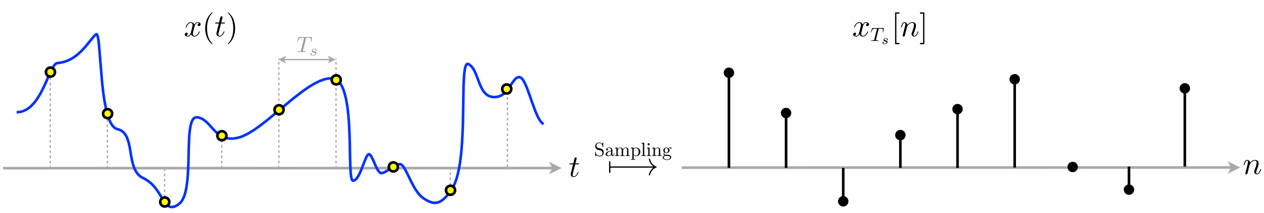
Harry Nyquist
1889-1976

Vladimir Kotelnikov
1908-2005

Denis Gabor
1900-1979

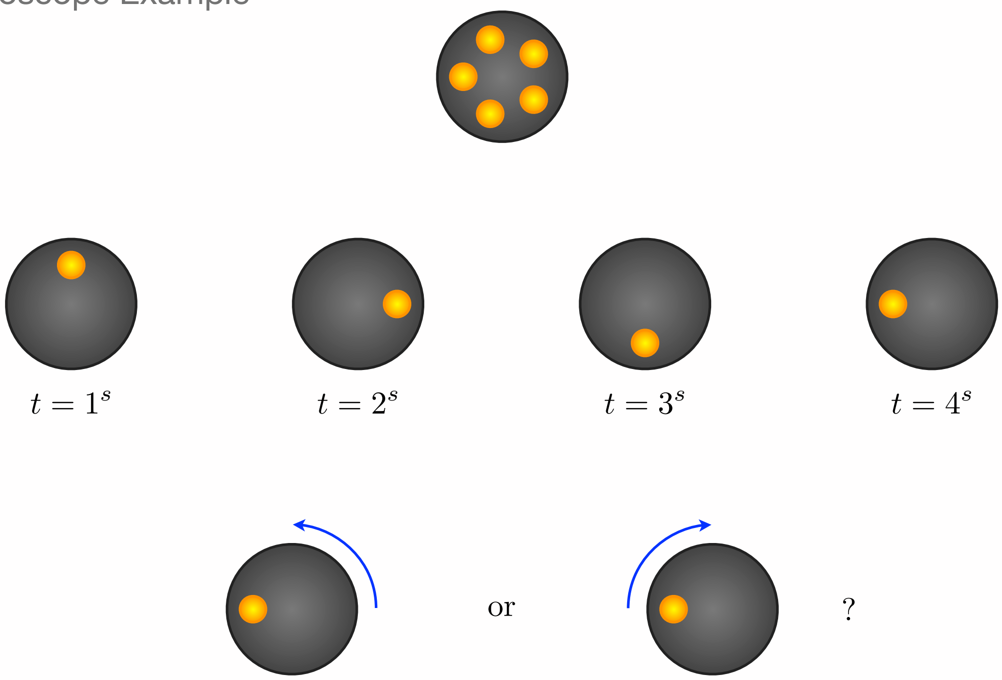


Claude Shannon
1916-2001



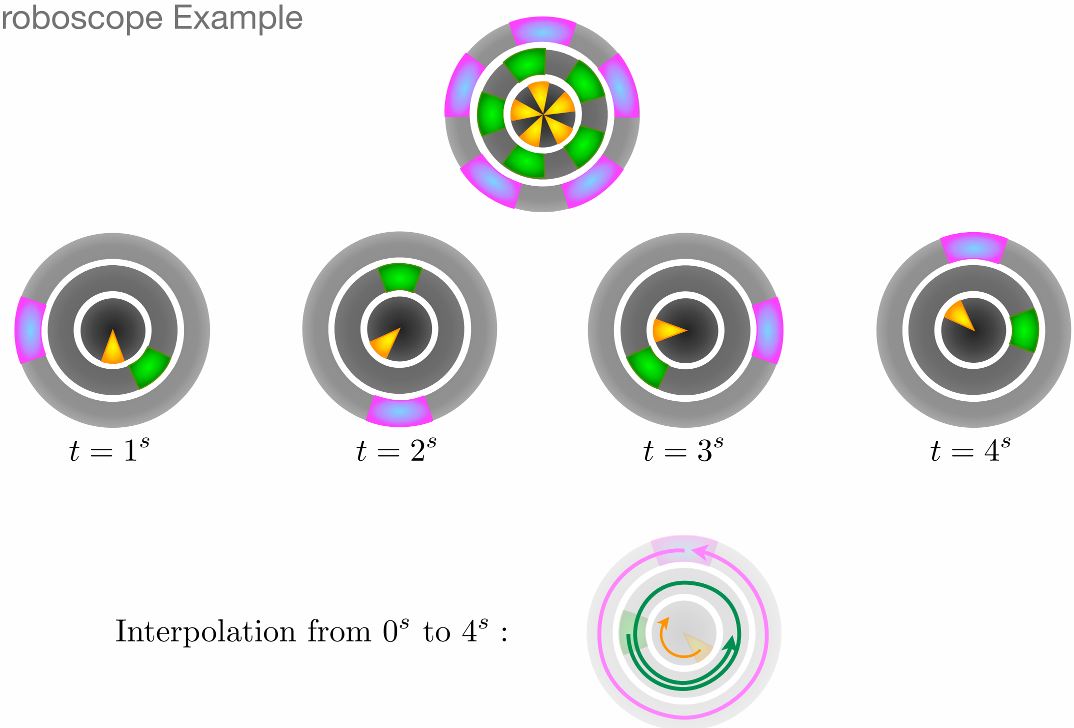
Sampling Dilemma

- Stroboscope Example



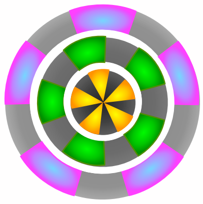
Sampling Dilemma

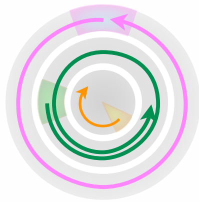
- Stroboscope Example



Sampling Rate

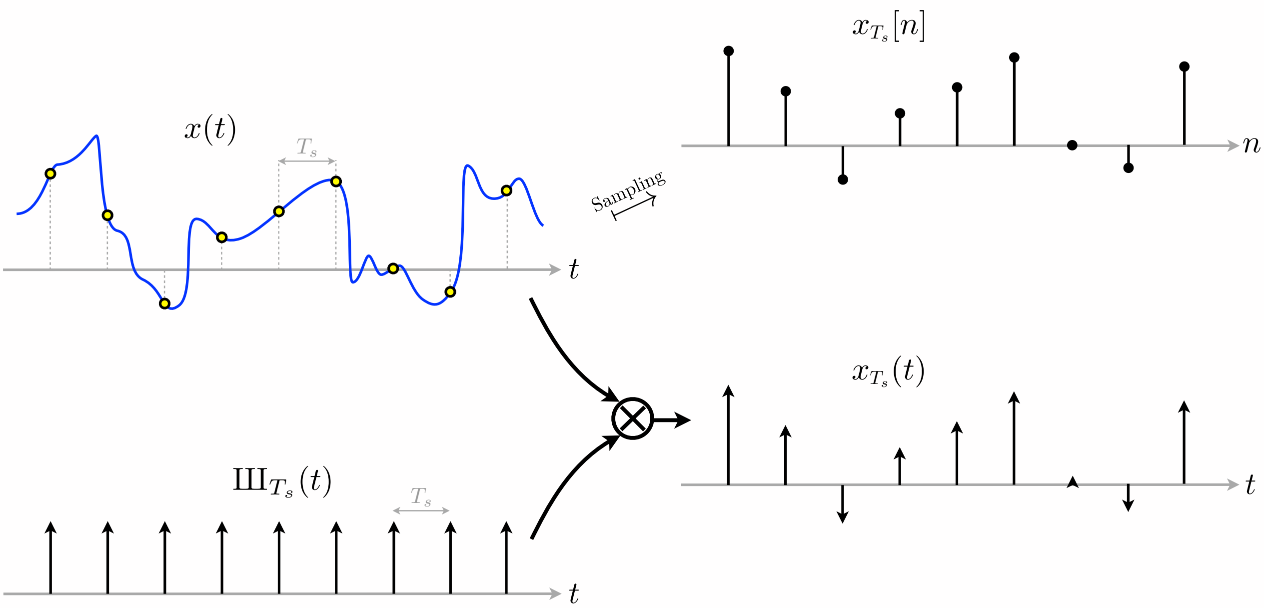
- Stroboscope Example





