# MATH3161 - Optimisation

Generatively made exam by Moae

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## QUESTION ONE

Consider the matrix

$$A = \begin{pmatrix} -6 & 6 \\ -4 & 4 \end{pmatrix}.$$

State whether this matrix is positive definite, negative definite, positive semi-definite, negative semi-definite, or indefinite.

## QUESTION TWO

Show that if $A$ is any matrix, then $K = A^T A$ and $L = AA^T$ are both symmetric positive definite matrices.

## QUESTION THREE

If $f$ is a convex function, show that $h(x) = \alpha f(x) + \beta$ is also convex for scalars $\alpha$ and $\beta$ and $\alpha \geq 0$.

## QUESTION FOUR

A construction company is tasked with creating guttering from a strip of metal 12 cm wide. To do this, let $x$ be the length of each of the sections that will form the slope, and let $\theta$ be the angle that each sloping side makes with the horizontal. Determine the values of $x$ and $\theta$ that maximise the capacity of the guttering.

## QUESTION FIVE

Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, where

$$f(x_1,\ x_2) = 7 - 5x_1 x_2 - 3x_2 + 10x_2 + 2x_1.$$

Find and classify all stationary point of this function.

## QUESTION SIX

Consider function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(\boldsymbol{x}) = 6x_1^2 + 7x_2^2 + 19x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 4x_2 x_3 - 2x_1 + 4x_2 + 4x_3,$$

where $\boldsymbol{x} = (x_1, x_2)$ and is subject to the constraint $3x_1 + 4x_2 + 4x_3 - 1 = 0$. Now consider the point $\boldsymbol{x}^* = \left[-1, \frac{1}{2}, \frac{1}{2}\right]^T$. Show that this point is a regular point and that it satisfies the first order necessary conditions. Also verify that $\boldsymbol{x}^*$ satisfies the second order sufficient conditions for a local minimiser of the problem. Is $\boldsymbol{x}^*$ a global minimiser?

## QUESTION SEVEN

Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, where

$$f(x_1, x_2) = 2x_1^2 + 4x_2^2 - 4x_1,$$

subject to the constraint $(x_1 - 1)^2 + (x_2 - 1)^2 \leq 16$. We aim to minimise this function in the given region defined by the constraints. Sketch the region and confirm whether or not this is a convex optimisation problem. If it is, write down the Kuhn-Tucker conditions for this problem and show that $\boldsymbol{x}^* = [1, -3]^T$ satisfies them. Is $\boldsymbol{x}^*$ a global minimum?

## QUESTION EIGHT

Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, where

$$f(x_1, x_2) = 2x_1^2 + x_2^2 + 2x_1 x_2 + 4x_1,$$

subject to the constraints $-3x_1 - x_2 + 2 \leq 0$ and $x_1 \geq 0$. We aim to minimise this function in the given region defined by the constraints. Write down the (Wolfe) dual problem, if applicable. Otherwise, show that the given problem is not convex.

## QUESTION NINE

Consider the function:
$$f(x_1, x_2) = 2x_1^2 + 3x_2^2 + 2x_1 - 2x_2.$$

Prove by induction that the method of steepest descent applied with an initial guess $\boldsymbol{x}^{(1)} = \boldsymbol{0}$ generates the sequence $\{\boldsymbol{x}^{(k)}\}$ where

$$\{\boldsymbol{x}^{(k+1)}\} = \left(-\left(\frac{1}{5}\right)^k + \frac{1}{2}, \ \left(-\frac{1}{5}\right)^k - \frac{1}{3}\right).$$

Hence, deduce the minimiser.

## QUESTION TEN

Consider minimising the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(\boldsymbol{x}) = x_1^2 + 3x_2^2 - x_1 - x_2 - 5$$

starting from $\boldsymbol{x}^{(1)} = [-1, 2]$. Show that $f$ is a strictly convex function and find its minimiser $\boldsymbol{x}^*$ and $f(\boldsymbol{x}^*)$. Calculate the Newton direction and confirm that it is a descent direction. Finally, use Newton's method to find the minimiser of $f$ starting from $\boldsymbol{x}^{(1)}$. Comment on your results.

## QUESTION ELEVEN

Consider minimising the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(\boldsymbol{x}) = 3x_1^2 + 3x_2^2 - 3x_1x_2 + x_1 + 1$$

starting from $\boldsymbol{x}^{(1)} = [0, 2]$. Show that $f$ is a strictly convex function and find its minimiser $\boldsymbol{x}^*$ and $f(\boldsymbol{x}^*)$. Use a conjugate gradient method with the Fletcher-Reeves update to minimise $f$ starting from $\boldsymbol{x}^{(1)}$. Verify that the search directions are descent directions and that the search directions are conjugate with respect to the Hessian of $f$.