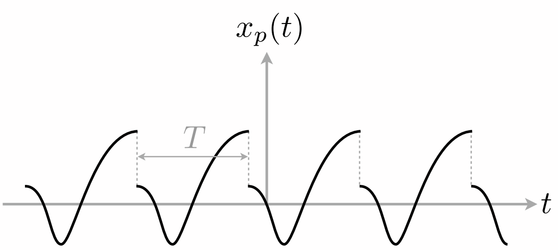
Continuous-domain Fourier Transform

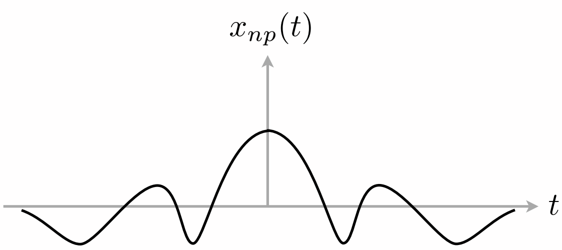
- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of differential equations

Non-periodic Signals

Jean B. J. Fourier
1768-1830

Non-periodic signals are periodic with $T = \infty$!!!

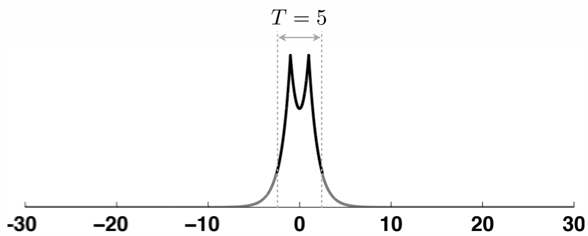


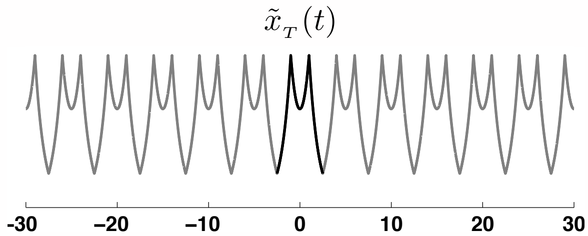


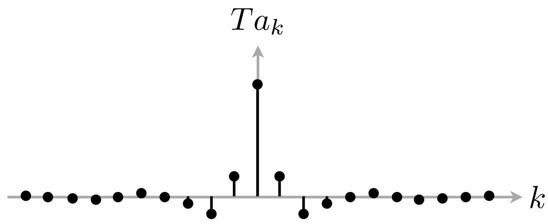
Periodized signals

- Non-periodic continuous-domain signal

$$x(t) = \frac{\mathrm{e}^{-|t-1|} + \mathrm{e}^{-|t+1|}}{2}$$







$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau$$

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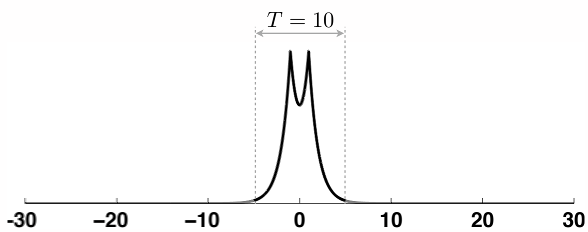

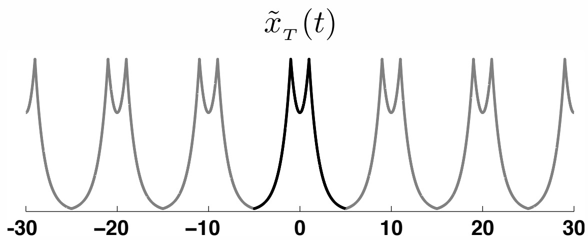

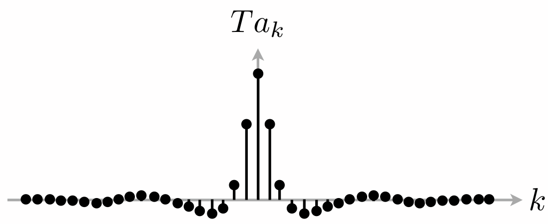

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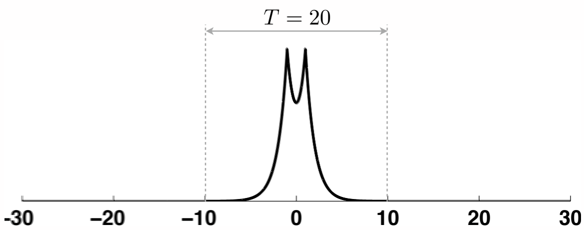

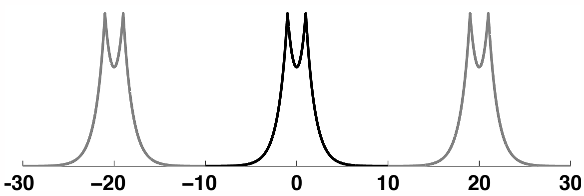

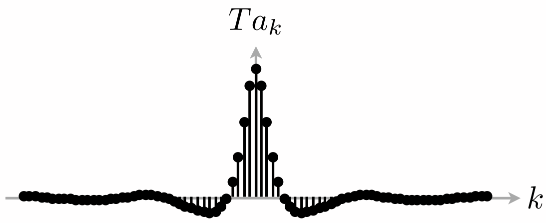

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Periodized signals

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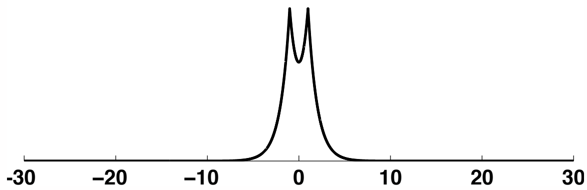
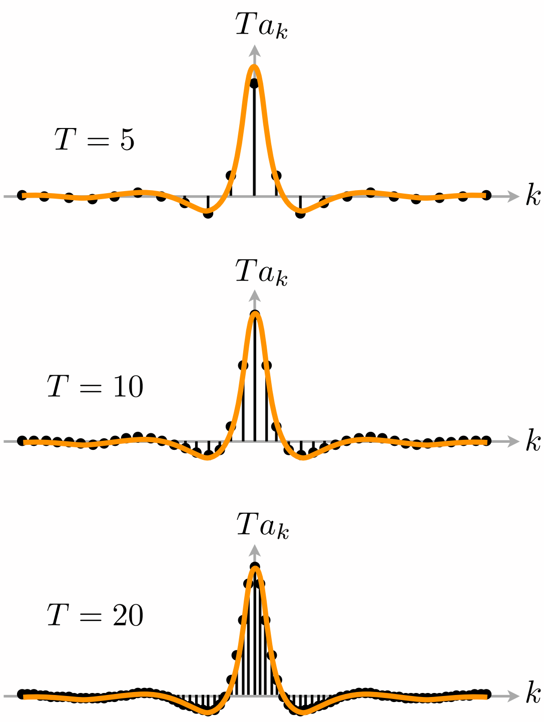




Fourier Transform (تبدیل فوریه)

$X(\mathrm{j}\omega)$

$\widehat{x}(\mathrm{j}\omega)$

$$\boxed{\widehat{x}(\omega) \triangleq \mathcal{F}\{x(t)\}(\omega) \triangleq \int_{-\infty}^{\infty} x(\tau)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau \triangleq \lim_{T\to\infty}\int_{-T}^{T} x(\tau)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau}$$