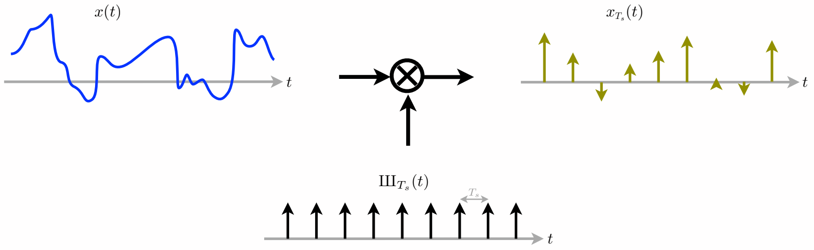
$$\text{Interpolation} = \text{exact} \quad \Leftrightarrow \quad \text{sampling rate} > 2 \times (\text{max. frequency})$$

Impulse-train Sampling

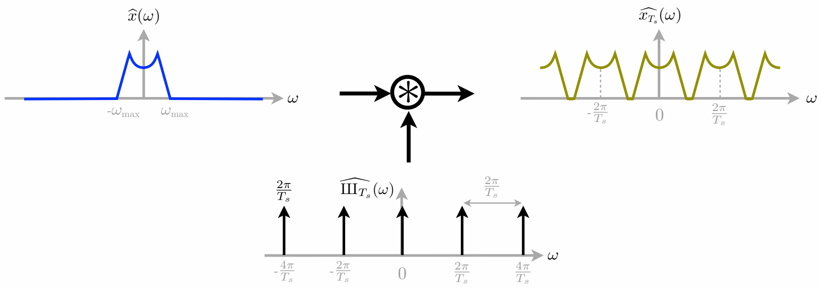


$$\Rightarrow \quad x_{T_s}(t) = \sum_{m\in\mathbb{Z}} x_{T_s}[m]\,\delta(t - mT_s) \quad \Rightarrow \quad \widehat{x_{T_s}}(\omega) = \sum_{m\in\mathbb{Z}} x_{T_s}[m]\mathrm{e}^{-\mathrm{j}\omega m T_s} = \widehat{x_{T_s}}(\mathrm{e}^{\mathrm{j}\omega T_s})$$

