Chapter 5

Arbitrage-Free Pricing in One-Period Finite Models

The simplest financial models involving random evolution of prices are those in which there are only two trading times and the prices of the basic securities are modelled as random variables on a finite probability space. Although such models are too simplistic to accurately describe real-world financial markets, they will allow us to introduce some important ideas (such as risk-neutral pricing) in the simplest possible setting. Moreover, these very basic models can be used as building blocks in more sophisticated models.

5.1 One-Period Binomial Model

In the *one-period binomial model* there are two times, $t = 0$ and $t = 1$, a bank, and a single stock that pays no dividends. We can borrow or invest any amount of money at the bank between $t = 0$ and $t = 1$ at the one-period interest rate $r \geq 0$. Capital in the bank account evolves according to the rule

$$B_1 = B_0(1 + r),$$

where B_0 is the initial capital; $B_0 > 0$ corresponds to an investment, whereas $B_0 < 0$ indicates a loan.

The initial price per share of the stock is S_0. It is assumed that we can buy or sell short as many shares of the stock as we please at time 0 at the price S_0 per share. The price S_1 of the stock at time 1 is not known at time 0. We assume that there are two possible values of S_1. Consequently it is appropriate to model S_1 as a random variable on a probability space $(\Omega, \mathbb{P})$ in which Ω contains two elements. We think of these two elementary events as "head" and "tail" resulting from the toss of a coin. An appropriate sample space is therefore $\Omega = \{H, T\}$. We do not assume that the coin is "fair", i.e. we do not assume that $\mathbb{P}(H) = \mathbb{P}(T)$.

We do, however, assume that

$$\mathbb{P}(H) > 0, \; \mathbb{P}(T) > 0.$$

We also assume that

$$S_1(H) \neq S_1(T);$$

otherwise the stock is just another bank account (and the no-arbitrage principle would tell us that the interest rate for the stock must also be r.) Without loss of generality, we assume that

$$S_1(H) > S_1(T). \tag{5.1}$$

(If we had $S_1(T) > S_1(H)$, we could relabel the sides of the coin to achive (5.1.) We refer to $\mathbb{P}$ as the *reference probability measure*, or the *actual probability measure*, in order to distinguish it from another probability measure $\tilde{\mathbb{P}}$, called a *pricing measure* (or *risk-neutral measure*), that will be introduced later to compute arbitrage-free prices.

This model is extremely simple. It is certainly a stretch to believe that the price of the stock at the terminal time can take only two possible values, and that these values are known at the initial time. However, it is very useful to study (and completely understand) the one-period binomial model. Indeed, many of the important features of this model have analogues in more sophisticated models. In particular, one-period binomial models can be strung together to form multi-period binomial models, which are somewhat more realistic. A continuous-time model can be obtained taking a suitable limit of multi-period binomial models. A key to understanding multi-period models, is the complete analysis of one-period models.

It is useful to introduce the random variable ρ^S defined by

$$\rho^S(\omega) = \frac{S_1(\omega) - S_0}{S_0} \quad \text{for all } \omega \in \{H, T\}$$

so that

$$S_1(\omega) = (1 + \rho^S(\omega))S_0 \quad \text{for all } \omega \in \{H, T\}.$$

We refer to ρ^S as the *return* of the stock. It is also convenient to let $u = 1 + \rho^S(H)$ and $d = 1 + \rho^S(T)$ so that

$$S_1(H) = uS_0, \quad S_1(T) = dS.$$

We refer to u and d as the up factor and the down factor, respectively, although it is possible to have $u > 1$ and $d > 1$, in which case the stock price will be sure to go up. (As we shall see shortly, since $r \geq 0$, a no-arbitrage hypothesis rules out the possibility of having $u < 1$ and $d < 1$.) In view of (5.1) we have

$$u > d > 0.$$

Indeed, it is essential that $u, d > 0$, because stock prices must be positive. (We could consider the possibility of bankruptcy and allow $d = 0$, but we shall not do so here.)

