Introduction for Robotics to Rigid Body Transformations and Differentiation over $SE(3)$

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Abstract

This document is a comprehensive, self-contained and practical introduction to rotations and Rigid Body Transformations (RBT) in three dimensions. The document starts defining the group of RBT, known as $SE(3)$ and developing some mathematical tools to describe the its structure, using Lie algebra. From this tools, we will develop other useful topics to robotics and state estimation such as random variables, differentiation over $SE(3)$ and manifold optimization. The material is gathered from lecture notes on the course *Perception in Robotics* at Skoltech.

1 Rotations and Rigid Body Transformations

We will start by describing the mathematical properties for the groups of rotations and RBT. We will also consider some interesting properties and how we can actually use these groups in multiple ways, such as state variables, vector transformations or frame operations.

1.1 Rotations

All possible matrix rotations in 3D (generalizes to any dimension) are included in the Special Orthogonal group

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, RR^\top = I \,,\, det(R) = 1\}, \tag{1}$$

where the binary operation between two elements of the group is matrix multiplication. Since matrix multiplication is *non-commutative*, the group is *non-commutative* as well.

Four axioms of groups hold as well fin $SO(3)$:

- Closure: $R_1, R_2 \in SO(3) \Longrightarrow R_1 \cdot R_2 \in SO(3)$
- Associativity: $R_1(R_2R_3) = (R_1R_2)R_3$
- Identity element: $\exists! I \in SO(3) : RI = IR = R$. There exists a unique rotation I that satisfies this condition, and this element is the matrix identity.

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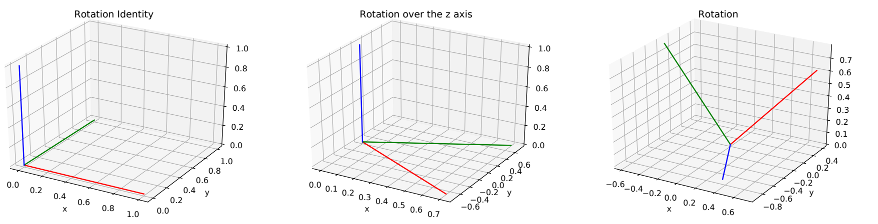

Figure 1: Examples of rotations.

- Inverse element: $\exists! R^{-1} \in SO(3) : RR^{-1} = I$. From the definition of the group one can derive the inverse element $R^{-1} = R^{\top}$.

The closure axiom implies that we can chain several different rotations and we will obtain a valid rotation as a result of this sequence of rotations. One has to be careful with the order, since the group operation is the matrix multiplication, meaning that in general

$$R_1 \cdot R_2 \neq R_2 \cdot R_1 .$$

The group of rotations $SO(3)$ can be used to 1) transform vectors and rotate them into new reference frames; 2) to transform reference frames as well (with coincident origins); 3) another valuable application is to express orientations.

1.2 Rigid Body Transformations

Similarly to $SO(3)$, all possible rigid body transformation (RBT) matrices conform the Special Euclidean group,

$$SE(3) = \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \mid R \in SO(3) \, , \, t \in \mathbb{R}^3 \right\} , \tag{2}$$

which is the result of a rotation followed by a translation and the group operation is the matrix multiplication.

Four axioms of groups are also satisfied:

- Closure: $T_1, T_2 \in SE(3) \Longrightarrow T_1 \cdot T_2 \in SE(3)$

$$T_1 \cdot T_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 \cdot R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix} \in SE(3)$$

- Associativity: $T_1(T_2 T_3) = (T_1 T_2) T_3$

- Identity element: $\exists! I \in SE(3) : TI = T$. There exists a unique identity element in the group which corresponds to a RBT. In particular, this is the 4×4 matrix identity.

- Inverse element: $\exists! T^{-1} \in SE(3) : TT^{-1} = I$. From the definition we can arrange terms such that the inverse corresponds to

$$T^{-1} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\top} & -R^{\top} t \\ 0 & 1 \end{bmatrix} \in SE(3) \, . \tag{3}$$

As a result of the *closure* axiom one can chain a sequence of RBT and obtain a valid transformation, very similarly to rotations. The physical meaning is a sequence of different frames compose a general frame.

For RBT the order matters as well,

$$T_1 \cdot T_2 \neq T_2 \cdot T_1,$$

where the left hand side and the right hand side are elements of the group, but in general they are not equal.

