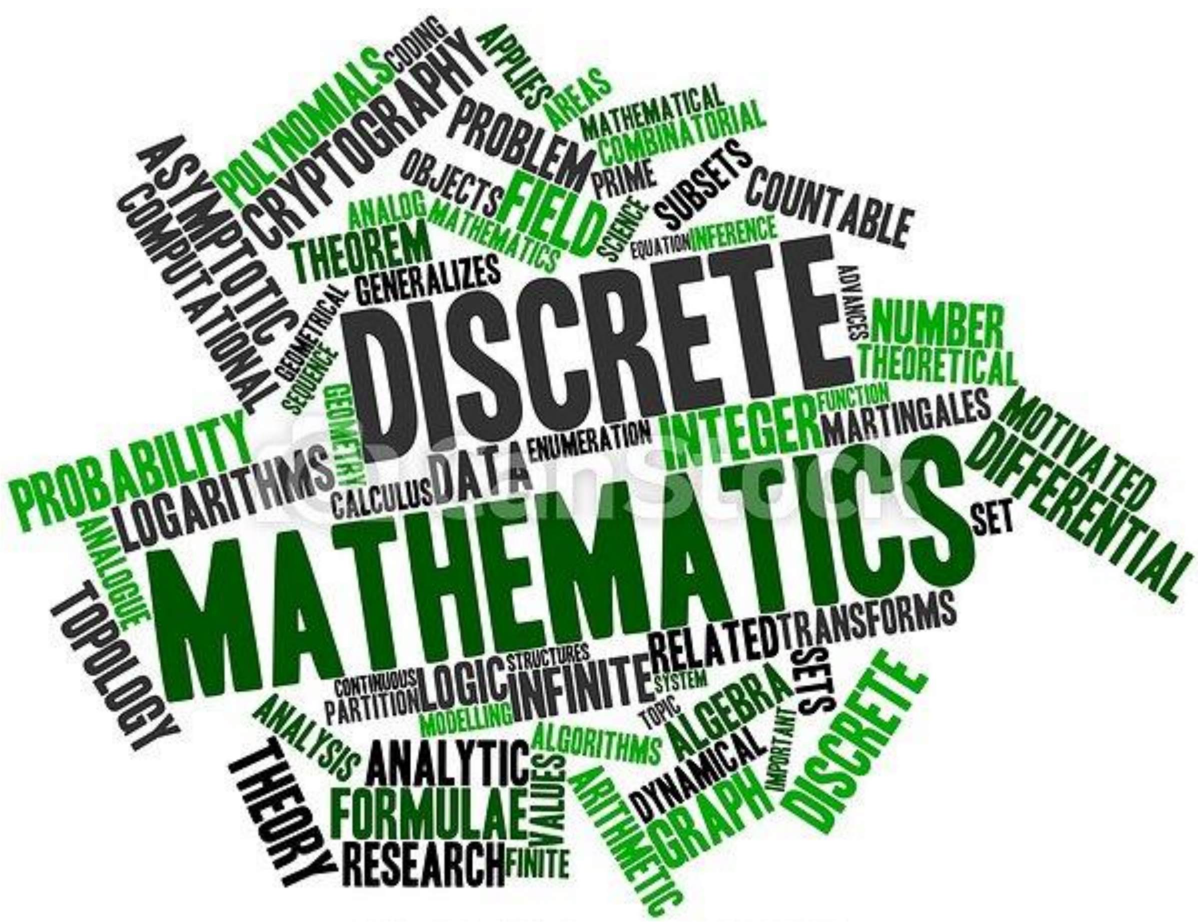
جامعة الفيوم
كلية الحاسبات والذكاء الاصطناعى
مؤسسة تعليمية معتمدة

الكتاب الجامعى
2023/2024

جامعة الفيوم

كلية الحاسبات والذكاء الاصطناعى

Discrete Mathematics
CS201



Prepared By:

CS&IS Department

Table of Contents

Goals of a Discrete Mathematics

Studing discrete mathematics has more than one purpose. Students should learn a particular set of mathematical facts and how to apply them; more importantly, such a course should teach students how to think logically and mathematically. To achieve these goals, this text stresses mathematical reasoning and the different ways problems are solved. Five important themes are interwoven in this text: mathematical reasoning, combinatorial analysis, discrete structures, algorithmic thinking, and applications and modeling. A successful discrete mathematics course should carefully blend and balance all five themes.

1. *Mathematical Reasoning:* Students must understand mathematical reasoning in order to read, comprehend, and construct mathematical arguments. This text starts with a discussion of mathematical logic, which serves as the foundation for the subsequent discussions of methods of proof. Both the science and the art of constructing proofs are addressed. The technique of mathematical induction is stressed through many different types of examples of such proofs and a careful explanation of why mathematical induction is a valid proof technique.

2. *Combinatorial Analysis:* An important problem-solving skill is the ability to count or enumerate objects. The discussion of enumeration in this book begins with the basic techniques of counting. The stress is on performing combinatorial analysis to solve counting problems and analyze algorithms, not on applying formulae.

3. *Discrete Structures:* A course in discrete mathematics should teach students how to work with discrete structures, which are the abstract mathematical structures used to represent discrete objects and relationships between these objects. These discrete structures include sets, permutations, relations, graphs, trees, and finite-state machines.

4. *Algorithmic Thinking:* Certain classes of problems are solved by the specification of an algorithm. After an algorithm has been described, a computer program can be constructed implementing it. The mathematical portions of this activity, which include the specification of the algorithm, the verification that it works properly, and the analysis of the computer memory and time required to perform it, are all covered in this text.Algorithms are described using both English and an easily understood form of pseudocode.

5. *Applications and Modeling:* Discrete mathematics has applications to almost every conceivable area of study. There are many applications to computer science and data networking in this text, as well as applications to such diverse areas as chemistry, biology, linguistics, geography, business, and the Internet. These applications are natural and important uses of discrete mathematics and are not contrived. Modeling with discrete mathematics is an extremely important problem-solving skill, which students have the opportunity to develop by constructing their own models in some of the exercises.

What is the discrete Mathematics?

It is the part of mathematics devoted to the study of discrete objects. It is intended to learn the discrete structures/techniques to solve problems (that handles discrete objects).

- More general, Discrete Math is used whenever:
    - Objects are counted
    - Relationships between finite sets are studied,
    - Processes involving a finite number of steps are analyzed.
- It is the science needed to learn:
    1. Mathematical reasoning
    2. Problem solving techniques
    3. How to attack problems that may be different than previous.

Why to study Discrete Math.?

1. Discrete Math contains the mathematical background for solving problems in Operations research, Chemistry, Engineering, Biology, computer science... etc.
2. To create the ability/skills to understand/create mathematical arguments.
3. To provide the mathematical foundation for many computer science courses e.g. Data structures, Algorithms, Database theory, Automata theory, Formal languages, Compiler theory, Computer security and operating systems.

The importance of studying Discrete Mathematics is grown because "The information is stored and manipulated by computing machines in a discrete fashion".

Topics in discrete mathematics will be important in many courses that you will take in the future:

• Computer Science: Computer Architecture, Data Structures, Algorithms, Programming Languages, Compilers, Computer Security, Databases, Artificial Intelligence, Networking, Graphics, Game Design, Theory of Computation, ……

• Mathematics: Logic, Set Theory, Probability, Number Theory, Abstract Algebra, Combinatorics, Graph Theory, Game Theory, Network Optimization, …

• Other Disciplines: You may find concepts learned here useful in courses in philosophy, economics, linguistics, and other departments.

CHAPTER 1:

The Foundations:

Logic and Proofs

CHAPTER 1:

The Foundations: Logic and Proofs

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Because a major goal of this book

is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic. Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically

1.1 Propositiona Logic

Our discussion begins with an introduction to the basic building blocks of logic—propositions. A proposition is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.

Are the following sentences propositions?

- Cairo is the capital of Egypt. ➔Yes
- Read this carefully. ➔No
- 1+2=3 ➔Yes
- x+1=2. ➔No
- What time is it? . ➔No

- Propositional Logic – the area of logic that deals with propositions
- Propositional Variables – variables that represent propositions: *p*, *q*, *r*, *s*

E.g. Proposition *p* – "Today is Friday."

- Truth values – T, F

We can denote a proposition by a variable for future reference:

p : "Cairo is the capital of Egypt

A. Negation (NOT): ¬p

DEFINITION

Let *p* be a proposition. The **negation** of *p*, denoted by ¬*p*, is the statement "It is not the case that *p*."

The proposition ¬*p* is read "not *p*." The truth value of the negation of *p*, ¬*p* is the opposite of the truth value of *p*.

p	$\neg p$
T	F
F	T

Examples

- Find the negation of the proposition "Today is Friday." and express this in simple English.

  Solution:

  The negation is "It is not the case that *today is Friday*."

  simply, "Today is not Friday." or "It is not Friday today."

- Find the negation of the proposition "At least 10 inches of rain fell today in Miami." and express this in simple English.

  Solution:

  The negation is "It is not the case that *at least 10 inches of rain fell today in Miami*." simply, "Less than 10 inches of rain fell today in Miami."

B. Conjunction (AND): p Λ q

DEFINITION

Let *p* and *q* be propositions. The ***conjunction*** of *p* and *q*, denoted by *p* Λ *q*, is the proposition "*p* and *q*". The conjunction *p* Λ *q* is true when both *p* and *q* are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples

Find the conjunction of the propositions *p* and *q* where *p* is the proposition "Today is Friday." and *q* is the proposition "It is raining today.", and the truth value of the conjunction.

Solution:

The conjunction is the proposition "Today is Friday and it is raining today." The proposition is true on rainy Fridays.

Hint: In language (not mathematics), we may use the word “but” to mean “and”. “Today is Friday but it is raining”.

C. Disjunction (OR) p v q

DEFINITION: ***disjunction*** “OR” , *p* v *q* , *inclusive or*

Let *p* and *q* be propositions. The ***disjunction*** of *p* and *q*, denoted by *p* v *q*, is the proposition “*p* or *q*”. The conjunction *p* v *q* is false when both *p* and *q* are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

inclusive or : The disjunction is true when at least one of the two propositions is true.

- E.g. “Students who have taken calculus or computer science can take this class.” – those who take one or both classes.
- The computer processor and any software action you do is translated to ONLY those three mathematical function NOT, OR, AND!!!

D. Exclusive or (p ⊕ q)

DEFINITION: *exclusive or ,p⊕ q*

Let *p* and *q* be propositions. The *exclusive or* of *p* and *q*, denoted by *p*⊕*q*, is the proposition that is true when exactly one of *p* and *q* is true and is false otherwise.

exclusive or : The disjunction is true only when one of the proposition is true.

E.g. “Students who have taken calculus or computer science, but not both, can take this class.” – only those who take one of them.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive *Or* (*XOR*) of Two Propositions.

p	q	p q
T	T	F
T	F	T
F	T	T
F	F	F

1.1.1 Conditional Statements

A. Implication "if *p*, then *q*." $p \to q$

DEFINITION 5: $p \to q$,
"if *p*, then *q*."

- Let *p* and *q* be propositions. The *conditional statement* $p \to q$, is the proposition "if *p*, then *q*." The conditional statement is false when *p* is true and *q* is false, and true otherwise.
- In the conditional statement $p \to q$, *p* is called the *hypothesis* (or *antecedent* or *premise*) and *q* is called the *conclusion* (or *consequence*).
- A conditional statement is also called an implication.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

"If I am elected, then I will lower taxes." $p \to q$

implication:

	p	q	$p \to q$
elected, lower taxes.	T	T	T
not elected, lower taxes.	F	T	T
not elected, not lower taxes.	F	F	T
elected, not lower taxes.	T	F	F

Sufficiency
"if p then q"
"p is sufficient for q"
"a sufficient condition for q is p"
"if p, q"
"q if p"
"q when p"
"q whenever p"
"p implies q"
"q follows from p"

Necessity
"p only if q"
"q is necessary for p"
"a necessary condition for p is q"

Another subtle way:
"q unless ¬ p" (if you exclude the case that ¬ p, then q is true)

- **Example:**

Let *p* be the statement "Maria learns discrete mathematics." and *q* the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Solution:

Any of the following -

"**If** Maria learns discrete mathematics, **then** she will find a good job.

"Maria will find a good job **when** she learns discrete mathematics."

"For Maria to get a good job, **it is sufficient for** her to learn discrete mathematics."

"Maria will find a good job **unless** she does **not** learn discrete mathematics."

B. Other conditional statements:

- *Contrapositive* of $p \rightarrow q : \neg q \rightarrow \neg p$ *takes the same truth value of the original statement, We say both are equivalent*
- *Converse of* $p \rightarrow q : q \rightarrow p$
- *Inverse* of $p \rightarrow q : \neg p \rightarrow \neg q$

• *Of course both converse and inverse are equivalent because they are the contrapositive of each other.*

• *Neither is the same as the original implication. E.g., when P=T, q=F , the conditional statement is false; however, both converse and inverse are true.*

<u>Example</u>

The conditional statement: "The home team wins whenever it is raining" find contrapositive, converse, inverse

solution

q: "The home team wins"; p: "it is raining" (q whenever p or if p then q)

contrapositive: $\neg q \rightarrow \neg p$ *"if the home team does not win then it is not raining".*

converse: $q \rightarrow p$ *"if the home team wins then it is raining".*

inverse: $\neg p \rightarrow \neg q$ *"if it is not raining, then the home team does not win"*

C. Biconditional $p \leftrightarrow q$, "p if and only if q."

DEFINITION:

"*p* if and only if *q*."

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "*p* if and only if *q*."
- The biconditional statement $p \leftrightarrow q$ is true when p and q have the **same truth** values, and is false otherwise.
- Biconditional statements are also called ***bi-implications.***
- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$
- "*if and only if*" can be expressed by "***iff***"

There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

Example:

- Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket."
- Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.

- Example:

Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

<u>Precedence of Logical Operators</u>

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

1.1.2 Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*.				
x	y	$x \vee y$	$x \wedge y$	$x \quad y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

DEFINITION 7

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

- Example:

Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110

11 0001 1101
----
11 1011 1111 bitwise *OR*
01 0001 0100 bitwise *AND*
10 1010 1011 bitwise *XOR*

1.1.3 Translating English Sentences

- Translation to logical expression removes this ambiguity.
- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.

Example:

Translate to logical expression:

“You can access the internet from campus only if you are a computer science major or you are not a freshman”.

Solution:

a: “You can access the internet from campus”.

c: “You are a computer science major”.

f : “You are a freshman”.

$a \rightarrow (c \wedge \neg f)$

System Specifications

- Very important to translate system requirements to formal specifications to avoid ambiguity.
- This is done by design engineers, e.g., software engineers, and others.

Example:

Using logical connectives express the following specifications:

“The automated reply cannot be sent when the file system is full”

Solution:

p: “The automated reply can be sent”.

q: “The file system is full”.

$q \rightarrow \neg p$

Example:

Determine whether these system specifications are consistent:

"The diagnosis message is stored in the buffer or it is retransmitted"

"The diagnostic message is not stored in the buffer"

"If the diagnostic message is stored in the buffer, then it is retransmitted".

Solution :

- p: "The diagnostic message is stored in the buffer".
- q: "The diagnostic message is retransmitted".

$$\dot{C} = (p \vee q) \bar{\wedge} (\neg p)(p \rightarrow \bar{q})$$

p	q	$(p \vee q)$	$(\neg p)$	$(p \rightarrow q)$	C
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	T	T	T
F	F	F	T	T	F

1.1.4 Boolean Searches

- Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages.
- Because these searches employ techniques from propositional logic, they are called Boolean searches.
- In Boolean searches, the connective AND is used to match records that contain both of two search terms, the connective OR is used to match one or both of two search terms, and the connective NOT (sometimes written as AND NOT) is used to exclude a particular search term.
- **Example** :

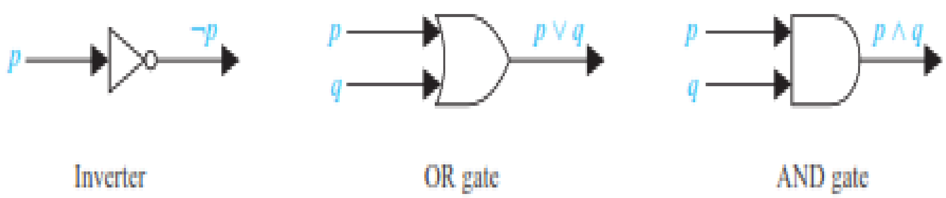
Search for universities in Cairo or Alexandria.

solution

(Cairo OR Alexandria) AND Universities

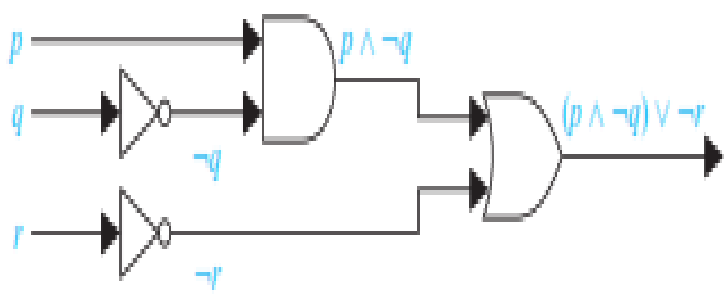
1.1.5 Logic Circuits

- Propositional logic can be applied to the design of computer hardware. In Chapter 12 we will study this topic in depth. We give a brief introduction to this application here.
- A logic circuit (or digital circuit) receives input signals p1,p2,...,pn, each a bit [either 0 (off) or 1 (on)], and produces output signals s1,s2,...,sn, each a bit.
- In this section we will restrict our attention to logic circuits with a single output signal; in general, digital circuits may have multiple outputs.



