Let f_1, f_2, g_1, g_2 be functions from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, and suppose that $f_1 \in O(g_1)$ and $f_2 \in O(g_2)$.

(a) Show that $f_1 + f_2 \in O(g_1 + g_2)$.

(b) Show that $f_1 \cdot f_2 \in O(g_1 \cdot g_2)$.

Solution.

(a) Since $f_1 \in O(g_1)$, there exist constants $c_1 > 0$ and $n_{01} \in \mathbb{Z}^+$ such that for all $n \geq n_{01}$, we have $f_1(n) \leq c_1 g_1(n)$. Similarly, since $f_2 \in O(g_2)$, there exist constants $c_2 > 0$ and $n_{02} \in \mathbb{Z}^+$ such that $\forall n \geq n_{02}$, we have $f_2(n) \leq c_2 g_2(n)$.

Let $n_0 = \max\{n_{01}, n_{02}\}$ and $c = \max\{c_1, c_2\}$. Then for all $n \geq n_0$, we have:

$$\begin{aligned} f_1(n) + f_2(n) &\leq c_1 g_1(n) + c_2 g_2(n) \\ &\leq c g_1(n) + c g_2(n) \\ &\leq c(g_1(n) + g_2(n)). \end{aligned}$$

Thus, by definition, we have $f_1 + f_2 \in O(g_1 + g_2)$.

(b) Consider the same constants from part (a). $\forall n \geq \max\{n_{01}, n_{02}\}$, we have

$$\begin{aligned} f_1(n) \cdot f_2(n) &\leq c_1 g_1(n) \cdot c_2 g_2(n) \\ &\leq c_1 c_2 (g_1(n) \cdot g_2(n)). \end{aligned}$$

Thus, by definition, we have $f_1 \cdot f_2 \in O(g_1 \cdot g_2)$.