Let $G = (V, E)$ be a *directed* graph with n vertices and m edges. Let $\mathscr{C} = \{C_1, \ldots, C_k\}$ be a collection of disjoint cycles in G. We say that $\mathscr{C}$ is a *cycle covering* if every vertex $v \in V$ in G is covered in *exactly* one cycle C_i. In other words, a cycle covering is a collection of disjoint cycles that cover all vertices in G.

(a) Let $E' \subseteq E$ be a set of edges in G. Prove that E' forms a cycle covering if and only if every vertex has exactly one incoming edge and one outgoing edge in E'.

(b) Describe an $O(m\sqrt{n})$ algorithm to decide whether G contains a cycle covering. You are not asked to return the covering (if one exists), just whether G contains one.

Hint. *Use bipartite matching to check for the existence of a set of edges that satisfy the condition from part (a).*

Solution.

(a) Firstly, assume that E' forms a cycle covering $C = \{C_1, \ldots, C_k\}$. This means each vertex $v \in V$ is part of exactly one cycle in C. Now, it's obvious that, in a cycle, each vertex has exactly one incoming edge and one outgoing edge. So, in the cycle C, each vertex will have exactly one incoming and outgoing edge in E'. Conversely, assume that $v \in V$ has exactly one incoming and outgoing edge in E'. Follow the outgoing edge to reach other vertex, and continue along all outgoing edges. Since the graph is finite, this process must eventually reach v again from our assumption, which will form a cycle. Thus, it must be so that ever vertex is part of exactly one cycle as there would be no other path, and so E' decomposes G into a collection of disjoint cycles that cover every vertex exactly once. This completes the proof.

(b) First, split each vertex v into v_{in} and v_{out}, with each edge $(u, v) \in E$ then being connected from u_{out} to v_{in}. This graph is bipartite, and with the Hopcroft-Karp algorithm, find the bipartite maximum matching. If the size of the maximum matching is $|V|$, then each vertex must have exactly one incoming edge and one outgoing edge, satisfying our condition from part (a). Otherwise, G will not contain a cycle covering. The most expensive part is the Hopcroft-Karp algorithm is $O(m\sqrt{n})$.