Chapter 2

Fixed-Income Securities and Interest Rates

We now begin a systematic study of fixed-income securities and interest rates. The literal definition of a *fixed-income security* is a financial instrument that promises fixed (or definite) payments at prescribed future dates. In some cases only one payment is made at maturity; in other cases, periodic payments are made. Such a security is really a loan made by the purchaser of the security to the issuer of the security. The purchase price of the security is the *principal* of the loan. In order for such a security to be desirable to investors, the total amount of the future payments to be received should be greater than the purchase price[1], so that these payments reflect repayment of the original principal plus *interest*. The interest represents the price paid for the use of the borrowed money.

The payment amounts of the securities are typically described in terms of an *interest rate*. Roughly speaking, by a *debt security* we mean a financial security that is characterized by a principal amount (the amount borrowed), together with a plan for repayment of the principal plus interest. Some debt securities have variable interest rates that depend on the future values of certain "economic variables" and are unknown when the security is issued.

In practice, there is sometimes ambiguity regarding what *is* or *is not* a fixed-income security. For some securities that are broadly classified as being fixed-income securities, the payments are not known with certainty at the time that the security is issued either because the payments are tied to a variable interest rate that fluctuates or because the security has certain features of optionality built in. For our purposes, a reasonable working definition of fixed-income security is *a security that is either a debt security (possible with a variable interest rate) or a derivative written on debt securities.*

[1]Sometimes economic conditions are such that investors are willing to purchase a security that promises to make fixed future future payments that add up to less than the purchase price. Such a security has a *negative yield*. As of February 1, 2017, there are some important economies, including Japan and Germany where government issued bonds have negative yields. On a few occasions, governemnt issued bonds with negative yields have existed in the US. However, negative yields have been rare and short-lived in the US.

In this chapter we focus mostly on situations where the payment amounts and payment dates are known with certainty at the time that the security is issued. We refer to such securities as securities with *fixed payments*, *deterministic payments*, or *deterministic cash flows*. We will also discuss an extremely important kind of contract called an *interest rate swap* which involves payments that are known when the contract is issued. The class 21-378 (Mathematics of Fixed Income Markets) will devote an entire semester to studying fixed income securities.

2.1 Zero-Coupon Bonds and Discount Factors

The most basic type of fixed-income security is a *zero-coupon bond*. This type of security makes a single payment of amount F (called the *face value*) at a prescribed future date T (called the *maturity date*). Every security with deterministic payments can be expressed as a portfolio of zero-coupon bonds.

Most people have some familiarity with interest rates because they are used in the descriptions of many common financial transactions. One tricky point concerning interest rates is that they cannot be used to determine payment amounts unless a *compounding convention* is specified. In other words, to compute payment amounts, one needs not only an interest rate, but also an additional convention (that is often hidden in the "fine print"). There is a more intrinsic concept, known as a *discount factor*, that does not rely on any additional conventions. The *discount factor* for time $T > 0$, denoted $d(T)$, is defined to be the time-zero price of a zero-coupon bond having maturity T and face value $F = 1$. The no-arbitrage principle implies that the time-0 price of a zero-coupon bond with face value F and maturity is simply $Fd(T)$.

Discount factors for various maturities can be inferred from the prices of traded bonds. Various kinds of interest rates can then be computed from discount factors.

2.2 Basic Interest Rate Mechanics

Suppose that at time 0, an amount $A > 0$ is invested in zero-coupon bonds maturing at time $T > 0$. The value of the investment at time T will be

$$\frac{A}{d(T)};$$

in other words, the capital will grow by a factor of $1/d(T)$. An interest rate is a way to describe the growth of capital in terms of a rate per unit of time. Unless stated otherwise, we measure time in years and interest rates will be expressed as annualized rates. There is more than one convention for associating interest rates with discount factors.

Suppose that some money is deposited in the bank at time 0 and is left in the bank to be withdrawn at a future date T. Under typical economic conditions, the customer will receive more than the initial deposit at the withdrawl date. The difference between the final account value and the initial deposit is *interest*. The amount of interest to be paid will be based on the amount deposited, an *interest rate*, and a *compounding convention*. (The interest rate offered by the bank will generally be different for deposits of different lengths.) Unless otherwise stated, we assume that the interest rate for deposits is the same as the rate for loans.

Interest may be either *simple* or *compound*. In order to describe the basic mechanics of interest calculations, we shall assume that an initial amount $A > 0$ is invested at time 0, that no additional investments (or withdrawals) are made, and that the annual interest rate is[2] $r \geq 0$. We are interested in the value V_T of the investment at the maturity date T. Analogous formulas apply to the case of a loan. (Indeed, a loan can be considered as an investment of an amount $A < 0$.)

Remark 2.1. Unless stated otherwise, all interest rates should be assumed to be nonnegative. Under extreme economic conditions it can happen that interest rates become slightly negative. (This may seem strange because with a negative interest a deposit would lose money, so it would be better to just hold on to the money. However, it would not be wise to store \$100,000,000 under your mattress or even in your office safe.) It is quite rare for this to happen in the US – and if it does – the rates generally are only negative by a few one hundredths of a percentage point, and become positive again after a short period of time.

2.2.1 Simple Interest

The interest earned up to time T is simply equal to rTA. (This is just the familiar formula “Interest = (Principal) × (rate) × (time)”.) Therefore, we have

$$\begin{aligned} V_T &= A + rTA \\ &= A(1 + rT). \end{aligned}$$

In practice, the simple interest convention is used only for investments or loans of relatively short maturity (one year or less).

Example 2.2. (a) Suppose that you invest \$100 for 3 months in an account paying 4% simple interest. At the end of three months, you will have

$$V_{.25} = \$100(1 + (.04) \times .25) = \$101$$

in the account

(b) Suppose that you invest \$2,500 for 9 months in account paying 6.6% simple interest. At the end of 9 months you will have

$$V_{.75} = \$2,500(1 + (.066) \times .75) = \$2,623.75$$

in the account.

[2] The formulas with interest rates are the same if $r < 0$. However, certain statements can become very confusing if we allow $r < 0$. If there is a situation in which the possibility of a negative interest rate is important, I will explicitly state that we are now allowing r to be negative.

2.2.2 Annual Compounding

Suppose that an amount A is to be left on deposit for N years, where N is a positive integer. If interest is computed according to the annual compounding convention at rate r, the account value is updated to be

$$A(1+r)$$

at the end of year one. The interest that will be credited to the account at the end of year two is

$$rA(1+r)$$

and consequently the account value is updated to

$$A(1+r) + rA(1+r) = A(1+r)^2.$$

For each $k = 1, 2, \cdots, n$ the account balance will be updated to

$$A(1+r)^k$$

at the end of year k. In particular, at the end of N years, the account balance will be

$$V_N = A(1+r)^N.$$

(For an $N - year$ certificate of deposit (CD) at a bank, an early withdrawl penalty is generally charged if the customer wishes to withdraw the money before the end of year N. For example, if you invest \$1,000 in a 5-year CD at 6% and you want to withdraw the money at the end of 4 years, you should expect to receive less than $\$1,000(1.06)^4$.)

Example 2.3. Suppose that you invest \$100 for 2 years in an account paying 8% interest compounded annually. Then, after two years the value of the account will be

$$V_2 = \$100(1+.08)^2 = \$116.64.$$

Example 2.4. Suppose that \$1 were invested 217 years ago at the interest rate r with annual compounding. (We take $t = 0$ to correspond to 217 years ago, so that today corresponds to $t = 217$.)

(a) If $r = 3.3\%$ then the value of the investment today would be

$$(1+.033)^{217} \approx \$1,147.55.$$

(b) If $r = 6.6\%$ then the value of the investment today would be

$$(1+.066)^{217} \approx \$1,055,148.09.$$

(c) If $r = 9.9\%$ then the value of the investment today would be

$$(1+.099)^{217} \approx \$787,951,082.00.$$

2.2.3 Fractional Compounding

With annual compounding, interest is credited to the account once a year. In practice, the interest is usually credited more frequently (e.g. quarterly, monthly, or daily).

Let m be a positive integer and let $T > 0$ be an integer multiple of $\frac{1}{m}$. Suppose that an amount A is deposited for T years at the rate r. Suppose further that interest is credited to the account m times per year at equally spaced times according to the following rule: For each $k \in \{0, 1, 2, \cdots\}$ the account value at time $\frac{k+1}{m}$ is updated to

$$V_{\frac{k+1}{m}} = V_{\frac{k}{m}} \left(1 + \frac{r}{m}\right),$$

where $V_{\frac{k}{m}}$ is the account value at time $\frac{k}{m}$.
After n periods, the value of the account will be

$$V_{\frac{n}{m}} = A\left(1 + \frac{r}{m}\right)^n.$$

To find the value after T years, we simply put $n = mT$ and arrive at

$$V_T = A\left(1 + \frac{r}{m}\right)^{mT}.$$

In order to compare situations with different frequencies of compounding, it is useful to introduce the notion of an *effective interest rate* R. The effective interest rate R is the rate under which annual compounding yields the same result at the end of each year as the rate r does with fractional compounding. More precisely, R is the unique solution of the equation

$$1 + R = \left(1 + \frac{r}{m}\right)^m,$$

i.e.

$$R = \left(1 + \frac{r}{m}\right)^m - 1.$$

It is also straightforward to check that $R \geq r$. In this context we refer to r as the *nominal interest rate.*

Example 2.5. Suppose that the nominal interest rate is $r = 8\%$ and that \$100 is deposited for 2 years at $t = 0$. The following table gives the effective rate and the value of the account after 2 years under various frequencies of compounding

m	1	2	4	12	365	8760	525600
R	8%	8.16%	8.2432%	8.3000%	8.3278%	8.3287%	8.3287%
V_2	116.64	116.9859	117.1659	117.2888	117.3490	117.3510	117.3511

2.2.4 Continuous Compounding

The above example suggests that if the nominal rate is held fixed and the number of compoundings tends to infinity, then the effective rate will approach a limit. This is indeed the case, by virtue of the well-known fact

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m = e^r$$

from basic calculus. We refer to the situation where the value of the account balance for a T-year investment is determined by the rule

$$V_T = Ae^{rT}$$

as *continuous compounding* at the nominal interest rate r. The corresponding effective interest rate R for continuous compounding is given by

$$R = e^r - 1.$$

Although some financial institutions advertise that they use continuous compounding, the primary value of this concept is that it leads to much cleaner mathematical formulations in many situations.

Example 2.6. Suppose that \$100 is deposited at $t = 0$ for 2 years and that interest is compounded continuously at the nominal rate $r = 8\%$. We find that

$$R = e^{.08} - 1 = 8.3287\%$$

and that

$$V_2 = \$100 \exp\,(.08 \times 2) = \$100e^{.16} = \$117.3511.$$

The following remark addresses the important issue of what happens if different banks are using different frequencies of compounding.

Remark 2.7. Let m_1 and m_2 be positive integers and assume that $T > 0$ is an integer multiple of $\frac{1}{m_1}$ and $\frac{1}{m_2}$. Suppose that there are two banks - Bank 1 and Bank 2. At Bank 1, interest is compounded m_1 times per year and we can borrow or invest for T years at the nominal rate r_1. At Bank 2, interest is compounded m_2 times per year and we can borrow or invest for T years at the nominal rate r_2. If there is no arbitrage, then the effective rates at the two banks must be the same; in other words, we must have

$$\left(1+\frac{r_1}{m_1}\right)^{m_1} = \left(1+\frac{r_2}{m_2}\right)^{m_2}.$$

Indeed, if the effective rates were not the same then we could borrow money at $t = 0$ from the bank with the lower effective rate and invest the money in the bank with the higher effective rate. At time T we could withdraw the money we deposited (together with the interest), pay back the loan, and have some money left over. The same analysis applies if one of the banks compounds continuously.

In order to avoid repeating some standard assumptions over and over, it is useful to describe something called an *ideal money market*. In an *ideal money market with constant effective rate* R we can, at any time we choose, borrow or invest any amount of money for any length of time at the effective rate R. It is possible to choose any number of compoundings per year we wish, including continuous compounding. No fees or commissions are charged. Separate transactions in an account are additive in their effect on future balances. Notice that in an ideal money market with constant effective rate, the rate is assumed to be independent of the date at which the transaction is initiated and also the length of the loan or deposit.

Notice that the effective rate R implies a corresponding nominal rate for each possible number of annual compoundings. It is convenient to denote the nominal rate for m compoundings per year by $r[m]$, so that

$$\left(1+\frac{r[m]}{m}\right)^m = 1+R.$$

The nominal rate corresponding to continuous compounding will be denoted by $r[\infty]$, so that

$$e^{r[\infty]} = 1+R.$$

Observe that in an ideal money market with constant effective rate R, if V_0 is deposited at $t=0$ then the value of the account at time T will be

$$V_T = V_0(1+R)^T,$$

even if T is not an integer. If an amount V_τ is deposited at time τ, then for each $t \geq \tau$ the value of the account at time t will be

$$V_t = V_\tau(1+R)^{t-\tau},$$

assuming that no additional deposits or withdrawls were made between time τ and time t. The situation is analogous for loans. In fact, when we can borrow or invest at the same rate, a loan can be treated as an investment in which the amount invested is negative. For this reason, many formulas will be stated only for the case of an investment (or only for a loan).

