Linear Shift-Invariant Systems

- LSI systems and Convolution
- Properties of LSI systems
- Differential and Difference systems
- Singularity functionals

LSI Systems (سیستمهای خطی تغییرناپذیر با زمان)

- Linearity

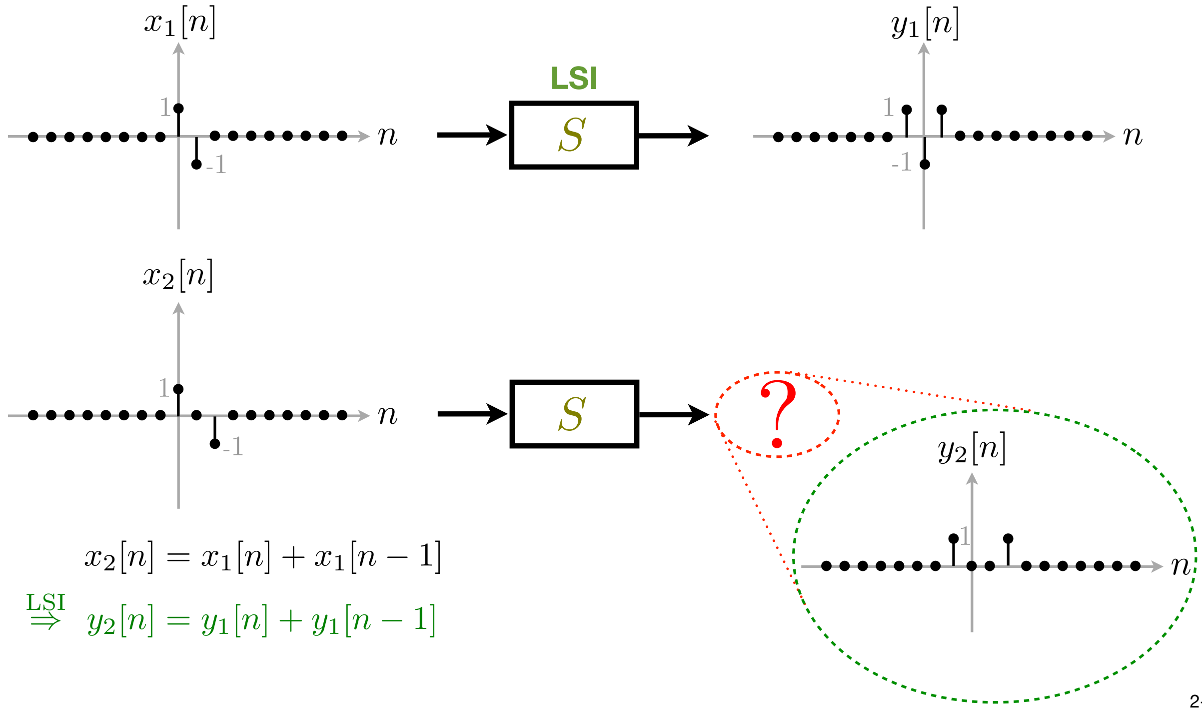
$$\begin{cases} x_1 \mapsto y_1 \\ x_2 \mapsto y_2 \end{cases} \Rightarrow \quad a\,x_1 + b\,x_2 \mapsto a\,y_1 + b\,y_2$$

- SI

$$x[n] \mapsto y[n] \quad \Rightarrow \quad x[n-n_0] \mapsto y[n-n_0]$$
$$x(t) \mapsto y(t) \quad \Rightarrow \quad x(t-t_0) \mapsto y(t-t_0)$$

LSI Systems (سیستمهای خطی تغییرناپذیر با زمان)

- Discrete-domain

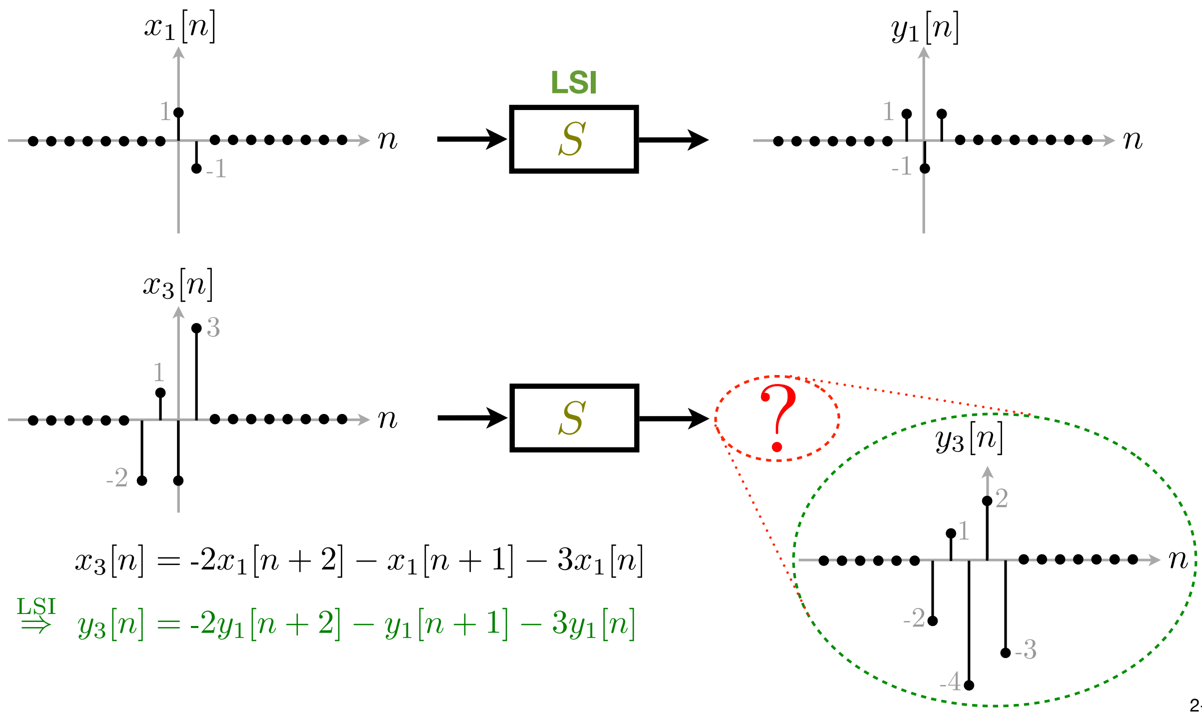


$$x_2[n] = x_1[n] + x_1[n-1]$$

$$\overset{\text{LSI}}{\Rightarrow} \; y_2[n] = y_1[n] + y_1[n-1]$$

LSI Systems (سیستمهای خطی تغییرناپذیر با زمان)

