$\mathcal{Z}$-Transform

- $\mathcal{Z}$-Transform and Region of Convergence (ROC)
- Properties of $\mathcal{Z}$-Transform
- Characterization of LSI Systems
- One-sided (unilateral) $\mathcal{Z}$-Transform

Generalization of DT Fourier Transform


Pierre-Simon Laplace
1749-1827


Witold Hurewicz
1904-1956


Lotfi A. Zadeh
1921-2017


Eliahu I. Jury
1923-2020

$$x[n] \longrightarrow \boxed{\text{LSI}} \longrightarrow y[n] \qquad \widehat{x}(e^{j\omega}) = \sum_{m\in\mathbb{Z}} x[m]e^{-j\omega m}$$

$$\begin{pmatrix}\text{general}\\ \text{eigen-signals}\end{pmatrix} z^n \quad \Rightarrow \quad \big(\mathcal{Z}\text{-Transform}\big) \quad X(z) = \sum_{m\in\mathbb{Z}} x[m]z^{-m}$$

$\mathcal{Z}$-Transform

$$\left\{\begin{array}{c} x[n]:\ \text{complex-valued} \\ z \in \mathbb{C} \\ \sum_{m\in\mathbb{Z}} x[m]z^{-m}\ \text{exists} \end{array}\right. \quad \Rightarrow \quad X(z) \triangleq \mathcal{Z}\{x[n]\}(z) \triangleq \sum_{m\in\mathbb{Z}} x[m]z^{-m}$$

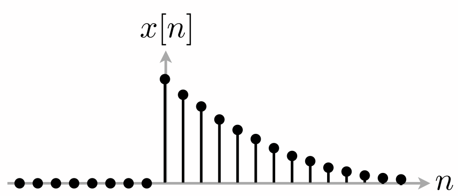
$$\widehat{x}(\mathrm{e}^{\mathrm{j}\omega}) = X(z)\big|_{z=\mathrm{e}^{\mathrm{j}\omega}}$$

$$x[n]\,r^{-n} \ \overset{\text{DTFT}}{\longmapsto}\ X(z)\big|_{z=r\mathrm{e}^{\mathrm{j}\omega}}$$

$\mathcal{Z}$-Transform

- Example

$$x[n] = \alpha^n u[n], \quad \alpha \in \mathbb{C}$$

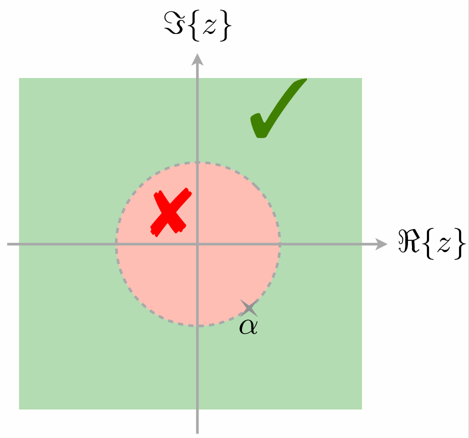


$$X(z) = \sum_{m\in\mathbb{Z}} x[m]z^{-m} = \sum_{m=0}^{\infty} \alpha^m z^{-m}$$

convergent $\Leftrightarrow |\alpha z^{-1}| < 1$

$$\overset{|z|>|\alpha|}{=} \frac{1}{1-\alpha z^{-1}}$$

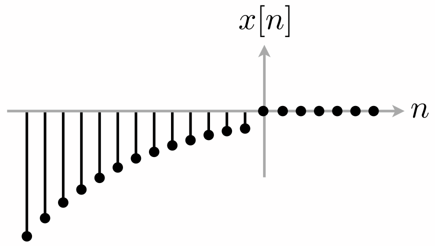
ROC



$\mathcal{Z}$-Transform

- Example

$$x[n] = -\alpha^n u[-n-1], \quad \alpha \in \mathbb{C}$$

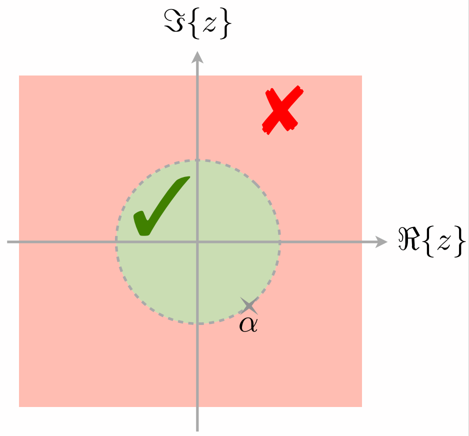


$$X(z) = \sum_{m\in\mathbb{Z}} x[m] z^{-m} = -\sum_{m=-\infty}^{-1} \alpha^m z^{-m}$$

convergent $\Leftrightarrow |\alpha z^{-1}| > 1$

$$\overset{|z|<|\alpha|}{=} \frac{1}{1-\alpha z^{-1}}$$

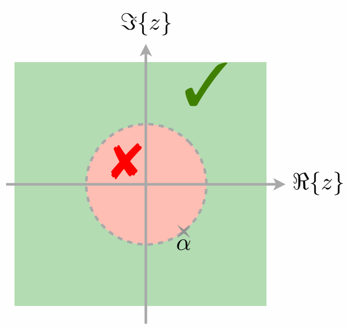
ROC



$\mathcal{Z}$-Transform

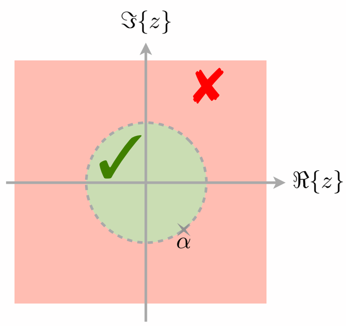
$$\alpha^n u[n] \overset{\mathcal{Z}.\text{T.}}{\longmapsto} \frac{1}{1-\alpha z^{-1}}$$

$|z| > |\alpha|$



$$-\alpha^n u[-n-1] \overset{\mathcal{Z}.\text{T.}}{\longmapsto} \frac{1}{1-\alpha z^{-1}}$$

