Let $G = (V, E)$ be an undirected graph on n vertices. A *clique* is a subset $S \subseteq V$ of vertices such that every pair of vertices in S are adjacent. The size of a clique is the number of vertices in the clique.

(a) Let $k \geq 1$ be an integer. How many distinct cliques of size k could there be in G?

(b) If G has a clique of size k, show that G has a clique of size ℓ for all $\ell \leq k$.

Solution.

(a) We assume that the question is asking for the **maximum** number of cliques possible for a graph with n vertices containing exactly k vertices. This obviously occurs when the graph is complete, so any subset (along with their connected edges) will also be included in order to achieve a clique. So, we must choose k vertices out of the n total vertices. Therefore, we have $\binom{k}{n}$ total cliques.

(b) Assume you remove a vertex, along with all edges connected to it. By definition of a clique, all other vertices must be connected and hence the result will also be a clique. This result follows mathematical induction can be repeated $k - \ell$ times.