MATHEMATICS (2)

DR. Heba Nagaty

ALGBRA

Chapter 1 Matrices

Matrix

A set of $m \times n$ numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called $m \times n$ matrix.

A $m \times n$ matrix is expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \cdots & \cdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

❖ **Note that**

a_{ij} is the element in the i^{th} row and j^{th} column of the matrix .Thus the matrix A is sometimes denoted by simplified form as $[a_{ij}]$.

Matrices are usually denoted by capital letters A, B, C etc. and its elements by small letters a, b, c etc.

Order of a Matrix

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix.

In general if m are rows and n are columns of a matrix, then its order is $(m \times n)$.

Examples

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is matrix of order (2×3).

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is matrix of order (3×1).

$\begin{bmatrix} 2 & 0 & 3 & 6 \\ 2 & 1 & 7 & 5 \\ 4 & 2 & 4 & 2 \\ 0 & 3 & 1 & 0 \end{bmatrix}$ is matrix of order (4×4).

Some types of matrices

1. Row Matrix and Column Matrix:

A matrix consisting of a single row is called a **row matrix** whereas a matrix having single column is called a **column matrix**.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d & e \end{bmatrix}$$

2. Null or Zero Matrix

A matrix in which each element is 0 is called a **Null** or **Zero matrix**. Zero matrices are generally denoted by the symbol O.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Square matrix

A matrix A having same numbers of rows and columns is called a **square matrix**. i.e. $m = n$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Particular cases of a square matrix

(a) Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a **diagonal matrix**.

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

(b) Scalar Matrix

A diagonal matrix, in which all the diagonal elements are same, is called a **scalar matrix**.

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \quad \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

(c) Identity Matrix or Unit matrix

A scalar matrix in which each diagonal element is 1(unity) is called a **unit matrix**. An identity matrix of order n is denoted by I_n.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Equal Matrices

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix} \rightarrow A = B$$

⌗ Example 1

Find the values of x , y , z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution

By the definition of equality of matrices, we have

$$x+3=0 \rightarrow x=-3$$

$$2y+x=-7 \rightarrow y=-2$$

$$z-1=3 \rightarrow z=4$$

$$4a-6=2a \rightarrow a=3$$

Operations on Matrices

1. **Multiplication by a scalar**

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow kA=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

⌗ Example 2

If $A=\begin{bmatrix} 4 & 8 \\ -3 & 2 \end{bmatrix} \rightarrow kA=\begin{bmatrix} 4k & 8k \\ -3k & 2k \end{bmatrix}$

2. **Addition and Subtraction of Matrices**

If $A=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, then

- $A+B=\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$
- $A-B=\begin{bmatrix} a_1-a_2 & b_1-b_2 \\ c_1-c_2 & d_1-d_2 \end{bmatrix}$

⌗ Example 3

If $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, then

- $A + B = \begin{bmatrix} 3+1 & 1+0 \\ 2+(-1) & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$
- $A - B = \begin{bmatrix} 3-1 & 1-0 \\ 2-(-1) & 1-3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$

3. Product of Matrix

The product of matrices is exists if the number of the column of the first matrix equal the number of the rows of the second matrix.

If $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{m\times n}$ $and\ B = \begin{bmatrix} x & y \\ r & t \end{bmatrix}_{n\times k}$, then

$$A.B = \begin{bmatrix} ax+dr & ay+dt \\ bx+er & by+et \\ cx+fr & cy+ft \end{bmatrix}_{m\times k}$$

⌗ Example 4

If $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}_{3\times 2}$ $and\ B = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}_{2\times 2}$, then

$$A.B = \begin{bmatrix} (2)(1)+(-1)(-4) & (2)(0)+(-1)(2) \\ (3)(1)+(0)(-4) & (3)(0)+(0)(2) \\ (4)(1)+(5)(-4) & (4)(0)+(5)(2) \end{bmatrix}_{3\times 2}$$

$$\therefore A.B = \begin{bmatrix} 6 & -2 \\ 3 & 0 \\ -16 & 10 \end{bmatrix}_{3\times 2}$$

❖ Note that

1- $B.A$ does not exist because the number of the column of B not equal the number of the rows of A.

2- $A.B = B.A$ if A and B are commute to each other.

❖ Properties of basic matrix operations

Each of the following statements is valid for any matrices A, B and C which the indicated operations are defined and for any scalar α :

i. $A + B = B + A$ additive commutativity

ii. $A + (B + C) = (A + B) + C$ additive associativity

iii. $A.(B.C) = (A.B).C$ multiplicative associativity

iv. $A.(B + C) = A.B + A.C$

v. $\alpha(A + B) = \alpha A + \alpha B$ distributivity

Exercises I

1- Write the following matrices in tabular form:

a) $A = [a_{ij}]$, where $i = 1{,}2 \; and \; j = 1{,}2{,}3$

b) $B = [b_{ij}]$, where $i = 1 \; and \; j = 1{,}2{,}3{,}4$

2- Solve each of the following matrix equations:

a) $X + \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

b) $X + 2I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

3- Find the product of the following matrices

a) $\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} . \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 3 & -2 & 2 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 1 \\ 5 & 0 & 2 \\ 3 & 1 & 7 \end{bmatrix}$

Determinant

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified operations, which is characteristic of the matrix.

The determinants are defined **only** for square matrices. It is denoted by $det\ A$ or $|A|$ for a square matrix A.

The determinant of 2 × 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $det\ A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

⌑ **Example 5**

$$\begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix} = (3)(6) - (-5)(4) = 18 + 20 = 38$$

The determinant of 3 × 3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is given by $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

⌑ **Example 6**

$$\begin{vmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2[6 - 8] - [9 - 10] - 2[12 - 10] = -7$$

⌗ Example 7

If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

Find $det\ A$ by expansion about (a) the first row (b) the first column.

Solution

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3[4-(3)(-2)] - 2[0-(1)(-2)] + 1[0-1]$$

$$= 30 - 4 - 1 = 25$$

(b)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 3[4-(3)(-2)] - 0[8-3] + 1[(2)(-2)-1]$$

$$= 30 - 0 - 5 = 25$$

⌗ Example 8

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find $det\ A$

Solution

$$det\ A = 2\begin{vmatrix} -5 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & -5 \\ 0 & 0 \end{vmatrix}$$

$= 2(-5)(3) - 0 + 0 = -30$, which is the product of diagonal elements.

Special Matrices

1. **Transpose of a Matrix**

If A = $[a_{ij}]$ is $m \times n$ matrix, then the matrix of order $n \times m$ obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted $A^{\,t}$.

⌑ **Example 9**

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ -1 & 2 & 1 \end{bmatrix}$, then $A^{\,t} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 5 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

2. Symmetric Matrix:

A square matrix A is called symmetric if $A = A^t$.

3. Skew Symmetric Matrix:

A square matrix A is called symmetric if $A = -A^t$.

4. Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$.

⌑ **Example 10**

Find the value of k if $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular

Solution

Since A is singular so $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$$(k-2)(k+2)-5=0$$

$$k^2-4-5=0$$

$$k^2-9=0$$

$$k=\pm 3$$

5. Adjoint of a Matrix

Adjoint of square matrix A = $[a_{ij}]$ is obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij}, then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A .

i) Adjoint of a 2×2 Matrix

Remember

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

For example

If $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$c_{11} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = d$$

$$c_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -c$$

$$c_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -b$$

$$c_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a$$

$$\therefore C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\text{adj.}\,A = \left(\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\right)^t = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ii) **Adjoint of a 3×3 Matrix**

Remember

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$,

$$c_{11} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = +\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = (6-1) = 5$$

$$c_{12} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = (2-0) = -2$$

$$c_{13} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = +\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = (1-0) = 1$$

$$c_{21} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = -\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = -\big(0-(-1)\big) = -1$$

$$c_{22} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = +\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = (2-0) = 2$$

$$c_{23} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -(1-0) = -1$$

$$c_{31} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = +\begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} = \big(0-(-3)\big) = 3$$

$$c_{32} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -\big(1-(-1)\big) = -2$$

$$c_{33} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = +\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = (3-0) = 3$$

$$\therefore C = \begin{bmatrix} 5 & -2 & 1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

$$\text{adj}.A = \left(\begin{bmatrix} 5 & -2 & 1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}\right)^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ 1 & -1 & 3 \end{bmatrix}$$

6. Inverse of a Matrix:

If A is a non-singular square matrix, then $A^{-1} = \frac{1}{|A|}\text{adj}.A$

⌗ Example 11

Find the inverse of the matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Solution

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Hence solution exists.

$$C = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} \rightarrow adj.A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$$

$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Note

$$A^{-1}.A = A.A^{-1} = I$$

⌗ **Example 12**

Find the inverse of the matrix $A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

Solution

$$|A| = \begin{vmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{vmatrix}$$

$$= 0 - (-2)(-2+3) + (-3)(-2+3) = 2 - 3 = -1 \neq 0$$

Hence solution exists.

$$C = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 2 \\ 3 & -3 & 2 \end{bmatrix}$$

$$adj.A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = -\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Exercises II

1. Find the value of the following determinants

a) $\begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix}$

b) $\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$

c) $\begin{vmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ -2 & 5 & 3 \end{vmatrix}$

2. If $\begin{vmatrix} a & d & x \\ b & e & y \\ c & f & z \end{vmatrix} = 20$, then $\begin{vmatrix} b & e & y \\ a & d & x \\ c & f & z \end{vmatrix} = \cdots$.

3. Find the solution of the equation;

$$\begin{vmatrix} x & 1 & 2 \\ 0 & x & 3 \\ 0 & 0 & x \end{vmatrix} - 8 = 0$$

4. Find the value of x if

$$\begin{vmatrix} 0 & -1 & x \\ x & 4 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 10$$

5. Find the inverse if it exists, of the following matrices

a) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

Chapter 2 Sequences and Series

Overview

Up to this point in calculus, we have focused on the derivative and integral of a function. Now we introduce a **third** key topic of interest in the analysis and computation of functions, called **infinite series**. Such series give us precise ways to express many numbers and functions, both familiar and new, as arithmetic sums with infinitely many terms.

Often scientists and engineers simplify a problem by replacing a function with an approximation using the first few terms of a series that expresses it. One method represents a known differentiable function $f(x)$ as an infinite series in powers of x, so it looks like a "polynomial with infinitely many terms,"

Sequences

A sequence is a list of numbers in a given order.

$$a_1, a_2, a_3, \dots \dots, a_n, \dots.$$

Each of a_1, a_2, a_3 and so on represents a number. These are the terms of the sequence.

The integer n is called the **index** of a_n.

For **example**, the sequence

$$2,4,6,8,10, \dots ,2n, \dots$$

The first term $a_1 = 2$

The second term $a_2 = 4$

Example

Write down the first five terms of this sequence.

$a_n = 3n + 5, for\ n = 1,2,3, \dots$

Solution

$$8,11,14,17,20$$

Try to solve

Write down the first five terms of the sequence given by $a_n = \frac{(-1)^{n+1}}{n}$

The sequence can be described by

1. writing rules that specify their terms

$$a_n = \sqrt{n}$$

2. listing terms

$$\{ a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$

3. using its rule

$$\{ a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

The n*th* term in Addition $a_n = a_1 + d(n-1)$

The n*th* term in Multiplication $a_n = a_1 \times m^{n-1}$

m multiplication number, d difference number

Examples Find the nth term of the following sequences

Sequences	Solution
1- $(2, 4, 6, 8, \ldots.)$	$a_n = 2 + 2(n-1) = 2n + 1$
2- $(5, 10, 15, \ldots.)$	$a_n = 5 + 5(n-1) = 5n$
3- $(5, 15, 25, \ldots.)$	$a_n = 5 + 10(n-1) = 10n - 5$
4- $\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots.\right)$	$a_n = \dfrac{1}{3 + 2(n-1)} = \dfrac{1}{1 + 2n}$
5- $\left(\frac{3}{5}, \frac{7}{4}, \frac{11}{3}, \ldots.\right)$	$a_n = \dfrac{3 + 4(n-1)}{5 - 1(n-1)} = \dfrac{4n - 1}{6 - n}$
6- $(1, 3, 9, \ldots.)$	$a_n = 1 \times 3^{n-1} = 3^{n-1}$
7- $(8, 16, 32, \ldots.)$	$a_n = 8 \times 2^{n-1} = 2^{n+2}$
8- $(32, 16, 8, \ldots.)$	$a_n = 32 \times (\frac{1}{2})^{n-1} = 2^{5-n+1} = 2^{-n+6}$
9- $(-3, 5, -7, \ldots.)$	$a_n = (-1)^n(1 + 2n)$
10- $(3, -5, 7, -9, \ldots$	$a_n = (-1)^{n+1}(1 + 2n)$

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. Then the sequence is **convergence**.

On the other hand, sequences like $\{1, -1, 1, \ldots, (-1)^n, \ldots\}$ never converging to a single value. Then the sequence is **divergence**.

Calculating Limits of Sequences

Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. **Sum Rule:** $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. **Difference Rule:** $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. **Constant Multiple Rule**: $\lim_{n\to\infty}(k.\ b_n) = k.B$

4. **Product Rule**: $\lim_{n\to\infty}(a_n.\ b_n) = A.B$

5. **Quotient Rule** $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}, B \neq 0$

Example

- $\lim_{n\to\infty}\left(\frac{-1}{n}\right) = (-1).\lim_{n\to\infty}\frac{1}{n} = (-1)(0) = 0$
- $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}1 - \lim_{n\to\infty}\frac{1}{n} = 1$
- $\lim_{n\to\infty}\left(\frac{4-7n^2}{n^2+3}\right) = \frac{\infty}{\infty}$

Divide the numerator and denominator by n^2 البسط و المقام

$$\lim_{n\to\infty}\left(\frac{\frac{4}{n^2}-7}{1+\frac{3}{n^2}}\right) = -\frac{7}{1} = -7$$

- $\lim_{n\to\infty}\frac{\ln n}{n} = \frac{\infty}{\infty}$

Using l'Hôpital's Rule

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$$

Commonly Occurring Limits

Theorem

The following six sequences converge to the limits listed below

1-	$\lim_{n\to\infty}\frac{\ln n}{n} = 0$	2-	$\lim_{n\to\infty}\sqrt[n]{n} = 1$
3-	$\lim_{n\to\infty}x^{1/n} = 1 \quad ,x > 0$	4-	$\lim_{n\to\infty}x^n = 0 \quad \lvert x\rvert < 1$
5-	$\lim_{n\to\infty}\frac{x^n}{n!} = 0$ any x	6-	$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ any x

Example

- $\lim_{n\to\infty}\frac{\ln n^2}{n} = \lim_{n\to\infty}\frac{2\ln n}{n} = 2(0) = 0$

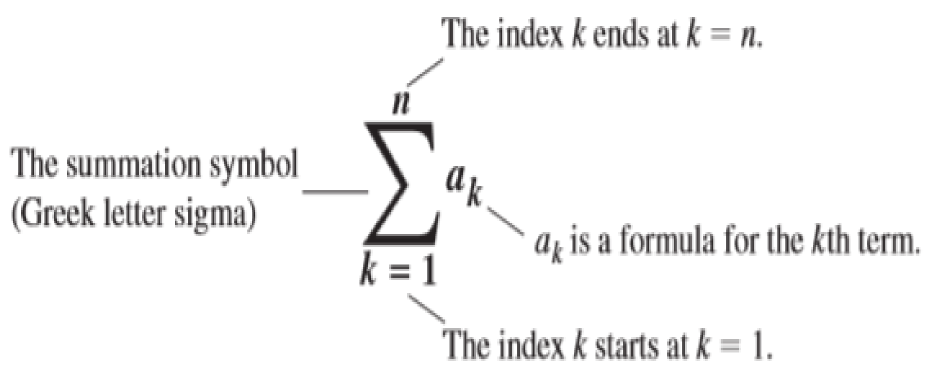
- $\lim_{n\to\infty} \sqrt[n]{n^2} = n^{2/n} = \left(n^{1/n}\right)^2 = (1)^2 = 1$
- $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} 3^{1/n} \sqrt[n]{n} = 1.1 = 1$
- $\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$
- $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2}$
- $\lim_{n\to\infty} \frac{10^n}{n!} = 0$

Try to solve

1. $\lim_{n\to\infty} \left(1 + \frac{7}{n}\right)^n$
2. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$
3. $\lim_{n\to\infty} \sqrt[n]{10n}$
4. $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n}$

Sigma notation

Sigma notation enables us to write a sum with many terms in the compact form.