- Discrete-domain

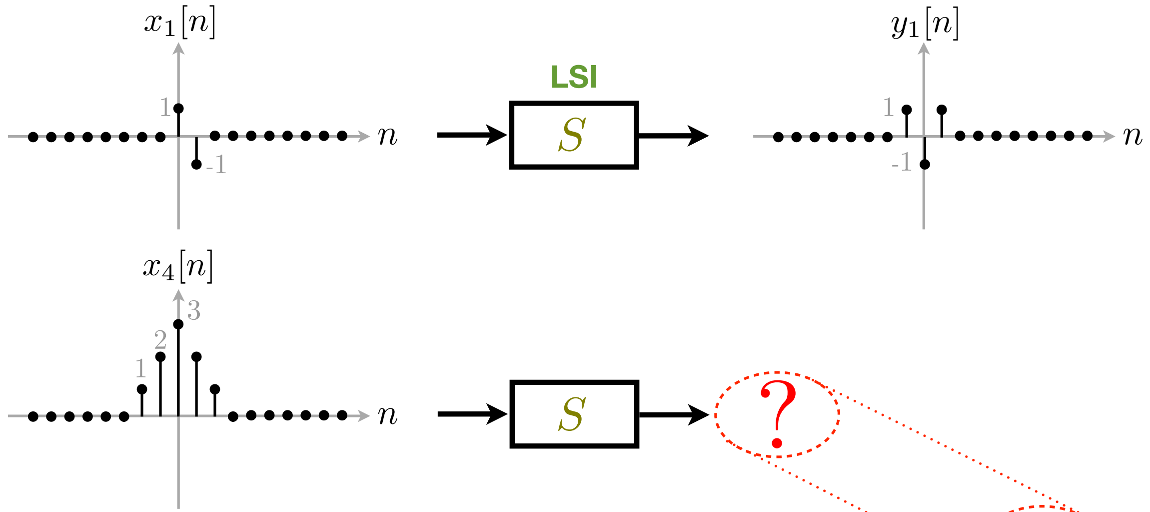


$$x_3[n] = \text{-}2x_1[n+2] - x_1[n+1] - 3x_1[n]$$

$$\overset{\text{LSI}}{\Rightarrow} \; y_3[n] = \text{-}2y_1[n+2] - y_1[n+1] - 3y_1[n]$$

LSI Systems (سیستم‌های خطی تغییرناپذیر با زمان)

- Discrete-domain

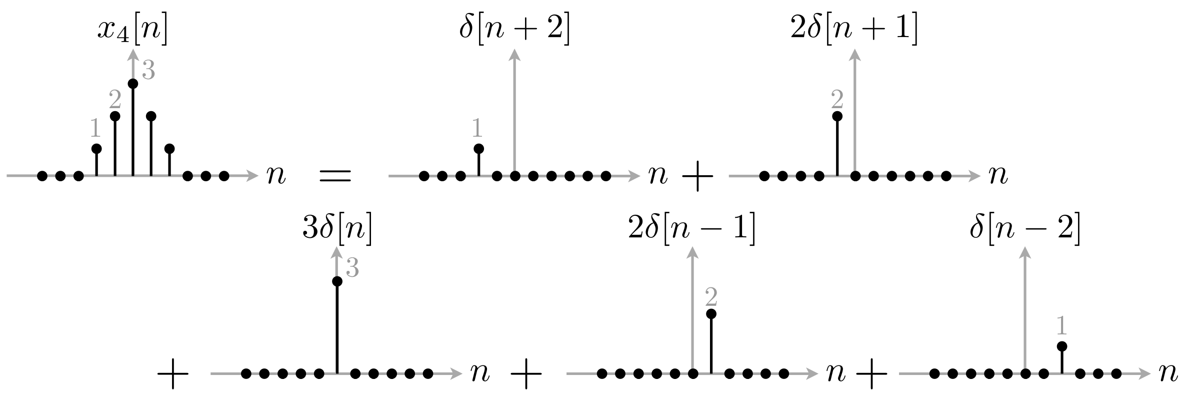


$$\sum_n x_1[n] = 0 \;\Rightarrow\; \sum_n \sum_m a_m\, x_1[n-m] = 0 \neq \sum_n x_4[n]$$

$$\Rightarrow\;\; x_4[n] \neq \sum_m a_m\, x_1[n-m]$$

Representation in terms of delta function

What is a suitable $x_1[\cdot]$?



$$x_4[n] = \underset{x_4[\text{-}2]}{1} \times \delta[n+2] + \underset{x_4[\text{-}1]}{2} \times \delta[n+1] + \underset{x_4[0]}{3} \times \delta[n] + \underset{x_4[1]}{2} \times \delta[n-1] + \underset{x_4[2]}{1} \times \delta[n-2]$$

Discrete-domain Convolution

- Representation

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

- Impulse response

$$\delta[n] \longrightarrow \overset{\text{LSI}}{\boxed{S}} \longrightarrow h_s[n]$$