Remark 2.8. It will be very convenient for us to use effective rates most of the time so we can avoid having to worry about compounding conventions. However, it is very important to realize that other conventions are frequently used to describe interest rates in practice (and in books on mathematical finance). It is common for a single bank to use different conventions to describe their different products so as to make the rates sound more attractive to potential customers. For example, bank CD rates are generally quoted as effective rates whereas loan rates are usually quoted as nominal rates. When comparing investment opportunities, it is crucial to ascertain the precise conventions used for computing the interest. If interest is actually being compounded m times per year, one must keep in mind that formulas involving effective interest rates should be applied only at times that are consistent with the compounding schedule. (In practice, a number of different conventions are used to compute the interest that is accrued for a fraction of an investment period. This issue will be discussed briefly in Remark 2.18.)

Remark 2.9. In an ideal money market with constant effective rate R, the account balance V_t at time t satisfies

$$\frac{d}{dt}V_t = (\ln(1+R))\, V_t = r[\infty]V_t$$

(at times when no money is being deposited or withdrawn). This means that if an amount α is deposited for a small time interval of length Δt then the interest over this time interval will be approximately

$$\alpha(\ln(1+R))\,\Delta t = \alpha r[\infty]\Delta t.$$

In such a situation, the description in terms of the nominal rate $r[\infty]$ is certainly cleaner. This is important to keep in mind during arguments in which a limit is taken as the length of an investment tends to zero. In such arguments the standard practice in the literature is to use the nominal rate $r[\infty]$ rather than the effective rate.

2.3 Term Structure of Interest Rates

Anyone who has studied bank CD rates or bond prices (or who has looked into getting a loan) knows that in reality, interest rates are not constant, but depend on factors such as the risk of default, the amount of money to be invested, the length of time (or term) of the investment, and the time at which the investment will be made. We now consider situations where the interest rate depends on time. We still assume that there is no risk of default, that the same rate is used for loans and investments, and that the same rate applies to transactions of any size. Throughout this discussion, we make the convention that the present time is $t = 0$, unless stated otherwise. It is very important to understand that the interest rates that will prevail at future times are unknown at the present time.

Suppose that we can borrow or invest any amount of money we want at $t = 0$ (the present time) up to any given maturity $T > 0$ at the effective annual rate $R_*(T)$. In other words, the interest rate is allowed to be a function of the maturity. It is assumed here that the full amount invested or borrowed is transferred at $t = 0$ and the only withdrawal or payment takes place at maturity (i.e. $t = T$). Early withdrawals and early payments are not allowed. If V_0 is invested at $t = 0$ (for a term of T years), then the terminal capital is given by

$$V_T = V_0(1 + R_*(T))^T.$$

Remark 2.10. If early withdrawals of deposits and early payments on loans were allowed "without penalty", then a no-arbitrage argument could be used to show that $R_*(T)$ must be constant in T. (Here by "without penalty" we mean that: for a deposit, the rate that was supposed to apply to the full-term deposit is also used for the shorter deposit; for a loan, we mean that the rate for the full term loan is used for the shorter term loan, but the interested is calculated only up until the time that the loan is repaid.)

We refer to $R_*(T)$ as the *effective (annual) spot rate* corresponding to maturity T. Here the term "spot" means current or present. It refers to the fact that the transaction will take place "on the spot". It is not known at the present time what the spot rates will be at future dates. The graph of $R_*(T)$ versus T is called the *effective spot-rate curve* or the[3] *effective yield curve.* The dependence of interest rates on the length of the investment (or loan) is referred to as the *term structure of interest rates.*

It is typically the case that longer maturities correspond to higher rates, although it sometimes does happen that short-term rates are higher than long-term rates. This latter situation occurs when investors expect interest rates to drop significantly in the future.

Remark 2.11. Spot interest rates are frequently described in terms of nominal rates. (This will occur in several exercises at the end of the chapter.) Throughout the text, spot rates will always be given as effective rates. In practice, it is essential to be sure what convention is being employed to describe the interest rates in question. It should also be kept in mind that the notion of compounding can be misleading when discussing spot rates, because interest is not actually added to an account balance at intermediate times – the principal and all interest are paid at time T. Consequently, the capital is well defined only at times 0 and T. The notion of a compounding convention simply tells us how to compute the terminal capital from the initial capital. In fact, the ratio V_0/V_T for a $T-year$ investment or loan is simply the discount factor $d(T)$ for maturity T. In particular, we have

$$d(T) = \frac{1}{(1+R_*(T))^T}, \quad R_*(T) = \left(\frac{1}{d(T)}\right)^{\frac{1}{T}} - 1.$$

For many purposes, it is much more convenient to use discount factors instead of spot rates. Indeed, discount factors are more intrinsic than spot rates because they are independent of any kind of compounding convention.

It is convenient to generalize the notion of an ideal money market to allow for term structure. We shall refer to this generalized notion as an ideal bank (because when there is term structure, the phrase money market is usually associated with short term investments). In an *ideal bank with effective spot rate function R_** we can choose any time $T > 0$ and borrow or invest any amount of money at time 0 (the present time) until maturity T at the effective rate $R_*(T)$. The full amount invested or borrowed is transferred at time 0 and settlement is made at time T with a single lump-sum payment (principal and interest). We can make multiple transactions at $t = 0$, each having different maturities. The effects of the individual transactions on future balances are additive. No fees are charged for making transactions.

[3]There is more than one type of yield curve. The graph of $R_*(T)$ versus T is the *effective zero-coupon yield curve.*

Example 2.12. Assume that there is an ideal bank with $R_*(.5) = 4\%$ and $R_*(8) = 5.2\%$.

(a) If an agent invests \$10,000 at $t = 0$ for six months, the value of this investment at maturity will be

$$\$10,000(1 + .04)^{.5} = \$10,000\sqrt{1.04} \approx \$10,198.04.$$

(b) If an agent borrows \$5,000 between $t = 0$ and $t = 8$, then at time 8 the agent will owe the bank

$$\$5,000(1 + .052)^8 \approx \$7,500.60.$$

It is important to understand that an ideal money market with constant effective rate R is not exactly the same an ideal bank with $R_*(T) = R$ for all $T > 0$ because in the ideal money market with constant effective rate R it is assumed that the effective spot rates that prevail in the future will also be equal to R for all maturities[4]. We shall use the term ideal bank to mean an ideal bank having some effective rate function R_*, possibly constant. Unless stated otherwise, from now on, we shall always assume the presence of an ideal bank.

Remark 2.13. In order to study term structure in a serious way, it is necessary to compare the spot rates that prevail on different dates. Therefore, it is necessary to use two time variables to describe interest rates. One of the variables will be the date on which the investment (or loan) will be initiated. The other variable is sometimes taken to be the length of the investment, and sometimes taken to be the date at which settlement is made. Of course, once the initiation date has been specified, one can determine the settlement date from the length of the investment, and vice-versa, so the two different notational conventions provide the same information. In practice, one must be very careful to check which convention is used to describe a particular interest rate. In this course, we use the notation $\mathcal{R}_{\tau,T}$ to denote the effective spot rate at time τ for investments (or loans) to be settled with a single lump-sum payment at time $T > \tau$. (In other words this rate corresponds to investments of duration $T - \tau$ that are initiated at time τ.) The rates $\mathcal{R}_{\tau,T}$ are known at time τ, but unknown prior to time τ. Notice that

$$R_*(T) = \mathcal{R}_{0,T}$$

for all $T > 0$. Nevertheless, we shall continue to use the notation $R_*(T)$ – even though it is not really needed any more – partly in the hope that the asterisk will serve as a reminder that we are making the convention that the present time is $t = 0$. (Indeed, there are situations in which one needs to talk about $\mathcal{R}_{0,T}$ at times other than $t = 0$.)

[4]It is possible to have arbitrage-free models for interest rates in which the yield curve is flat at $t = 0$ but does not remain flat in the future. In such a model, interest rates for investments made at $t = 0$ will be independent of maturity, but interest rates for investments initiated at other times can depend on the maturity of the investment.

2.4 Forward Interest Rates

It is not known today what the spot rates will be at future dates. However, if we want to agree today on an interest rate for a loan that will be initiated 6 months from today and repaid with a single lump-sum payment two years later (i.e., 2.5 years from today), we can use the no-arbitrage principle to determine an appropriate interest rate in terms of the spot rates $R_*(.5)$ and $R_*(2.5)$. Such an interest rate is known as a *forward rate*. It is evident that the idea of locking a rate now for an investment or loan to be made in the future could be extremely useful. Moreover, forward rates will play a key role in the development of a full theory of term structure. For these reasons, we give the definition here in a bit more generality than is required for immediate use of this concept, namely we allow for the interest rate to be agreed on at an arbitrary time τ rather than only at time 0.

Let τ, η, T with $\tau \leq \eta < T$ be given. Let us denote by $\mathcal{R}^{for}_{\tau,\eta,T}$ an effective interest rate agreed upon at time τ for an investment or loan (of a prescribed amount) initiated at time η and settled with a single lump-sum payment at time T. In particular, if V_η is invested at time η under the terms of an agreement made at time τ then the value of the investment at time T will be

$$V_T = V_\eta(1 + \mathcal{R}^{for}_{\tau,\eta,T})^{T-\eta}.$$

We refer to $\mathcal{R}^{for}_{\tau,\eta,T}$ as the *effective forward interest rate* for the time interval $[\eta, T]$ as seen from time τ.

Given $\eta \geq 0$ and $T > \eta$, let us determine a general expression for $\mathcal{R}^{for}_{0,\eta,T}$ using the no-arbitrage principle. The easiest way to do this is to replicate a forward loan (or deposit) using zero-coupon bonds, expressing the bond prices in terms of discount factors, and then expressing the discount factors in terms of spot rates.

Let us replicate a forward loan in which it is agreed at time 0 to borrow \$1 at time η and repay the loan with a single payment of $\$(1 + \mathcal{R}^{for}_{0,\eta,T})^{T-\eta}$ at time T. We replicate the position of the agent borrowing the money. The agent needs to

(i) recieve \$1 at time η and

(ii) owe $\$(1 + \mathcal{R}^{for}_{0,\eta,T})^{T-\eta}$ at time T.

To acheive (i) a zero-coupon bond with face value 1 and maturity η should purchased at time 0 and (ii) can be acheived by shortselling a zero-coupon bond with face value $(1 + \mathcal{R}^{for}_{0,\eta,T})^{T-\eta}$ and maturity T at time 0. The initial capital of this strategy is

$$X_0 = d(\eta) - (1 + \mathcal{R}^{for}_{0,\eta,T})^{T-\eta} d(T).$$

Since nothing is paid at time 0 to enter the agreement we must have $X_0 = 0$ which gives

$$(1 + \mathcal{R}^{for}_{0,\eta,T})^{T-\eta} = \frac{d(\eta)}{d(T)} = \frac{(1 + R_*(T))^T}{(1 + R_*(\eta))^\eta}.$$

After a bit of algebra we find that

$$\mathcal{R}^{for}_{0,\eta,T} = \left(\frac{(1 + R_*(T))^T}{(1 + R_*(\eta))^\eta} \right)^{\frac{1}{T-\eta}} - 1. \tag{2.1}$$

It is not necessary (nor is it even a good idea) to "memorize" equation (2.1). When you need a forward interest rate in practice, the compounding convention will likely be different. What you should do is be sure that you know how to replicate a forward loan using zero-coupon bonds. For many calculations with forward loans, it will be simpler to use zero-coupon bonds and discount factors than it will be to introduce forward rates.

Remark 2.14. The same argument shows that for each time τ we have

$$\mathcal{R}^{for}_{\tau,\eta,T} = \left(\frac{(1 + \mathcal{R}_{\tau,T})^{T-\tau}}{(1 + \mathcal{R}_{\tau,\eta})^{\eta-\tau}} \right)^{\frac{1}{T-\eta}} - 1.$$

Example 2.15. Today's date is $t = 0$. Suppose that we want to borrow \$10,000 six months from now (at $t = .5$) and repay the loan with a single lump-sum payment 2 years later (at $t = 2.5$). Suppose further that we go to a bank today to agree on an interest that will be used for this loan. Given that $R_*(.5) = 3\%$ and $R_*(2.5) = 4\%$, let us determine the appropriate effective interest rate and the amount that will be needed at $t = 2.5$ to repay the loan.

Putting $\eta = .5$ and $T = 2.5$, we find that

$$\mathcal{R}^{for}_{0,.5,2.5} = \left(\frac{(1 + .04)^{2.5}}{(1 + .03)^{.5}} \right)^{\frac{1}{2}} - 1 = 4.2515\%.$$

The amount we will need to pay at $t = 2.5$ is given by

$$\$10,000 \times (1 + .042515)^2 = \$10,868.375.$$

Remark 2.16. It is important to understand that even though forward interest rates do not depend on the amount of money to be borrowed or invested, when entering into an agreement to borrow or invest money at a future date it is essential that the amount of money to be borrowed or invested is agreed upon at the same time that the interest rate is agreed upon. It is also very important to realize that the spot rate $\mathcal{R}_{\eta,T}$ that will prevail at time η will be different from the forward interest rate $\mathcal{R}^{for}_{\tau,\eta,T}$, except by chance.