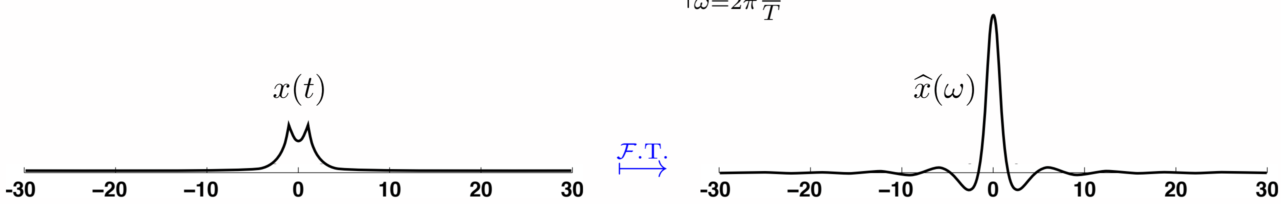
$x(t)$: decaying,

$\tilde{x}_T(t)$: T-periodized version of $x(t)$,

a_k : $\mathcal{F}$.S. of $\tilde{x}_T(t)$

$$\Rightarrow \quad T a_k = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau \overset{T\gg 1}{\approx} \int_{-\infty}^{\infty} x(\tau)\,\mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\,\mathrm{d}\tau$$

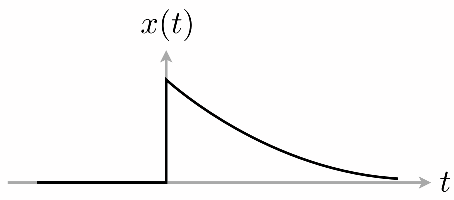
$$= \widehat{x}(\omega)\Big|_{\omega=2\pi\frac{k}{T}}$$



Fourier Transform (تبدیل فوریه)

- Example

$$x(t) = \mathrm{e}^{-a\,t}u(t), \quad a \in \mathbb{R}^+$$

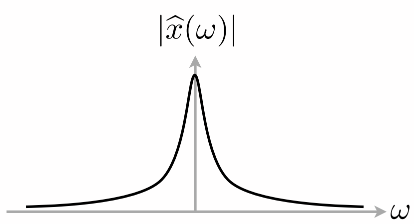


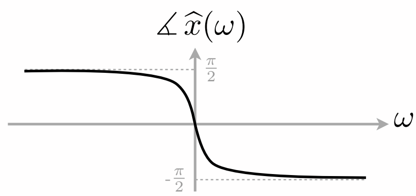
$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_0^{\infty} \mathrm{e}^{-a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$= -\frac{\mathrm{e}^{-(a+\mathrm{j}\omega)\tau}}{a+\mathrm{j}\omega}\Bigg|_{\tau=0}^{\infty}$$

$$\boxed{= \frac{1}{a+\mathrm{j}\omega}}$$

$$= \frac{1}{\sqrt{a^2+\omega^2}}\,\measuredangle\tan^{-1}\left(\frac{-\omega}{a}\right)$$





Fourier Transform (تبدیل فوریه)

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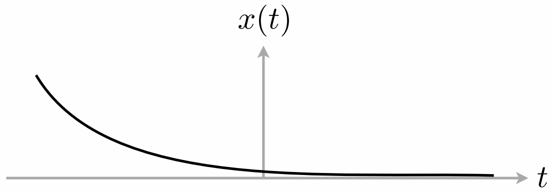
$$x(t) = \mathrm{e}^{-a\,t}, \quad a \in \mathbb{R}^+$$



$$\int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{\mathbb{R}} \mathrm{e}^{-a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

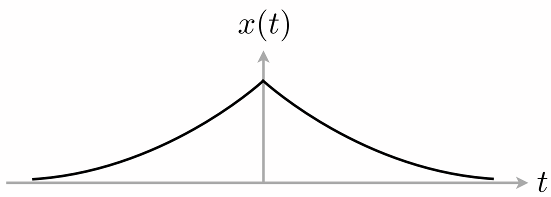
$$= \underbrace{\int_{-\infty}^{0} \mathrm{e}^{-a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau}_{\text{infinite}} + \underbrace{\int_{0}^{\infty} \mathrm{e}^{-a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau}_{\text{finite}}$$

$\Rightarrow$ no Fourier transform!!! ✘

Fourier Transform (تبدیل فوریه)

- Example

$$x(t) = \mathrm{e}^{-a\,|t|}, \quad a \in \mathbb{R}^+$$

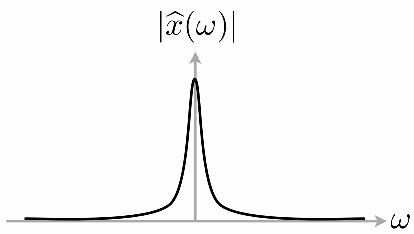


$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{\mathbb{R}} \mathrm{e}^{-a|\tau|}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$= \int_{-\infty}^{0} \mathrm{e}^{a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau + \int_{0}^{\infty} \mathrm{e}^{-a\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$= \frac{\mathrm{e}^{(a-\mathrm{j}\omega)\tau}}{a-\mathrm{j}\omega}\Big|_{-\infty}^{0} - \frac{\mathrm{e}^{-(a+\mathrm{j}\omega)\tau}}{a+\mathrm{j}\omega}\Big|_{0}^{\infty}$$

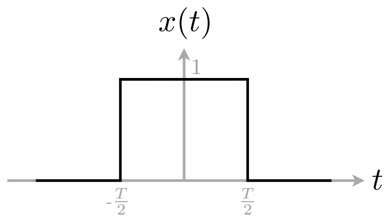
$$= \frac{1}{a-\mathrm{j}\omega} + \frac{1}{a+\mathrm{j}\omega} = \frac{2a}{a^2+\omega^2}$$



Fourier Transform (تبدیل فوریه)

- Example

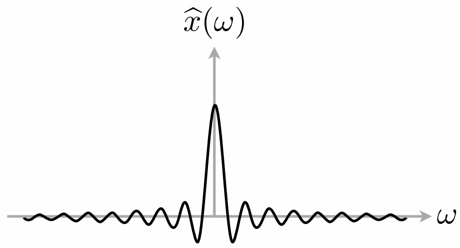
$$x(t) = \Pi\left(\tfrac{t}{T}\right) = \begin{cases} 1 & |\frac{t}{T}| < \frac{1}{2}, \\ \frac{1}{2} & |\frac{t}{T}| = \frac{1}{2}, \\ 0 & |\frac{t}{T}| > \frac{1}{2} \end{cases}$$



$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$= -\frac{\mathrm{e}^{-\mathrm{j}\omega\tau}}{\mathrm{j}\omega}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= T\,\frac{\sin(\frac{\omega T}{2})}{\frac{\omega T}{2}} = T\,\mathrm{sinc}(\tfrac{\omega T}{2\pi})$$

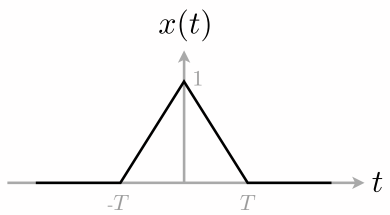


$$\mathrm{sinc}(t) \triangleq \begin{cases} \frac{\sin(\pi t)}{\pi t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Fourier Transform (تبدیل فوریه)

- Example

$$x(t) = \wedge(\tfrac{t}{T}) = \begin{cases} 1 - |\frac{t}{T}| & |\frac{t}{T}| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

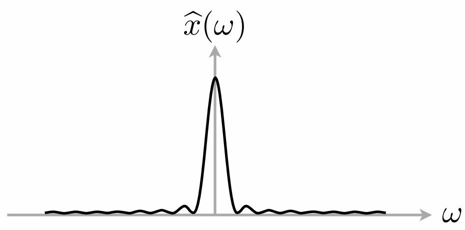


$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{-T}^{T} (1 - |\tfrac{\tau}{T}|)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

$$= 2\int_0^T (1 - \tfrac{\tau}{T})\cos(\omega\tau)\mathrm{d}\tau$$

$$= 2\frac{(T-\tau)\omega\sin(\omega\tau) - \cos(\omega\tau)}{T\omega^2}\Bigg|_0^T$$

$$= 2\frac{1-\cos(T\omega)}{T\omega^2} = T\,\mathrm{sinc}^2(\tfrac{\omega T}{2\pi})$$