- Arbitrary response

$$x[m]\,\delta[\cdot - m] \overset{\text{LSI}}{\longmapsto} x[m]\,h_s[\cdot - m]$$

$$\sum_{m=-M}^{M} x[m]\,\delta[\cdot - m] \longmapsto \sum_{m=-M}^{M} x[m]\,h_s[\cdot - m]$$

$$x[\cdot] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[\cdot - m] \overset{\text{often}}{\longmapsto} \sum_{m=-\infty}^{\infty} x[m]\,h_s[\cdot - m]$$

Discrete-domain Convolution

- Discrete-domain Convolution

$$\begin{cases} z \in \ell_p \\ w \in \ell_q \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \Rightarrow \quad (z * w)[n] \triangleq \sum_{m=-\infty}^{\infty} z[n-m]\,w[m]$$

$$= \sum_{m=-\infty}^{\infty} w[n-m]\,z[m] \triangleq (w * z)[n]$$

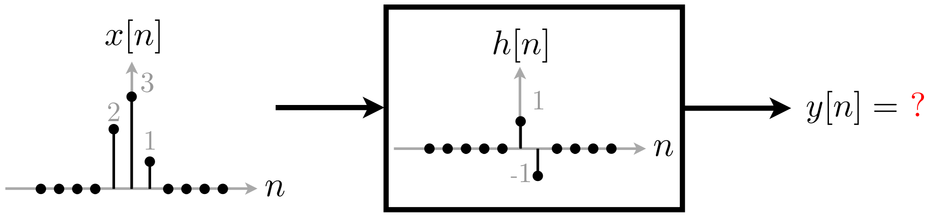
- Discrete-domain LSI system

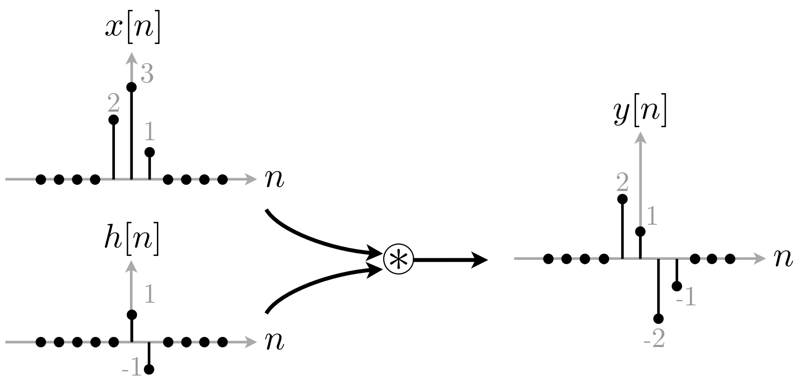
$$x[n] \longrightarrow \boxed{h[\cdot]} \longrightarrow y[n] = (x * h)[n]$$

$$\exists\, h,\ \forall\, x, \quad y[n] = (x * h)[n] \quad \Rightarrow \quad \text{LSI system}$$

Discrete-domain Convolution

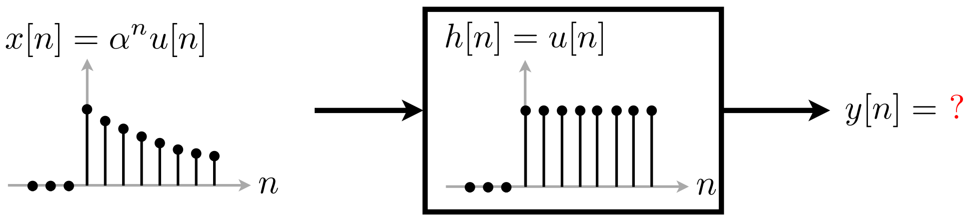
- Example





Convolution

- Example



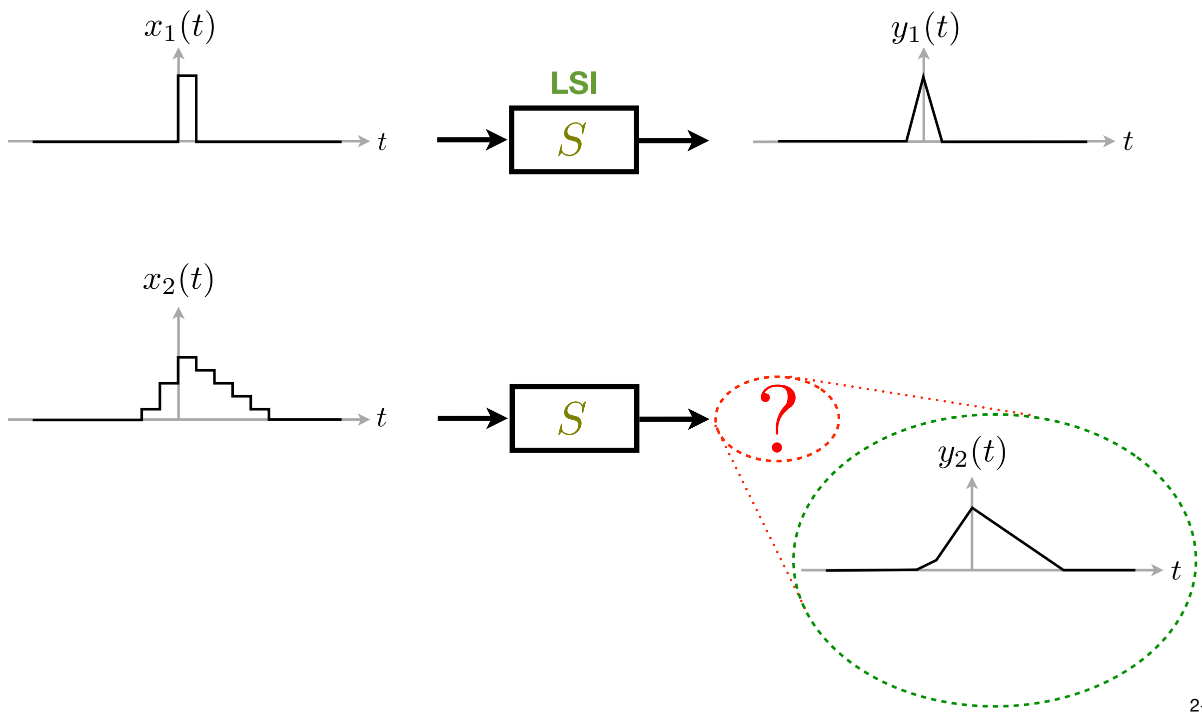
$$n < 0: \quad y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = 0$$