2.5 Basic Types of Securities with Deterministic Payments

We now describe some important securities having fixed payments (or deterministic cash flows). Such a security is completely described by a payment schedule (i.e. payment amounts and dates). We assume that the payment amounts and dates are known with certainty at the time that the security is issued. In practice, such securities are often described using interest rates or *yields*. We shall talk about yields or *rates of return* of securities with deterministic payments later. Unless stated otherwise, we assume that the present time is $t = 0$, that there is no risk of default, and that all securities can be bought or sold short in any desired quantities (including fractional numbers of securities).

2.5.1 Zero-Coupon Bonds

Although we have already introduced zero-coupon bonds, we repeat some information here for the sake of completeness. A *zero-coupon bond* (also called a *pure discount bond*) is characterized by a *face value* $F > 0$ and a *maturity date* $T > 0$. The holder of the bond receives a single payment of amount F at time T. Zero-coupon bonds are extremely simple securities. However, they are building blocks that can be used to replicate any security with fixed payments. In other words, any given security with fixed payments can be expressed as a portfolio of zero coupon bonds. The face value of a zero-coupon bond is also referred to as the *par value*.

2.5.2 General Security with Fixed Payments

A *general security with fixed payments* will be characterized by a list $(F_i, T_i)_{1 \le i \le N}$ where the T_i are times with $0 < T_1 < T_2 < \cdots < T_N$ and the F_i are payments received by the holder of the security at the times T_i. It is convenient to allow some (or all) of the F_i to be negative, but then we need to be careful about a sign convention:

(a) If $F_i > 0$ then the holder of the security receives F_i at time T_i.

(b) If $F_i < 0$ then the holder of the security must pay $|F_i|$ at time T_i.

(c) If $F_i = 0$, then no money changes hands at time T_i.

Although it is convenient to allow the F_i to be negative, some caution must be exercised. This can lead to situations in which the phrase "holder of the security" as used above may conflict with the usage of this phrase in ordinary parlance. If all of the F_i have the same sign, then it is clear which party must pay the other initially. If the F_i change sign some interesting and surprising situations can occur. In the vast majority of cases encountered in this course, the securities will be such that the F_i all have the same sign. Of course, payments of amount zero can be ignored. We are allowing for the possibility that some of the F_i are zero simply because this can sometimes simplify the indexing of the payment times.

In order to avoid cumbersome language we shall usually not distinguish between a general security with fixed payments and the list $(F_i, T_i)_{1 \le i \le N}$ of payment times and amounts.

A general security with fixed payments is often referred to as a *deterministic cash flow*.

2.5.3 Coupon Bond

A coupon bond is characterized by a face value $F > 0$ (also called the *par value*), a maturity date $T > 0$, a positive integer m giving the number of coupon payments per year, and a nominal annual coupon rate $q[m]$. The holder of the security will receive payments m times per year with equal spacing between payment times. To simplify the indexing, we assume that the maturity date is $T = N$, for some positive integer N, and that the payments will be received at the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots, mN$. The holder of the bond receives coupon payments

$$C = F\frac{q[m]}{m}$$

at the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots, mN$ plus the face value F at maturity. Notice that holder receives $F + C$ at time N. The coupon rate is usually chosen so that at the time the bond is originally issued, the price of the bond will be very close to the face value. Consequently the coupon payments are effectively interest payments on the principal F. The term "coupon payments" comes from the fact that in the past, physical coupons were attached to the bond documentation, and the holder of the bond would send in the coupons by mail in order to receive payments. Today, bonds generally do not include actual paper coupons, but the terminology persists. Coupon bonds issued by the US Treasury pay coupons twice per year, i.e. $m = 2$.

2.5.4 Annuity

An annuity is a security that pays the holder the same amount periodically (at equally spaced times) over some interval of time. An annuity is characterized by a payment size $A > 0$, the number m of payments per year, and a maturity date $T > 0$. Again, to simplify the indexing, we assume that the maturity date is $T = N$, for some positive integer N and that the payments will be received at the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots$, mN. The holder of the annuity receives the amount A at each T_i, $i = 1, 2, \ldots, mN$.

Remark 2.17. In practice, some coupon bonds are *callable*. This means that at certain times prior to maturity, the issuer of the bond can decide to pay the face value F (or some other contractually specified amount) plus any coupon that is due at that date to the holder thereby relieving the issuer from the obligation to make any further payments to the bond holder. Callable bonds will not be treated here.

Remark 2.18. In our description of coupon bonds, the assumption that T is a positive integer could easily be replaced by the assumption that T is an integer multiple of $\frac{1}{m}$, necessitating only minor changes in formulas. However, the assumption that the first payment will be received at time $\frac{1}{m}$ is much more significant. This indicates that we are considering a newly issued bond or that we are considering a previously issued bond immediately after a coupon payment has been made. As soon as a coupon is paid the value of a bond must drop by the amount of the coupon payment. For several reasons, it is inconvenient to quote prices that are discontinuous.

In practice, even when one ignores the bid-ask spread, there will be two relevant prices for coupon bonds at times after their initial issue – a so-called *flat price* and a so-called *full price*. The flat price is the one that is quoted in places such as *The Wall Street Journal*. The full price (or *invoice price*) is the one that a purchaser of the bond must actually pay. These two prices are the same immediately after a coupon payment has been made. At times between coupon payments, the full price is higher. The difference between the two prices is the interest I that has accrued on the face value since the last coupon payment. The formula for computing the accrued interest is

$$I = q[m] \times \Delta T,$$

where ΔT is the time in years since the last coupon payment. In other words, the accrued interest is computed using the simple interest convention.

It should be noted that the formula used to compute the accrued interest is just a market convention designed to make quoted prices of bonds continuous in time. One could use a different convention if desired and this would slightly change the values of the flat price and the accrued interest. This is not an issue because it is the full price that is determined by market conditions; by definition the flat price price plus the accrued interest must add up to the full price.

We know that $I < C$ because $\Delta T < \frac{1}{m}$. Since most years have an odd number of days, and because payments generally are not made on weekends and holidays, the calculation of ΔT is not completely clearcut. The manner in which the interval between two calendar dates is interpreted as a fraction of a year is called a *day count convention*. In practice, different day count conventions are used for different types of fixed-income securities. Although such market details fall outside the scope of the course, and we will not pursue these issue further, it is important to be aware that one must be careful when talking about prices of coupon bonds at times between the coupon payments. The flat price is sometimes called the *clean price* and the full price is sometimes called the *dirty price*.

Remark 2.19. In practice, there are lifetime annuities that pay the holder the same amount periodically for the remainder of his or her life, and annuities in which the amount of the payments is variable. Annuities involving variable maturity or variable payments will not be treated here. Unless stated otherwise, when we use the term "annuity" it is to be understood tacitly that the payment amounts are constant and that the maturity date is finite and specified at time 0.

Remark 2.20. A mortgage (or any other loan with constant payments at equally spaced times) can be viewed as an annuity with the roles of the customer and the bank being interchanged. Consequently the formulas we obtain for annuities can also be used to analyze loans of this type. However, there are often important practical differences between mortgage and annuity contracts. In particular, most mortgages (and other types of loans) have pre-payment provisions that allow the client to pay the outstanding balance of the loan early and avoid interest charges. Annuities typically do not have such provisions. Most mortgages and consumer loans involve monthly payments. The standard day count convention for such loans is that all months are treated as $\frac{1}{12}$ of a year.

2.6 Discount Factors and Present Value of Future Payments

Suppose that we have access to an ideal bank with effective spot rate function R_*. If we invest an amount A between time 0 (the present time) and time T, we can easily compute the value of this investment at time T. This process can also be carried out in reverse. If we want the value of our investment at some given future time T to be B, we can easily determine an amount A that if invested now will have value B at time T. In this case, we say that A is the *present value* of a payment B that will be made at time T. For example if $R_*(1) = 5\%$ and you are going to receive a rebate of \$210 from a company one year from now, the present value of the rebate is \$200, because an investment of \$200 now will be worth $\$200 \times (1.05) = \$210$ one year from now. It may seem that receiving \$210 one year from now is different than receiving \$200 now, because in one case you have to wait a year until you receive any money and in the other case you get some money right away. However, since borrowing from the bank is allowed, the two situations are completely equivalent. Indeed, you can borrow \$200 from the bank (between now and time 1) and do with it as you please. One year from now, when you receive \$210 you can repay the loan.

Although the notion of present value is very simple, it provides us with a powerful tool for evaluating investments involving payments at different times. In particular, if an investor truly has access to an ideal bank then it is sensible for the investor to decide between two investment possibilities having fixed payments by simply computing the *net present values* (i.e., the sum of the present values of all payments) for each alternative and choosing the one whose net present value is most favorable to the investor. Indeed, if two payment streams have the same net present value, then one can be "converted" to the other via transactions with the bank.

Recall that the *discount factor* $d(T)$ for time T is the price at time 0 to receive \$1 at time T and it is given by

$$d(T) = \frac{1}{(1 + R_*(T))^T}. \tag{2.2}$$

Observe that if A is the present value of a payment of size B that will occur at time T then

$$A = Bd(T).$$

Remark 2.21. A very important property of discount factors is that they remove the need to worry about the convention that is being used to describe interest rates. The formula to compute $d(T)$ will look different if the interest is computed using a rate other than the effective rate, but the numerical value of $d(T)$ will be the same – no matter what convention is used to describe interest rates. The discount factor $d(T)$ is simply the ratio of the initial capital to the capital at time T for a T-year investment in the money market. For this reason, discount factors are more fundamental than interest rates.

2.7 Arbitrage-Free Prices of Securities with Fixed Payments in the Presence of an Ideal Bank

Suppose that there is an ideal bank with effective spot rate function R_*. It is straightforward to determine the arbitrage-free prices of the securities discussed above. Securities with fixed payments can be (and frequently are) traded at times after the issue date. In computing prices for such trades we must be careful to make sure that the payment times are measured relative to the present time (rather than the issue date) and we must be sure to realize that the new owner of the security receives only those payments made after the trade is executed. We generally use the symbol $\mathcal{P}$ (possibly embellished) to denote the current price of a security with fixed payments. If a security is being traded on a date at which the security is to make a payment, we assume that the trade takes place immediately after the party who is selling the security has received the payment in question. We make a similar convention in pricing (or valuing) securities that were issued in the past and are currently being held as part of a portfolio – the price on a payment date is understood to be the price just after the payment has been made.

2.7.1 Zero-coupon Bond

Consider a zero-coupon bond with face value F and maturity T. In order to determine the arbitrage-free price of the bond at $t = 0$ we can construct a replicating strategy using the bank account. If we invest X_0 at $t = 0$ until $t = T$, then the value of the investment at time T will be

$$X_T = X_0(1 + R_*(T))^T.$$

Since the terminal capital must satisfy $X_T = F$, we must have

$$X_0 = \frac{F}{(1 + R_*(T))^T} = Fd(T).$$

Therefore, the arbitrage-free price $\mathcal{P}$ of the bond at $t = 0$ is

$$\mathcal{P} = Fd(T) = \frac{F}{(1 + R_*(T))^T}. \tag{2.3}$$

Since zero-coupon bonds can always be replicated by simple bank deposits, we now consider zero-coupon bonds of all possible face values and maturities to be basic (or traded) securities.

2.7.2 General Security with Fixed Payments

Consider a general security with fixed payments $(F_i, T_i)_{1 \le i \le N}$. We have not yet defined exactly what a replicating strategy is for securities that make multiple payments. (We shall do so shortly.)However, it seems pretty clear what a replicating strategy should be for this security. Let X be a portfolio consisting of zero-coupon bonds with maturities T_i and face values $|F_i|$, for each i with $1 \leq i \leq N$; if $F_i > 0$ then one bond with maturity T_i and face value $|F_i|$ is purchased in the portfolio, while if $F_i < 0$ then one bond with maturity T_i and face value $|F_i|$ is sold short in the portfolio. (If $F_i = 0$, the there is nothing to worry about.) Notice that the strategy X will not be self-financing. We shall refer to this strategy as a replicating strategy for the general security with fixed payments $(F_i, T_i)_{1 \leq i \leq N}$. The initial capital of the replicating strategy is

$$X_0 = \sum_{i=1}^{N} F_i d(T_i).$$

(Notice that if the i^{th} bond has been sold short then $F_i < 0$ and the corresponding contribution to the initial capital of the portfolio is $-|F_i| = F_i$.) The arbitrage-free price of the security at $t = 0$ is therefore given by

$$\mathcal{P} = \sum_{i=1}^{N} F_i d(T_i) = \sum_{i=1}^{N} \frac{F_i}{(1 + R_*(T_i))^{T_i}}. \tag{2.4}$$

Our official definition of a replicating strategy for a security making multiple payments is as follows: A strategy X is a replicating strategy for a security making multiple payments provided that the strategy Y obtained by implementing X and selling short the security is a self-financing strategy with 0 terminal caplital. Notice that Y is self-financing and has terminal capital $Y_T = 0$. If we let Z be the zero strategy, (i.e. the strategy in which no money is ever invested in anything) then Z is also self-financing and has terminal capital $Z_T = 0$. Consequently, $Y_0 = Z_0 = 0$, which implies that initial price of the security must be the same as the initial capital of a replicating strategy.