Example

$$a)\ \sum_{k=1}^{5} k = 1+2+3+4+5 = 15$$

$$b)\ \sum\nolimits_{k=1}^{2} (-1)^k k = (-1)^1(1) + (-1)^2(2) = -1+2 = 1$$

$$c)\ \sum_{i=1}^{2} \frac{i}{i+1} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

Example

Suppose that $\sum_{k=1}^{n} a_k = 0$. Find

a- $\sum_{k=1}^{n} 8a_k$	b- $\sum_{k=1}^{n} (a_k + 1)$

Algebra Rules for Finite Sums

1. Sum Rule

$$\sum_{k=1}^{n} (a_n + b_n) = \sum_{k=1}^{n} a_n + \sum_{k=1}^{n} b_n$$

2. Difference Rule

$$\sum_{k=1}^{n} (a_n - b_n) = \sum_{k=1}^{n} a_n - \sum_{k=1}^{n} b_n$$

3. Constant Multiple Rule

$$\sum_{k=1}^{n} ca_n = c.\sum_{k=1}^{n} a_n$$

4. Constant Value Rule

$$\sum_{k=1}^{n} c = n.c$$

Example

$$a)\ \sum_{k=1}^{n}(3k-2)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}2=3\sum_{k=1}^{n}k-2n$$

$b)\ \sum_{i=1}^{n}\frac{1}{n}=n.\frac{1}{n}=1$

try to solve

Evaluate

a) $\sum_{k=1}^{2}\frac{6k}{k+1}$ b) $\sum_{k=1}^{3}\frac{k-1}{k}$

c) $\sum_{k=1}^{2}\cos k\pi$ d) $\sum_{k=0}^{3}2^{k}$

Series

A series is the **sum** of all the terms in a sequence.

1. If $a_1, a_2, a_3, \dots \dots, a_n$ is a finite sequence, then the sum

$$a_1+a_2+a_3+\cdots+a_n=\sum_{i=1}^{n}a_i$$

is a finite series corresponding to these sequence.

2. If $a_1, a_2, a_3, \dots \dots, a_n, \dots$ is infinite sequence, then the sum

$$a_1+a_2+a_3+\cdots+a_n+\cdots=\sum_{i=1}^{\infty}a_i$$

is an infinite series corresponding to these sequence.

Example

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{n-1}}=1-\frac{1}{2}+\frac{1}{4}-\cdots$$

- If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its sum is L. In this case, we also write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

- If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

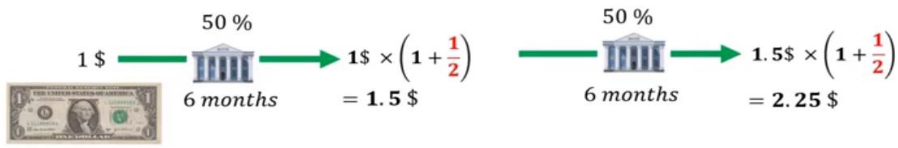
Difference between Sequences and Series

Sequences	Series
Set of elements that follow a pattern	Sum of elements of the sequence
Order of elements is important	Order of elements is not so important
Finite sequence: $\{a_i\}_{i=1}^{n}$	Finite series: $\sum_{i=1}^{n} a_i$
Infinite sequence: $\{a_i\}_{i=1}^{\infty}$	Infinite Series: $\sum_{i=1}^{\infty} a_i$

للفهم فقط

Euler's constant

Bernoulli $\mathbf{2 < e < 3}$



$$1\$ \times \left(1+\frac{1}{2}\right)\times\left(1+\frac{1}{2}\right) = 1\$ \times \left(1+\frac{1}{2}\right)^{2} = 2.25$$

Every Month :

$$1\$ \times \underbrace{\left(1+\frac{1}{12}\right)\times\cdots\left(1+\frac{1}{12}\right)}_{12\ times} = 1\$ \times \left(1+\frac{1}{12}\right)^{12} = 2.613$$

Every Weak :

$$1\$ \times \underbrace{\left(1+\frac{1}{52}\right)\times\cdots\left(1+\frac{1}{52}\right)}_{52\ times} = 1\$ \times \left(1+\frac{1}{52}\right)^{52} = 2.693$$

Every Day :

$$1\$ \times \underbrace{\left(1+\frac{1}{365}\right)\times\cdots\left(1+\frac{1}{365}\right)}_{365\ times} = 1\$ \times \left(1+\frac{1}{365}\right)^{12} = 2.715$$

Every Second :

$$1\$ \times \underbrace{\left(1+\frac{1}{31536000}\right)\times\cdots\left(1+\frac{1}{31536000}\right)}_{31536000\ times} = 1\$ \times \left(1+\frac{1}{31536000}\right)^{31536000} = 2.7182$$

n …… numbers of subintervals .

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

Total Money after 1 year of continuous interest if:

1- start with 1$

2- 100 % interest / year

2. 71828182846 … … … … …. .

Leonhard Euler (1707-1783)

$e \rightarrow$ **Euler**

$e \rightarrow$ **Exponential**

$e \rightarrow$ **A → E**

$$\boldsymbol{y(t) = e^{t}}$$

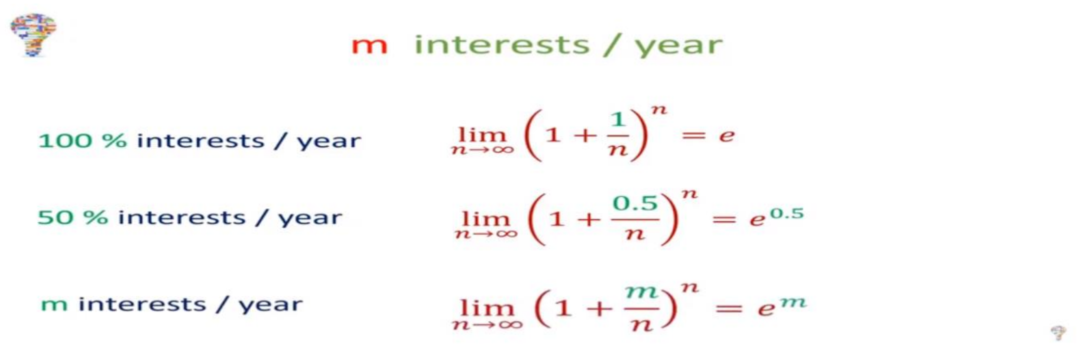
$t \in (-\infty, \infty)$

$y \in (0, \infty)$

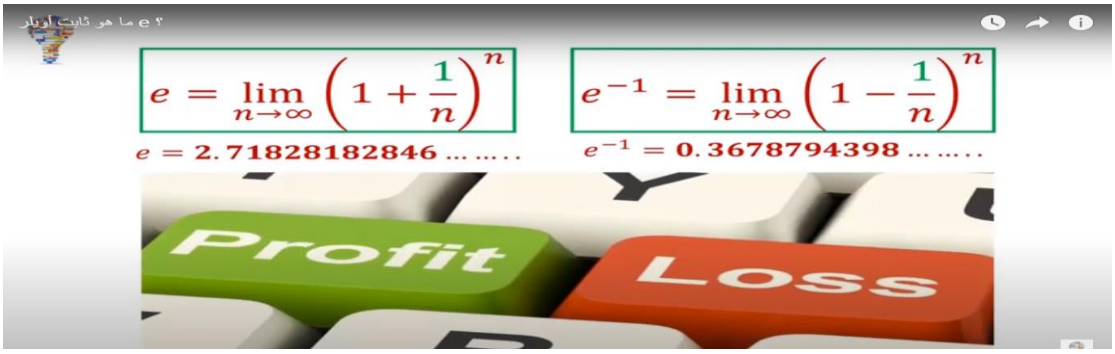
Money after (t) years of continuous interests if :

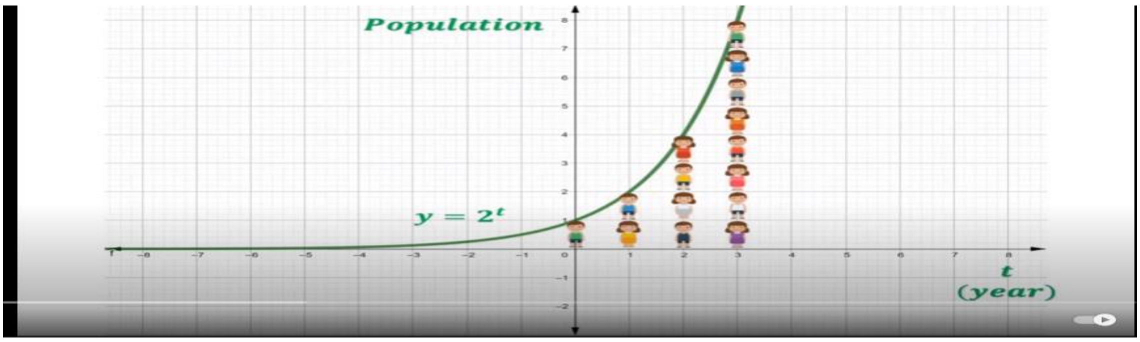
1- start with1$ at t=0 ⟶ c $

2- 100 % interests / year ⟶ m


m interests / year
100 % interests / year
$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$
50 % interests / year
$\lim_{n\to\infty}\left(1+\frac{0.5}{n}\right)^n = e^{0.5}$
m interests / year
$\lim_{n\to\infty}\left(1+\frac{m}{n}\right)^n = e^{m}$


ما هو ثابت أويلر e ؟
$y(t) = Ce^{mt}$
Money after (t) years of continuous interests if :
1- start with C\$
2- m interests / year


ما هو ثابت أويلر e ؟
$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
$e^{-1} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$
$e = 2.71828182846 \dots\dots..$
$e^{-1} = 0.3678794398 \dots\dots..$
Profit
Loss


Population
$y = 2^t$
t
(year)

Chapter 3 Partial Fractions

A rational function $f(x) = \frac{p(x)}{q(x)}$ can be expressed as a sum of simpler fractions, called **Partial Fractions.**

1. **Improper rational function**

$$\text{degree of } p(x) \geq \text{degree of } p(x)$$

In the case, we must first perform long division to rewrite the quotient $\frac{p(x)}{q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$.

⌗ Example

Transform the following fraction into partial fraction

$$f(x) = \frac{x^3 + 4x^2 + 3}{x^2 + 2x + 1}$$

Solution

by using long division

$$\begin{array}{r} x + \mathbf{2} \\ x^2 + 2x + 1 \overline{) x^3 + 4x^2 + 3} \\ \underline{x^3 + 2x^2 + x} \\ 2x^2 - x + 3 \\ \underline{2x^2 + 4x + 2} \\ -5x + 1 \end{array}$$

$$\therefore f(x) = \mathrm{x} + 2 + \frac{-5\mathrm{x}+1}{x^2+2x+1}$$

2. **Proper rational function**

$$\text{degree of } p(x) < \text{degree of } q(x)$$

There are different cases for proper rational function.

Case I $q(x)$ is a product of distinct linear factors

$$f(x)=\frac{p(x)}{(a_1x+b_1)(a_2x+b_2).....(a_mx+b_m)}$$

Then, to this factor, assign the sum of the m partial fractions:

$$\boxed{f(x)=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+..+\frac{A_m}{(a_mx+b_m)}}$$

⌑ **Example 2**

Transform the following fraction into partial fraction

$f(x)=\frac{3x-1}{x^2-x-3}$, and then find $\int f(x)\,dx$

Solution

Method 1 Solve by substitution

$$\frac{3x-1}{x^2-2x-3}=\frac{3x-1}{(x-3)(x+1)}=\frac{A}{(x-3)}+\frac{B}{(x+1)}=\frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$\therefore 3x-1=A(x+1)+B(x-3)$$

Put $x=-1 \Rightarrow -4=-4B \Rightarrow B=1$

Put $x=3 \Rightarrow 8=4A \Rightarrow A=2$

$$\therefore f(x)=\frac{2}{(x-3)}+\frac{1}{(x+1)}$$

Method 2 Solve by equating corresponding coefficients

Coefficients of $x \Rightarrow 3=A+B$ (1)

Coefficients of $x^0 \Rightarrow -1=A-3B$ (2)

solving two equations, we get $A=2, B=1$

$$\int\left(\frac{2}{(x-3)}+\frac{1}{(x+1)}\right)dx=2\ln|x-3|+\ln|x+1|+c$$

Case II $q(x)$ is a product of linear factors, some of which are repeated

$$f(x) = \frac{p(x)}{(a_1x+b_1)(a_1x+b_1)^2 \,..(a_1x+b_1)^m}$$

Then, to this factor, assign the sum of the m partial fractions:

$$\boxed{f(x) = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_1x+b_1)^2} + .. + \frac{A_m}{(a_1x+b_1)^m}}$$

⌗ **Example 3**

Transform the following fraction into partial fraction

$f(x) = \frac{3x^2}{(x-1)^3}$,and then find $\int f(x)\,dx$

Solution

$$\frac{3x^2}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{\mathrm{A}(x-1)^2+\mathrm{B}(\mathrm{x}-1)+\mathrm{C}}{(x-1)^3}$$

$$\therefore 3x^2 = \mathrm{A}(x-1)^2 + \mathrm{B}(\mathrm{x}-1) + \mathrm{C}$$

Put $\mathrm{x} = 1 \Rightarrow 3 = C \Rightarrow C = 3$

Put $\mathrm{x} = 0 \Rightarrow 0 = A - B + C \Rightarrow A - B = -3$ (1)

Put $\mathrm{x} = -1 \Rightarrow 3 = 4A - 2B + C \Rightarrow 4A - 2B = 0$

$$2A = B \qquad (2)$$

Substitute from equation (2) in equation (1)

$A - 2A = -3 \Rightarrow -A = -3 \Rightarrow A = 3$ and $B = 6$

$$\therefore f(x) = \frac{3}{(x-1)} + \frac{6}{(x-1)^2} + \frac{3}{(x-1)^3}$$

$$\int f(x)\,dx = \int \left(\frac{3}{(x-1)} + \frac{6}{(x-1)^2} + \frac{3}{(x-1)^3}\right) dx$$

$$= 3\ln|x-1| - \frac{6}{(x-1)} - \frac{3}{2(x-1)^2} + c$$

Case III $q(x)$ is irreducible quadratic factors and non-repeated

$$f(x) = \frac{p(x)}{(ax^2+bx+c)}$$

Then, to this factor, the partial fraction is:

$$f(x) = \frac{Ax+B}{(ax^2+bx+c)}$$

⌗ Example 4

Transform the following fraction into partial fraction

$$f(x) = \frac{x-1}{(x+1)(x^2+2x+2)}$$

Solution

$$\frac{x-1}{(x+1)(x^2+2x+2)} = \frac{Ax+B}{(x^2+2x+2)} + \frac{C}{(x+1)}$$

$$\therefore x-1 = (Ax+B)(x+1) + C(x^2+2x+2)$$

Put $\mathbf{x} = -\mathbf{1} \Rightarrow -2 = C \Rightarrow C = -2$

Put $\mathbf{x} = \mathbf{0} \Rightarrow -1 = B + 2C \Rightarrow B = 3$

Put $\mathbf{x} = \mathbf{1} \Rightarrow 0 = 2A + 2B + 5C \Rightarrow A = 2$

$$\frac{x-1}{(x+1)(x^2+2x+2)} = \frac{2x+3}{(x^2+2x+2)} + \frac{-2}{(x+1)}$$

Remember

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Example 5

Transform the following fraction into partial fraction

$$f(x) = \frac{1}{(x^3+1)}$$

Solution

$$\frac{1}{(x^3+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)}$$

$$\therefore 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$$

Put $\mathbf{x} = \mathbf{-1} \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$

Put $\mathbf{x} = \mathbf{0} \Rightarrow 1 = A + C \Rightarrow C = \frac{2}{3}$

Coefficients of $x^2 \Rightarrow 0 = A + B \Rightarrow B = \frac{-1}{3}$

$$\frac{1}{(x^3+1)} = \frac{1}{3}\frac{1}{(x+1)} + \frac{1}{3}\frac{(-x+2)}{(x^2-x+2)}$$

Try to solve

Transform the following fraction into partial fraction

1- $\frac{2+x}{1-x^2}$

2- $\frac{x^3+4x^2+3}{x^2+2x+1}$

3- $\frac{2x+5}{x^2+5x+6}$

4- $\frac{x^4-3x^3-3}{x^2-4}$

5- $\frac{8x^3+13x}{(x^2+2)^2}$

6- $\frac{3x+11}{x+6-x^2}$

7- $\frac{x^2+4}{3x^3+4x^2-4x}$

8- $\frac{x^2}{x^2-1}$

9- $\frac{x^2+x+1}{x^2(x^3-1)}$

10- $\frac{x^2-29x+5}{(x-4)^2(x^2+3)}$

Geometry

Chapter 4 Vectors

Overview

Vectors provide simple ways to define equations for lines, planes, curves, and surfaces in space.

Scalars and Vectors

Scalars

Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure.

Vectors

We need more information to describe a force, displacement, or velocity.

- To describe a **force**, we need to record the direction in which it acts as well as how large it is.
- To describe a body's **displacement**, we have to say in what direction it moved as well as how far.
- To describe a body's **velocity**, we have to know where the body is headed as well as how fast it is going.

In this section we show how to represent things that have both magnitude and direction in the plane or in space.

Definition

Vector is a quantity that has magnitude and direction.

Types of Space Coordinates

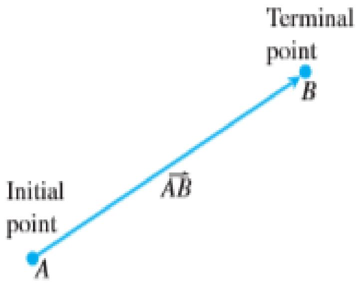
1. Cartesian coordinates $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.
2. Spherical coordinates $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$.
3. Cylindrical coordinates $(\boldsymbol{\rho}, \boldsymbol{\varphi}, \boldsymbol{z})$**.**

Component form

A quantity such as force, displacement, or velocity is called a **vector** and is represented by a **directed line segment**.

