Recall the recursive definition of the Fibonacci sequence:

$$F_i = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = 2, \\ F_{i-1} + F_{i-2} & \text{otherwise.} \end{cases}$$

It turns out that any positive integer can be written as a sum of *non-consecutive* Fibonacci numbers; this is known as *Zeckendorf's theorem*. For example, we can write the integer 83 as

$$83 = 55 + 21 + 5 + 2 = F_{10} + F_8 + F_5 + F_3.$$

We will prove this theorem with a greedy algorithm. Let n be a positive integer, and consider the following greedy algorithm.

Always choose the largest Fibonacci number that is at most equal to n, subtract the integer from n, and recurse until there is no remainder.

(a) Prove that the algorithm is correct.

Hint. *You should prove that if you take the largest Fibonacci number F_k, then largest Fibonacci number that can fit $n - F_k$ is at most F_{k-2}.*

(b) Prove that the Fibonacci representation produced by the algorithm is *unique*.

Rubric.

- This task will form part of the portfolio.
- Ensure that your argument is clear and keep reworking your solutions until your lab demonstrator is happy with your work.