The potential uses of $SE(3)$ are very similar to rotations:

1. Transform points from one reference from to another.

$$^{W}p = {}^{W}T_{A} \cdot {}^{A}p$$

2. Transform reference frames:

$$^{W}T_{B} = {}^{W}T_{A} \cdot {}^{A}T_{B}$$

3. Express 3D poses (position and orientation). This is similar to the XYT parametrization for 2D poses ($SE(2)$) where 3 state variables $[x, y, \theta]$ completely define a RBT in 2D. However, we need to define which is the minimal representation for 3D poses which is the topic of the next section.

2 Lie Algebra for Rotations $SO(3)$

This section is devoted to explain Lie algebra for rotations. Later, the same intuition can be used to derive similar results to Rigid Body Transformations $SE(3)$ (Sec. 3)

2.1 Infinitesimal increments over Rotations $SO(3)$

First we need to understand the structure of infinitesimal variations in a rotation matrix.

As discussed before, R is orthonormal and has positive determinant, which constrains the space of solution in the differential form. A natural question arises regarding the group of rotations and RBT: What is the minimal representation? How many degrees of freedom?

To illustrate this, let's consider a rotation $R \in SO(3)$, and we are looking for a smooth rotation that provides an infinitesimal update to R over time in the following way:

$$\dot{R} = WR \qquad s.t. \quad RR^{\top} = I \tag{4}$$

$$\begin{aligned} \dot{R}R^{\top} + R(\dot{R})^{\top} &= 0 \\ W\underbrace{RR^{\top}}_{I} + \underbrace{RR^{\top}}_{I}W^{\top} &= 0 \iff W = -W^{\top}. \end{aligned} \tag{5}$$

The group of matrices that satisfy (5) is known as *Skew symmetric matrices.*

For 2D, the group looks like this $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ and for 3D rotations

$$W = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \boldsymbol{\omega}^\wedge \tag{6}$$

The hat operator $(\cdot)^\wedge$ denotes the construction of the skew symmetric matrix. In fact, this group is also a Lie Group $\boldsymbol{\omega}^\wedge \in \mathfrak{so}(3)$ around the identity element ($R = I$). The group operation is the Lie bracket operation, but we will not use it in this document. Lie groups in general need an additional property to those described in 1.1, which is smoothness (previously shown).

2.2 The Exponential Map

We have derived a differential form for rotations $\boldsymbol{\omega}^\wedge \in \mathfrak{so}(3)$ and the solution to the differential equation is of the form

$$R(t) = e^{\boldsymbol{\omega}^\wedge t} \cdot R(t_0), \tag{7}$$

where the resultant rotation is a function of time t. Now the question is, can we solve this equation for matrices as well? There is an analogous derivation from kinematics, presented in [5], that uses the angular velocity of each frame. The notation of using $\boldsymbol{\omega}$ is drawn from the fact that this corresponds to a natural angular rotation, if we focus on the physical meaning of what smoothness entails for rotation matrices.

The integration of (7) requires a constant $\boldsymbol{\omega}$ and a final time that we set to $t = 1$, for instance. We will follow the following convention: distinguish between the derivative of rotations, with units $[rad/s]$ and the "angle" of the rotation, still to be properly defined in 3D. Accordingly, $\boldsymbol{\theta}^\wedge = \boldsymbol{\omega}^\wedge t\Big|_{t=1}$ corresponds to the skew-symmetric matrix of the "angle" $\boldsymbol{\theta}$.

Accordingly, the **exponential map** solves this differential equation and it can be calculated by the Taylor expansion of $\boldsymbol{\theta}$, in the following case, if the rotation R in (7) is the identity rotation:

$$\exp(\boldsymbol{\theta}^\wedge) = I + \boldsymbol{\theta}^\wedge + \frac{1}{2!}(\boldsymbol{\theta}^\wedge)^2 + \frac{1}{3!}(\boldsymbol{\theta}^\wedge)^3 + \ldots = \sum_n^\infty \frac{1}{n!}(\boldsymbol{\theta}^\wedge)^n. \tag{8}$$