Definition

The vector represented by the **directed line segment** $\overrightarrow{AB}$ has initial point A and terminal point B and its **length** is denoted by $\left|\overrightarrow{AB}\right|$.



Definition

If v is a three-dimensional vector equal to the vector with initial point at the origin and terminal point $\langle v_1, v_2, v_3 \rangle$,then the component form of v is

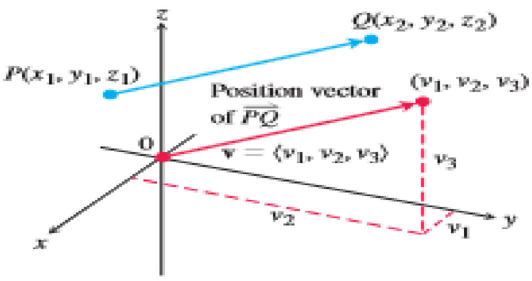
$$v = \langle v_1, v_2, v_3 \rangle$$

The length (magnitude) of vector

Let $v \neq 0$,

$v = \overrightarrow{PQ} = \langle v_1, v_2, v_3 \rangle$

$P = \langle x_1, y_1, z_1 \rangle$ is the initial point and $Q = \langle x_2, y_2, z_2 \rangle$ is the terminal point, then the **length** (**magnitude**) of v is



$$|v| = \sqrt{\mathrm{v}_1{}^2 + \mathrm{v}_2{}^2 + \mathrm{v}_3{}^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The Direction of vector

$$\hat{v} = \frac{v}{|v|} = \frac{\mathrm{v}_1 i + \mathrm{v}_2 j + \mathrm{v}_3 k}{\sqrt{\mathrm{v}_1{}^2 + \mathrm{v}_2{}^2 + \mathrm{v}_3{}^2}}$$

Note

1. Two vectors are equal if they have the same length and direction.
2. The only vector with length 0 is the zero vector $0 = \langle 0,0,0 \rangle$.
3. Zero vector is also the only vector with no specific direction.

Example 1

If the initial point is $\mathrm{P} = \langle -3,4,1 \rangle$ and the terminal point is $\mathrm{Q} = \langle -5,2,2 \rangle$, Find a) The component form. b) The length of the vector.

Solution

$$v = \overrightarrow{PQ} = \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle, \ \mathrm{v}_1 = -5 - (3) = -2$$

$$\mathrm{v}_2 = 2 - 4 = -2$$

$$\mathrm{v}_3 = 2 - 1 = 1$$

a) The component form of $\overrightarrow{PQ}$ is

$$v = \langle -2, -2, 1 \rangle$$

b) The length of the vector v is

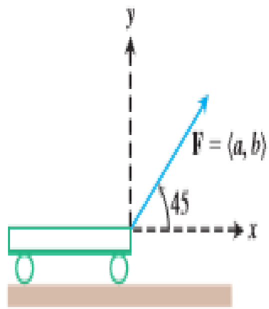
$$|v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3$$

Example 2

A small cart is being pulled along a smooth horizontal floor with a 20-lb force F making a 45° angle to the floor. What is the effective force moving the cart forward?

Solution

The effective force is the horizontal component of



$$F = \langle a, b \rangle, \text{ given by}$$

$$a = |F| \cos 45^{\circ} = 20\left(\frac{1}{\sqrt{2}}\right)$$

$$\cong 14.14 - Ib$$

Vector Algebra Operations

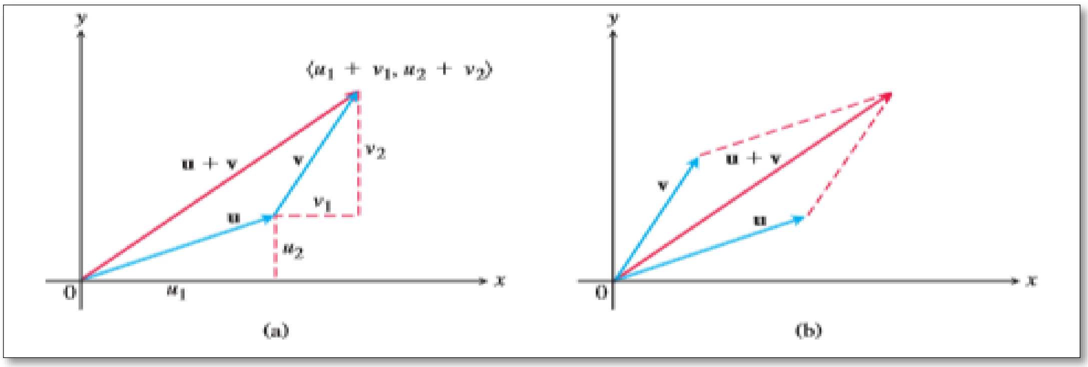
Two principal operations involving vectors are vector **addition** and **scalar multiplication**. A scalar is simply a real number, and is called such when we want to draw attention to its differences from vectors. Scalars can be **positive**, **negative**, or **zero** and are used to “scale” a vector by multiplication.

Definition

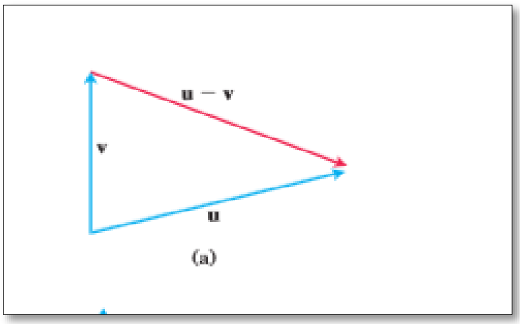
Let $u = \langle \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3 \rangle$ and $v = \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle$ be vectors with k a scalar.

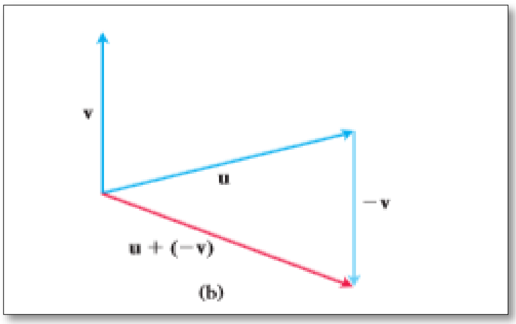
Addition: $u + v = \langle \mathrm{u}_1 + \mathrm{v}_1, \mathrm{u}_2 + \mathrm{v}_2, \mathrm{u}_3 + \mathrm{v}_3 \rangle$

Vector Addition	
Triangle law	Parallelogram law

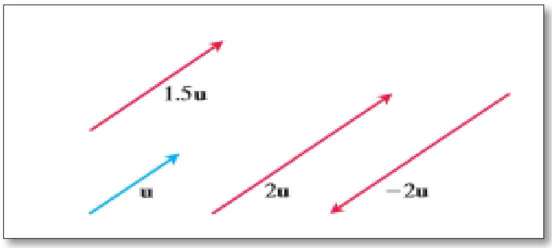


Vector subtraction	





Scalar multiplication: $ku = \langle k\mathrm{u}_1, k\mathrm{u}_2, \mathrm{ku}_3 \rangle$



- The length of ku is the absolute value of the scalar k times the length of u.
- The vector $(-1)u = -u$ has the same length as u but points in the opposite direction.
- The difference $u - v$ of two vectors is defined by

$$u - v = u + (-v)$$

Example 3

Let $u = \langle -1,3,1 \rangle$ and $v = \langle 4,7,0 \rangle$. Find the components of

a) $2u + 3v$ b) $u - v$ c) $\left|\frac{1}{2}u\right|$

Solution

a) $2u + 3v = 2\langle -1,3,1 \rangle + 3\langle 4,7,0 \rangle = \langle -2,6,2 \rangle + \langle 12,21,0 \rangle$

$= \langle -2 + 12, 6 + 21, 2 + 0 \rangle = \langle 10,27,2 \rangle$

b) $u - v = \langle -1,3,1 \rangle - \langle 4,7,0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle =$

$\langle -5, -4, 1 \rangle$

c) $\left|\frac{1}{2}u\right| = \left|\frac{1}{2}\langle -1,3,1 \rangle\right| = \left|\frac{-1}{2}, \frac{3}{2}, \frac{1}{2}\right| =$

$\sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$

When three or more space vectors lie in the same plane, we say they are **coplanar** vectors.

For example, the vectors u , v and $u + v$ are always coplanar

Properties of Vector Operations

Let u, v, w be vectors and a, b be scalars.

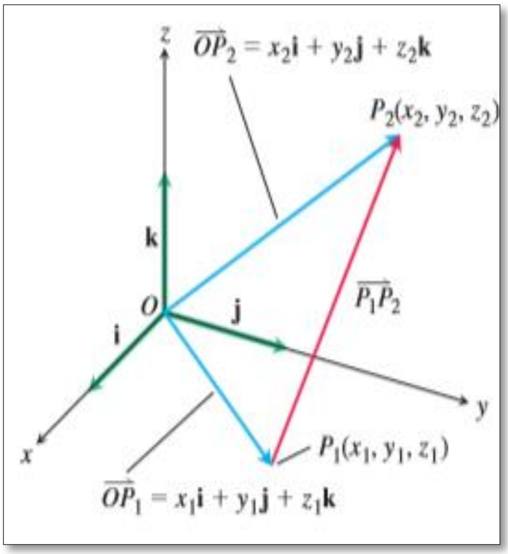
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

A vector v of length 1 is called a **unit vector**. The standard unit vectors are

$$\boldsymbol{i} = \langle 1,0,0 \rangle \qquad \boldsymbol{j} = \langle 0,1,0 \rangle \qquad \boldsymbol{k} = \langle 0,0,1 \rangle$$

Any vector $v = \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle$ can be written as a linear combination of the standard unit vectors as follows:



$$\begin{aligned} v &= \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle \\ &= \langle \mathrm{v}_1, 0,0 \rangle + \langle 0, \mathrm{v}_2, 0 \rangle \\ &\qquad + \langle 0,0, \mathrm{v}_3 \rangle \\ &= \mathrm{v}_1 \langle 1,0,0 \rangle + \mathrm{v}_2 \langle 0,1,0 \rangle \\ &\qquad + \mathrm{v}_3 \langle 0,0,1 \rangle \\ &= \mathrm{v}_1 \mathbf{i} + \mathrm{v}_2 \mathbf{j} + \mathrm{v}_3 \boldsymbol{k} \end{aligned}$$

In component form, the vector from $P_1 = \langle \mathrm{x}_1, \mathrm{y}_1, \mathrm{z}_1 \rangle$ to $P_2 = \langle x_2, y_2, z_2 \rangle$ is

$$\overrightarrow{P_1 P_2} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$$

Example 4

Find a unit vector u in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution

$$\overrightarrow{P_1 P_2} = (3-1)\boldsymbol{i} + (2-0)\boldsymbol{j} + (0-1)\boldsymbol{k} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

$$\left|\overrightarrow{P_1 P_2}\right| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{\mathbf{9}} = \mathbf{3}$$

The unit vector u in the direction of $\overrightarrow{P_1P_2}$.

$$u = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}}{3} = \frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} - \frac{1}{3}\boldsymbol{k}$$

Example 5

If $v = 3\boldsymbol{i} - 4\boldsymbol{j}$ is a velocity vector, express v as a product of its speed length and its direction of motion**.**

Solution

Speed is the length of v:

$$|v| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

The unit vector u is the direction of

$$\frac{v}{|v|} = \frac{3\boldsymbol{i} - 4\boldsymbol{j}}{5} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$$

So

$$v = 3\boldsymbol{i} - 4\boldsymbol{j} = 5\left(\frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}\right) = |v|\frac{v}{|v|}$$

Example 6

A force of 6 newton's is applied in the direction of the vector

$v = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$. Express the force F as a product of its magnitude and direction.

Solution

The force vector has magnitude 6 and direction $\frac{v}{|v|}$, so

$$F = 6\frac{v}{|v|} = 6\frac{2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}}{\sqrt{(2)^2 + (2)^2 + (-1)^2}}$$

$$= 6\frac{2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}}{3} = 6\left(\frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} - \frac{1}{3}\boldsymbol{k}\right)$$

Midpoint of a Line Segment

The midpoint M of the line segment joining points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 7

The midpoint of the segment joining $P_1 = (3, -2, 0)$ and $P_2 = (7, 4, 4)$ is……

Solution

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5,1,2)$$

Definition

The dot product $u.v$ (u dot v) of vectors $u = \langle \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3 \rangle$ and $v = \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle$ is the scalar

$$u.v = \mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2 + \mathrm{u}_3\mathrm{v}_3$$

Dot products are also called **inner** or **scalar** products because the product results in a scalar, not a vector.

Theorem 1(Angle between two vectors)

The angle θ between two nonzero vectors $u = \langle \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3 \rangle$ and

$v = \langle \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u.v}{|u||v|}\right)$$

Example 8

Find the angle between $u = \mathbf{i} - 2\mathbf{j} - 2\boldsymbol{k}$ **and** $\mathrm{v} = 6\mathbf{i} + 3\mathbf{j} + 2\boldsymbol{k}$.

Solution

$$u.v = (1)(6) + (-2)(3) + (-2)(2) = -4$$

$$|u| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = 3$$

$$|v| = \sqrt{(6)^2 + (3)^2 + (2)^2} = 7$$

$$\theta = \cos^{-1}\left(\frac{u.v}{|u||v|}\right) = \theta = \cos^{-1}\left(\frac{-4}{|3||7|}\right) \cong 100.98°$$

Definition

Vectors u and v are orthogonal if $u.v = 0$.

$i.j = i.k = j.k = 0$ i.e. i, j *and* k are orthogonal.

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

Example 9

Find the measures of the angles of the triangle whose vertices are $A = (0,0)$, $B = (3,5)$ and $C = (5,2)$

Solution

$\overrightarrow{CA} = \langle -5, -2 \rangle$ and $\overrightarrow{CB} = \langle -2,3 \rangle$

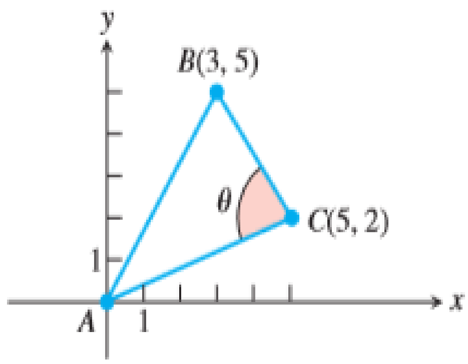
First we calculate the dot product and magnitudes of these two vectors

$$\overrightarrow{CA}.\overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA}.\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right)$$

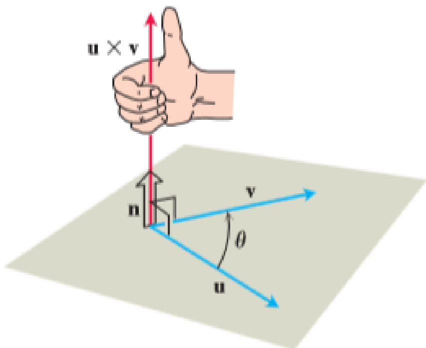


$$=\cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right) \cong 78.1°$$

Definition

The cross product $u \times v$ (u cross v) is the vector

$$u \times v = (|u||v| \sin\theta)\, n$$



Unlike the dot product, the cross product is a vector. For this reason it's also called the **vector product** of u and v, and applies only to vectors in space.

Definition

Nonzero vectors u and v are **parallel** if $u \times v = 0$.

Properties of the Cross Product

If **u**, **v**, and **w** are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

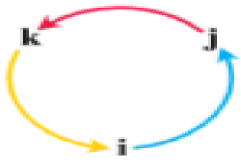


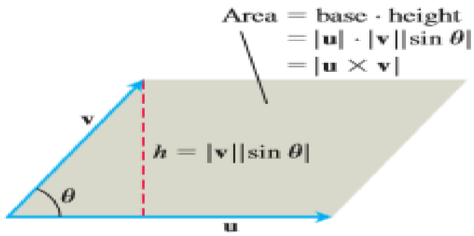
Diagram for recalling cross products

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

The Area of a Parallelogram

Because n is a unit vector, the magnitude of $u \times v$ is

$$|u \times v| = |u||v||\sin\theta||n| = |u||v|\sin\theta$$



Calculating the Cross Product as a Determinant

If $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\boldsymbol{k}$ and $v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\boldsymbol{k}$, then

$$u \times v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example 11

Find $u \times v$ and $v \times u$ if u $= 2\mathbf{i} + \mathbf{j} + \boldsymbol{k}$ and $v = -4\mathbf{i} + 3\mathbf{j} + \boldsymbol{k}$

Solution

- $u \times v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \boldsymbol{k}$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\boldsymbol{k}$$

- $v \times u = -(u \times v) = 2\mathbf{i} + 6\mathbf{j} - 10\boldsymbol{k}$

Example 12

If $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$, find

a) The vector perpendicular to the plane.
b) The area of the triangle.
c) The unit vector perpendicular to the plane.