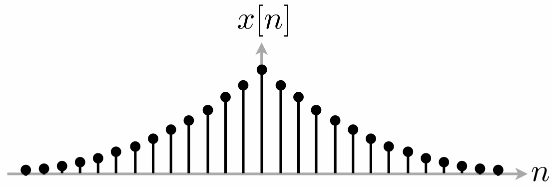
$|z| < |\alpha|$



$\mathscr{Z}$-Transform

- Example

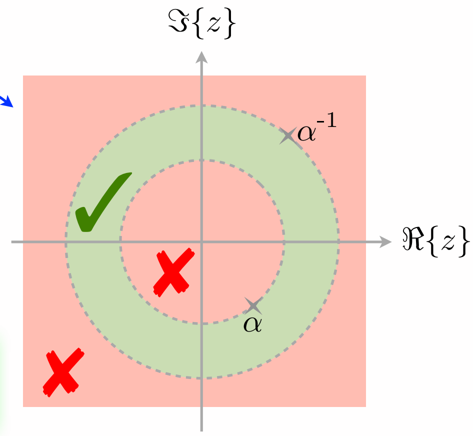
$$x[n] = \alpha^{|n|}, \quad \boxed{|\alpha| < 1}$$



$$X(z) = \sum_{m\in\mathbb{Z}} x[m]z^{-m} = \sum_{m=-\infty}^{-1} \alpha^{-m} z^{-m} + \sum_{m=0}^{\infty} \alpha^{m} z^{-m}$$

ROC

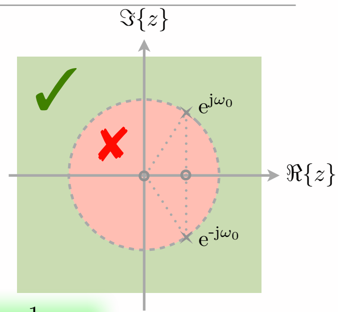
$$\text{convergent} \Leftrightarrow \begin{cases} |\alpha^{-1}z^{-1}| > 1 \\ |\alpha z^{-1}| < 1 \end{cases}$$



$$= \frac{-1}{1-\alpha^{-1}z^{-1}} + \frac{1}{1-\alpha z^{-1}} = \frac{(\alpha-\alpha^{-1})z^{-1}}{(1-\alpha^{-1}z^{-1})(1-\alpha z^{-1})}$$

$\mathscr{Z}$-Transform

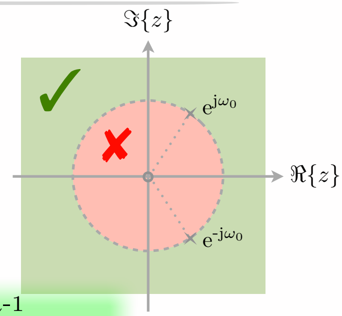
- Example $\quad x_1[n] = \cos(\omega_0 n)u[n], \quad \omega_0 \in \mathbb{R}$



$$X_1(z) = \sum_{m=0}^{\infty} \cos(\omega_0 m) z^{-m} = \sum_{m=0}^{\infty} \frac{e^{j\omega_0 m}+e^{-j\omega_0 m}}{2} z^{-m}$$

$$\stackrel{|z|>1}{=} \frac{1}{2}\left(\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

- Example $\quad x_2[n] = \sin(\omega_0 n)u[n], \quad \omega_0 \in \mathbb{R}$



$$X_2(z) = \sum_{m=0}^{\infty} \sin(\omega_0 m) z^{-m} = \sum_{m=0}^{\infty} \frac{e^{j\omega_0 m}-e^{-j\omega_0 m}}{2j} z^{-m}$$

$$\stackrel{|z|>1}{=} \frac{1}{2j}\left(\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right) = \frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$\mathcal{Z}$-Transform

- Example

$$x[n] = \Big(3 - 2^{n+1}\cos(\tfrac{\pi}{4}n)\Big)u[n]$$

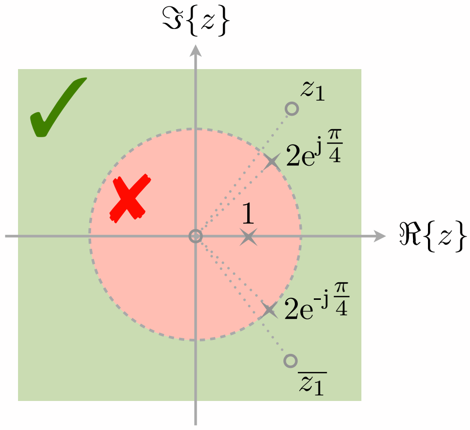
$$X(z) = \sum_{m\in\mathbb{Z}} x[m]z^{-m} = \sum_{m=0}^{\infty}\Big(3\times 1^m - (2\mathrm{e}^{\mathrm{j}\frac{\pi}{4}})^m - (2\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}})^m\Big)z^{-m}$$

convergent $\Leftrightarrow |z| > 2$

$$= \frac{3}{1-z^{-1}} - \frac{1}{1-2\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}z^{-1}} - \frac{1}{1-2\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}z^{-1}}$$

$$= \frac{(1-z_1z^{-1})(1-\overline{z_1}z^{-1})}{(1-z^{-1})(1-2\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}z^{-1})(1-2\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}z^{-1})}$$

$$z_1 = 2\sqrt{2} - 1 + \mathrm{j}\big(1+\sqrt{2}\big)$$



Poles and Zeros (قطب و صفر)

$$X(z) = \frac{N(z^{-1})}{D(z^{-1})} = \frac{\sum_{k=0}^{n_N} a_k\, z^{-k}}{\sum_{k=0}^{n_D} b_k\, z^{-k}} = \alpha\,\frac{\prod_{k=1}^{n_N}(1-z_kz^{-1})}{\prod_{k=1}^{n_D}(1-p_kz^{-1})}$$

zeros

poles

$$a_k, b_k \in \mathbb{C}$$

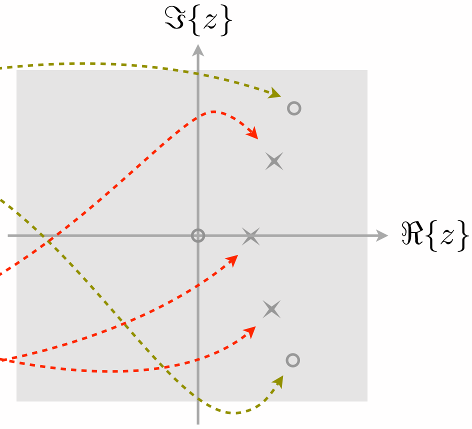
- Example

zeros

$$X(z) = \frac{(1-z_1z^{-1})(1-\overline{z_1}z^{-1})}{(1-z^{-1})(1-2\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}z^{-1})(1-2\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}z^{-1})}$$