Since there is only one investment period and only one risky asset (namely the stock), a trading strategy is completely described by the amount of stock purchased and the amount of money invested in the bank at time 0. (Strategies are automatically self-financing because it is not possible to introduce or remove capital at intermediate times.) It is customary to describe a strategy by specifying the total initial capital and the number of shares of stock. (The amount of money initially invested in the bank can be easily computed by subtracting the cost of the stock from the total initial capital.)

More formally, a *strategy* or *portfolio* X is described by a pair (X_0, Δ_0), where X_0 is the total initial capital and Δ_0 is the number of shares of stock purchased at $t = 0$. If $\Delta_0 > 0$ then the portfolio is long on the stock, whereas if $\Delta_0 < 0$ then the portfolio is short on the stock. Notice that the initial capital of the bank account is $X_0 - \Delta_0 S_0$. The total capital X_1 at time 1 is a random variable given by

$$\begin{aligned} X_1(\omega) &= (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_1(\omega) \\ &= (X_0 - \Delta_0 S_0)(1+r) + \Delta_0(1+\rho^S(\omega))S_0 \\ &= X_0(1+r) + \Delta_0 S_0(\rho^S(\omega) - r) \ \text{ for all } \omega \in \{H, T\}. \end{aligned}$$

The *return* of a portfolio with $X_0 > 0$ is defined by

$$\rho(\omega) = \frac{X_1(\omega) - X_0}{X_0} \text{ for all } \ \omega \in \{H, T\}.$$

Notice that for a portfolio in which all of the capital is invested in stock we have $\rho(\omega) = \rho^S(\omega)$ and for a portfolio in which all of the capital is invested in the bank we have $\rho(\omega) = r$. In situations involving more than one portfolio, we shall use the notation ρ^X to indicate the return of the portfolio X.

Definition 5.1. By an *arbitrage strategy* we mean a strategy (X_0, Δ_0) with $X_0 = 0$ such that

(i) $\mathbb{P}[X_1 \geq 0] = 1$ (i.e min $\{X_1(H), X_1(T)\} \geq 0$) and

(ii) $\mathbb{P}[X_1 > 0] > 0$ (i.e. max $\{X_1(H),\ X_1(T)\} > 0$).

Proposition 5.2. *In the one period binomial model there are no arbitrage strategies if and only if*

$$d < 1 + r < u. \tag{5.2}$$

Proof. Assume first that (5.2) holds. Consider a strategy with $X_0 = 0$. The terminal capital X_1 must satisfy

$$\begin{aligned} X_1(H) &= \Delta_0 S_0(\rho^S(H) - r) = \Delta_0 S_0(u - 1 - r), \\ X_1(T) &= \Delta_0 S_0(\rho^S(T) - r) = \Delta_0 S_0(d - 1 - r). \end{aligned}$$

Notice that $u - 1 - r > 0$ and $d - 1 - r < 0$. If $\Delta_0 = 0$, then $X_1(H) = X_1(T) = 0$, and the strategy is not an arbitrage. If $\Delta_0 > 0$ then $X_1(H) > 0$ and $X_1(T) < 0$, and the strategy is not an arbitrage. If $\Delta_0 < 0$ then $X_1(H) < 0$ and $X_1(T) > 0$, and, again, the strategy is not an arbitrage. It follows that there are no arbitrage strategies.

To prove that (No Arbitrage $\Rightarrow d < 1 + r < u$), we shall show that if $1 + r \leq d$ or if $1 + r \geq u$, then there is arbitrage. Suppose that $1 + r \leq d$. Then, at $t = 0$, we borrow S_0 from the bank and buy 1 share of stock. The initial capital of this strategy is $X_0 = 0$ and the terminal capital is $X_1(\omega) = S_0(\rho^S(\omega) - r)$ for all $\omega \in \{H, T\}$. Notice that $X_1(T) = S_0(d - 1 - r) \geq 0$ and $X_1(H) = S_0(u - 1 - r) > 0$. This is an arbitrage.

Suppose that $1+r \geq u$. Then, at time $t = 0$ we sell short 1 share of stock and invest S_0 in the bank. The initial capital of this strategy is $X_0 = 0$ and the terminal capital is $X_1(\omega) = S_0(r - \rho^S(\omega))$ for all $\omega \in \{H, T\}$. Notice that $X_1(H) = S_0(1 + r - u) \geq 0$ and $X_1(T) = S_0(1 + r - d) > 0$. This is an arbitrage. $\square$

Definition 5.3. By a *derivative security* or *contingent claim*, in the one-period binomial model we mean a security V with payoff at time 1 a random variable V_1 on Ω.

