Solution.

(a) Consider the contrapositive. Let $\hat{C}(S,T)$ be the minimum cut of $\hat{F}$. From the new capacity function, we must have

$$\hat{C}(S,T) = C(S,T) \cdot (m+1) + k,$$

where k is the number of edges the cut passes through. Our function is linear, with a positive coefficient $m+1$, which means that the minimum of this occurs at the minimum $C(S,T)$, and so $C(S,T)$ is minimised. The value of k does not affect the minimality of $C(S,T)$ because the scaling factor $m+1$ dominates the additive term k. Even if two cuts have different numbers of crossing edges, the cut with the smaller capacity in F will still have a smaller capacity in $\hat{F}$ due to the linear scaling and the fact that $k \leq m$. Thus, by contrapositive, $C(S,T)$ must also be a minimum cut of F.

(b) Suppose that we have two distinct cuts yet equal capacity cuts $C(S,T)$ and $C(S',T')$. Let's also assume that $C(S',T')$ passes through more edges. Their capacity in $\hat{F}$ will be $C(S,T) \cdot (m+1)+k_1$ and $C(S',T') \cdot (m+1)+k_2$ respectively. Now, k_1 represents the number of edges $C(S,T)$ passes through, and k_2 represents the number of edges $C(S',T')$ passes through. It's clear that $k_1 < k_2$, and since $C(S,T) = C(S',T')$, we have that the minimum must also pass through the fewest number of edges.