$$z_1 = 2\sqrt{2} - 1 + \mathrm{j}\big(1+\sqrt{2}\big)$$

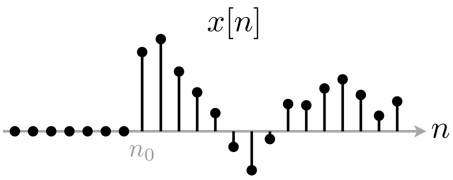
poles

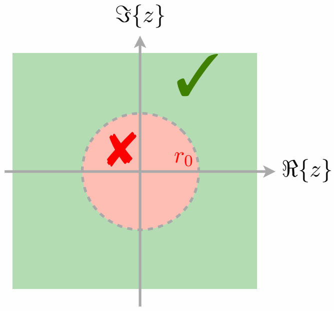


ROC (ناحیه همگرایی)

- Region of Convergence

$$\textbf{Property 1.}\quad \begin{cases} x[n]: & \text{right-sided,} \\ \mathcal{Z}\{x\}(z): & \text{absolutely convg.} \\ & \text{at } z_0 = r_0 e^{j\omega_0} \end{cases} \Rightarrow \quad X(z) \text{ converges at all } \quad r_0 < |z|$$



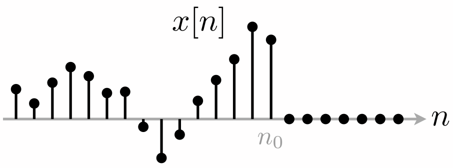


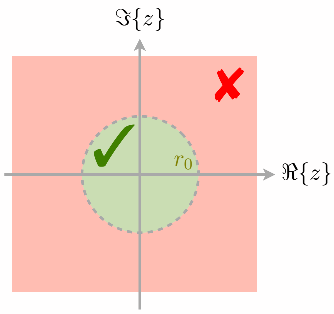
$$|X(z)| = \left|\sum_{m=n_0}^{\infty} x[m] z^{-m}\right| = \left|\sum_{m=n_0}^{\infty} x[m] z_0^{-m} \left(\tfrac{z}{z_0}\right)^{-m}\right| \underset{\substack{\uparrow \\ r_0 < |z|}}{\leq} \left(\tfrac{|z|}{r_0}\right)^{-n_0} \sum_{m=n_0}^{\infty} \left|x[m] z_0^{-m}\right| < \infty$$

ROC (ناحیه همگرایی)

- Region of Convergence

$$\textbf{Property 2.}\quad \begin{cases} x[n]: & \text{left-sided,} \\ \mathcal{Z}\{x\}(z): & \text{absolutely convg.} \\ & \text{at } z_0 = r_0 e^{j\omega_0} \end{cases} \Rightarrow \quad X(z) \text{ converges at all } \quad |z| < r_0$$



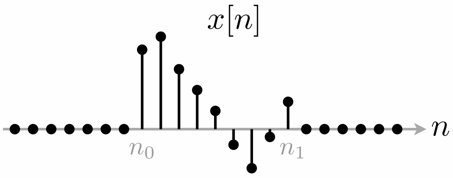


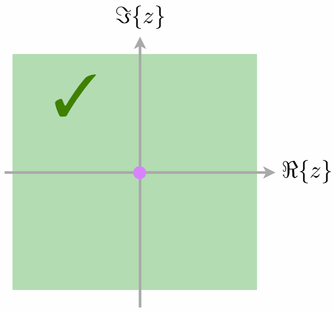
$$|X(z)| = \left|\sum_{m=-\infty}^{n_0} x[m] z^{-m}\right| = \left|\sum_{m=-\infty}^{n_0} x[m] z_0^{-m} \left(\tfrac{z}{z_0}\right)^{-m}\right| \underset{\substack{\uparrow \\ |z| < r_0}}{\leq} \left(\tfrac{|z|}{r_0}\right)^{-n_0} \sum_{m=-\infty}^{n_0} \left|x[m] z_0^{-m}\right| < \infty$$

ROC (ناحیه همگرایی)

- Region of Convergence

Property 3. $x[n]$: compact support $\Rightarrow$ $X(z)$ converges everywhere
(except possibly at $z = 0$)





$$|X(z)| = \left|\sum_{m=n_0}^{n_1} x[m]z^{-m}\right| \underset{z\neq 0}{\leq} (n_1 - n_0 + 1)\left(\max_{n_0\leq m\leq n_1} |z^{-m}x[m]|\right) < \infty$$

ROC (ناحیه همگرایی)

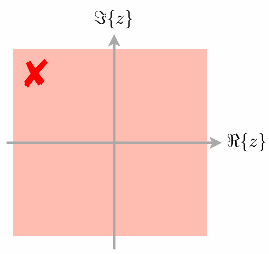
- Region of Convergence

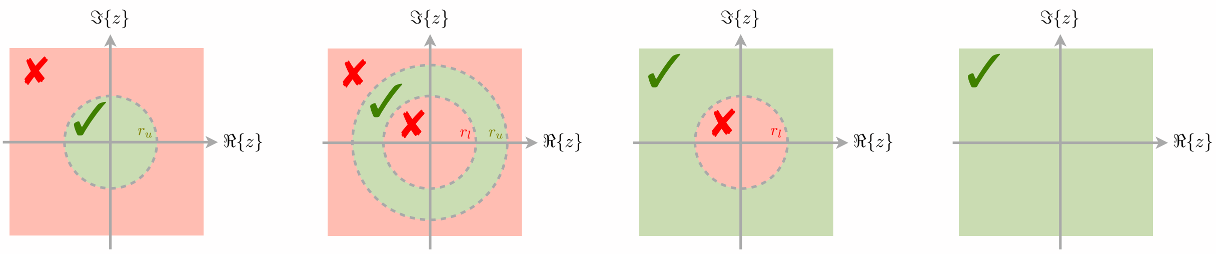
or $\leq$

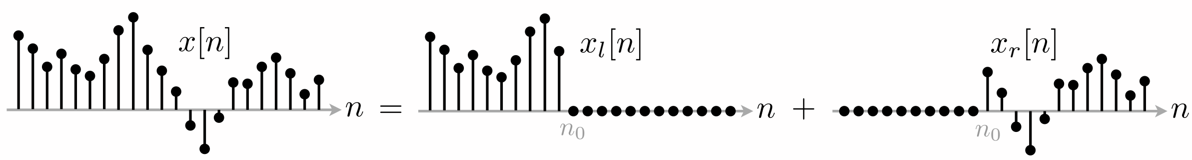
Property 4. arbitrary $x[n]$ $\Rightarrow$ ROC: $b_l < |z| < b_u$