Some simple examples of derivative securities are

1. Zero-Coupon Bond with Face Value F and Maturity $T = 1$:

$$V_1(\omega) = F \text{ for all } \omega \in \{H, T\}.$$

2. The Stock:

$$V_1(\omega) = S_1(\omega) \text{ for all } \omega \in \{H, T\}.$$

3. Standard Call Option on the Stock with Strike Price K and Expiration Date $T = 1$:

$$V_1(\omega) = (S_1(\omega) - K)^+ = \max\{S_1(\omega) - K, 0\} \text{ for all } \omega \in \{H, T\}.$$

4. Standard Put Option on the Stock with Strike Price K and Expiration Date $T = 1$:

$$V_1(\omega) = (K - S_1(\omega))^+ = \max\{K - S_1(\omega), 0\} \text{ for all } \omega \in \{H, T\}.$$

5. Standard Straddle Option on the stock with strike price K and Expiration Date $T = 1$:

$$V_1(\omega) = |S_1(\omega) - K| \text{ for all } \omega \in \{H, T\}.$$

6. Long Position on a Forward Contract with Forward Price $\mathcal{F}$ and Delivery Date $T = 1$:

$$V_1(\omega) = S_1(\omega) - \mathcal{F} \text{ for all} \omega \in \{H, T\}.$$

Remark 5.4. In the one-period binomial model, straddle options are completely redundant because they can be expressed as the sum of a put and a call with the same strike price. (In fact, even when there are trading times strictly between $t = 0$ and $t = T$, holding a European straddle is equivalent to holding a European put and a European call with the same exercise date and strike price.) However, as we shall see, in the multiperiod binomial model, the situation regarding the relationship between puts, calls, and straddles becomes interesting when the options are American.

A very important problem is to compute prices for derivative securities. The price of a derivative security should be chosen so that arbitrage is not introduced if the security is added to the market comprised of the stock and the bank account. If it is possible to replicate a security by trading in the stock and the bank account, then the price of the security must match the initial capital fo the replicating portfolio – otherwise the mismatch in pricing can be used to creat an arbitrage. Within the context of the one-period binomial model, it is straightforward to give a precise definition of replication.

Definition 5.5. A strategy (X_0, Δ_0) is said to be a *replicating strategy* for a derivative security V provided that the terminal capital of the strategy is the same as the payoff of V, i.e.

$$X_1(\omega) = V_1(\omega) \text{ for all } \omega \in \{H, T\}.$$

Let V be a given derivative security. We shall try to construct a replicating strategy (X_0, Δ_0). Notice that

$$\begin{aligned} X_1(H) &= (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 u S_0 \\ X_1(T) &= (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 d S_0. \end{aligned}$$

We must find X_0 and Δ_0 (if possible) such that

$$(X_0 - \Delta_0 S_0)(1+r) + \Delta_0 u S_0 = V_1(H) \tag{5.3}$$

$$(X_0 - \Delta_0 S_0)(1+r) + \Delta_0 d S_0 = V_1(T). \tag{5.4}$$

Subtracting (5.4) from (5.3) we find that

$$\Delta_0 S_0 (u - d) = V_1(H) - V_1(T), \tag{5.5}$$

which yields

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{(u-d)S_0} = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{5.6}$$

Substituting (5.6) into (5.3) we obtain

$$\begin{aligned} X_0(1+r) &= V_1(H) + \Delta_0 S_0 (1 + r - u) \\ &= V_1(H) + \left(\frac{V_1(H) - V_1(T)}{u - d} \right)(1 + r - u) \\ &= \left(\frac{1+r-d}{u-d} \right) V_1(H) + \left(\frac{u-1-r}{u-d} \right) V_1(T). \end{aligned} \tag{5.7}$$