Fourier Adjoint (عملگر الحاقی تبدیل فوریه)

$$\langle \mathcal{F}z\,,\,w\rangle = \langle z\,,\,\mathcal{F}^*w\rangle \quad\Rightarrow\quad \mathcal{F}^* = ?$$

$$\langle \mathcal{F}z\,,\,w\rangle = \int_{\mathbb{R}} \mathcal{F}_t\{z(t)\}(\omega)\,w(\omega)\mathrm{d}\omega = \int_{\mathbb{R}}\int_{\mathbb{R}} z(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t\;w(\omega)\mathrm{d}\omega$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} z(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,w(\omega)\,\mathrm{d}t\,\mathrm{d}\omega \overset{?}{=} \int_{\mathbb{R}} z(t)\underbrace{\int_{\mathbb{R}} w(\omega)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}\omega}_{\mathcal{F}_\omega\{w(\omega)\}(t)}\mathrm{d}t$$

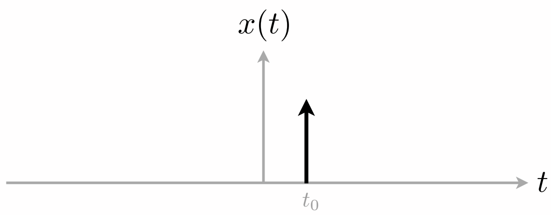
$$= \langle z(t)\,,\,(\mathcal{F}w)(t)\rangle \quad\Rightarrow\quad \mathcal{F}^* \equiv \mathcal{F}$$

Fourier transform is self-adjoint.

Fourier Transform (تبدیل فوریه)

- Example

$$x(t) = \delta(t - t_0)$$



$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{\mathbb{R}} \delta(\tau - t_0)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau = \mathrm{e}^{-\mathrm{j}\omega t_0}$$

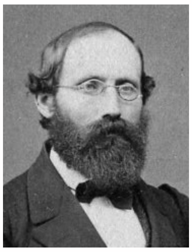
$$\langle z(\omega)\,,\,\widehat{x}(\omega)\rangle = \langle z(\omega)\,,\,\mathcal{F}\{\delta(t-t_0)\}(\omega)\rangle = \langle \mathcal{F}\{z(\omega)\}(t)\,,\,\delta(t-t_0)\rangle$$

$$= \mathcal{F}\{z(\omega)\}(t_0) = \int_{\mathbb{R}} z(\omega)\mathrm{e}^{-\mathrm{j}\omega t_0}\mathrm{d}\omega = \langle z(\omega)\,,\,\mathrm{e}^{-\mathrm{j}\omega t_0}\rangle$$

$$\boxed{t_0 = 0:\quad \delta(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{\delta}(\omega) \equiv 1}$$

Convergence of Fourier Transform

$$x(t) \in L_1(\mathbb{R}) \;\Rightarrow\; \begin{cases} \widehat{x}(\omega) \text{ exists,} \\ \widehat{x}(\omega) \text{ is continuous,} \\ \widehat{x}(\omega) \in L_\infty(\mathbb{R}). \end{cases} \qquad x(t) \in L_2(\mathbb{R}) \;\Rightarrow\; \begin{cases} \widehat{x}(\omega) \text{ exists,} \\ \widehat{x}(\omega) \in L_2(\mathbb{R}). \end{cases}$$



Bernhard Riemann
1826-1866

&

Henri Lebesgue
1875-1941

$$x(t) \in L_1(\mathbb{R}) \quad \Rightarrow \quad \lim_{|\omega|\to\infty} \widehat{x}(\omega) = 0$$

Inverse Fourier Transform(تبدیل عکس فوریه)

$$x(t) \in L_2(\mathbb{R}) \quad \Rightarrow \quad \tilde{x}_T(t) \in L_2([0,T[) \quad \Rightarrow \quad \tilde{x}_T(t) \overset{\text{a.e.}}{=} \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)\, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}k\tau}\, \mathrm{d}\tau \underset{T\gg 1}{\approx} \frac{1}{T}\widehat{x}(\omega)\Big|_{\omega = \frac{2\pi}{T}k}$$

$$\Rightarrow \quad \tilde{x}_T(t) \underset{T\gg 1}{\overset{\text{a.e.}}{\approx}} \frac{1}{T}\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\, \widehat{x}(\omega)\Big|_{\omega = \frac{2\pi}{T}k} \underset{T\gg 1}{\approx} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{x}(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

$$|t| < \tfrac{T}{2} \quad \Rightarrow \quad x(t) = \tilde{x}_T(t) \quad \Rightarrow \quad \forall\, t,\; x(t) = \lim_{T\to\infty} \tilde{x}_T(t)$$

$$\boxed{x(t) \overset{\text{a.e.}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{x}(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega}$$

Inverse Fourier Transform(تبدیل عکس فوریه)

$$x(t) \stackrel{?}{=} \mathcal{F}_\omega^{-1}\{\widehat{x}(\omega)\}(t)$$

$$\mathcal{F}_\omega^{-1}\{\widehat{x}(\omega)\}(t) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \widehat{x}(\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T}\int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\tau\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{\mathbb{R}} x(\tau)\left(\int_{-T}^{T}\mathrm{e}^{\mathrm{j}\omega(t-\tau)}\,\mathrm{d}\omega\right)\mathrm{d}\tau$$

$$= \frac{T}{\pi}\lim_{T\to\infty}\int_{\mathbb{R}} x(\tau)\mathrm{sinc}\big(\tfrac{T(t-\tau)}{\pi}\big)\mathrm{d}\tau = \lim_{T\to\infty}\big(x * h_T\big)(t)$$

$$h_T(t) \triangleq \frac{T}{\pi}\,\mathrm{sinc}(\tfrac{T}{\pi}t) \;\Rightarrow\; \begin{cases} \int_{\mathbb{R}} h_T(\tau)\mathrm{d}\tau = 1 \\ \lim_{|t|\to\infty} h_T(t) = 0 \end{cases} \;\Rightarrow\; \lim_{T\to\infty} h_T(t) = \delta(t)$$

$$\Rightarrow\; \mathcal{F}^{-1}\{\widehat{x}(\omega)\}(t) = \lim_{T\to\infty}(x * h_T)(t) = (x*\delta)(t) = x(t) \quad ✓$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{j}\omega\tau}\mathrm{d}\tau = \delta(\omega)$$

Continuous-domain Fourier Transform

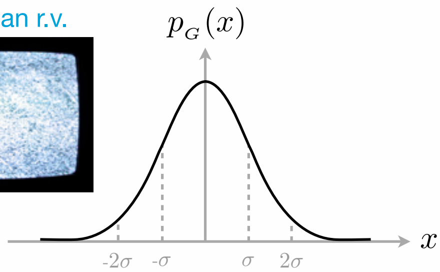
- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of differential equations

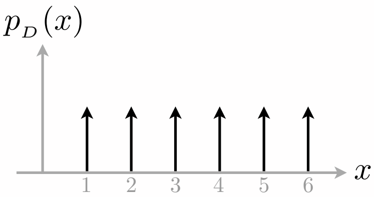
Analogy with Probability Density Functions

Discrete-random variable	Continuous random variable

$$\begin{cases} P_D(\text{⚀}) = \frac{1}{6} \\ P_D(\text{⚁}) = \frac{1}{6} \\ P_D(\text{⚂}) = \frac{1}{6} \\ P_D(\text{⚃}) = \frac{1}{6} \\ P_D(\text{⚄}) = \frac{1}{6} \\ P_D(\text{⚅}) = \frac{1}{6} \end{cases}$$

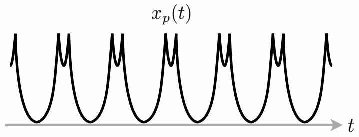
Gaussian r.v.