- **Example**:

  Determine the output for the following combinatorial circuit



  solution:

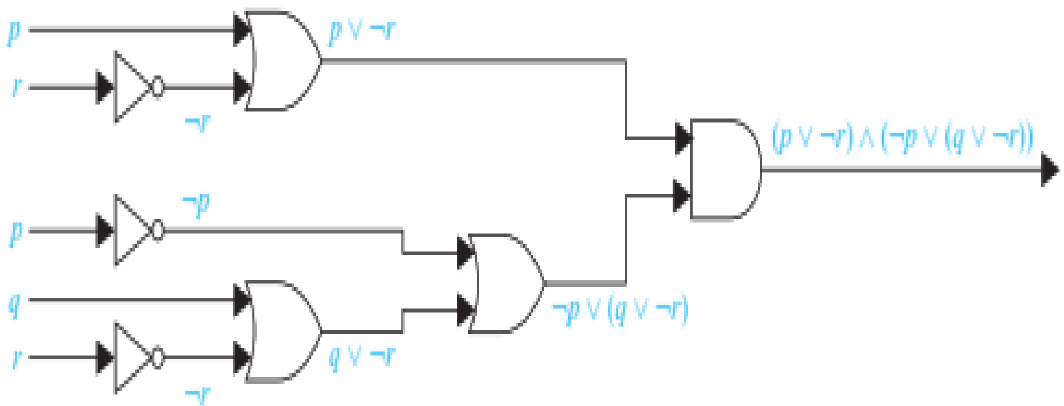
(p∧¬ q) ν ¬ r

- **Example**:

Build a digital circuit that produces the output

(p ∨¬ r)∧ (¬ p ∨ (q ∨ ¬r)) when given input bits p, q, and r

solution



Exercises

1- Which of these sentences are propositions?

a.Today is Friday.

b. Where is the library?

2- Let p and q be the prepositions

p: It is below freezing

q: It is snowing

Write these prepositions using p and q and logical connectives

a) It is not below freezing and it is not snowing

b) If it is below freezing, it is also snowing.

3- Let p, q, and r be the prepositions

p: You have the flu

q: You miss the final examination

r: You pass the course

4. Express the prepositions as an English sentence.

$(p \to \neg r) \vee (q \to \neg r)$

5- Determine whether these biconditionals are true or false.

a) 2+2 =4 if and only if 1+1=2.

b) 1+1=2 if and only if 2 + 3=4.

6- Determine whether each of these conditional statements is true or false.

a) If 1 + 1 = 2, then 2 + 2 = 5.

c) If monkeys can fly, then 1 + 1 = 3.

7- How many rows appear in a truth table for each of these compound prepositions?

$(p \vee \neg t) \wedge (p \vee \neg s)$

8- Construct a truth table for the compound preposition.

$(p \to q) \leftrightarrow (\neg q \to \neg p)$

1.2 <u>Propositional Equivalences</u>

DEFINITION:

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.

- **<u>Example</u>**:

Show that $\neg p \vee q$ and $p \to q$ are logically equivalent.

solution

Truth Tables for $\neg p \vee q$ and $p \to q$.					
p	q	$\neg p$	$\neg p \vee q$	$p \to q$	$(\neg p \vee q) \leftrightarrow (p \to q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

- In general, 2^n rows are required if a compound proposition involves n propositional variables in order to get the combination of all truth values.

- **Example** :

Show that: $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are equivalent.

solution

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Logical Equivalences.	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences.	
Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg (p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

© The McGraw-Hill Companies, Inc. all rights reserved.

TABLE 8 Logical Equivalences Involving Biconditionals.
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

- **Example**:

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$

$\equiv \neg(\neg p) \wedge \neg q$ *by the second De Morgan law*

$\equiv p \wedge \neg q$ *by the double negation law*

- ***Example****:*

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ *by example*

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ *by the first De Morgan law*

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ *by the associative and communicative law for disjunction*

$\equiv \mathrm{T} \vee \mathrm{T}$

$\equiv \mathrm{T}$

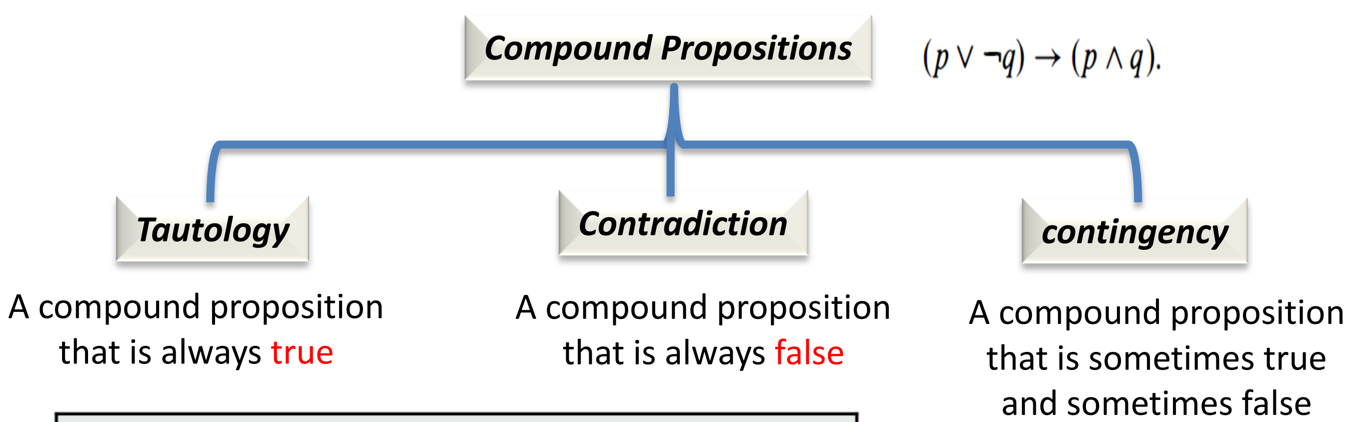
- *Note: The above examples can also be done using truth tables.*

1.3 <u>Compound Propositions Classification</u>

DEFINITION

- A compound proposition that is always **true**, no matter what the truth values of the propositions that occurs in it, is called a ***tautology***.
- A compound proposition that is always **false** is called a ***contradiction***.
- A compound proposition that is neither a tautology or a contradiction is called a ***contingency***.
-

Examples of a Tautology and a Contradiction.			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F



Examples of a Tautology and a Contradiction and contingency.				
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \wedge p$
T	F	T	F	T
F	T	T	F	F

Example:

Show that following conditional statement is a tautology by using truth table

$$p \wedge q \rightarrow p$$

solution

P	q	$p \wedge q$	$p \wedge q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Exercises:

1] Use truth tables to verify the following equivalences.

$p \vee F \equiv p$

2] Use truth table to verify the associative law

3] Use De Morgan's laws to find the negation of the following statements.

Jan is rich and happy.

5] Show that the conditional statements is a tautology by using truth tables.

$(p \wedge q) \rightarrow (p \rightarrow q)$

10] Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logical equivalent

11] Prove that $(p \wedge \neg q) \vee q \equiv p \vee q$

12] Prove that $\neg (p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are equivalent.

CHAPTER 2 :

Predicates and Quantifiers

CHAPTER 2 :

Predicates and Quantifiers

Propositional logic, studied in the previous chapter , cannot adequately express the meaning of all statements in mathematics and in natural languag. In this chapter we will introduce a more powerful type of logic called **predicate logic**.

We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects. To understand predicate logic, we first need to introduce the concept of a predicate. Afterward, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property

2.1 Predicates

Statements involving variables are neither true nor false.

E.g. "$x > 3$", "$x = y + 3$", "$x + y = z$"

- "x is greater than 3"
    - "x": subject of the statement
    - "is greater than 3": the ***predicate***
- We can denote the statement "x is greater than 3" by ***P(x)***, where *P* denotes the predicate and *x* is the variable.
- Once a value is assigned to the variable *x*, the statement *P(x)* becomes a **proposition** and has a truth value.
- **Example**:

Let *P(x)* denote the statement "$x > 3$." What are the truth values of *P*(4) and *P*(2)?

Solution:

P(4) – "$4 > 3$", *true*
P(3) – "$2 > 3$", *false*

- **Example:**

Let $Q(x,y)$ denote the statement "x = y + 3." What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

Solution:

$Q(1,2)$ – "1 = 2 + 3" , *false*
$Q(3,0)$ – "3 = 0 + 3", *true*

- **Example:**

Let *A(c,n)* denote the statement "**Computer *c* is connected to network *n***", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

solution:

A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false
A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

- **Example:**

Let A(x): "Computer x is under attack by an intruder". Suppose that only CS2 and MATH1 are currently under attack by intruders. What are the values of A(CS1) , A(CS2) , and A(MATH1)

solution
A(CS1): "Computer CS1 is under attack by an intruder" is False
A(CS2): is True
A(MATH1) is True.

In programming, all conditions are predicates:
Example:

```
if (x > 0)
        y = 3; //C - language code .
```

Example:

" if x > 0 then x:= x + 1"
When the statement is encountered, the value if x is inserted into *P(x).*
If *P(x)* is true, *x* is increased by 1.
If *P(x)* is false, *x* is not changed.

Extension

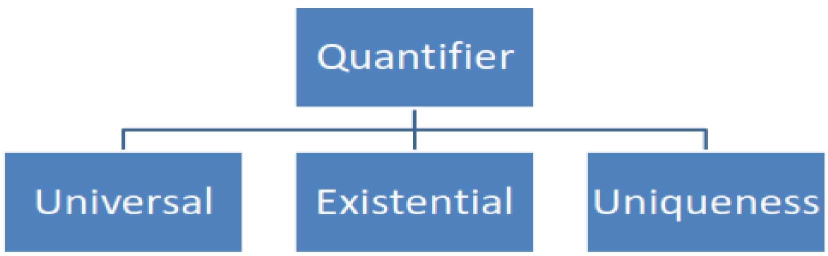
- A statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a ***n*-place predicate** or a ***n*-ary predicate.**

2.2 <u>Quantifiers</u>

Express the extent to which a predicate is true over a range of elements.

- **Universal quantification**: a predicate is true for **every** element under consideration
- **Existential quantification**: a predicate is true for **one or more** element under consideration
- **Uniqueness quantification**! or $\exists i x P(x)$ or $\exists_1 x P(x)$ states "There exists a unique x such that $P(x)$ is true."

A domain must be specified.



2.2.1 Universal Quantification ∀:

DEFINITION

- The *universal quantification* of $P(x)$ is the statement
  "$P(x)$ for all values of x in the domain."
- The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here is called the **Universal Quantifier**. We read $\forall x P(x)$ as "**for all $xP(x)$**" or "**for every $xP(x)$.**"
- An element for which $P(x)$ is false is called a **<u>counterexample</u>** of $xP(x)$.
- **<u>Example:</u>**

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution:

Because P(x) is true for all real numbers, the quantification is true.

- A statement $\forall x P(x)$ is <u>false</u>, if and only if $P(x)$ is not always true where x is in the domain. One way to show that is to find a <u>counterexample</u> to the statement $\forall x P(x)$.

Example:

Let *Q(x)* be the statement "$x < 2$". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution:

Q(x) is not true for every real numbers, e.g. Q(3) is false. x = 3 is a counterexample for the statement $\forall x Q(x)$.

- *Thus the quantification is false.*
- **Note:**

When all elements in a domain can be listed, i.e., $x_1, x_2, \ldots, x_n$, we can say $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \ldots. \wedge P(x_n)$$

Or $$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \ldots \wedge P(x_n)$$

Example:

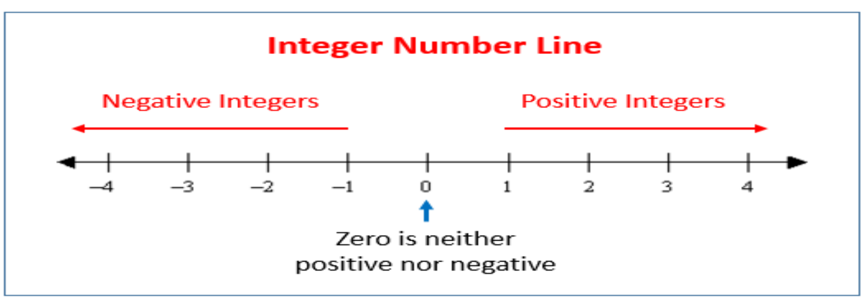
What does the statement $\forall x N(x)$ mean if *N(x)* is "Computer *x* is connected to the network" and the domain consists of all computers on campus?

Solution:

"Every computer on campus is connected to the network."

Example:

- P(x): "$x^2>0$" and the domain is all integers $\forall x P(x)$ " is false by counterexample x=0.
- P(x): "$x^2>0$" and the domain is all positive integers." $\forall x P(x)$" is true.



Note:

Real numbers consist of zero (0), the positive and negative integers (-3, -1, 2, 4), and all the fractional and decimal values in between (0.4, 3.1415927, 1/2).

Example :

What is the truth value of $\forall x(x^2 \geq x)$for the real numbers domain and integers domain?

Solution:

The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \ngeq \frac{1}{2}$.

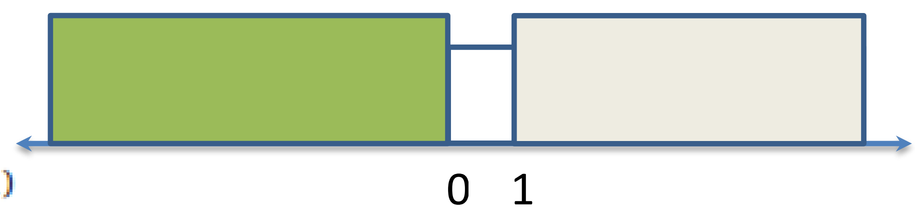
Note that

$x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$.

It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$).

However, if the domain consists of the integers, is true, because there are no integers x with $0 < x < 1$.

$$x^2 \geq x$$
$$x^2 - x \geq 0$$
$$x(x-1) \geq 0$$
$$(x \leq 0) \ or \ (x \geq 1)$$



Integers can also **include negative numbers** ... but still no fractions allowed!

2.2.2 Existential Quantifier ∃:

DEFINITION

The *existential quantification* of *P(x)* is the statement

"There exists an element x in the domain such that *P(x)*."

We use the notation ∃*xP(x)* for the existential quantification of *P(x)*. Here ∃ is called the **Existential Quantifier**.

- The existential quantification ∃*xP(x)* is read as

"There is an x such that $P(x)$," or

"There is at least one x such that $P(x)$," or
"For some x, $P(x)$."

- Other readings "for at least one", etc.

Example:

Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists xP(x)$, where the domain consists of all real numbers?

Solution:

"$x > 3$" is sometimes true – for instance when

$x = 4$. The existential quantification is true.

$\exists xP(x)$ is false if and only if $P(x)$ is false for every element of the domain.

Example:

Let $Q(x)$ denote the statement "$x = x + 1$". What is the true value of the quantification $\exists xQ(x)$, where the domain consists for all real numbers?

Solution:

$Q(x)$ is false for every real number. The existential quantification is false.

Example:

$R(x)$: "$x^2>10$", and the domain is positive integers not exceeding 4. What is the true value of the quantification $\exists xR(x)$

solution:

$R(x)$ is true for $x=4$. Then, "$\exists xR(x)$" is true.