Skew symmetric matrices present a recursive property that turn out to be very useful, where $\theta = ||\boldsymbol{\theta}||_2$

$$(\boldsymbol{\theta}^\wedge)^2 = \omega \cdot \omega^\top - \theta^2 I, \qquad \theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{9}$$

$$(\boldsymbol{\theta}^\wedge)^3 = (\omega\omega^\top - \theta^2 I)\,\boldsymbol{\theta}^\wedge = 0 - \theta^2\,\boldsymbol{\theta}^\wedge \tag{10}$$

and so forth. One can calculate the closed form for the series arising after the simplification given by skew symmetric matrices and will obtain the well known Rodrigues' formula:

$$R = \exp(\boldsymbol{\theta}^\wedge) = I + \frac{\sin(\theta)}{\theta}\,\boldsymbol{\theta}^\wedge + \frac{1 - \cos(\theta)}{\theta^2}(\boldsymbol{\theta}^\wedge)^2 \tag{11}$$

An alternative interpretation of the Rodrigues' formula is drawn by using the angle-axis rotation:

$$R = I + \frac{\sin(\theta)}{\theta} a^\wedge + \frac{1-\cos(\theta)}{\theta^2} (a^\wedge)^2 = \cos(\theta) I + (1-\cos(\theta)) a a^\top + \sin(\theta) a^\wedge \quad (12)$$

where $a = \frac{\boldsymbol{\theta}}{\theta}$ is a unit vector, the axis of rotation, and the angle of rotation around this axis is θ.

The exponent is a surjective function, since a unique rotation can be obtained from different values of $\boldsymbol{\theta}$. The analogy with a 1D angle α is clear, where multiples values of $\alpha' = \alpha + i2\pi$, $\forall i \in \mathbb{Z}$ represent the same angle α. However, under some conditions, this will not be a problem, as we will see later.

Now that we have found an analytical solution to the exponential map, we can analyze in more detail the structure of the Lie algebra $\boldsymbol{\theta}^\wedge$, spanned by three variables:

$$\boldsymbol{\theta}^\wedge = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{G_1} \theta_1 + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{G_2} \theta_2 + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{G_3} \theta_3. \quad (13)$$

The elements created by linear combinations of the generator matrices G_i, span a vector space. Under this conditions, we have found the minimal representation for rotation matrices, in this case by using these three angles $[\theta_1, \theta_2, \theta_3]$. The Lie algebra $\mathfrak{so}(3)$ around the identity element in the Lie group quite resembles a 3D Euclidean space $\mathbb{R}^3$, at it is also referred as the *tangent space of rotations around the identity.*

There is a sequence of operations from rotations to the tangent space:

$$(\cdot)^\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$$
$$\exp(\boldsymbol{\theta}^\wedge) : \mathfrak{so}(3) \to SO(3)$$

In an abuse of notation we can define the (capital) exponent as a composition of the functions above, which directly maps the manifold to rotations:

$$\mathrm{Exp}(\boldsymbol{\theta}) : \mathbb{R}^3 \to SO(3) \quad (14)$$

Useful properties of the exponent $R = \mathrm{Exp}(\omega)$

$$\mathrm{Exp}(-\boldsymbol{\theta}) = R^{-1} = R^\top \quad (15)$$
$$\mathrm{Exp}(\tau\boldsymbol{\theta}) = \mathrm{Exp}(\boldsymbol{\theta})^\tau \quad (16)$$

Is there an inverse solution? Yes, the logarithm, which can be easily obtained by writing the series of a rotation and its inverse (transpose):

$$\begin{aligned} R &= I + \frac{\sin(\theta)}{\theta} \boldsymbol{\theta}^\wedge + \frac{1-\cos(\theta)}{\theta^2} (\boldsymbol{\theta}^\wedge)^2 \\ R^{-1} = R^\top &= I - \frac{\sin(\theta)}{\theta} \boldsymbol{\theta}^\wedge + \frac{1-\cos(\theta)}{\theta^2} (\boldsymbol{\theta}^\wedge)^2 \\ \hline R - R^\top &= 0 + 2 \cdot \frac{\sin(\theta)}{\theta} \boldsymbol{\theta}^\wedge + 0 \end{aligned}$$

$$\begin{tikzcd}
SO(3) \arrow[r, bend left, "\ln()"] \arrow[rr, bend left=60, "\mathrm{Exp}()"] & \mathfrak{so}(3) \arrow[l, bend left, "\exp()"] \arrow[r, bend left, "(\cdot)^\vee"] & \mathbb{R}^3 \arrow[l, bend left, "(\cdot)^\wedge"] \arrow[ll, bend left=60, "\mathrm{Ln}()"]
\end{tikzcd}$$