Impulse-train Sampling

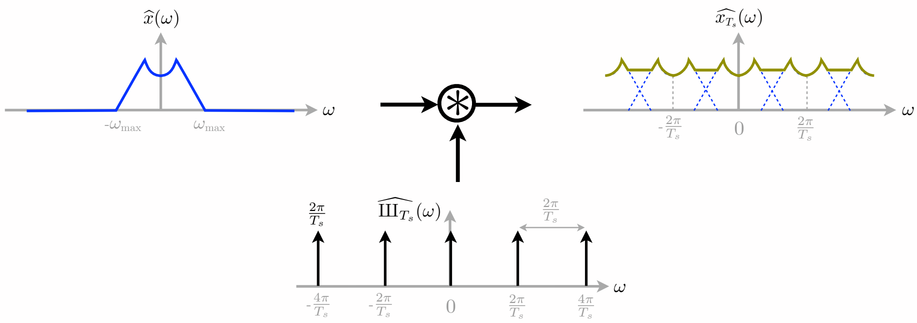


$$x_{T_s}(t) = x(t)\,\mathrm{III}_{T_s}(t) \quad \Rightarrow \quad \widehat{x_{T_s}}(\omega) = \frac{1}{2\pi}\Big(\widehat{x} * \underbrace{\widehat{\mathrm{III}_{T_s}}}_{\frac{2\pi}{T_s}\mathrm{III}_{\frac{2\pi}{T_s}}}\Big)(\omega) = \frac{1}{T_s}\sum_{k\in\mathbb{Z}} \widehat{x}\Big(\omega - \frac{2\pi}{T_s}k\Big)$$

