جامعـــة الفيـــوم
كلية الحاسبات والذكاء الاصطناعى
مؤسسة تعليمية معتمدة

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جامعـة الفيوم

كلية الحاسبـــات والذكاء الاصطناعى

FAYOUM UNIVERSITY- FACULTY OF COMPUTERS AND ARTIFICIAL INTELLIGENCE

MATHEMATICS (3)

Second Year

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Part I

Chapter 1

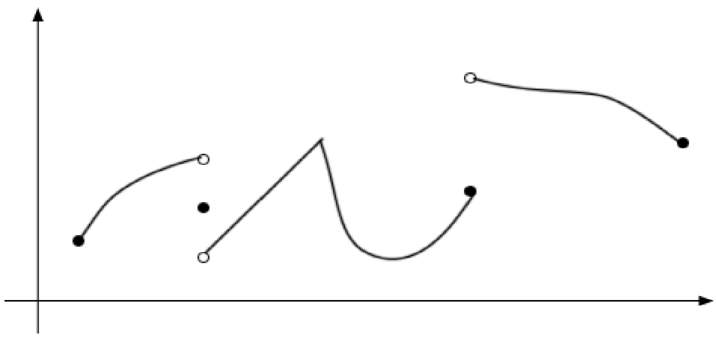
Laplace transform

Introduction

Laplace transform is named in honour of the great French mathematician, Pierre Simon De Laplace (1749-1827). Like all transforms, the Laplace transform changes one signal into another according to some fixed set of rules or equations. The best way to convert differential equations into algebraic equations is the use of Laplace transformation.

Definition piecewise continuous

A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



For example,

$$f(x) = \begin{cases} 7x - 5 & x \leq 1 \\ 3x^2 - x & 1 < x \leq 3 \\ x^3 + 4 & x > 3 \end{cases}$$

Definition of Laplace transform

Assume that the function $f(t)$ is a piecewise continuous function, then $f(t)$ is defined using the Laplace transform by the integral transform of the given derivative function with real variable t to convert into a complex function with variable s.

The Laplace transform of $f(t)$,that is denoted $L\{f(t)\}\ or\ F(s)$.
It is defined by the Laplace transform formula:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Table of Laplace Transform

$f(t)\,, t \geq 0$	$F(S) = L[f(t)]$
a constant	$\frac{a}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
t^n	$\frac{n!}{s^{n+1}}, s > 0$
$sinat$	$\frac{a}{s^2 + a^2}$
$cosat$	$\frac{s}{s^2 + a^2}$
$sinhat$	$\frac{a}{s^2 - a^2}, s > \lvert a \rvert$
$coshat$	$\frac{s}{s^2 - a^2}, s > \lvert a \rvert$

Properties of Laplace Transform

1- Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then

$$L\{Af_1(t) + Bf_2(t)\} = AF_1(s) + BF_2(s)$$

Example 1 Find:

1. $L[5sinh2t - 6\,e^{4t} + 2]$
2. $L[3cos3t + t^3 - e^{-2t}]$

Solution:

1. $L[5sinh2t] - 6L[\,e^{4t}] + L[2]$

$$= 5.\frac{2}{s^2 - 4} - \frac{6}{s - 4} + \frac{2}{s}$$

2. $3L[cos3t] + L[t^3] - L[e^{-2t}]$

$$= 3.\frac{s}{s^2 + 9} + \frac{3!}{s^4} - \frac{1}{s + 2}$$

2- Shifting

Theorem 1 (The shift theorem)

If $L\{f(t)\} = F(s)$, then

$$L\{e^{-at}.f(t)\} = F(s + a)$$

Example 2 Find:

1. $L[t^2\,e^{-4t}]$
2. $L[cos5t\,e^{-2t}]$
3. $L[t\,e^{2t}]$

Solution:

1. Since $L[t^2\ ] = \frac{2}{s^3}$, then

$$L[t^2\ e^{-4t}] = \frac{2}{(s+4)^3}$$

2. Since $L[cos5t\ ] = \frac{s}{s^2+25}$, then

$$L[cos5t\ e^{-2t}] = \frac{(s+2)}{(s+2)^2+25}$$

3. Since $L[t\ ] = \frac{1}{s^2}$, then

$$L[t\ e^{2t}] = \frac{1}{(s-2)^2}$$

3- Multiplication by Time

Theorem 2

If $L\{f(t)\} = F(s)$, then

$$L\{t.f(t)\} = -\frac{d}{ds}F(s)$$

and

$$L\{t^n.f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$$

Example 3 Find:

1. $L[t\ sin3t]$
2. $L[t^2\ cosh2t\ ]$

Solution:

1. Since $L[sin3t\ ] = \frac{3}{s^2+9}$, then

$$L[t\ sin3t] = -\frac{d}{ds}\frac{3}{s^2+9} = \frac{6s}{(s^2+9)^2}$$

2. Since $L[cosh2t\ ] = \frac{s}{s^2-4}$, then

$$L[t^2\ cosh2t] = (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right) = \frac{s^3+24s}{(s^2-4)^3}$$

Try to solve

Find the Laplace transforms of the given functions

1- $f(t) = 5t^4 - 9 + 6e^{-5t} + te^{3t}$

2- $g(t) = 4e^{3t}cos4t - 9sin4t + 2cosht$

3- $h(t) = 2t + 3tsin2t$

The inverse Laplace Transform

If the Laplace transform of $f(t)$ is $F(s)$, then $f(t)$ is called the inverse of Laplace transform of $F(s)$ and is written as

$$f(t) = L^{-1}[F(s)].$$

Example 4 Find:

1. $L^{-1}\left[\frac{1}{s^2+9}\right]$
2. $L^{-1}\left[\frac{5}{3s-1}\right]$

Solution:

1. $L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}L^{-1}\left[\frac{3}{s^2+3^2}\right] = \frac{1}{3}sin3t$
2. $L^{-1}\left[\frac{5}{s-1}\right] = 5L^{-1}\left[\frac{1}{s-1}\right] = 5e^t$

Example 5 Find:

1. $L^{-1}\left[\frac{-3s}{s^2+64} + \frac{12}{s-2} + \frac{4}{s^2-4} + \frac{6}{s^4}\right]$
2. $L^{-1}\left[\frac{4}{s^2+8s+25}\right]$

Solution:

1. $L^{-1}\left[\frac{-3s}{s^2+64}\right] + L^{-1}\left[\frac{12}{s-2}\right] + L^{-1}\left[\frac{4}{s^2-4}\right] + L^{-1}\left[\frac{6}{s^4}\right]$

   $= -3cos8t + 12e^{2t} + 2sinh2t + t^3$

2. $F(s) = \frac{4}{s^2+8s+25} = \frac{4}{(s+4)^2+9}$, (shift theorem)

Then, $L^{-1}\{F(s)\} = \frac{4}{3}e^{-4t}\sin 3t$

Try to solve

Find the inverse of Laplace transforms

1- $\frac{s}{s^2+16} + \frac{7}{s}$

2- $\frac{2s}{s^2+4} + \frac{5}{s^3}$

3- $\frac{15}{3s^2-27} + \frac{6}{(s-2)^3}$

4- $\frac{4}{s^2+6s+13}$

The inverse Laplace Transform by using partial fractions

Consider the ratio of two polynomials P(s) & Q(s) s.t. the degree of P(s) is lower than that of Q(s) and let

1- $(\mathrm{s}-\mathrm{a})$ **unrepeated factor** of Q(s), then

$$F(s) = \frac{\mathrm{P(s)}}{\mathrm{Q(s)}} = \frac{A}{(\mathrm{s}-\mathrm{a})} + \frac{B}{(\mathrm{s}-\mathrm{b})} + \frac{C}{(\mathrm{s}-\mathrm{c})}$$

2- $(\mathrm{s}-\mathrm{a})^m$ **repeated factor** of Q(s), then

$$F(s) = \frac{\mathrm{P(s)}}{\mathrm{Q(s)}} = \frac{A}{(\mathrm{s}-\mathrm{a})} + \frac{B}{(\mathrm{s}-\mathrm{a})^2} + \frac{C}{(\mathrm{s}-\mathrm{a})^m}$$

3- Q(s) of the **second degree**, then

$$F(s) = \frac{\mathrm{P(s)}}{\mathrm{Q(s)}} = \frac{As+B}{s^2+as+b}$$

Example 6 Find:

1- $L^{-1}\left[\frac{s^2+2s-4}{(s+1)(s-2)(s-4)}\right]$

2- $L^{-1}\left[\frac{s}{(s-1)^3(s-2)}\right]$

3- $L^{-1}\left[\frac{s}{(s-1)(s^2+2s+5)}\right]$

Solution:

$$1\text{- } F(s)=\frac{s^2+2s-4}{(s+1)(s-2)(s-4)}=\frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-4)}$$

$$F(s)=\frac{A(s-2)(s-4)+B(s+1)(s-4)+C(s+1)(s-2)}{(s+1)(s-2)(s-4)}$$

$$A(s-2)(s-4)+B(s+1)(s-4)+C(s+1)(s-2)$$
$$=s^2+2s-4$$

- **Put $s=-1$**, $15A=-5,\quad A=-\frac{1}{3}$
- **Put $s=2$**,$-6B=4,\quad B=-\frac{2}{3}$
- **Put $s=4$**, $10C=20,\quad C=2$

$$F(s)=\frac{-\frac{1}{3}}{(s+1)}+\frac{-\frac{2}{3}}{(s-2)}+\frac{2}{(s-4)},\ \text{then}$$

$$f(t)=-\frac{1}{3}e^{-t}-\frac{2}{3}e^{2t}+2e^{4t}$$

$$2\text{- } F(s)=\frac{s}{(s-1)^3(s-2)}=\frac{A}{(s-2)}+\frac{B}{(s-1)}+\frac{C}{(s-1)^2}+\frac{D}{(s-1)^3}$$