$$n \geq 0: \quad y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

$$= \sum_{m=-\infty}^{n} x[m] = \sum_{m=0}^{n} \alpha^m = \frac{1-\alpha^{n+1}}{1-\alpha}$$

LSI Systems (سیستم‌های خطی تغییرناپذیر با زمان)

- Continuous-domain

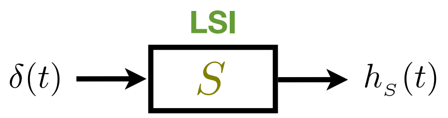


Continuous-domain Convolution

- Representation

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \qquad x(t) = \int_{\mathbb{R}} x(\tau)\delta(t-\tau)\mathrm{d}\tau$$

- Impulse response



- Arbitrary response

$$x(\tau)\,\delta(\cdot-\tau) \xmapsto{\text{LSI}} x(\tau)\,h_S(\cdot-\tau)$$

$$x(\cdot) = \int_{\mathbb{R}} x(\tau)\,\delta(\cdot-\tau)\mathrm{d}\tau \xmapsto{\text{often}} \int_{\mathbb{R}} x(\tau)\,h_S(\cdot-\tau)\mathrm{d}\tau$$

Continuous-domain Convolution

- Continuous-domain Convolution

$$\begin{cases} z \in L_p \\ w \in L_q \\ \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \Rightarrow \quad (z * w)(t) \triangleq \int_{\mathbb{R}} z(t-\tau)\, w(\tau)\, \mathrm{d}\tau = \int_{\mathbb{R}} w(t-\tau)\, z(\tau)\, \mathrm{d}\tau \triangleq (w * z)(t)$$

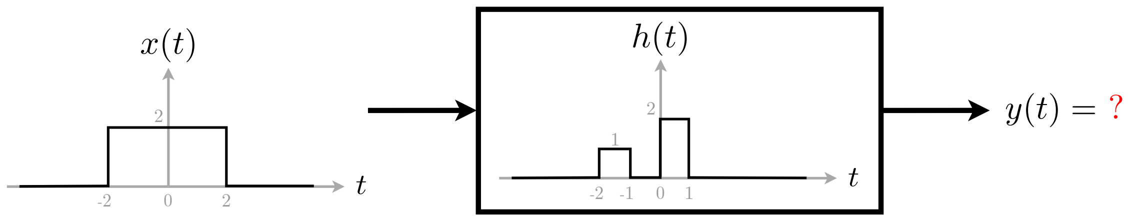
- Continuous-domain LSI system

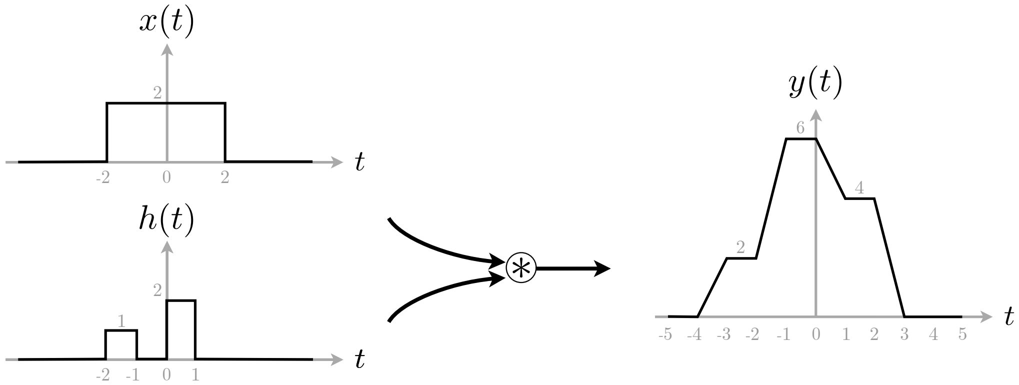
$$x(t) \longrightarrow \boxed{h(\cdot)} \longrightarrow y(t) = (x * h)(t)$$

$$\exists\, h,\ \forall\, x, \quad y(t) = (x * h)(t) \quad \Rightarrow \quad \text{LSI system}$$

Continuous-domain Convolution

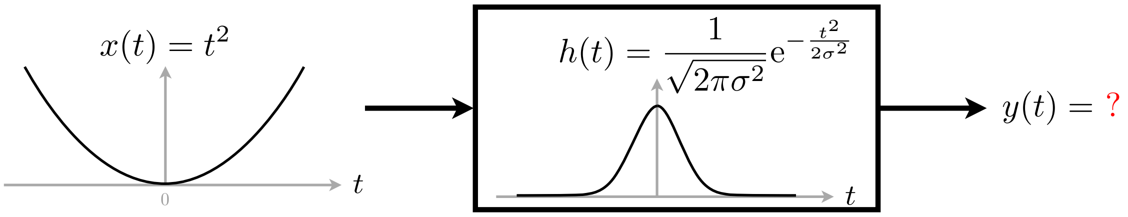
- Example





Continuous-domain Convolution

- Example



$$y(t) = (x * h)(t) = \int_{\mathbb{R}} x(t-\tau)h(\tau)\mathrm{d}\tau = \int_{\mathbb{R}} (t-\tau)^2 \frac{\mathrm{e}^{-\frac{\tau^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\mathrm{d}\tau$$