Figure 2: Mapping functions

which after some manipulation the following expression can be obtained:

$$\boldsymbol{\theta}^\wedge = \frac{\theta}{2\sin\theta}(R - R^\top). \tag{17}$$

The value of θ can be obtained similarly if we sum both expressions:

$$\begin{aligned}
R + R^\top &= 2I + 0 + 2\frac{1-\cos(\theta)}{\theta^2}(\boldsymbol{\theta}^\wedge)^2 \\
\mathrm{Tr}(R + R^\top) &= 2\,\mathrm{Tr}(I) + 2\frac{1-\cos(\theta)}{\theta^2}\,\mathrm{Tr}((\boldsymbol{\theta}^\wedge)^2) \\
2\,\mathrm{Tr}(R) &= 2\cdot 3 + 2\frac{1-\cos(\theta)}{\theta^2}\,\mathrm{Tr}(\omega\omega^\top - \theta^2 I) \\
\mathrm{Tr}(R) &= 3 + \frac{1-\cos(\theta)}{\theta^2}(\theta^2 - 3\theta^2) \implies 2\cos(\theta) = \mathrm{Tr}(R) - 1
\end{aligned}$$

which can be rearranged into the following equation to obtain θ:

$$\theta = arc\cos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right). \tag{18}$$

The inverse operation is known as the logarithm, and it first maps a rotation to the Lie algebra:

$$\ln(R) : SO(3) \to \mathfrak{so}(3)$$

and then to map from the Lie algebra, to the manifold:

$$(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3.$$

Similarly as what we proposed to the Exp(), we can define a function that first maps a rotation to the Lie algebra and then to the manifold $\mathbb{R}^3$:

$$\mathrm{Ln}(R) : SO(3) \to \mathbb{R}^3.$$

3 Lie Algebra for RBT $SE(3)$

3.1 Infinitesimal increments over RBT $SE(3)$: Twists

A similar reasoning from Sec. 2 can be done, now for RBT

$$\dot{T} = \mathcal{W}T, \quad \text{s.t.} \quad T \in SE(3), \tag{19}$$

where $\mathcal{W}$ is a *Twist* of 3D poses. If we expand (19) further, we obtain

$$\mathcal{W} = \dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{t} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^\top & -\dot{R}R^\top t + \dot{t} \\ 0 & 0 \end{bmatrix} \tag{20}$$

One can identify the same result previously derived for $SO(3)$ plus a term related to the rotated derivative of the translation.

$$\mathcal{W} = \begin{bmatrix} \boldsymbol{\omega}^\wedge & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

We have denoted the components related to the rotation, representing an orientation, with $\boldsymbol{\omega}$ (angular velocities) and the components related to the translation vector as $\boldsymbol{v}$ (linear velocities).

This Twist can be integrated to obtain a RBT which solve the differential equation (19)

$$T(t) = e^{\mathcal{W}t} \cdot T(t_0). \tag{22}$$

The integration of the *Twist*, which is composed of angular and linear velocities, is assumed constant over a fixed amount of time $t = 1$

$$\mathcal{W}|_{t=1} = \xi^\wedge = \begin{bmatrix} \theta^\wedge & \rho \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 & \rho_1 \\ \theta_3 & 0 & -\theta_1 & \rho_2 \\ -\theta_2 & \theta_1 & 0 & \rho_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

where there are 6 elements on the 4×4 matrix of generators.

The Lie algebra $\mathfrak{se}(3)$ associated with the group of RBT $SE(3)$ represents the group of infinitesimal RBT around the identity ($\mathcal{W} = \dot{T}$). There exist operators that relate both groups. In particular, the exponent operator $\exp : \mathfrak{se}(3) \to SE(3)$ and the logarithm $\ln : SE(3) \to \mathfrak{se}(3)$.

