Let $G = (V, E, w)$ be a weighted graph, where $w(e)$ denotes the weight of a given edge e. A *matching* is a subset $M \subseteq E$ of edges such that no two edges in M share a common vertex. A matching is *perfect* if every vertex is included in exactly one edge in M.

Consider the following two problems:

- MINCOSTPERFECTMATCHING (MCPM): given a weighted graph $G = (V, E, w)$, where the edge weights can be positive, negative, or zero, find a minimum-cost perfect matching M (if there exist one; otherwise, report that none exists), where the cost of a matching M is given by

$$w(M) = \sum_{e \in M} w(e).$$

- MAXWEIGHTEDPERFECTMATCHING (MWPM): given a weighted graph $G = (V, E, w)$, where the edge weights are *non-negative*, find a maximum-weighted perfect matching M (if there exist one; otherwise, report that none exists), where the weight of a matching M is given by

$$w(M) = \sum_{e \in M} w(e).$$

Consider the following polynomial-time reduction from MCPM to MWPM.

Reduction: Let n be the number of vertices in G. If n is odd, then there are no perfect matchings so report that none exists. We will now assume that the number of vertices is even. For each edge $e \in E$, define $\hat{w}(e) = w_{\max} - w(e) + 1$, where $w_{\max} = \max_{e \in E} w(e)$. Let $\Delta = \sum_{e \in E} \hat{w}(e)$.

We construct $G^* = (V^*, E^*, w^*)$ as follows: we keep the same vertices and edges; that is, $V^* = V$, $E^* = E$. For each edge $e \in E^*$, we have $w^*(e) = \hat{w}(e) + \Delta$. Clearly, G^* is a graph that has *non-negative* edge weights so G^* is a valid instance to MWPM. Moreover, G^* can be constructed in polynomial-time.

Your task is to prove the correctness of the reduction by proving each of the following claims.

(a) Firstly, argue that G has a perfect matching if and only if G^* has a perfect matching.

(b) Suppose that G^* has a perfect matching. Prove that a maximum weighted matching in G^* is also a maximum weighted *perfect* matching.

(c) Let M be a matching. Prove that M is a minimum-cost perfect matching in G *if and only if* M is a maximum-weighted perfect matching in G^*.

(d) Hence, conclude that the reduction is correct.

Solution.

(a) A perfect matching does not depend on w. G by definition contains the exact same vertices and edges as G^*, so if there exists some set $U \subseteq E' =^*$ with $U \subseteq E$ that satisfies the conditions of being a perfect matching. The converse is also true for the exact same reason.

(b) For the general case, the maximum weighted matching of some graph does not necessarily have to be a perfect matching. In this case, however, we can use the construction of G^* to see why this is the case under the proposed reduction method. We know that $\Delta > n > 0$ since each $\hat{w}(e) >\geq 1$ from its definition. Our reduction to $w^*(e)$ is bounded on the bottom by $1 + \Delta$ when $w(e) = w_{\text{max}}$, and can be no larger than $w_{\text{max}} - w_{\text{min}} + 1 + \Delta$. It's trivial to see that $w_{\text{max}} - w_{\text{min}} \leq \Delta$ by the definition of Δ.

Now, let's assume that we have a matching M that is not a perfect matching. This would mean that, since n is even, there are at least two vertices that are not covered. According to our result, the size of each edge is bounded by $w_{\text{max}} - w_{\text{min}} + 1 + \Delta$. If we were to use the perfect matching we know G^* has, we would be able increase the total weight by $\Delta + 1$ in the worst case. In other words, it would be better to have more edges in M due to the bounds on w^*.

(c) Consider rewriting out $w^*(M)$ in terms of $w(M)$.

$$\begin{aligned} w^*(M) &= \sum_{e \in M} w^*(e) \\ &= \sum_{e \in M} [w_{\text{max}} - w(e) + 1 + \Delta] \\ &= |M| (\cdot w_{\text{max}} + \Delta + 1) - \sum_{e \in M} w(e) \\ &= |M| (\cdot w_{\text{max}} + \Delta + 1) - w(M). \end{aligned}$$

In order to maximise $w^*(M)$, we need to minimise $w(M)$. $|M|$ will need to be the maximum and a fixed size from the result in part (b). The converse is obviously true.

(d) From part (c), we can see that M is a minimum-cost perfect matching in G if and only if M is a maximum-weighted perfect matching in G^*, tThus, solving the MaxWeightedPerfectMatching in G^* is equivalent to solving the MinCostPerfectMatching in G.