- If the domain is empty, $\exists xQ(x)$ is false because there can be no element in the domain for which $Q(x)$ is true.
- The existential quantification $\exists xP(x)$ is the same as the disjunction $P(x_1)VP(x_2)V\ldots VP(x_n)$.

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots P(x_n)$$

Quantifiers		
Statement	***When True?***	***When False?***
xP(x) *xP(x)*	*xP(x) is true for every x.* *There is an x for which P(x) is true.*	*There is an x for which xP(x) is false.* *P(x) is false for every x.*

2.2.3 Precedence of Quantifiers

All quantifiers have higher precedence than logical operators:

$$\forall x\, P(x) \vee Q(x) \equiv \left(\forall x\, P(x)\right) \vee Q(x)$$
$$\not\equiv \forall x \left(P(x) \vee Q(x)\right)$$

Quantifiers with restricted domains

We can define quantifier for Restricted domain

Restricted domain

- $\forall x < 0\ (x^2 > 0)$.
- $\forall y \neq 0\ (y^3 \neq 0)$
- $\exists z > 0\ (z^2 = 2)$.

Statement

Example:

What do the following statements mean? The domain in each case consists of real numbers.

solution:

- $x < 0$ ($x^2 > 0$): For every real number x with $x < 0$, $x^2 > 0$. "The square of a negative real number is positive." It's the same as $x(x < 0 \rightarrow x^2 > 0)$
- $y \neq 0$ ($y^3 \neq 0$): For every real number y with $y \neq 0$, $y^3 \neq 0$. "The cube of every non-zero real number is non-zero." It's the same as $y(y \neq 0 \rightarrow y^3 \neq 0\ )$.
- $z > 0$ ($z^2 = 2$): There exists a real number z with $z > 0$, such that $z^2 = 2$. "There is a positive square root of 2." It's the same as $z(z > 0\ \Lambda\ z^2 = 2)$.

Binding Variables

Example :

In the statement $\exists x(x + y = 1)$ we say

- x is bound by the quantifier $\exists$.
- y is free
- x+y=1 is the scope of the quantifier $\exists$.

However, for the statement $\exists x(P(x)\Lambda Q(x))\vee\forall xR(x)$

- All variables are bound
- The scope of the quantifier $\exists$ is $(P(x)\Lambda Q(x))$
- The scope of the quantifier $\forall$ is R(x).

The existential quantifier binds the variable *x* in *P (x)* ∧ *Q(x)* and the universal quantifier ∀*x* binds the variable *x* in *R(x)*

2.2.4 Logical Equivalences Involving Quantifiers

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
- We use the notation **S ≡ T** to indicate that two statements S and T involving predicates and quantifiers are logically equivalent

2.2.5 Negating Quantified Expressions (De Morgan's laws for quantifiers)

Negate Q:"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely, ∀xP (x),

P(x) :"x has taken a course in calculus"

the domain is "the students in your class."

¬ Q:"**It is not the case that every student in your class has taken a course in calculus**."

This is equivalent to "**There is a student in your class who has not taken a course in calculus.**" And this is simply the existential quantification of the negation of the original propositional function ∃x¬P(x)

- So these are logically equivalent **$\neg\forall xP(x) \equiv \exists x\neg P(x)$**

Example:

Negate :"There is a student in the class who has taken a course in calculus"

solution

Q(x): "x has taken a course in calculus."

- The **negation** of this statement is the proposition "It is not the case that there is a student in this class who has taken a course in calculus."
- This is equivalent to "Every student in this class has not taken calculus," which is just the **universal** quantification of the negation of the original propositional function, $\forall x \neg Q(x)$.
- This example illustrates the equivalence

$$\neg \exists xQ(x) \equiv \forall x \neg Q(x).$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

When the domain has n elements $x_1, x_2,\ldots,x_n$, it follows that

$$\forall xP(x) \equiv P(x1) \wedge P(x2) \wedge\cdots\wedge P(xn)$$

$$\neg\forall xP(x) \equiv \neg [P(x1) \wedge P(x2) \wedge\cdots\wedge P(xn)]$$

$$\equiv \neg P(x1) \vee\neg P(x2) \vee \cdots\vee\neg P(xn)$$

$$\equiv \exists x \neg P(x).,$$

And

$$\exists xP(x) \equiv P(x1)\vee P(x2) \vee\cdots\vee P(xn)$$

$$\neg\exists xP(x) \equiv \neg [P(x1) \vee P(x2) \vee\cdots\vee P(xn)]$$

$$\equiv \neg P(x1) \wedge\neg P(x2) \wedge\cdots\wedge\neg P(xn)$$

$$\equiv \forall x \neg P(x).$$

What are the negations of the statements

$\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$

Solution:

$\neg \forall x (x^2 > x) \equiv \exists x \neg (x^2 > x) \equiv \exists x(x^2 \leq x)$

$\neg \exists x (x^2 = 2) \equiv \forall x \neg (x^2 = 2) \equiv \forall x(x^2 \neq 2)$

Example:

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent

solution:

$$\begin{aligned} \neg\forall x(P(x) \to Q(x)) &\equiv \exists x\, \neg(P(x) \to Q(x)) \\ &\equiv \exists x\, \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x\, (P(x) \wedge \neg Q(x)) \end{aligned}$$

Translating from English into Logical Expressions

Example:

Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution:

- If the domain consists of students in the class

  $C(x)$ is the statement "x has studied calculus"

$\forall x C(x)$

- If the domain consists of all people

  $S(x)$ represents that person x is in this class.

$\forall x(S(x) \to C(x))$

- Other possible courses are involved:

  $Q(x,y)$ for the statement "student x has studies subject y."

$\forall x Q(x, \text{calculus})$

$\forall x(S(x) \to Q(x, \text{calculus}))$

Example :

Express the following two statements using predicates and quantifiers:

P : “Some student in this class has visited Mexico”

Q:“Every student in this class has visited either Canada or Mexico”.

Solution:

Domain is students in the class:

M(x) : “x has visited Mexico”.

C(x) : “x has visited Canada”.

P: $\exists x M(x)$

Q: $\forall x(M(x) \vee C(x))$

Domain is all people:

S(x): “Person x is in class”.

P: $\exists x(S(x) \wedge M(x))$

Q: $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$

Example

Express, using predicates and quantifiers:

P : “Every mail message larger than one megabyte will be compressed”.

Q : “If a user is active, at least one network link will be available”.

Solution:

Domain of m is all message:

S(m,1) : “Size of message m is larger than 1 MB”.

C (m) : “message m will be compressed”.

P : $\forall m(S(m,1) \rightarrow C(m))$.

Domain of u is all users:

A(u) : “User u is active”.

S(n) : “State of link n is available”.

Q : $\exists u\, A(u) \rightarrow \exists n\, S(n)$.

2.2.6 Nested Quantifiers

- Nested quantifiers, where one quantifier is within the scope of another, such as

$\forall x \exists y(x+y=0)$.

- Note that everything within the scope of a quantifier can be thought of as a propositional function.
- Nested quantifiers commonly occur in mathematics and computer science. For example, $\forall x\boldsymbol{\exists y(x+y=0)} \equiv \forall x\mathbf{Q(x)}$, where Q(x) is $\exists yP(x,y)$, where P(x,y) is x+y=0.
- $\forall x\forall yP(x, y)$ is true, if we find that P(x,y) is true for all values for x and y, and it is false , If we ever hit a value x for which we hit a value y for which P(x,y) is false.
- $\forall x\exists yP(x,y)$ is false if for some x we never hit such a y.
- $\exists x\forall yP(x,y)$ is false If we never hit such an x.
- $\exists x\exists yP(x,y)$ is false only if we never hit an x for which we hit a y such that P(x,y) is true.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x\forall y P(x,y)$ $\forall y\forall x P(x,y)$	$P(x,y)$ is true for every pair x, y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x\exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x\forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x\exists y P(x,y)$ $\exists y\exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x, y.

Example:

Translate into English the statement

$\forall x\forall y((x > 0)\wedge(y < 0)\rightarrow(xy < 0))$,

where the domain for both variables consists of all real numbers.

Solution:

- “for every real number x and for every real number y, if x >0 and y <0, then xy < 0”.
- “for every real numbers x and y, if x is positive and y is negative, then xy is negative”.
- “The product of a positive real number and a negative real number is always a negative real number.”

Example:

let P(x, y) denote x+y=0, What are the truth values of the quantifications ∀x∀yP(x,y), ∀x∃yP(x,y), ∃x∀yP(x,y), ∃x∃yP(x,y) where the domain for all variables consists of all real numbers?

Solution

If P(x,y) : x+y=0

- ∀x∀yP(x,y) “for all real numbers x and y , x+y=0” ex: 1+3≠0 then ∀x∀yP(x,y) is false
- ∀x∃yP(x,y) “for every real number x there exist y such that x+y=0” let x=1 there are y=-1 , x=-1/2 y=1/2 so on then ∀x∃yP(x,y) is True
- ∃x∀yP(x,y) “there exist some real numbers x for every real number y such that x+y=0” Let x=1, P(1,y)➔1+y=0

Now plug in all real numbers y

P(1,1/2) is false ,P(1,1) is false and so on

No matter what y you choose p(x,y) is always false for all real numbers

- ∃x∃yP(x,y) “there exist some real numbers x any y such that x+y=0” ex: x=1, y=-1 then 1-1=0 then ∃x∃yP(x,y) is true

Example:

let P(x, y) denote x.y=0, What are the truth values of the quantifications ∀x∀yP(x,y), ∃x∀yP(x,y), ∀x∃yP(x,y), ∃x∃yP(x,y)

where the domain for all variables consists of all real numbers?

Solution

- If P(x,y) : x.y=0
  - ∀x∀yP(x,y) “for all real numbers x and y , x.y=0” ex 1.3≠0 therefore ∀x∀yP(x,y) is false

- $\forall x\exists yP(x,y)$ "for all real numbers x there exist a real number y , x.y=0". Ex: x=1,y=0 xy=0 therefore $\forall x\exists yP(x,y)$ is true.

No matter what value of x is chosen , one can always take y=0 to make xy=0

- $\exists x\forall yP(x,y)$ "there exist some real number x such that for every real number y xy=0"

Let x=0 then x.y=0.y=0 so $\exists x\forall yP(x,y)$ is true

- $\exists x\exists yP(x,y)$ "there exist some real numbers x any y such that xy=0

Let x=1 y=0 then xy=0 therefore $\exists x\exists yP(x,y)$ is true

2.2.7 The Order of Quantifiers

- It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

$$\forall x\forall yP(x, y) \equiv \forall y\forall xP(x, y)$$

$$\exists x\exists yP(x, y) \equiv \exists y\exists xP(x, y)$$

Example:

Let Q(x,y,z) be the statement "x+y=z." What are the truth values of the statements $\forall x\forall y\exists zQ(x,y,z)$ and $\exists z\forall x\forall yQ(x,y,z)$, where the domain of all variables consists of all real numbers?

Solution:

- $\forall x\forall y\exists zQ(x,y,z)$ "For all real numbers x and for all real numbers y there is a real number z such that x+y=z",

$\forall x\forall y\exists zQ(x,y,z)$ true

- $\exists z\forall x\forall yQ(x,y,z)$ "There is a real number z such that for all real numbers x and for all real numbers y, x +y=z",

$\exists z\forall x\forall yQ(x,y,z)$ false

because there is no value of z that satisfies the equation x+y=z for all values of x and y.

Translating Mathematical Statements into Statements Involving Nested Quantifiers:

Example:

Translate the statement :

“The sum of two positive integers is always positive”

Solution

$\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x+y > 0))$,

Example: Translate the statement:

“Every real number except zero has a multiplicative inverse.” (A multiplicative inverse of a real number x is a real number y such that xy=1.)

Solution

We first rewrite this as “For every real number x except zero, x has a multiplicative inverse.” We can rewrite this as “For every real number x, if $x \neq 0$, then there exists a real number y such that xy=1.”

$\forall x((x \neq 0) \rightarrow \exists y(xy=1))$.

Translating from Nested Quantifiers into English

Expressions with nested quantifiers expressing statements in English can be quite complicated.

- The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean.
- The next step is to express this meaning in a simpler sentence.

EXAMPLE:

Translate the following statement into English,

$\forall x(C(x) \lor \exists y(C(y) \land F(x,y)))$

where C(x) is “x has a computer,” F(x, y) is “x and y are friends,” and the domain for both x and y consists of all students in your school.

Solution:

- The statement says that for “every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends”.
- In other words, “every student in your school has a computer or has a friend who has a computer.

Example:

Translate the following statement into English

$\exists x \forall y \forall z((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$

where $F(x,y)$ means x and y are friends and the domain for x, y, and z consists of all students in your school.

Solution:

We first examine the expression $(F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z)$.

- This expression says that **"if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends"**.
- It follows that the original statement, which is triply quantified, says that "**there is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends, then y and z are not friends"**.
- In other words, **"there is a student none of whose friends are also friends with each other"**.

Translating English Statements into Logical Expressions

Example:

Express "**If a person is female and is a parent, then this person is someone's mother**" using predicates and quantifiers, where domain is all people.

Solution

The statement "**If a person is female and is a parent, then this person is someone's mother"** can be expressed as **"For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y."**

- F (x) : "x is female".
- P(x) : "x is parent".
- M(x, y) : x is the mother of y.

$\forall x((F(x) \land P(x)) \rightarrow \exists y M(x,y)) \equiv \forall x \exists y((F(x) \land P(x)) \rightarrow M(x,y))$.

example:

Express the statement “Everyone has exactly one best friend” as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

- “Everyone has exactly one best friend” can be expressed as “For every person x, person x has exactly one best friend.” where the domain consists of all people.
- To say that x has exactly one best friend means that there is a person y who is the best friend of x.
- B(x,y): “y is the best friend of x,”

$\forall x \exists ! y B(x,y)$, where $\exists !$ is the “uniqueness quantifier”

Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

- Example:

Express the negation of the statement $\forall x \exists y(xy=1)$ so that no negation precedes a quantifier.

Solution:

By successively applying De Morgan’s laws for quantifiers ,

$\neg \forall x \exists y(xy=1) \equiv \exists x \neg \exists y(xy=1)$

$\equiv \exists x \forall y \neg(xy=1)$.

As $\neg(xy=1)$ can be expressed more simply as $xy \neq 1$,

$\neg \forall x \exists y(xy=1) \equiv \exists x \forall y(xy \neq 1)$.

Exercises:

1. Let P (x) i s th e statement " x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.
   a. $\exists x\, P(x)$.
   b. $\forall x P(x)$
   c. $\exists x\, \neg P(x)$
   d. $\forall x \neg P(x)$
2. Translate $\forall x(R(x) \rightarrow H(x))$ into English, where R(x) is "x is a rabbit "and H(x) is "x hops "and the domain consists of all animals.
3. Let C(x) be the statement "x has a cat," let D(x) be the statement "x has a dog," and let F(x) be the statement "x has a ferret." Express each of these statements in terms of C(x), D(x), F(x), quantifiers, and logical connectives. Let the domain consist of all students in your class.
   a. A student in your class has a cat, a dog, and a ferret.
   b. Some student in your class has a cat and a ferret, but not a dog.
4. If P(x) is defined for x in {0, 1, 2, 3, and 4} express with disjunctions, conjunctions, negations.
   a. $\exists x\, \neg P(x)$
   b. $\forall x\, \neg P(x)$
5. Express the negation of these propositions using quantifiers, and then express the negation in English.
   a) Some drivers do not obey the speed limit.
   b)No one can keep a secret.
6. Translate $\exists x \forall y(xy=y)$into English, where the domain for each variable consists of all real numbers.
7. Use quantifiers and predicates with more than one variable to express these statements.
   a) There is a student in this class who can speak Hindi.
   b) All students in this class have learned at least one programming language.

CHAPTER 3 :

Rules Of Inference & Proof Techniques

CHAPTER 3 :

Rules Of Inference & Proof Techniques

3.1 Rules of Inference

Rules of Inference provide the templates or guidelines for constructing valid arguments from the statements that we already have. Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

Example:

"If you have a current password, then you can log onto the network"
"You have a current password"
Therefore,
"You can log onto the network"
solution
P: you have a current password
q: you can log onto the network
Argument Form:

$p \to q$
p

———

$\therefore q$

This argument is valid if $((p \to q) \wedge p) \to q$ is a tautology

note: $((p \to q) \wedge p) \to q = ((\neg p \vee q) \wedge p) \to q$

$(q \wedge p) \to q = \neg (q \wedge p) \vee q$

$\neg q \vee \neg p \vee q$=which is always True, So it is a tautology

Modus Ponens

It is the basis: $((p \to q) \wedge p) \to q$ is a tautology. We can always use a truth table to show that an argument form is valid

		Premise 1	Premise 2		conclusion	
p	q	$P \to q$	p	$(P \to q) \wedge p$	q	$((P \to q) \wedge p) \to q$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	F	F	F	T

$((p \to q) \wedge p) \to q$ is a tautology

Valid Arguments in Propositional Logic

Definition (Argument) :

- Valid Arguments in Propositional Logic An argument in propositional logic is a sequence of propositions. All the proposition in the argument are called premises and the final proposition is called the conclusion.
- An **argument is valid** if <u>the truth of all its premises</u> implies the <u>truthiness of its conclusion</u>
- However, if any <u>premises</u> is false, even a valid argument can lead to an incorrect conclusion.
- i.e., $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots p_n) \to q$ is a tautology.

