Given a positive integer n, the *complexity* of n is the minimum number of ones that can be used to represent n, using only the operations of addition and multiplication, as well as parenthesisation.

For example, we have the following representations:

$$\begin{aligned}
6 &= (1+1+1)\times(1+1). \\
8 &= (1+1)\times(1+1)\times(1+1). \\
9 &= (1+1+1)\times(1+1+1). \\
12 &= (1+1+1+1)\times(1+1+1). \\
19 &= (1+1+1)\times(1+1+1)\times(1+1)+1.
\end{aligned}$$

The first twenty entries are given for you.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	2	3	4	5	5	6	6	6	7	8	7	8	8	8	8	9	8	9	9

(a) Show that every positive integer can be represented by a string of ones, along with addition, multiplication, and parenthesisation operations; that is, the complexity of n is always finite.

(b) Given a positive integer n, describe an $O(n^2)$ algorithm to compute the minimum number of one's (1's) using only the operations of addition and multiplication, as well as parentheses, whose expression equals n.

Note. *This is also known as the Mahler-Popken complexity. Here is the OEIS entry.*

Solution.

(a) Any integer can be represented as a sum of ones, so we have an upper bound as simply $n = \sum_{i=1}^{n} 1$.

(b) Begin by initialising a Dynamic Programming table of size $n+1$, DP, with $DP[i] = i$ for each element. Now, we must explore all pairs (a, b) such that $a \times b = n$ or $a + b = n$ to ensure that we cover all different ways to arrive at that number.

- For each value a from 1 up until $\lceil \sqrt{n} \rceil$, we check to see if $i = 0 \bmod a$. If so, we take $b = i \div a$ and update $DP[i] = \min\{DP[i], DP[a] + DP[b]\}$.
- For each value a from 1 up until $\left\lceil \frac{n}{2} \right\rceil$, we take $b = i - a$ and again update $DP[i] = \min\{DP[i], DP[a] + DP[b]\}$.

Finally, we return the result $DP[n]$ as the ideal value. Our multiplication and addition checks take $O\left(\sqrt{n} + \frac{n}{2}\right) = O(n)$, and since we do this for all numbers up until n, we have an overall time complexity of $O(n^2)$.