We note that $\mathcal{P}$ (as defined by (2.4)) is also called the *net present value* of the security. This is very appropriate terminology because $F_i d(T_i)$ is the present value of a payment of F_i received at time T_i. The sum of these payments is simply the total or net present value of the future payments. If $\mathcal{P} > 0$ then one must pay money initially to hold the security, while if $\mathcal{P} < 0$ then the holder of the security should initially receive money from the counterparty. (Of course, if $\mathcal{P} = 0$ then no money changes hands at $t = 0$.) Notice that the sign convention for $\mathcal{P}$ is opposite to the sign convention for the F_i; $\mathcal{P} > 0$ means that the holder pays money, while $F_i > 0$ means that the holder receives money. The reason for this is that the amount $\mathcal{P}$ paid by the holder at $t = 0$ is supposed to balance out the net present value of the payments that the holder will receive in the future. When there is an ideal bank, we may treat general securities with fixed payments as basic (or traded) instruments because they can easily be replicated in terms of transactions with the bank.

2.7.3 Coupon Bond

Consider a coupon bond with face value $F > 0$, and maturity $T > 0$, making coupon payments m times per year at the nominal annual coupon rate $q[m]$. As discussed previously, we assume that $T = N$, for some positive integer N, and that the holder of the bond receives coupon payments

$$C = F\frac{q[m]}{m} \tag{2.5}$$

at the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots, mN$ plus an additional payment of amount F at time N. The arbitrage-free price at $t = 0$ (net present value) of the bond is therefore

$$\mathcal{P} = Fd(N) + \sum_{i=1}^{mN} Cd\left(\frac{i}{m}\right) = \frac{F}{(1 + R_*(N))^N} + \sum_{i=1}^{mN} \frac{C}{(1 + R_*(\frac{i}{m}))^{i/m}}. \tag{2.6}$$

2.7.4 Annuity

Consider an annuity that makes payments of size $A > 0$ m times per year, and has maturity date $T > 0$. We assume that $T = N$, for some positive integer N and that the holder of the annuity receives payments of size A at each of the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots,\ mN$. Therefore, the arbitrage-free price at $t = 0$ (net present value) is given by

$$\mathcal{P} = \sum_{i=1}^{mN} Ad\left(\frac{i}{m}\right) = \sum_{i=1}^{mN} \frac{A}{(1 + R_*(\frac{i}{m}))^{i/m}}. \tag{2.7}$$

Remark 2.22. When we analyze mortgages (or other loans with constant payments at equally spaced times), we shall usually replicate the position of the *creditor* (i.e., the lender) in order to cut down on the number of minus signs needed. At $t = 0$, the creditor will pay a positive amount $\mathcal{P}$ to the *debtor* (i.e., the borrower) in exchange for the debtor's promise to pay the creditor a positive amount A at each of the times $T_i = \frac{i}{m}$, $i = 1, 2, \ldots,\ mN$.

Example 2.23. Assume that $R_*(3) = 5\%$ and consider a zero coupon bond with maturity $T = 3$ and face value $F = \$10,000$. Suppose that we have an opportunity to purchase one of these bonds at the price $\hat{\mathcal{P}} = \$8,500$. What should we do?

To answer this question, we should compute the arbitrage-free price of the bond. The discount factor $d(3)$ is given by

$$d(3) = \frac{1}{(1 + R_*(3))^3} = \frac{1}{(1 + .05)^3} = .863838,$$

which yields an arbitrage-free price of

$$\mathcal{P} = Fd(3) = \$10,000 \times (.863838) = \$8,638.38.$$

Since the price at which we can purchase the bond is lower than the arbitrage-free price, we should buy the bond. This would lead to an immediate profit of \$138.38 at $t = 0$ via the following strategy:

(a) Borrow \$8,638.38 from the bank between $t = 0$ and $t = 3$.

(b) Buy the bond for \$8,500 at $t = 0$.

The difference, \$138.38, constitutes a profit that we can safely consume now (or invest for later). At $t = 3$ we will receive \$10,000 (the face value of the bond) and we must pay the debt $\$8,638.38 \times (1 + .05)^3 = \$10,000$ on the bank account. The debt on the bank account will be perfectly offset by the face value of the bond.

We could also implement a slightly different strategy:

(a′) Borrow \$8,500 from the bank between $t = 0$ and $t = 3$.

(b′) Buy the bond for \$8,500 at $t = 0$.

There is no money left over at $t = 0$. However, if we follow this strategy, then at $t = 3$ we will receive \$10,000 (face value of the bond), but we will owe only $\$8,500 \times (1 + .05)^3 = \$9,839.81$ to the bank, leaving us a profit of $\$10,000 - \$9,839.81 = \$160.19$. This is perfectly reasonable because \$138.38 is the present value of \$160.19 at time 3. Observe that the opportunity to buy one of these bonds for \$8,500 is an arbitrage opportunity. However, the arbitrage is limited in scope since we have the opportunity to buy only one bond. Such an opportunity might very well exist in the real world. However, if a dealer were offering to sell many of these bonds at \$8,500 per bond (and if transaction costs were significantly less than \$138.38 per bond) investors would flock to purchase the bond and the high demand would drive the price upward.

Example 2.24. Assume that

$$R_*(.5) = 5\%, \;\; R_*(1) = 5.25\%, \;\; R_*(1.5) = 5.5\%, \;\; R_*(2) = 6\%. \tag{2.8}$$

Consider a coupon bond with maturity $T = 2$ and face value $F = \$10,000$ that pays coupons twice per year at the nominal coupon rate $q[2] = 4\%$ Suppose that we have an opportunity to buy one of these bonds at the price $\hat{\mathcal{P}} = \$9,750$ What should we do?

To answer this question, we should compute the arbitrage-free price $\mathcal{P}$ of the bond. Before computing $\mathcal{P}$, let us observe that we should expect $\mathcal{P}$ to be less than the face value F because the effective coupon rate is $(1 + .02)^2 - 1 = 4.04\%$ and this is noticeably less than all of the relevant effective spot rates.

The coupon payments are given by

$$C = F\frac{q[2]}{2} = \$10,000 \times \left(\frac{.04}{2}\right) = \$200.$$

Using (2.2) and (2.8), we find that the discount factors are given by

$$d(.5) = \frac{1}{(1 + .05)^{.5}} = .9759001, \;\; d(1) = \frac{1}{(1 + .0525)^1} = .9501188,$$

$$d(1.5) = \frac{1}{(1 + .055)^{1.5}} = .9228292, \;\; d(2) = \frac{1}{(1 + .06)^2} = .8899964.$$

The arbitrage-free price of the bond is given by

$$\mathcal{P} = \$200 \times d(.5) + \$200 \times d(1) + \$200 \times d(1.5) + \$10,200 \times d(2) = \$9,647.73.$$

We should not purchase a bond at the price $\hat{\mathcal{P}} = \$9,750$ because this price is higher than the arbitrage-free price.

Example 2.25. Assume that $R_*(15) = 10\%$. Annuities with maturity $T = 15$ that make payments of \$500 per month are trading at the current price of \$50,000. Consider a coupon bond with maturity $T = 15$ and face value $F = \$1,000$ that pays coupons monthly at the nominal coupon rate $q[12] = 12\%$. Determine the price at $t = 0$ of the bond.

Since we have not been given all of the relevant spot rates, we cannot use (2.6) directly. Instead, we can find a simple replicating strategy. The coupon payments are given by

$$C = F\frac{q[m]}{m} = \$1,000 \times \frac{.12}{12} = \$10.$$

We can replicate the stream of coupon payments by purchasing $\frac{1}{50}$ annuities at $t = 0$. The payment of F at $T = 15$ can be replicated by investing

$$\$1,000d(15) = \frac{\$1,000}{(1 + R_*(15))^{15}} = \frac{\$1,000}{(1 + .1)^{15}} = \$239.39$$

in the bank between $t = 0$ and $t = 15$. The initial capital of this strategy is

$$\$50,000 \times \frac{1}{50} + \$239.39 = \$1,239.39,$$

and consequently the arbitrage-free price of the bond at $t = 0$ is $\mathcal{P} = \$1,239.39$.

2.8 Prices of Coupon Bonds and Annuities in the Presence of an Ideal Bank with Constant Effective Spot Rate Function

When interest rates are the same for all maturities, the formula for summing a geometric series can be used to simplify the expressions (2.6) and (2.7) for the prices of coupon bonds and annuities. In order to pursue this issue, we assume, temporarily, that there is an ideal bank with constant effective spot rate function $R_*(T) = R$.

Recall that

$$\sum_{i=0}^{k-1} \lambda^i = 1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1} = \frac{1 - \lambda^k}{1 - \lambda} \text{ for } \lambda \neq 1,$$

from which it follows immediately that

$$\sum_{i=1}^{k} \lambda^i = \lambda + \lambda^2 + \lambda^3 + \cdots + \lambda^k = \lambda \left(\frac{1-\lambda^k}{1-\lambda} \right) \text{ for } \lambda \neq 1. \tag{2.9}$$

Observe that the relevant discount factors are given by

$$d\left(\frac{i}{m}\right) = \frac{1}{(1+R)^{i/m}} = \left(\frac{1}{(1+R)^{1/m}}\right)^i, \tag{2.10}$$

which suggests that we put

$$\lambda = \frac{1}{(1+R)^{1/m}} = \frac{1}{1+\frac{r[m]}{m}}. \tag{2.11}$$

Here $r[m]$ is the nominal rate (for compounding m-times per year) that is consistent with the effective rate R. Notice that the expression for λ in terms of the nominal rate $r[m]$ is slightly cleaner because it does not involve an m^{th} root.

Combining (2.9), (2.10), and (2.11) we find that

$$\sum_{i=1}^{mN} d\left(\frac{i}{m}\right) = \lambda \left(\frac{1-\lambda^{mN}}{1-\lambda} \right). \tag{2.12}$$

Observe also that

$$d(N) = \lambda^{mN}. \tag{2.13}$$

Using (2.12) and (2.13) we obtain the following two very important remarks.

Remark 2.26. If $R_*(T) = R$, for all $T > 0$ the formula (2.6) for the arbitrage-free price of a coupon bond simplifies to

$$\mathcal{P} = F\lambda^{mN} + C\lambda \left(\frac{1-\lambda^{mN}}{1-\lambda} \right), \tag{2.14}$$

where λ is given by (2.11).

Remark 2.27. If $R_*(T) = R$ for all $T > 0$, the formula (2.7) for the arbitrage-free price of an annuity simplifies to

$$\mathcal{P} = A\lambda \left(\frac{1-\lambda^{mN}}{1-\lambda} \right), \tag{2.15}$$

where λ is given by (2.11).

2.9 Internal Rate of Return or Yield to Maturity of a Security with Fixed Payments

The assumption that interest rates are independent of maturity is not very realistic – especially for securities such as coupon bonds and annuities that have a long period of time remaining until maturity. Since payments will be received at numerous times (with each payment time possibly corresponding to a different spot rate), it is not evident how to assign a single interest rate to such a security. One way that this can be accomplished is by means of something called an *internal rate of return*.

Definition 2.28. A real number $R_I > -1$ is called an *effective internal rate of return* for the general security with fixed payments $(F_i, T_i)_{1\leq i\leq N}$ provided that

$$\hat{\mathcal{P}} = \sum_{i=1}^{N} \frac{F_i}{(1+R_I)^{T_i}}, \tag{2.16}$$

where $\hat{\mathcal{P}}$ is the price of the security at $t = 0$ (the present time).

Remark 2.29. We have used the symbol $\hat{\mathcal{P}}$, rather than $\mathcal{P}$ to denote the price at $t = 0$ because there are certain fixed income securities (such as corporate bonds) that are traded in the real world and for which it is not appropriate to use equation (2.4) to compute an arbitrage-free price. The reason for this is that (2.4) was obtained under the assumption that there is no risk of default. Although bonds issued by companies having strong credit ratings are usually pretty safe investments, it does sometimes happen that such bonds default. It is standard practice to use (2.16) (with $\hat{\mathcal{P}}$ equal to the market price) to determine effective internal rates of return for corporate (and other) bonds. In such cases it is important to realize that returns computed in this way are returns that are **promised, but not guaranteed**. Of course, for a bond having risk of default, the rate of return should be higher than for a US Treasury bond promising the same payments. In fact, listings of corporate bond prices usually indicate the difference between the return of the bond and the return of a US Treasury bond with the same promised payments.