Rules of Inference for Propositional Logic

The general form of a rule of inference is:

$$\begin{array}{l} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \\ \hline \therefore\ q \end{array}$$

The rule states that if p_1 **and** p_2 **and** … **and** p_n are all true, then q is true as well.

Each rule is an established tautology of

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \to q$$

These rules of inference can be used in any mathematical argument and do not require any proof.

Basic rules of inference:

TABLE 1 Rules of Inference.		
Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

example:

State which rule of inference is the basis of the following argument: “It is below freezing and raining now”. Therefore, “it is below freezing now.”

Solution:

- Let p be the proposition “It is below freezing now,” and
- q be the proposition “It is raining now.”

This argument is of the form.

$p \wedge q$

$\therefore p$

- This argument uses **the simplification rule.**

Example:

State which rule of inference is the basis of the following argument: “It is below freezing now. Therefore, it is either below freezing or raining now.”

Solution:

Let p :“It is below freezing now” and

q :“It is raining now.”

Then this argument is of the form

P

$\therefore p \vee q$

This is an argument that uses the **addition rule.**

example:

State which rule of inference is used in the argument:

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution:

- Let p be the proposition “It is raining today,”
- Let q be the proposition “We will not have a barbecue today,”
- Let r be the proposition “We will have a barbecue tomorrow.”

Then this argument is of the form

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

- Hence, this argument is a hypothetical syllogism.

Example:

Gary is either intelligent or a good actor.

If Gary is intelligent, then he can count from 1 to 10.

Gary can only count from 1 to 3.

Therefore, Gary is a good actor.

solution:

i: “Gary is intelligent.”

a: “Gary is a good actor.”

c: “Gary can count from 1 to 10.”

$$
\begin{array}{l}
a \vee i \\
i \rightarrow c \\
\neg c \\
\hline
a
\end{array}
$$

Step 1:	$\neg c$	Hypothesis
Step 2:	$i \rightarrow c$	Hypothesis
Step 3:	$\neg i$	Modus tollens Steps 1 & 2
Step 4:	$a \vee i$	Hypothesis
Step 5:	a	Disjunctive Syllogism Steps 3 & 4

Conclusion: **a** (“Gary is a good actor.”)

3.1.1 Using Rules of Inference to Build Arguments

Example:

Show that **the hypotheses** “It is not sunny this afternoon and it is colder than yesterday”, “We will go swimming only if it is sunny”, “If we do not go swimming, then we will take a canoe trip”, and “If we take a canoe trip, then we will be home by sunset” lead to the **conclusion** “We will be home by sunset”.

solution:

“p: it is sunny this afternoon ”.

“q: it is colder than yesterday”.

“r : We will go swimming ”.

“s: We will take a canoe trip”.

“t: We will be home by sunset”.

$\neg p \wedge q$
$r \to p$
$\neg r \to s$
$s \to t$
----
t

The premises become $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$.

The conclusion is simply t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t, such a truth table would have 32 rows

$$(\neg p \wedge q) \wedge (r \to p) \wedge (\neg r \to s) \wedge (s \to t) \to t$$

Example:

Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”

Solution:

Let **p :“You send me an e-mail message,” q :“I will finish writing the program,”**

r :“I will go to sleep early,” and **s :“I will wake up feeling refreshed.”**

Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\boldsymbol{\neg q \rightarrow s}$.

$p \rightarrow q$
$\neg p \rightarrow r$
$r \rightarrow s$

$\neg q \rightarrow s$

Step	**Reason**
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

Example:

Show that the premises : $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$ imply the conclusion : $\neg q \rightarrow s$

Solution

using

$$\boxed{\begin{array}{ll} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}}$$

Step	**Reason**
1. $p \rightarrow q = \neg p \vee q$	Premise
2. $\neg p \rightarrow r = p \vee r$	Premise
3. $q \vee r$	Resolution using (1) and (2)

4. r →s = ¬r ∨s=s ∨ ¬r   Premise

5. q ∨s = ¬q →s   Resolution using (3) and (4)

3.1.2 Rules of Inference for Quantified Statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

universal modus ponens

$$\begin{array}{l} \forall x\,(P(x) \rightarrow Q(x)) \\ P(a) \\ \hline \therefore Q(a) \end{array}$$

universal modus tollens

$$\begin{array}{l} \forall x\,(P(x) \rightarrow Q(x)) \\ \neg Q(a) \\ \hline \therefore \neg P(a) \end{array}$$

universal transitivity, (hypothetical syllogism)

$$\begin{array}{l} \forall x\,(P(x) \rightarrow Q(x)) \\ \forall x\,(Q(x) \rightarrow R(x)) \\ \hline \therefore \forall x\,(P(x) \rightarrow R(x)) \end{array}$$

where a is a particular element in the domain

- **Universal instantiation**

It is used to conclude that P(c)is true, where c is a particular member of the domain, given the premise ∀xP (x). used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

- **Universal generalization**

It is used when we show that ∀xP(x) is true by taking an arbitrary element c from the domain and showing that P(c) is true. The element c that we select must be an arbitrary, and not a specific, element of the domain.

- **Existential instantiation**

It is the rule that allows us to conclude that there is an element c in the domain for which P(c)is true if we know that ∃xP(x) is true. We cannot select an arbitrary value of c here, but rather it must be a c for which P(c)is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

- **Existential generalization**

It is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with P(c)true is known. That is, if we know one element c in the domain for which P(c) is true, then we know that $\exists x P(x)$ is true.

example:

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution:

- Let D(x) :"x is in this discrete mathematics class," and
- let C(x) :"x has taken a course in computer science."
- Then the premises are $\forall x(D(x) \rightarrow C(x))$ and D(Marla).
- The conclusion is C(Marla).

$\forall x(D(x) \rightarrow C(x))$
D(Marla)

C(Marla).

The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. D(Marla)	Premise
3. C(Marla)	Modus ponens from (1) and (2)

Example:
Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."
Solution:
Let C(x) :"x is in this class,"
B(x) :"x has read the book," and
P(x):"x passed the first exam."
The premises are
$\exists x(C(x) \land \neg B(x))$

$\forall x(C(x) \to P(x))$.

The conclusion is $\exists x(P(x) \land \neg B(x))$.

These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\neg B(a)$	Simplification from (2)
5. $\forall x(C(x) \to P(x))$	Premise
6. $P(a)$	Modus ponens from (3) and (5)
7. $P(a) \land \neg B(a)$	Conjunction from (6) and (4)
8. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (7)

Example:

Anyone performs well is either intelligent or a good actor.

If someone is intelligent, then he/she can count from 1 to 10.

Gary performs well.

Gary can only count from 1 to 3.

Therefore, not everyone is both intelligent and a good actor

solution:

$P(x)$: x performs well

$I(x)$: x is intelligent

$A(x)$: x is a good actor

$C(x)$: x can count from 1 to 10

Hypotheses:

1. Anyone performs well is either intelligent or a good actor.
   $\forall x\,(P(x) \to I(x) \lor A(x))$
2. If someone is intelligent, then he/she can count from 1 to 10.
   $\forall x\,(I(x) \to C(x))$
3. Gary performs well.
   $P(G)$
4. Gary can only count from 1 to 3.
   $\neg C(G)$

Conclusion: not everyone is both intelligent and a good actor

$\neg\forall x(I(x) \land A(x))$

Promises:

$\forall x\,(P(x) \to I(x) \lor A(x))$

$\forall x\,(I(x) \to C(x))$

$P(G)$

$\neg C(G)$

$\neg\forall x(I(x) \wedge A(x))$

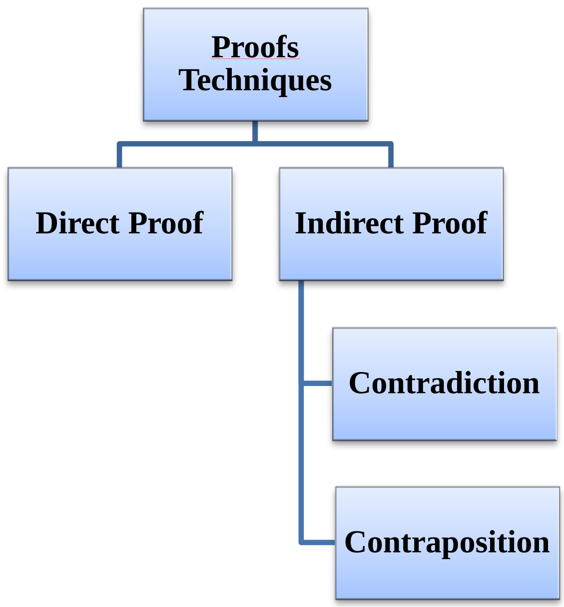
- Direct proof:

Step 1: $\forall x\ (P(x) \rightarrow I(x) \vee A(x))$	Hypothesis
Step 2: $P(G)$	Hypothesis
Step 3: $I(G) \vee A(G)$	Modus ponens Steps 1 & 2
Step 5: $\forall x\ (I(x) \rightarrow C(x))$	Hypothesis
Step 6: $\neg C(G)$	Hypothesis
Step 7: $\neg I(G)$	Modus tollens Steps 5 & 6
Step 8: $\neg I(G) \vee \neg A(G)$	Addition Step 7
Step 9: $\neg(I(G) \wedge A(G))$	Equivalence Step 8
Step 10: $\exists x \neg(I(x) \wedge A(x))$	Exist. general. Step 9
Step 11: $\neg\forall x\ (I(x) \wedge A(x))$	Equivalence Step 10

Conclusion: $\neg\forall x\ (I(x) \wedge A(x))$, not everyone is both intelligent and a good actor.

3.2 Proofs Techniques

- A theorem (fact/result) is a statement that can be shown to be true. We demonstrate that a theorem is true with a proof.
- A proof is a valid argument that establishes the truth of a theorem.
- A lemma is a 'helping theorem' or a result which is needed to prove a theorem. Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually.



notes:

- ➢ Even Integer
  - $2 * Any\ Integer = even$
  - if a is an even number, so you can write it as follows:

$a = 2n$, where n is integer

  - Even + 1 = Odd
  - Odd + 1 = Even
  - ¬Even = Odd
  - ¬Odd = Even
- ➢ Odd Integer

- if a is an odd number, so you can write it as follows:

$a = 2m + 1$, where m is integer

- Prefect Square

if a is a prefect square, so you can write it as follows:

$a = n^2$, where n is integer

- Rational Number

if a is a $rational\ number$, so you can write it as follows:

$a = n/m$, where n, and m are integers with NO common factor, and $m \neq 0$

- ¬Rational = Irrational
- ¬Irrational = Rational

3.2.1 Direct Proof

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

$p \rightarrow q$

1. We assume that p is true

2. We try to prove that q is also true

3. Then $p \rightarrow q$ is true

Example:

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd".

Solution

Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd). n is odd.

1. We assume that p (n is an odd integer)is true $n = 2m + 1$, where m is integer.
2. We try to prove that q (n^2 is odd) is also true

$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2\ (2m^2 + 2m) + 1$

$= even + 1 = odd$

3. $\therefore p \rightarrow q$ is true

3.2.2 Indirect Proof

Indirect Proof (Contraposition)

An implication p → q is equivalent to its **contra-positive** ¬q → ¬p. Therefore, we can prove p → q by showing that whenever q is false, then p is also false.

$p \rightarrow q$

$\neg q \rightarrow \neg p$

1. We assume that $\neg q$ is true

2. We try to prove that $\neg p$ is also true

3. Then $\neg q \rightarrow \neg p$ is true.

4. The $p \rightarrow q$ is also true

Example:

Give an indirect proof of the theorem "If 3n + 2 is odd, then n is odd."

Solution

Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false (3n + 2 is even).

$\neg q$ is n is even

$\neg p$ is $3n + 2$ is even

1. we assume that n is even $n = 2m$, where m is integer.
2. $3n + 2 = 3(2m) + 2$

$3n + 2 = 6m + 2 = 2(3m + 1) =$ even

$\therefore 3n + 2$ is even

$\therefore \neg q \rightarrow \neg p$ is true, then $p \rightarrow q$ is also true.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If 3n + 2 is odd, then n is odd).

Indirect Proof (Contradiction)

A – We want to prove $\boldsymbol{p}$.

We show that:

1. $\neg p \rightarrow F$ (i.e., a False statement)

2. We conclude that $\neg p$ is False since (1) is True and therefore p is True

B – We want to show $\boldsymbol{p} \rightarrow \boldsymbol{q}$

1. Assume the negation of the conclusion, i.e., $\neg q$

2. Show that $(p \land \neg q) \rightarrow F$

3. Since $((p \land \neg q) \rightarrow F)$ therefore $(p \rightarrow q)$ we are done

- $((p \land \neg q) \rightarrow F) \leftrightarrow \neg(p \land \neg q) \leftrightarrow p \rightarrow q$

Example:

Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd".

Solution :

- p: $3n + 2$ is odd ,q: n is odd

1. we assume that n is even $n = 2m$, where m is integer.

2. $3n + 2 = 3(2m) + 2 = 6m + 2 = 2\ (3m + 1) =$ even

$\therefore$ $3n + 2$ is even, then p is false.

3. $\because$ $p \land \neg q$ is false, then by contradiction $p \rightarrow q$ is true

example:

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution:

- Let p the proposition "$\sqrt{2}$ is irrational". Suppose that $\neg p$ is true $\rightarrow$ then $\sqrt{2}$ is rational
- Rational Number if a is a $rational\ number$, so you can write it as follows: $a = n/m$, where n, and m are integers with NO common factor, and $m \neq 0$

- ¬Rational = Irrational
- ¬Irrational = Rational
- Let p the proposition "$\sqrt{2}$ is irrational".

1) To start a proof by contradiction, we suppose that ¬p is true, then $\sqrt{2}$ is rational.

2) $\sqrt{2}$ = a/b , where a and b are intergers without common factor and $b \neq 0$

3) 2= a^2 / b^2

4) a^2 = $2b^2$, then a is even, and you can write a = $2m$, where m is integer.

5) $(2m)^2$ = 2 b^2, then 4 m^2 = 2 b^2.

6) b^2 = 2 m^2, then b is also even.

7) Because a and b are even, then a and b have a common factor 2.

8) Therefore, $\sqrt{2}$ = a /b is not rational, then ¬p is false.

9) Therefore, p is true and $\sqrt{2}$ is irrational

CHAPTER 4:

Basic Structures: Sets, Functions, Sequences, Sums

CHAPTER 4:

Basic Structures: Sets, Functions, Sequences, Sums

4.1 Sets :

Sets are used to group objects together. Often the objects in a set have similar properties.

DEFINITION

A set is an unordered collection of objects, called **elements** or members of the set. A set is said to contain its elements.

We write a ∈ A to denote that a is an element of the set A.

The notation a ∉ A denotes that a is not an element of the set A.

DEFINITION

The objects in a set are called the ***elements***, or *members*, of the set. A set is said to contain its elements.

A = $\{a, b, c, d\}$

- a ∈ A: a is an element of the set A.
- v∉A: a is not an element of the set A.
- Note: lower case letters are used to denote elements.

Ways to describe a set:

- **Use { … }**
  - E.g. {a, b, c, d} – A set with four elements.
    - V = {a, e, i, o, u} – The set V of all vowels in English alphabet.
    - O = {1, 3, 5, 7, 9} – The set O of odd positive integers less than 10.
    - X={1, 2, 3, …, 99} – The set X of positive integers less than 100.
- Another way to describe a set is to use **set builder** notation : characterize all the elements in the set by stating the property or properties.
  - E.g. O = { x | x is an odd positive integer less than 10}

- $O = \{ x \quad Z^+ \mid x$ is odd and $x < 10\}$

➢ Commonly accepted letters to represent sets

- **N** = {0, 1, 2, 3, …}, the set of natural numbers
- **Z** = {…, -2, -1, 0, 1, 2, …}, the set of integers
- **Z**$^+$ = {1, 2, 3, …}, the set of positive integers
- **Q** = {p/q | p Z, q Z, and q $\neq$ 0}, the set of rational numbers
- **R**, the set of real numbers

➢ Sets can have other sets as members

Example:

The set {N, Z, Q, R} is a set containing four elements, each of which is a set.