Solution

$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\boldsymbol{k} = \mathbf{i} + 2\mathbf{j} - \boldsymbol{k}$

$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\boldsymbol{k} = \mathbf{-2i} + 2\mathbf{j} + 2\boldsymbol{k}$

a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \boldsymbol{k}$

$$= \mathbf{6i} + 6\mathbf{k}$$

b) The area of the parallelogram is

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\mathbf{6i} + 6\mathbf{k}| = \sqrt{(6)^2 + (6)^2} = 6\sqrt{2}$$

Such that the area of the triangle is **half** of the area of the parallelogram

∴The area of the triangle is equal $3\sqrt{2}$

c) Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction n is a unit vector perpendicular to the plane.

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{\mathbf{6i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{k}$$

Calculating the Triple Scalar Product as a Determinant

$$(u \times v).w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Exercises

Vectors in the Plane

1. Let $u = \langle 3, -2\rangle$ and $v = \langle -2,5\rangle$. Find the (a) component form and (b) magnitude (length) of the vector.
   i) $3u$
   ii) $u + v$
   iii) $-2v$
   iv) $2u - 3v$
2. Find the component form of the vector $\overrightarrow{PQ}$, where $P = (1,3)$ and $Q = (2, -1)$.

Vectors in space

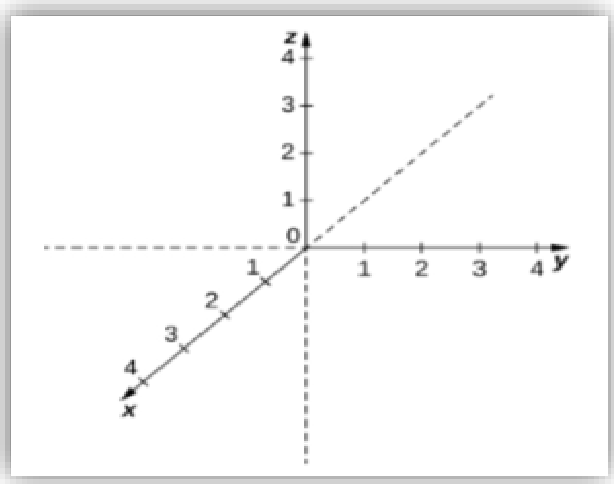
3. Express vector $\overrightarrow{P_1P_2}$ in the form $v_1\mathbf{i} + v_2\mathbf{j} + v_3\boldsymbol{k}$ if $P_1(5,7,-1)$ to $P_2(2,9,-2)$.
4. Find the length and direction of the vector
   i) $2\mathbf{i} + \mathbf{j} - 2\boldsymbol{k}$
   ii) $5\boldsymbol{k}$

5. Find the midpoint of the line segment $P_1 = (-1,1,5)$ and $P_2 = (2,5,0)$
6. Find the angle between the vectors $u = \mathbf{2i + j}$ **and** $\mathrm{v} = \mathbf{i} + 2\mathbf{j} - \boldsymbol{k}$.
7. Find the measures of the angles of the triangle whose vertices are $A = (-1,0)$, $B = (2,1)$ and $C = (1,-2)$
8. If $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$, find
   a) The vector perpendicular to the plane.
   b) The area of the triangle.
   c) The unit vector perpendicular to the plane.

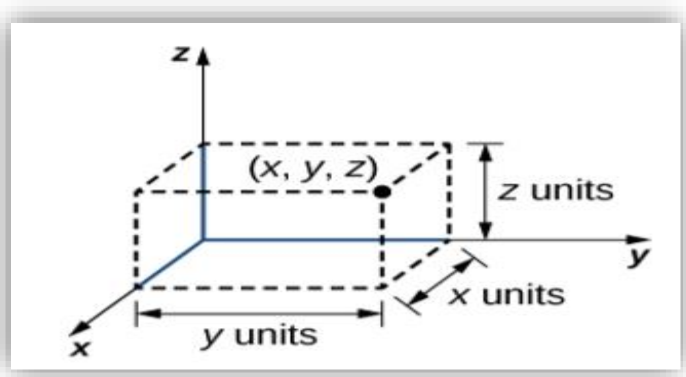
Three-Dimensional Coordinate Systems

Definition

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y -axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

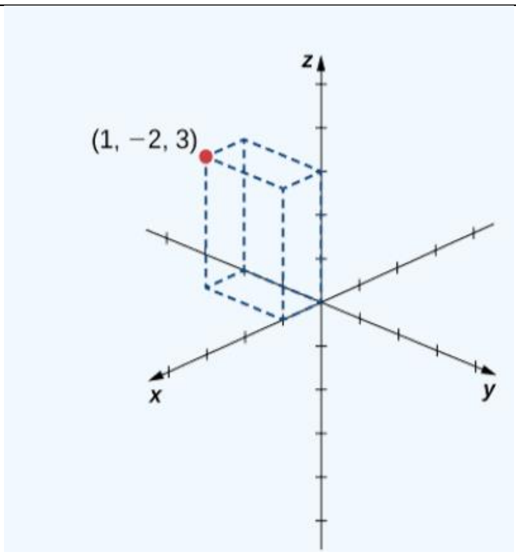
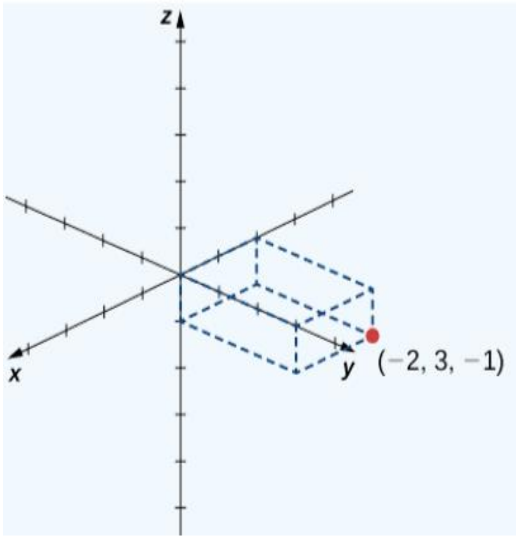


A point in space is identified by all three coordinates. To plot the point (x, y, z), go x units along the $x -$ axis, then y units in the direction of the y $-$axis, then z units in the direction of the z $-$axis.



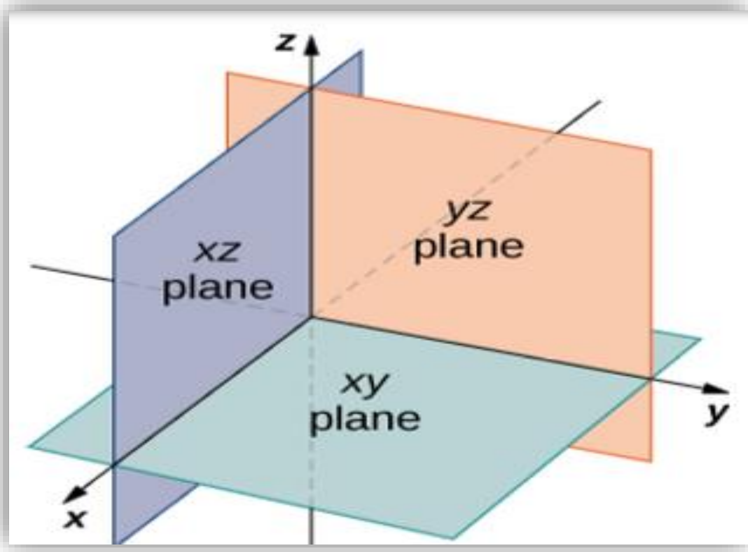
⌗ Example 1

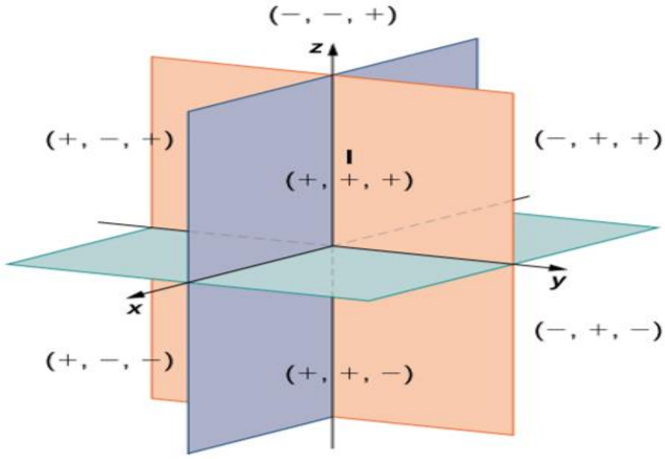
Sketch the following point in three-dimensional space.

$(1,-2,3)$	$(-2,3,-1)$
	

In two dimensions, the coordinate axes partition the plane into four quadrants.

Similarly, the coordinate planes divide space between them into eight regions about the origin, called octants.





Distance in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

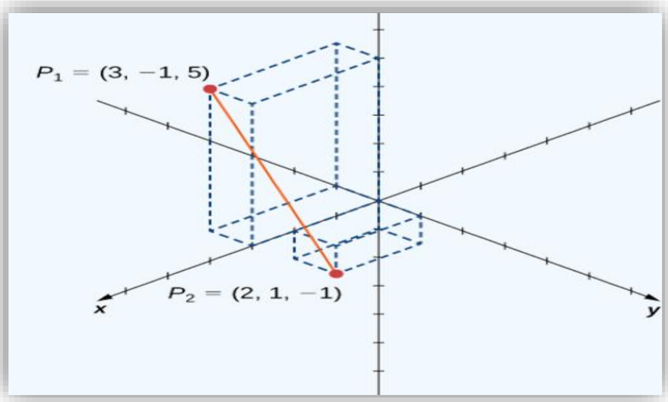
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ .$$

⌗ Example 2

Find the distance between points $P_1(3, -1, 5)$ and $P_2(2, 1, -1)$

Solution

$$d = \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} = \sqrt{41}$$

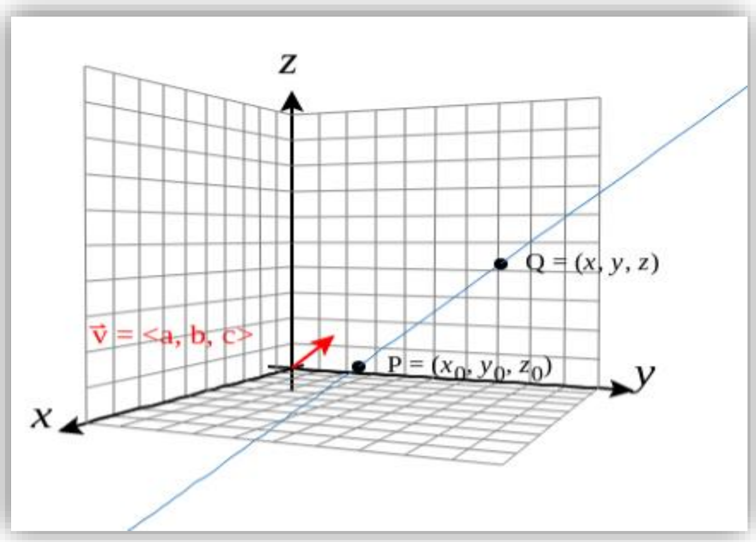


The Line

Equations for a Line in Space

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the direction vector. Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{v} = \langle a, b, c\rangle$ be a vector parallel to$L$.

Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{v}$. Thus, as we just discussed, there is a scalar t , such that $\overrightarrow{PQ} = t\overrightarrow{v}$, which give $\overrightarrow{PQ} = t\overrightarrow{v}$



$$\langle x - x_0, y - y_0, z - z_0\rangle = t\langle a, b, c\rangle$$

$$\langle x - x_0, y - y_0, z - z_0\rangle = \langle ta, tb, tc\rangle$$

$$\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle = t\langle a, b, c\rangle$$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

Then **the vector equation of a line**

$$\vec{r} = \overrightarrow{r_0} + t\,\vec{v}$$

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\}$$

This set of three equations forms a set of **parametric equations** of a line.

If we solve each of the equations for t assuming$a$, b and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$

$$\frac{y - y_0}{b} = t$$

$$\frac{z - z_0}{c} = t$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line**:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

⌗ Example 3

Find parametric and symmetric equations of the line passing through points $(1,4,-2)$ and $(-3,5,0)$.

Solution

$$\vec{v} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle$$

Use either of the given points on the line to complete the parametric equations

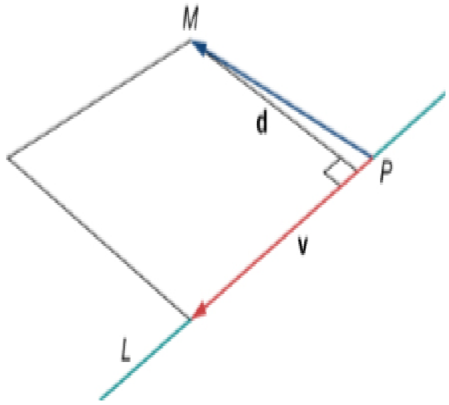
$x = 1 - 4t \;, y = 4 + t \;\;, \;\; z = -2 + 2t$

Solve each equation for t to create the symmetric equation of the line

$$\frac{x-1}{-4} = \frac{y-4}{1} = \frac{z+2}{2}$$

Distance between a Point and a Line

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L.



In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance.

Therefore, let P be an arbitrary point on line L and let $\vec{v}$ be a direction vector for

Vectors $\overrightarrow{PM}$ and $\vec{v}$ form two sides of a parallelogram with base $|\vec{v}|$ and height d , which is the distance between a line and a point in space.

- Vectors $\overrightarrow{PM}$ and $\vec{v}$ form two sides of a parallelogram with area $|\overrightarrow{PM} \times \vec{v}|$
- Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height.

$$|\overrightarrow{PM} \times \vec{v}| = |\vec{v}|\, d$$

$$\therefore d = \frac{|\overrightarrow{PM} \times \vec{v}|}{|\vec{v}|}$$

⌗ Example 4

Find the distance between point $M = (1,1,3)$ and line

$\frac{x-3}{4} = \frac{y+1}{2} = z - 3.$

Solution

From the symmetric equations of the line , we have
$\vec{v} = \langle 4,2,1 \rangle$ and $P = (3, -1,3)$ lies on the line.
$\therefore \overrightarrow{PM} = (1-3, 1-(-1), 3-3) = (-2,2,0)$

$$\overrightarrow{PM} \times \vec{v} = \begin{vmatrix} i & j & k \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} = 2i + 2j - 12k$$

$$d = \frac{|\overrightarrow{PM} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}} = \frac{2\sqrt{798}}{21}$$

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are **parallel** but not equal, or they **intersect** in a single point.

In three dimensions, a **fourth** case is possible. If two lines in space are not **parallel,** but do not **intersect**, then the lines are said to be **skew** lines.

⌗ **Example 5**

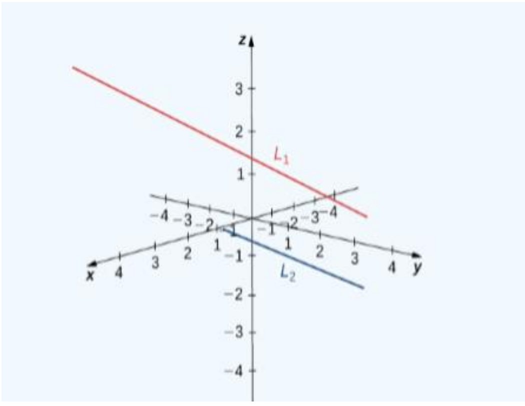
For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

1-Two lines are parallel

$L_1: x = 6s - 1, y = -2s, z = 3s + 1$

$L_2: \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

$$\overrightarrow{v_1} = \overrightarrow{v_2} = \langle 6, -2, 3 \rangle$$



2- Two lines are intersecting

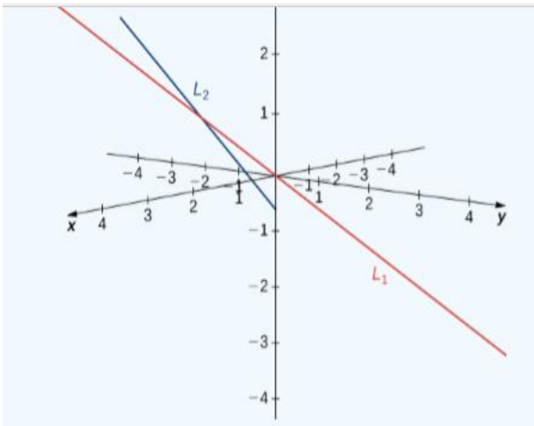
$L_1: x = -y = z$

$L_2: \dfrac{x-3}{2} = y = z - 2$

$$\overrightarrow{v_1} = \langle 1, -1, 1 \rangle$$

$$\overrightarrow{v_2} = \langle 2, 1, 1 \rangle$$

Two lines are **not parallel**; the lines are either intersecting or skew.



$\frac{x-3}{2} = y = z - 2 = s$ and $x = -y = z = t$

$$x = 2s + 3 \ , x = t$$

$$y = s \ , \qquad y = -t$$

$$z = s + 2 \ , z = t$$

Solve the system of equations $\rightarrow s = -1 \ \ and \ \ t = 1$

If we need to find the point of intersection we can substitute these parameters into the original equations to get $(1, -1, 1)$.