$$\therefore A(s-1)^3+B(s-1)^2(s-2)+C(s-1)(s-2)+D(s-2)=s$$

- **Put $s=1$**, $-D=1,\quad D=-1$
- **Put $s=2$**, $A=2$
- **Coefficients of s^3**, $A+B=0,\quad B=-2$
- **Coefficients of constants**,$A-2B+2C-2D=0,\ c=-2$

$$F(s)=\frac{2}{(s-2)}-\frac{2}{(s-1)}-\frac{2}{(s-1)^2}-\frac{1}{(s-1)^3},\ \text{then}$$

$$f(t) = 2e^{2t} - 2e^{t} - 2te^{t} - \frac{1}{2}t^2e^{t}$$

3- $F(s) = \frac{s}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$

$$A(s^2 + 2s + 5) + (Bs + C)(s - 1) = s$$

- **Put $\boldsymbol{s} = \mathbf{1}$**, $8A = 1, A = \frac{1}{8}$
- **Coefficients of $\boldsymbol{s^2}$**, $A + B = 0, B = \frac{-1}{8}$
- **Coefficients of $\boldsymbol{constants}$**, $5A - C = 0,\ c = \frac{5}{8}$

$$F(s) = \frac{\frac{1}{8}}{(s-1)} + \frac{-\frac{1}{8}s+\frac{5}{8}}{(s^2+2s+5)} = \frac{1}{8}\left[\frac{1}{(s-1)} + \frac{-s+5}{(s^2+2s+5)}\right]$$

$$= \frac{1}{8}\left[\frac{1}{(s-1)} - \frac{(s+1)}{(s+1)^2+4} + 3\frac{2}{(s+1)^2+4}\right], \text{ then}$$

$$f(t) = \frac{1}{8}[e^{t} - e^{-t}cos2t + 3e^{-t}sin2t$$

Try to solve Find

1- $L^{-1}\left[\frac{11-3s}{s^2+2s-3}\right]$

2- $L^{-1}\left[\frac{5s^2-2s-19}{(s-1)^2(s+3)}\right]$

3- $L^{-1}\left[\frac{s^2-2}{s(s^2+2)}\right]$

Laplace Transform of Derivatives

Let $y'(t)$ denote the first derivatives of $y(t)$ whose Laplace transform is $Y(s)$, then

$$L[y'(t)] = \int_0^\infty y'(t)e^{-st}dt$$

$$L\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-st}\frac{dy}{dt}dt = \int_0^\infty e^{-st}dy$$

By integrating by parts, we have

$$L\left[\frac{dy}{dt}\right] = [ye^{-st}]_{t=0}^{\infty} + s\int_0^{\infty} y(t)e^{-st}dt$$

$$\therefore L[y'(t)] = sY(s) - y(0)$$

and

$$L[y''(t)] = s^2Y(s) - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

$$L[y^n(t)] = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots\ldots\ldots - y^{n-1}(0)$$

Example 7

Solve the initial value problem

$$y'' - y = 4t \quad , y(0) = 1, y'(0) = 3$$

Solution:

Taking the Laplace transform for both sides

$$L[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 3$$

$$L[y'(t)] = sY(s) - y(0) = sY(s) - 1$$

$$s^2Y(s) - s - 3 - Y(s) = 4.\frac{1}{s^2}$$

$$(s^2 - 1)Y(s) = (s + 3) + \frac{4}{s^2}$$

$$Y(s) = \frac{(s+3)}{(s^2-1)} + \frac{4}{s^2(s^2-1)}$$

Partial Fraction

$$y(t) = L^{-1}[\frac{(s+3)}{(s^2-1)} + \frac{4}{s^2(s^2-1)}]$$

$$y(t) = -4t + 2e^t - 2e^{-t} + cosht + 3sinht$$

Example 8

Solve the initial value problem

$$y'' - 3y' = 9 \quad , when\ x = 0, y = 0\ and\ y' = 0$$

Solution:

Taking the Laplace transform for both sides

$$L[y''(t)] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$L[y'(t)] = sY(s) - y(0) = sY(s)$$

$$s^2Y(s) - 3sY(s) = \frac{9}{s}$$

$$(s^2 - 3s)Y(s) = \frac{9}{s}$$

$$Y(s) = \frac{9}{s(s^2 - 3s)} = \frac{9}{s^2(s-3)}$$

$$y(t) = L^{-1}\left[\frac{9}{s^2(s-3)}\right] = L^{-1}[\frac{A}{(s-3)} + \frac{B}{s} + \frac{c}{s^2}]$$

$$y(t) = -1 - 3t + e^{3t}$$

Try to solve

1. $y'' - 10y' + 9y = 5t \quad , when\ y(0) = -1\ and\ y'(0) = 2$
2. $y' - 5y = -e^{-2t} \quad , y(0) = 3$
3. $y'' + 7y = 10e^{2t} \quad , y(0) = 0, y'(0) = 3$

Chapter 2

Fourier Series

Periodic Functions & Orthogonal Functions

Periodic Function The first topic we need to discuss is that of a periodic function. A function is said to be periodic with period T if the following is true, $f(x+T) = f(x)$, for all x

Even and Odd functions

A function is said to be **even** if, $f(-x) = f(x)$,

$$f(x) = \begin{cases} f(x) & for \quad 0 < x < \pi \\ f(-x) & for -\pi < x < 0 \end{cases}$$

and a function is said to be **odd** if, $f(-x) = -f(x)$ **,**

$$f(x) = \begin{cases} f(x) & for\ 0 < x < \pi \\ -f(-x) & for -\pi < x < 0 \end{cases}$$

<u>**Note that**</u>

1. If $f(x)$ is an even function then,

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$$

2. If $f(x)$ is an odd function then,

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

Orthogonal Functions

1- Two non-zero functions, $f(x)$ and$g(x)$, are said to be orthogonal on $a \le x \le b$ if,

$$\int_{a}^{b} f(x)g(x)dx = 0$$

2- A set of non-zero functions, $\{f_i(x)\}$is said to be mutually orthogonal on $a \le x \le b$ if,

$$\int_{a}^{b} f_m(x)f_n(x)dx = 0, \qquad m \neq n$$

Remember

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Example 1

Show that $f(x) = x, g(x) = x^2$ are orthogonal on $(-2,2)$.

Solution

$$\int_a^b f(x)g(x)dx = \int_{-2}^{2} x.x^2dx = \int_{-2}^{2} x^3dx = 0$$

Example 2

Show that a set $\{sinx, sin2x, \dots, sinnx\}$ are mutually orthogonal on $(0,\pi)$.

Solution

$$sinn\pi = 0$$

$$cosn\pi = (-1)^n$$

$$\int_a^b f_m(x)f_n(x)dx = \int_0^\pi sinmx.sinnxdx$$

$$= \frac{1}{2}\int_0^\pi \cos(m-n)\,x - \cos(m+n)\,xdx$$

$$= \frac{1}{2}\left[\frac{sin(m-n)x}{(m-n)} - \frac{sin(m+n)x}{(m+n)}\right]_0^\pi = 0$$

Example 3

Show that a set $\{1, sinx, cosx, sin2x, cos2x, \dots, sinnx, cosnx\}$ are mutually orthogonal on $(-\pi,\pi)$. Is this set orthogonal over $(0,\pi)$.

Solution

We must show that every pair of distinct functions of the given system is orthogonal over $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} 1. \sin nx \; dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} 1. \cos nx \; dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin mx \, . \sin nx \; dx =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)\,x - \cos(m+n)\,xdx$$

$$= \frac{1}{2}\left[\frac{sin(m-n)x}{(m-n)} - \frac{sin(m+n)x}{(m+n)}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos mx \, . \cos nx \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)\,x + \cos(m+n)\,xdx$$

$$= \frac{1}{2}\left[\frac{sin(m-n)x}{(m-n)} + \frac{sin(m+n)x}{(m+n)}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin mx \, . \cos nx \; dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)\,x + \sin(m-n)\,xdx$$

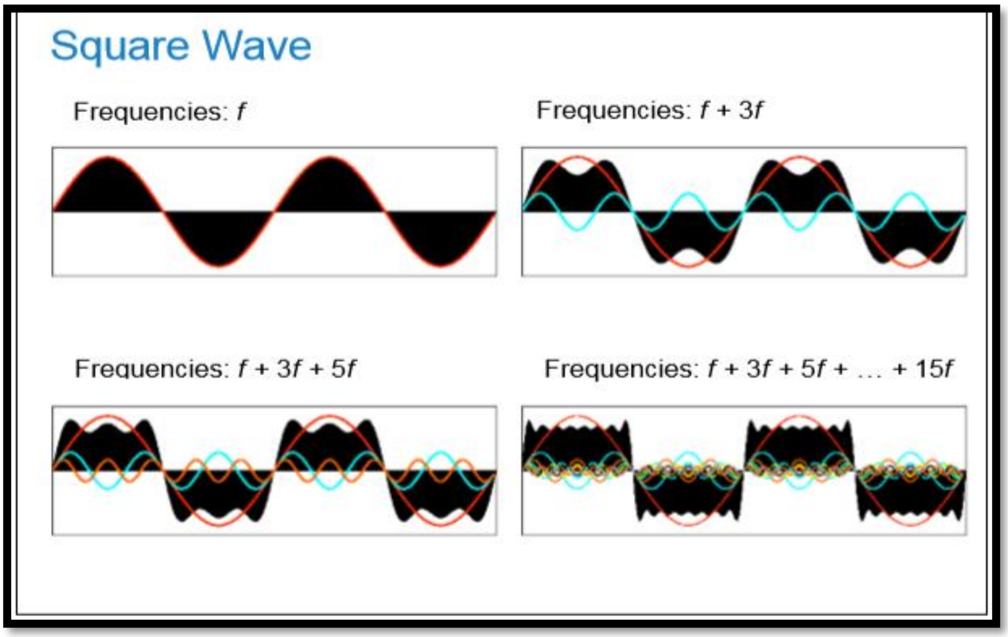
$$= \frac{1}{2}\left[\frac{cos(m+n)x}{(m+n)} + \frac{cos(m-n)x}{(m-n)}\right]_{-\pi}^{\pi} = 0$$

Try to solve

Is this set $\{1, sinx, cosx, sin2x, cos2x, \dots, sinnx, cosnx\}$ are mutually orthogonal over $(0, \pi)$.