It is convenient to define $\tilde{p}$ and $\tilde{q}$ by

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d} \tag{5.8}$$

so that (5.7) becomes

$$X_0(1+r) = \tilde{p} V_1(H) + \tilde{q} V_1(T). \tag{5.9}$$

The initial capital X_0 is given by

$$X_0 = \frac{1}{1+r}(\tilde{p}V_1(H) + \tilde{q}V_1(T)). \tag{5.10}$$

We conclude that V is replicable and the arbitrage-free price V_0 is given by

$$V_0 = \frac{1}{1+r}(\tilde{p}V_1(H) + \tilde{q}V_1(T)). \tag{5.11}$$

Indeed, if V could be traded at a price higher than (5.11) then an arbitrage could be created by selling V and buying the replicating protfolio. Similarly, if V could be traded at a price lower than (5.11) then an arbitrage could be created by purchasing V and selling the replicating portfolio. Moreover, it is not difficult to show that if V is added to the market consisting of the stock and the bank account, and if (5.2) and (5.11) hold, then the extended model is arbitrage free. (See Proposition .)

The formula for V_0 has an extremely important interpretation. Notice that the no-arbitrage condition (5.2) implies that $\tilde{p} > 0$, $\tilde{q} > 0$. Observe further $\tilde{p} + \tilde{q} = 1$. Consequently, if we define $\tilde{\mathbb{P}} : \Omega \to [0, 1]$ by

$$\tilde{\mathbb{P}}(H) = \tilde{p}, \quad \tilde{\mathbb{P}}(T) = \tilde{q},$$

then $\tilde{\mathbb{P}}$ is a probability measure on Ω. It is called a *pricing measure* or *risk-neutral measure*. The term pricing measure is probably a better choice, but the term risk-neutral measure is used much more frequently in practice. We shall give an explanation of the terminiolgy "risk-neutral measure" later. It is best for now to simply think of $\tilde{\mathbb{P}}$ as a useful device to compute arbitrage-free prices of derivative securities.

The numbers $\tilde{p}$ and $\tilde{q}$ are called, *risk-neutral probabilities*, *pricing probablilities*, or *pricing weights*. It is important to understand that they do not represent the actual probabilities of events occuring within the model. However, since they are positive and sum to one, they can be interpreted as probabilities and this interpretation will allow us to invoke some very powerful tools from probability theory.

If we use $\tilde{\mathbb{E}}$ to denote expected value with respect to $\tilde{\mathbb{P}}$ then (5.11) can be rewritten as

$$V_0 = \frac{1}{1+r}\tilde{\mathbb{E}}(V_1). \tag{5.12}$$

In other words:

THE ARBITRAGE-FREE PRICE OF A DERIVATIVE SECURITY IS EQUAL TO THE DISCOUNTED RISK-NEUTRAL EXPECTED VALUE OF THE PAYOFF.

Remark 5.6. There is no need to memorize the formulas (5.8) for $\tilde{p}$ and $\tilde{q}$. Once we know that these probabilities exist we can find them quickly by using the observations that (5.11) must price the stock correctly and that $\tilde{q} = 1 - \tilde{p}$. This leads to the equation

$$(1 + r)S_0 = \tilde{p}uS_0 + (1 - \tilde{p})dS_0,$$

which can easily be solved for $\tilde{p}$ and then $\tilde{q}$ can be found from the relation $\tilde{q} = 1 - \tilde{p}$.

Example 5.7. We now revisit Example 1.23, described in terms of the binomial model. Let $u = 2,\ d = .5,\ r = .25$, and $S_0 = \$50$. Notice that $S_1(H) = 100$ and $S_1(T) = 25$. The risk-neutral probabilities, or pricing weights, are given by

$$\tilde{p} = \frac{1 + .25 - .5}{2 - .5} = .5 \text{ and } \tilde{q} = 1 - \tilde{p} = .5.$$

(a) Consider a standard put option P on the stock with strike price $K = \$40$ and exercise date $T = 1$. Observe that

$$P_1(\omega) = (40 - S_1(\omega))^+ \text{ for all } \omega \in \{H, T\},$$

so that $P_1(H) = 0$ and $P_1(T) = 15$. It follows that

$$P_0 = \frac{1}{1 + .25}(0 \times .5 + (15) \times (.5)) = \frac{4}{5}\left(\frac{15}{2}\right) = 6.$$