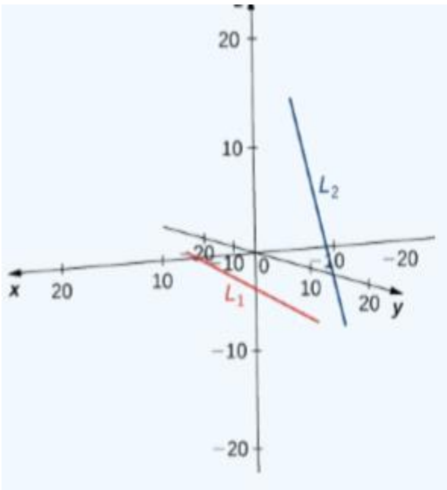
3- Two lines are skew

$L_1: x = 2s - 1, y = s - 1, z = s - 4$
$L_2: x = t - 3, y = 3t + 8, z = 5 - 2t$

$$\overrightarrow{v_1} = \langle 2,1,1 \rangle$$

$$\overrightarrow{v_2} = \langle 1,3,-2 \rangle$$

Because the direction vectors are **not parallel** vectors, the lines are either intersecting or skew.



To determine whether the lines intersect, we see if there is a point that lies on both lines.

$$2s - 1 = t - 3$$
$$s - 1 = 3t + 8$$
$$s - 4 = 5 - 2t$$

Solve the system of equations → $s = -3 \ \ or \ \ s = 1$

There is no single point that satisfies the parametric equations

These lines do not intersect, so they are skew

Skew lines

two lines that are not parallel but do not intersect

Exercises

1- Find the Vector, parametric and symmetric equations of line L passing through points

a. $P(4,0,5), Q(2,3,1)$
b. $P(4,0,5), Q(2,3,1)$
c. $P(1,-2,3), \vec{v} = \langle 1,2,3 \rangle$
d. $P(3,1,5), \vec{v} = QR \ where\ Q\ (2,2,3)\ and\ R(3,2,3)$

2- Find the distance from the origin to line **L**

$$x = 1 + t, y = 3 + t, z = 5 + 4t, t \in R$$

3- Find the distance between point A(−3,1,1) and the line of symmetric equations $x = -y = -z$

4- Find the vector form of the equation of the straight line passing through the point $A(2,5,5)$ and parallel to the straight line passing through the two points $B(-3,-2,-6)\ and\ C(5,0,-9)$.

5- For what value of a do the lines $\frac{x}{-5} = \frac{y-2}{-1} = \frac{z}{-2}\ and$ $\frac{x-1}{a} = \frac{y+2}{4} = \frac{z+1}{4}$ intersect?

6- Determine whether the lines are ***equal***, ***parallel but not equal***, ***skew***, or ***intersecting***.

a- $L_1: x = y - 1 = -z$

$L_2: x - 2 = -y = \frac{z}{2}$

b- $L_1: x = 2t, y = 0, z = 3$

$L_2: x = 0, y = 8 + s, z = 7 + s$

c- $L_1: x = -1 + 2t, y = 1 + 3t, z = 7t$

$L_2: x - 1 = \frac{2}{3}(y - 4) = \frac{2}{7}Z - 2$

The Plane

Equations for a Plane

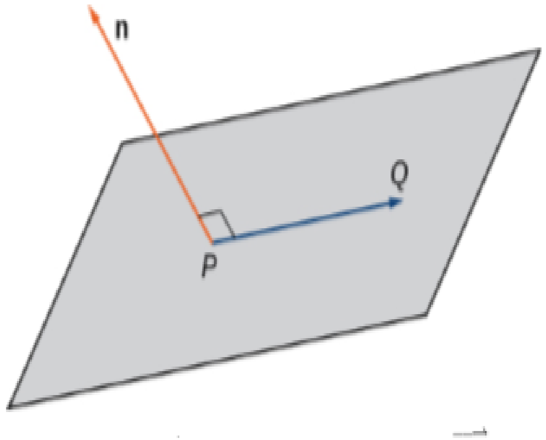
We know that a line is determined by two points.

In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three.

Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

Let $\vec{n} = \langle a, b, c \rangle$ be a vector and $P(x_0, y_0, z_0)$ be a point. Then the set of all points $Q(x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to forms a plane. We say that $\vec{n}$ is a **normal vector**, or perpendicular to the plane.

Remember, the dot product of orthogonal vectors is zero.



The vector equation of a plane:

$$\vec{n}.\overrightarrow{PQ} = 0$$

The scalar equation of a plane

$\langle a, b, c \rangle . \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

The general form of equation of a plane

$$ax + \mathrm{by} + \mathrm{cz} + \mathrm{d} = 0$$

Where $\mathrm{d} = -ax_0 - by_0 - cz_0$

⌗ Example 6

Write an equation for the plane containing points $P = (1,1,-2)$, $Q = (0,2,1)$ and $R(-1,-1,0)$ in both scalar and general forms.

Solution

$\overrightarrow{PQ} = (-1,1,3)$ **and** $\overrightarrow{PR} = (-2,-2,2)$

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 3 \\ -2 & -2 & 2 \end{vmatrix} = 8i - 4j + 4k$$

$8(x-1) - 4(y-1) + 4(z+2) = 0$

$8x + 4y + 4z + 4 = 0$

⌗ Example 7

Write an equation for the plane that passes through point $(1,4,3)$ and contains the line given by $x = \frac{y-1}{2} = z + 1$

Solution

Symmetric equations describe the line that passes through point $(0,1,-1)$ parallel to vector $\overrightarrow{v_1} = \langle 1,2,1 \rangle$

Use this point and the given point $(1,4,3)$, to identify a second vector parallel to the plane $\overrightarrow{v_2} = \langle 1,3,4 \rangle$

$$\vec{n} = \overrightarrow{v_1} \times \overrightarrow{v_2}$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} = 5i - 3j + k$$

The scalar equations for the plane are

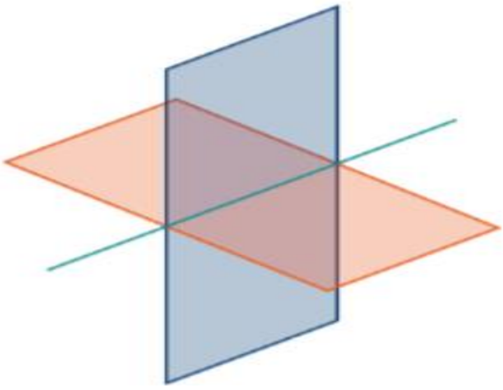
$5(x-0) - 3(y-1) + (z+1) = 0$

$5x - 3y + z + 4 = 0$

Parallel and Intersecting Planes

When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are **parallel** or they **intersect**.

When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line.



⌗ Example 8

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z$=0 and $2x - y + z = 0$

Solution

$$x + y + z = 0$$
$$2x - y + z = 0$$
$$\overline{}$$
$$3x + 2z = 0.$$

This gives us $x = -\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$x + y + z = 0$$
$$-\frac{2}{3}z + y + z = 0$$
$$y + \frac{1}{3}z = 0$$
$$y = -\frac{1}{3}z$$

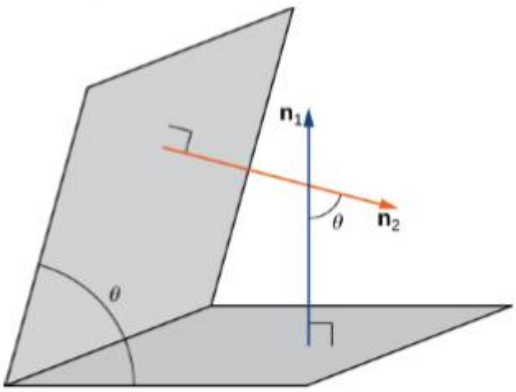
The parametric equations for the line of intersection, choose parameter , put $y = t \to z = -3t \to x = 2t$

The symmetric equations for the line

$$\frac{x}{2} = y = \frac{z}{-3}$$

The Angle between Two Planes

$$\cos\theta = \frac{|\overrightarrow{n_1}.\overrightarrow{n_2}|}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$



We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

⌗ Example 9

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them.

a) $x + 2y - z = 8$ and $2x + 4y - 2z = 10$

b) $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$

c) $x + y + z = 4$ and $x - 3y + 5z = 1$

Solution

a.
$$\overrightarrow{n_1} = \langle 1,2,-1\rangle$$
$$\overrightarrow{n_2} = \langle 2,4,-2\rangle = 2\langle 1,2,-1\rangle$$

These two vectors are scalar multiples of each other.

The normal vectors are parallel, so the planes are parallel.

b.
$$\overrightarrow{n_1} = \langle 2,-3,2\rangle$$
$$\overrightarrow{n_2} = \langle 6,2,-3\rangle$$

$$\overrightarrow{n_1}.\overrightarrow{n_2} = 0$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal.

c. $\overrightarrow{n_1} = \langle 1,1,1 \rangle$

$$\overrightarrow{n_2} = \langle 1,-3,5 \rangle$$

$$\overrightarrow{n_1}.\overrightarrow{n_2} = 3$$

$$\cos\theta = \frac{|\overrightarrow{n_1}.\overrightarrow{n_2}|}{|\overrightarrow{n_1}||\overrightarrow{n_2}|} = \frac{3}{\sqrt{3}\sqrt{35}}$$

$$\theta = \cos^{-1}\frac{3}{\sqrt{3}\sqrt{35}} \sim 73^{\circ}$$

Exercises

1- Find the general form of the equation of the plane that passes through $P(0,0,0)$ and has normal vector $\vec{n}=3\hat{\imath}-2\hat{\jmath}+4\hat{k}$

2- Find the general form of the equation of the plane that passes through $P(-3,2,-1)$ and has normal vector $\vec{n}=\hat{\imath}-2\hat{\jmath}+\hat{k}$

3- Find parametric equations of the line passing through point $P(-2,1,3)$ that is perpendicular to the plane of equation $2x-3y+z=7$.

4- Suppose the equations of two planes are given,

i) $x+y+z=0, \quad 2x-y+z-7=0$

ii) $5x-3y+z=4, x+4y+7z=15$

iii) $x-5y-z=1, 5x-25y-5z=-3$

a. Determine whether the planes are parallel, orthogonal, or neither.

b. If the planes are neither parallel nor orthogonal, then find the measure of the angle between the planes. Express the answer in degrees rounded to the nearest integer.

c. If the planes intersect, find the line of intersection of the planes, providing the parametric equations of this line.

The Circle

Circle is the set of all points in a plane that are equidistant from the center represents a circle.

In a circle, the distance from the center to a point on the sphere is called the **radius**.

The equation of a circle

The circle with center (a, b) and radius r can be represented by the equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Polar coordinates

⌑ Convert from Polar to Cartesian

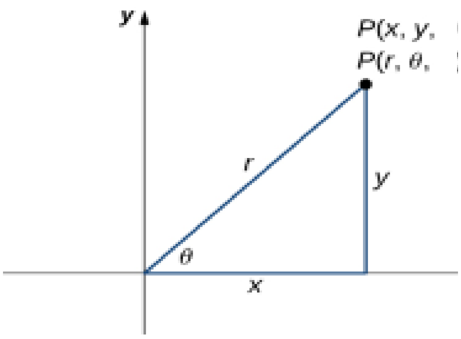
$x = r\cos\theta$

$y = r\sin\theta$

⌑ Convert from Cartesian to Polar

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$



⌑ Example 10

Change the Polar into Cartesian

1- $P\left(\sqrt{2}, \frac{5\pi}{4}\right)$

$$x = \sqrt{2}\cos(\frac{5\pi}{4}) = -1$$

$$y = \sqrt{2}\sin\left(\frac{5\pi}{4}\right) = -1$$

2- $P(-2, -2)$

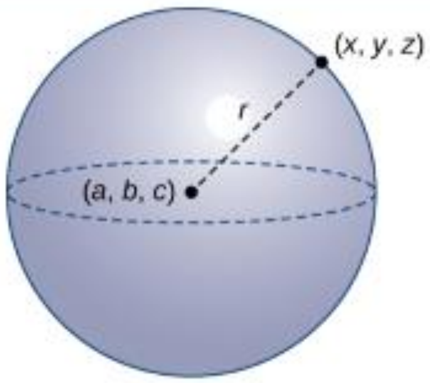
$$r^2 = (-2)^2 + (-2)^2 = 8 \rightarrow \quad r = 2\sqrt{2}$$

$$tan\,\theta = \frac{-2}{-2} = 1 \rightarrow \theta = \tan^{-1} 1 = \frac{5\pi}{4}$$

The Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere.

In a sphere, the distance from the center to a point on the sphere is called the **radius**.



Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

⌗ **Example 11**

Find the standard equation of the sphere with center $(10,7,4)$ and radius 5

Solution

$$(x - 10)^2 + (y - 7)^2 + (z - 4)^2 = 25$$

⌗ **Example 12**

Find the standard equation of the sphere with center $(1,3,4)$ and point $(-1,3,-2)$

Solution

$$r = \sqrt{(-1 - 1)^2 + (3 - 3)^2 + (-2 - 4)^2} = \sqrt{36} = 6$$

$$(x - 1)^2 + (y - 3)^2 + (z - 4)^2 = 36$$

Spherical coordinates

⌗ Convert from Spherical to Cartesian

$x = \rho\cos\theta\ \sin\varphi$

$y = \rho\sin\theta\sin\varphi$

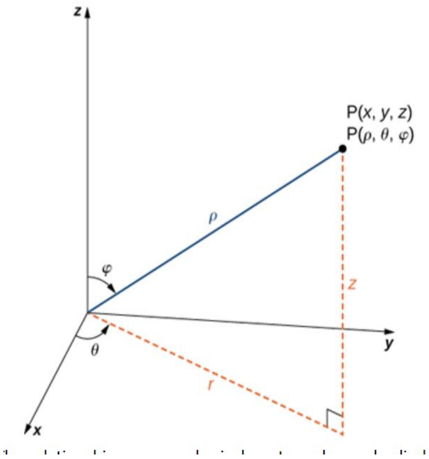
$z = \rho\cos\varphi$

⌗ Convert from Cartesian to Spherical

$\rho^2 = x^2 + y^2 + z^2$

$\tan\theta = \frac{y}{x} \Rightarrow\ \ \theta = \tan^{-1}\frac{y}{x}$

$\cos\varphi = \frac{z}{\rho} \Rightarrow \varphi = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$



$\boldsymbol{\rho}$ is the distance between point P and the origin $\boldsymbol{\rho \neq 0}$.

$\boldsymbol{\theta}$ is the same angle to describe cylindrical coordinates

$$0 \leq \theta \leq 2\pi$$

$\boldsymbol{\varphi}$ is the angle formed by the positive z -axis and line segment OP , where O is the origin and $0 \leq \varphi \leq \pi$

⌗ Example

Change the Spherical into Cartesian $P\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$

Solution

$x = \rho\cos\theta\ \sin\varphi = 8\cos\frac{\pi}{3}\ \sin\frac{\pi}{6} = 2$

$y = \rho\sin\theta\sin\varphi = 8\sin\frac{\pi}{3}\sin\frac{\pi}{6} = 2\sqrt{3}$

$z = \rho\cos\varphi = 8\cos\frac{\pi}{6}\ = 4\sqrt{3}$

Example

Change the equation to Spherical Coordinates $x^2 + y^2 + z^2 = 4$

Solution

$$(\rho\cos\theta\ \sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2 + (\rho\cos\varphi)^2 = 4$$

$$\rho^2 = 4$$

Cylindrical coordinates

⌗ Convert from Cylindrical to Cartesian

$x = r\cos\theta$

$y = r\sin\theta$

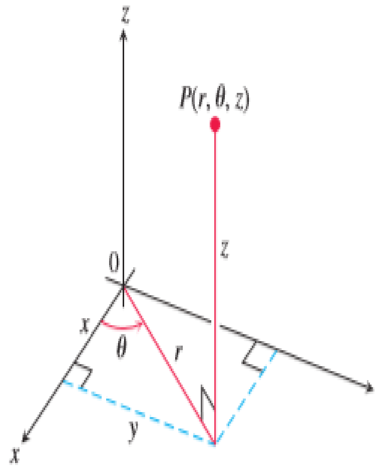
$z = z$

⌗ Convert from Cartesian to Cylindrical

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x} \Rightarrow \quad \theta = \tan^{-1}\frac{y}{x}$

$z = z$



⌗ **Example**

Change the Cylindrical into Cartesian $P\left(4, \frac{2\pi}{3}, -2\right)$

Solution

$x = 4\cos(\frac{2\pi}{3}) = -2$

$y = 4\sin\left(\frac{2\pi}{3}\right) = 2\sqrt{3}$

$z = -2$

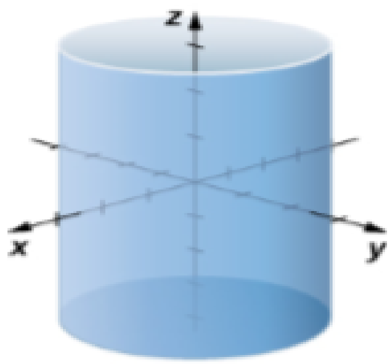
⌗ **Example**

Change the equation to Cylindrical Coordinates

$$x^2 + y^2 + z^2 = 4$$

Solution

$r^2 + z^2 = 4$



Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

$$\cos^2\theta + \sin^2\theta = 1.$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Even

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x$$

Odd

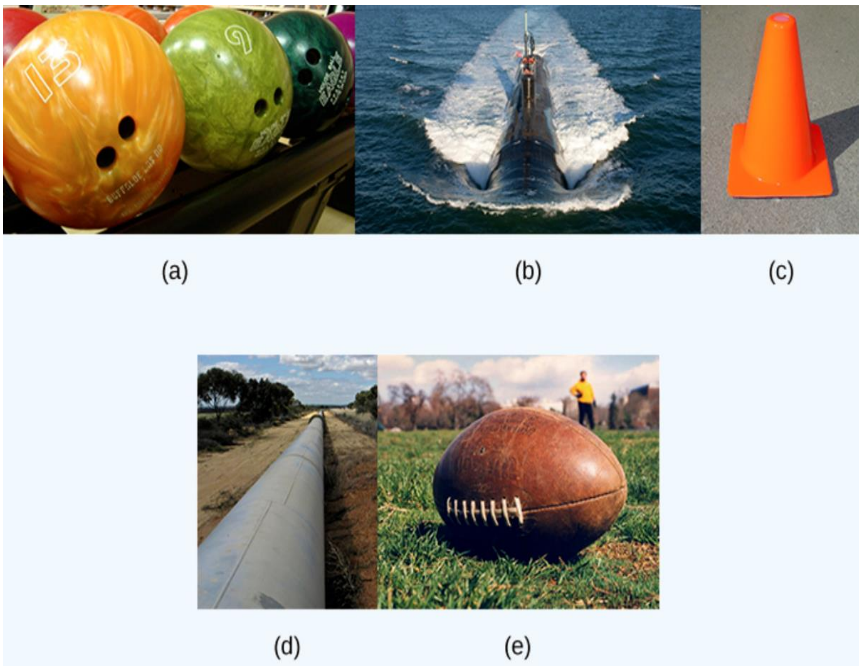
$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\cot(-x) = -\cot x$$

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

⌗ Example

In each of the following situations, we determine which coordinate system is most appropriate

a) Find the center of gravity of a bowling ball. spherical
b) Determine the velocity of a submarine subjected to an ocean current. rectangular
c) Calculate the pressure in a conical water tank. cylindrical
d) Find the volume of oil flowing through a pipeline. cylindrical
e) Determine the amount of leather required to make a football. cylindrical



Exercises

For problems 1 & 2 convert the Cartesian coordinates for the point into Cylindrical coordinates.