From now on, we use the standard abbreviation IRR for internal rate of return. Notice that an effective IRR has the following simple interpretation: It is a constant effective interest rate R_I such that the actual price of the security coincides with the net present value of the security's future payments as computed under the assumption of an ideal bank with constant effective spot rate function $R_*(T) = R_I$.

Remark 2.30. In the presence of an ideal bank with constant effective spot rate function $R_*(T) = R$, if there is no arbitrage then R is an effective IRR for all securities with fixed payments. The notion of IRR is of most significance in situations where interest rates depend on maturity.

In view of the remark (and examples) below, one must be very cautious in using IRRs in situations when the F_i are allowed to change sign.

Remark 2.31. If the F_i are not all of the same sign, it is possible that a given fixed-income security may have exactly one effective IRR, more than one effective IRR, or no effective IRR.

Example 2.32 (More than one IRR)**.** Consider the fixed income security $(F_i, T_i)_{1\leq i\leq 2}$ with issue date $t = 0$, $F_1 = \$3$, $F_2 = -\$2$, and price $\hat{\mathcal{P}} = \$1$. In other words, the holder pays \$1 at $t = 0$ to purchase the security, receives \$3 at $t = 1$, and pays \$2 at $t = 2$. An effective IRR for this security is a solution R_I of the equation.

$$1 = \frac{3}{1 + R_I} - \frac{2}{(1 + R_I)^2}.$$

To solve this equation, we make the substitution $x = \frac{1}{1+R_I}$ to obtain

$$2x^2 - 3x + 1 = 0,$$

which is equivalent to

$$(2x - 1)(x - 1) = 0.$$

We find that $x = \frac{1}{2}$ or $x = 1$, which yields $R_I = 100\%$ or $R_I = 0\%$. It is straightforward to check that both rates are consistent with the payment schedule of the security.

(a) Suppose that $R_I = 100\%$. The holder of the security pays \$1 at $t = 0$, and therefore he is owed \$2 at $t = 1$. Since he receives \$3 at $t = 1$, he then owes \$1. This means that he will owe \$2 at $t = 2$. Since he will pay \$2 at $t = 2$, the books will be clear.

(b) Suppose that $R_I = 0\%$. The holder of the security pays \$1 at $t = 0$, and therefore he is owed \$1 at $t = 0$ (since the interest rate is 0%). He receives \$3 at $t = 1$ and therefore he now owes \$2. He pays back \$2 at $t = 2$ and the books will be clear.

Example 2.33 (Nonexistence of IRR)**.** Consider the fixed income security $(F_i, T_i)_{1 \le i \le 2}$ with issue date $t = 0$, $F_1 = \$3$, $F_2 = -\$2$, and price $\hat{\mathcal{P}} = \$1.25$. This is the same security as in the previous example, but the price has been increased to \$1.25. An effective IRR for this security is a solution R_I of the equation.

$$1.25 = \frac{3}{1 + R_I} - \frac{2}{(1 + R_I)^2}.$$

To solve this equation, we make the substitution $x = \frac{1}{1+R_I}$ to obtain

$$2x^2 - 3x + 1.25 = 0. \tag{2.17}$$

It is straightforward to check that (2.17) has no real roots and consequently the security in question does not have an effective IRR.

If $F_i > 0$ for $i = 1, 2, \ldots, N$ and $\hat{\mathcal{P}} > 0$ (or if $F_i < 0$ for $i = 1, 2, \ldots, N$ and $\hat{\mathcal{P}} < 0$) then the general security with fixed payments $(F_i, T_i)_{1 \le i \le N}$ has exactly one IRR. More precisely, we have the following theorem.

Theorem 2.34. *Consider the security with fixed payments $(F_i, T_i)_{1\le i\le N}$ with current price $\hat{\mathcal{P}} > 0$ at $t = 0$. Assume that $F_i > 0$ for all $i = 1, 2, \ldots, N$. Then there is exactly one effective IRR, $R_I > -1$, for this security. Moreover, $R_I > 0$ if and only if*

$$\sum_{i=1}^{N} F_i > \hat{\mathcal{P}}, \tag{2.18}$$

i.e. if and only if the sum of the payments received by the holder exceeds the purchase price.

Proof. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = -\hat{\mathcal{P}} + \sum_{i=1}^{N} F_i x^{T_i} \quad \text{for all } x \ge 0.$$

Observe that f is continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and

$$f'(x) = \sum_{i=1}^{N} F_i T_i x_i^{T_i - 1} > 0 \text{ for all } x > 0.$$

It follows that f is strictly increasing. Observe also that $f(0) = -\hat{\mathcal{P}} < 0$ and $\lim\limits_{x\to\infty} f(x) = \infty$. Using the Intermediate Value Theorem and the fact that f is strictly increasing we conclude that there is exactly one number $x^* > 0$ such that $f(x^*) = 0$. If we put

$$R_I = \frac{1}{x^*} - 1$$

then $x^* = \frac{1}{1+R_I}$ and it is easy to see that R_I is an effective IRR. If $R_I^* > -1$ is also an effective IRR then $f\left(\frac{1}{1+R_I^*}\right) = 0$ which implies $\frac{1}{1+R_I^*} = x^* = \frac{1}{1+R_I}$ and consequently $R_I^* = R_I$. Finally, observe that $R_I > 0$ if and only if $x^* < 1$. This is the case if and only if $f(1) > 0$, i.e. if and only if (2.18) holds. $\square$

Definition 2.35. For a security with fixed payments $(F_i, T_i)_{1\le i\le N}$ with $F_i > 0$ for $i = 1, 2, \ldots, N$ and current price $\hat{\mathcal{P}} > 0$ (or with $F_i < 0$ for $i = 1, 2, \ldots, N$ and $\hat{\mathcal{P}} < 0$), the unique effective IRR is also called the *effective yield to maturity* of the security. In fact, the terminology "yield to maturity" is more commonly used than "IRR"for such securities.

2.9.1 Yield to Maturity of Zero-Coupon Bond

The effective yield to maturity R_I of a zero-coupon bond with face value F, maturity T years from today, and current price $\hat{P}$ can be computed easily. In particular, we have

$$\hat{\mathcal{P}} = \frac{F}{(1 + R_I)^T}.$$

Solving for R_I we obtain

$$R_I = \left(\frac{F}{\hat{\mathcal{P}}}\right)^{\frac{1}{T}} - 1. \tag{2.19}$$

If the bond is priced at the arbitrage-free price of a guarenteed payment of amount F at time T, (i.e., $\hat{\mathcal{P}} = Fd(T)$), it follows easily from (2.19) that $R_I = R_*(T)$ as we should expect.

Example 2.36. (a) Consider a zero-coupon bond with face value $F = \$10,000$, current price $\hat{\mathcal{P}} = \$9,500$, and maturity $T = 1.5$ years from today. We find that

$$R_I = \left(\frac{10,000}{9,500}\right)^{\frac{1}{1.5}} - 1 = 3.48\%.$$

(b) For a zero-coupon bond with face value $F = \$50,000$, current price $\hat{\mathcal{P}} = \$49,600$, and maturity 13 weeks from today, we have $T = \frac{1}{4}$ so that

$$R_I = \left(\frac{50,000}{49,600}\right)^{4} - 1 = 3.27\%.$$

2.9.2 Yield to Maturity of a Coupon Bond

Consider a coupon bond with face value F and maturity $T = N$ years from today for some positive integer N. Assume that the bond pays coupons m times per year at the nominal coupon rate $q[m]$ and that the current price of the bond is $\hat{\mathcal{P}}$. (We assume that if the bond was issued in the past, then the price $\hat{\mathcal{P}}$ is the price just after today's coupon payment was made.) The amount of each coupon payment is

$$C = \frac{Fq[m]}{m}.$$

The yield to maturity R_I of the bond satisfies

$$\hat{\mathcal{P}} = \frac{F}{(1+R_I)^N} + \sum_{i=1}^{mN} \frac{C}{(1+R_I)^{i/m}}. \tag{2.20}$$

We assume

$$F + mNC > \hat{\mathcal{P}},$$

which ensures that $R_I > 0$ by virtue of Theorem 2.34.

If we make the substitution $\lambda = \frac{1}{(1+R_I)^{1/m}}$ then (2.20) becomes

$$F\lambda^{mN} + C\sum_{i=1}^{mN} \lambda^i - \hat{\mathcal{P}} = 0. \tag{2.21}$$

This is a polynomial equation of degree mN for λ. (This polynomial has $mN+1$ nonzero coefficients.) If we use (2.9) then, (2.21) becomes

$$F\lambda^{mN} + C\lambda\left(\frac{1-\lambda^{mN}}{1-\lambda}\right) - \hat{\mathcal{P}} = 0, \tag{2.22}$$

which can be rearranged to obtain

$$(F+C)\lambda^{mN+1} - F\lambda^{mN} - (\hat{\mathcal{P}}+C)\lambda + \hat{\mathcal{P}} = 0. \tag{2.23}$$

This is a polynomial equation of degree $mN+1$ for λ, but there are only four nonzero coefficients.

In practice, one would generally need to solve (2.21) or (2.23) by a numerical method such as bisection or Newton's method. The corresponding value of R_I can be found from the equation $R_I = \lambda^{-m} - 1$. It is worth noting that although the polynomial in (2.21) may have more nonzero terms, this polynomial is strictly increasing for $\lambda > 0$ and has exactly one positive root. On the other hand, the polynomial in (2.23) has two positive roots: an extraneous root of $\lambda = 1$ that was introduced by multiplying through by $\lambda - 1$ and the desired root (which will probably be very close to 1, but less than 1). One must therefore be a bit more careful when solving (2.23).

2.9.3 Annuity Yield

For annuities, the effective IRR is often called the *effective annuity yield* or *effective annuity rate*. Consider an annuity with maturity $T = N$ years from today for some positive integer N. Assume that the annuity makes m payments of size A per year and that the current price is $\hat{\mathcal{P}}$. (We assume that if the annuity was issued in the past, then $\hat{\mathcal{P}}$ is the price just after today's payment was made.) The effective annuity yield R_I satisfies

$$\hat{\mathcal{P}} = \sum_{i=1}^{mN} \frac{A}{(1+R_I)^{i/m}}. \tag{2.24}$$

We assume that

$$mNA > \hat{\mathcal{P}},$$

which ensures that $R_I > 0$. If we let $\lambda = \frac{1}{(1+R_I)^{1/m}}$ then (2.24) becomes

$$A\sum_{i=1}^{mN} \lambda^i - \hat{\mathcal{P}} = 0. \tag{2.25}$$

Using (2.9), we find, after simplification, that

$$A\lambda^{mN+1} - (\hat{\mathcal{P}}+A)\lambda + \hat{\mathcal{P}} = 0. \tag{2.26}$$

We note that (2.25) is a polynomial equation of degree mN (with $mN+1$ nonzero coefficients) for λ. The polynomial is strictly increasing for $\lambda > 0$ and has exactly one positive root. On the other hand (2.26) is a polynomial equation of degree $mN + 1$ for λ, but there are only three nonzero coefficients. This equation has an extraneous root of $\lambda = 1$, as well as the desired root which will be less than 1, but probably close to 1. We can solve either one of these equations numerically for λ and then obtain the corresponding value of R_I from the relation $R_I = \lambda^{-m} - 1$.

Remark 2.37. For mortgages, the IRR is generally called the *mortgage rate*. Even though most mortgages allow prepayments (without penalty), the mortgage rate is computed assuming that there will be no prepayments.

2.9.4 Nominal IRR

The IRR can also be expressed as a nominal rate $r_I[m]$, corresponding to compounding m times per year, via the formula

$$\left(1 + \frac{r_I[m]}{m}\right)^m = 1 + R_I, \tag{2.27}$$

or as a nominal rate $r_I[\infty]$, corresponding to continuous compounding, via the formula

$$e^{r_I[\infty]} = 1 + R_I. \tag{2.28}$$

In cases where payments occur at regular times $T_i = \frac{i}{m}$, the IRR is usually given as a nominal rate $r_I[m]$. In particular, mortgage rates are typically quoted as nominal rates $r_I[12]$. The yield to maturity of bond paying coupons semiannually is usually given as a nominal rate $r_I[2]$. The continuously compounded nominal rate $r_I[\infty]$ leads to cleaner formulas in many situations and is often used by mathematicians for this reason.

Example 2.38. Consider a coupon bond (issued today) with maturity $T = 10$ and face value $F = \$10,000$ that pays coupons twice per year at the nominal coupon rate $q[2] = 4\%$. Given that the price of the bond is \$9,000 find the yield to maturity.

