In a graph $G = (V, E)$, a *loop* is a sequence of vertices $\{v_1, v_2, \ldots, v_n, v_1\}$ such that the first and last vertices are the same and that $v_i \to v_{i+1}$ for each $i = 1, \ldots, n-1$ and $v_n \to v_1$. However, vertices may repeat in the sequence; that is, we may visit the same vertex multiple times.

A directed graph G is said to be *strongly-connected* if, for every pair $u, v \in V$ of vertices, there exist a directed path from u to v. A vertex w is said to be *divisible* if there exist an integer $k > 1$ such that every loop containing w has length that is divisible by k. Prove that, if there exist a divisible vertex on a strongly-connected graph, then every vertex is divisible.

Solution. Using the parameters defined in the question, there is a divisible vertex w in graph G with corresponding integer $k > 1$. Let u be an arbitrary vertex in G. Since G is strongly connected, there exists a path from u to w (denoted as P_{uw}) and another path from w to u (denoted as P_{wu}). Now, consider any loop containing u (denoted as L_u). We can construct a new loop that starts at w, follows P_{uw} to u, then follows L_u, and finally returns to w through P_{wu}. This new loop passes through w and thus its length must be divisible by k. The length of this new loop is the sum of the lengths of P_{uw}, L_u, and P_{wu}. Since the new loop's length is divisible by k and the sum of the lengths of P_{uw} and P_{wu} is fixed, the length of L_u must also be divisible by k. Therefore, every loop L_u containing u has a length divisible by k. Since u was chosen arbitrarily, this holds for every vertex in G, thus every vertex in G is divisible.