$b_u \in \mathbb{R}^+ \cup \{+\infty\}$

$b_l \in \mathbb{R}^+ \cup \{0\}$



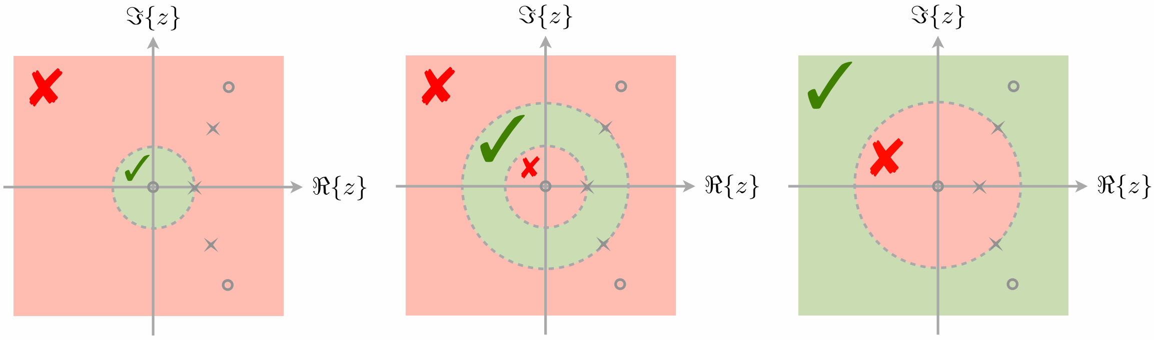




ROC (ناحیه همگرایی)

- Region of Convergence

Property 5. $X(z) = \dfrac{N(z^{-1})}{D(z^{-1})} = \alpha \dfrac{\prod_{k=1}^{n_N}(1 - z_k z^{-1})}{\prod_{k=1}^{n_D}(1 - p_k z^{-1})} \Rightarrow \begin{cases} p_k \notin \text{ROC} \\ \text{there is at least one pole on each finite boundary of ROC} \end{cases}$



Inverse $\mathcal{Z}$-Transform

$$x[n] \overset{\mathcal{Z}.\text{T.}}{\longmapsto} X(z)$$
$$X(z) \overset{?}{\longmapsto} x[n]$$

$$x_1[n] = \alpha^n u[n] \overset{\mathcal{Z}.\text{T.}}{\longmapsto} X_1(z) = \frac{1}{1 - \alpha z^{-1}}$$

$$x_2[n] = -\alpha^n u[-n-1] \overset{\mathcal{L}.\text{T.}}{\longmapsto} X_2(z) = \frac{1}{1 - \alpha z^{-1}}$$

$$X_1(z) \cancel{\overset{?!?}{=}} X_2(z)$$

$\text{ROC}_1 \neq \text{ROC}_2$

$$X(z),\ \text{ROC} \overset{\mathcal{Z}^{-1}.\text{T.}}{\longmapsto} x[n]$$

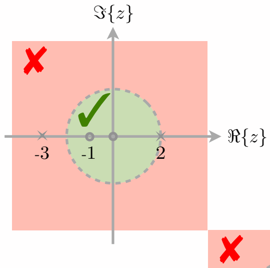
$$\mathcal{Z}^{-1}.\text{T.}: \begin{cases} x[n] \overset{\mathcal{Z}.\text{T.}}{\longmapsto} X(z) \\ \{z,\ |z| = r_0\} \subset \text{ROC} \end{cases} \Rightarrow X(z)\big|_{z=r_0 e^{j\omega}} \overset{(\text{DTFT})^{-1}}{\longmapsto} x[n] r_0^{-n}$$

$$\Rightarrow x[n] = \boxed{\frac{r_0^n}{2\pi} \int_{2\pi} X(z)\big|_{z=r_0 e^{j\omega}} e^{j\omega n} d\omega}$$

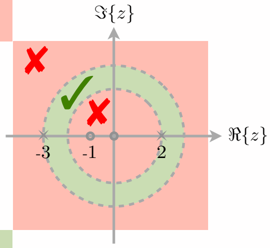
Inverse $\mathcal{Z}$-Transform

- Example

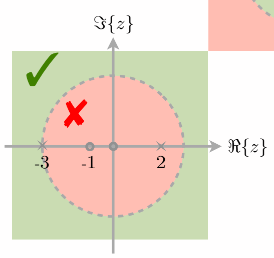
$$X(z) = \frac{1+z^{-1}}{(1-2z^{-1})(1+3z^{-1})} = \frac{\frac{3}{5}}{1-2z^{-1}} - \frac{\frac{2}{5}}{1+3z^{-1}}$$



$$\Rightarrow \quad x[n] = \tfrac{-3}{5}2^n u[-n-1] + \tfrac{2}{5}(-3)^n u[-n-1]$$



$$\Rightarrow \quad x[n] = \tfrac{3}{5}2^n u[n] + \tfrac{2}{5}(-3)^n u[-n-1]$$



$$\Rightarrow \quad x[n] = \tfrac{3}{5}2^n u[n] - \tfrac{2}{5}(-3)^n u[n]$$

Geometrical Perspective of DT Fourier Transform

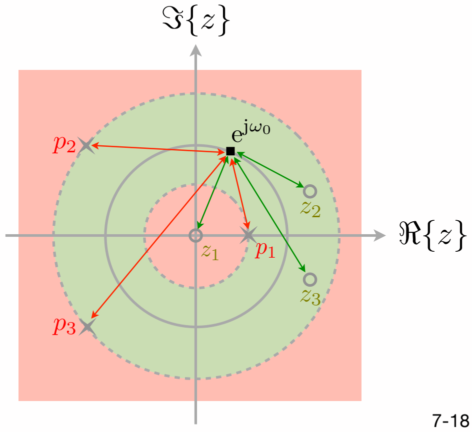
DT Fourier Transform $\quad \widehat{x}(e^{j\omega}) = \sum_{m\in\mathbb{Z}} x[m]e^{-j\omega m}$

$$\rightarrow \quad \text{DTFT exists} \;\Leftrightarrow\; \{\text{unit circle}\} \subseteq \text{ROC}$$

