Let $G = (V, E)$ be an undirected graph. A vertex v is a *cut vertex* if $G \setminus \{v\}$ has more connected components than G. In other words, v splits G into more connected components than G.

(a) Prove that every graph with at least two vertices has at least two vertices that are not cut vertices.

(b) If G is a simple and connected graph with *exactly* two vertices that are not cut vertices, then G must be a path.

Hint. *Firstly, show that every spanning tree in G has degree at most 2 (i.e. if T is a spanning tree in G, then every vertex in T has degree at most 2). Then, show that G cannot be a cycle.*

Solution.

(a) Consider two cases. If G is a tree, for $V \geq 2$, there must be at least 2 leaf vertices, which we can remove easily without cutting the graph. If G is not a tree, then there must exist a cycle with at least 3 vertices (by definition of a cycle). In a cycle, no vertex is a cut vertex, as removing any vertex from a cycle does not disconnect the graph. We must then be able to pick at least two points in our cycle (for a cycle with more than 3 vertices, these will be adjacent points). In both cases, we are able to find at least 2 vertices that are not cut vertices.

(b) Consider a spanning tree T of G. If there is a vertex v in T with degree greater than 2, then removing v would disconnect the tree into three or more components, making v a cut vertex in G. However, this contradicts the assumption that G has exactly two vertices that are not cut vertices. Therefore, every vertex in T must have degree at most 2. Furthermore, G cannot be a cycle since in a cycle, no vertex is a cut vertex, which contradicts having exactly two vertices that are not cut vertices. Since every spanning tree of G has vertices of degree at most 2 and G is not a cycle, the only possibility left is that G is a path.