$$= t^2 \underbrace{\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{\tau^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\mathrm{d}\tau}_{1} - 2t \underbrace{\int_{\mathbb{R}} \tau \frac{\mathrm{e}^{-\frac{\tau^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\mathrm{d}\tau}_{0} + \underbrace{\int_{\mathbb{R}} \tau^2 \frac{\mathrm{e}^{-\frac{\tau^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\mathrm{d}\tau}_{\sigma^2}$$

$$= t^2 + \sigma^2$$

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Commutativity (جابه جایی)

$$(w * z)[n] = (z * w)[n] \qquad (w * z)(t) = (z * w)(t)$$

$$x[n] \longrightarrow \boxed{h[\cdot]} \longrightarrow y[n] = (x * h)[n]$$

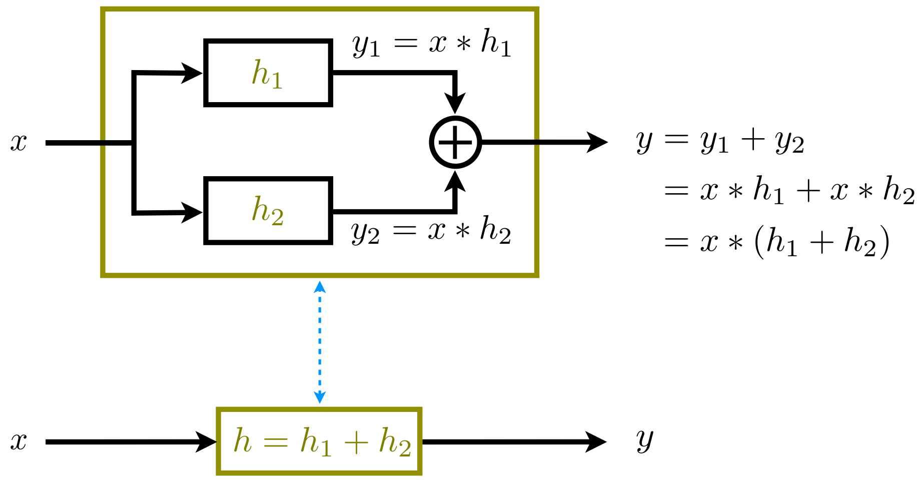
$$h[n] \longrightarrow \boxed{x[\cdot]} \longrightarrow y[n] = (h * x)[n]$$

$$h(t) \longrightarrow \boxed{x(\cdot)} \longrightarrow y(t) = (h * x)(t)$$

$$x(t) \longrightarrow \boxed{h(\cdot)} \longrightarrow y(t) = (x * h)(t)$$

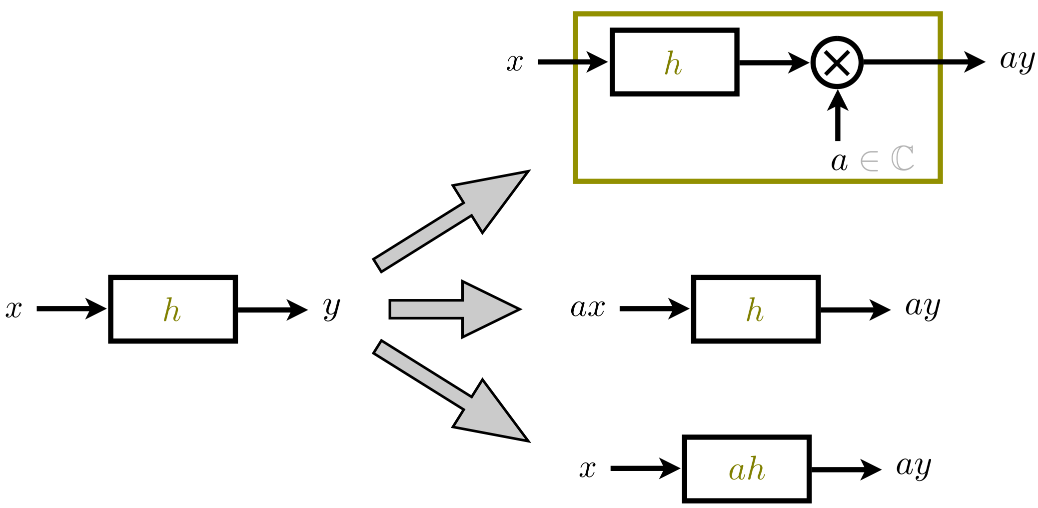
Distributivity (پخشی)

$$x * (w + z) = x * w + x * z$$



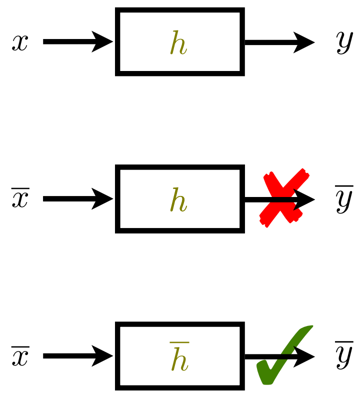
Associativity with scalars

$$a(w * z) = \big((aw) * z\big), \quad \forall\, a \in \mathbb{C}$$



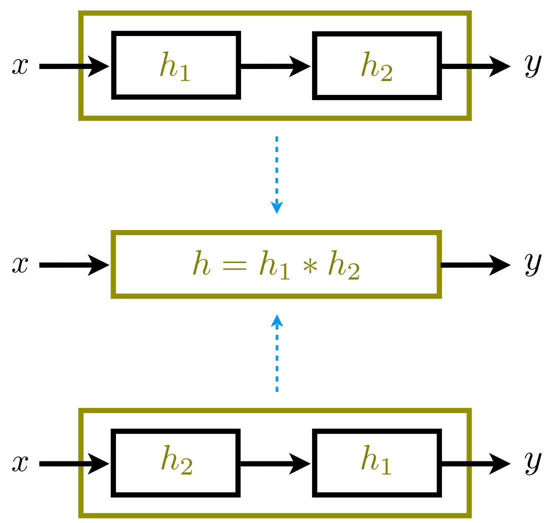
Conjugation (مزدوج گیری)

$$\overline{(w * z)[n]} = (\overline{w} * \overline{z})[n] \qquad \overline{(w * z)(t)} = (\overline{w} * \overline{z})(t)$$