Fourier Series

Any periodic function can be expressed as the sum of a series of sins and cosines.



Definition of Fourier series

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be an orthogonal set of functions on the interval (a, b) and $f(x)$ be any given function in (a, b), assume that a function $f(x)$ can be expanded in a series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \ \ldots\ldots(1)$$

If we multiply (1) by $f_n(x)$ and integrate both sides from a to b.

$$\int_a^b f(x)\, f_n(x) dx = \int_a^b c_n\, {f_n}^2(x) dx$$

$$\therefore\ c_n = \frac{\int_a^b f(x) f_n(x) dx}{\int_a^b {f_n}^2(x) dx} \ldots\ldots.(2)$$

The coefficient c_n is called **Fourier coefficient**.

The series (1) with c_n is called **Fourier series** of $f(x)$ w.r.t. the given orthogonal set of functions.

Example 3

Find the Fourier series of the function $f(x) = x$ on the interval $(0, \pi)$ w.r.t. the orthogonal set $\{sinx, sin2x, \dots, sinnx, \dots\}$

Solution

$f(x) = x$ and $f_n(x) = sinnx$

$$\therefore c_n = \frac{\int_a^b f(x) f_n(x) dx}{\int_a^b {f_n}^2(x)\, dx} = \frac{\int_0^\pi x\, sinnxdx}{\int_0^\pi sinnx^2\, dx}$$

$\int_0^\pi x\, sinnxdx = \left[\frac{-x}{n} cosnx + \frac{1}{n^2} sinnx\right]_0^\pi = \frac{\pi}{n}(-1)^{n+1}$

$\int_0^\pi sinnx^2\, dx = \frac{1}{2}\int_0^\pi (1 - cos2nx) dx = \frac{1}{2}\left[x + \frac{1}{2n} sin2nx\right]_0^\pi = \frac{\pi}{2}$

$$\therefore c_n = \frac{2}{n}(-1)^{n+1}$$

Then the Fourier series of the function is

$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} sinnx \ = 2sinx - sin2x + \frac{2}{3} sin3x - \cdots.$

The Trigonometric Fourier series

Asymmetric Functions

Since the set $\{1, sinx, cosx, sin2x, cos2x, \dots, sinnx, cosnx\}$ is orthogonal over the interval$(-\pi, \pi)$, we have

$$f_n(x) = 1 \qquad \to \int_{-\pi}^{\pi} {f_n}^2(x)\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$$

$$f_n(x) = sinnx \quad \to \int_{-\pi}^{\pi} sinn^2 x\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - cos2nx) dx = \pi$$

$$f_n(x) = cosnx \quad \to \int_{-\pi}^{\pi} cosn^2x\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 + cos2nx)dx = \pi$$

Hence by using (2), the Fourier series of $f(x)$ w.r.t. this trigonometric orthogonal set is

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx)}$$

Where

- $\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$

The sufficient conditions for a function to have a Fourier series representation are given in the following theorem.

Dirichlet's Theorem

Let f(x) be a periodic function with period 2π satisfies the following conditions:

1- f(x) is continuous in the interval $-\pi < x < \pi$

2- f(x) is bounded. i.e. $|f(x)| \leq M \;\; for\ some\ positive\ M$

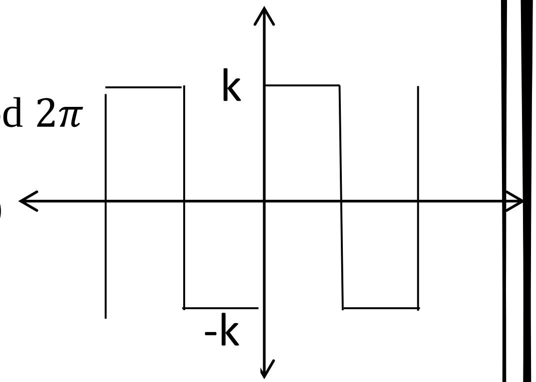
3- f(x) has a finite number of maximum and minimum points (extreme value)in each period.

Example 4

Find the Fourier series of the function $f(x)$, where

$$f(x) = \begin{cases} -k & for -\pi < x < 0 \\ k & for\ 0 < x < \pi \end{cases}, \text{ where } f(x+2\pi) = f(x)$$

Solution



Since $f(x)$ is a periodic function with period 2π

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

where

- $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx] = \frac{1}{\pi}[-k\pi + k\pi] = 0$$

- $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)cosnxdx$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} -kcosnxdx + \int_{0}^{\pi} kcosnxdx]$$

$$= \frac{1}{\pi}\left[-k\frac{sinnx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[k\frac{sinnx}{n}\right]_{0}^{\pi} = 0$$

- $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)sinnxdx$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} -ksinnxdx + \int_{0}^{\pi} ksinnxdx]$$

$$= \frac{1}{\pi}\left[k\frac{cosnx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-k\frac{cosnx}{n}\right]_{0}^{\pi}$$

$$= \frac{k}{n\pi}[1 - \cos(-n\pi) - \cos(n\pi) + 1]$$

$$= \frac{2k}{n\pi}(1 - cosn\pi) = \frac{2k}{n\pi}(1 - (-1)^n)$$

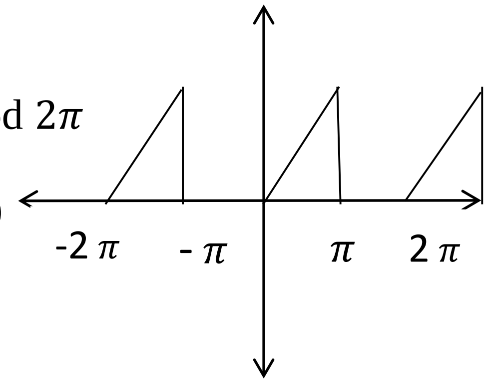
$$\therefore f(x) = \sum_{n=1}^{\infty}(b_n sinnx) = \sum_{n=1}^{\infty}\frac{2k}{n\pi}(1 - (-1)^n)sinnx$$

Example 5

Find the Fourier series of the function $f(x)$, where

$$f(x) = \begin{cases} 0 & for -\pi < x < 0 \\ x & for\ 0 < x < \pi \end{cases}, \text{ where } f(x + 2\pi) = f(x)$$

Solution



Since $f(x)$ is a periodic function with period 2π

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}[\int_0^{\pi} xdx] = \frac{\pi}{2}$
- $a_n = \frac{1}{\pi}\int_0^{\pi} xcosnxdx = \frac{1}{\pi}\left[x\frac{sinnx}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{sinnx}{n}dx]$

$$= \frac{1}{\pi}\left[\frac{cosnx}{n^2}\right]_0^{\pi} = \frac{1}{n\pi^2}(cosn\pi - 1) = \frac{1}{n\pi^2}((-1)^n - 1)$$

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = \frac{1}{\pi}[\int_0^{\pi} xsinnxdx]$

$$= \frac{1}{\pi}\left[-x\frac{cosnx}{n}\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{cosnx}{n}dx] = \frac{-1}{n}\cos(n\pi) = \frac{-(-1)^n}{n}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n\pi^2}((-1)^n - 1)cosnx - (\frac{(-1)^n}{n})sinnx$$

Symmetric Functions

Fourier series of even and odd functions

1- Let $f(x)$ be an even function

In this case Fourier series is called **Cosine harmonics** and

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)cosnxdx$
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = 0$

∴The Fourier series for the even function $f(x)$is

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnx)}$$

2- Let $f(x)$ be an odd function

In this case Fourier series is called **sine harmonics** and

- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$
- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = 0$
- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)sinnxdx$

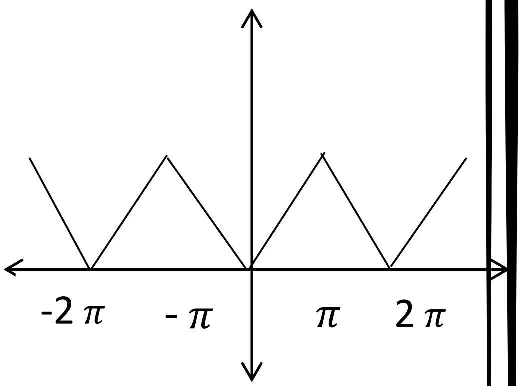
∴The Fourier series for the odd function $f(x)$is

$$\boxed{f(x) = \sum_{n=1}^{\infty}(b_n sinnx)}$$

Example 6

Expand the function $f(x) = |x|, \quad -\pi < x < \pi$ into Fourier series.

Solution



Since $f(x)$ is a periodic function.

Since the function is even

- $b_n = 0$

$$f(x) = \begin{cases} -x & for -\pi < x < 0 \\ x & for\ 0 < x < \pi \end{cases}$$

- $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)dx = \dfrac{2}{\pi}\int_0^{\pi} xdx = \pi$

- $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)cosnxdx = \dfrac{2}{\pi}\int_0^{\pi} x.cosnxdx$

Integrate by parts

$u = x \qquad dv = cosnxdx$

$du = dx \qquad v = \dfrac{sinnx}{n}$

$$\therefore \mathrm{a_n} = \frac{2}{\pi}\left[x\frac{sinnx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{sinnx}{n}dx = \frac{2}{\pi n^2}cosnx]_0^{\pi}$$

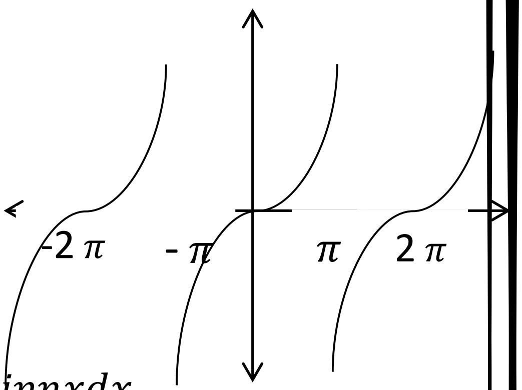
$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}[(-1)^n - 1]cosnx$$

Example 7

Expand the function $f(x) = \begin{cases} -x^2 & for -\pi < x < 0 \\ x^2 & for\ 0 < x < \pi \end{cases}$ into Fourier series.