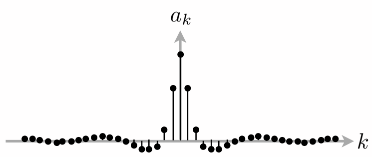


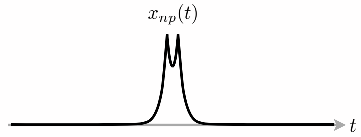
Analogy with Probability Density Functions

Periodic Continuous-domain Signals	Non-periodic Continuous-domain Signals

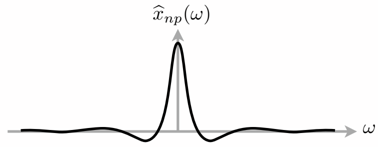


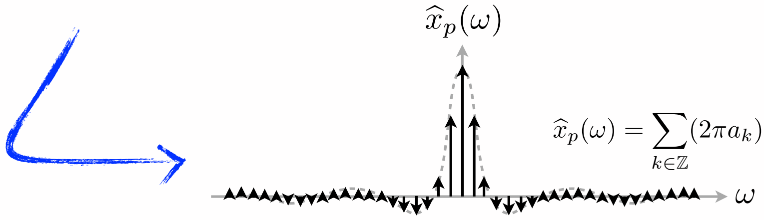
$$x_p(t) = \sum_{k\in\mathbb{Z}} a_k\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$





$$x_{np}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{x}_{np}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$





$$\widehat{x}_p(\omega) = \sum_{k\in\mathbb{Z}} (2\pi a_k)\, \delta(\omega - \tfrac{2\pi}{T}k)$$

Fourier Transform of CDPS

$$x(t) \xmapsto{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega) = \int_{\mathbb{R}} x(\tau)\, \mathrm{e}^{-\mathrm{j}\omega\tau}\, \mathrm{d}\tau \qquad \widehat{x}(\omega) \xmapsto{\mathcal{F}^{-1}.\mathrm{T}.} x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{x}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega$$

$$x(t) = \delta(t - t_0) \;\Rightarrow\; \widehat{x}(\omega) = \int_{\mathbb{R}} \delta(\tau - t_0)\mathrm{e}^{-\mathrm{j}\omega\tau}\, \mathrm{d}\tau = \mathrm{e}^{-\mathrm{j}\omega t_0}$$

$$\Rightarrow\; \boxed{\frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{j}\omega(t-t_0)} \mathrm{d}\omega = \delta(t - t_0)}$$

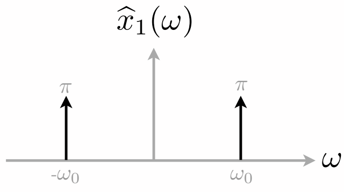
$$\left\{ \begin{array}{lcl} x_p(t) & = & T\text{-periodic}, \\ x_p(t) & = & \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt} \end{array} \right. \Rightarrow \widehat{x}_p(\omega) = \mathcal{F}\left\{\sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\right\}(\omega) = \sum_{k\in\mathbb{Z}} a_k\, \mathcal{F}\left\{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\right\}(\omega)$$

$$= \sum_{k\in\mathbb{Z}} a_k \underbrace{\int_{\mathbb{R}} \mathrm{e}^{\mathrm{j}\tau\left(\frac{2\pi}{T}k-\omega\right)} \mathrm{d}\tau}_{2\pi\delta\left(\frac{2\pi}{T}k-\omega\right)} = \boxed{\sum_{k\in\mathbb{Z}} (2\pi a_k)\delta(\omega - \tfrac{2\pi}{T}k)}$$

Fourier Transform of CDPS

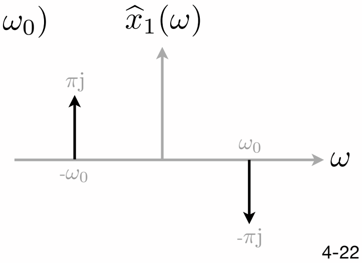
- Example $\quad x_1(t) = \cos(\omega_0 t)$

$$x_1(t) = \tfrac{1}{2}\mathrm{e}^{\mathrm{j}\omega_0 t} + \tfrac{1}{2}\mathrm{e}^{\text{-}\mathrm{j}\omega_0 t} \quad \Rightarrow \quad \widehat{x}_1(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



- Example $\quad x_2(t) = \sin(\omega_0 t)$

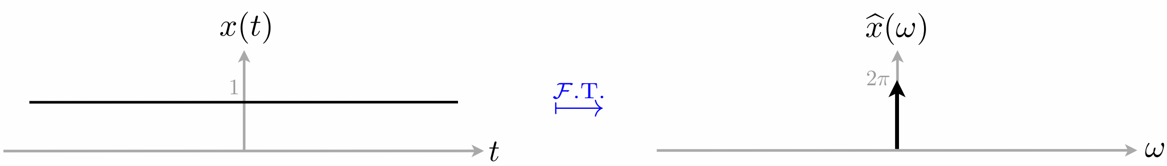
$$x_2(t) = \tfrac{1}{2\mathrm{j}}\mathrm{e}^{\mathrm{j}\omega_0 t} - \tfrac{1}{2\mathrm{j}}\mathrm{e}^{\text{-}\mathrm{j}\omega_0 t} \quad \Rightarrow \quad \widehat{x}_2(\omega) = \text{-}\mathrm{j}\pi\delta(\omega - \omega_0) + \mathrm{j}\pi\delta(\omega + \omega_0)$$



Fourier Transform of CDPS

- Example $\quad x(t) \equiv 1$

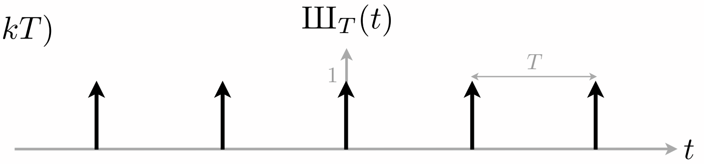
$$x(t) = \mathrm{e}^{\mathrm{j}\omega_0 t}\Big|_{\omega_0 = 0} \quad \Rightarrow \quad \widehat{x}(\omega) = 2\pi\,\delta(\omega - \omega_0)\Big|_{\omega_0=0} = 2\pi\,\delta(\omega)$$



Fourier Transform of CDPS

- Example
  (قطار ضربه)

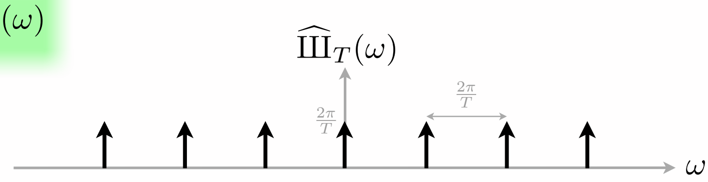
$$Ш_T(t) = \sum_{k\in\mathbb{Z}} \delta(t - kT)$$



$$Ш_T(t) = T\text{-periodic}, \quad a_k = \tfrac{1}{T}, \ \forall\, k \in \mathbb{Z} \quad \Rightarrow \quad Ш_T(t) = \sum_{k\in\mathbb{Z}} \tfrac{1}{T}\, \mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}$$

$$\Rightarrow \quad \widehat{Ш}_T(\omega) = \sum_{k\in\mathbb{Z}} \tfrac{1}{T}\, \mathcal{F}\left\{\mathrm{e}^{\mathrm{j}\frac{2\pi}{T}kt}\right\}(\omega) \ = \tfrac{1}{T} \sum_{k\in\mathbb{Z}} 2\pi\, \delta(\omega - \tfrac{2\pi}{T}k)$$

$$= \boxed{\tfrac{2\pi}{T}\, Ш_{\frac{2\pi}{T}}(\omega)}$$



$\mathcal{F}\{$قطار ضربه$\} =$ قطار ضربه

Continuous-domain Fourier Transform

- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of differential equations

Properties of Continuous-domain FT

- Linearity

$$x(t),\, w(t) \in L_1, \begin{cases} x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega) \\ w(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{w}(\omega) \end{cases}$$

$$\Rightarrow \quad \forall \alpha, \beta \in \mathbb{C}, \quad \alpha\, x(t) + \beta\, w(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \alpha\, \widehat{x}(\omega) + \beta\, \widehat{w}(\omega)$$