1. $(-3, 5, -8)$

2. $(4, 1, 7)$

3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates. $\dfrac{x-y}{x^2+y^2+1} = xyz$

For problems 4 – 6 convert the equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

4. $zr^3\cos(\theta) = 4r + 8$

5. $r^2 - 3\sin(\theta) = z^3 + \sqrt{r^2+1}$

6. $\tan(\theta) + 2z = 1 - r^2$

For problems 7 – 9 identify the surface generated by the given equation.

7. $z = -4r, \quad z < 0$

8. $2r + 6\cos(\theta) + 18\sin(\theta) = \dfrac{51}{r}$

For problems 1 – 3 convert the Cartesian coordinates for the point into Spherical coordinates.

1. $(6, 2, -8)$

2. $(-1, 5, 2)$

3. $(-3, -2, 1)$

4. Convert the Cylindrical coordinates for the point $(5, 1.294, 6)$ into Spherical coordinates.

5. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

$$\frac{xz}{y} = 2 - x$$

For problems 6 – 8 convert the equation written in Spherical coordinates into an equation in Cartesian coordinates

6. $\rho \cos \varphi \sin \varphi \sin \theta = 3$

7. $\rho - \cos \varphi = 2 + \cos^2 \varphi$

8. $\tan \varphi \left(\cos \theta - \sin \theta\right) = 4$

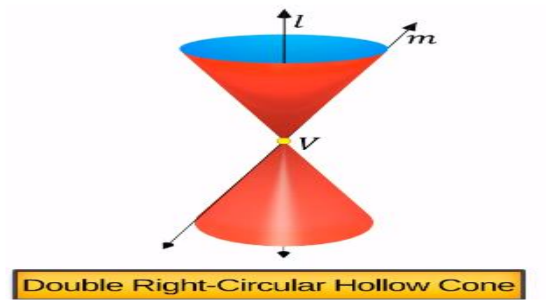
For problems 9 & 10 identify the surface generated by the given equation.

9. $\cos^2 \varphi - \sin^2 \varphi = 0$

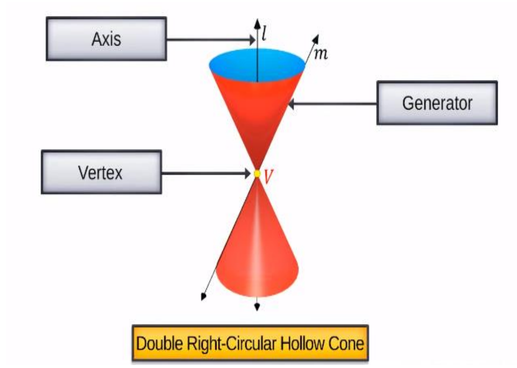
10. $\sin \varphi cos\theta + sin\varphi sin\theta + \cos \varphi = \dfrac{1}{\rho}$

Conic Sections

If a straight line intersects a fixed vertical line at a fixed point and rotates such that the angle between the two lines remains constant, then the resulting surface is called a double right-circular cone.

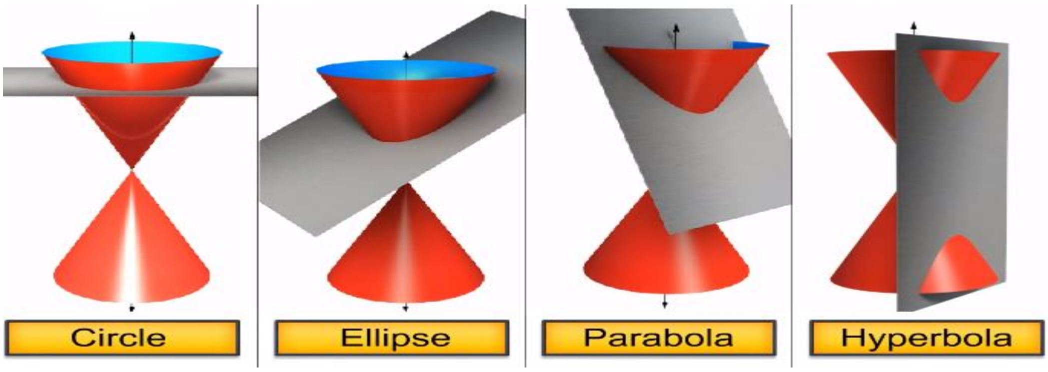


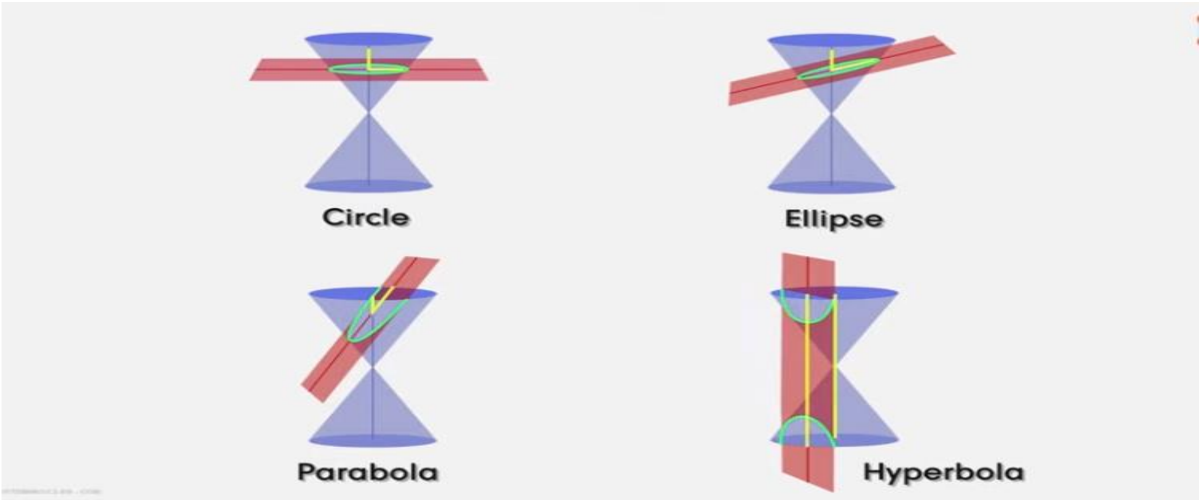
The parts of a cone include a vertical axis, a generator line, a vertex and an upper nappe and a lower nappe.



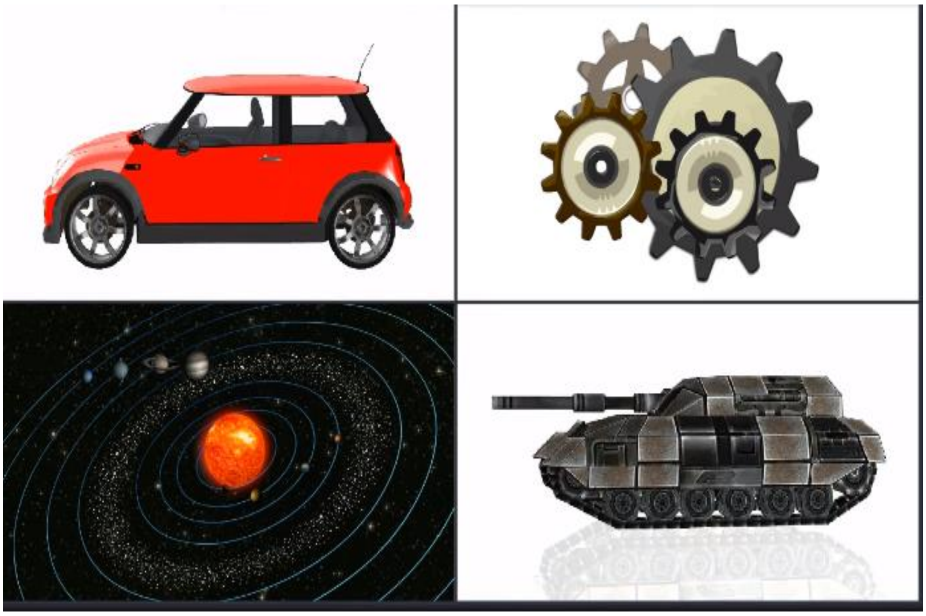
When a plane intersects a cone, it cuts a section from the cone. This section is called a conic section or a conic.

Depending on the position of the intersecting plane and the angle it makes with the axis, a conic section can be a circle, an ellipse, a parabola or a hyperbola.





Applications



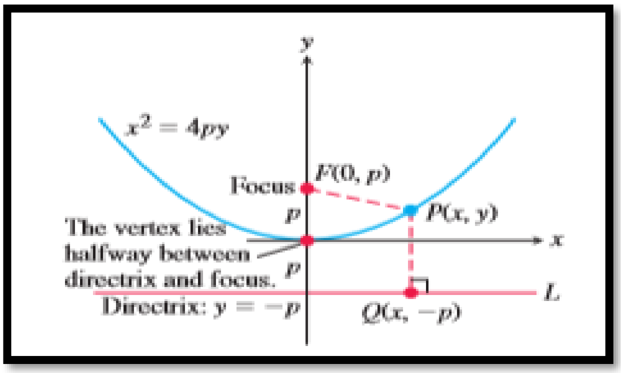
Parabolas

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix.**

The equation of the parabola

y-axis parabola

$$\boldsymbol{x^2 = 4py} \qquad \textbf{(1)}$$

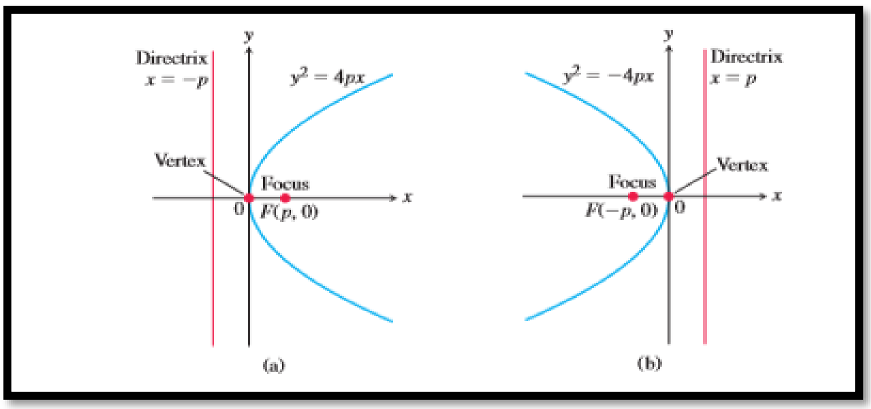


* If the parabola opens **downward**, with its **focus** at $(0, -p)$ and its **directrix** the line $y = p$, then Equations (1) become

$$\boldsymbol{x^2 = -4py}$$

x-axis parabola

$$\boldsymbol{y^2 = \pm 4px}$$



Example 1

Find the focus and directrix of the parabola $\boldsymbol{y^2 = 12x}$.

Solution

since it's $x-$ $axis$ parabola $4p = 12 \rightarrow p = 3$,

focus $(p, 0) = (3,0)$

directrix $x = -3$

Example 2

Find the vertex, focus and directrix of the parabola

$\boldsymbol{x^2 + 12y = 0}$

Solution

$$\therefore \quad \boldsymbol{x^2 = -12y}$$

since it's $y-$ $axis$ parabola $4p = 12 \rightarrow p = 3$,

focus $(0, -p) = (0, -3)$

directrix $y = 3$

Try to Solve

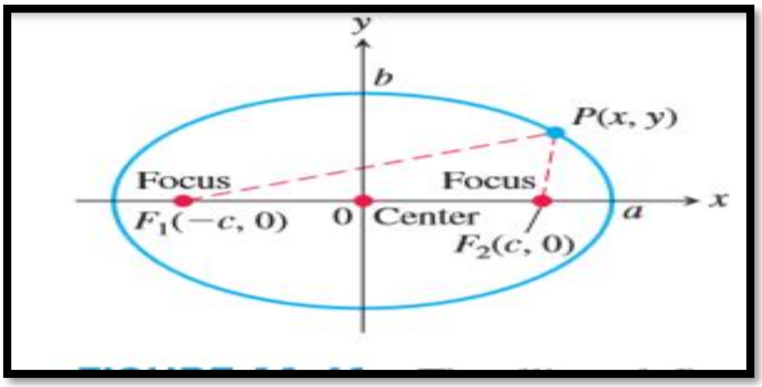
Find each parabola's vertex, focus and directrix

1. $\boldsymbol{y^2 = -12x}$
2. $\boldsymbol{x^2 = 6y}$
3. $\boldsymbol{y^2 = -2x}$
4. $\boldsymbol{4x^2 = y}$
5. $\boldsymbol{x = -3y^2}$
6. $\boldsymbol{x^2 = -8y}$

Ellipses

An ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices**



$$PF_1 + PF_2 = 2a$$

The equation of the ellipse

$x - axis\ ellipse$

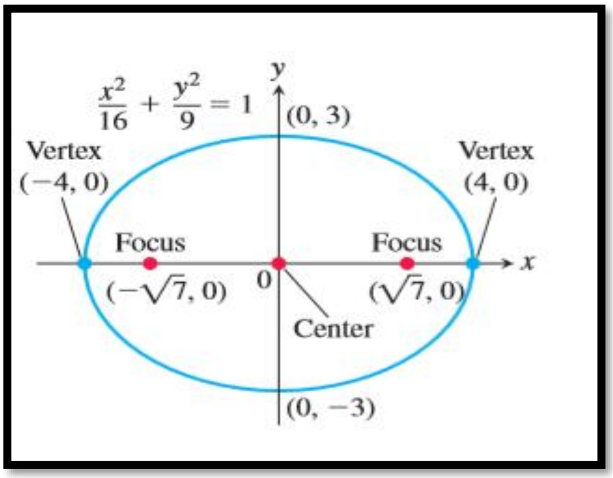
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \mathrm{a} > \mathrm{b}$$

- The **major axis** of the ellipse is the line segment of length $2a$
- The **minor axis** of the ellipse is the line segment of length$2b$
- The number a itself is the **semi major axis**
- The number b the **semi minor axis**.
- Vertices: $(\pm a, 0)$
- Foci: $(\pm ae, 0)$
- $e = \sqrt{1 - \frac{b^2}{a^2}} \quad , \ e < 1$

Example 3

Find the foci and vertices of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution



- Semi major axis: $a = \sqrt{16} = 4$
- Semi mino axis: $b = \sqrt{9} = 3$

since $a > b \rightarrow x - ellipse$

- $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$
- Foci $(\pm ae, 0) = \left(\pm \frac{\sqrt{7}}{4}.4,0\right) = (\pm\sqrt{7}, 0)$
- Vertices $(\pm a, 0) = (\pm 4,0)$
- center (0,0)

$y - axis\ ellipse$

$$\frac{\boldsymbol{y^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{x^2}}{\boldsymbol{b^2}} = \mathbf{1}, \text{a} > \text{b}$$