$\mathcal{Z}$-Transform $\quad X(z) = \sum_{m\in\mathbb{Z}} x[m]z^{-m}$

$$X(z) = \frac{N(z^{-1})}{D(z^{-1})} = \alpha \frac{\prod_{k=1}^{n_N}(1-z_k z^{-1})}{\prod_{k=1}^{n_D}(1-p_k z^{-1})}$$

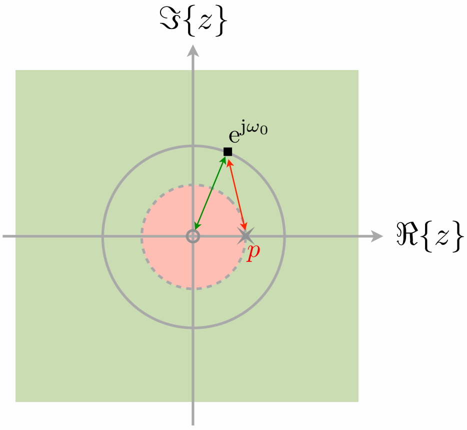
$$\Rightarrow \quad \widehat{x}(e^{j\omega}) = \alpha\, e^{j\omega(n_D-n_N)} \frac{\prod_{k=1}^{n_N}(e^{j\omega}-z_k)}{\prod_{k=1}^{n_D}(e^{j\omega}-p_k)}$$

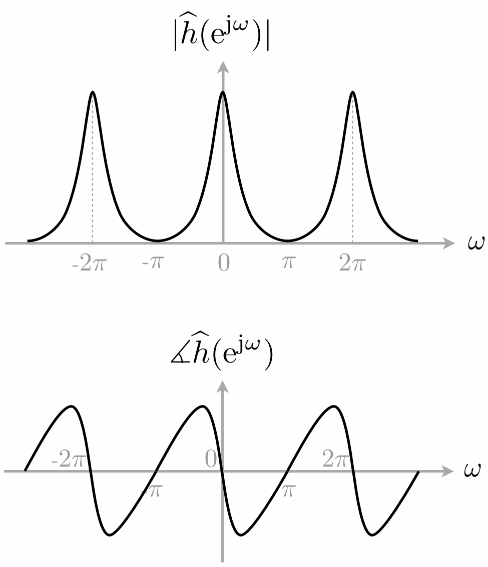


Geometrical Perspective of DT Fourier Transform

- 1st-order Systems

$$h[n] = p^n u[n], \quad |p| < 1 \quad \Rightarrow \quad H(z) = \frac{1}{1 - pz^{-1}}$$





$\mathscr{Z}$-Transform

- $\mathscr{Z}$-Transform and Region of Convergence (ROC)
- Properties of $\mathscr{Z}$-Transform
- Characterization of LSI Systems
- One-sided (unilateral) $\mathscr{Z}$-Transform

Properties of $\mathscr{Z}$-Transform

- Linearity

$$\begin{cases} x[n] \xmapsto{\mathscr{Z}.\mathrm{T}.} X(z), & z \in \mathrm{ROC}_x \\ w[n] \xmapsto{\mathscr{Z}.\mathrm{T}.} W(z), & z \in \mathrm{ROC}_w \end{cases}$$

$$\Rightarrow \quad \forall \alpha, \beta \in \mathbb{C}, \quad \alpha\, x[n] + \beta\, w[n] \xmapsto{\mathscr{Z}.\mathrm{T}.} \alpha\, X(z) + \beta\, W(z), \; z \in \mathrm{ROC}_x \cap \mathrm{ROC}_w \text{ or larger}$$

- Shift in Time-domain

$$x[n] \xmapsto{\mathscr{Z}.\mathrm{T}.} X(z), \; z \in \mathrm{ROC}_x$$

$$\Rightarrow \quad x[n-n_0] \xmapsto{\mathscr{Z}.\mathrm{T}.} z^{-n_0}\, X(z), \; z \in \mathrm{ROC}_x$$

$$\sum_{m\in\mathbb{Z}} x[m-n_0] z^{-m} = \sum_{\tilde{m}\in\mathbb{Z}} x[\tilde{m}] z^{-(\tilde{m}+n_0)} = z^{-n_0}\, X(z)$$

Properties of $\mathcal{Z}$-Transform

- Differencing

$$x[n] \xrightarrow{\mathcal{Z}.\mathrm{T.}} X(z),\ z \in \mathrm{ROC}_x$$

$$\Rightarrow\ \underbrace{x[n]-x[n-1]}_{\text{difference}} \xrightarrow{\mathcal{Z}.\mathrm{T.}} \underbrace{(1-z^{-1})}_{\text{equivalent of } s} X(z),\ z \in \mathrm{ROC}_x \text{ or larger}$$

- Accumulation

$$x[n] \xrightarrow{\mathcal{Z}.\mathrm{T.}} X(z),\ z \in \mathrm{ROC}_x$$

$$\Rightarrow\ \sum_{m=-\infty}^{n} x[m] \xrightarrow{\mathcal{Z}.\mathrm{T.}} \frac{X(z)}{1-z^{-1}},\ z \in \mathrm{ROC}_x \cap \{z,\ |z|>1\}$$

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m=-\infty}^{n} x[m]\Big) z^{-n} = \sum_{m,n\in\mathbb{Z}} x[m]\,u[n-m] z^{-n} = \sum_{m\in\mathbb{Z}} x[m]\Big(\sum_{n=m}^{\infty} z^{-n}\Big) = \tfrac{1}{1-z^{-1}} \sum x[m] z^{-m}$$