- Shift

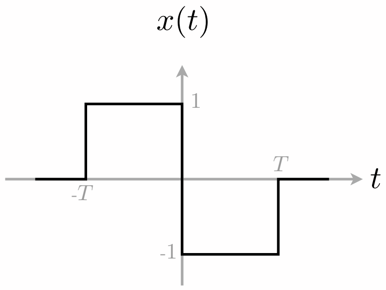
$$x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega)$$

$$\Rightarrow \quad x(t - t_0) \xrightarrow{\mathcal{F}.\mathrm{T}.} \mathrm{e}^{-\mathrm{j}\omega t_0}\, \widehat{x}(\omega)$$

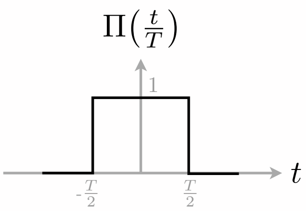
$$\int_{\mathbb{R}} x(\tau - t_0)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{\mathbb{R}} x(\tilde{\tau})\mathrm{e}^{-\mathrm{j}\omega(\tilde{\tau}+t_0)}\mathrm{d}\tilde{\tau} = \mathrm{e}^{-\mathrm{j}\omega t_0}\, \widehat{x}(\omega)$$

Properties of Continuous-domain FT

- Example



$$\Rightarrow \quad \widehat{x}(\omega) = ?$$



$$z(t) \triangleq \Pi\left(\tfrac{t}{T}\right) \qquad \Rightarrow \quad x(t) = z(t + \tfrac{T}{2}) - z(t - \tfrac{T}{2})$$

$$\Rightarrow \quad \widehat{x}(\omega) = \widehat{z}(\omega)\mathrm{e}^{\mathrm{j}\frac{T}{2}\omega} - \widehat{z}(\omega)\mathrm{e}^{-\mathrm{j}\frac{T}{2}\omega} = T\mathrm{sinc}\left(\tfrac{\omega T}{2\pi}\right)\left(\mathrm{e}^{\mathrm{j}\frac{T}{2}\omega} - \mathrm{e}^{-\mathrm{j}\frac{T}{2}\omega}\right) = \boxed{2\mathrm{j}\,\frac{1-\cos(\omega T)}{\omega}}$$

Properties of Continuous-domain FT

- Time-reversal

$$x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega) \qquad \Rightarrow \quad x(\text{-}t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\text{-}\omega)$$

$$\int_{\mathbb{R}} x(\text{-}\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \int_{\mathbb{R}} x(\tilde{\tau})\mathrm{e}^{\mathrm{j}\omega\tilde{\tau}}\mathrm{d}\tilde{\tau} = \widehat{x}(\text{-}\omega)$$

$$x(t) = \text{even} \quad \Rightarrow \quad \widehat{x}(\omega) = \text{even} \qquad\qquad x(t) = \text{odd} \quad \Rightarrow \quad \widehat{x}(\omega) = \text{odd}$$

- Conjugation

$$x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega) \qquad \Rightarrow \quad \overline{x(t)} \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \overline{\widehat{x}(\text{-}\omega)}$$

$$\int_{\mathbb{R}} \overline{x(\tau)}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \overline{\int_{\mathbb{R}} x(\tau)\mathrm{e}^{\mathrm{j}\omega\tau}\mathrm{d}\tau} = \overline{\widehat{x}(\text{-}\omega)}$$

$$x(t) = \text{real-valued} \quad \Rightarrow \quad \widehat{x}(\text{-}\omega) = \overline{\widehat{x}(\omega)} \qquad\qquad x(t) = \begin{matrix}\text{real-valued}\\ \&\\ \text{even}\end{matrix} \quad \Rightarrow \quad \widehat{x}(\omega) = \begin{matrix}\text{real-valued}\\ \&\\ \text{even}\end{matrix}$$

Properties of Continuous-domain FT

$$x(t) = \text{real-valued}, \quad x(t) = \underbrace{\mathcal{E}v\{x(t)\}}_{x_e(t)} + \underbrace{\mathcal{O}d\{x(t)\}}_{x_o(t)}$$

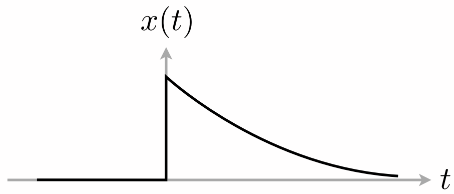
$$\Rightarrow \quad \widehat{x}(\omega) = \widehat{x_e}(\omega) + \widehat{x_o}(\omega)$$

real-valued

purely imaginary

- Example

$$x(t) = \mathrm{e}^{-a\,t}u(t), \quad a \in \mathbb{R}^+$$



$$\mathcal{F}\Big\{\underbrace{\mathcal{E}v\{x(t)\}}_{\frac{1}{2}\mathrm{e}^{-a|t|}}\Big\}(\omega) = \Re\Big\{\underbrace{\widehat{x}(\omega)}_{\frac{1}{a+\mathrm{j}\omega}}\Big\} \quad \Rightarrow \quad \mathcal{F}\Big\{\mathrm{e}^{-a|t|}\Big\}(\omega) = \frac{2a}{a^2+\omega^2}$$

Properties of Continuous-domain FT

- Differentiation

$$x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega) \qquad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \boxed{\mathrm{j}\omega\,\widehat{x}(\omega)}$$

$$\int_{\mathbb{R}} \left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\right)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \underbrace{x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\Big|_{\tau=-\infty}^{\infty}}_{=0} - \int_{\mathbb{R}} x(\tau)\left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\right)\mathrm{d}\tau = \mathrm{j}\omega\int_{\mathbb{R}} x(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

- Integration

$$x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega) \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \boxed{\widehat{x}(\omega)\left(\frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)\right)}$$

inverse derivative

DC compensation

Properties of Continuous-domain FT

- Integration property

$$\forall\, a>0,\quad z_a(t) \triangleq x(t) - \tfrac{a}{2}\,\widehat{x}(0)\,\mathrm{e}^{-a|t|} \quad\Rightarrow\quad \widehat{z}_a(\omega) = \widehat{x}(\omega) - \widehat{x}(0)\tfrac{a^2}{a^2+\omega^2}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} z_a(\tau)\mathrm{d}\tau = z_a(t) \quad\Rightarrow\quad \mathrm{j}\omega\,\mathcal{F}\left\{\int_{-\infty}^{t} z_a(\tau)\mathrm{d}\tau\right\}(\omega) = \underbrace{\widehat{x}(\omega) - \widehat{x}(0)\tfrac{a^2}{a^2+\omega^2}}_{\text{at }\ \omega=0\text{ equals }0}$$

$$\Rightarrow\ \mathcal{F}\left\{\int_{-\infty}^{t} z_a(\tau)\mathrm{d}\tau\right\}(\omega) = \frac{\widehat{x}(\omega) - \widehat{x}(0)\tfrac{a^2}{a^2+\omega^2}}{\mathrm{j}\omega}$$

$$\int_{-\infty}^{t} z_a(\tau)\mathrm{d}\tau = \int_{-\infty}^{t} x(\tau)\mathrm{d}\tau - \tfrac{\widehat{x}(0)}{2} - \tfrac{\widehat{x}(0)}{2}\big(1-\mathrm{e}^{-a|t|}\big)\mathrm{sign}(t)$$