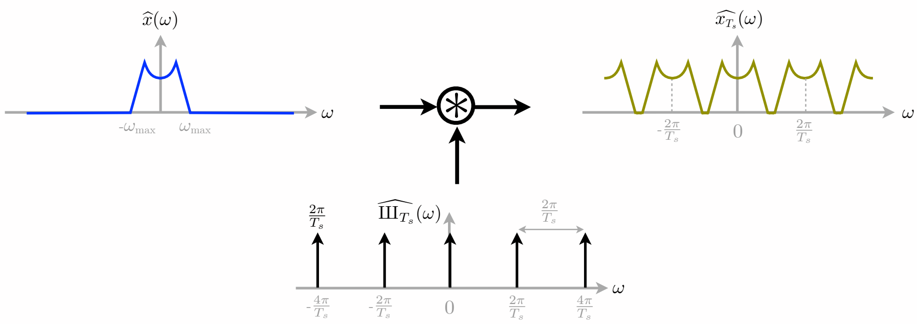


Impulse-train Sampling

Case 1: $\omega_{\max} \geq \frac{\pi}{T_s}$

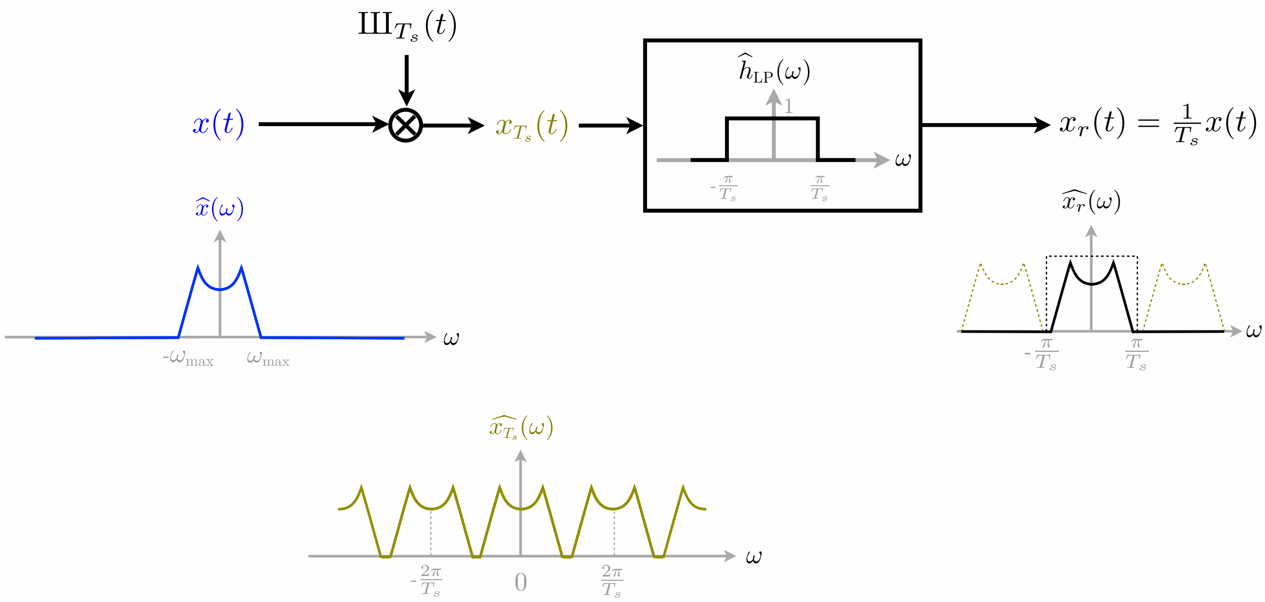


Case 2: $\omega_{\max} < \frac{\pi}{T_s}$



Impulse-train Sampling

Case 2: $\omega_{\max} < \frac{\pi}{T_s}$