Interval Notation

Recall the notation for intervals of real numbers.

Closed interval $[a, b]$ interval from a to b including a and b

Open interval $(a, b) \equiv]a, b[$ interval from a to b not including a and b

When a and b are real numbers with a <b, we write

➢ $[a, b] = \{x \mid a \leq x \leq b\}$

➢ $[a, b) = \{x \mid a \leq x < b\}$

➢ $(a, b] = \{x \mid a < x \leq b\}$

➢ $(a, b) = \{x \mid a < x < b\}$

DEFINITION

Two sets are **equal** if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if

$x(x \in A \leftrightarrow x \in B)$.

We write A = B if A and B are equal sets.

Example:

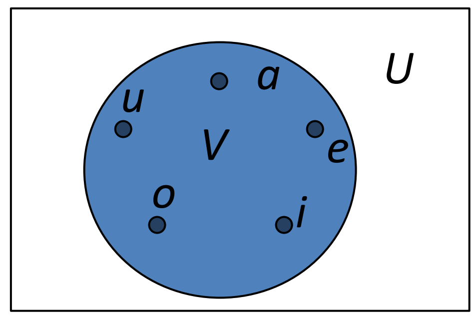
- Are sets {1, 3, 5} and {3, 5,1} equal?
- Are sets {1, 3, 3, 3, 5, 5, 5, 5} and {1, 3, 5} equal?

Solution:

- The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. The order in which the elements of a set are listed does not matter.
- It does not matter if an element of a set is listed more than once, so {1,3,3,3,5,5,5,5} is the same as the set{1, 3, 5} because they have the same element

Venn Diagrams

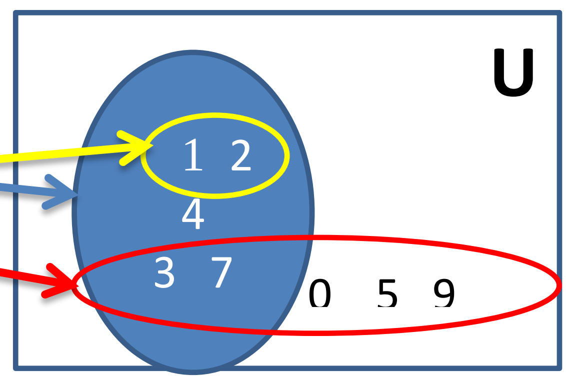
- Represent sets graphically
- The universal set U, which contains all the objects under consideration, is represented by a **rectangle**. The set varies depending on which objects are of interest.
- Inside the rectangle, **circles** or other geometrical figures are used to represent sets.



- Sometimes **points** are used to represent the particular elements of the set.

Example:

A = {1,2,3,4,7}
B = {0,3,5,7,9}
C = {1,2 }



- Empty Set (null set): a set that has no elements, denoted by ϕ or {}.

Example: The set of all positive integers that are greater than their squares is an empty set.

- Singleton set: a set with one element
- Compare: ϕ and {ϕ}
  - Φ: an empty set. Think of this as an empty folder

- $\{\phi\}$: a set with one element. The element is an empty set. Think of this as an folder with an empty folder in it.

4.1.1 subset

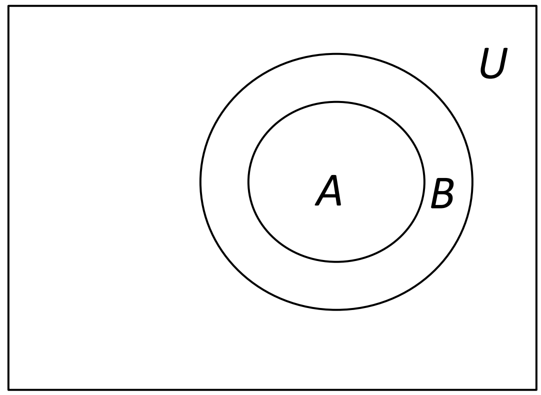
DEFINITION

The set A is said to be a **subset** of B if and only if every element of A is also an element of B.

We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

- $A \subseteq B$ if and only if the quantification

  $x(x\in A \rightarrow x\in B)$ is true



Example:

Prove that if sets A and B are equal is equivalent to both are subsets of each others.

solution

$p \leftrightarrow q=(p \rightarrow q) \land (q \rightarrow p)$

$$
\begin{aligned}
A = B &\equiv \forall x\,(x \in A \leftrightarrow x \in B)\\
&\equiv \forall x\,[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]\\
&\equiv \forall x\,(x \in A \rightarrow x \in B) \land \forall x\,(x \in B \rightarrow x \in A)\\
&\equiv (A \subseteq B) \land (B \subseteq A).
\end{aligned}
$$

For every non-empty set S is guaranteed to have at least two subset, the empty set and the set S itself, that is $\phi\subseteq S$ and $S\subseteq S$.

THEOREM

For every set S,

(i) $\phi\subseteq S$ and *(ii)* $S\subseteq S$

4.1.2 Proper subset

- If A is a subset of B but A ≠ B, then A B or A is a proper subset of B.
- For A B to be true, it must be the case that A B and there must exist an element x of B that is not an element of A, i.e.

$$x(x \in A \rightarrow x \in B) \Lambda \exists x(x \in B \Lambda x \notin A)$$

finite & infinite set

DEFINITION
Let *S* be a set. If there are exactly *n* distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is a ***finite*** set and that *n* is the *cardinality* of *S*.

The cardinality is the number of distinct elements in *S*. The **cardinality** of *S* is denoted by |*S*|.

- Example:
    - Let A be the set of odd positive integers less than 10. Then |A| = 5.
    - Let A be the set of letters in the English alphabet. Then |A| = 26.
    - Null set has no elements, | ϕ | = 0.
    - Let A be the set of $A = \{1, 2, 3, 7, 9\}$. Then $|A| = 5$
    - Let A be the set of A = $\{a, b, c, d, \{2\}\}$ Then $|A| = 5$
    - Let A be the set of A = {1, 2, 3, {2,3}, 9}. Then $|A| = 5$
- DEFINITION
- A set is said to be ***infinite*** if it is not finite.
- Example: The set of positive integers is infinite.

4.1.3 the power set P(S)

DEFINITION
Given a set, the *power set* of *S* is **<u>the set of all subsets of the set</u>** *S*. The power set of *S* is denoted by *P(S)*.
If a set has *n* elements, its power set |P(S)| has 2^n elements or $2^{\|S\|}$.

Example:

- **What is the power set of the set {0,1,2}?**

Solution:

P({0,1,2}) = {ϕ, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}}

- **What is the power set of the empty set? What is the power set of the set {ϕ}?**

Solution:

The empty set has exactly one subset, namely, itself.

$$P(\phi) = \{\phi\}$$

The set {ϕ} has exactly two subsets, namely, ϕ and the set {ϕ}.

$$P(\{\phi\}) = \{\phi, \{\phi\}\}$$

Sets are unordered, a different structure is needed to represent an ordered collections – ordered n-tuples.

DEFINITION

The *ordered n-tuple* **(**$a_1, a_2, \ldots, a_n$**)** is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.

- $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, n$

4.1.4 Cartesian product

DEFINITION

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}.$$

Example:

What is the Cartesian product of A = {1,2} and B = {a,b,c}?

Solution:

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

$|A| \times |B| = |A| * |B| = 2 * 3 = 6$

- Cartesian product of $A \times B$ and $B \times A$ are not equal, unless $A = \phi$ or $B = \phi$ (so that $A \times B = \phi$) or $A = B$.

$B \times A = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

```
for (i =1 , i <=| A|, i ++)
    for (j =1 , j <=| B|, j ++)
              (A[i], B[j]) ;
```

The Cartesian product of more than two sets.

DEFINITION 10

The *Cartesian product* of sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1,2, \ldots,n$. In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1,2, \ldots, n\}.$$

- Example:

What is the Cartesian product of A × B × C where A= {0,1}, B = {1,2}, and C = {0,1,2}?

Solution:

$|A| \times |B| \times |C| = |A| * |B| * |C| = 2*2*3 = 12$

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Using Set Notation with Quantifiers

- $\forall x \in S(P(x))$ denotes the universal quantification of P(x) over all elements in the set S.

$\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$.

- $\exists x \in S(P(x))$ denotes the existential quantification of P(x) over all elements in S.

$\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

Example:

What do the statements $\forall x \in R\ (x^2 \geq 0)$ and $\exists x \in Z\ (x^2 = 1)$ mean?

Solution:

- The statement $\forall x \in R(x^2 \geq 0)$. states that for every real number x, $x^2 \geq 0$. This statement can be expressed as **"The square of every real number is nonnegative."** This is a true statement.
- The statement $\exists x \in Z(x^2=1)$ states that there exists an integer x such that $x^2=1$.This statement can be expressed as **"There is an integer whose square is 1."** This is also a true statement because x=1 is such an integer (as is -1).

Truth Sets of Quantifiers

- Given a predicate P , and a domain D, we define the truth set of P to be the set of elements x in D for which P(x)is true.
- The truth set of P(x)is denoted by $\{ x \in D \mid P(x)\}$.

Example:

What are the truth sets of the predicates P(x), Q(x), and R(x), where the domain is the set of integers and P(x) is " |x| =1," Q(x) is "$x^2=2$," and R(x) is "|x| = x."

Solution:

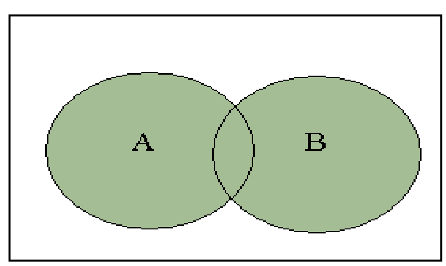
- The truth set of P , $\{ x \in Z \mid |x| =1\}$, is the set of integers for which |x|=1. Because |x| = 1 when x = 1or x = -1, and for no other integers x, we see that the truth set of P is the set {-1,1}.
- The truth set of Q,$\{x \in Z \mid x^2=2\}$, is the set of integers for which $x^2=2$. This is the empty set because there are no integers x for which $x^2=2$.
- The truth set of R,$\{x \in Z \mid |x|=x\}$, is the set of integers for which |x| = x. Because |x| = x if and only if $x \geq 0$, it follows that the truth set of R is N, the set of nonnegative integers.

4.1.5 Set Operations

Union A U B

DEFINITION : Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A U B*, is the set that contains those elements that are either in *A* or in *B*, or in both.

- $A \cup B = \{ x \mid x \in A \vee x \in B\}$
- Shaded area represents *A* U *B*.



Example:

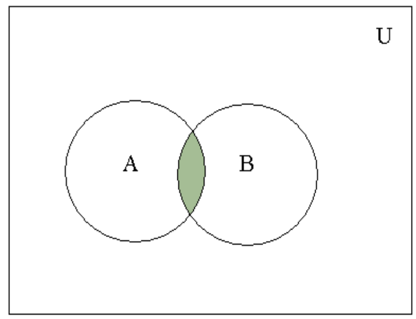
- The union of the sets {1,3,5} and {1,2,3} is the set {1,2,3,5};

that is {1,3,5} U {1,2,3} = {1,2,3,5}

- The union of **the set of all computer science majors at your school** and **the set of all mathematics majors at your school** is the set of students at your school who are majoring either in mathematics **or** in computer science (or in both).

Intersection $A \cap B$

DEFINITION : Let *A* and *B* be sets. The *intersection* of the sets *A* and *B*, denoted by $A \cap B$, is the set containing those elements in both *A* and *B*.



- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$
- *Shaded area represents* $A \cap B$.

Example:

- *The intersection of the sets {1,3,5} and {1,2,3} is the set {1,3}; that is* $\{1,3,5\} \cap \{1,2,3\} = \{1,3\}$
- *The intersection of t**he set of all computer science majors at your school** and **the set of all mathematics majors at your school** is the set of students at your school who are joint majors in mathematics and in computer science.*

Disjoint $A \cap B = \phi$

DEFINITION : Two sets are called ***disjoint*** if their intersection is the empty set.

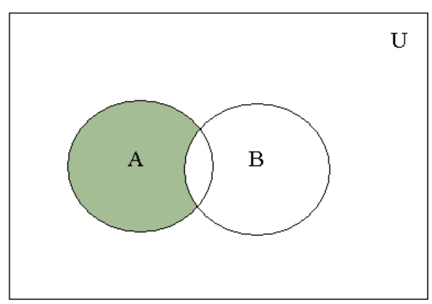
- **Example**:

Let $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$.

Because $A \cap B = \phi$, *A* and *B* are disjoint.

Difference A – B= complement of B with respect to A

DEFINITION : Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by $A - B$, is the set containing those elements that are **in *A* but not in *B***.



The difference of *A* and *B* is also called the *complement* of *B* *with respect to A.*

- $A - B = \{ x \mid x \in A \land x \notin B\}$
- $A - B$ is shaded.

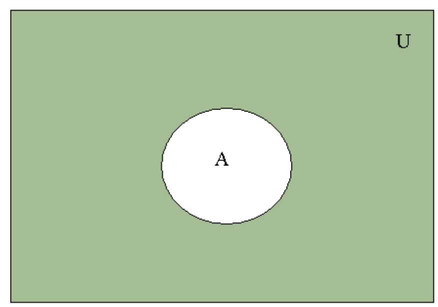
Example:

- $\{1,3,5\} - \{1,2,3\} = \{5\}$
- $\{1,2,3\} - \{1,3,5\} = \{2\}$
- The difference of **the set of computer science majors at your school** and **the set of mathematics majors at your school** is <u>the set of all computer science majors at your school who are not mathematics majors</u>.

Complement $\bar{A}$:

the complement of *A* with respect to ***U***

DEFINITION : Let ***U*** be the universal set. The *complement* of the set *A*, denoted by $\bar{A}$, is the complement of *A* with respect to ***U***. In other words, the containing those complement of the set *A* is ***U*** – *A*.

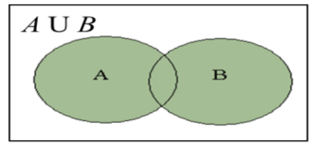
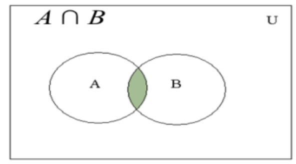
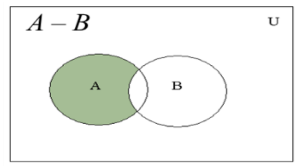
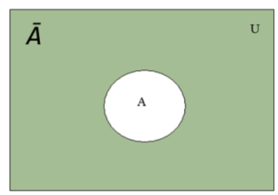
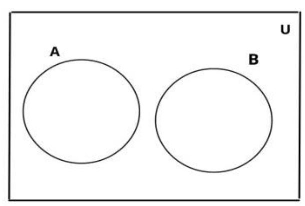


- $\bar{A} = \{ x \mid x \quad A \} = \{x \mid \neg x \in A\}$
- $\bar{A}$ is shaded.

Example:

Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers.)

Then $\bar{A} = \{1,2,3,4,5,6,7,8,9,10\}$

Set Operations		
union	$A \cup B = \{ x \mid x \in A \vee x \in B\}$	
intersection	$A \cap B = \{ x \mid x \in A \wedge x \in B \}$	
difference	$A - B = \{ x \mid x \in A \wedge \ x \notin B\}$ $=\{ x \mid x \in A \wedge \neg x \in B\}$	
complement	$\bar{A} = \{ x \mid x \notin A \} = \{x \mid \neg x \in A\}$ $= \boldsymbol{U} - A$	
disjoint	$A \cap B = \phi$, A and B are disjoint.	

<u>Finding the cardinality of |A U B|:</u>

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example:

$A = \{1,3,5,7,9\}$, $B = \{5,7,9,11\}$ find the cardinality of $|A \cup B|$

solution

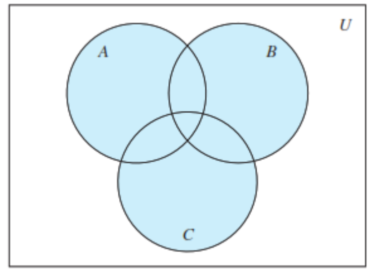
$$|A \cup B| = |A| + |B| - |A \cap B| = 5 + 4 - 3 = 6$$

Generalized Unions and Intersections

DEFINITION:

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection. We use the notation

to denote the union of the sets A1,A2,...,An.