Properties of $\mathcal{Z}$-Transform

- Scaling in z-domain

$$x[n] \xrightarrow{\mathcal{Z}.\mathrm{T.}} X(z),\ z \in \mathrm{ROC}_x$$

$$\overset{z_0\in\mathbb{C}}{\Rightarrow}\ z_0^n\, x[n] \xrightarrow{\mathcal{Z}.\mathrm{T.}} X\big(\tfrac{z}{z_0}\big),\ z \in |z_0|\,\mathrm{ROC}_x$$

$$\sum_{m\in\mathbb{Z}} z_0^m x[m] z^{-m} = \sum_{m\in\mathbb{Z}} x[m]\big(\tfrac{z}{z_0}\big)^{-m} = X\big(\tfrac{z}{z_0}\big)$$

- $\mathcal{Z}$-domain differentiation

$$x[n] \xrightarrow{\mathcal{Z}.\mathrm{T.}} X(z),\ z \in \mathrm{ROC}_x$$

$$-(n-1)x[n-1] \xrightarrow{\mathcal{Z}.\mathrm{T.}} \tfrac{\mathrm{d}}{\mathrm{d}z} X(z),\ z \in \mathrm{ROC}_x$$

$$\tfrac{\mathrm{d}}{\mathrm{d}z} X(z) = \sum_{m\in\mathbb{Z}} x[m]\Big(\tfrac{\mathrm{d}}{\mathrm{d}z} z^{-m}\Big) = \sum_{m\in\mathbb{Z}} \big(-m x[m]\big) z^{-m-1} = \sum_{\tilde{m}\in\mathbb{Z}} \big(-(\tilde{m}-1)x[\tilde{m}-1]\big) z^{-\tilde{m}}$$

Properties of $\mathcal{Z}$-Transform

- Time-reversal

$$x[n] \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(z),\ z \in \text{ROC}_x$$

$$x[\text{-}n] \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(\tfrac{1}{z}),\ z \in (\text{ROC}_x)^{-1}$$

$$\sum_{m\in\mathbb{Z}} x[\text{-}m]z^{-m} = \sum_{\tilde{m}\in\mathbb{Z}} x[\tilde{m}]\left(\tfrac{1}{z}\right)^{-\tilde{m}} = X\left(\tfrac{1}{z}\right)$$

- Conjugation

$$x[n] \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(z),\ z \in \text{ROC}_x$$

$$\overline{x[n]} \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} \overline{X(\overline{z})},\ z \in \text{ROC}_x$$

$$\overline{X(\overline{z})} = \overline{\sum_{m\in\mathbb{Z}} x[m]\overline{z}^{-m}} = \sum_{m\in\mathbb{Z}} \overline{x[m]}z^{-m}$$

Properties of $\mathcal{Z}$-Transform

- Time expansion $\quad x[n] \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(z),\ z \in \text{ROC}_x$

$$M \in \mathbb{N}^+,\ x_{\uparrow M}[n] \triangleq \begin{cases} x[n/M] & n \in M\,\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \quad \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(z^M),\ z \in \sqrt[M]{\text{ROC}_x}$$

$$\sum_{m\in\mathbb{Z}} x_{\uparrow M}[m]z^{-m} = \sum_{\tilde{m}\in M\mathbb{Z}} x[\tilde{m}]z^{-M\tilde{m}} = X(z^M)$$

- Time shrinkage $\quad x[n] \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} X(z),\ z \in \text{ROC}_x$

$$x_{\downarrow M}[n] \triangleq x[M\,n] \quad \stackrel{\mathcal{Z}.\text{T.}}{\longmapsto} \quad \frac{1}{M}\sum_{k=0}^{M-1} X\left(\sqrt[M]{z}\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{M}k}\right),\ z \in (\text{ROC}_x)^M$$

$$\sum_{k=0}^{M-1} X\left(\sqrt[M]{z}\,\mathrm{e}^{\mathrm{j}\frac{2\pi}{M}k}\right) = \sum_{m\in\mathbb{Z}} x[m]z^{-\frac{m}{M}} \underbrace{\sum_{k=0}^{M-1} \mathrm{e}^{\text{-}\mathrm{j}2\pi k\frac{m}{M}}}_{=\begin{cases} M, & \frac{m}{M}\in\mathbb{Z}, \\ 0, & \text{otherwise} \end{cases}} = M \sum_{m\in M\mathbb{Z}} x[m]z^{-\frac{m}{M}}$$

Properties of $\mathscr{Z}$-Transform

- Convolution

$$\begin{aligned} x[n] &\xmapsto{\mathscr{Z}.\mathrm{T}.} X(z),\ z \in \mathrm{ROC}_x \\ w[n] &\xmapsto{\mathscr{Z}.\mathrm{T}.} W(z),\ z \in \mathrm{ROC}_w \\ (x * w)[n] &\xmapsto{\mathscr{Z}.\mathrm{T}.} ? \end{aligned}$$

$$y[n] \triangleq (x * w)[n] = \sum_{m\in\mathbb{Z}} x[n-m]\, w[m] \quad \Rightarrow \quad Y(z) = \sum_{m,n\in\mathbb{Z}} x[n-m]\, w[m] z^{-n}$$

$$= \sum_{m\in\mathbb{Z}} w[m] \Bigg(\underbrace{\sum_{n\in\mathbb{Z}} x[n-m] z^{-n}}_{\mathscr{Z}\{x[\cdot - m]\}(z)} \Bigg) = X(z) \sum_{m\in\mathbb{Z}} w[m]\, z^{-m} = X(z)W(z)$$

$$\Rightarrow \quad (x * w)[n] \xmapsto{\mathscr{Z}.\mathrm{T}.} X(z)\, W(z),\ z \in \mathrm{ROC}_x \cap \mathrm{ROC}_w$$

or larger

$\mathcal{Z}$-Transform

- $\mathcal{Z}$-Transform and Region of Convergence (ROC)
- Properties of $\mathcal{Z}$-Transform
- Characterization of LSI Systems
- One-sided (unilateral) $\mathcal{Z}$-Transform

Transfer Function (تابع تبدیل)

- LSI Systems

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = (x * h)[n]$$

$$\Rightarrow \quad Y(z) = \underbrace{H(z)}_{\text{Transfer Function}} X(z)$$

$$\underset{z \in \mathrm{ROC}_h}{x[n] = z^n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z) z^n$$