The number of shares of stock in the replicating portfolio for P is

$$\Delta_0 = \frac{P_1(H) - P_1(T)}{S_1(H) - S_1(T)} = \frac{0 - 15}{100 - 25} = -\frac{1}{5}.$$

(b) Consider a standard call option C on the stock with strike price $K = \$40$ and exercise date $T = 1$. Observe that

$$C_1(\omega) = (S_1(\omega) - 40)^+ \text{ for all } \omega \in \{H, T\}$$

so that $C_1(H) = 60$ and $C_1(T) = 0$. It follows that

$$C_0 = \frac{1}{1 + .25}(60 \times (.5) + 0 \times .5) = \frac{4}{5}(30) = 24.$$

The number of shares of stock in the replicating portfolio for C is

$$\Delta_0 = \frac{C_1(H) - C_1(T)}{S_1(H) - S_1(T)} = \frac{60 - 0}{100 - 25} = \frac{4}{5}.$$

(c) Let us use put-call parity together with the result of part (a) to compute the price C_0 of the call option from part (b). Recall that

$$P_0 - C_0 = D(1)(K - \mathcal{F}),$$

where $\mathcal{F}$ is the forward price for delivery of the stock at time 1. Since the stock does not pay dividends, we know that $\mathcal{F} = S_0(1 + r)$. It follows that

$$\begin{aligned} C_0 &= P_0 + \frac{1}{1 + r}(S_0(1 + r) - K) \\ &= 6 + 50 - 40 \times \left(\frac{4}{5}\right) \\ &= 56 - 32 = 24, \end{aligned}$$

which agrees with the result of part (b).

Example 5.8. Consider the one-period binomial model with $u = 1.2$, $d = .8$, $r = .05$ and $S_0 = 100$. Consider a standard call option V on the stock with strike price $K = 100$ and exercise date $T = 1$. Find the arbitrage-free price V_0 of the option and the number of shares of stock in a replicating portfolio.

Assuming that a broker sells 1,000 of these options at 2% above the arbitrage-free price, what should he do with the money obtained from the sale to hedge his position, and what will be the time 0 value of his profit. (Ignore any commissions on the stock transaction.)

The broker has exposed himself to significant risk by selling the options. He stands to lose more money than he collects from the sale of the options if the stock price goes up. Even if the broker believes very strongly that the stock price will go down, he should hedge (or protect) his short position on the calls by purchasing some stock. If he creates a perfect hedge for the option sale, he will be certain to make a profit because he is selling the options for more than the arbitrage-free price. Recall that the replicating strategy for the long position provides a hedging strategy for the short position.

Observe that

$$\begin{aligned} \tilde{p} &= \frac{1 + .05 - .8}{1.2 - .8} = .625 \\ \tilde{q} &= 1 - \tilde{p} = .375. \end{aligned}$$

Observe further that

$$V_1(H) = 20, \quad V_1(T) = 0,$$

so that

$$V_0 = \frac{1}{1.05}((.625) \times 20 + (.375) \times 0) \approx 11.90476$$

Observe also that

$$\Delta_0 = \frac{20 - 0}{120 - 80} = .5.$$

In order to hedge the sale of one option, the broker should purchase Δ_0 shares of stock and invest $V_0 - \Delta_0 S_0$ in the bank. (Notice that $V_0 - \Delta_0 S_0 < 0$, so the broker should borrow $\Delta_0 S_0 - V_0$ per option.) Consequently, in order to hedge the sale of 1,000 options, the broker needs to borrow

$$1,000(.5 \times \$100 - \$11.90476) = \$38,095.24$$

and purchase 500 shares of stock.

If the broker sells 1,000 options at $V_0 \times (1.02)$ he will collect \$12,142.86. If he borrows \$38,095.24 from the bank and purchases 500 shares of stock for \$50,000 he will be able to meet his obligation with regard to the options whether the stock goes up or down.

Indeed, if the stock goes up, he will have to pay the option holder \$20,000 and pay the bank \$40,000. His stock will be worth \$60,000 which will exactly cover his obligations.