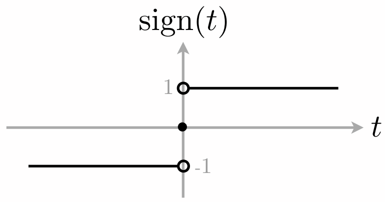
$$\Rightarrow\quad \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau\right\} = \frac{\widehat{x}(\omega)-\widehat{x}(0)\tfrac{a^2}{a^2+\omega^2}}{\mathrm{j}\omega} + \pi\widehat{x}(0)\delta(\omega) + \tfrac{\widehat{x}(0)}{2}\mathcal{F}\Big\{\underbrace{\big(1-\mathrm{e}^{-a|t|}\big)\mathrm{sign}(t)}_{(-at)\sum_{n=0}^{\infty}\frac{(-a|t|)^n}{(n+1)!}}\Big\}(\omega)$$

$$a\to 0 \quad\Rightarrow\quad ✓$$

Properties of Continuous-domain FT

- Example

$$x(t) = \mathrm{sign}(t) \quad\Rightarrow\quad \widehat{x}(\omega) = ?$$



$$\mathrm{sign}(t) = 2\,u(t) - 1 = 2\int_{-\infty}^{t}\delta(\tau)\mathrm{d}\tau - 1$$

$$\Rightarrow\quad \widehat{\mathrm{sign}}(\omega) = 2\Big(\underbrace{\tfrac{1}{\mathrm{j}\omega} + \pi\delta(\omega)}_{\widehat{u}(\omega)}\Big) - 2\pi\delta(\omega) = \boxed{\tfrac{2}{\mathrm{j}\omega}}$$

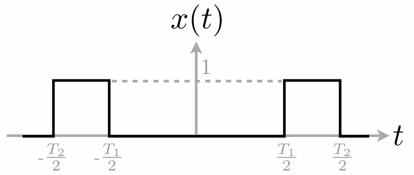
Properties of Continuous-domain FT

- Dilation (Time-Scaling)

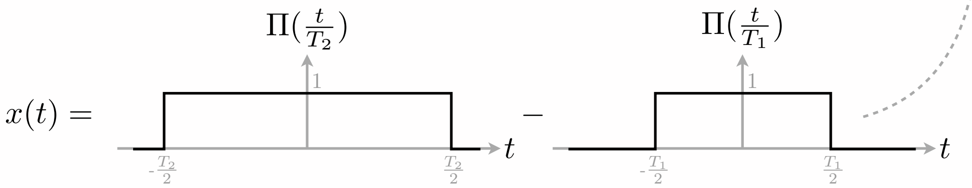
$$x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega) \quad \Rightarrow \quad \underset{\alpha\in\mathbb{R}}{x(\alpha\, t)} \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \boxed{\tfrac{1}{|\alpha|}\widehat{x}(\tfrac{\omega}{\alpha})}$$

$$\int_{\mathbb{R}} x(\underbrace{\alpha\tau}_{\tilde{\tau}})\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \frac{1}{|\alpha|}\int_{\mathbb{R}} x(\tilde{\tau})\mathrm{e}^{-\mathrm{j}\frac{\omega}{\alpha}\tilde{\tau}}\mathrm{d}\tilde{\tau}$$

- Example



$$\Rightarrow \quad \widehat{x}(\omega) = T_2\,\mathrm{sinc}(\tfrac{\omega T_2}{2\pi}) - T_1\,\mathrm{sinc}(\tfrac{\omega T_1}{2\pi})$$



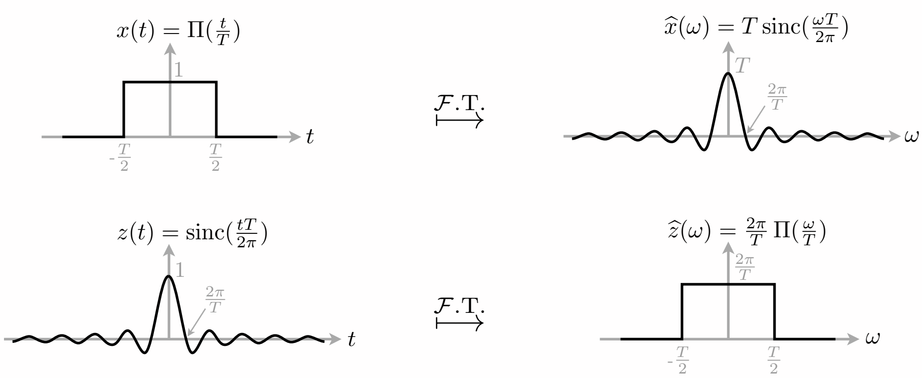
$$\Pi(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \mathrm{sinc}(\tfrac{\omega}{2\pi})$$

Properties of Continuous-domain FT

- Duality

$$\widehat{x}(\omega) = \mathcal{F}_t\{x(t)\}(\omega) = \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \qquad x(t) = \mathcal{F}_\omega^{-1}\{\widehat{x}(\omega)\}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{x}(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

$$\Rightarrow \quad \boxed{\mathcal{F}_\omega\{\widehat{x}(\omega)\}(t) = 2\pi\mathcal{F}_\omega^{-1}\{\widehat{x}(\omega)\}(-t) = 2\pi\, x(-t)}$$



Properties of Continuous-domain FT

- Example

$$z(t) = \frac{1}{1+t^2} \quad \xrightarrow{\mathcal{F}.\mathrm{T}.} \quad \widehat{z}(\omega) = \pi\, \mathrm{e}^{-|\omega|}$$

$$\mathrm{e}^{-a|t|} \quad \xrightarrow{\mathcal{F}.\mathrm{T}.} \quad \frac{2a}{a^2+\omega^2}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \mathrm{j}\omega\, \widehat{x}(\omega)$$

$$\Rightarrow \quad t\, x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \mathrm{j}\tfrac{\mathrm{d}}{\mathrm{d}\omega}\widehat{x}(\omega)$$

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega)\Big(\tfrac{1}{\mathrm{j}\omega} + \pi\delta(\omega)\Big)$$

$$\frac{x(t)}{t} \xrightarrow{\mathcal{F}.\mathrm{T}.} -\mathrm{j}\int_{-\infty}^{\omega} \widehat{x}(\eta)\mathrm{d}\eta + \mathrm{j}\pi x(0)$$

Properties of Continuous-domain FT

- Proof of $\; t\, x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \mathrm{j}\tfrac{\mathrm{d}}{\mathrm{d}\omega}\widehat{x}(\omega)$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} z(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \mathrm{j}\omega\widehat{z}(\omega)$$

$$\Rightarrow \quad \mathcal{F}_\omega\Big\{\mathrm{j}\omega\widehat{z}(\omega)\Big\}(t) = 2\pi \underbrace{\Big(\tfrac{\mathrm{d}}{\mathrm{d}\tau} z(\tau)\Big)\Big|_{\tau=-t}}_{-\tfrac{\mathrm{d}}{\mathrm{d}t} z(-t)} = -\tfrac{\mathrm{d}}{\mathrm{d}t}\big(2\pi z(-t)\big) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{F}_\omega\{\widehat{z}(\omega)\}(t)\Big)$$

$$\Rightarrow \quad \mathcal{F}_\omega\Big\{\omega\widehat{z}(\omega)\Big\}(t) = \mathrm{j}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{F}_\omega\{\widehat{z}(\omega)\}(t)\Big)$$

$$\Rightarrow \quad \mathcal{F}_t\Big\{\underbrace{t\;\widehat{z}(t)}_{\triangleq\, x(t)}\Big\}(\omega) = \mathrm{j}\tfrac{\mathrm{d}}{\mathrm{d}\omega}\Big(\underbrace{\mathcal{F}_t\{\widehat{z}(t)\}(\omega)}_{=\,\widehat{x}(\omega)}\Big)$$