Solution



Since $f(x)$ is a periodic function.

Since the function is odd

- $a_0 = 0$ and • $a_n = 0$

- $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) sinnxdx = \frac{2}{\pi}\int_0^{\pi} x^2 sinnxdx$

Integrate by parts

$$u = x^2 \qquad dv = sinnxdx$$

$$du = 2xdx \qquad v = \frac{-cosnx}{n}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left[-x^2\frac{cosnx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} 2x.\frac{cosnx}{n}dx$$

$$u = x \qquad dv = cosnxdx$$

$$du = dx \qquad v = \frac{sinnx}{n}$$

$$\therefore \mathrm{b_n} = \frac{2}{\pi}\left[\left[-x^2\frac{cosnx}{n}\right]_0^{\pi} + \frac{2x}{n^2}sinnx + \frac{2}{n^3}cosnx\right]_0^{\pi}$$

$$= -\frac{2(-1)^n\pi}{n} + \frac{4[(-1)^n - 1]}{\pi n^3}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} -\frac{2(-1)^n\pi}{n} + \frac{4[(-1)^n - 1]}{\pi n^3} sinnx$$

The Fourier series of functions having arbitrary period

Consider the set

$\{1, sin\frac{\pi x}{L}, cos\frac{\pi x}{L}, sin\frac{2\pi x}{L}, cos\frac{2\pi x}{L}, \dots, sin\frac{n\pi x}{L}, cos\frac{n\pi x}{L}\}$.

Each function of this set has period $2L$ and orthogonal over the interval$(-L, L)$ and over any interval of length$2L$. We have

$$f_n(x) = 1 \qquad \rightarrow \int_{-L}^{L} f_n{}^2(x)\,dx = \int_{-L}^{L} 1\,dx = 2L$$

$$f_n(x) = sin\frac{n\pi x}{L} \quad \rightarrow \int_{-L}^{L} sinn^2\frac{n\pi x}{L}dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left(1 - cos\frac{2n\pi x}{L}\right)dx = L$$

$$f_n(x) = cos\frac{n\pi x}{L} \quad \rightarrow \int_{-L}^{L} cosn^2\frac{n\pi x}{L}dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left(1 + cos\frac{2n\pi x}{L}\right)dx = L$$

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L})}$$

Where

- $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$
- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx$
- $b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx$

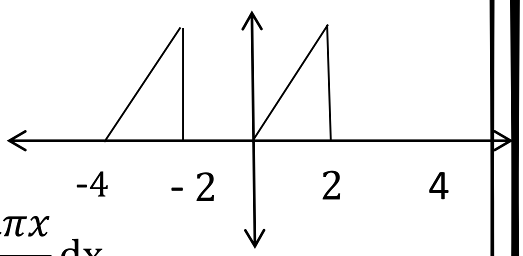
Example 8

Find the Fourier series of the periodic function $f(x)$ of period 4, for

$$f(x) = \begin{cases} 0 & for -2 < x < 0 \\ x & for\ 0 < x < 2 \end{cases}$$

Solution

Since $f(x)$ is a periodic function.



- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{2}\int_{0}^{2} x.\cos\frac{n\pi x}{2}dx$

Integrate by parts

$u = x \qquad dv = cos\frac{n\pi x}{2}dx$

$du = dx \quad - \quad v = sin\frac{n\pi x}{2}$

$$a_n = \frac{1}{2}\left[\frac{2}{n\pi}\left[xsin\frac{n\pi x}{2}\right]_0^{\pi} - \frac{1}{n\pi}\int_0^2 sin\frac{n\pi x}{2}dx\right] = \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

- $b_n = \frac{1}{2}\int_{0}^{2} x.\sin\frac{n\pi x}{2}dx$

Integrate by parts

$u = x \qquad dv = sin\frac{n\pi x}{2}dx$

$du = dx \quad - \quad v = -cos\frac{n\pi x}{2}$

$$b_n = \frac{1}{2}\left[\frac{-2}{n\pi}xcos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}sin\frac{n\pi x}{2}\right]_0^2 = \frac{-2}{n\pi}(-1)^n$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]cos\frac{n\pi x}{2}\frac{-2}{n\pi}(-1)^n sin\frac{n\pi x}{2}$$

1- Let $f(x)$ be an even function

In this case Fourier series is called **Cosine harmonics** and

- $a_0 = \frac{2}{L}\int_0^L f(x)dx$
- $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$
- $b_n = 0$

$\therefore$The Fourier series for the even function $f(x)$is

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{L})}$$

2- Let $f(x)$ be an odd function

In this case Fourier series is called **sine harmonics** and

- $a_0 = 0$
- $a_n = 0$
- $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

$\therefore$The Fourier series for the odd function $f(x)$is

$$\boxed{f(x) = \sum_{n=1}^{\infty}(b_n sin\frac{n\pi x}{L})}$$

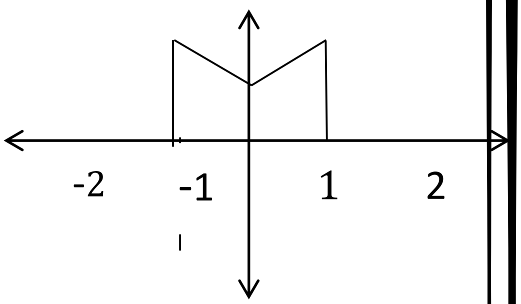
Example 9

Expand in a Fourier series, the function

$$f(x) = |x| + 1 \quad for -1 < x < 1$$

Solution

Since $f(x)$ is an even function.



$\therefore \mathrm{b_n} = 0$

$\mathrm{L} = \frac{1-(-1)}{2} = 1$

$$f(x) = \begin{cases} 1 - x & for -1 < x \leq 0 \\ 1 + x & for\ 0 < x \leq 1 \end{cases}$$

$$a_0 = \frac{2}{\mathrm{L}} \int_0^{\mathrm{L}} \mathrm{f(x)dx} = \frac{2}{1} \int_0^1 (1 + x)\mathrm{dx} = 2\left[x + \frac{x^2}{2}\right]_0^1 = 3$$

$$\bullet\ a_n = \frac{2}{\mathrm{L}} \int_0^{\mathrm{L}} \mathrm{f(x)cos} \frac{n\pi x}{L} \mathrm{dx} = 2 \int_0^1 (1 + x)\mathrm{cos}n\pi x\mathrm{dx}$$

Integrate by parts

$u = 2(1 + x) \qquad dv = cosn\pi xdx$

$du = 2dx \qquad v = \frac{1}{n\pi} sinn\pi x$

$$\therefore \mathrm{a_{n=}} \left[\frac{2(1 + x)}{n\pi} sinn\pi x\right]_0^1 - \frac{2}{n\pi} \int_0^1 sinn\pi x\mathrm{dx} = \frac{2}{n^2\pi^2} [cosn\pi x]_0^1$$

$$= \frac{2}{n^2\pi^2} [(-1)^n - 1]$$

$$\therefore f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} [(-1)^n - 1]cosn\pi x$$

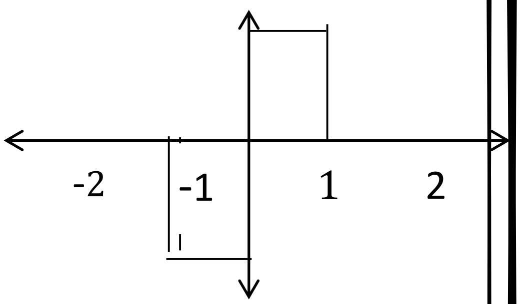
Example 10

Find the Fourier series of the periodic function $f(x)$ of period 4, for

$$f(x) = \begin{cases} -2 & for -1 < x \leq 0 \\ 2 & for\ 0 < x \leq 1 \end{cases}$$

Solution

Since $f(x)$ is an odd function.



$\therefore a_0 = a_n = 0$

$$L = \frac{1-(-1)}{2} = 1$$

- $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{1}\int_0^1 2.\sin\frac{n\pi x}{1}dx$$

$$= 4\int_0^1 \sin n\pi x dx = \frac{-4}{n\pi}[cos n\pi x]_0^1 = \frac{-4}{n\pi}[(-1)^n - 1]$$

$$= \frac{4}{n\pi}[1-(-1)^n]$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}[1-(-1)^n] sin n\pi x\right)$$

Exercises

1) **Expand the periodic function $\boldsymbol{f(x)}$ with period $\boldsymbol{2\pi}$ into Fourier series where**

   1- $f(x) = x^2 \quad , \quad -\pi < x < \pi$

   2- $f(x) = \begin{cases} 0 & for -\pi < x < 0 \\ \pi & for\ 0 < x < \pi \end{cases}$

   3- $f(x) = \begin{cases} -\frac{\pi}{2} & for -\pi < x < 0 \\ \frac{\pi}{2} & for\ 0 < x < \pi \end{cases}$

   4- $f(x) = \begin{cases} x & for\ \ 0 < x < \pi \\ 0 & for\ \pi < x < 2\pi \end{cases}$

2) **Expand the function $\boldsymbol{f(x) = 2 - x, 0 < x < 2}$ into cosine harmonics.**

3) **Expand the function $\boldsymbol{f(x)} = \begin{cases} 1 & for\ \ 0 < x < 1 \\ 0 & for\ 1 < x < 2 \end{cases}$ into sine harmonics.**

Chapter 4

Fourier Transform

Introduction

Fourier transform is widely used in theory of communication engineering, wave propagation and other branches of applied mathematics.