If the stock goes down, he will owe nothing to the option holder, but will owe \$40,000 to the bank. His stock will be worth \$40,000 which will exactly cover his obligations.

The difference

$$\$12,142.86 - (\$50,000 - \$38,095.24) = \$238.10$$

between what he collects and the initial capital of the hedging strategy is the time-0 value of his profit. Of course,we should expect that the time 0 value of the profit will be

$$(.02) \times 1000 \times (\$11.90476) = \$238.10.$$

5.2 General One-Period Finite Models

We now study a general one-period finite model with two times, $t = 0$ and $t = 1$, and $k + 1$ financial instruments, where k is some given positive integer. One of the instruments is a riskless asset B that we shall refer to as a bank account. The remaining k instruments $S^1, S^2, \ldots, S^k$ are "risky" assets whose prices at time $t = 0$ are given, but whose prices at $t = 1$ are random variables on a finite probability space. For definiteness we shall usually refer to the risky assets as stocks, but it is important to realize that some (or all) of them could be other types of assets.

We assume that a finite probability space $(\Omega, \mathbb{P})$ is given with

$$\mathbb{P}(\omega) > 0 \text{ for all } \omega \in \Omega.$$

We shall refer to $\mathbb{P}$ as the *reference probability measure* or *actual probability measure* on Ω.

It is assumed that one can borrow or invest any amount of money at the bank between $t = 0$ and $t = 1$ at the one-period interest rate $r \geq 0$. Capital in the bank account evolves according to the rule

$$B_1 = B_0(1 + r).$$

$B_0 > 0$ corresponds to an investment, whereas $B_0 < 0$ corresponds to a loan.

The initial prices of the stocks are described by a list $(S_0^i)_{1 \leq i \leq k}$ where the entry S_0^i is simply the price at time 0 of the i^{th} stock. It is assumed that we can buy or sell short any number of shares (including fractional shares) of each of the stocks at $t = 0$. The prices of the stocks at $t = 1$ are described by a list $(S_1^i)_{1 \leq i \leq k}$ where each S_1^i is a random variable on Ω. A *strategy* or *portfolio* X in the model is described by a total initial capital X_0 and a list of real numbers $(\Delta_0^i)_{1 \leq i \leq k}$ where Δ_0^i represents the number of shares of S^i purchased at $t = 0$. If $\Delta_0^i > 0$ then the i^{th} stock has been purchased in the portfolio, whereas if $\Delta_0^i < 0$ then the i^{th} stock has been sold short in the portfolio. The total initial cost of the stocks is

$$\sum_{i=1}^{k} \Delta_0^i S_0^i,$$

so that the initial capital in the bank account is given by

$$B_0 = X_0 - \sum_{i=1}^{k} \Delta_0^i S_0^i.$$

It is convenient to introduce the *standard inner product* of two lists. If $(x^i)_{1\le i\le k}$ and $(y^i)_{1\le i\le k}$ are lists of length k we define

$$\langle x, y\rangle = \sum_{i=1}^{k} x^i y^i.$$

The initial capital in the bank account can be expressed as

$$B_0 = X_0 - \langle \Delta_0, S_0\rangle .$$

The terminal capital in the bank account is

$$B_1 = (X_0 - \langle \Delta_0, S_0\rangle)(1+r),$$

and the terminal capital of the strategy is given by

$$X_1(\omega) = (X_0 - \langle \Delta_0, S_0\rangle)(1+r) + \langle \Delta_0, S_1(\omega)\rangle \text{ for all } \omega \in \Omega.$$

The return of a portfolio with initial capital $X_0 > 0$ is the random variable ρ defined by

$$\rho(\omega) = \frac{X_1(\omega) - X_0}{X_0} \text{ for all } \omega \in \Omega.$$

In situations involving more than one portfolio, we shall use the notation ρ^X to denote the return of the portfolio X.

In order to be sure that the notational conventions are clear, we consider a simple example.