Properties of Continuous-domain FT

- Parseval's Theorem

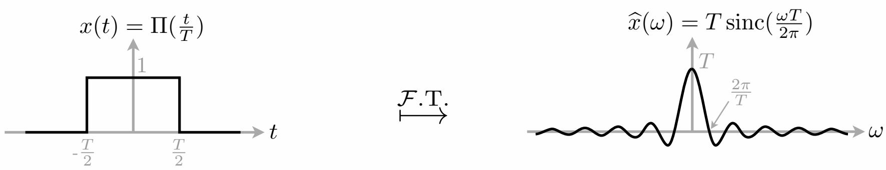
$$x(t) \in L_2(\mathbb{R}), \quad x(t) \overset{\mathcal{F}.\mathrm{T}.}{\longmapsto} \widehat{x}(\omega)$$

$$\Rightarrow \quad \boxed{\int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t = \tfrac{1}{2\pi} \int_{\mathbb{R}} |\widehat{x}(\omega)|^2 \mathrm{d}\omega}$$

$$\begin{aligned}
\tfrac{1}{2\pi} \int_{\mathbb{R}} |\widehat{x}(\omega)|^2 \mathrm{d}\omega &= \tfrac{1}{2\pi} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} x(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t \right|^2 \mathrm{d}\omega = \tfrac{1}{2\pi} \iiint_{\mathbb{R}^3} x(t) \overline{x(\tau)} \mathrm{e}^{\mathrm{j}(\tau - t)\omega} \mathrm{d}\tau \, \mathrm{d}t \, \mathrm{d}\omega \\
&= \tfrac{1}{2\pi} \iint_{\mathbb{R}^2} x(t) \overline{x(\tau)} \Big(\underbrace{\int_{\mathbb{R}} \mathrm{e}^{\mathrm{j}(\tau - t)\omega} \mathrm{d}\omega}_{2\pi\delta(\tau - t)} \Big) \mathrm{d}\tau \, \mathrm{d}t \\
&= \iint_{\mathbb{R}^2} x(t) \overline{x(\tau)} \delta(\tau - t) \mathrm{d}\tau \, \mathrm{d}t = \int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t
\end{aligned}$$

Properties of Continuous-domain FT

- Example



$$\Rightarrow \quad \int_{\mathbb{R}} \frac{\sin^2(\omega)}{\omega^2} \mathrm{d}\omega = \pi$$

Properties of Continuous-domain FT

- Convolution

$$x(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega)$$
$$z(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{z}(\omega)$$
$$(x * z)(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \ ?$$

$$y(t) \triangleq (x * z)(t) = \int_{\mathbb{R}} x(t-\tau)z(\tau)\,\mathrm{d}\tau \quad \Rightarrow \quad \widehat{y}(\omega) = \iint_{\mathbb{R}^2} x(t-\tau)z(\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}\tau\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} z(\tau)\Big(\underbrace{\int_{\mathbb{R}} x(t-\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t}_{\mathcal{F}\{x(\cdot-\tau)\}(\omega)}\Big)\mathrm{d}\tau = \widehat{x}(\omega)\int_{\mathbb{R}} z(\tau)\,\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau = \widehat{x}(\omega)\widehat{z}(\omega)$$

$$\Rightarrow \quad (x * z)(t) \xrightarrow{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega)\,\widehat{z}(\omega)$$

Properties of Continuous-domain FT

- Example

frequency response پاسخ فرکانسی

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \qquad \Rightarrow \quad \widehat{y}(\omega) = \widehat{h}(\omega)\,\widehat{x}(\omega)$$

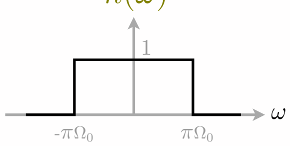
$$\rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}t}} \rightarrow \qquad\qquad \rightarrow \boxed{\int_{-\infty}^{t}} \rightarrow$$

$$\widehat{h}(\omega) = \mathrm{j}\omega \qquad\qquad \widehat{h}(\omega) = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)$$

ideal lowpass filter

$$\rightarrow \boxed{h(t) = \Omega_0\,\mathrm{sinc}(\Omega_0 t)} \rightarrow$$

$\widehat{h}(\omega)$



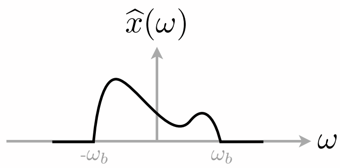
Properties of Continuous-domain FT

- Multiplication

$$\begin{cases} x(t) & \xmapsto{\mathcal{F}.\mathrm{T}.} & \widehat{x}(\omega) \\ z(t) & \xmapsto{\mathcal{F}.\mathrm{T}.} & \widehat{z}(\omega) \\ \big(x * z\big)(t) & \xmapsto{\mathcal{F}.\mathrm{T}.} & \widehat{x}(\omega)\,\widehat{z}(\omega) \end{cases} \quad \overset{\text{duality}}{\Longrightarrow} \quad \boxed{x(t)z(t) \xmapsto{\mathcal{F}.\mathrm{T}.} \tfrac{1}{2\pi}\big(\widehat{x} * \widehat{z}\big)(\omega)}$$

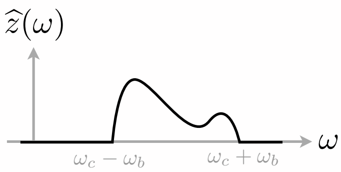
- Example

$$x(t) \xmapsto{\mathcal{F}.\mathrm{T}.} \widehat{x}(\omega)$$



$$z(t) \triangleq x(t)\mathrm{e}^{\mathrm{j}\omega_c t} \qquad \Rightarrow \qquad \widehat{z}(\omega) = \widehat{x}(\omega - \omega_c)$$

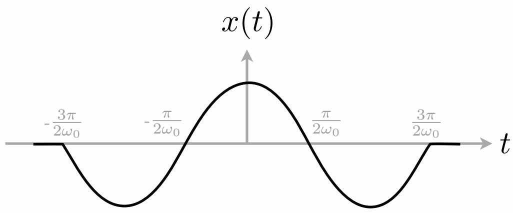
$\widehat{x}(\omega)$ $\qquad$ $2\pi\delta(\omega - \omega_c)$



Properties of Continuous-domain FT

- Example

$$x(t) = \begin{cases} \cos(\omega_0 t) & |t| \le \frac{3\pi}{2\omega_0}, \\ 0 & \text{otherwise.} \end{cases}$$