Associativity (شرکت پذیری)

$$x * (w * z) = (x * w) * z$$



LSI Systems with[out] memory

- Memoryless systems

$$\begin{array}{ccc} \vdots & & \vdots \\ x[-1] & \longrightarrow & y[-1] \\ x[0] & \longrightarrow & y[0] \\ \vdots & & \vdots \end{array}$$

- LSI systems

$$y[n] = (x * h)[n] = \sum_m x[n-m]h[m]$$

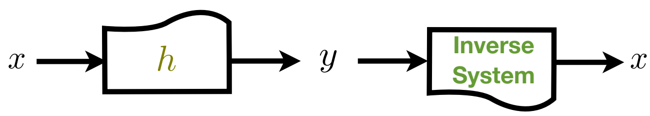
$$y(t) = (x * h)(t) = \int_{\mathbb{R}} x(t-\tau)h(\tau)\mathrm{d}\tau$$

- Memoryless LSI systems

$$h[n] = k\,\delta[n], \quad k \in \mathbb{R} \text{ or } \mathbb{C}$$

$$h(t) \overset{\text{a.e.}}{=} k\,\delta(t), \quad k \in \mathbb{R} \text{ or } \mathbb{C}$$

Invertible LSI Systems



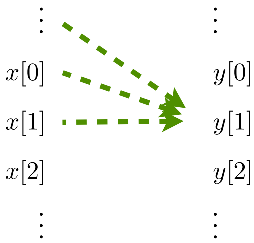
$$\begin{cases} x[n-n_0] \overset{h[\cdot]}{\mapsto} y[n-n_0] \\ x(t-t_0) \overset{h(\cdot)}{\mapsto} y(t-t_0) \end{cases} \Rightarrow \begin{cases} y[n-n_0] \underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} x[n-n_0] \\ y(t-t_0) \underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} x(t-t_0) \end{cases} \Rightarrow \boxed{\underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} = \text{SI}}$$

$$a\,x_1 + b\,x_2 \overset{h}{\mapsto} a\,y_1 + b\,y_2 \Rightarrow a\,y_1 + b\,y_2 \underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} a\,x_1 + b\,x_2 \Rightarrow \boxed{\underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} = \text{Linear}}$$

$$\Rightarrow \underset{\text{sys.}}{\overset{\text{inv.}}{\mapsto}} \equiv \longrightarrow \boxed{h_{\text{inv}}} \longrightarrow \quad \& \quad h * h_{\text{inv}} = \delta$$

Causal LSI Systems

- Causal systems



- LSI systems

$$y[n] = (x * h)[n] = \sum_m x[n-m]h[m]$$

$$y(t) = (x * h)(t) = \int_{\mathbb{R}} x(t-\tau)h(\tau)\mathrm{d}\tau$$

- Causal LSI systems

$$h[n] = 0, \quad n < 0$$

$$h(t) \overset{\text{a.e.}}{=} 0, \quad t < 0$$

Stable LSI Systems

- Stable systems

$$x[\cdot] \in \ell_\infty \quad \mapsto \quad y[\cdot] \in \ell_\infty$$

$$x(\cdot) \in L_\infty \quad \mapsto \quad y(\cdot) \in L_\infty$$

- LSI systems

$$y[n] = (x * h)[n] = \sum_m x[n-m]h[m]$$

$$y(t) = (x * h)(t) = \int_{\mathbb{R}} x(t-\tau)h(\tau)\mathrm{d}\tau$$

- Stable LSI system

$$x[n] \triangleq \mathrm{sign}(h[\text{-}n]) \in \ell_\infty \quad \Rightarrow \quad y[0] = \sum_m \big|h[m]\big| < \infty \quad \Rightarrow$$

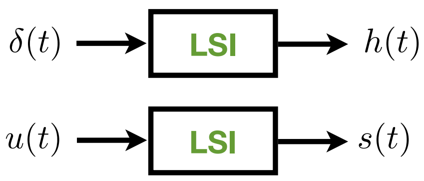
- LSI system with $h \in \ell_1$

$$\big|y[n]\big| \leq \sum_m \big|x[n-m]\big| \cdot \big|h[m]\big| \leq \sup_n \big|x[n]\big| \underbrace{\sum_m \big|h[m]\big|}_{<\infty} \quad \Rightarrow$$

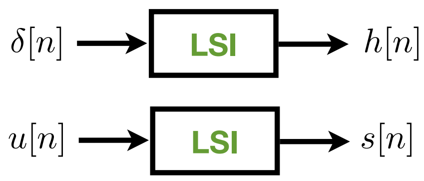
$$\boxed{h \in \ell_1 \; \Updownarrow \; \text{Stability}} \qquad \boxed{h \in L_1 \; \Updownarrow \; \text{Stability}}$$