The coupon payments are given by

$$C = \left(\frac{.04}{2}\right) \times (\$10,000) = \$200,$$

so that (after division by 100) (2.23) becomes

$$102\lambda^{21} - 100\lambda^{20} - 92\lambda + 90 = 0.$$

Solving this equation numerically (and ignoring the root $\lambda = 1$) we find that $\lambda = .974178$. This yields

$$R_I = \left(\frac{1}{\lambda}\right)^2 - 1 = 5.3715\%.$$

The corresponding nominal yield to maturity is given by

$$r_I[2] = 2((1 + R_I)^{\frac{1}{2}} - 1) = 2\left(\frac{1}{\lambda} - 1\right) = 5.3013\%.$$

Example 2.39 (Yield to Maturity of a Coupon Bond that was Previously Issued)**.** Today's date is $t = 0$. Consider a coupon bond issued three years ago today with face value $F = \$1,000$. Assume that the bond pays coupons twice per year at the nominal coupon rate $q[2] = 4\%$. The original maturity of the bond was 5 years, i.e. the bond matures 2 years from today. Suppose that the bond will be traded today just after the original holder receives today's coupon payment. Assume that the effective spot rates for maturities .5, 1, 1.5, and 2 years are $R_*(.5) = 2.4\%$, $R_*(1) = 2.5\%$, $R_*(1.5) = 2.6\%$, and $R_*(2) = 2.7\%$. Find the arbitrage-free price today (just after the coupon payment is made). Assuming that the bond trades at its arbitrage-free price, find the yield to maturity of the bond at the time of the trade?

Observe that the coupon payments are given by $C = \$1000 \times (\frac{.04}{2}) = \$20$. We next compute the relevant discount factors:

$$d(.5) = \frac{1}{(1.024)^{.5}} = .988212, \quad d(1) = \frac{1}{1.025} = .975610,$$

$$d(1.5) = \frac{1}{(1.026)^{1.5}} = .962230, \quad d(2) = \frac{1}{(1.027)^2} = .948111.$$

The arbitrage-free price of the bond today (just after the coupon payment) is given by

$$\mathcal{P} = 20d(.5) + 20D(1) + 20d(1.5) + 1020d(2) = \$1025.59.$$

To determine the effective yield to maturity, we make the substitution $\lambda = \frac{1}{(1+R_I)^{.5}}$. Equations (2.21) and (2.23) become

$$1020\lambda^4 + 20\lambda^3 + 20\lambda^2 + 20\lambda - 1025.59 = 0 \tag{2.29}$$

and

$$1020\lambda^5 - 100\lambda^4 - 1045.59\lambda + 1025.59 = 0. \tag{2.30}$$

The reader is urged to factor out $\lambda - 1$ from equation (2.30) and verify that this procedure results in equation (2.29).

Using a numerical method such as bisection to solve (2.29) we find that $\lambda = .98679$, which yields

$$R_I = \left(\frac{1}{\lambda}\right)^2 - 1 = 2.695\%.$$

The nominal yield to maturity is given by

$$r_I[2] = 2((1 + R_I)^{\frac{1}{2}} - 1) = 2\left(\frac{1}{\lambda} - 1\right) = 2.677\%.$$

Remark 2.40. It is true in general that the current price of a coupon bond (when it is issued, or immediately after a coupon payment) is greater than the face value of the bond if and only if the nominal coupon rate is greater than the nominal yield to maturity. (See Exercise 29 of this chapter.)

Remark 2.41 (Current Yield). Another kind of yield for coupon bonds that is frequently used in practice is the so-called *current yield* which expresses the annual coupon payments as a fraction of the current bond price (rather than the face value). The nominal current yield $r_{cur}[m]$ of a coupon bond that pays coupons m times per year at the nominal coupon rate $q[m]$ is given by

$$r_{cur}[m] = \frac{q[m]F}{\hat{\mathcal{P}}},$$

where $\hat{\mathcal{P}}$ is the current price of the bond and F is the face value. Notice that the amount of the coupon payments can be expressed as

$$C = \hat{\mathcal{P}}\frac{r_{cur}[m]}{m}.$$

2.10 Floating Rate Bonds and Interest Rate Swaps

In practice, the interest payments for certain types of loans and investments are computed using variable interest rates that "float" with the market. A complete analysis of such securities cannot be carried out until we have more machinery available; however, there are a couple of important situations that we can analyze now, namely floating rate bonds and interest rate swaps. We assume that the present time is $t = 0$ and that there is an ideal bank. For each $t \geq 0$ and $s > t$ we let $\mathcal{R}_{t,s}$ denote the effective spot rate that will prevail at time t for investments or loans initiated at time t and settled with a single lump-sum payment at time s. In other words, if an amount A is invested at time t, then at maturity s, the value of the investment will be

$$A(1 + \mathcal{R}_{t,s})^{(s-t)}.$$

Let us fix an integer $m = 1, 2, 3, \cdots$ and for each $i = 0, 1, 2, \cdots$, we let

$$T_i = \frac{i}{m}.$$

The integer m will represent the number of interest payments per year. It is convenient to define

$$p_i[m] = m\left(\left(1 + \mathcal{R}_{T_i,T_{i+1}}\right)^{\frac{1}{m}} - 1\right),$$

so that

$$1 + \frac{p_i[m]}{m} = \left(1 + \mathcal{R}_{T_i,T_{i+1}}\right)^{\frac{1}{m}}.$$

With this definition of $p_i[m]$, the value at T_{i+1} of an amount A invested between times T_i and T_{1+1} will be

$$A\left(1 + \frac{p_i[m]}{m}\right).$$

Notice that $p_i[m]$ is known at time T_i, but not earlier.

2.10.1 Floating Rate Bonds

A *floating rate bond* (also called a *float note*) is characterized by a face value $F > 0$, a maturity date $T > 0$, and a positive integer m giving the number of coupon payments per year. For simplicity we assume that $T = N$ for some positive integer N (although no significant changes are required so long as the maturity is an integer multiple of $\frac{1}{m}$). The holder of the bond receives the coupon payment

$$C_i = F\frac{p_{i-1}[m]}{m}$$

at each of the times $T_i = \frac{i}{m}$, $i = 1, 2, \cdots, mN$ plus the face value F at maturity. (Notice that the holder receives $F + C_{mN}$ at maturity. It seems quite intuitive that the arbitrage-free price of such a bond should simply be the face value F because each coupon payment is the interest that would be obtained by investing F in the bank at the time of the previous coupon payment. We can use this intuition to create a replicating strategy for the bond; the initial capital of this strategy will be F justifying the idea the initial price of the bond should be F.

Let X be the strategy with terminal time T that is created as follows:

(a) At time 0, the amount F is invested in the bank until time T_1;

(b) At each time T_i, with $1 \le i < mN$, the amount $F\left(1 + \frac{p_{i-1}[m]}{m}\right) = F + C_i$ is withdrawn from the bank and the amount F is reinvested until time T_{i+1};

(c) At time T_{mN}, the amount $F\left(1 + \frac{p_{mn-1}[m]}{m}\right) = F + C_{mN}$ is withdrawn from the bank.

We can see that an agent who implements the strategy X receives exactly the same payments at each of the times T_i, $i = 1, 2, \cdots, mN$ as an agent who buys the float note. (In other words, X replicates the float note.) Notice that the initial capital of this strategy is $X_0 = F$. The strategy that is created by implementing X and selling one floating rate bond at time 0 is self-financing and has terminal capital 0; consequently the initial capital of the strategy must also be zero. We conclude that in order to avoid arbitrage the initial price of the floating rate bond must be F, as expected.

2.10.2 Interest Rate Swaps

An interest rate swap is an agreement made between two parties A and B at the present time $t = 0$. The agreement is characterized by a notional principal amount $F > 0$, a maturity date $T > 0$, a positive integer m representing the number of swap dates per year, and a nominal swap rate $q[m]$. We assume that $T = N$ for some positive integer N. At each of the times $T_i = \frac{i}{m}$, $i = 1, 2, \cdots, mN$:

(i) Party A pays the amount

$$F\frac{p_{i-1}[m]}{m}$$

to party B; and

(ii) Party B pays the amount

$$F\frac{q[m]}{m}$$

to party A.

Neither party pays anything at time 0 to enter the contract, and the notional principal is never paid. In other words, party A is making variable interest rate payments on some hypothetical amount F and receiving fixed interest rate payments on the same hypothetical amount. (Of course, party B is making fixed interest payments and receiving variable interest payments.) Such agreements can be extremely useful in many situations. Since the initial cost of both positions on the contract is specified to be zero, it is the swap rate $q[m]$ that we need to calculate. In situations where there may be other coupon rates that may be relevant we write $q^{swap}[m]$ to denote the rate used in interest rate swaps.

We can replicate A's payments to B by purchasing one float note with face value F and maturity T and selling short one zero coupon bond with face value F and maturity T at $t = 0$; let us call this strategy Y. We can replicate B's payments to A by purchasing one coupon bond with face value F maturity T that pays coupons m times per year at the nominal coupon rate $q[m]$ and selling short one zero coupon bond with face value F and maturity T; let us call this strategy Z. Observe that the initial capital of Y is given by

$$Y_0 = F(1 - d(T)) = F(1 - d(N)),$$

and that the initial capital of Z is given by

$$Z_0 = F\frac{q[m]}{m}\sum_{i=1}^{mN} d\left(\frac{i}{m}\right).$$

Since neither party pays the other initially, we need to have $Y_0 = Z_0$ in order to avoid arbitrage. (If we let X be the strategy obtained by purchasing the portfolio corresponding to Y, selling the portfolio corresponding to Z, and taking B's position on the contract, then X is self-financing and has terminal capital $X_T = 0$ and initial capital $X_0 = Y_0 - Z_0$. Since X_0 must be zero, we conclude that $Y_0 = Z_0$.) Setting $Y_0 = Z_0$ and solving for $q[m]$, we find that

$$q^{swap}[m] = \frac{m(1 - d(N))}{\sum_{i=1}^{mN} d\left(\frac{i}{m}\right)}. \tag{2.31}$$

The value of $q^{swap}[m]$ given by (2.31) is called the swap rate for maturity T with m payment dates per year.

Observe that A's position on the contract can be replicated by selling short one float note with face F value and maturity T and purchasing one coupon bond with face value F maturity T and nominal coupon rate $q[m]$. In order for the initial capital of this strategy, the coupon rate $q[m]$ must be such that coupon bond has time-0 price equal to the face value F. Using the formula for the arbitrage-free price of a coupon bond, we see that the value of $q[m]$ given by (2.31) is the unique nominal coupon rate for which the arbitrage-free price of a coupon bond will be equal to the face value.

2.11 Exercises for Chapter 2

Exercise 2.1. An investor deposits \$15,000 in a bank today for 23 years at 6% annual interest. Assuming that no additional deposits are made and that no money is withdrawn, how large will the account balance be 23 years from today if

(a) the interest is computed using the simple interest convention?

(b) the interest is compounded annually?

(c) the interest is compounded quarterly?

(d) the interest is compounded monthly?

Exercise 2.2. In 1461 King Edward IV of England borrowed the equivalent of \$384 from New College of Oxford. He promptly repaid \$160, but never repaid the remaining \$224. In 1996, an administrator at New College discovered a record of this debt and contacted the queen asking for repayment of the original \$224 together with interest compounded annually for 535 years.

(a) The administrator originally suggested an annual rate of $R = 4\%$. (Here we are writing R because with annual compounding there is no distinction between nominal and effective rates.) Calculate the value of the debt after 535 years using this interest rate.

(b) After the queen refused to pay the amount calculated in part (a), the administrator suggested using the rate $R = 2\%$ instead, indicating that this greatly reduced amount would be enough to help the college with needed renovations. Calculate the value of the debt using annual compounding at this rate for 535 years. The queen refused to pay this amount as well, and to the best of our knowledge, the debt still remains unpaid.

Exercise 2.3. Consider a simple financial model with 2 banks, each with interest rates at time 0 that do not depend on the length of the deposit or loan.

(i) At bank 1, interest is compounded monthly at the nominal rate $r[12] = .08$ and transactions can be made at times $t = \frac{k}{12}$, $k = 0, 1, 2, 3, \cdots$. You can borrow or invest at this rate between $t = 0$ and $t = \frac{k}{12}$ for any positive integer k.

(ii) At bank 2, interest is compounded quarterly at the nominal rate $r[4]$ and transactions can be made at times $t = \frac{k}{4}$, $k = 0, 1, 2, 3, \cdots$. You can borrow or invest at this rate between $t = 0$ and $t = \frac{k}{4}$ for any positive integer k.

(a) Assuming that there is no arbitrage, determine $r[4]$.

(b) A third bank that compounds interest continuously at the nominal rate $r[\infty]$ is added to the model. This rate applies to deposits or loans between time 0 and any time $t > 0$. You can borrow or invest at this rate. Assuming that the extended model is arbitrage free, find $r[\infty]$.

Exercise 2.4. Suppose that there is an ideal money market with constant effect rate R. A customer calls the bank and asks: "If I deposit \$1,000 today, deposit an additional \$2,000 6 months from today, and make no other deposits or withdrawals, what will my account balance be one year from today?" The bank answers \$3128.98. Determine R.

Exercise 2.5. Suppose that there is an ideal bank, but the spot rates are given as nominal rates $r_*(T), T > 0$, corresponding to continuous compounding, i.e. the discount factors are given by

$$d(T) = e^{-Tr_*(T)}.$$

At $t = 0$, Alice agrees to borrow \$100,000 at $t = 1$ and repay the loan with a single lump-sum payment of \$112,000 at $t = 3$. Given that $r_*(1) = .05$ and there is no arbitrage, find $r_*(3)$.