This chapter discusses the integral transform and their applications for the solution of partial differential equations.

Fourier Transform

Assume the following conditions:

1. If $f(x)$ is piecewise continuous, has piecewise continuous derivatives in every finite interval.
2. If $\int_{-\infty}^{\infty}|f(x)|dx$ converges i.e. the integral of $f(x)$ in$(-\infty,\infty)$ is exists.

Then

1. A function $F[f(x)] = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$ is called the **Fourier transform** of $f(x)$.
2. A function $F^{-1}\left[\bar{f}(s)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{isx}ds$ is called the **inverse Fourier transform** of $f(x)$.

Remember

$$\cos \mathrm{ax} = \frac{\mathrm{e}^{\mathrm{iax}} + \mathrm{e}^{-\mathrm{iax}}}{2}$$

$$\sin \mathrm{ax} = \frac{\mathrm{e}^{\mathrm{iax}} - \mathrm{e}^{-\mathrm{iax}}}{2i}$$

Example 1

Find the Fourier transform of the function

$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$ and find the value of $\int_0^\infty \frac{\sin x}{x} \mathrm{dx}$

Solution

By definition

$$\bar{\mathrm{f}}(\mathrm{s}) = \int_{-\infty}^{\infty} \mathrm{f}(\mathrm{x}) \mathrm{e}^{-\mathrm{isx}} \mathrm{dx} = \int_{-\mathrm{a}}^{\mathrm{a}} 1.\, \mathrm{e}^{-\mathrm{isx}} \mathrm{dx}$$

$$= \left[\frac{\mathrm{e}^{-\mathrm{isx}}}{-\mathrm{is}}\right]_{-\mathrm{a}}^{\mathrm{a}} = \left[\frac{\mathrm{e}^{\mathrm{ias}} - \mathrm{e}^{-\mathrm{ias}}}{\mathrm{is}}\right]$$

Since $sin\,\mathrm{as} = \frac{\mathrm{e}^{\mathrm{ias}} - \mathrm{e}^{-\mathrm{ias}}}{2i}$

$$\therefore \bar{\mathrm{f}}(\mathrm{s}) = 2\frac{sinas}{s}, \mathrm{s} \neq 0$$

Taking the inverse Fourier transform

$$F^{-1}\left(\bar{\mathrm{f}}(\mathrm{s})\right) = F^{-1}\left[2\frac{sinas}{s}\right] = f(x)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{sinas}{s} e^{isx} ds = f(x)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{sinas}{s} (\cos sx + i \sin sx)\, ds = f(x)$$

$$\frac{2}{\pi}\int_{0}^{\infty} \frac{1}{s} \sin as \cos sx\, ds = f(x)$$

$$\int_{0}^{\infty} \frac{1}{s} \sin as \cos sx\, ds = \frac{\pi}{2} f(x) = \frac{\pi}{2}\begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\int_0^\infty \frac{1}{s}\sin as\cos sx\,ds = \begin{cases} \frac{\pi}{2} & |x| < a \\ 0 & |x| > a \end{cases}$$

Put $a = 1\ and\ x = 0$

$$\therefore \int_0^\infty \frac{1}{s}\sin s\,ds = \frac{\pi}{2}$$

Example 2

Find the Fourier transform of the function

$$f(x) = \begin{cases} e^{-ax} & x > 0, a > 0 \\ 0 & x < 0 \end{cases}$$

Solution

By definition

$$\bar{\mathrm{f}}(\mathrm{s}) = \int_{-\infty}^{\infty} \mathrm{f(x)e^{-isx}dx} = \int_0^\infty e^{-ax}.\mathrm{e^{-isx}dx}$$

$$= \int_0^\infty \mathrm{e^{-(a+is)x}dx} = \left[\frac{\mathrm{e^{-(a+is)x}}}{-\mathrm{(a+is)}}\right]_0^\infty$$

$$\therefore \bar{\mathrm{f}}(\mathrm{s}) = \frac{1}{\mathrm{(a+is)}}$$

Try to solve

1. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

2. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

Fourier Sine and Cosine Transforms

1. If $f(x)$ is an even function, then $F_c[f(x)] = \overline{f_c}(s) = \int_0^\infty f(x) \cos sx \, dx$ is called the **Fourier Cosine transform** of $f(x)$ and

   $F_c^{-1}[\overline{f_c}(s)] = \frac{2}{\pi} \int_0^\infty \overline{f_c}(s) \cos sx \, ds$ is called the **inverse Fourier Cosine transform** of $f(x)$.

2. If $f(x)$ is an odd function, then $F_s[f(x)] = \overline{f_s}(s) = \int_0^\infty f(x) \sin sx \, dx$ is called the **Fourier Sine transform** of $f(x)$ and

   $F_s^{-1}[\overline{f_s}(s)] = \frac{2}{\pi} \int_0^\infty \overline{f_s}(s) \sin sx \, ds$ is called the **inverse Fourier Sine transform** of $f(x)$.

Example 3

Find the Fourier cosine transform of the function

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Solution

By definition

$$F_c[f(x)] = \overline{f_c}(s) = \int_0^\infty f(x) \cos sx \, dx$$

$$= \int_0^a 1. \cos sx \, dx = \left[\frac{\sin sx}{s}\right]_0^a = \frac{\sin as}{s}$$

Example 4

Find the Fourier sine transform of the function $f(x) = \begin{cases} \sin x & 0 < x < a \\ 0 & x > a \end{cases}$

Solution

By definition

$$F_s[f(x)] = \bar{f_s}(s) = \int_0^\infty f(x) \sin sx\, dx$$

$$= \int_0^a \sin x \sin sx\, dx = \frac{1}{2}\int_0^a \cos(s-1)x - \cos(s+1)x\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)x}{(s-1)} - \frac{\sin(s+1)x}{(s+1)}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)a}{(s-1)} - \frac{\sin(s+1)a}{(s+1)}\right]$$

Transform of Derivatives

If $f(x)$ is continuous, $f'(x)$ is piecewise continuously differentiable, $f(x)$ and $f'(x)$ are absolutely integrable in $(-\infty, \infty)$ and $\lim_{x \to \pm\infty} [f(x)] = 0$, then

$$F\big(f'(x)\big) = is\, \bar{f}(s), \ \ where \quad \bar{f}(s) = F[f(x)]$$

In general,

If $f, f', f'', \ldots, f^{(n)}$ are continuous, $f^{(n)}(x)$ is piecewise continuously differentiable, $f, f', f'', \ldots, f^{(n)}$ are absolutely integrable in $(-\infty, \infty)$ and $f, f', f'', \ldots, f^{(n)} \to 0$ as $x \to \pm\infty$, then

$$F\left(f^{(n)}(x)\right) = (is)^n\, \bar{f}(s), where \quad \bar{f}(s) = F[f(x)]$$

✘ Example 5

Solve the differential equation $y'' + 3y' + 2y = e^{-x} \;\; x > 0$

Use the Fourier transform

Solution

Take the Fourier transform

$$(is)^2\,\bar{y}(s) + 3(is)^1\,\bar{y}(s) + 2\,\bar{y}(s) = \int_0^{\infty} e^{-x}.\mathrm{e}^{-\mathrm{isx}}\mathrm{dx}$$

$$(is)^2\,\bar{y}(s) + 3(is)^1\,\bar{y}(s) + 2\,\bar{y}(s) = \int_0^{\infty} e^{-(1+is)x}\mathrm{dx}$$

$$\bar{y}(s)[(is)^2 + 3(is) + 2] = \left[\frac{e^{-(1+is)x}}{-(1+is)}\right]_0^{\infty} = \frac{1}{(1+is)}$$

$$\bar{y}(s) = \frac{1}{(1+is)}.\frac{1}{((is)^2 + 3(is) + 2)} = \frac{1}{(1+is)^2}.\frac{1}{(2+is)}$$

By using partial fraction

$$\bar{y}(s) = \frac{-1}{(1+is)} + \frac{1}{(1+is)^2} + \frac{1}{(2+is)}$$

Take the inverse of Fourier transform

$$F^{-1}[\bar{y}(s)] = F^{-1}\left[\frac{-1}{(1+is)}\right] + F^{-1}\left[\frac{1}{(1+is)^2}\right] + F^{-1}\left[\frac{1}{(2+is)}\right]$$

$$\therefore\; y(x) = -e^{-x} + x.e^{-x} + e^{-2x}$$

Table of Fourier transform

f(x)	$\bar{f}(s)$
$f(x) = \begin{cases} 1 & \lvert x \rvert < a \\ 0 & \lvert x \rvert > a \end{cases}$	$2\dfrac{sinas}{s}$
$f(x) = e^{-ax}$	$\dfrac{1}{a+is}$
$f(x) = x.e^{-ax}$	$\dfrac{1}{(a+is)^2}$
$f(x) = e^{-a\lvert x \rvert}$	$\dfrac{2a}{a^2+s^2}$

Try to solve

Solve the following differential equations by use the Fourier transforms

1. $y'' - 4y' + 4y = x.e^{-2x} \quad x > 0$
2. $y'' - 4y' + 3y = e^{-x} \quad x > 0$

Chapter 5

Numerical Analysis

Introduction to Numerical Methods

Numerical Methods:

Algorithms that are used to obtain numerical solutions of a mathematical problem.

Why do we need them?

1- No analytical solution exists
2- An analytical solution is difficult to obtain or not practical

Solution of Nonlinear Equations

- Some simple equations can be solved analytically:

$$x^2 + 4x + 3 = 0$$

$$\text{Analytic solution } roots = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$$

$$x = -1 \quad and \quad x = -3$$

- Many other equations have no analytical solution:

$$\left.\begin{array}{c} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{array}\right\} \text{No analytic solution}$$

Numerical methods properties

Practical

- Can be computed in a reasonable amount of time

Accurate

- Good approximate to the true value
- Information about the approximate error

I- Solution of Algebraic and Transcendental Equations

The equation of the form

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ …….(1) is called ***rational integral equation***.