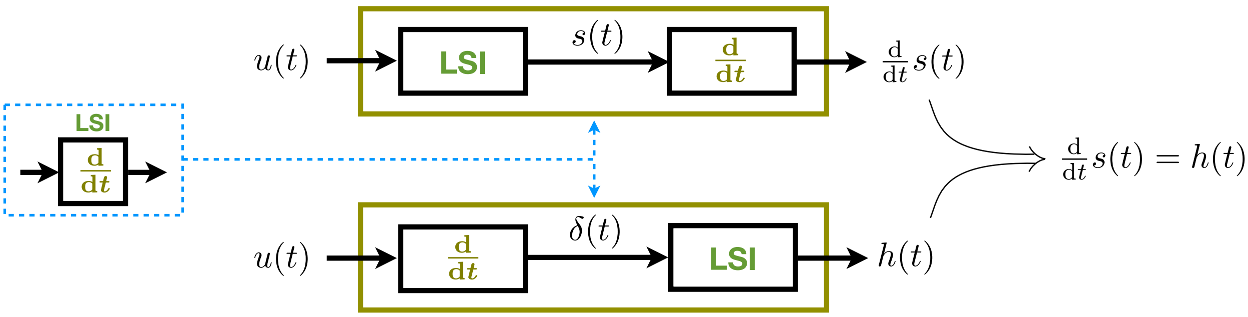
Step Response

- Continuous-domain



- Discrete-domain





Linear Shift-Invariant Systems

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Differential Systems

- Non-linear differential systems

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + y(t)^2 = x(t) \qquad\qquad y(t)\times\frac{\mathrm{d}}{\mathrm{d}t}y(t) = (x * u)(t)$$

- Linear differential systems

$$t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + t\frac{\mathrm{d}}{\mathrm{d}t}y(t) + (t^2 - \alpha^2)y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

- Linear constant-coefficient differential systems

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+2\frac{\mathrm{d}}{\mathrm{d}t}y(t)-3y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)-\frac{1}{3}x(t)$$

Differential Systems

- Linear constant-coefficient differential systems

$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y(t) = \sum_{k=0}^{N_x} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t)} \longrightarrow y(t)$$

N_y boundary conditions

$$y(t) = y_h(t) + y_p(t)$$

homogeneous solution (جواب عمومی)

particular solution (جواب خصوصی)

$$\sum_{k=0}^{N_y} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} y_h(t) = 0$$

Linear Constant-coefficient Differential Systems

- Example

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) + 4y(t) = 2u(t)$$

$$\text{b.c.} \begin{cases} y(0) & = & 1 \\ y(\frac{\pi}{4}) & = & \text{-}1 \end{cases}$$

$$y_h(t) = \alpha \sin(2t) + \beta \cos(2t), \quad \forall \alpha, \beta$$

$$y_p(t) = \frac{1 - \cos(2t)}{2} u(t)$$

$$\begin{cases} \alpha & = & \text{-}\frac{3}{2} \\ \beta & = & 1 \end{cases}$$

Linear Constant-coefficient Differential Systems

- Example

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + \frac{\mathrm{d}}{\mathrm{d}t}y(t) - 2y(t) = x(t)$$

$$y_h(t) = \alpha \mathrm{e}^{t} + \beta \mathrm{e}^{-2t}, \quad \forall\, \alpha, \beta$$

- LSI?

Boundary condition	Linear	SI
$y(0) = 1,\ \dot{y}(0) = -1$	✗	✗
$y(0) = 0,\ \dot{y}(0) = 0$	✓	✗
$y(0) = 0,\ y(\pi) = 0$	✓	✗
initial rest: $\boxed{y(t) = 0 \text{ for all } t \le t_0 \text{ if } x(t) = 0 \text{ for all } t \le t_0}$	✓	✓

Linear Constant-coefficient Difference Systems

- Continuous-domain differentiation

$$z(t) \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}z(t)$$

- Discrete-domain differentiation

$$z[n] \longmapsto z[n] - z[n-1]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) - 3\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t) \longleftrightarrow y[n] + y[n-1] + y[n-2] = x[n]$$

$$x[n] \longrightarrow \boxed{\sum_{k=0}^{N_y} a_k y[n-k] = \sum_{k=0}^{N_x} b_k x[n-k]} \longrightarrow y[n] = y_h[n] + y_p[n]$$

N_y boundary conditions

homogeneous solution (جواب عمومی)

$$\sum_{k=0}^{N_y} a_k y_h[n-k] = 0$$

particular solution (جواب خصوصی)

Linear Constant-coefficient Difference Systems

- Example

$$y[n] - y[n-1] - y[n-2] = u[n]$$

$$\text{b.c.} \begin{cases} y[0] & = & 0 \\ y[\text{-}1] & = & 1 \end{cases}$$

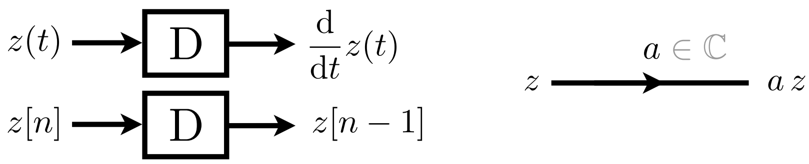
$$y_h[n] = \alpha \left(\tfrac{1+\sqrt{5}}{2}\right)^n + \beta \left(\tfrac{1-\sqrt{5}}{2}\right)^n, \quad \forall\, \alpha, \beta$$

$$y_p[n] = \left((1+\tfrac{2}{\sqrt{5}}) \left(\tfrac{1+\sqrt{5}}{2}\right)^n + (1-\tfrac{2}{\sqrt{5}}) \left(\tfrac{1-\sqrt{5}}{2}\right)^n - 1\right) u[n]$$

$$\begin{cases} \alpha & = & \frac{1-\sqrt{5}}{2\sqrt{5}} \\ \beta & = & \text{-}\frac{1+\sqrt{5}}{2\sqrt{5}} \end{cases}$$