- The number a itself is the **semi major axis**
- The number b the **semi minor axis**.
- Vertices: $(0, \pm a)$
- Foci: $(0, \pm ae)$
- $e = \sqrt{1 - \frac{\boldsymbol{b^2}}{\boldsymbol{a^2}}}$, $e < 1$

Example 4

Find the foci and vertices of the ellipse $\frac{\boldsymbol{x^2}}{\mathbf{9}} + \frac{\boldsymbol{y^2}}{\mathbf{16}} = \mathbf{1}$

Solution

- Semi major axis**:** $a = \sqrt{16} = 4$
- Semi mino axis**:** $b = \sqrt{9} = 3$

since $a > b \to y - ellipse$

- $e = \sqrt{1 - \frac{\boldsymbol{b^2}}{\boldsymbol{a^2}}} = \sqrt{1 - \frac{\mathbf{9}}{\mathbf{16}}} = \frac{\sqrt{7}}{4}$
- Foci $(0, \pm ae) = \left(0, \pm \frac{\sqrt{7}}{4}.4\right) = (0, \pm\sqrt{7})$
- Vertices $(0, \pm a) = (0, \pm 4)$
- center (0,0)

Try to Solve

Find each ellipse's foci and vertices

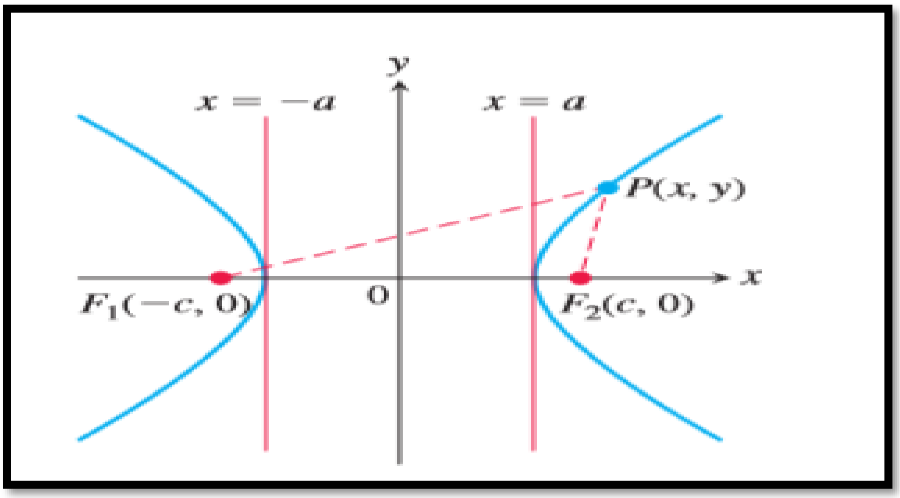
1- $\frac{\boldsymbol{x^2}}{\mathbf{2}} + \boldsymbol{y^2} = \mathbf{1}$

2- $\boldsymbol{x^2} + \mathbf{4}\ \boldsymbol{y^2} = \mathbf{16}$

Hyperbola

A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices**



$$PF_2 - PF_1 = 2a$$

The equation of the hyperbola

x − axis hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- The **major axis** is the line segment of length $2a$
- The **minor axis** is the line segment of length$2b$
- The number a itself is the **semi major axis**
- The number b the **semi minor axis**.
- **Vertices**: $(\pm a, 0)$
- **Foci**: $(\pm ae, 0)$

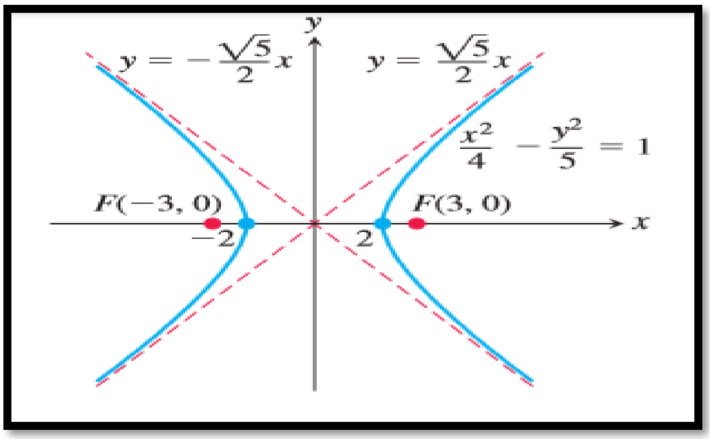
- $e = \sqrt{1 + \frac{b^2}{a^2}}$, $e > 1$
- **asymptotes** (خطوط التقارب) $y = \pm \frac{b}{a} x$
- The fastest way to find the equations of the asymptotes is to replace the 1 in Equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \mathbf{1}$ by 0

  i.e. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \rightarrow y = \pm \frac{b}{a} x$

Example 5

Find the foci and vertices of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = \mathbf{1}$

Solution



- Semi major axis**:** $a = \sqrt{4} = 2$
- Semi mino axis**:** $b = \sqrt{5}$

since $x -$ hyperbola

- $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{5}{4}} = \frac{3}{2} > 1$
- Foci $(\pm ae, 0) = \left(\pm \frac{3}{2} * 2,0\right) = (\pm 3,0)$
- Vertices $(\pm a, 0) = (\pm 2,0)$
- asymptotes $y = \pm \frac{\sqrt{5}}{2} x$

$y-axis\ hyperbola$

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

- The number a itself is the **semi major axis**
- The number b the **semi minor axis**.
- Vertices: $(0, \pm a)$
- Foci: $(0, \pm ae)$
- $e=\sqrt{1+\frac{b^2}{a^2}}$, $e>1$
- asymptotes $y=\pm\frac{a}{b}x$

Example 6

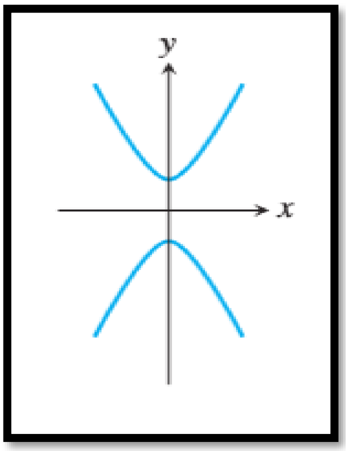
Find the foci and vertices of the hyperbola $\frac{y^2}{9}-\frac{x^2}{16}=1$

Solution

- Semi major axis**:** $a=\sqrt{9}=3$
- Semi mino axis**:** $b=\sqrt{16}=4$

since $y-$ hyperbola

- $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$
- Foci $(0, \pm ae)=(0, \pm 5)$
- Vertices $(0, \pm a)=(0, \pm 3)$
- asymptotes $y=\pm\frac{3}{4}x$



Try to Solve

Find each ellipse's foci and vertices

3- $\frac{x^2}{4}-\frac{y^2}{9}=1$ **4-** $\frac{y^2}{4}-x^2=1$

DIFFERENTIAL EQUATIONS

Differential Equations

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve a first order - first degree differential equation and some applications of differential equations in different areas.

An equation involving derivative (derivatives) of the dependent variable (variables) with respect to independent variable (variables) is called a **differential equation**.

A differential equation involving derivatives of the dependent variable with respect to **only** one independent variable is called an **ordinary differential equation (ODE).**

$$x\frac{dy}{dx}+y=0$$

Order and degree of a differential equation

Order of a differential equation is defined as the order of **the highest order derivative** of the dependent variable with respect to the independent variable involved in the given differential equation.

Degree is **the power of the highest order derivative** involved in the given differential equation.

Consider the following differential equations:

$\frac{dy}{dx}=e^x$ first order & first degree

$\frac{d^2y}{dx^2}+y=0$ second order & first degree

$\frac{d^3y}{dx^3}+x^2\left(\frac{d^2y}{dx^2}\right)^3=0$ third order &first degree

$y''' + y^2 + e^{y'} = 0$ third order & its degree is not defined.

Denote

$$\frac{dy}{dx} = y' \ \& \ \frac{d^2y}{dx^2} = y'' \& \ \frac{d^3y}{dx^3} = y''' \text{ and so on……}$$

Try to solve

Determine order and degree (if defined) of differential equations given in Exercises 1 to 10.

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$
2. $y' + 5y = 0$
3. $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$
4. $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$
5. $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$
6. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$
7. $y''' + 2y'' + y' = 0$
8. $y' + y = e^x$
9. $y'' + (y')^2 + 2y = 0$
10. $y'' + 2y' + \sin y = 0$
11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

(A) 3 (B) 2 (C) 1 (D) not defined

12. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is}$$

(A) 2 (B) 1 (C) 0 (D) not defined

General and Particular Solutions of a Differential Equation

Example 1

Verify $y = e^{-3x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution

since $y = e^{-3x} \Rightarrow y' = -3e^{-3x} \Rightarrow y'' = 9e^{-3x}$

Substituting in DE

L.H.S= $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 9e^{-3x} - 3e^{-3x} - 6(e^{-3x})$
$= 0$=R.H.S

Therefore, the given function is a general solution of the given differential equation.

Example 2

Verify $y = a\cos x + b\sin x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

Solution

since $y = a\cos x + b\sin x \Longrightarrow$

$$y' = -\text{a}\sin x + bcos\ x$$

$$y'' = -a\cos x - b\sin x$$

Substituting in DE

L.H.S=$\frac{d^2y}{dx^2} + y = -a\cos x - b\sin x + (a\cos x + b\sin x) =$ 0=R.H.S

Formation of a Differential Equation whose General Solution is given

Suppose $x^2 + y^2 + 2x - 4y + 4 = 0$, represents a circle having center at (–1, 2) and radius 1 unit.

Differentiating equation of circle with respect to x, we get

$\frac{dy}{dx} = \frac{x+1}{2-y}$, which is a differential equation (DE).

Example 3

Form the differential equation representing the family of curves $y = mx$, where, m is arbitrary constant.

Solution

$y = mx$, differentiating both sides of equation with respect to x, we get

$$\frac{dy}{dx} = m$$

Substituting the value of m in equation, we get

$$y = \frac{dy}{dx} . x$$

which is free from the parameter m and hence this is the required DE.

Example 4

Form the differential equation representing the family of curves $y = a \sin x + b \cos x$, where, a, b are arbitrary constants.

Solution

$y = a \sin x + b \cos x$,

Differentiating both sides of equation with respect to x, we get

$y' = \text{a}\cos x - bsin\ x$

$y'' = -a \sin x - b \cos x$

$y'' = -(a \sin x + b \cos x)$

Eliminating a and b from equations $y'' = -y$

Which is free from the arbitrary constants a and b and hence this required differential equation.

Try to solve

Form the differential equation representing the family of curves

1- $y = Ax + Bx^2$

2- y =A e^{2x}

Methods of Solving First Order, First Degree Differential Equations

1. Separation of Variables

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \qquad (1)$$

If $F(x, y)$ can be expressed as a product $g(x)\ h(y)$, where, $g\ (x)$ is a function of x and $h(y)$ is a function of y, then the differential equation (1) is said to be of variable separable type.

The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y).\, g(x) \qquad (2)$$

If $h(y) \neq 0$, separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)} dy = g(x)dx \qquad (3)$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)} dy = \int g(x)dx \qquad (4)$$

Thus, (4) provides the solutions of given differential equation in the form

$$H(y) = G(x) + c$$

Example 1

Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}$

Solution

Separating the variables $\quad (2-y)dy = (1+x)dx$

Integrating both sides $\; 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + c_1$

or $\quad x^2 + y^2 + 2x - 4y + c = 0, \;\; c = 2c_1$

which is the general solution of equation.

Example 2

Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Solution

Separating the variables $\quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides $\; \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$$\tan^{-1} y = \tan^{-1} x + c$$

which is the general solution of equation.

Example 3

Find the general solution of the differential equation

$\frac{dy}{dx} = -4xy^2$ given that y = 1, when x = 0.

Solution

Separating the variables $\quad \frac{dy}{y^2} = -4x\,dx$

Integrating both sides $\int \frac{dy}{y^2} = \int -4x\,dx$

$$\frac{-1}{y} = -2x^2 + c$$

or $$y = \frac{1}{2x^2 - c}$$

Substituting y = 1 and x = 0 in equation, we get $c = -1$

we get the particular solution of the given differential equation as $y = \frac{1}{2x^2+1}$

Example 4

Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\,dy = (2x^2 + 1)\mathrm{dx}$

Solution

Separating the variables $dy = \frac{(2x^2+1)}{x}\,dx$

Integrating both sides $\int dy = \int (2x + \frac{1}{x})\,dx$

$$y = (x^2 + \ln|x| + c)$$

at the point (1, 1) $1 = 1 + \ln 1 + c \Rightarrow c = 0$

$$\therefore y = (x^2 + \ln|x|)$$

Try to solve

1- $y' = \frac{1+\cos x}{1-\cos y}$

2- $x^5 \frac{dy}{dx} = -y^5$

3- $e^y dx = x^2 y dy$

4- $y' = y \sec x$

5- $y' = 1 - x + y - xy$

2. Homogeneous Differential Equations

The following function in x and y , $f(x,y)$ is a homogeneous function if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

A differential equation of the form $\frac{dy}{dx} = f(x,y)$is said to be *homogenous* if $f(x,y)$is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = f(x,y) \quad \ldots(1)$$

We make the substitution $y = v.x$ .....(2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x.\frac{dv}{dx} \quad \ldots.(3)$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get $v + x.\frac{dv}{dx} = g(v)$ or $x.\frac{dv}{dx} = g(v) - v$ ....(4)

Separating the variables in equation (4), we get

$$\frac{dv}{(g(v)-v)} = \frac{dx}{x} \quad \ldots..(5)$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{(g(v)-v)} = \int \frac{dx}{x} + C \quad \ldots(6)$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by y/x

Example 5

Solve the following differential equation $(x-y)\frac{dy}{\mathrm{dx}} = \mathrm{y}$

Solution $\frac{dy}{\mathrm{dx}} = \frac{\mathrm{y}}{x-y}$ Equation is homogeneous

$$y = v.x \quad \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx}$$

$$(v + x.\frac{dv}{dx}) = \frac{v.x}{(x-v.x\quad)}$$

$$v + x.\frac{dv}{dx} = \frac{v.x}{x(1-v\quad)}$$

$$x.\frac{dv}{dx} = \frac{v}{(1-v\quad)} - v = \frac{v-v+v^2}{(1-v\quad)}$$

$$x.\frac{dv}{dx} = \frac{v^2}{(1-v\quad)}$$ separate variables

$$\frac{(1-v)dv}{v^2} = \frac{dx}{x}$$

$$\int \frac{(1-v)dv}{v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{v} - \ln|v| = \ln|x| + c$$

put $v = \frac{y}{x} \Longrightarrow -\frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$

Example 6

Solve the following differential equation $(x - y)\frac{dy}{\mathrm{dx}} = \mathrm{x} + \mathrm{y}$

Solution

$$\frac{dy}{\mathrm{dx}} = \frac{\mathrm{x+y}}{x-y}$$

Let $f(x, y) = \frac{\mathrm{x+y}}{x-y}$

$$f(\lambda x, \lambda y) = \frac{\lambda\mathrm{x}+\lambda\mathrm{y}}{\lambda x-\lambda y} = \frac{\lambda(\mathrm{x+y})}{\lambda(x-y)} = \lambda^0 f(x, y)$$

Therefore, $f(x, y)$ is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

$$y = v.x \quad \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx}$$

$$\frac{dy}{\mathrm{dx}} = \frac{\mathrm{x+y}}{x-y}$$

$$\Rightarrow v + x.\frac{dv}{dx} = \frac{\mathrm{x}+v\mathrm{x}}{x-vx} = \frac{\mathrm{x}(1+v)}{x(1-v)}$$

$$\Rightarrow v + x.\frac{dv}{dx} = \frac{(1+v)}{(1-v)}$$

$$x.\frac{dv}{dx} = \frac{(1+v)}{(1-v)} - v$$

$$x.\frac{dv}{dx} = \frac{1+v-v+v^2}{(1-v)}$$ Separate variables

$$\frac{(1-v)dv}{(1+v^2)} = \frac{dx}{x}$$

$$\int \frac{(1-v)dv}{(1+v^2)} = \int \frac{dx}{x}$$

$$\int \frac{dv}{(1+v^2)} - \int \frac{vdv}{(1+v^2)} = \int \frac{dx}{x}$$

$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln x + \ln c$$

$$\tan^{-1} v = \ln cx\sqrt{(1+v^2)}$$

put $v = \frac{y}{x} \Longrightarrow \tan^{-1}\frac{y}{x} = \ln c\sqrt{(x^2+y^2)}$