Exercise 2.6.

For $\eta, T > 0$ such $T > \eta$ and both T and η are multiples of 3 months (i.e., multiples of $\frac{1}{4}$ years) let $\tilde{r}^{for}_{0,\eta,T}$ be the forward rate, with quarterly compounding, agreed upon at time 0 for a loan to be initiated at time η and repaid with a single payment at time T. (Nothing is paid to enter into a forward loan agreement.) For every dollar borrowed at time η, the amount to be repaid at time T is

$$\left(1 + \frac{\tilde{r}^{for}_{0,\eta,T}}{4}\right)^{4(T-\eta)}.$$

Assume that $d(3) = .88888$ and $d(3.75) = .86322$. Determine $\tilde{r}^{for}_{0,3,3.75}$.

Exercise 2.7 (Continuously Compounded Spot and Forward Interest Rates)**.** Suppose that there is an ideal bank, but that the spot rates are described by nominal rates $r_*(T), T > 0$, corresponding to continuous compounding. In other words, if an amount V_0 is invested between $t = 0$ and $t = T$, the value of the investment at time T will be

$$V_T = V_0 e^{Tr_*(T)}.$$

We assume that the function r_* is differentiable and for each $T > 0$ we let $R_*(T)$ denote the corresponding effective spot rate.

(a) Let $T > 0$ be given. Show that $R'_*(T) > 0$ if and only if $r'_*(T) > 0$.

(b) Given $\eta \geq 0$ and $T > \eta$, let $r^{for}_{0,\eta,T}$ denote a continuously compounded nominal interest rate agreed upon at $t = 0$ for an investment or loan to be initiated at $t = \eta$ and settled with a single lump sum payment at $t = T$. If the amount to be invested time η is V_η then the value of the investment at time T will be

$$V_T = V_\eta \exp((T - \eta) r^{for}_{0,\eta,T}).$$

Find an expression for $r^{for}_{0,\eta,T}$ in terms of $\eta, T, r_*(\eta)$ and $r_*(T)$.

(c) Express

$$\lim_{T \to \eta^+} r^{for}_{0,\eta,T}$$

in terms of $\eta, r_*(\eta)$, and $r'_*(\eta)$. (The value of this limit has a special name. It is called the instantaneous forward rate at time η as seen from time 0.)

Exercise 2.8 (Spot Rates and Forward Rates Under a Simple Interest Convention)**.** Suppose that there is an ideal money market, but that the spot rates are described by simple interest rates $\hat{R}_*(T),\ T > 0$. In other words if an amount V_0 is invested between $t = 0$ and $t = T$, the value of the investment at time T will be

$$V_T = V_0(1 + T\hat{R}_*(T)).$$

We assume that the function $\hat{R}_*$ is differentiable and for each $T > 0$ we let $R_*(T)$ denote the corresponding effective spot rate.

(a) Let $T > 0$ be given. Find a formula that expresses $\hat{R}_*(T)$ in terms of T and $R_*(T)$. Suppose that $R_*(.83) = 6.02\%$. Find the corresponding value of $\hat{R}_*(.83)$.

(b) Given $\eta \geq 0$ and $T > 0$ let $\hat{R}^{for}_{0,\eta,T}$ be a simple interest rate agreed upon at $t = 0$ for an investment or loan to be initiated at $t = \eta$ and settled with a single lump sum payment at time T. If the amount to be invested at time η is V_η then the value of the investment at time T will be

$$V_T = V_\eta(1 + (T - \eta)\hat{R}^{for}_{0,\eta,T}).$$

Find an expression for $\hat{R}^{for}_{0,\eta,T}$ in terms of $\eta, T, \hat{R}_*(\eta),\ \hat{R}_*(T)$.

(c) Express

$$\lim_{T \to \eta^+} \hat{R}^{for}_{0,\eta,T}$$

in terms of $\eta, \hat{R}_*(\eta),\ \hat{R}'_*(\eta)$.

Note: In practice, the simple interest convention is rarely used to describe investments or loans of duration longer than one year.

Exercise 2.9.

Let $T > \eta > 0$ be given and assume that $R_*(T) > R_*(\eta)$. Show that

$$\mathcal{R}^{for}_{0,\eta,T} > R_*(T).$$

(Here $\mathcal{R}^{for}_{0,\eta,T}$ is the effective forward rate agreed upon at time 0 for a loan to be initiated at time η and settled with a single payment at time T.)

Exercise 2.10. Today's date is $t = 0$. The ABC Drug Company is issuing a zero-coupon bond today. It has a face value of $F = \$1,000$, purchase price of \$900, and maturity $T = 1$. The company is waiting for a decision by the FDA concerning approval of a new drug. The decision will be made and announced 6 months from today. If the drug is approved, the company will be in great shape financially and all bond holders will receive \$1,000 per bond at maturity. On the other hand, if approval is denied, the company will go bankrupt and bond holders will receive nothing. It is known that the probability the drug will be approved is .8. The current effective spot rate for risk-free borrowing or investing with maturity 1 year is $R_*(1) = 5\%$. Many investors are tempted by the ABC bond because of its high yield of 11.11%. However, they are concerned about the possibility of losing their entire investment. A major investment bank (with no significant risk of default) has decided to sell insurance policies for the ABC bond. The policies will pay nothing if the bond holders receive their payments from ABC, but they will pay \$1,000 per bond at $t = 1$ if ABC declares bankruptcy and defaults on the bonds.

How much should the bank charge (per bond) for these policies? What assumptions did you make to reach your conclusion?

Exercise 2.11. (A Currency Swap) The current exchange rate for dollars and Japanese yen is $E^{\$}_{y,0} = 120$, i.e. it costs 120 yen to purchase one dollar at time 0. The effective spot rates for investments in dollars and yen are

$$R^{\$}(.25) = 5.00\%, \quad R^{\$}(.5) = 5.00\%, \quad R^{\$}(.75) = 5.15\%, \quad R^{\$}(1) = 5.25\%,$$

$$R^{y}(.25) = 1.00\%, \quad R^{y}(.5) = 1.00\%, \quad R^{y}(.75) = 1.25\%, \quad R^{y}(1) = 1.30\%.$$

A US company and a Japanese company are making an agreement today to exchange dollars and yen at each of the dates $T_i = \frac{i}{4}$, $i = 1, 2, 3, 4$. At each T_i, the Japanese company will receive \$10,000,000 from the US company in exchange for a payment of $10,000,000\mathcal{F}$ yen. Assuming that nothing is paid by either company to enter the agreement, find the appropriate value of $\mathcal{F}$. (The same value of $\mathcal{F}$ is to be used for each payment.)

Exercise 2.12. Annuities that will pay \$500 twice per year for the next 5 years (i.e., 10 payments in all) are being issued today at the arbitrage-free price of \$4,000 per annuity. Coupon bonds having maturity 5 years and face value $F = \$1,000$ are being issued today at the arbitrage-free price \$1,100. These bonds pay coupons twice per year at the nominal coupon rate $q[2] = 10\%$. Find the effective 5-year spot rate $R_*(5)$.

Exercise 2.13. Two different annuities $A^{(1)}$ and $A^{(2)}$ and a coupon bond B are being issued today. Their characteristics are given below.

(i) $A^{(1)}$ has maturity 4 years and will make payments of \$2,000 twice per year. The current price of this annuity is \$14,629.85.

(ii) $A^{(2)}$ has maturity 3 years and will make payments of \$1,000 twice per year. The current price of this annuity is \$5,635.00

(iii) B has maturity 4 years, face value \$10,000 and will make coupon payments twice per year at the nominal coupon rate $q[2] = 5\%$. The current price of this bond is \$10,118.75.

(a) Find the effective spot rate $R_*(4)$.

(b) Find the effective spot rate $R_*(3.5)$.

Exercise 2.14. Coupon bonds with maturity $T = 10$ years and face value \$10,000 that pay coupons twice per year at the nominal rate $q[2] = .04$ are currently trading at \$9,060.00 per bond. Coupon bonds with maturity $T = 10$ and face value \$5,000 that pay coupons twice per year at the nominal rate $q[2] = .08$ are trading at \$6,146.50 per bond. Assuming that there is no arbitrage, determine the prices of each of the following fixed-income securities:

(a) A zero-coupon bond with face value \$20,000 and maturity 10 years.

(b) An annuity that has maturity 10 years and makes payments of \$500 twice per year.

(c) A coupon bond with maturity 10 years and face value \$7,500 that pays coupons twice per year at the nominal rate $q[2] = .06$.

Exercise 2.15. A portfolio of a company has the following structure.

(i) There are -5 zero-coupon bonds each with maturity 2 years and face value \$10,000. (The minus sign means that 5 bonds have been sold short.)

(ii) There are 3 annuities. Each annuity pays \$2,000 per month over the next year at the times $\frac{i}{12}$, $i = 1, 2, \ldots, 12$.

(iii) There are two coupon bonds. Each bond has maturity 2 years and face value \$10,000. Each bond pays coupons twice per year at the nominal coupon rate $q[2] = 3\%$. (The first coupon payment will be received at $t = .5$.)

In the framework of an ideal bank with constant spot rate function $R_*(T) = R = 4\%$ for all $T > 0$, find the net present value of the portfolio.

Exercise 2.16. A former CMU student is working for a large investment bank. She has invented a new derivative security on shares of Amazon stock. The security has maturity one year and its payoff equals the average of the monthly prices of Amazon shares. In other words, at time $T = 1$ year the security will pay its holder the amount

$$\frac{1}{12}\sum_{i=1}^{12} S_{\frac{i}{12}},$$

where $S_{\frac{i}{12}}$ is the price of one share of Amazon at the end of ith month.Compute the arbitrage-free price of the derivative security at $t = 0$. Assume that the initial price of one share of stock is $S_0 = \$100$, that Amazon stock does not pay dividends, and that there is an ideal money market with constant effective rate $R = (1.01)^{12} - 1$ (so that the nominal rate for monthly compounding is r[12]=.12).

Exercise 2.17. Find the effective yield to maturity (effective IRR) of a zero coupon having face value \$5,000 and maturity $T = 7$ assuming that the bond is currently trading at \$3,637.44.

Exercise 2.18. A 3-year coupon bond was issued two years ago today (i.e., the bond matures one year from today). The face value of the bond is $F = \$1,000$ and the bond pays coupons twice per year at time nominal coupon rate $q[m] = 8\%$. The effective spot rates for maturities 6 months and one year are $R_*(.5) = 4\%$ and $R_*(1) = 6\%$. The bond will be sold today just after the original holder receives today's coupon payment.

(a) Find the arbitrage-free price of the bond just after today's coupon payment.

(b) Assuming that the bond is sold at the arbitrage-free price find the effective yield to maturity (effective IRR) of the bond.

Exercise 2.19. Find the effective yield to maturity of each of the securities in Exercise 2.14. (You will need to solve Exercise 2.14 first to obtain the prices. You may wish to use a numerical method to find the roots of a polynomial to determine R_I for (b) and (c).)

Exercise 2.20. An annuity is purchased at $t = 0$ for \$50,000. The maturity of the annuity is $T = 15$ years and payments are to be made monthly. Assuming that the nominal yield to maturity (i.e. the nominal IRR) is $r_I[12] = 8\%$, find the value A of the monthly payments.

Exercise 2.21. A coupon bond is purchased at $t = 0$ for \$9,500. The maturity of the bond is $T = 10$ years and coupons are paid twice per year. The nominal coupon rate is $q[2] = 4\%$. Assuming that the effective yield to maturity (i.e. the effective IRR) is $R_I = 6\%$, find the face value F of the bond.

Exercise 2.22. Consider a 15-year mortgage for \$125,000 with monthly payments and nominal mortgage rate (i.e., nominal IRR) $r_I[12] = 7.5\%$. Find the effective mortgage rate R_I and the amount A of the monthly payments.

Exercise 2.23. Consider a 30-year mortgage for $200,000 with monthly payments of $1,310.52. Find the nominal mortgage rate (i.e., nominal IRR) $r_I[12]$. (You may want to use a numerical method to find the roots of a polynomial.)

Exercise 2.24. A 15-year annuity (issued today) is selling for $50,000. The annuity will make monthly payments of $500. The effective spot rate for maturity $T = 15$ years is $R_*(15) = .1$. Assuming that there is no arbitrage, determine the price of a coupon bond being issued today and having the following characteristics: The face value is $1,000, the maturity is 15 years, and the bond will pay coupons monthly at the nominal coupon rate $q[12] = 12\%$.

Exercise 2.25. Suppose that you have access to an ideal bank with constant effective spot rate $R_*(T) = R = 6\%$ for all $T > 0$. You are about to purchase a new car. You have negotiated a price of $21,000 with the dealer. You have also negotiated a trade-in value of $3,000 for your current car. You have no other money to put down on the car so you will have to borrow the balance. (For simplicity, we shall ignore taxes, title, registration, dealer preparation fees, etc.) The dealer tells you that he has some good news. The car manufacturer is offering customers a choice of $3,000 cash back or 0%-financing for 5 years. If you take the cash back you will have to pay the dealer $15,000 when you take possession of the car. (You will therefore need to borrow this amount from the money market.) If you take the 0%-financing, you will not need to pay anything when you get the car, but you will have to pay a total of $18,000 to the car manufacturer in equal monthly payments over the next five years. (In theory, you could pay the loan off over a shorter period of time, but this would not be advantageous.)