The vee $^\vee$ and hat $^\wedge$ operators simply encode (23) into a vector, whose space is called the manifold and from the manifold back to the Lie group. One can map a RBT $T \in SE(3)$ to $\xi \in \mathbb{R}^6$ by $\xi = \ln(T)^\vee$ and vice-versa $T = \exp(\xi^\wedge)$. In general, this mapping is surjective, but if $||w|| < \pi$, then we can consider it bijective.

$$\xi = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \tag{24}$$

Useful properties of the exponent $T = \text{Exp}(\xi)$

$$\text{Exp}(-\xi) = T^{-1} \tag{25}$$

$$\text{Exp}(\tau\xi) = \text{Exp}(\xi)^\tau \tag{26}$$

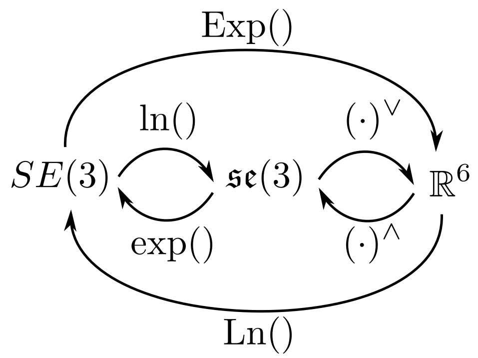

Figure 3: Mapping functions

The topic of Lie algebra for RBT is vast and well documented. We just reviewed those concepts that are used on the sections below. For a more complete discussion on Lie algebra and its applications please check [4, 3, 6, 2].

4 Adjoint

4.1 $SO(3)$

$$R \cdot \mathrm{Exp}(\theta) = \mathrm{Exp}(Adj_R \cdot \theta) \cdot R \tag{27}$$

$$Adj_R = R \tag{28}$$

4.2 $SE(3)$

$$T \cdot \mathrm{Exp}(\xi) = \mathrm{Exp}(Adj_T \cdot \xi) \cdot T \tag{29}$$
$$\mathrm{Exp}(Adj_T \cdot \xi) = T \cdot \mathrm{Exp}(\xi) T^{-1} \tag{30}$$

Where ξ is a vector in the manifold and expresses a transformation.

$$Adj_T = \begin{bmatrix} R & 0 \\ t^\wedge R & R \end{bmatrix}_{6\times 6} \tag{31}$$

The physical meaning of the adjoint for 3D RBT is a linear transformation of coordinates in the tangent space around the identity to coordinates in the tangent space around T.

5 Random Variables in $SE(3)$

5.1 Normal Random Variable in $SE(3)$

For state estimation, we are interested in obtaining distributions of variables, and that will be true for RBT as well. Then, the question is how to propose a distribution over a group of matrices which has some redundancies? In order to solve that, we need variables that are the minimal representation for $SE(3)$.

We have discussed before that Lie algebra provides us the tools to do that. The peculiarity around groups is that if some randomness is added into an element of the group, then the result must be an element of the group too. For that, we can define a *normal* random $SE(3)$ variable as a composition of a fixed transformation $\bar{T} \in SE(3)$ and a Gaussian random variable $\delta \in \mathbb{R}^3$ in the manifold

$$T = \mathrm{Exp}(\delta) \cdot \bar{T}, \qquad \delta \sim \mathcal{N}(0, \Sigma_T). \tag{32}$$

We will follow a left-hand-side convention, i.e. the perturbation $\mathrm{Exp}(\delta)$ is multiplying element $\bar{T}$ by its left-hand side. Other works follow a right-hand-side convention. We could draw samples δ_i in the manifold and obtain a random RBT.

5.2 Sample Mean and Sample Covariance

As we have presented before, we can define a random variable in the manifold. Now the problem is the opposite, we have different samples of a RBT and we want to calculate mean and covariance of a normal distribution given some samples of a transformation drawn from $T_i \sim p(T)$. The calculation of the sample mean $\bar{T} \in SE(3)$ is an iterative process, where we need a starting value of the mean $\bar{T}_{[0]}$. Then

$$\begin{aligned} \bar{T}_{[k+1]} &= E\{T\} = E\{T \cdot \bar{T}_{[k]}^{-1} \cdot \bar{T}_{[k]}\} = E\{\mathrm{Exp}(\delta)\} \cdot \bar{T}_{[k]} \\ &= \mathrm{Exp}\left(\frac{1}{N}\sum_{n=0}^{N} \delta_n\right) \cdot \bar{T}_{[k]} \end{aligned} \tag{33}$$

until convergence to $\bar{T}$. Sample covariance can be calculated directly by

$$\Sigma_T = E\{\underbrace{\mathrm{Ln}(T\bar{T}^{-1})}_{\delta} \mathrm{Ln}(T\bar{T}^{-1})^{\top}\} = \frac{1}{N-1}\sum_{n=0}^{N} \delta_n \cdot \delta_n^{\top}. \tag{34}$$

