Consider the following problem, which we call $\exists_2$-SAT.

Input. A boolean formula $\varphi(\mathbf{x}, \mathbf{y})$ in $2n$ variables, where $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$.

Task. Decide if there exist an assignment for $\mathbf{x}$ such that, for all $\mathbf{y} \in \{0,1\}^n$, $\varphi(\mathbf{x}, \mathbf{y})$ is true.

For example, consider the following boolean formula $\varphi(\mathbf{x}, \mathbf{y}) = (x_1 \wedge x_2) \vee (y_1 \wedge y_2)$. Assigning $x_1 = x_2 = 1$ produces a tautological formula and hence, any binary string for $\mathbf{y}$ satisfies $\varphi(\mathbf{x}, \mathbf{y})$.

Prove that, if $\mathsf{P} = \mathsf{NP}$, then $\exists_2$-SAT $\in \mathsf{P}$.

Hint. *If $\mathsf{P} = \mathsf{NP}$, there exist a polynomial-time algorithm that solves SAT...*

Solution. Solution begins here...