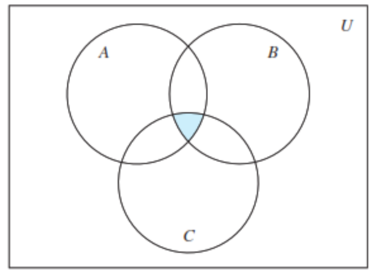


(a) $A \cup B \cup C$ is shaded.

DEFINITION:

- The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection. We use the notation

- to denote the intersection of the sets A1,A2,...,An.



(b) $A \cap B \cap C$ is shaded.

EXAMPLE :

Let A = {0, 2, 4, 6, 8} , B= {0, 1, 2, 3, 4}, and C= {0, 3, 6, 9}.

What are A∪B∪C and A∩B∩C?

Solution:

- A∪B∪C = {0, 1, 2, 3, 4, 6, 8, 9}.
- A∩B∩C= {0}

Example:

For i=1, 2,..., let A_i= {i, i+1,i+2,...}. Then, what will be the value of $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$

solution

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ..... \cup \ A_n$

$\bigcup_{i=1}^{n} A_i = \ \{1,2,3, ... \}\cup\{2,3,4, ... \}\cup\{3,4,5, ... \} ... \cup\{n, n+1, n+2, ... \}$

$=\{1,2,3,\ldots\}=A_1$

$$\bigcap_{i=1}^{n} A_i = \{1,2,3, \ldots.\} \cap \{2,3,4, \ldots\} \cap \{3,4,5, \ldots\} \ldots \cap \{n, n+1, n+2, \ldots\}$$

$$= \{n, n+1, n+2, \ldots\} = A_n$$

More notation: suppose that I={1,5,6}

$\bigcup_{i=I} A_i = A_1 \cup A_5 \cup A_6$

$$\bigcap_{i=I} A_i = A_1 \cap A_5 \cap A_6$$

Example:

Suppose that A_i= {1, 2, 3, . . . , i} for i = 1, 2, 3, ... Then find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

Solution

- $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}\{1, 2, 3, \ldots, i\}$= {1, 2, 3, . . .} = Z+
- $\bigcap_{i=1}^{\infty} A_i = \{1,2,3, \ldots., i\} = \{1\}$

Set Identities

- Very parallel to logic identities,
- As in logical equivalence, can be proven by:

1. Membership tables
2. set builder
3. both are subsets of each other
4. algebraically

TABLE 1 Set Identities.	
Identity	***Name***
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Membership table:

- Set identities can be proved using **membership tables**.
- We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity.
- To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Example:

prove that $\overline{\boldsymbol{A} \cap \boldsymbol{B}} = \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$

solution

A	**B**	$\boldsymbol{A \cap B}$	$\overline{\boldsymbol{A \cap B}}$	$\overline{\boldsymbol{A}}$	$\overline{\boldsymbol{B}}$	$\overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Set builder notation and logical equivalences

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Both are subsets of each other

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

example:

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Example:

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Computer Representation of Sets

Represent a subset A of U with the bit string of length n, where the i^{th} bit in the string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

- The bit string for the union is the bitwise *OR* of the bit string for the two sets.
- The bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.
- **Example:**

The bit strings for the sets {1,2,3,4,5} and {1,3,5,7,9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution:

Union:

11 1110 0000 V 10 1010 1010 = 11 1110 1010, {1,2,3,4,5,7,9}

Intersection:

11 1110 0000 Λ 10 1010 1010 = 10 1010 0000, {1,3,5}

- **Example:**
    - Let U = {1,2,3,4,5,6,7,8,9,10}, and the ordering of elements of U has the elements in increasing order; that is $a_i = i$.

What bit string represents the subset of all odd integers in U?

Solution: 10 1010 1010

What bit string represents the subset of all even integers in U?

Solution: 01 010 10101

What bit string represents the subset of all integers not exceeding 5 in U?

Solution: 11 1110 0000

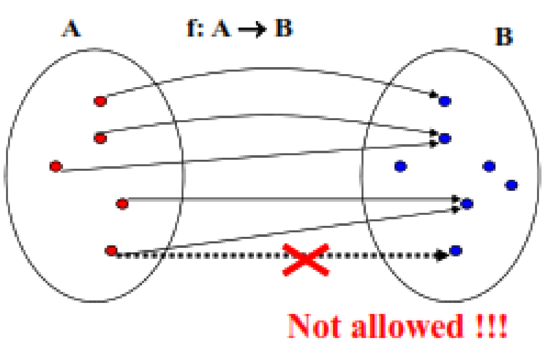
What bit string represents the complement of the set {1,3,5,7,9}?

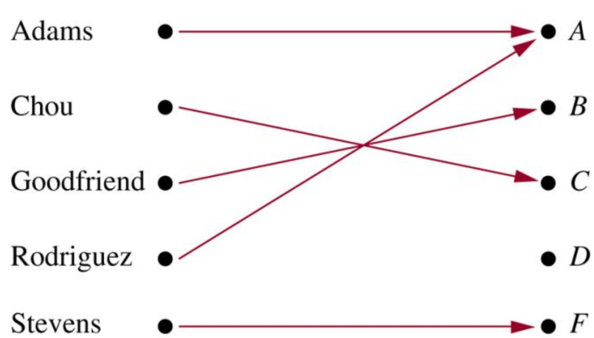
Solution: 01 0101 0101

4.2 Functions

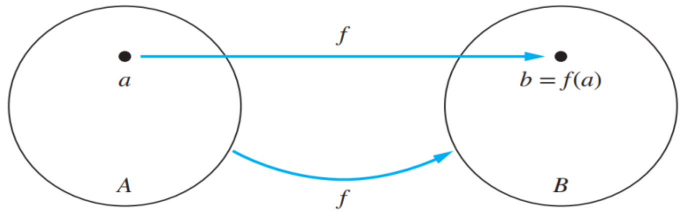
DEFINITION

- Let A and B to be **nonempty** sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.





DEFINITION :If f is a function from A to B, we write f: $A \rightarrow B$ (also called mapping or transformation, or f maps A to B) we say that:



The function f maps A to B.

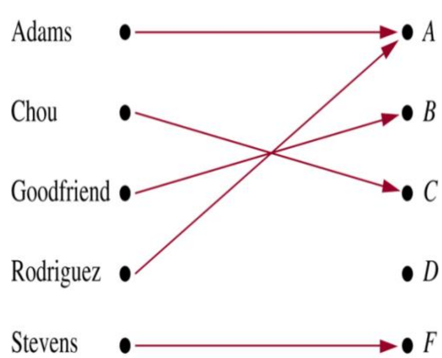
- **Domain**: A
- **Co-Domain**: B
- $f(a)= b$
  b is the ***image*** of a
  a is a ***preimage*** of b
- The **range** of f is the ***set of all images*** of elements of A.
- When we define a function, we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain.
- Two functions are **equal** when they have the same domain and codomain, and map elements of their common domain to the same elements in their common codomain.
- If we change either the domain or the codomain of a function, we obtain a **different function**. If we change the mapping of elements, we also obtain a **different function**.

Example:

What are the domain, codomain, and range of the function that assigns grades to students?

Solution:

- domain: {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- codomain: {A, B, C, D, F}
- range: {A, B, C, F}$\neq$ codomain
- Range $\subseteq$ $Codomain$



Example:

Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, f(11010) = 10.

What is the domain and codomain?

Solution

The domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00,01,10,11}

What is the domain and codomain of the function

int floor(float real){…}?

Solution: domain: the set of real numbers

codomain: the set of integer numbers

DEFINITION

If f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to R defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Example:

Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4$$

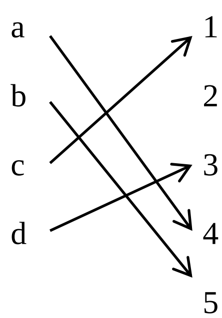
4.2.1 One-to-One and Onto Functions

DEFINITION

- A function f is said to be ***one-to-one***, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function is said to be an *injection* if it is one-to-one.
- $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$ (If it's a different element, it should map to a different value.)

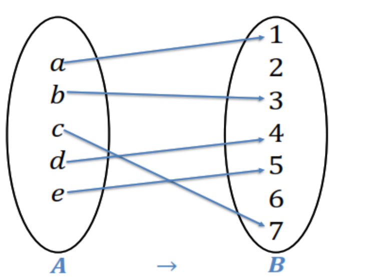
Example:

Determine whether the function f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one-to-one.



Solution: Yes.

***One-to-One* function (injective)**



$f(a) = 1$

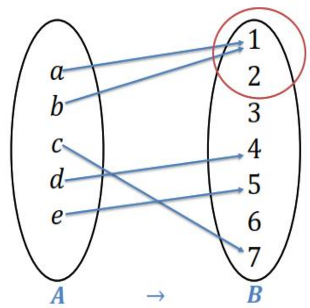
$f(b) = 3$

$f(c) = 7$

$f(d) = 4$

$f(e) = 5$

NOT *One-to-One* function (Not injective)



$\boxed{f(a) = 1 \quad f(b) = 1}$

$f(c) = 7$

$f(d) = 4$

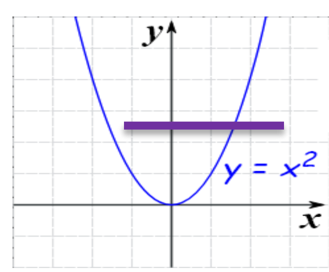
$f(e) = 5$

Example:

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

  Solution:

$f(1) = f(-1) = 1$, not one-to-one

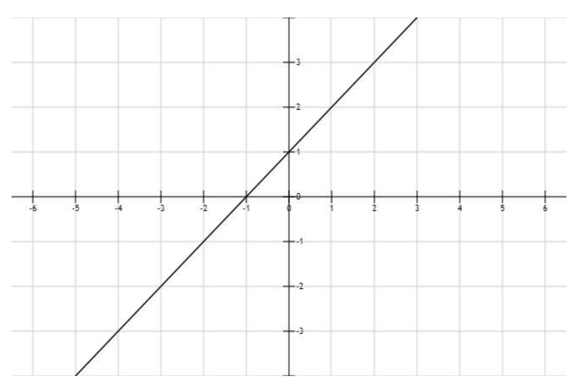


$f(x) = x^2$

- Determine whether the function $f(x) = x + 1$ from the set of integers to the set of integers is one-to-one

Solution:

it is one-to-one



DEFINITION

A function *f* whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and *strictly increasing* if $f(x)<f(y)$, whenever $x < y$ and x and y are in the domain of *f*.

$$\forall x \forall y (x < y \rightarrow f(x) \leq f(y)) \qquad \text{(increasing)}$$

$$\forall x \forall y (x < y \rightarrow f(x) < f(y)) \qquad \text{(strictly increasing)}$$

Similarly, *f* is called ***decreasing*** if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x)>f(y)$, whenever $x < y$ and x and y are in the domain of *f*.

$$\forall x \forall y (x < y \rightarrow f(x) \geq f(y)) \qquad \text{(decreasing)}$$

$$\forall x \forall y (x < y \rightarrow f(x) > f(y)) \qquad \text{(strictly decreasing)}$$

- A function that is either strictly increasing or strictly decreasing must be one-to-one.

Example:

- **f(x)=x+1.**

This is a **strictly increasing** function, because for all numbers a and b with a<b, we have a+1<b+1. Because f(a)=a+1 and f(b)=b+1, this means that we have f(a)<f(b).

- **f(x)=−2x**

This is a **strictly decreasing** function because for all numbers a and b with a>b, we have a+1>b+1. Because f(a)=-2a and f(b)=-2b, this means that we have f(a)>f(b).

- **f(x)=x^2.**

This is neither strictly increasing nor strictly decreasing, because −2<−1 but f(−2)>f(−1).

DEFINITION

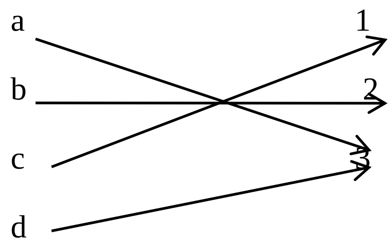
A function *f* from *A* to *B* is called *onto*, or *surjective*, if and only if for every element *b* *B* there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.

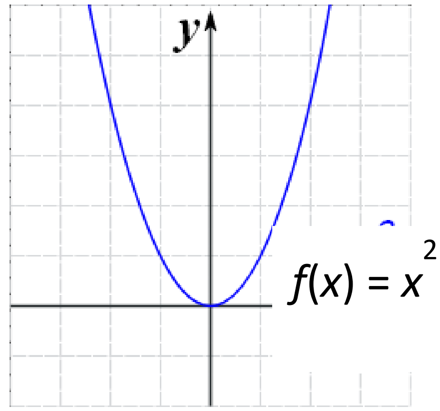
Example: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No. There is no integer x with $x^2 = -1$, for instance.

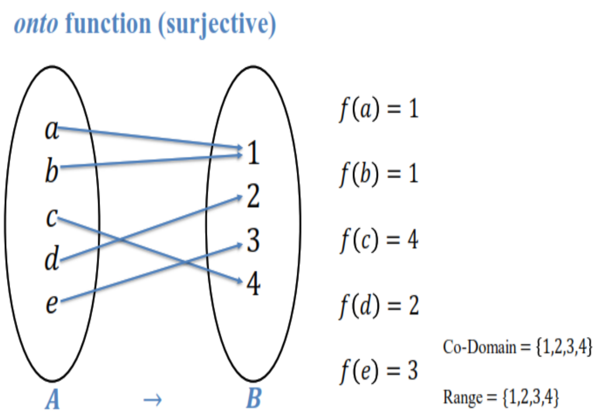
- **Example**:

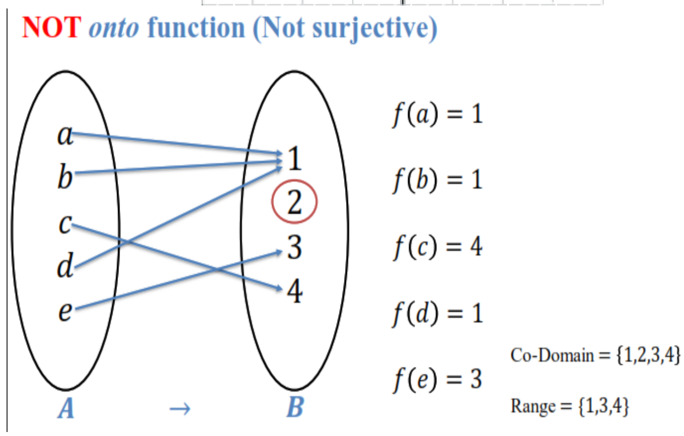
Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?





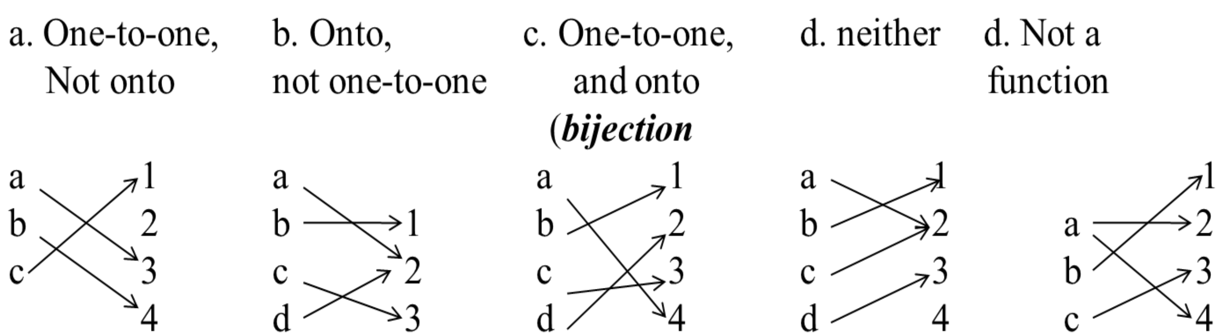
Solution: Yes.