5.3 Covariance Propagation over a Function

Given a function of a RBT $f(T) : SE(3) \to \mathbb{R}^m$, then how does the covariance from the normal r.v. propagate to the image of the function in $\mathbb{R}^m$? We can obtain the following result after applying first order approximation

$$\Sigma_f = F\,\Sigma_\delta F^{\top}. \tag{35}$$

5.4 Covariance Propagation after Transformation

How to propagate the covariance of a r.v. after transforming it by $f(T) : SE(3) \to SE(3)$?

For example, T_r is a random normal variable $T_r = \mathrm{Exp}(\delta)\bar{T}_r$, where $\delta \sim \mathcal{N}(0, \Sigma_\delta)$ and the transformation is $T_{new} = T \cdot T_r$

$$T \cdot T_r = T \cdot \mathrm{Exp}(\delta)\bar{T}_r = \underbrace{\mathrm{Exp}(Adj_T \cdot \delta) \cdot T}_{(30)} \cdot \bar{T}_r$$

In this particular example, mean transformation is $T \cdot \bar{T}_r$ and covariance $\Sigma_{new} = Adj_T \Sigma_\delta Adj_T^{\top}$.

6 Differentiation

The purpose of this section is to obtain a derivative of any function w.r.t to a matrix transformation.

$$\frac{\partial f(T)}{\partial T} = \lim_{\Delta T \to 0} \frac{f(T + \Delta T) - f(T)}{\Delta T}.$$

In general this is an ill-posed operation, since the definition of differentiation (directional derivative) does not consider matrices. To alleviate this problem, we will change variables between a transformation matrix and its Lie algebra coordinates expressed in the manifold.

More formally, this change is known as a *Retraction*, and it is a smooth mapping from the tangent space around the T element to the manifold: $R_T(\xi) : \mathbb{R}^6 \to SE(3)$ [1].

The retraction map, in this document, is defined with the exponential map

$$R_t(\xi) = \operatorname{Exp}(\xi)T. \tag{36}$$

There are many other valid retractions, but we have chosen the exponential map since it is the most natural choice and it has already been discussed its properties and relation to $SE(3)$.

A property of the retractions is $R_T(0) = T$, which indicates that ξ is a small variation around the element T.

Two alternative procedures for differentiation of functions w.r.t. RBT are discussed as follows.

6.1 Differentiation: First order approximation

We want to differentiate over the function $f(T) : SE(3) \to \mathcal{K}$, where the image of this function $\mathcal{K}$ could be a vector, a matrix, $SE(3)$, etc. The solution is to use the retraction map, and do function composition $f_T = f \circ R_T$, such that

$$f_T(\xi) = f(R_T(\xi)) = f(\operatorname{Exp}(\xi)T).$$

Then, one can apply standard differentiation rules on an input that does no longer belong to $SE(3)$, but to the manifold of RBTs $\mathbb{R}^6$.

We will make use of the following approximation $\operatorname{Exp}(\xi) \simeq I + \xi^\wedge$, a first order Taylor expansion. This approximation is accurate enough since we are proposing increments around the identity element ($\xi = 0$), and hence, .

In order to obtain a derivative of the exponential map, we will use the Taylor expansion. The derivative of the retraction is only evaluated around the identity element, i.e. $\xi = 0$, and hence higher order terms vanish when evaluated at this point:

$$\left.\frac{\partial \operatorname{Exp}(\xi)}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial}{\partial \xi}\left(I + \xi^\wedge + o(||\xi||^2)\right)\right|_{\xi=0} = \frac{\partial}{\partial \xi}\begin{bmatrix} 0 & -\theta_3 & \theta_2 & \rho_1 \\ \theta_3 & 0 & -\theta_1 & \rho_2 \\ -\theta_2 & \theta_1 & 0 & \rho_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{G}, \tag{37}$$

where $\mathbf{G}$ is a 3D tensor and each component $\boldsymbol{G}(i) = G_i$ is a Lie algebra matrix generator (see (13)). This method could be easily adopted to other input spaces, such as $SO(3)$, $SE(2)$, $SO(2)$, etc. just by considering their matrix generators spanning their corresponding tangent spaces.