LSI System Properties

- Causality

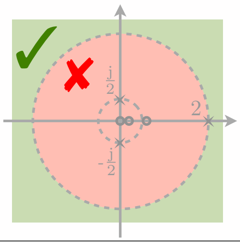
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\text{Causality} \iff h[n] = 0,\ \forall n < 0 \quad \Rightarrow \quad h[n] = \text{right-sided}$$

$$\Rightarrow \quad \boxed{\text{ROC}_h = \text{exterior of a circle}}$$

- Example

$$\text{Causal System,} \quad H(z) = \frac{(12 - 7z^{-1})(5 - z^{-1})}{(1 - 2z^{-1})(4 + z^{-2})} \quad \Rightarrow \quad h[n] = ?$$



$$H(z) = \frac{9}{1-2z^{-1}} + \frac{3+\frac{1}{4}\mathrm{j}}{1+\frac{\mathrm{j}}{2}z^{-1}} + \frac{3-\frac{1}{4}\mathrm{j}}{1-\frac{\mathrm{j}}{2}z^{-1}}$$

$$\Rightarrow \quad h[n] = \left(9\times 2^n + \frac{12\cos(\frac{\pi}{2}n)+\sin(\frac{\pi}{2}n)}{2^{n+1}}\right)u[n]$$

LSI System Properties

- Stability

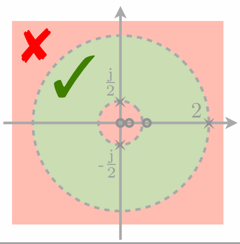
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\text{Stability} \iff h[n] \in \ell_1 \quad \Leftrightarrow \quad \widehat{h}(\mathrm{e}^{\mathrm{j}\omega}) \text{ exists (simple definition)}$$

$$\Leftrightarrow \quad \boxed{\{z,\ |z| = 1\} \subseteq \text{ROC}_h}$$

- Example

$$\text{Stable System,} \quad H(z) = \frac{(12 - 7z^{-1})(5 - z^{-1})}{(1 - 2z^{-1})(4 + z^{-2})} \quad \Rightarrow \quad h[n] = ?$$



$$H(z) = \frac{9}{1-2z^{-1}} + \frac{3+\frac{1}{4}\mathrm{j}}{1+\frac{\mathrm{j}}{2}z^{-1}} + \frac{3-\frac{1}{4}\mathrm{j}}{1-\frac{\mathrm{j}}{2}z^{-1}}$$

$$\Rightarrow \quad h[n] = -9\times 2^n u[-n-1] + \frac{12\cos(\frac{\pi}{2}n)+\sin(\frac{\pi}{2}n)}{2^{n+1}}u[n]$$

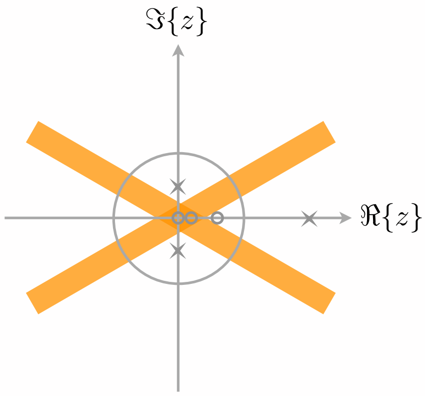
LSI System Properties

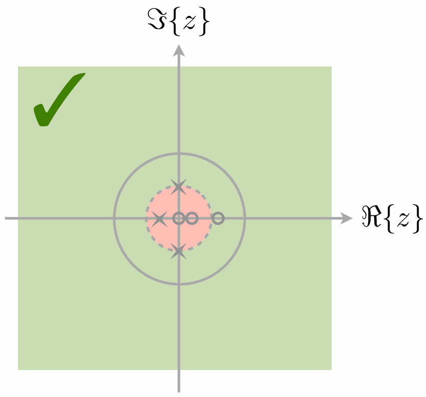
- Causal and Stable

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\text{Causal+Stable} \quad \Rightarrow \quad \{z,\ |z| \geq 1\} \subseteq \mathrm{ROC}_h$$

$$\Rightarrow \quad \text{all poles strictly inside the unit circle}$$





Difference Systems

- Linear constant-coefficient difference systems

$$x[n] \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k y[n-k] = \sum_{k=0}^{N_x} b_k x[n-k]} \longrightarrow$$

$$\text{proper boundary conditions} \quad \Rightarrow \quad \text{LSI} \quad \Rightarrow \quad Y(z) = \underbrace{H(z)}_{?}\, X(z)$$

$$\sum_{k=0}^{N_y} a_k y[n-k] = \sum_{k=0}^{N_x} b_k x[n-k] \quad \Rightarrow \quad \mathcal{Z}\left\{\sum_{k=0}^{N_y} a_k y[n-k]\right\}(z) = \mathcal{Z}\left\{\sum_{k=0}^{N_x} b_k x[n-k]\right\}(z)$$

$$\Rightarrow \quad \sum_{k=0}^{N_y} a_k z^{-k}\, Y(z) = \sum_{k=0}^{N_x} b_k z^{-k}\, X(z) \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N_x} b_k z^{-k}}{\sum_{k=0}^{N_y} a_k z^{-k}}$$

System Identification

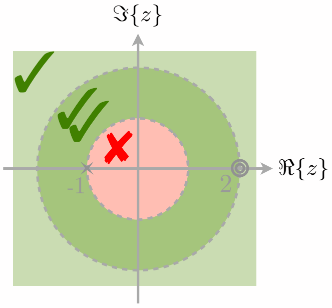
- Example

$$x[n] = n\,2^n u[\text{-}n] \longrightarrow \boxed{\text{LSI}} \longrightarrow y[n] = (\text{-}1)^n u[n]$$

$$\Rightarrow \quad h[n] = ?$$

$$X(z) = \mathcal{Z}\{n\,2^n u[\text{-}n]\}(z) = (\text{-}z)\tfrac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}\{2^n u[\text{-}n]\}(z) = \tfrac{\text{-}2z^{\text{-}1}}{(1-2z^{\text{-}1})^2} \qquad |z| < 2$$

$$Y(z) = \mathcal{Z}\{(\text{-}1)^n u[n]\}(z) = \tfrac{1}{1+z^{\text{-}1}} \qquad |z| > |\text{-}1| = 1$$



$$H(z) = \frac{Y(z)}{X(z)} = \frac{\text{-}(1-2z^{\text{-}1})^2}{2z^{\text{-}1}(1+z^{\text{-}1})}$$