Should you take the cash back or the 0%-financing? Give a convincing quantitative explanation.

Exercise 2.26. Consider a coupon bond with face value $F = \$10,000$ and maturity $T = 5$ that pays coupons twice per year at the nominal rate $q[2] = .04$. Brokers are currently selling this bond at an ask price $\hat{\mathcal{P}}^a$ and purchasing the bond at a bid price $\hat{\mathcal{P}}^b$. The effective yield to maturity computed using the ask price is $R_I^a = .045$ and the effective yield to maturity computed using the bid price is $R_I^b = .048$. Compute the bid-ask spread $\hat{\mathcal{P}}_a - \hat{\mathcal{P}}_b$.

Exercise 2.27. A coupon bond was issued 2 years ago today. The bond has face value $F = \$1,000$ and matures 8 years from today. It pays coupons twice per year at the nominal coupon rate $q[2]$. Just after today's coupon payment, the price of the bond is $\mathcal{P} = \$987$ and the effective yield to maturity is $R_I = 5.83\%$. Find the nominal coupon rate $q[2]$.

Exercise 2.28. A coupon bond with face value $F = \$1,000$ and maturity $T = 2$ years is being issued today. The bond pays coupons twice per year at the nominal coupon rate $q[2] = 6\%$. The current spot rates for maturities .5,1,1.5, and 2 are $R_*(.5) = 5\%,\ R_*(1) = 5.5\%,\ R_*(1.5) = 5.8\%$ and $R_*(2) = 6.1\%$.

(a) Find the arbitrage-free price of the bond.

(b) Find the effective yield to maturity, assuming that the bond is sold today at its arbitrage-free price. (You may wish to use a numerical method to find the roots of a polynomial.)

Exercise 2.29. Consider a coupon bond with face value F and maturity $T = n$, for some positive integer n. Assume that the bond pays coupons m times per year at the nominal coupon rate $q[m]$ and that the current market price of the bond is $\hat{\mathcal{P}}$. Prove each of the following three statements:

(i) The bond is trading at par (i.e., $\hat{\mathcal{P}} = F$) if and only if the nominal coupon rate is the same as the nominal yield to maturity (i.e., $q[m] = r_I[m]$).

(ii) The bond is trading above par (i.e., $\hat{\mathcal{P}} > F$) if and only if the nominal coupon rate is greater than the nominal yield to maturity (i.e., $q[m] > r_I[m]$).

(iii) The bond is trading below par (i.e., $\hat{\mathcal{P}} < F$) if and only if the nominal coupon rate is lower than the nominal yield to maturity (i.e., $q[m] < r_I[m]$).

(Suggestion: Use equation (2.22).)

Exercise 2.30. A student is planning to study at CMU for four years. She has \$40,000 set aside to cover her living expenses during that period. She wants to purchase a 4-year annuity for \$40,000 in which the payments will increase yearly in order to cover cost-of-living increases. More precisely she wants to purchase an annuity in which payments are made monthly for four years (48 payments total); the first twelve payments will be for some amount A; the next twelve payments will be for the amount $A(1.04)$; the payments during the third year will be for the amount $A(1.04)^2$; the final twelve payments will be for the amount $A(1.04)^3$. A broker tells her that the effective yield to maturity of such a security will be $R_I = .06$. Determine A.

Exercise 2.31. You are a representative of a financial company. The company can borrow and lend through an ideal bank having constant spot rate function $R_*(T) = R = 5\%$ for all $T > 0$.

A client (who does not have access to this bank) wants to borrow \$100,000 for the period of one year. The amount \$100,000 and the time one year will be called, respectively, the principal and the maturity of the loan. The client will readily accept any of the schemes below:

(1) In the first scheme all the interest and the principal will be paid at maturity. The interest rate for this scheme is $r_1[1] = 6\%$. (Notice that since the only payment takes place at $t = 1$, there is no distinction between the nominal rate and the effective.)

(2) In the second scheme there will be equal monthly interest payments (12 payments in total) plus payment of the principal at maturity. The nominal interest rate for this scheme is $r_2[12] = 6\%$.

(3) In the third scheme the loan will be amortized or repayed by equal monthly payments (12 payments in total). The nominal interest rate for this scheme is $r_3[12] = 6\%$.

You naturally want to maximize the profit of the financial company. Which type of loan will you sell to the client? What will the time-0 valuee of your profit be?

Exercise 2.32. Consider an interest rate swap with maturity $T = 2$ with two swap dates per year. Given that $R_*(.5) = 3\%$, $R_*(1) = 3.5\%$, $R_*(1.5) = 3.8\%$, and $R_*(2) = 4\%$, find the swap rate $q[2]$.

Exercise 2.33. The 10-year effective spot rate is $R_*(10) = 4.87\%$ and annuities that have maturity 10 years and make payments of \$100 four times per year are trading at \$3,187.31. Find the swap rate $q^{swap}[4]$ for an interest rate swap having maturity 10 years and 4 swap dates per year.

Exercise 2.34. (Mortgage Points) Loan fees for mortgages are often described using *points*. (There are actually two kinds of points used in practice *origination points* and *discount points*. In this exercise we are talking about discount points.) For each point, the borrower must pay the lender 1% of the amount borrowed. This loan fee is paid at the start of the loan and no part of it will be refunded if the mortgage is paid off early. The lender will offer a lower mortgage rate in exchange for the points. Consider a 30-year mortgage for \$400,000 with monthly payments.

- If no points are purchased, the nominal mortgage will be $r_I^{0p}[12] = .05$.
- If one point is purchased, the nominal mortgage rate will be $r_I^{1p}[12] = .0475$.

(a) Compute the monthly payment amount assuming that no points are purchased. Call this amount A^{0p}.

(b) Compute the monthly payment amount assuming that 1 point is purchased. Call this amount A^{1p}. (To determine A^{1p}, you disregard the \$4,000 paid to purchase the point and simply compute the payment amount for an annuity with $m = 12$, maturity 30 years, nominal yield to maturity $r_I^{1p}[12]$ and initial price 400,000.)

(c) Put $A = A^{0p} - A^{1p}$. Find the time-0 price of an annuity with $m = 12$, maturity 30 years, monthly payment amount A, and nominal yield to maturity $r_I[12] = .05$. (This is a measure of the time-0 value of the money saved in monthly payments due to purchasing 1 point. This number will be substantially greater than the \$4,000 cost to purchase the point. The “catch” here is that most mortgages are not held until maturity. When a mortgage is paid off early, no part of the purchase price of the points will be refunded.)

Remark: Suppose that 1 point is purchased. If the mortgage is paid off immediately, the borrower simply loses \$4,000. If the mortgage is held until maturity, the borrower saves a lot of money. Banks often quote a “break even” time for purchasing points.

This is a time τ such that if the mortgage is paid off before time τ, the customer will have lost money by purchasing the point, and if the mortgage is paid off after time τ the borrower will have saved money by purchasing the point. A careful analysis of the "break even" time is a bit tricky. If you are interested, I suggest that you try to determine it for the mortgage of this exercise. I would be glad to discuss this with you.

Exercise 2.35. The table below gives time-0 prices of three different bonds. Each bond has face value \$100 and pays coupons every 6 months at the nominal coupon rate $q[2]$ shown in the table. (The first bond makes a single payment of coupon plus face at time .5).

Maturity	$q[2]$	**Price**
.5 years	.06	100.982
1 year	.04	99.712
1.5 years	.08	105.471

(a) Find the discount factors $d(.5)$, $d(1)$, and $d(1.5)$.

(b) Find the effective spot rates $R_*(.5)$, $R_*(1)$, and $R_*(1.5)$.

(c) Find the effective forward rate $\mathcal{R}^{for}_{0,.5,1.5}$ agreed upon at time 0 for a loan initiated at time .5 and settled with a single lump-sum payment at time 1.5.

(d) Find a nominal coupon rate $q^*[2]$ such that the time-0 price of a bond that pays coupons every 6 months and has maturity 1.5 years, face value \$100 and nominal coupon rate $q^*[2]$ will be equal to \$100.

Exercise 2.36. Two coupon bonds $B^{(1)}$ and $B^{(2)}$ are being issued today. Both bonds have maturity $T = 10$ years and pay coupons twice per year. It is known that $R_*(10) = .06$. It is also known that

(i) $B^{(1)}$ has face value $F^{(1)} = \$10,000$, nominal coupon rate $q^{(1)}[2] = .06$ and the price of this bond at $t = 0$ is \$10,104.95.

(ii) $B^{(2)}$ has face value $F^{(2)} = \$5,000$ and the price of this bond at time 0 is \$5,523.41.

(a) Find the price at time 0 of a zero coupon bond with maturity 10 years and face value \$10,000.

(b) Find the nominal coupon rate $q^{(2)}[2]$ for $B^{(2)}$.

Exercise 2.37. For $T > \eta \geq 0$, let $\mathcal{R}^{for}_{0,\eta,T}$ be the effective forward agreed upon at time 0 for borrowing between time η and time T. Given that

$$\mathcal{R}^{for}_{0,1,2} = .052, \quad \mathcal{R}^{for}_{0,2,4} = .057, \quad \mathcal{R}^{for}_{0,4,8} = .062,$$

and that the time-0 price of a zero-coupon bond with face value 100 and maturity 1 year is \$96, find the time-0 price of a zero-coupon bond with maturity 8 years and face value 100.

Exercise 2.38. In this problem we use the notation

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ \\ 0 & \text{if } x < 0. \end{cases}$$

Assume that $R_*(9) = .05$ and $R_*(10) = .06$. Let $\mathcal{R}_{9,10}$ denote the effective spot rate that will prevail at time 9 for loans (or investments) between times 9 and 10. (This rate is not known until time 9. If an amount A is borrowed at time 9, then the amount to be repaid at time 10 is $A(1 + \mathcal{R}_{9,10})$.) Let U be a security that makes a single payment of

$$U_{10} = 10,000(\mathcal{R}_{9,10} - .05)^+$$

at time 10 and V be a security that makes a single payment of amount

$$V_{10} = 10,000(.05 - \mathcal{R}_{9,10})^+$$

at time 10. It is known that arbitrage free price of U at time 0 is $U_0 = 750$. Determine the arbitrage-free price V_0 of V at time 0. (Suggestion: Look at $U_{10} - V_{10}$.)

Exercise 2.39. Let $T_i = \frac{i}{2}$ for $i = 0, 1, 2, \cdots, 20$ and let $p_i[2]$ denote the nominal spot rate that will prevail at time T_i for borrowing and investing between times T_i and T_{i+1}. Consider a nonstandard interest rate swap between two parties A and B in which the notional principal will double half way through the swap. More precisely:

- At each of the times T_i with $i = 1, 2, \cdots, 10$

$$A \text{ pays } B \text{ the variable amount } \; F\frac{p_{i-1}[2]}{2} \text{ and}$$

$$B \text{ pays } A \text{ the fixed amount } \; F\frac{q[2]}{2}.$$

- At each of the times T_i with $i = 11, 12, \cdots, 20$

$$A \text{ pays } B \text{ the variable amount } \; Fp_{i-1}[2] \text{ and}$$

$$B \text{ pays } A \text{ the fixed amount } \; Fq[2].$$

The (constant) swap rate $q[2]$ is chosen that neither party pays anything to enter the agreement and the notional principal F is never paid. Find an expression for the swap rate $q[2]$ in terms of the discount factors $d(.5), d(1), \cdots, d(10)$.

Exercise 2.40. Let $T_i = \frac{i}{2}$ for $i = 0, 1, 2, \cdots, 20$ and let $p_i[2]$ denote the nominal spot rate that will prevail at time T_i for borrowing and investing between times T_i and T_{i+1}. Consider a nonstandard interest rate swap between two parties A and B in which the roles of receiving fixed and floating payments will interchange half way through the swap. More precisely:

- At each of the times T_i with $i = 1, 2, \cdots, 10$

$$A \text{ pays } B \text{ the variable amount } \; F\frac{p_{i-1}[2]}{2} \text{ and}$$

$$B \text{ pays } A \text{ the fixed amount } \; F\frac{q[2]}{2}.$$

- At each of the times T_i with $i = 11, 12, \cdots, 20$

$$A \text{ pays } B \text{ the fixed amount } \; F\frac{q[2]}{2} \text{ and}$$

$$B \text{ pays } A \text{ the variable amount } \; F\frac{p_{i-1}[2]}{2}.$$

The (constant) swap rate $q[2]$ is chosen that neither party pays anything to enter the agreement and the notional principal F is never paid. Find an expression for the swap rate $q[2]$ in terms of the discount factors $d(.5), d(1), \cdots, d(10)$.