6.1.1 Example: Transforming a vector

To better illustrate this, let's consider the following example: Transforming a vector.

The function $f(T) = T \cdot p$, a standard transformation T of a vector p in homogeneous coordinates. Then

$$f_T(\xi) = \mathrm{Exp}(\xi) \cdot T \cdot p \tag{38}$$

$$\left.\frac{\partial f_T}{\partial \xi}\right|_{\xi=0} = \frac{\partial}{\partial \xi}\left.\big(\,\mathrm{Exp}(\xi) \cdot T \cdot p\big)\right|_{\xi=0} = \mathbf{G} \cdot T \cdot p \tag{39}$$

In particular, for the first coordinate of our increments ξ_1 we obtain

$$\left.\frac{\partial f_T}{\partial \xi_1}\right|_{\xi_1=0} = \boldsymbol{G}(1) \cdot T \cdot p = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{G_1} \cdot T \cdot p.$$

If we solve element by element and remove the last row where all matrices generators have zeros, then the resultant Jacobian can be contracted into

$$\left.\frac{\partial f_T}{\partial \xi}\right|_{\xi=0} = [-(T \cdot p)^{\wedge} \mid I\,]_{3\times 6}. \tag{40}$$

6.2 Differentiation: Small perturbations

Let the function $g : SE(3) \to SE(3)$ whose input and output are RBT (Sec. 3). The same derivation could be done for other transformations, such as $SO(3)$, $SE(2)$, etc.

Every small perturbation on the input of g, where we apply the retraction $R_T(\xi)$, is expressed as $g(\mathrm{Exp}(\xi)T) = \mathrm{Exp}(\epsilon)g(T)$, and results as well in a small perturbation of the output. Here a left-hand-side conventions has been taken. If we expand further the previous term we obtain

$$\epsilon = \mathrm{Ln}\left(g(\mathrm{Exp}(\xi)T)g(T)^{-1}\right). \tag{41}$$

Note that we have defined a new function of the perturbation of $g(\cdot)$ expressed in the coordinates of the tangent space, so as a result the Jacobian that we should obtain if the new input $\xi \in \mathbb{R}^6$ and the output $\epsilon \in \mathbb{R}^6$ is a 6×6 matrix.

Now, the derivative of the function $g(\cdot)$ w.r.t. the transformation T is equivalent to the derivative of ϵ w.r.t. ξ

$$\frac{\partial g}{\partial T} = \frac{\partial \epsilon}{\partial \xi} = \frac{\partial}{\partial \xi}\Big(\,\mathrm{Ln}\left[g(\mathrm{Exp}(\xi)T) \cdot g(T)^{-1}\right]\Big) \tag{42}$$

We will see in the following examples how this notation serves useful.

6.2.1 Example: Direct Observation of a Pose

Consider the following function $g(T) : SE(3) \to SE(3)$

$$g(T) = T_{obs} \cdot T^{-1} \tag{43}$$

This function outputs the difference between a transformation T and a transformation T_{obs}, in the case when they perfectly match $T = T_{obs}$, then the output will be the identity element.

Now, if we apply (42)

$$\begin{aligned}
\frac{\partial g}{\partial T} &= \frac{\partial}{\partial \xi}\Big(\operatorname{Ln}\left[T_{obs} \cdot (\operatorname{Exp}(\xi)T)^{-1} \cdot \left(T_{obs} \cdot T^{-1}\right)^{-1}\right]\Big) \\
&= \frac{\partial}{\partial \xi}\Big(\operatorname{Ln}\big[\underbrace{T_{obs} \cdot T^{-1} \operatorname{Exp}(-\xi) \cdot \left(T_{obs} \cdot T^{-1}\right)^{-1}}_{\text{Adjoint (30)}}\big]\Big) \\
&= \frac{\partial}{\partial \xi}\Big(\operatorname{Ln}\big[\operatorname{Exp}\left(Adj_{\{T_{obs}T^{-1}\}} \cdot (-\xi)\right)\big]\Big) \\
&= -Adj_{\{T_{obs}T^{-1}\}},
\end{aligned} \tag{44}$$

we obtain a compact result for the derivative, a 6×6 matrix.