Here, $a_0 \neq 0$, n is a positive integer, $a_0, a_1, \dots, a_n$ are constants.

The rational integral equation is classified into two parts

1. Algebraic Equation
2. Transcendental Equation

Algebraic Equation

In this equation, $f(x)$ is a polynomial purely in x as in the equation (1)

Example: $x^3 - 3x + 1 = 0$

Transcendental Equation

In this equation, $f(x)$ contains some other functions such as trigonometric, logarithmic or exponential etc.

Example: $3x - \cos x - 1 = 0$, $x \log_{10} x = 0$

Properties:

1. If $f(a)$ and $f(b)$ have opposite signs then one root of $f(x) = 0$ lies between a and b.
2. Every equation of an **odd** degree has at least one real root whose sign is opposite to that of its last term.
3. Every equation of an **even** degree with last term negative has at least a pair of real roots one positive and other negative.

Methods for solving Algebraic and Transcendental Equations

1. Fixed point iteration method.
2. Newton's Raphson method.

1- Fixed point iteration method.

Let $f(x) = 0$ be the given equation whose roots are to be determined.

Steps for fixed method

1. Use the first property find "a" and "b" where the roots lies between.
2. Write the given equation in the form $x = \varphi(x)$with the condition $|\varphi'(x)| < 1$
3. Let the initial approximation be $x_0$which is lies in the interval (a, b)
4. Continue the process using $x_n = \varphi(x_{n-1})$
5. If the difference between the two consecutive values of x_n is very small then we stop the process and the value is the root of the equation.

Convergence of fixed point iteration method

The iteration process converges quickly if $|\varphi'(x)| < 1$.If $|\varphi'(x)| > 1$, the iteration process is not converge.

❖ **Example 1**

Consider the equation $f(x) = x^3 + x - 1 = 0,$

Find its forms and which form is converge.

Solution

We can write $x = \varphi(x)$ in three forms

1- $x = 1 - x^3$

2- $x = \frac{1}{1+x^2}$

3- $x = (1-x)^{1/3}$

But we take the form which has convergence property $|\varphi'(x)| < 1$

$$f(x) = x^3 + x - 1 = 0$$

1- $f(0) = -ve$ and $f(1) = +ve$

$$\therefore \text{ The root lies between 0 and 1.}$$

2- Consider $x = \varphi(x)$

$$x = 1 - x^3 \quad (\text{First form})$$

$$\therefore \varphi(x) = 1 - x^3 \quad \Rightarrow \varphi'^{(x)} = -3x^2$$

$at\ x = 0.7 \quad \Rightarrow \quad |\varphi'(0.7)| = |-3(0.7)^2| = 1.47$

$\Rightarrow \quad |\varphi'(x) > 1|$

$\Rightarrow$ This equation $x = 1 - x^3$ will not converge

Consider $x = \varphi(x)$

$$x = \frac{1}{1+x^2} \quad (\text{Second form})$$

$$\therefore \varphi(x) = \frac{1}{1+x^2} \quad \Rightarrow \varphi'^{(x)} = \frac{-2x}{(1+x^2)^2}$$

$at\ x = 0.9 \quad \Rightarrow \quad |\varphi'(0.9)| = \left|\frac{-2(0.9)}{(1+0.9^2)^2}\right| = 0.5494$

$\Rightarrow \quad |\varphi'(x) < 1| \ \Rightarrow$ This equation $x = \frac{1}{1+x^2}$ is converge

❖ **Example 2**

Find a real root of the equation

$f(x) = x^3 + x^2 - 1 = 0$, using iteration method

Solution

1- $f(0) = -ve$ and $f(1) = +ve$

$\therefore$ The root lies between 0 and 1.

2- Consider

$$x = \frac{1}{\sqrt{1+x}} = \varphi(x)$$

$\varphi'^{(x)} = -\frac{1}{2}(1+x)^{-3/2} \Longrightarrow |\varphi'(x)| < 1$ in interval (0,1)

3- Let the initial approximation be $x_0 = 0.5$

4- Then , $x_n = \varphi(x_{n-1})$

$$x_1 = \varphi(x_0) = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.81649$$

$$x_2 = \varphi(x_1) = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.81649}} = 0.74196$$

$$x_3 = \varphi(x_2) = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.74196}} = 0.75767$$

$$x_4 = \varphi(x_3) = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1+0.75767}} = 0.75427$$

$$x_5 = \varphi(x_4) = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1+0.75427}} = 0.75500$$

$$x_6 = \varphi(x_5) = \frac{1}{\sqrt{1+x_5}} = \frac{1}{\sqrt{1+0.7500}} = 0.75485$$

$$x_7 = \varphi(x_6) = \frac{1}{\sqrt{1+x_6}} = \frac{1}{\sqrt{1+0.75427}} = 0.75488$$

5- Here the difference between the two values of x_6 and x_7 is very small, therefore the root of the equation is $\mathbf{0.75488}$

❖ Example 3

Find the real root of the equation

$f(x) = \cos x = 3x - 1$, using iteration method

Solution

1. $f(0) = +ve$ and $f(1) = -ve$

$$\therefore \text{ The root lies between 0 and } \frac{\pi}{2}.$$

2. Consider

$$x = \frac{1}{3}(\cos x + 1) = \varphi(x)$$

$$\varphi'(x) = -\frac{1}{3}\sin x \quad |\varphi'(x)| < 1 \text{ in interval } (0, \frac{\pi}{2})$$

3. Let the initial approximation be $x_0 = 0$
4. Then , $x_n = \varphi(x_{n-1})$

$$x_1 = \varphi(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$

$$x_2 = \varphi(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.59529$$

$$x_3 = \varphi(x_2) = \frac{1}{3}(\cos 0.59529 + 1) = 0.60933$$

$$x_4 = \varphi(x_3) = \frac{1}{3}(\cos 0.60933 + 1) = 0.60668$$

$$x_5 = \varphi(x_4) = \frac{1}{3}(\cos 060668 + 1) = 0.60718$$

$$x_6 = \varphi(x_5) = \frac{1}{3}(\cos 0.60718 + 1) = 0.60709$$

$$x_7 = \varphi(x_6) = \frac{1}{3}(\cos 0.60709 + 1) = 0.60710$$

$$x_8 = \varphi(x_7) = \frac{1}{3}(\cos 0.60710 + 1) = 0.60710$$

5. Here the difference between the two values of x_7 and x_8 is very small, therefore the root of the equation is $\mathbf{0.75488}$

2-Newton's Raphson method.

Let $f(x) = 0$ be the given equation whose roots are to be determined.

$$\text{Formula } \boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Steps for Newton's Raphson method

1. Use the first property find "a" and "b" where the roots lies between.
2. Let the initial approximation be $x_0 = a$ if $|f(a)| < |f(b)|$, the initial approximation be $x_0 = b$ if $|f(b)| < |f(a)|$in the interval (a, b)
3. Use formula and continue the process
4. If the difference between the two consecutive values of x_{n+1} is very small then we stop the process and the value is the root of the equation.

Convergence of Newton's Raphson iteration method

The iteration process converges if

$$\boldsymbol{|f(x)f''(x)| < |f'(x)|^2}$$

- **Example 4**

Compute the real root of the function

$$f(x) = x\ \log_{10} x = 1.2\ .$$

Correct to three decimal places using Newton's Raphson method

Solution

$$\text{Let } (x) = x \log_{10} x - 1.2 .$$

1. $f(2) = -0.59$ and $f(3) = +0.23$

$$\therefore \text{ The root lies between 2 and 3.}$$

Hence $|f(3)| < |f(2)|$ the root nearer to 3.

2. Let the initial approximation be $x_0 = 3$

$$f'(x) = \log_{10} x + x.\frac{1}{x ln\ 10} = \log_{10} x + 0.4343$$

3. Initial iteration $x_0 = 3$

Iteration $$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.4343}}$$

1	$\boldsymbol{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.746}$
2	$\boldsymbol{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.741}$
3	$\boldsymbol{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.741}$

The real root of $f(x) = 0$, correct to three decimal places is

2.741

❖ Example 5

Evaluate $\sqrt{12}$ to four decimal places by Newton's Raphson method

Solution

$$\text{Let } x = \sqrt{12} \Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0.$$

Let $f(x) = x^2 - 12$ and $f'(x) = 2x$

1- $f(3) = -3$ and $f(4) = +4$

$\therefore$ The root lies between 3 and 4.

Hence $|f(3)| < |f(4)|$ the root nearer to 3.

2- Let the initial approximation be $x_0 = 3$

3- Initial iteration $x_0 = 3$

Iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^2 - 12}{2\,x_n}$$

1 $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5000$$

2 $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.4642$$

3 $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4641$$

4 $$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3.4641$$

The value of $\sqrt{12}$ is **3.4641**

II- Solution of Linear Algebraic Equations

Definition: Linear System

A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \textbf{Linear system}$$

where

$$1 \le i \le n \quad and \quad 1 \le j \le m \qquad a_{ij}, b_j \in R.$$

Linear system of equations is called **Homogeneous** if $b_1 = b_2 = \cdots = b_m = 0$ and **Non- Homogeneous** otherwise.

Linear system of equations can be written in the form

$$A\,X = B, \quad where$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}.$$

The matrix A is called the **coefficient** matrix and the matrix $[A\!:B]$ is called the **augmented matrix** of the linear system.

** Linear system of equations $A\,X = B$, is called **Associated Homogeneous system** if $B = 0$. **i.e.** $A\,X = 0$**

There are two methods to solve such a system by numerical method

- ✓ Direct methods (Gauss elimination& Gauss-Jordan)
- ✓ Iterative method (Gauss -Jacobi)

Definition: Elementary Row Operations

Let A be a $m \times n$ matrix. Then the elementary row operations are defined as follows:

1. $r_i \leftrightarrow r_j$: Interchange of the i^{th} row of A and the j^{th} row of A.
2. $r_i \leftrightarrow cr_i$: Replace the i^{th} row of A by c times of the i^{th} row of A.
3. $r_i \leftrightarrow r_i + cr_j$: Replace the i^{th} row of A by the i^{th} row of A plus c times of the j^{th} row of A.