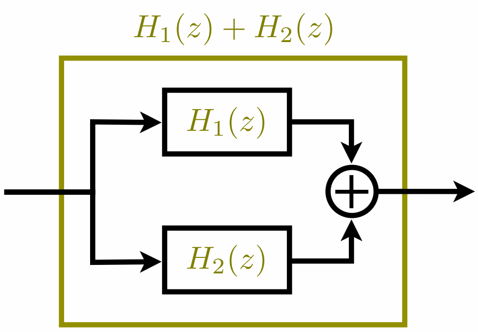
$$= 2 + \frac{1}{2z^{\text{-}1}} - \frac{\frac{9}{2}}{1+z^{\text{-}1}}$$

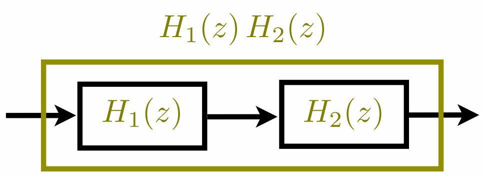
$$\Rightarrow \quad h[n] = 2\delta[n] + \tfrac{1}{2}\delta[n+1] - \tfrac{9}{2}(\text{-}1)^n u[n]$$

Transfer Function Algebra

- Block diagram representation

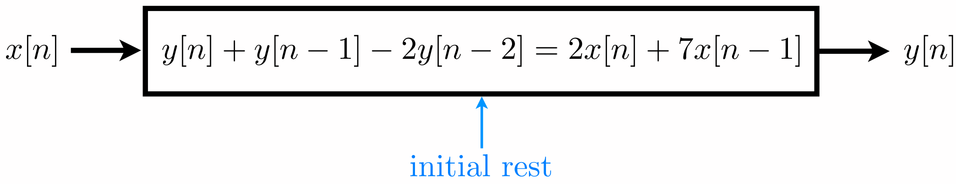
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \quad \equiv \quad x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$



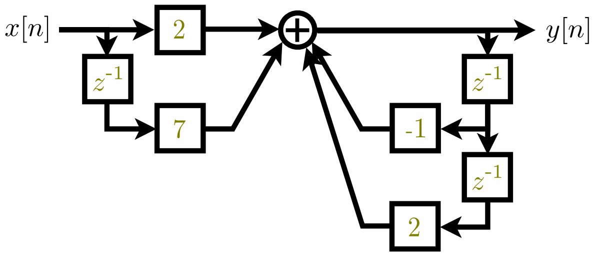


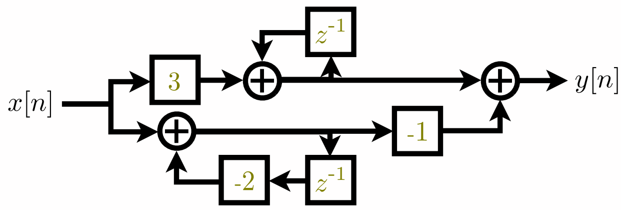
Transfer Function Algebra

- Example



$$H(z) = \frac{2 + 7z^{-1}}{1 + z^{-1} - 2z^{-2}} = \frac{2 + 7z^{-1}}{(1 - z^{-1})(1 + z^{-1})} = \frac{3}{1 - z^{-1}} - \frac{1}{1 + 2z^{-1}}$$





$\mathcal{Z}$-Transform

- $\mathcal{Z}$-Transform and Region of Convergence (ROC)
- Properties of $\mathcal{Z}$-Transform
- Characterization of LSI Systems
- One-sided (unilateral) $\mathcal{Z}$-Transform

Unilateral $\mathcal{Z}$-Transform

$$\underline{X}(z) = \mathcal{UZ}\{x[n]\}(z) = \sum_{m=0}^{\infty} x[m] z^{-m}$$

$$\Rightarrow \quad \text{ROC} = \text{exterior of a circle}$$

$$\underline{X}(z) = \mathcal{Z}\Big\{x[n]\, u[n]\Big\}(z)$$

- Example $\quad x[n] = \alpha^{|n|}, \quad |\alpha| < 1$

$$X(z) = \mathcal{Z}\{x[n]\}(z) = \frac{(\alpha - \alpha^{-1})z^{-1}}{(1-\alpha^{-1}z^{-1})(1-\alpha z^{-1})}$$

$$\underline{X}(z) = \mathcal{UZ}\{x[n]\}(z) = \mathcal{Z}\{\alpha^n u[n]\}(z) = \frac{1}{1-\alpha z^{-1}}$$

Application

- Example

$$x[n] \longrightarrow \boxed{3y[n] - 5y[n-1] - 2y[n-2] = x[n]} \longrightarrow y[n]$$

$$y[\text{-}1] = \beta,\ \ y[\text{-}2] = \gamma$$

$$\begin{cases} \mathcal{UZ}\{y[n-1]\}(z) & = & \sum_{m=0}^{\infty} y[m-1]z^{\text{-}m} = \underbrace{y[\text{-}1]}_{\beta} + z^{\text{-}1}\,\underline{Y}(z) \\ \mathcal{UZ}\{y[n-2]\}(z) & = & \sum_{m=0}^{\infty} y[m-2]z^{\text{-}m} = \underbrace{y[\text{-}2]}_{\gamma} + \underbrace{y[\text{-}1]}_{\beta}\,z^{\text{-}1} + z^{\text{-}2}\,\underline{Y}(z) \end{cases}$$

$$\Rightarrow \quad \underline{Y}(z) = \underbrace{\frac{\underline{X}(z)}{3 - 5z^{\text{-}1} - 2z^{\text{-}2}}}_{\text{zero-state response}} + \underbrace{\frac{5\beta + 2\gamma + 2\beta z^{\text{-}1}}{3 - 5z^{\text{-}1} - 2z^{\text{-}2}}}_{\text{zero-input response}}$$

$$\hookrightarrow \left(\tfrac{2\gamma-\beta}{21}\left(\tfrac{\text{-}1}{3}\right)^n + \tfrac{12\beta+4\gamma}{7}2^n\right)u[n]$$