Example 7

Solve the following differential equation

$(x^3 + y^3)\mathrm{dx} - 3\mathrm{x}y^2 dy = 0$

Solution

$(x^3 + y^3)\mathrm{dx} = 3\mathrm{x}y^2 dy$

$\frac{dy}{dx} = \frac{(x^3+y^3)}{3\mathrm{x}y^2}$ Equation is homogeneous

Put $y = v.x \quad \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx}$

$$v + x.\frac{dv}{dx} = \frac{x^3+(vx)^3}{3\text{x}(xv)^2} = \frac{x^3(1+v^3)}{3x^3v^2}$$

$$x.\frac{dv}{dx} = \frac{(1+v)}{3v^2} - v = \frac{(1+v^3)-3v^3}{3v^2}$$

$x.\frac{dv}{dx} = \frac{1-2v^3}{3v^2}$ Separate variables

$$\frac{3v^2dv}{1-2v^3} = \frac{dx}{x}$$

$$\int \frac{3v^2dv}{1-2v^3} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln(|1-2v^3|) = \ln|x| + \ln c$$

$$\ln(1-2v^3)^{-1/2} = \ln cx$$

$$(1-2v^3)^{-1/2} = cx$$

put $v = \frac{y}{x} \Longrightarrow (1-2(\frac{y}{x})^3)^{-1/2} = cx$

Try to solve

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$
2. $y' = \frac{x+y}{x}$
3. $(x - y)\, dy - (x + y)\, dx = 0$
4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$
5. $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$
6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

3. Exact Differential Equations

The ordinary differential equation $P(x,y)dx + Q(x,y)dy = 0$ is Exact if

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

The general solution is

$$\int P(x,y)dx + \int Q(x,y)dy = c$$

Example 8

Solve the following differential equation

$$(2x^3 + 3\mathrm{y})\mathrm{dx} + (3\mathrm{x} + \mathrm{y} - 1)dy = 0$$

Solution

$P(x,y) = 2x^3 + 3\mathrm{y}$ **and** $Q(x,y) = 3\mathrm{x} + \mathrm{y} - 1$

$$\frac{\partial P}{\partial y} = 3 \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = 3$$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ the differential equation is exact

$\int P(x,y)dx = \frac{1}{2}x^4 + 3xy$

$\int Q(x,y)dy = 3xy + \frac{1}{2}y^2 - y$

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The general solution is

$$\frac{1}{2}x^4 + 3xy + \frac{1}{2}y^2 - y = c$$

Example 9

Solve the following differential equation

$$\left(2x\, e^{x^2}\mathrm{y} + 3x^2\right)\mathrm{dx} + (e^{x^2} - 2y)dy = 0$$

Solution

$P(x,y) = 2x\, e^{x^2}\mathrm{y} + 3x^2$ **and** $Q(x,y) = e^{x^2} - 2y$

$$\frac{\partial P}{\partial y} = 2x\, e^{x^2} \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2x\, e^{x^2}$$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ the differential equation is exact

$$\int P(x,y)dx = y\, e^{x^2} + x^3$$

$$\int Q(x,y)dy = y\, e^{x^2} - y^2$$

The general solution is

$$y\, e^{x^2} + x^3 - y^2 \;\; = c$$

4. Reducible to Exact Differential Equations

Let the ordinary differential equation is in the form

$$P(x,y)dx + Q(x,y)dy = 0 \ldots (1)$$

If $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then the differential equation is not exact. We calculate the integrating factor $\mu(x,y) \neq 0$ to transform differential equation to exact.

Multiplication equation (1) by $\mu(x,y)$, we get

$$\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy = 0 \ldots (2)$$

Then the equation (2) is exact.

We have two cases for integrating factor:

Case 1 $$\frac{d\mu}{\mu} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}\, dx$$

Case 2 $$\frac{d\mu}{\mu} = \frac{-\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]}{P}\, dy$$

Example 10

Solve the following differential equation

$$4\text{xy}dx + (4x^2 + 3y)dy = 0 \ (*)$$

Solution

$P(x, y) = 4\text{xy}$ **and** $Q(x, y) = 4x^2 + 3y$

$$\frac{\partial P}{\partial y} = 4\text{x} \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = 8\text{x}$$

$\therefore \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ the differential equation is not exact.

$$\therefore \frac{d\mu}{\mu} = -\frac{\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]}{P} dy = \frac{-(4x - 8x)}{4xy} dy$$

$$= \frac{4x}{4xy} dy = \frac{1}{y} dy$$

$$\therefore \frac{d\mu}{\mu} = \frac{1}{y} dy \ \text{ Integrate both sides}$$

$\ln \mu = \ln y \quad \rightarrow \mu = y$

Multiplication equation (*) by μ

$$y(4\text{xy})dx - 3y(4x^2 + 3y)dy = 0$$
$$4\text{x}y^2 dx + (4x^2 y + 3y^2)dy = 0$$

$$\int P(x, y)dx = \int (4\text{x}y^2)dx = \frac{x^2}{2} y^2$$

$$\int Q(x, y)dy = \int (4x^2 y + 3y^2)dy = \frac{y^2}{2} x^2 + y^3$$

The general solution is

$$\frac{y^2}{2} x^2 + y^3 = c$$

Example 11

Solve the following differential equation

$$(x^3+y^3)\mathrm{dx}-3\mathrm{x}y^2dy=0\ (*)$$

Solution

$P(x,y)=(x^3+y^3)$ **and** $Q(x,y)=-3\mathrm{x}y^2$

$$\frac{\partial P}{\partial y}=3y^2 \quad \text{and} \quad \frac{\partial Q}{\partial x}=-3y^2$$

$\therefore \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$ the differential equation is not exact.

$$\therefore \frac{\mathrm{d}\mu}{\mu}=\frac{\left[\frac{\partial \mathrm{P}}{\partial \mathrm{y}}-\frac{\partial \mathrm{Q}}{\partial \mathrm{x}}\right]}{\mathrm{Q}}\mathrm{dx}=\frac{3y^2-(3y^2)}{-3xy^2}dx$$

$$=\frac{6y^2}{-3xy^2}dx=-\frac{2}{x}dx$$

$$\therefore \frac{\mathrm{d}\mu}{\mu}=-\frac{2}{x}dx \text{ Integrate both sides}$$

$$\ln\mu=-2\ln x\rightarrow\ln\mu=\ln x^{-2}\quad\rightarrow\mu=x^{-2}=\frac{1}{x^2}$$

Multiplication equation (*) by μ

$$\frac{1}{x^2}(x^3+y^3)\mathrm{dx}-3\frac{1}{x^2}\mathrm{x}y^2dy=0$$

$$\left(\mathrm{x}+\frac{y^3}{x^2}\right)\mathrm{dx}-3\frac{1}{x}y^2dy=0$$

$$\int P(x,y)dx=\int\left(\mathrm{x}+\frac{y^3}{x^2}\right)\mathrm{dx}=\frac{x^2}{2}-\frac{y^3}{x}$$

$$\int Q(x,y)dy=\int -3\frac{1}{x}y^2dy=-\frac{y^3}{x}$$

The general solution is

$$\frac{x^2}{2}-\frac{y^3}{x}=c$$

Try to solve

1. $2xydx + (x^2 + y^2)dy = 0$
2. $(y + 2x)dx + xdy = 0$
3. $e^{-y}dx - (2y + xe^{-y})dy = 0$
4. $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$
5. $(1 - xy)dx + (xy - x^2)dy = 0$
6. $y(x^2 + y^2 + 2x)dx + (x^2 + 3y^2)dy = 0$
7. $(x^2 + y^2 + x)dx + xydy = 0$
8. $(2y - 3x)dx + xdy = 0$

5. Linear Differential Equations

Let the ordinary differential equation is in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

First, we determine

$$I(x) = e^{\int P(x)dx}, \text{ it is called } \textit{Integrating factor}$$

The general solution is

$$I(x)y = \int I(x).Q(x)dx + c$$

Example 12

Solve the following differential equation

$$\frac{dy}{dx} + \frac{1}{x}y = 2x, \text{ with } x = -1 \; at \; y = 1$$

Solution

$$P(x) = \frac{1}{x} \quad \textbf{and} \quad Q(x) = 2x$$

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

The general solution is

$$x.y = \int x.(2x)dx + c$$

$$x.y = \frac{2}{3}x^3 + c$$

$x = -1 \ at \ y = 1$

$-1 = \frac{2}{3} + c \ \rightarrow \therefore c = -\frac{\mathbf{5}}{\mathbf{3}}$

$$x.y = \frac{2}{3}x^3 - \frac{5}{3}$$

Example 13

Solve the following differential equation

$$\frac{dy}{dx} - y = x$$

Solution

$$P(x) = -1 \quad \textbf{and} \quad Q(x) = x$$

$$I(x) = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$$

The general solution is

$$e^{-x}.y = \int e^{-x}.xdx + c$$

We integrate by parts

$$u = x, \quad dv = e^{-x}\ dx$$

$$du = dx, \quad v = -e^{-x}$$

$$e^{-x}.y = -e^{-x}x - e^{-x} + c$$

Example 14

Solve the following differential equation

$$\frac{dy}{dx} + \tan x\, y = \sec x$$

Solution

$P(x) = \tan x$ **and** $Q(x) = \sec x$

$$I(x) = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

The general solution is

$$\sec x\, .\, y = \int \sec x\, .\sec x\, dx + c$$

$$\sec x\, .\, y = \tan x + c$$

Try to solve

1. $y'(x-1) - y = 2(x-1)^3$
2. $y' - \frac{2}{(x+1)} y = (x+1)^2$
3. $\frac{dy}{dx} - y\cot x = 2x\sin x$
4. $y' + 2xy = 2x$
5. $y' \cos x + y\sin x = 1$

Second order differential equations

In this chapter we will be looking exclusively at linear second order differential equations

$$ay'' + by' + cy = f(x) \qquad (1)$$

where $a, b \; and \; c$ are constants

When $f(x) = 0$, we call the differential equation **homogeneous** and when $f(x) \neq 0$,we call the differential equation **nonhomogeneous**.

The general solution equals the sum of the homogeneous solution and the particular solution.

$$y_G = y_H + y_P$$

I - The homogeneous solution

$$ay'' + by' + cy = 0 \qquad (2)$$

Let $y = e^{\lambda x}$ is a solution of equation (2), then

$y' = \lambda e^{\lambda x}$ and

$y'' = \lambda^2 e^{\lambda x}$

$$a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) = 0$$

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

since $e^{\lambda x} \neq 0 \quad \therefore (a\lambda^2 + b\lambda + c) = 0$

This equation is typically called the **characteristic equation**.

This characteristic equation is quadratic and so will have two roots λ_1 and λ_2. The roots will have three possible forms. These are

1. **Real and distinct roots** $\lambda_1 \neq \lambda_2$

The general solution is

$$y_H = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

2. **Real and repeated roots**, $\lambda_1 = \lambda_2 = \lambda$

The general solution is

$$y_H = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

3. **Complex root** $\lambda_{1,2} = \propto \pm \beta i$

The general solution is

$$y_H = e^{\propto x}(c_1 \cos \beta x + c_1 \sin\beta\, x)$$

Example 1

Solve the following differential equations

1- $y'' - 2y' - 3y = 0$

Solution

The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda_1 = 3$ or $\lambda_2 = -1$ **real & distinct**

$$y_H = c_1 e^{3x} + c_2 e^{-x}$$

2- $y'' - 2y' + y = 0$

Solution

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda_1 = 1 \text{ or } \lambda_2 = 1 \quad \textbf{real\& repeat}$$

$$y_H = c_1 e^x + c_2 x e^x$$

3- $y'' + y = 0$

Solution

The characteristic equation is

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1 = i^2$$

$$\lambda = \pm \boldsymbol{i} \quad \textbf{complex}$$

$$y_H = c_1 \cos x + c_2 \sin x$$

4- $y'' + 2y' + 2y = 0$

Solution

The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0$$

$\mathrm{A} = 1 \;, \mathrm{B} = 2 \;, \mathrm{C} = 2$

$$\lambda = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= -\frac{2 \pm \sqrt{2^2 - 4 * 2 * 2}}{2 * 1} = \frac{-2 \pm 2i}{2}$$

$$\lambda = 1 \pm i$$

$$y_H = e^x (c_1 \cos x + c_2 \sin x)$$

Try to solve

Solve the following differential equations

3- $y'' - 10y' + 16y = 0$

4- $y'' - 14y' - 48y = 0$

5- $y'' + 2y' + 5y = 0$

6- $y'' + 4y' + 13y = 0$

7- $y'' + y' = 0$

II- The Particular solution

In this section we will take a look at the first method that can be used to find a particular solution to a nonhomogeneous differential equation.

$$y'' + by' + cy = f(x) \neq 0$$

1- IF $f(\mathrm{x}) = \mathrm{e}^{\lambda \mathrm{x}}$

Example 2

Solve the following differential equations

$$y'' - y' - 2y = 6\mathrm{e}^{2\mathrm{x}}$$

Solution

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$

$$(\lambda - 2)(\lambda + 1) = 0$$

$\lambda_1 = 2$ or $\lambda_2 = -1$ **real & distinct**

$$y_H = c_1 e^{2x} + c_2 e^{-x} \to 1$$

Suppose the particular solution is $y_P = \mathrm{Ae}^{2\mathrm{x}}$

Then, compare y_P with y_H if there is a similar term, multiply y_P by x

$$\therefore y_P = Axe^{2x}$$

Get the first and second derivatives

$$\therefore y'_P = \mathrm{A}(e^{2x} + 2xe^{2x}) = \mathrm{A}e^{2x}(1 + 2x)$$

$$\therefore y''_P = \mathrm{A}(2e^{2x} + 4xe^{2x} + 2e^{2x}) = \mathrm{A}e^{2x}(4 + 4x)$$

Plugging into the differential equation given

$$\mathrm{A}e^{2x}(4 + 4x) - \mathrm{A}e^{2x}(1 + 2x) - 2Axe^{2x} = 6\mathrm{e}^{2\mathrm{x}}$$
$$\mathrm{A}e^{2x}(4 + 4x - 1 - 2x - 2x) = 6\mathrm{e}^{2\mathrm{x}}$$
$$3Ae^{2x} = 6\mathrm{e}^{2\mathrm{x}}$$
$$3A = 6 \rightarrow A = 2$$

$$\therefore y_P = 2xe^{2x} \rightarrow 2$$

From equations 1& 2

$$y_G = c_1e^{2x} + c_2e^{-x} + 2xe^{2x}$$

2- IF $f(x) = ax^n + bx^{n-1} + \cdots + c$

Example 3

Solve the following differential equations

$$y'' - 2y' - 3y = x^2 - 3$$

Solution

The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda_1 = 3$ or $\lambda_2 = -1$ **real & distinct**

$$y_H = c_1e^{3x} + c_2e^{-x} \rightarrow 1$$

Suppose the particular solution is $y_P = \mathrm{a}x^2 + b\mathrm{x} + \mathrm{c}$

Get the first and second derivatives

$$\therefore y'_P = 2\mathrm{ax} + \mathrm{b}$$

$$\therefore y''_P = 2\mathrm{a}$$

Plugging into the differential equation given

$$2a - 2(2ax + b) - 3(\mathrm{a}x^2 + b\mathrm{x} + \mathrm{c}) = x^2 - 3$$

Compare the coefficient of x^2

$-3a = 1 \rightarrow a = -\frac{1}{3}$

Compare the coefficient of x

$-4\mathrm{a} - 3\mathrm{b} = 0 \rightarrow -4\mathrm{a} = 3\mathrm{b} \rightarrow b = \frac{4}{9}$

Compare the coefficient of x^0

$2\mathrm{a} - 2\mathrm{b} - 3\mathrm{c} = -3 \rightarrow 3\mathrm{c} = -\frac{2}{3} - \frac{8}{9} \rightarrow c = \frac{-14}{9}$

$$\therefore y_P = -\frac{1}{3}x^2 + \frac{4}{9}\mathrm{x} - \frac{14}{9} \rightarrow 2$$

from equations 1& 2

$$y_G = c_1 e^{3x} + c_2 e^{-x} + -\frac{1}{3}x^2 + \frac{4}{9}\mathrm{x} - \frac{14}{9}$$

3- IF $f(x) = \cos\lambda\mathrm{x}$ or $\sin\lambda\mathrm{x}$

Example 4

Solve the following differential equations

$$y'' - y = 10\cos 2x$$

Solution

The characteristic equation is $\lambda^2 - 1 = 0$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda_1 = 1 \text{ or } \lambda_2 = -1 \textbf{ real \& distinct}$$

$$y_H = c_1 e^x + c_2 e^{-x} \to 1$$

Suppose the particular solution is $y_P = \mathrm{A}\cos 2x + B\sin 2x$

Get the first and second derivatives

$$\therefore y'_P = -2\mathrm{A}\sin 2x + 2B\cos 2x$$

$$\therefore y''_P = -4\mathrm{A}\cos 2x - 4B\sin 2x$$

Plugging into the differential equation given

$$-4\mathrm{A}\cos 2x - 4B\sin 2x - (\mathrm{A}\cos 2x + B\sin 2x) = 10\cos 2x$$

$$-5\mathrm{A}\cos 2x - 5B\sin 2x = 10\cos 2x$$

Compare the coefficient of $\cos 2x$

$$-5\mathrm{A} = 10 \to A = -2$$

Compare the coefficient of $\sin 2x$

$$-5\mathrm{B} = 0 \to B = -2\cos 2x$$

$$\therefore y_P = -2\cos 2x \to 2$$

from equations 1& 2

$$y_G = c_1 e^x + c_2 e^{-x} - 2\cos 2x$$

Try to solve

Solve the following differential equations

1. $y'' - 6y' + 9y = e^{3x}$
2. $2y'' - y' - y = xe^{x}$
3. $y'' - y' + 5y = \sin 2x$
4. $y'' + 4y = \sin 2x$
5. $y'' + 2y' = x^2 + 2$