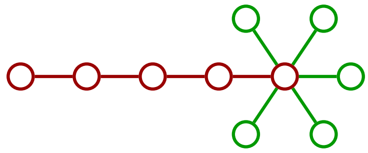
A *dandelion* of length k is an undirected graph that consists of a simple path of length k, followed by k additional vertices connected to one of the endpoints.



A dandelion of length 5.

Given a graph G, prove that it is NP-hard to find the length of the longest dandelion subgraph of G.

Hint. *HamiltonianPath is known to be* NP-C.

Solution. We aim to reduce the problem to HamiltonianPath to prove it is NP-hard. Let $G = (V, E)$ and let $n = |V|$. Construct a new graph $G' = (V', E')$ where V' contains all the vertices in V and $n + 1$ new ones, x and $y_1, \dots y_n$, and E' contains all the edges in E, along with new edges for each $i = 1, \dots n$ of (x, y_i) and (x, v) for all $v \in V$. Since there are $2n + 1$ edges, our graph G' can have a longest dandelion of n. G' can be constructed in linear time.

- Now, suppose that G contains a Hamiltonian path. The end points will connect to x, which has a degree greater than n. If this is the case, then G' will contain a dandelion of length n.
- Conversely, if H contains a dandelion of length n, the hub of the dandelion must be x since it is the only vertex with degree greater than n. Obviously the path cannot contain y_i as once you cannot leave from those vertices, so the stem of the dandelion must then be in G. This clearly implies that we have a Hamiltonian path in G.

In other words, G contains a Hamiltonian path if and only if G' contains a dandelion of length n. The difficulty of solving the dandelion problem is directly encompassed as at least the difficulty of solving the Hamiltonian path problem, as otherwise it would be such that HamiltonianPath is not NP-C. Thus, our problem must be NP-hard.