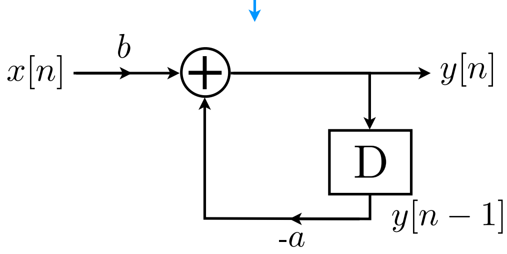
Block Diagram Representation

- Basic blocks

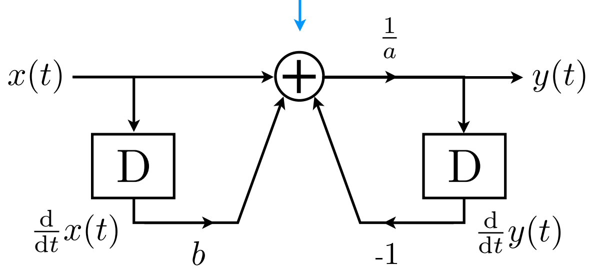


- Examples

$$y[n] + a\,y[n-1] = b\,x[n]$$



$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + a y(t) = b\,\frac{\mathrm{d}}{\mathrm{d}t} x(t) + x(t)$$



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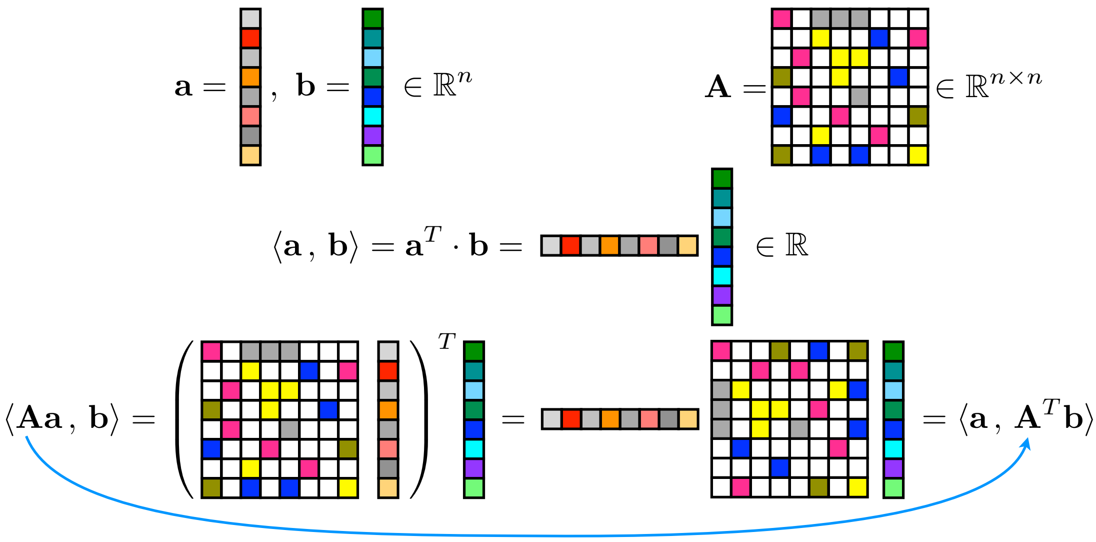
Adjoint Operator (عملگر الحاقی)



$$z(\cdot),\, w(\cdot) \;=\; \text{signals}: \quad \langle z, w\rangle \triangleq \int_{\mathbb{R}} z(\tau)w(\tau)\mathrm{d}\tau$$

$$\mathrm{L} \;=\; \underline{\text{linear}}\text{ sys.} \quad \Rightarrow \quad \exists\, \underbrace{\mathrm{L}^* : \text{linear sys.}}_{\text{adjoint of L}},\; \forall\, z, w : \quad \langle \mathrm{L}z\,,\, w\rangle = \langle z\,,\, \mathrm{L}^* w\rangle$$

Singular Functions

- Derivative

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{L}^* = ?$$

$$z, w : \underset{\text{fast decaying}}{\underset{\text{smooth \&}}{\mathbb{R} \mapsto \mathbb{R}}} \quad \Rightarrow \quad \left\langle \tfrac{\mathrm{d}}{\mathrm{d}t} z(t) \,,\, w(t) \right\rangle = \int_{\mathbb{R}} \tfrac{\mathrm{d}}{\mathrm{d}\tau} z(\tau)\, w(\tau)\, \mathrm{d}\tau$$

$$= -\int_{\mathbb{R}} z(\tau)\, \tfrac{\mathrm{d}}{\mathrm{d}\tau} w(\tau)\, \mathrm{d}\tau = \left\langle z(t) \,,\, -\tfrac{\mathrm{d}}{\mathrm{d}t} w(t) \right\rangle$$

- Doublet functional

$$\dot{\delta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t) \;{\color{red}=?}$$

$$\boxed{\langle \dot{\delta} \,,\, x \rangle} = \langle \tfrac{\mathrm{d}}{\mathrm{d}t}\delta(t) \,,\, x(t) \rangle \triangleq -\langle \delta(t) \,,\, \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) \rangle = \boxed{-\tfrac{\mathrm{d}}{\mathrm{d}t} x(t)\Big|_{t=0}}$$

Singular Functions

- High-order derivatives

$$z, w : \underset{\text{fast decaying}}{\underset{\text{smooth \&}}{\mathbb{R} \mapsto \mathbb{R}}} \quad \Rightarrow \quad \left\langle \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} z(t) \,,\, w(t) \right\rangle = -\left\langle \tfrac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} z(t) \,,\, \tfrac{\mathrm{d}}{\mathrm{d}t} w(t) \right\rangle = \dots$$

$$= (-1)^n \left\langle z(t) \,,\, \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} w(t) \right\rangle$$

$$\mathrm{L} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \quad \Rightarrow \quad \mathrm{L}^* = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}$$

$$x(t) \longrightarrow \boxed{\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}} \longrightarrow \frac{\mathrm{d}^n}{\mathrm{d}t^n} x(t)$$

$$\delta(t) \longrightarrow \boxed{\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}} \longrightarrow h(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta(t)$$

$$\big(x * h\big)(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} x(t)$$