DEFINITION 8

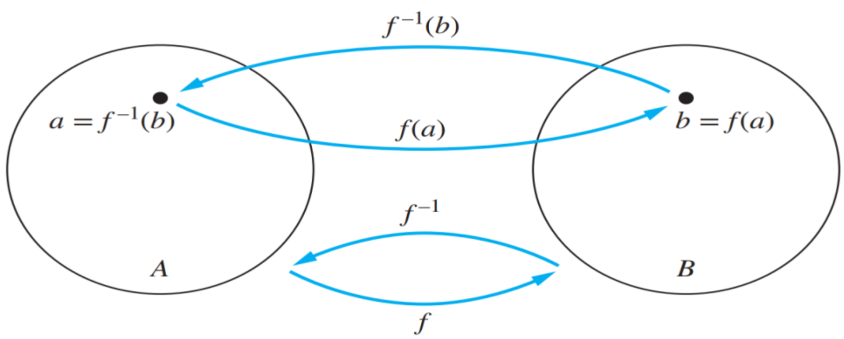
The function f is a ***one-to-one correspondence*** or a ***bijection***, if it is both one-to-one and onto.



4.2.2 Inverse Functions

- Let f be a *one-to-one correspondence* ***(bijection)*** from the set A to the set B.
- The **inverse** function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.

- The inverse function of f is denoted by $\boldsymbol{f^{-1}}$.
- Hence, f^{-1} (y) =x when f(x) = y

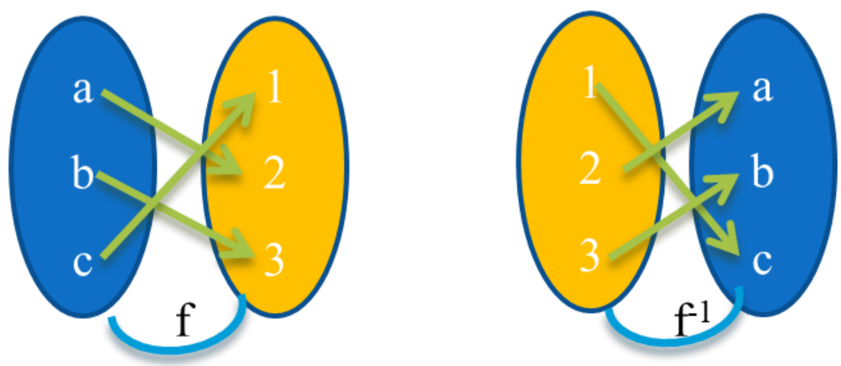


Invertible

A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example
Let f be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a)$=2,$f(b)$=3, and $f(c)$ = 1. Is f invertible, and if it is, what is its inverse?

Solution:
The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, f^{-1} (2)= a, and $f^{-1}(3) = b$.



Example:

Let f: Z➔Z be such that f(x)=x+1. Is f invertible, and if so , what is its inverse?

Solution:

The function f is invertible because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that y =x+1. Then x = y–1. The inverse function f^{-1} reverses the correspondence $f^{-1}(y) = y–1$.

Example:

Let f be the function from the set of all nonnegative real numbers to the set of all nonnegative real numbers with $f(x) = x^2$. Is f invertible?

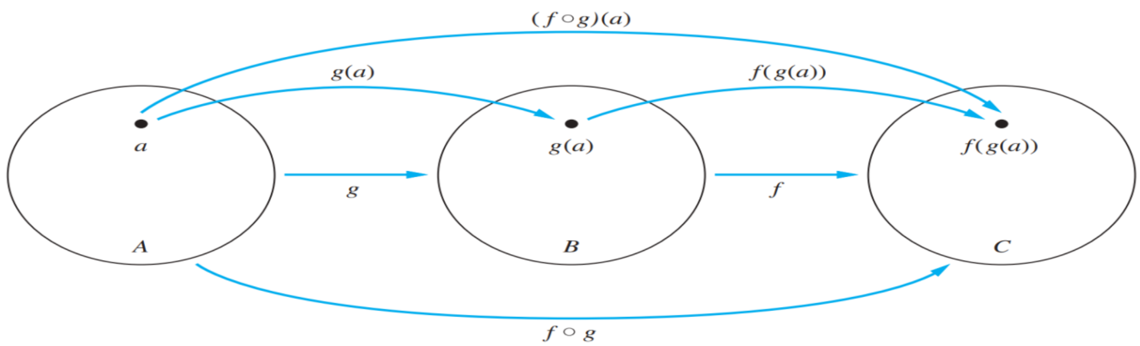
Solution:

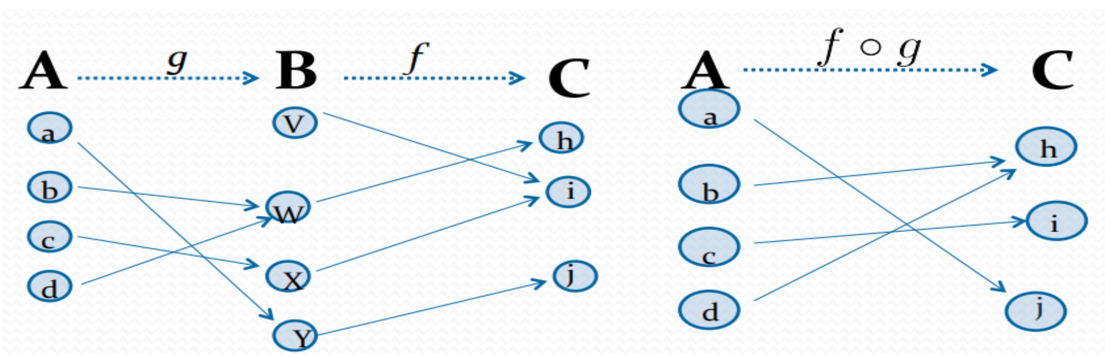
Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$.

- **Composition of the Functions *f* and *g***

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $\boldsymbol{f \circ g}$, is defined by: $\boldsymbol{(f \circ g)\, a = f(g(a))}$

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.





Example

Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a)= b$, $g(b)=c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a)=3$, $f(b)=2$, and $f(c)=1$. What is the composition of f and g, and what is the composition of g and f ?

Answer:

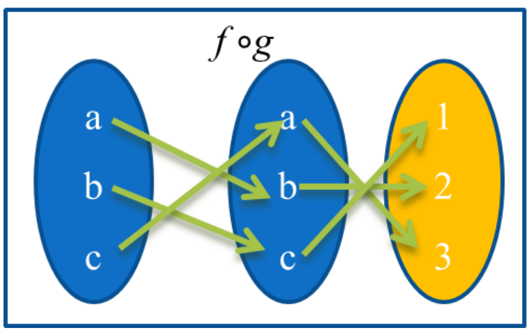
1) The composition of f and g (i.e., $(f \circ g)$)

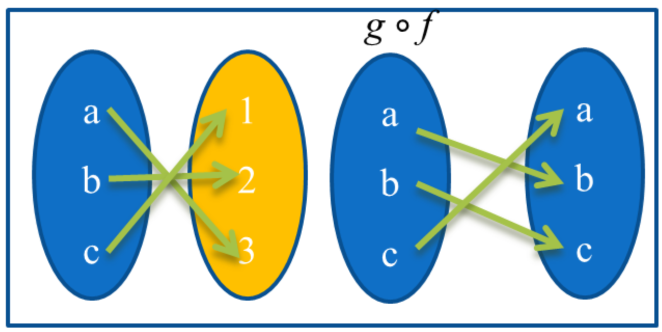
$$(f \circ g)(a)= 2,\ (f \circ g)(b) = 1,\ (f \circ g)(c) = 3$$

2) The composition of g and f (i.e., $(g \circ f)$) **cannot be defined** because the range of f is NOT a subset of the domain of g .

Range f= {1,2,3}

Domain g= {a,b,c}





Example:

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Solution

1) The composition of f and g (i.e., $(f \circ g)$)

$$(f \circ g)(x) = f(g(x)) = 2\,(3x + 2) + 3 = 6x + 7$$

2) The composition of g and f (i.e., $(g \circ f)$)

$$(g \circ f)(x) = g(f(x)) = 3(2x + 3) + 2 = 6x + 11$$

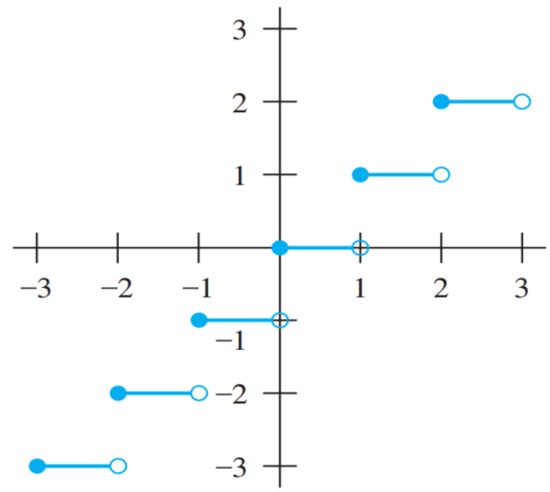
- **The Graphs of Functions**

Let f be a function from A to B. The graph of the function f is the set of ordered pairs $\{(a, b) |\ a \in A \text{ and } b \in B\}$

The graph of $f(x) = x^2$ from Z to Z.

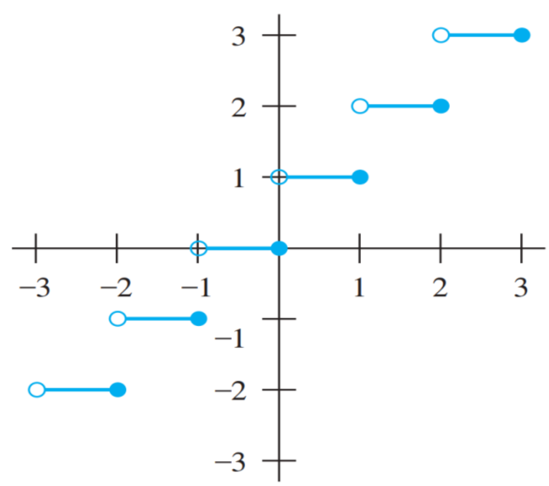
- **Floor function $y = \lfloor x \rfloor$**

is the largest integer less than or equal to x.



- **Ceiling function $y = \lceil x \rceil$**

  is the smallest integer greater than or equal to x



- **Useful Properties**

$$\lfloor -x \rfloor = -\lceil x \rceil$$
$$\lceil -x \rceil = -\lfloor x \rfloor$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$
$$\lceil x + n \rceil = \lceil x \rceil + n$$

Example:

$\lfloor 0.5 \rfloor$=0
$\lceil 0.5 \rceil$=1
$\lceil 3 \rceil$ =3
$\lfloor -0.5 \rfloor$=-1

$\lceil -1.2 \rceil$=-1
$\lfloor 1.1 \rfloor$=1

$\lfloor 0.3 + 2 \rfloor$=2

$\lceil 1.1 + \lceil 0.5 \rceil \rceil$=3

4.3 Sequences

A sequence is a discrete structure used to represent an ordered list

- Example: 1,2,3,5,8

1,3,9,27,81,…,30,…

DEFINITION

A sequence is a function from a subset of the set of integers (usually either

the set {0,1,2,…} or the set {1,2,3,…}) to a set *S*. f : N+ →S

We use the notation a_n to denote the image of the integer *n*. We call a_n a term of the sequence.

- We use the notation $\{a_n\}$ to denote the sequence. n is called the index; and domain is discrete.

Example:

Consider the sequence $\{a_n\}$, where $a_n = 1/n$.

The list of the terms of this sequence, beginning with a_1, namely

$a_1, a_2, a_3, a_4, \ldots$, starts with 1, 1/2, 1/3, 1/4, …

geometric progression

DEFINITION

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term ***a*** and the common ratio ***r*** are real numbers.

Example:

The following sequence are geometric progressions.

$\{b_n\}$ with $b_n = (-1)^n$ starts with 1, -1, 1, -1, 1, …

initial term: 1, common ratio: -1

$\{c_n\}$ with $c_n = 2*5^n$ starts with 2, 10, 50, 250, 1250, …

initial term: 2, common ratio: 5

$\{d_n\}$ with $d_n = 6 *(1/3)^n$ starts with 6,2, 2/3, 2/9, 2/27, …

initial term: 6, common ratio: 1/3

arithmetic progression

DEFINITION

A *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the initial term ***a*** and the common difference ***d*** are real numbers.

Example:

The following sequence are arithmetic progressions.

$\{s_n\}$ with $s_n = -1 + 4n$ starts with -1, 3, 7, 11,…

initial term: -1, common difference: 4

$\{t_n\}$ with $t_n = 7 – 3n$ starts with 7, 4, 1, -2, …

initial term: 7, common difference: -3

Example:

Find formulae for the sequences with the following first five terms

(a). 1, 1/2, 1/4, 1/8, 1/16

Solution: $a_n = 1/2^n$

(b). 1, 3, 5, 7, 9

Solution: $a_n = 2^* n + 1$

(c). 1, -1, 1, -1, 1

Solution: $a_n = (-1)^n$

Example:
Find $\boldsymbol{a}$, $\boldsymbol{r}$? $\mathbf{3 * 4^n}$, $\boldsymbol{n}$ = 0,1,2,3,4 …

Solution
ar^n , $n = 1,2,3,4, \ldots$
$a = 3$
$r = 4$

recurrence relation

- A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Example:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$.

Example

Let $\{an\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n =2, 3, 4, . . . , and suppose that $a_0 = 3$ and $a_1= 5$. What are a_2 and a_3?

Solution:
We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find $a4$, $a5$, and each successive term in a similar way

Fibonacci sequence,

The ***Fibonacci sequence***, $f_0, f_1, f_2, f_3, \ldots$, is defined by the initial conditions $f_0= 0$, $f_1= 1$, and the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

- Example:

Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 =0$ and $f_1 =1$, using the recurrence relation in the definition we find that.

$f_2 = f_1 + f_0 = 1 + 0 = 1,$
$f_3 = f_2 + f_1 = 1 + 1 = 2,$
$f_4 = f_3 + f_2 = 2 + 1 = 3,$
$f_5 = f_4 + f_3 = 3 + 2 = 5,$
$f_6 = f_5 + f_4 = 5 + 3 = 8.$

the sequence is 0, 1, 1, 2, 3, 5, 8, …

4.4 Summations

- The sum of the terms from the sequence $\{a_n\}$ $a_m + a_{m+1}, \ldots, a_n$ can be expressed as

$$\sum_{j=m}^{n} a_j \qquad \sum_{1\le j\le n} a_j$$

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n = \sum_{j=m}^{n} a_j = \sum_{m\le i\le n} a_i$$

```
s=0;
for(int i=m; i <= n; i++)
    s+= a[i];
```

Example:

- Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for n=1,2,3,.

  Solution: $$\sum_{j=1}^{100} \frac{1}{j}$$

- What is the value of $\sum_{j=1}^{5} j^2$?

  Solution:

  $$= 1 + 4 + 9 + 16 + 25 = 55$$

Expressed with a *for* loop:

```
int sum = 0;
for (int i=1; i<=5; i++){
    sum = sum +i*i;
}
```

What is the value of $\sum_{s\in\{0,2,4\}} S$?

Solution: Because $\sum_{s\in\{0,2,4\}} S$ represents the sum of the values of s for all the members of the set {0, 2, 4}, it follows that

$\sum_{s\in\{0,2,4\}} S$=0+2+4=6

Example:

Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

But want the index of summation to run between 0 and 4 j=k+1, $\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$

It is easily checked that both sums are 1+4+9+16+25=55.

example:

- What is the value of the double summation $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$?

Solution:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i+2i+3i)$$

$$\sum_{i=1}^{4} 6i = 6+12+18+24=60$$

- Expressed with two *for* loops:

```
int sum1 = 0;
int sum2 = 0;
for (int i=1; i<=4; i++){
    sum2 = 0;
    for (int j=1; j<=3; j++){
        sum2 = sum2 + i*j;
    }
    sum1 = sum1 + sum2; }
```

Chapter 5

Induction and Recursion

Chapter 5

Induction and Recursion

5.1 Mathematical Induction:

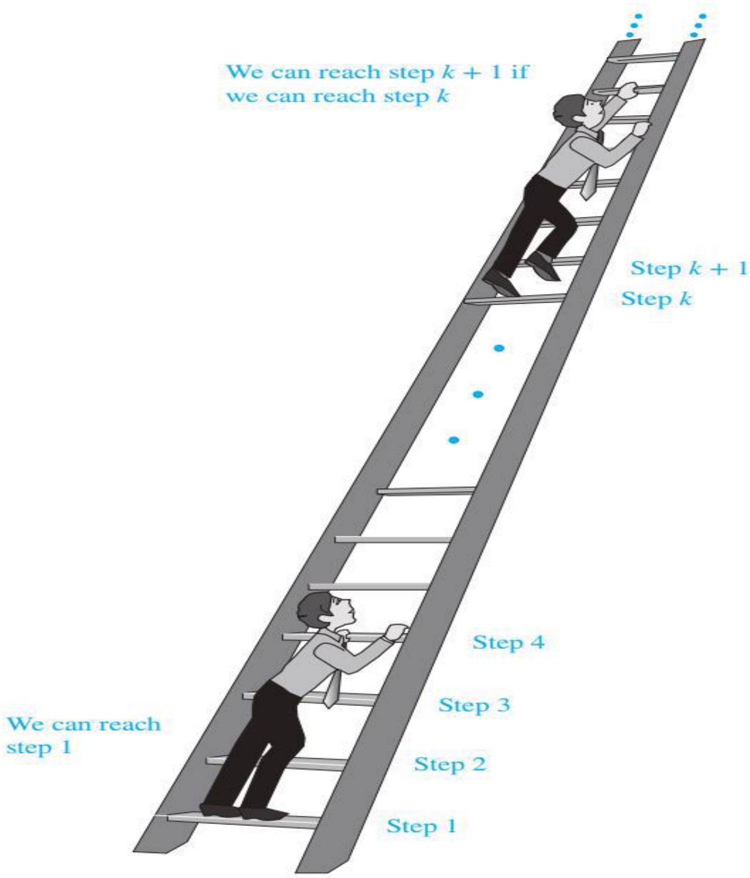
Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.