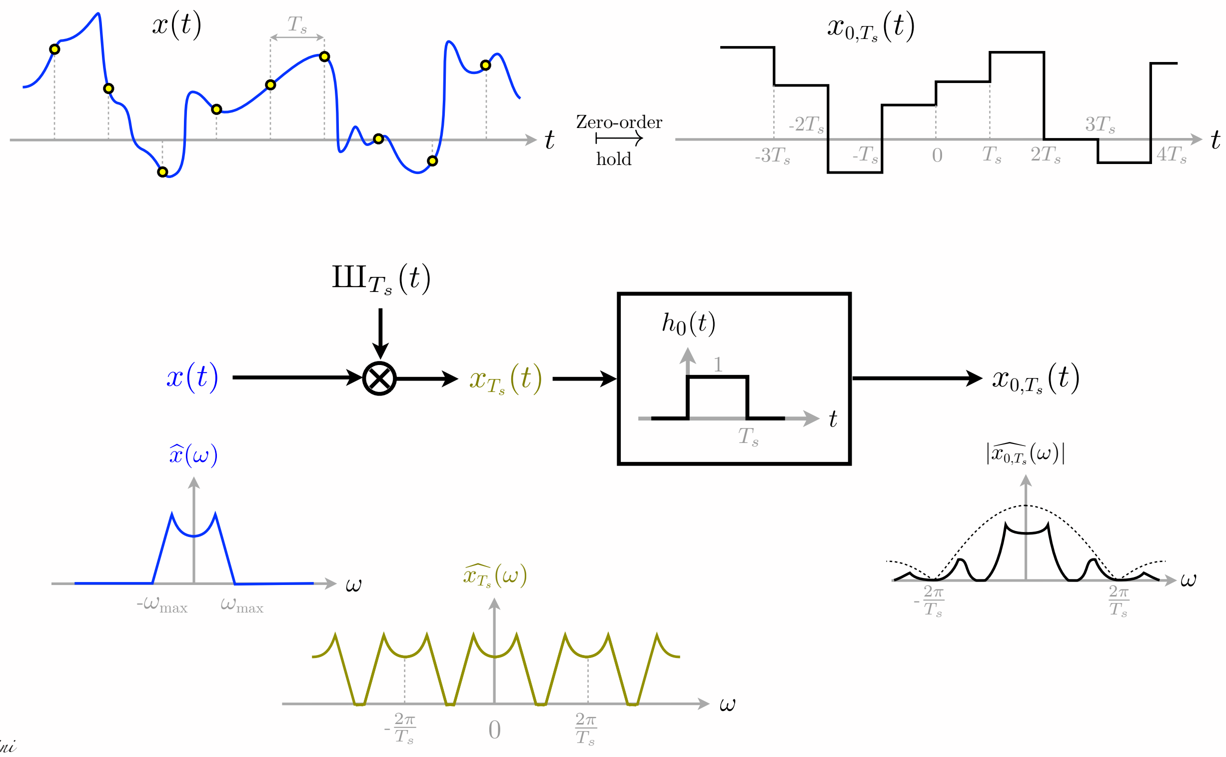
Sampling Theorem

$$\begin{cases} x(t) & = & \text{band-limited with } \omega_{\max}, \\ T_s & = & \text{Sampling period}, \\ \omega_s & \triangleq & \frac{2\pi}{T_s}, \\ \omega_s & > & 2\omega_{\max} \end{cases}$$