Direct Methods

1- Gauss Elimination Method

The Gaussian elimination method is a procedure for solving a linear system $A\,X = B$(consisting of m equations in n unknowns variables) bringing the augmented matrix $[A\!:B] =$

$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} & & b_1 \\ a_{21} & a_{22} \cdots a_{2n} & \vdots & b_2 \\ a_{m1} & a_{m2} \cdots a_{mn} & & b_m \end{bmatrix}$ to an upper triangle form

$\begin{bmatrix} 1 & c_{12} & \ldots c_{1m} & & d_1 \\ 0 & 1 & \ldots c_{2m} & \vdots & d_2 \\ 0 & 0 & \ldots\ 1 & & d_m \end{bmatrix}$ by application of elementary row operations.

- **Example 6**

Solve the following linear system by Gauss elimination method

$$20x_1 + 15x_2 + 12x_3 = 0$$

$$6x_1 + 4x_2 + 3x_3 = 0$$

$$6x_1 + 3x_2 + 2x_3 = 6$$

Solution

$$\begin{bmatrix} 20 & 15 & 12 & \vdots & 0 \\ 6 & 4 & 3 & \vdots & 0 \\ 6 & 3 & 2 & & 6 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow \frac{r_1}{20}} \begin{bmatrix} 1 & \frac{3}{4} & \frac{3}{5} & \vdots & 0 \\ 6 & 4 & 3 & \vdots & 0 \\ 6 & 3 & 2 & & 6 \end{bmatrix} \xrightarrow[r_3 \leftrightarrow r_3 - 6r_1]{r_2 \leftrightarrow r_2 - 6r_1}$$

$$\begin{bmatrix} 1 & 3/4 & 3/5 & 0 \\ 0 & -1/2 & -3/5 & 0 \\ 0 & -3/2 & -8/5 & 6 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow -2r_2}$$

$$\begin{bmatrix} 1 & 3/4 & 3/5 & & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & -3/2 & -8/5 & & 6 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_3 + 3/2 r_2}$$

$$\begin{bmatrix} 1 & 3/4 & 3/5 & & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & 0 & 1/5 & & 6 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow 5r_3} \begin{bmatrix} 1 & 3/4 & 3/5 & & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & 0 & 1 & & 30 \end{bmatrix}$$

The last matrix is equivalent to the following linear system

$$x_1 + 3/4x_2 + 3/5x_3 = 0 \qquad (1)$$

$$x_2 + 6/5x_3 = 0 \qquad (2)$$

$$x_3 = 30 \qquad (3)$$

$\therefore x_2 = -\frac{6}{5} * 30 = -36$ and

$$\therefore x_1 = -\frac{3}{4} * (-36) - \frac{3}{5} * (30) = 9$$

The system has a unique solution

- **Example 7**

Solve the following linear system by Gauss elimination method

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 2x_3 = 5$$
$$3x_1 + 4x_2 + 4x_3 = 11$$

Solution

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{array}\right] \xrightarrow[r_3 \leftrightarrow r_3 - 3r_1]{r_2 \leftrightarrow r_2 - r_1}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_3 \leftrightarrow r_3 - r_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is equivalent to the following linear system

$$x_1 + x_2 + x_3 = 3 \quad (1)$$
$$x_2 + x_3 = 2 \quad (2)$$

Let $x_3 = u$

$\therefore x_2 = 2 - u$ and

$x_1 = 3 - u - (2 - u) = 1$

The system has an infinite number of solutions

- **Example 8**

Solve the following linear system by Gauss elimination method

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 2x_3 = 5$$
$$3x_1 + 4x_2 + 4x_3 = 12$$

Solution

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 \vdots & 5 \\ 3 & 4 & 4 & 12 \end{array}\right] \xrightarrow[r_3 \leftrightarrow r_3 - 3r_1]{r_2 \leftrightarrow r_2 - r_1} \left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 \vdots & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{r_3 \leftrightarrow r_3 - r_2} \left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 \vdots & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last matrix is equivalent to the following linear system

$$x_1 + x_2 + x_3 = 3 \quad (1)$$

$$x_2 + x_3 = 2 \quad (2)$$

$$0 * x_1 + 0 * x_2 + 0 * x_3 = 1$$

The system has no solution

Remark

1. If there exist row of all elements of it are zero, then the system has infinite number of solutions.
2. If there exist row of all elements of it are zero expect the last element, then the system has no solutions.

2- Gauss- Jordan Method

This method is a modified from Gaussian elimination method. In this method, the coefficient matrix is reduced to a diagonal matrix rather than a triangular as in the Gaussian method.

- **Example 9**

Solve the following linear system by Gauss-Jordan method

$$20x_1 + 15x_2 + 12x_3 = 0$$

$$6x_1 + 4x_2 + 3x_3 = 0$$

$$6x_1 + 3x_2 + 2x_3 = 6$$

Solution

$$\begin{bmatrix} 20 & 15 & 12 & \vdots & 0 \\ 6 & 4 & 3 & \vdots & 0 \\ 6 & 3 & 2 & & 6 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow \frac{r_1}{20}} \begin{bmatrix} 1 & \frac{3}{4} & \frac{3}{5} & \vdots & 0 \\ 6 & 4 & 3 & \vdots & 0 \\ 6 & 3 & 2 & & 6 \end{bmatrix} \xrightarrow[r_3 \leftrightarrow r_3 - 6r_1]{r_2 \leftrightarrow r_2 - 6r_1}$$

$$\begin{bmatrix} 1 & 3/4 & 3/5 & \vdots & 0 \\ 0 & -1/2 & -3/5 & \vdots & 0 \\ 0 & -3/2 & -8/5 & & 6 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow -2r_2}$$

$$\begin{bmatrix} 1 & 3/4 & 3/5 & \vdots & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & -3/2 & -8/5 & & 6 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_3 - 3/4 r_1]{r_3 \leftrightarrow r_3 + 3/2 r_2}$$

$$\begin{bmatrix} 1 & 0 & -3/10 & \vdots & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & 0 & 1/5 & & 6 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow 5r_3} \begin{bmatrix} 1 & 0 & -3/10 & \vdots & 0 \\ 0 & 1 & 6/5 & \vdots & 0 \\ 0 & 0 & 1 & & 30 \end{bmatrix}$$

$$\xrightarrow[r_2 \leftrightarrow r_2 - 6/5 r_3]{r_1 \leftrightarrow r_1 + 3/10 r_3} \begin{bmatrix} 1 & 0 & 0 & \vdots & 9 \\ 0 & 1 & 0 & \vdots & -36 \\ 0 & 0 & 1 & & 30 \end{bmatrix}$$

The solution of the given system is

$x_1 = 9 \qquad x_2 = -36 \qquad x_3 = 30$

The system has a unique solution

Computing the inverse of a matrix

Let A be an $n \times n$ matrix. Apply elementary row operations to the matrix $[A: I_n]$ to obtain the matrix$[B: C]$, If

1. $B = I_n$, then $C = A^{-1}$.

2. $B \neq I_n$, then A is not invertible.

- **Example 10**

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solution

Apply elementary row operations to the matrix $[A{:}\, I_3]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\boldsymbol{r_3 \leftrightarrow r_3 - 3r_1}]{\boldsymbol{r_2 \leftrightarrow r_2 - 2r_1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{\boldsymbol{r_3 \leftrightarrow r_2}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[\boldsymbol{r_3 \leftrightarrow -r_3}]{\boldsymbol{r_2 \leftrightarrow -r_2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\boldsymbol{r_2 \leftrightarrow r_2 - 3r_3}]{\boldsymbol{r_1 \leftrightarrow r_1 - 3r_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{\boldsymbol{r_1 \leftrightarrow r_1 - 2r_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

- **Example 11**

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix}$ or show that it does not exist.

Solution

Apply elementary row operations to the matrix $[A: I_3]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 \leftrightarrow r_3 - r_1]{r_2 \leftrightarrow r_2 + 2r_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow \frac{1}{5} r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 \leftrightarrow r_3 + 4r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 3/5 & 4/5 & 1 \end{array}\right]$$

The row of zeros on the left means we can't get the identity matrix there and ***A* is singular (no inverse exist).**

Definition: Rank of a Matrix

The rank of matrix is the number of nonzero rows of reduced row Echelon form of A and denoted by $R(\boldsymbol{A})$.

Definition: Echelon form of a Matrix

1. Any rows of all zeros are below any other rows.
2. Each leading entry of a row is in a column to the right of the leading entry in the row above it.
3. All entries in a column below a leading entry are zeroes.

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 1 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: Reduced row Echelon form

A matrix is in reduced row echelon form if the above three conditions hold and in addition, we have

4. The leading entry in each nonzero row is 1.
5. The leading entry in each row is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3- Gauss –Jacobi Method

Example 12

Solve the following linear system by Gauss-Jacobi method

$$5x + y - z = 4$$

$$x + 4y + 2z = 15$$

$$x - 2y + 5z = 12$$

Solution

We have

$$
\begin{aligned}
x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n.
\end{aligned}
$$

Using

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

we obtain

$$x_1 = 0.8, y_1 = 3.0, z_1 = 2.6,$$

$$x_2 = 0.72, y_2 = 2.25, z_2 = 3.44,$$

$$x_3 = 1.038, y_3 = 1.85, z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, y = 2, z = 3,$$

when rounded to the nearest whole number.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

Exercises

1. Is $A \sim B$, where $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 4 & 6 \\ 2 & 5 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 7 & 8 \\ 0 & 1 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{bmatrix}$.