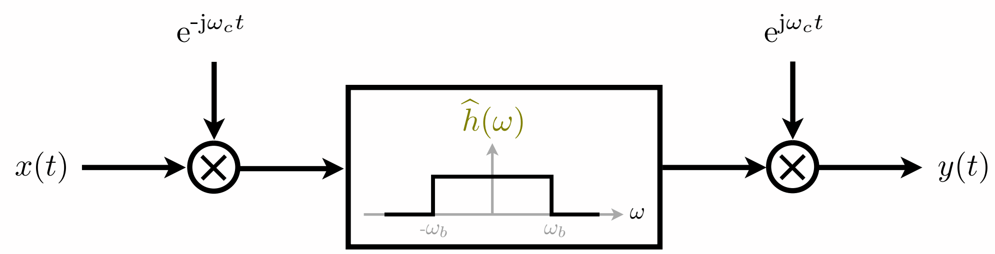


$$x(t) = \cos(\omega_0 t)\,\Pi\big(\tfrac{\omega_0}{3\pi}t\big)$$

$$\Rightarrow \widehat{x}(\omega) = \frac{1}{2\pi}\Big(\underbrace{\mathcal{F}\{\cos(\omega_0 t)\}}_{\pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)} * \underbrace{\mathcal{F}\{\Pi(\tfrac{\omega_0}{3\pi}t)\}}_{\frac{3\pi}{\omega_0}\mathrm{sinc}\left(\frac{3\omega}{2\omega_0}\right)} \Big)(\omega)$$

$$= \frac{3\pi}{2\omega_0}\Big(\mathrm{sinc}\big(\tfrac{3\omega}{2\omega_0} - \tfrac{3}{2}\big) + \mathrm{sinc}\big(\tfrac{3\omega}{2\omega_0} + \tfrac{3}{2}\big)\Big)$$

Tunable Bandpass Filter



Proof of CLT (اثبات قضیه حد مرکزی)

$$\begin{cases} X_i \overset{\text{i.i.d.}}{\sim} p_X(x) \\ \mathbb{E}[X_i] = 0 \\ \mathbb{E}\big[|X_i|^2\big] = \sigma^2 < \infty \end{cases} \qquad Y_n \triangleq X_1 + X_2 + \cdots + X_n, \qquad Z_n \triangleq \frac{Y_n}{\sqrt{n}}$$

$$\Rightarrow \lim_{n\to\infty} Z_n \sim N(0, \sigma^2)$$

$$p_X(x) = \text{pdf} \in L_1 \quad \Rightarrow \quad \widehat{p_X}(\omega) \text{ exists and is continuous}$$

$$\mathbb{E}\big[|X_i|^2\big] = \int_{\mathbb{R}} x^2\, p_X(x) \mathrm{d}x < \infty \quad \Rightarrow \quad x^2\, p_X(x) \in L_1$$

$$\Rightarrow \quad \mathcal{F}_x\left\{x^2\, p_X(x)\right\}(\omega) = \text{continuous}$$

$$\Rightarrow \quad \widehat{p_X}(\omega) = \text{cont. twice diff.}$$

$$\Rightarrow \quad \widehat{p_X}(\omega) = \underbrace{\widehat{p_X}(0)}_{1} + \omega\, \underbrace{\dot{\widehat{p_X}}(0)}_{\text{-j}\mathbb{E}[X_i]} + \frac{\omega^2}{2}\, \underbrace{\ddot{\widehat{p_X}}(0)}_{\text{-}\mathbb{E}[|X_i|^2]} + o(\omega^2) = 1 - \frac{\sigma^2}{2}\omega^2 + o(\omega^2)$$

Proof of CLT (اثبات قضیه حد مرکزی) (Cont'd)

$$\begin{cases} X_i \overset{\text{i.i.d.}}{\sim} p_X(x) \\ \mathbb{E}[X_i] = 0 \\ \mathbb{E}\big[|X_i|^2\big] = \sigma^2 < \infty \end{cases} \qquad Y_n \triangleq X_1 + X_2 + \cdots + X_n, \qquad Z_n \triangleq \frac{Y_n}{\sqrt{n}}$$

$$\Rightarrow \lim_{n\to\infty} Z_n \sim N(0, \sigma^2)$$

$$p_{Y_n}(y) = \big(p_{X_1} * \cdots * p_{X_n}\big)(y) \quad \Rightarrow \quad \widehat{p_{Y_n}}(\omega) = \prod_{i=1}^{n} \widehat{p_{X_i}}(\omega) = \Big(\widehat{p_X}(\omega)\Big)^n$$

$$p_{Z_n}(z) = \sqrt{n}\; p_{Y_n}(\sqrt{n}z) \quad \Rightarrow \quad \widehat{p_{Z_n}}(\omega) = \widehat{p_{Y_n}}\big(\tfrac{\omega}{\sqrt{n}}\big)$$

$$\Rightarrow \quad \widehat{p_{Z_n}}(\omega) = \Big(1 + \frac{-\frac{\sigma^2}{2}\omega^2}{n} + o(\tfrac{\omega^2}{n})\Big)^n$$

$$\lim_{n\to\infty} (1 + \tfrac{a_n}{n})^n = \exp\big(\lim_{n\to\infty} a_n\big) \quad \Rightarrow \quad \lim_{n\to\infty} \widehat{p_{Z_n}}(\omega) = e^{-\frac{\sigma^2}{2}\omega^2} = \mathcal{F}_z\left\{\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\right\}(\omega)$$

Continuous-domain Fourier Transform

- Fourier series to Fourier transform
- Fourier transform of periodic signals
- Properties of Fourier transform
- Systems of differential equations

LSI Differential Systems

- Linear constant-coefficient differential systems

$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t) = \sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t)} \longrightarrow y(t)$$

$$\text{proper boundary conditions} \quad \Rightarrow \quad \text{LSI} \quad \Rightarrow \quad \widehat{y}(\omega) = \underbrace{\widehat{h}(\omega)}_{?}\, \widehat{x}(\omega)$$

$$\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t) = \sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t) \quad \Rightarrow \quad \mathcal{F}\left\{\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t)\right\}(\omega) = \mathcal{F}\left\{\sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t)\right\}(\omega)$$

$$\Rightarrow \quad \sum_{k=0}^{N_y} a_k \left(\mathrm{j}\omega\right)^k \widehat{y}(\omega) = \sum_{k=0}^{N_x} b_k \left(\mathrm{j}\omega\right)^k \widehat{x}(\omega) \quad \Rightarrow \quad \widehat{h}(\omega) = \frac{\widehat{y}(\omega)}{\widehat{x}(\omega)} = \frac{\sum_{k=0}^{N_x} b_k \left(\mathrm{j}\omega\right)^k}{\sum_{k=0}^{N_y} a_k \left(\mathrm{j}\omega\right)^k}$$

LSI Differential Systems

- Example

$$x(t) \longrightarrow \boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 4y(t) = x(t)} \longrightarrow y(t)$$

initial rest

frequency response

$$\Rightarrow \quad \widehat{h}(\omega) = \frac{\widehat{y}(\omega)}{\widehat{x}(\omega)} = \frac{1}{(\mathrm{j}\omega)^2 + 4}$$
$$= \frac{1}{4\mathrm{j}}\left(\frac{1}{\mathrm{j}\omega + (\text{-}2\mathrm{j})} - \frac{1}{\mathrm{j}\omega + (2\mathrm{j})}\right)$$

impulse response

$$\Rightarrow \quad h(t) = \frac{1}{4\mathrm{j}}\left(\mathrm{e}^{2\mathrm{j}t}u(t) - \mathrm{e}^{\text{-}2\mathrm{j}t}u(t)\right)$$
$$= \frac{\sin(2t)}{2}u(t)$$

step response

$$x(t) = u(t) \quad \Rightarrow \quad \widehat{y}(\omega) = \widehat{h}(\omega)\widehat{x}(\omega) = \tfrac{1}{(\mathrm{j}\omega)^2+4}\left(\tfrac{1}{\mathrm{j}\omega} + \pi\delta(\omega)\right)$$
$$= \tfrac{1}{4}\left(\tfrac{\text{-}\mathrm{j}\omega}{(\mathrm{j}\omega)^2+4} + \tfrac{1}{\mathrm{j}\omega}\right) + \tfrac{\pi}{4}\delta(\omega)$$
$$\Rightarrow \quad y(t) = \tfrac{1}{4}\left(\text{-}\cos(2t)u(t) + \tfrac{1}{2}\mathrm{sign}(t)\right) + \tfrac{1}{8}$$
$$= \tfrac{1-\cos(2t)}{4}u(t)$$