$\Rightarrow$ $x(t)$ is uniquely determined by its samples $\{x(nT_s)\}_{n\in\mathbb{Z}}$

$$x(t) = \sum_{m\in\mathbb{Z}} x(mT_s)\,\text{sinc}\left(\frac{t}{T_s} - m\right)$$

Zero-order Hold



Linear Interpolator

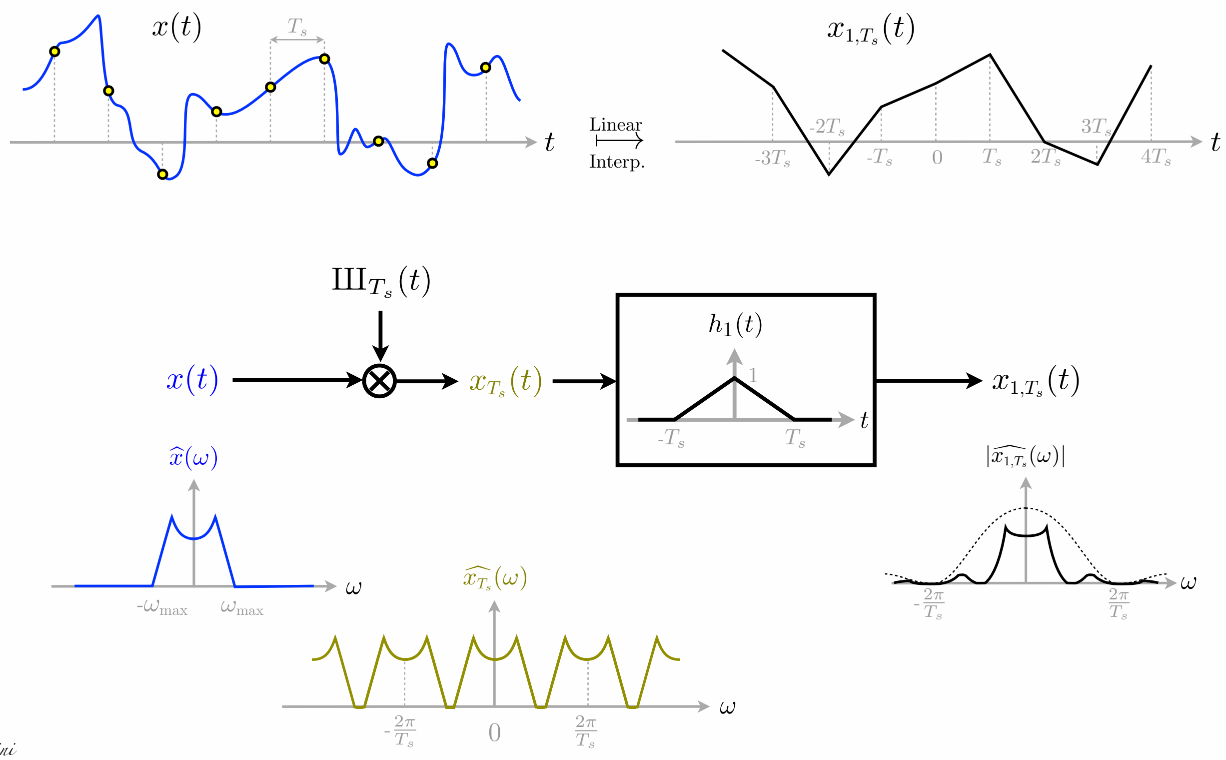


Image Enlargement

Original Image: 

?

