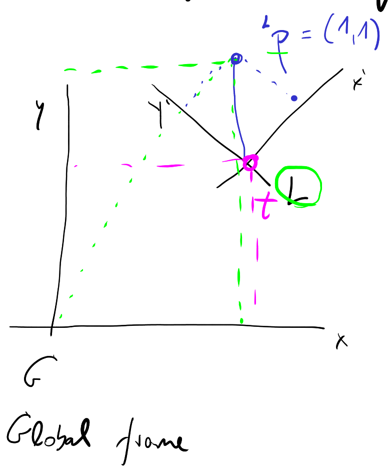
L13: Rigid Body Transformations



$$^G p = {}^G T_L \, {}^L p$$

for 2D. $(XYT \longrightarrow T)$

$$\underbrace{\begin{bmatrix} R(\theta) & t \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_L$$

* 3D Rotations

$$SO(3) = \{ R \in \mathbb{R}^{3\times 3} \mid RR^T = I \, , \, \det(R) = 1 \}$$

$$R_1, R_2 \in SO(3)$$

$$R_1 \cdot R_2 \in SO(3)$$

* 3D Rigid Body Transformation (RBT)

$$SE(3) = \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}_{4\times 4} \mid R \in SO(3) \, , \, t \in \mathbb{R}^3 \right\}$$

$$^G p = {}^G T_L \cdot {}^L p \quad , \quad p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

∘ Properties $\longrightarrow$ Group ~~Group~~ of SE(3)

$$T_1, T_2 \in SE(3) \quad , \quad T_1 \cdot I = T_1$$

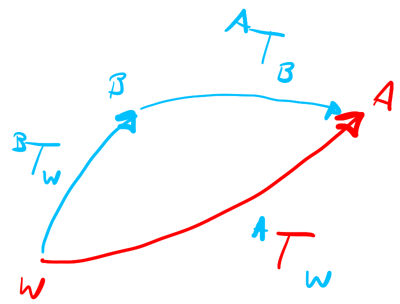
$$T_1 \cdot T_2 \in SE(3)$$

* Uses of RBT.

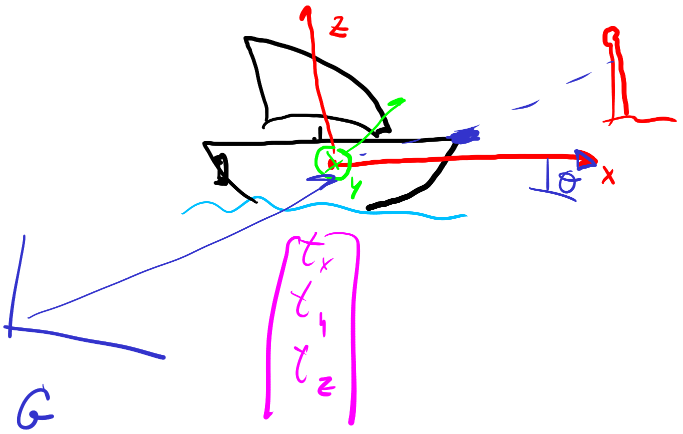
1) Transform points $\qquad {}^{G}p = {}^{G}T_{L} \cdot {}^{L}p \qquad$ (L05)

2) Transform frames

$${}^{A}T_{W} = {}^{A}T_{B} \cdot {}^{B}T_{W}$$



3) 3D Poses as SE(3)



$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

(2D) $T_{3\times3} \longrightarrow \underline{xyT}$

(3D) $\underline{T_{4\times4}} \longrightarrow \underline{6}$ DoF

* Lie Algebra for Rotations

$$\begin{cases} \dot{R}(t) = \underline{W}\, R(t) \\ \text{s.t.} \quad R \cdot R^T = I \end{cases}$$

$$\frac{\partial}{\partial t}(R R^T) = \frac{\partial}{\partial t}(I)$$

$$\dot{R} R^T + R \cdot (\dot{R})^T = 0$$

$$\underline{W} R \cdot R^T + R \cdot (W R)^T = 0 \quad ,$$

$$W \cdot \underbrace{RR^T}_{I} + \underbrace{RR^T}_{I} \cdot W^T = 0 \qquad -\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} -1 & 0 \\ -3 & -2 \end{pmatrix}$$

$$W + W^T = 0 \iff W = -W^T$$

Skew symetric

$$W = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = w^\wedge \quad , \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

(hat operator)

$y(t_0)=0$, $\dot{y} = b$
$y(t) = e^{bt}$

$$R(t) = \int_{t_0}^{t} \dot{R}\, dt = \exp(wt) \cdot R_0$$

* Exponential map

$$R(t) = e^{w^\wedge \cdot t} \cdot R(t_0)$$

$t_0 = 0$, $t = 1$ $\Rightarrow$ $\theta^\wedge = w^\wedge \cdot t \,\Big|_{t=1}$, w [rad/s], θ [rad]

Taylor

exp. map: $\left[\exp(\theta^\wedge) = I + \theta^\wedge + \frac{1}{2!}(\theta^\wedge)^2 + \frac{1}{3!}(\theta^\wedge)^3 + \dots\right.$

$$(\theta^\wedge)^2 = \theta\theta^T - \|\theta\|^2 \cdot I \qquad \left(\begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \cdot \Big[\quad \Big] = \right.$$

$$(\theta^\wedge)^3 = (\theta^\wedge)^2 \cdot \theta^\wedge = -\|\theta\|^2 \cdot \theta^\wedge$$

$\theta \times \theta$

Rodrigues formula:

$$\boxed{\underline{R} = \exp(\underset{\smile}{\theta}^\wedge) = I + \frac{\sin(|\theta|)}{|\theta|}\cdot\theta^\wedge + \frac{1-\cos(|\theta|)}{|\theta|^2}(\theta^\wedge)^2}$$

$$\theta^\wedge = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{G_1}\theta_1 + G_2\cdot\theta_2 + G_3\theta_3$$

$(\cdot)^\wedge : \underset{\smile}{\mathbb{R}}^3 \longrightarrow so(3)$

$\exp(\theta^\wedge) : so(3) \longrightarrow SO(3)$ → rotation group.

$\mathrm{Exp}(\theta) : \mathbb{R}^3 \longrightarrow SO(3)$

¿ yes . the Logarithm.

$\mathrm{Ln}(R) : SO(3) \longrightarrow \underline{\mathbb{R}}^3$

for 1D : $\alpha + \underset{\sim}{K\cdot 2\pi}$,

* for $\underline{SE(3)}$

$$\boxed{\begin{aligned} &\mathrm{Exp}(\xi) : \mathbb{R}^6 \longrightarrow SE(3) \\ &\mathrm{Ln}(T) : SE(3) \longrightarrow \mathbb{R}^6 \end{aligned}}$$