6.2.2 Example: Two RBT

Let the function $g(T_o, T_t) : SE(3) \times SE(3) \to SE(3)$ be equal to $g(T_o, T_t) = T_o T_{obs} T_t^{-1}$. This function measures how well our observation matches this pair of input poses, for instance, an odometry pose matching a pair of consecutive poses in our trajectory. Then, one can calculate the derivative w.r.t. the origin pose T_o

$$\begin{aligned}
\frac{\partial g}{\partial T_o} &= \frac{\partial}{\partial \xi}\Big(\operatorname{Ln}\big[\operatorname{Exp}(\xi)T_o T_{obs} T_t^{-1} \cdot \left(T_o T_{obs} T_t^{-1}\right)^{-1}\big]\Big) \\
&= \frac{\partial}{\partial \xi}\Big(\operatorname{Ln}\big[\operatorname{Exp}(\xi)\big]\Big) \\
&= I.
\end{aligned} \tag{45}$$

6.3 Chain Rule in differentiation

The chain rule applies as usual after applying the corresponding retractions to the original functions and then applying standard differentiation rules (Sec. 6.1 and Sec. 6.2).

6.3.1 Example: Anchor Factor

An anchor factor is a transformation function that directly observes a 3D pose (or RBT). If we were perfectly observing this pose, then

$$T_{obs} \cdot T^{-1} = T \cdot T_{obs}^{-1} = I.$$

This is a continuation of the example in Sec. 6.2.1, where we go one step further and define a function that measures the correct matching or alignment of the previous sequence of transformations.

Accordingly.

$$||r(T)||^2 = ||\operatorname{Ln}(T_{obs} \cdot T^{-1})||^2_{\Sigma}, \tag{46}$$

Intuitively, we want this residual to be as small as possible, ideally 0,

$$\begin{aligned} \frac{\partial}{\partial T}\Big(||r(T)||^2_{\Sigma}\Big) &= \frac{\partial}{\partial T}\Big(r(T)^{\top} \cdot \Sigma \cdot r(T)\Big) \\ &= r(T)^{\top}\Sigma\frac{\partial r(T)}{\partial T}, \end{aligned} \tag{47}$$

where we have obtained before a compact derivation for the derivative of r.

7 Updating a RBT

Now that we have obtained a derivative w.r.t the coordinates of the Lie algebra of $SE(3)$, we want to update the RBT accordingly. Usually, in vector spaces, increments to our state variables can be done in the following manner

$$x' = x + \Delta x$$

However, $SE(3)$ is a particular group and requires special update. For updating a pose $T \in SE(3)$, we will use the retraction we have defined earlier $R_T(\xi)$, such that

$$T' = R_T(\xi) = \operatorname{Exp}(\xi) \cdot T, \tag{48}$$

where we are following a left-hand side convention for updating the transformation, since we have followed a left-hand convention for obtaining derivatives. One can not follow a convention for obtaining derivatives and then use a different convention to update the result.

8 Interpolation

The continious time interpolated trajectory $T(\tau) : [0, 1] \to SE(3)$, and there are two points known in this trajectory $T(0) = T_o$ and $T(1) = T_f$.

In our first derivation, we will express the relation between the pair of poses $T_o, T_f \in SE(3)$ as

$$\Delta T \cdot T_o = T_f, \tag{49}$$

where ΔT corresponds to a global observation or left-hand-side expansion of T_o. Then, we can define the continious-time transfomation

$$\Delta T(\tau) = (T_f \cdot T_o^{-1})^{\tau}, \tag{50}$$

where $T(\tau = 0) = I$ and $T(\tau = 1) = \Delta T$. Then, it is straighforward to define the continuous time trajectory as an interpolation directly in the manifold

$$T(\tau) = \operatorname{Exp}\big(\tau \operatorname{Ln}(T_f \cdot T_o^{-1})\big)T_o. \tag{51}$$

Analogously, one can derive a different expansion for the right-hand-side and obtain:

$$T(\tau) = T_0 \operatorname{Exp}\big(\tau \operatorname{Ln}(T_o^{-1} \cdot T_f)\big). \tag{52}$$

Each of the two interpolation variants (51)(52) provides an identical solution.

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