Zero-order Hold
4x

Linear Interpolator
4x

Sinc Interpolator
4x

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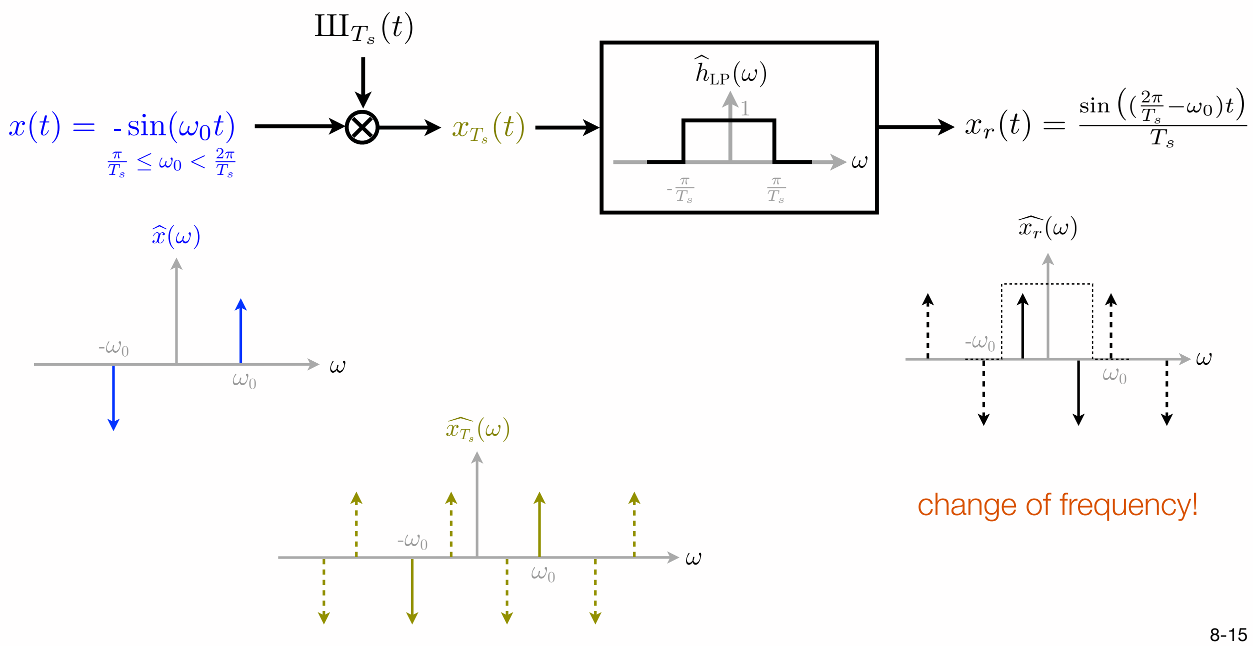
Aliasing

- Sampling Below Nyquist rate $\frac{1}{T_s} \le \boxed{\frac{\omega_{\max}}{\pi}}$

Nyquist rate

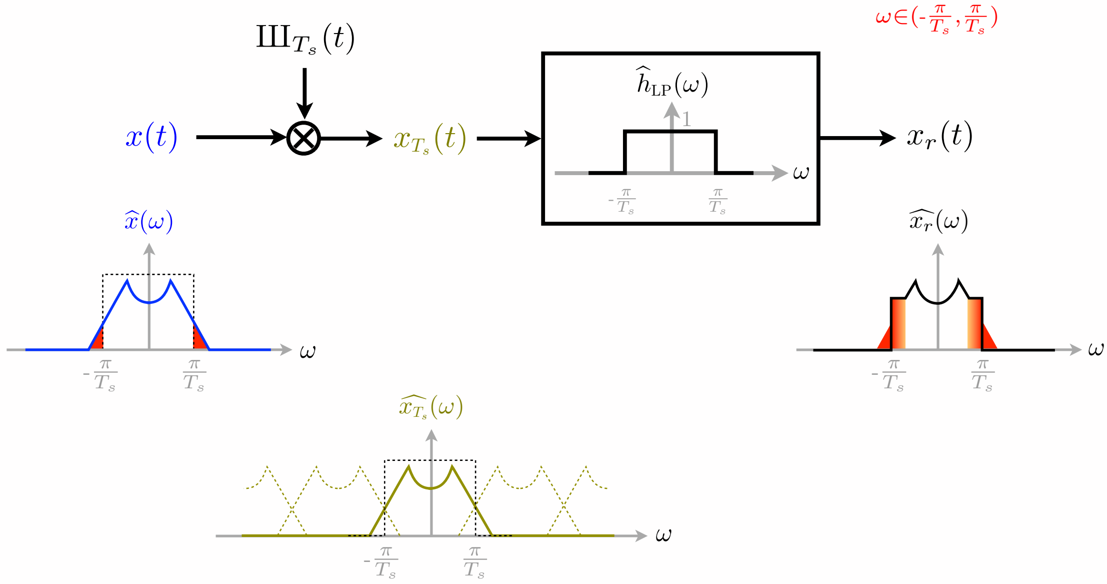


change of frequency!