Example 5.9. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1$, $k = 2$, $S_0^1 = S_0^2 = 20$ and $S_1^1(\omega_1) = 24$, $S_1^1(\omega_2) = 18$, $S_1^1(\omega_3) = 16$, $S_1^2(\omega_1) = 24$, $S_1^2(\omega_2) = 24$, $S_1^2(\omega_3) = 8$. We write

$$\begin{aligned} S_0 &= (S_0^i)_{1\le i\le 2} = (20, 20), \\ S_1(\omega_1) &= (S_1^i(\omega_1))_{1\le i\le 2} = (24, 24), \\ S_1(\omega_2) &= (S_1^i(\omega_2))_{1\le i\le 2} = (18, 24), \\ S_1(\omega_3) &= (S_1^i(\omega_3))_{1\le i\le 2} = (16, 8). \end{aligned}$$

Consider a portfolio with $X_0 = 100$ and $\Delta_0^1 = 3$, $\Delta_0^2 = -1$, so that

$$\Delta_0 = (\Delta_0^i)_{1\le i\le 2} = (3, -1).$$

Notice that

$$\langle \Delta_0, S_0\rangle = 3 \times 20 + (-1) \times 20 = 40.$$

The terminal capital of this portfolio is given by

$$\begin{aligned} X_1(\omega_1) &= (100 - 40) \times (1.1) + (3 \times 24 + (-1) \times 24) = 114, \\ X_1(\omega_2) &= (100 - 40) \times (1.1) + (3 \times 18 + (-1) \times 24) = 96, \\ X_1(\omega_3) &= (100 - 40) \times (1.1) + (3 \times 16 + (-1) \times 8) = 106. \end{aligned}$$

Definition 5.10. By an *arbitrage strategy* we mean a strategy X with

(i) $X_0 = 0$,

(ii) $\mathbb{P}[X_1 \geq 0] = 1$, i.e. $\min\{X_1(\omega) : \omega \in \Omega\} \geq 0$,

(iii) $\mathbb{P}[X_1 > 0] > 0$, i.e. $\max\{X_1(\omega) : \omega \in \Omega\} > 0$.

The model is said to be arbitrage free if there are no arbitrage strategies.

It is crucial to understand when the model is arbitrage-free. A very elegant and practical answer to this question will be provided by the First Fundamental Theorem of Asset Pricing, which will be introduced in the next section.

Definition 5.11. By a *derivative security*, or *contingent claim*, we mean a security V with payment payment function V_1, a random variable on Ω.

Some important examples of derivative securities are

1. Zero-Coupon Bond with Face Value F and Maturity $T = 1$:
$$V_1(\omega) = F \text{ for all } \omega \in \Omega.$$

2. The i^{th} stock:
$$V_1(\omega) = S_1^i(\omega) \text{ for all } \omega \in \Omega.$$

3. Standard Call Option on the i^{th} Stock with Strike Price K and Expiration Date $T = 1$:
$$V_1(\omega) = (S_1^i(\omega) - K)^+ \text{ for all } \omega \in \Omega.$$

4. Standard Put Option on the i^{th} Stock with Strike Price K and Expiration Date $T = 1$:
$$V_1(\omega) = (K - S_1^i(\omega))^+ \text{ for all } \omega \in \Omega.$$

5. Standard Straddle Option on the i^{th} Stock with Strike Price K and Expiration Date $T = 1$:
$$V_1(\omega) = |S_1^i(\omega) - K| \text{ for all } \omega \in \Omega.$$

6. Long Position on a Forward Contract for the i^{th} Stock with Forward Price $\mathcal{F}$ and Delivery Date $T = 1$:
$$V_1(\omega) = S_1^i(\omega) - \mathcal{F} \text{ for all } \omega \in \Omega.$$

   Recall that the forward price $\mathcal{F}$ is chosen that it costs nothing initially to enter into a forward contract.

Of course, it is possible to consider derivative securities with more than one underlying asset. For example, we could consider the derivative security V with payoff

$$V_1(\omega) = \max\{S_1^i(\omega) : 1 \leq i \leq k\} \text{ for all } \omega \in \Omega.$$

This is an example of a *basket option*.

Definition 5.12. Consider a general one period finite model that is arbitrage free. A real number ξ is said to be an *arbitrage-free price* for a derivative security V providing that the extended model (consisting of the bank account, the original k risky assets, and an additional risky asset having initial price ξ and payment function V_1) is arbitrage free.