2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung.

we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called **mathematical induction**.



- Mathematical induction is not a tool for discovering formulae or theorems.

- **Mathematical Induction definition:**

Mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function

- **Principle of Mathematical Induction :**

To prove that $P(n)$ is true for all positive integers n, Where $P(n)$ is a propositional function, We complete **two** steps:

Basis Step

We verify that $P(1)$ is true.

Inductive Step
We show that the conditional statement
$P(k) \rightarrow P(k+1)$ is true for all positive integers k

- **Principle of Mathematical Induction**

✓ To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true.

✓ The assumption that $P(k)$ is true is called the ***inductive hypothesis*** **(IH)**.

$$\forall k(P(k) \rightarrow P(k+1))$$

Remark**:** In a proof by mathematical induction, it is **not** assumed that $P(k)$ is true for all positive integers! It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.

- Expressed as a rule of inference, this proof technique can be stated as: $[P(1) \wedge \forall k(P(k) \rightarrow P(k+1)] \rightarrow \forall n P(n)$ when the domain is the set of positive integers.

Remark**:**

In a proof by mathematical induction, for basis step, we **not always** start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer

Notes for Proofs by Mathematical Induction:

➢ Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.

✓ for all positive integers n, let $b = 1$, and
✓ for all nonnegative integers n, let $b = 0$, and so on …

- Write out the words "**Basis Step**." Then show that $P(b)$ is true.
- Write out the words "**Inductive Step**" and state, and clearly identify, the inductive hypothesis, in the form "Assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$.
- State what needs to be proved under the assumption that the inductive hypothesis (IH) is true.
  ✓ That is, write out what $P(k + 1)$ says.
- Show that $P(k + 1)$is true under the assumption that $P(k)$ is true.
  ✓ The most difficult part of a mathematical induction proof.
  ✓ This completes the inductive step.
- After completing the basis step and the inductive step, state the conclusion, namely,

"By mathematical induction, $P(n)$ is true for all integers n with $n \geq b$"

Example:

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i = 1 + 2 \ + 3 \ \cdots \ + n \ = \frac{n(n+1)}{2}$$

For all positive integers n . (i.e., $n \geq 1$)

Solution:

Let $P(n)$ be the proposition that $1 + 2 + 3 \cdots + n = n(n + 1)/2$

1) Basis Step:
If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is **true**, because $1 = (1)(2)/2$
This completes the basis step.

2) Inductive Step:
We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i. e. : $P(k)$
"$1 + 2 + 3 \cdots + k = k(k + 1)/2$ ".

- $P(k)$: "$1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}$".

- We **need to show** that if $P(k)$ is true, then $P(k + 1)$ is true. i. e. , we need to show that $P(k + 1)$ is also true.

$1+2+3 \cdot \cdot \cdot +k+k+1= \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$

- We **add** $(\boldsymbol{k} + \mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$1 + 2 + 3 \ldots + k + k + 1 = \frac{k(k+1)}{2}$+k+1

$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

• This equation show that $\boldsymbol{P(k + 1)}$ is true under the assumption that $\boldsymbol{P(k)}$ is true.
• This completes the inductive step.

So, the conclusion , by mathematical induction we know that $P(n)$ is true for all positive integers n.
That is, we proven that

$1 + 2 + 3 \cdot \cdot \cdot +n =\frac{n(n+1)}{2}$

for all positive integers n.

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 \ + 3^2 \ \cdots + n^2 \ = \frac{n(n+1)(2n+1)}{6}$$

For all positive integers n . (i.e., $n \geq 1$)

solution

- Let $P\ n$ be the proposition that
- $1^2 + 2^2 + 3^2 \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

1) Basis Step:
If $\boldsymbol{n} = \mathbf{1}$. $P\ 1$ is $\mathbf{true}$, because $1^2 = 1 = \frac{(1)(2)(3)}{6}$
This completes the basis step.

2) Inductive Step:
We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i. e. : $P(k)$

$$1^2 + 2^2 + 3^2 \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

- We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.
  i. e. : we need to show that $P(k+1)$ is also true.

$$1^2 + 2^2 + 3^2 \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$LHS = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+k+6k+6]}{6} = \frac{(k+1)[2k^2+7k+6]}{6} = \frac{(k+1)(2k+3)(k+2)}{6} =$$

$$= \frac{(k+1)\ [(k+1)+1][2(k+1)+1]}{6}$$

This equation show that $\boldsymbol{P(k+1)}$ is true under the assumption that $\boldsymbol{P(k)}$ is true.
This completes the inductive step

- So, by mathematical induction we know that $P(n)$ is true for all positive integers n.
  That is, we proven that

$$1^2 + 2^2 + 3^2 \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

  for all positive integers n.

E xample:

Use mathematical induction to prove that $n < 2^n$ For all positive integers n . (i.e., $n \geq 1$)

Solution

Let $P\ n$ be the proposition that $n < 2^n$

1) **Basis Step:**
   If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is $\mathbf{true}$, because $1 < 2^1$
   This completes the basis step.

2) **Inductive Step:**
   We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i. e. : $P(k)$
   $k < 2^k$

3) We **need to show** that if $P(k)$is true, then $P(k+1)$ is true.
   i. e. , we need to show that $P(k+1)$ is also true.

$(k+1) < 2^{k+1}$

We **add** ($\mathbf{1}$) to both sides of the equation in $P(k)$, we obtain

$k + 1 < 2^k + 1$

Because the integer $\boldsymbol{k} \geq \mathbf{1}$**. Therefore,** $2^k > \mathbf{1}$
$k + 1 < 2^k + 2^k$
$k + 1 < 2^k\,(1+1)$

$k + 1 < 2.2^k$

$k + 1 < 2^{k+1}$

This equation show that $\boldsymbol{P(k+1)}$ is true under the assumption that $\boldsymbol{P(k)}$ is true.
This completes the inductive step.

So, by mathematical induction we know that $\boldsymbol{P(n)}$ is true for all positive integers $\boldsymbol{n}$.

That is, we proven that $\boldsymbol{n < 2^n}$ for all positive integers $\boldsymbol{n}$.

Example:

Use mathematical induction to prove that $2^n < n!$
For every integer integers n with $n \geq 4$

Solution:
Let $P(n)$ be the proposition that $2^n < n!$, where $n \geq 4$
1) Basis Step:
If $\boldsymbol{n} = \mathbf{4}$. $P(4)$ is **true**, because ($2^4 = 16$) < ($4! = 24$)
This completes the basis step.

2) Inductive Step:
We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer $k \geq 4$, i. e. :
$P(k)$: $2^k < k!$
We **need to show** that if $P(k)$ is true, then $P(k + 1)$ is true.
i. e. , we need to show that $P(k + 1)$ is also true.
$2^{k+1} < (k + 1)!$

- We are **multiple** both sides of the equation in $P(k)$ by ($\mathbf{2}$), we obtain
  $2^k < k!$
  $2 \cdot 2^k < 2 \cdot k!$

$2^{k+1} < (k+1) \cdot k!$, **Because the integer** $\boldsymbol{k} \geq \mathbf{4}$**. Therefore,** $\mathbf{2} < \boldsymbol{k} + \mathbf{1}$

- $2^{k+1} < (k+1)!$

- **This equation show that $\boldsymbol{P(k+1)}$ is true under the assumption that $\boldsymbol{P(k)}$ is true.**

- **This completes the inductive step.**

Example

Use mathematical induction to prove that n^3 - n is divisible by 3
For every positive integer integers n. (i.e., $n \geq 1$)

Solution:

Let $P(n)$ be the proposition that
" n^3 - n is divisible by 3 "
1) Basis Step:
If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is $\mathbf{true}$, because 1^3 - 1 = 0 is divisible by 3.
This completes the basis step.
2) Inductive Step:
We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer $k \geq 1$, i. e. :
$P(k)$
k^3 - k is divisible by 3

- We **need to show** that if $P(k)$ is true, then $P(k + 1)$ is true.
  i. e. , we need to show that $P(k + 1)$ is also true.
  $(k + 1)^3$-$(k + 1)$ is divisible by 3
  Note that
  $(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1)$
  $= k^3 + 3k^2 + 3k - k$
  $= k^3 - k + 3k^2 + 3k$
  $= (k^3 - k) + 3(k^2 + k)$

  - **Using the inductive hypothesis, we conclude that the first term $\boldsymbol{k^3 - k}$ is divisible by 3**

  - $3(k^2 + k)$ **The second term is divisible by 3 because it is 3 times an integer.**

- **So,** $(k + 1)^3$- $k + 1$ **is divisible by 3**
  • **This completes the inductive step.**

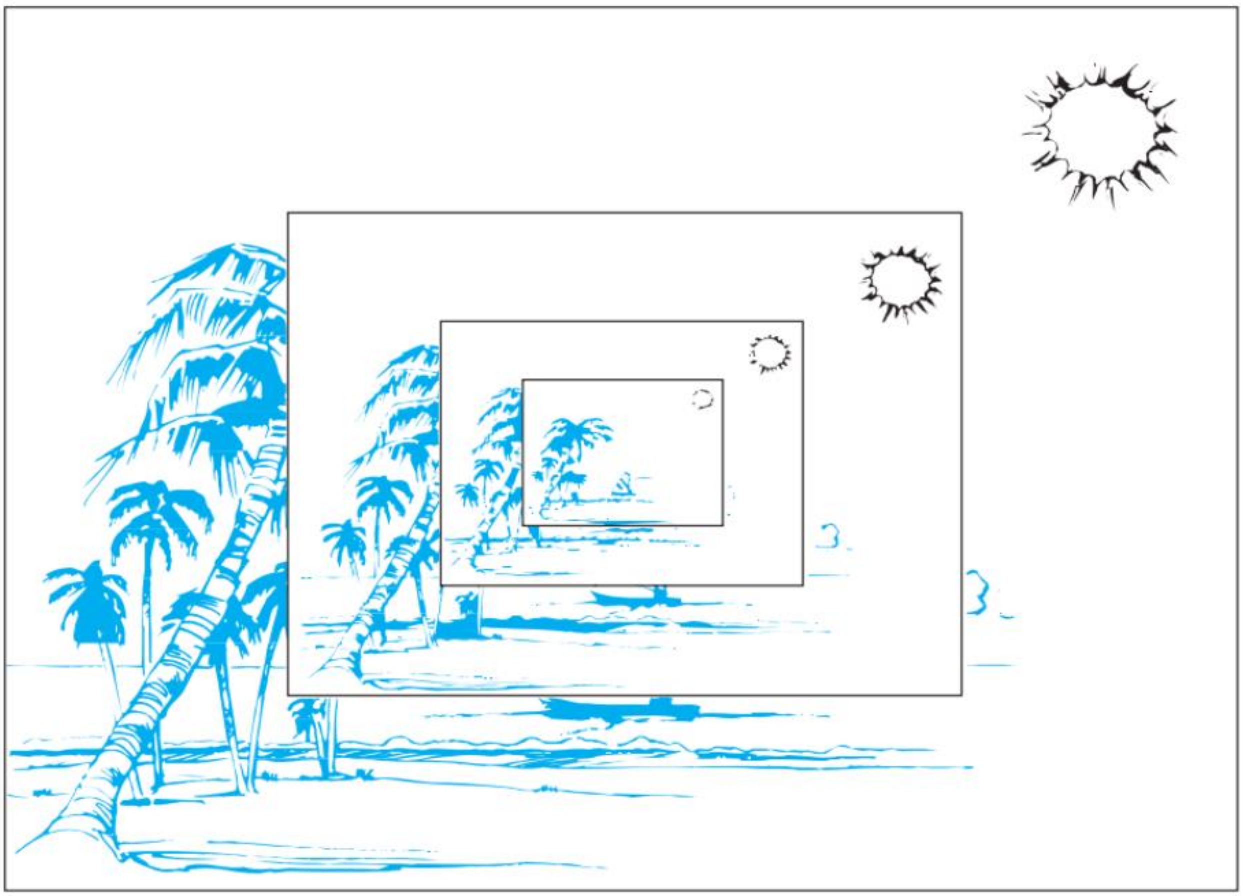
5.2 Recursion

Recursive Definitions :
The process of defining an object in terms of itself.

Recursively Defined Functions:

- **Basis Step**
  Specify the value of the function at the first point.

- **Recursive Step**
  Specifying how terms in the function are found from previous terms.



We use two steps to define a function with the set of *nonnegative integers* as its domain:

1) **Basis Step:**
   Specify the value of the function at *zero.*
   $f(0) = 0$

2) **Recursive Step:**
Give a rule for finding its value at an integer from its values at smaller integers.
$f(n + 1) = f(n) + 1$, for integer $n \geq 0$ (i.e., nonnegative integers)

Example:

The sequence of powers of 2 is given by $a_n = 2^n$ (**Explicit Formula**) for $n = 0, 1, 2, \ldots$

1) **Basis Step:**
Specify the value of the sequence at *zero*.
$a_0 = 2^0 = 1$

2) **Recursive Step:**
Give a rule for finding a term of the sequence from the previous one.
$a_{n+1} = 2a^n$ (**Recursive Formula**) , for $n = 0, 1, 2, \ldots$

Example:

Suppose that f is defined recursively by

f(0)=3

f(n+1)=2f(n)+3

find f(1),f(2),f(3), and f(4)

Solution

- F(1)=2f(0)+3=2.3+3=9
- F(2)=2f(1)+3=2.9+3=21
- F(3)=2f(2)+3=2.21+3=45
- F(4)=2f(3)+3=2.45+3=93

example:

Give a recursive definition of the factorial function n!

1) **Basis Step:**
Specify the value of the function at *zero.*
$f(0) = 1$

2) **Recursive Step:**
Give a rule for finding its value at an integer from its values at smaller integers.
$f(n + 1) = (n + 1) \cdot f(n)$, for $n = 0, 1, 2, \ldots$

remember that:

Factorial

$$n! = n * (n-1) * (n-2) * (n-3) * \cdots * 3 * 2 * 1$$

0! = 1
1! = 1
2! = 2 x 1 = 2
3! = 3 x 2 x 1 = 6
4! = 4 x 3 x 2 x 1 = 24

Example:

Recall from Chapter 2 that the Fibonacci numbers, $f_0, f_1, f_2, \ldots$, are defined by the equations $f_0 = 0$, $f_1 = 1$, and
$f_n = f_{n-1} + f_{n-2}$
Find:
f_2 ,f_3 ,f_4, f_5 ,

Solution

$f_2 = f_1 + f_0 = 1+0=1$

$f_3 = f_2 + f_1 = 1+1=2$

$f_4 = f_3 + f_2 = 2+1=3$

$f_5 = f_4 + f_3 = 3+2=5$

example:

Give a recursive definition of $\sum_{k=0}^{n} a_k$

Solution

The first part of the recursive definition is

- $\sum_{k=0}^{0} a_k = a_0$

The second part is

$\sum_{k=0}^{n+1} a_k = (\sum_{k=0}^{n} a_k) + an + 1$

References:

1. Rosen, Kenneth H. *Discrete mathematics and its applications*. TheMcGraw Hill Companies,, 2007.
2. Discrete Mathematics with Applications by Susanna S. Epp
3. Student's Solutions Guide to accompany Discrete Mathematics and Its Applications by Kenneth H. Rosen
4. Schaum's Outline of Discrete Mathematics, 3rd Ed. by Seymour Lipschutz, Marc Lipson