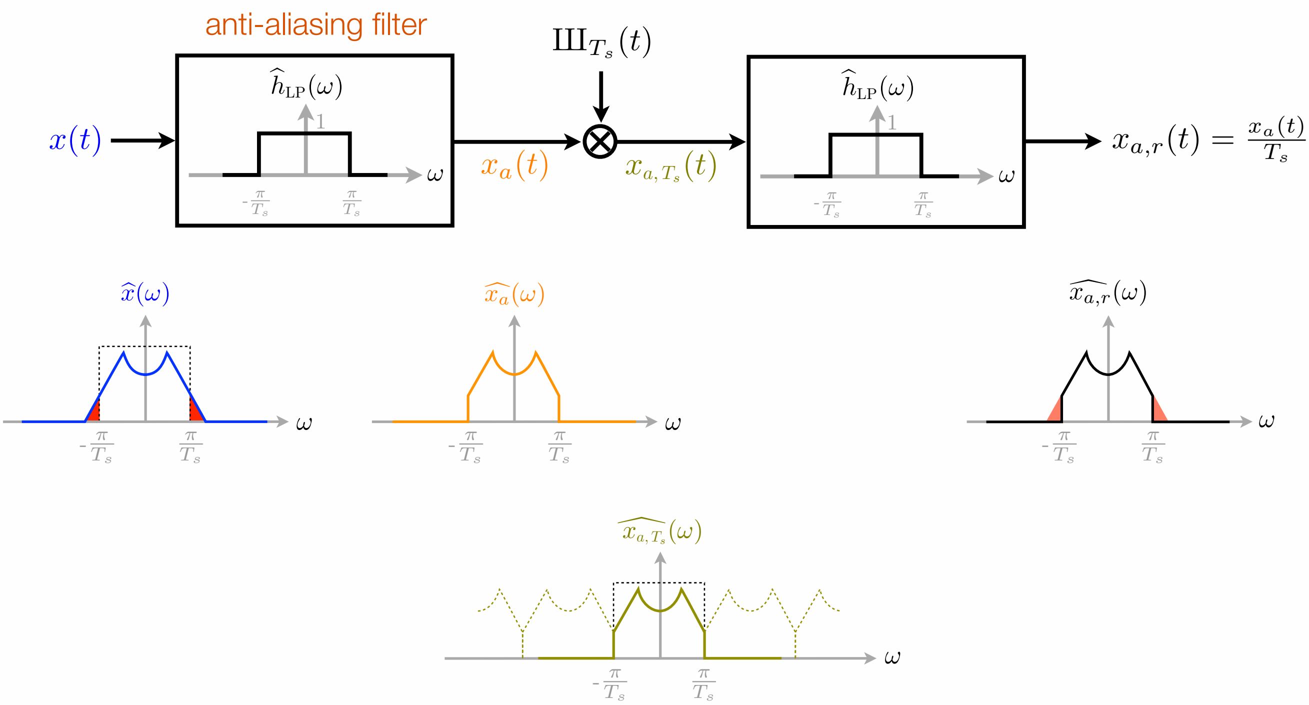
Aliasing

- Sampling Below Nyquist rate $\frac{1}{T_s} \leq \frac{\omega_{\max}}{\pi}$

$\underbrace{\widehat{x_r} \equiv \widehat{x}}_{\omega \in (-\frac{\pi}{T_s}, \frac{\pi}{T_s})}$? No!



Anti-Aliasing Filter



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Digital Systems

Digital Systems

- Example: Differentiator