2. Solve the following linear system of equations by Gauss and Gauss-Jordan elimination method:

i) $$x_1 + x_2 + x_3 = 2$$
$$2x_1 + x_2 + 3x_3 = 4$$
$$x_1 - 2x_2 + x_3 = 5$$

ii) $$5\,x_1 + 3x_2 + 11x_3 = 6$$
$$x_1 + x_2 + 3x_3 = 4$$
$$2x_1 + x_2 + 4x_3 = -5$$

3. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ or show that it does not exist.

4. Decide whether or not each of the following matrices has reduced row echelon form:

i) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

ii) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5. Find Rank $R(A)$, where

i) $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 5 \\ -1 & 2 & 3 & 4 \end{bmatrix}$

Vector Space

Definition "Vector Space"

Let V a nonempty set with two operations (+ vector addition) ($\cdot$ scalar multiplication). Then V is called a vector space if the following axioms hold for any vector $u, v, w \in V$ and for any scalars $\alpha, \beta \in R$:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

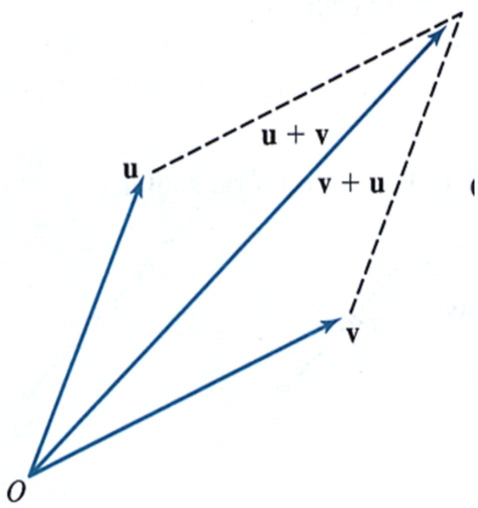
(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(g) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$



For example,

1- $\mathbb{R}^2$ is the collection of all sets of two ordered real numbers.

For example, (0, 0), (1, 2) and (-2, -3) are elements of $\mathbf{R}^2$.

2- $\mathbb{R}^3$ is the collection of all sets of three ordered real numbers.

For example, (0,0, 0) and (-1,3, 4) are elements of $\mathbf{R}^3$.

3- $\mathbb{R}^n$ is a vector space.

4- Any matrix $\boldsymbol{M}_{mxn}$ is a vector space.

5- Complex number $\mathbb{C}$ is a vector space.

Linear Combinations of Vectors

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V. We say that $\mathbf{v}$, a vector of V, is a linear **combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, if there exist scalars $c_1, c_2, \ldots, c_m$ such that $\mathbf{v}$ can be written $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$.

Example 1

Determine whether or not the vector (-1, 1, 5) is a linear combination of the vectors (1, 2, 3), (0, 1, 4), and (2, 3, 6).

Solution

Suppose

$$c_1(1,2,3)+c_2(0,1,4)+c_3(2,3,6)=(-1,1,5)$$

$$(c_1, 2c_1, 3c_1)+(0, c_2, 4c_2)+(2c_3, 3c_3, 6c_3)=(-1,1,5)$$

$$(c_1+2c_3, 2c_1+c_2+3c_3, 3c_1+4c_2+6c_3)=(-1,1,5)$$

$$\Rightarrow \begin{cases} c_1 \qquad\quad +2c_3 = -1 \\ 2c_1 + \ c_2 + 3c_3 = \ 1 \\ 3c_1 + 4c_2 + 6c_3 = \ 5 \end{cases} \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1$$

Solve the system of equations **by Gauss-Jordan method**.

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & 1 \\ 3 & 4 & 6 & 5 \end{array}\right] \xrightarrow[r_3 \leftrightarrow r_3 - 3r_1]{r_2 \leftrightarrow r_2 - 2r_1} \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 4 & 0 & 8 \end{array}\right] \xrightarrow{r_3 \leftrightarrow r_3 - 4r_2}$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -4 \end{array}\right] \xrightarrow{r_3 \leftrightarrow \frac{1}{4} r_3} \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The system has a unique solution.

Thus (-1, 1, 5) is a linear combination of (1, 2, 3), (0, 1, 4), and (2, 3, 6), where

$$(-1,1,5)=(1,2,3)+2(0,1,4)-1(2,3,6).$$

Example 2

Show that the vector (3, -4, -6) cannot be expressed as a linear combination of the vectors (1, 2, 3), (-1, -1, -2), and (1, 4, 5).

Solution

$$c_1(1,2,3)+c_2(-1,\ -1,-2)+c_3(1,4,5)=(3,-4,-6)$$

$$\begin{cases} c_1-\ c_2+\ c_3=\ 3 \\ 2c_1-\ c_2+4c_3=-4 \\ 3c_1-2c_2+5c_3=-6 \end{cases}$$

Solve the system of equations **by Gauss-Jordan method**

This system has no solution.

Thus (3, -4, -6) is not a linear combination of the vectors

(1, 2, 3), (-1, -1, -2), and (1, 4, 5).

Try to solve

Determine whether or not vector (5, 4, 2) is a linear combination of the vectors (1, 2, 0), (3, 1, 4), and (1, 0, 3)?

Definition "SubSpace"

Let V be a vector space. W is said to be a subspace of V if W is a subset of V and the following hold:

1- If $w_1, w_2 \in W$, then $w_1+w_2 \in$W

2- For any scalar c ,if $w \in W$ then $cw \in W$.

Example 3

Suppose that $(R^3,+,.)$ is a vector space. And let $w \in R^3$such that $W = \{(x,y,0), x,y \in R\}$. Prove that $(W,+,.)$ is a subspace on V?

Solution

Suppose $w_1 = (x_1,y_1,0)$ *and* $w_2 = (x_2,y_2,0)$

1- If w_1,$w_2 \in W$, then

w_1+w_2=$(x_1,y_1,0)$ +$(x_2,y_2,0) = (x_1+x_2,y_1+y_2,0))$∈W

2- For any scalar c ,if $w \in W$ then

$$cw = c(x,y,0) = (cx,cy,0) \in W.$$

$\therefore$ $(W,+,.)$ is a subspace on V

Example 4

Suppose that $V = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\}$ is a vector space. And let $w \in V$ such that $W = \{\begin{bmatrix} a & b \\ 2 & d \end{bmatrix}, a,b,d \in R\}$. Is $(W,+,.)$ a subspace on V?

Solution

Suppose $w_1 = \begin{bmatrix} a_1 & b_1 \\ 2 & d_1 \end{bmatrix}$ *and* $w_2 = \begin{bmatrix} a_2 & b_2 \\ 2 & d_2 \end{bmatrix}$

1- If w_1,$w_2 \in W$, then

$$w_1+w_2=\begin{bmatrix} a_1 & b_1 \\ 2 & d_1 \end{bmatrix}+\begin{bmatrix} a_2 & b_2 \\ 2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 4 & d_1+d_2 \end{bmatrix} \notin W$$

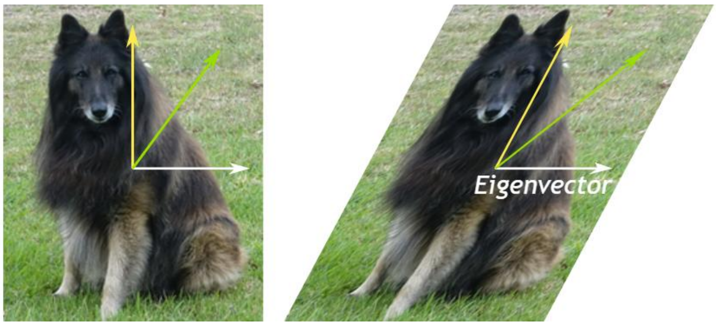
$\therefore$ $(W,+,.)$ is not a subspace on V

Eigenvalues and eigenvector

Definition

Let A be any square matrix. A scalar λ is called an eigenvalue of A if there exists nonzero column vector v such that

$$A\,v = \lambda\,v$$



A simple example is that an eigenvector **does not change direction** in a transformation.

One of the cool things is we can use **matrices** to do **transformations** in space, which is used a lot in **computer graphics.**

$$A\,v = \lambda\,v \Longrightarrow Av - \lambda\,v = 0$$

$$\Longrightarrow |A - \lambda I|v = 0$$

Since $v \neq 0 \Longrightarrow |A - \lambda I| = 0$

Steps of computing eigenvalue and eigenvectors

1- Find eigenvalue by solve $|A - \lambda I| = 0$

2- For each λ, find the basic eigenvectors $v \neq 0$ by finding the basic solutions to $(A - \lambda I)v = 0$

Example 3

Find the eigenvalue and the eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\lambda = 5 \text{ and } \lambda = -1$$

The corresponding eigenvector for $\lambda = 5$

$$\begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -4x + 4y &= 0 \\ 2x - 2y &= 0 \end{aligned}\right\}$$

Solve the two equations $x = y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; c_1 \text{ a constant}$$

The corresponding eigenvector for $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 4 \\ 2 & 3-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x + 4y = 0$$

Solve the two equations $x = -2y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ c_2 \text{ a constant}$$

Example 4

Find the eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(3 - \lambda) + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2$$

Example 5

Find the eigenvalue and the eigenvector of the matrix

$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ in 1-$\mathbb{R}$ real number 2- $\mathbb{C}$ complex number ?

Solution

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & -1 \\ 2 & -1 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(-1 - \lambda) + 2 = 0$$

$$\lambda^2 + 1 = 0$$

1- In $\mathbb{R}$ there is no eigenvalues.

2- $\lambda = \pm i$ in $\mathbb{C}$ complex number

The corresponding eigenvector for $\lambda = i$

$$\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)x - y = 0$$

$$2x - (1+i)y = 0$$

Solve the two equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}, \ c_1 \text{ a constant}$$

The corresponding eigenvector for $\lambda = -i$

$$\begin{bmatrix} 1-(-i) & -1 \\ 2 & -1-(-i) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1+i)x - y = 0$$

$$2x + (-1+i)y = 0$$

Solve the two equations $x = -2y$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \ c_2 \text{ a constant}$$