Given a derivative security V in an arbitrage-free model, we want to answer the following questions:

(i) Does an arbitrage-free price exist?

(ii) Can there be more than one arbitrage-free price?

(iii) How can we compute an arbitrage-free price?

Recall the basic principle:

PRICING = **REPLICATION**.

Definition 5.13. By a *replicating strategy* for a derivative security V we mean a strategy whose terminal capital is the same as the payment function for V, i.e.

$$X_1(\omega) = V_1(\omega) \text{ for all} \omega \in \Omega.$$

We say that V is *replicable* if it has at least one replicating strategy.

If a derivative security V is replicable (and the model is arbitrage free) then all replicating strategies for V have the same initial capital X_0 and X_0 is the unique arbitrage-free price for V.

Proposition 5.14. *Consider a general one-period finite model that is arbitrage free and let V be a replicable derivative security. Then*

(i) All replicating strategies for V have the same initial capital.

(ii) There is exactly one arbitrage-free price for V, namely the initial capital of any replicating strategy.

Proof. To prove (i), let X and Y be replicating strategies for V. Suppose, for the purpose of obtaining a contradiction, that $X_0 \neq Y_0$. Without loss of generality we may assume that $X_0 > Y_0$. Consider the strategy Z constructed by purchasing Y, selling X, and investing $X_0 - Y_0$ in the bank at time 0. The initial capital of this strategy is $Z_0 = 0$ and the terminal capital is given by

$$Z_1(\omega) = (X_0 - Y_0)(1 + r) > 0 \text{ for all } \omega \in \Omega.$$

Consequently, Z is an arbitrage strategy. This is impossible, since the model is arbitrage free. Therefore, we must have $X_0 = Y_0$.

To prove (ii), we assume first that the extended model is arbitrage free. Let X be a replicating strategy for V in the original model. Suppose that $V_0 > X_0$. The strategy $\hat{Z}$ constructed by selling V, purchasing X, and investing $V_0 - X_0$ in the bank at time 0 will be an arbitrage in the extended model since $\hat{Z}_0 = 0$ and

$$\hat{Z}_1(\omega) = (V_0 - X_0)(1 + r) > 0 \text{ for all } \omega \in \Omega.$$

Suppose that $V_0 < X_0$. Then the strategy $\bar{Z}$ constructed by selling X, purchasing V, and investing $X_0 - V_0$ in the bank at time 0 will be an arbitrage in the extended model since $\bar{Z}_0 = 0$ and

$$\bar{Z}_1(\omega) = (X_0 - V_0)(1 + r) > 0 \text{ for all } \omega \in \Omega.$$

We conclude that $V_0 = X_0$.

To complete the proof of (ii), we assume that $V_0 = X_0$, where X is a replicating strategy for V in the original model. We shall show that the extended model is arbitrage free by showing that the existence of an arbitrage in the extended model implies the existence of an arbitrage strategy in the original model.

For each $i = 1, 2, \cdots, k$, let Δ_0^i denote the number of shares of S^i in the replicating strategy X, so that

$$V_1(\omega) = \left(V_0 - \sum_{i=1}^{k} \Delta_0^i S_0^i \right)(1 + r) + \sum_{i=1}^{k} \Delta_0^i S_1^i(\omega) \text{ for all } \omega \in \Omega. \tag{5.13}$$

Suppose that W is an arbitrage strategy in the extended model. For each $i = 1, 2, \cdots, k$, let Γ_0^i denote the number of shares of S^i held in the strategy W and let μ denote the number of shares of V held in W. Observe that terminal capital W_1 is given by

$$W_1(\omega) = \left(0 - \mu V_0 - \sum_{i=1}^{k} \Gamma_0^i S_0^i \right)(1+r) + \mu V_1(\omega) + \sum_{i=1}^{k} \Gamma_0^i S_1^i(\omega) \text{ for all } \omega \in \Omega. \tag{5.14}$$

For each $i = 1, 2, \cdots, k$ let us put

$$\Lambda_0^i = \Gamma_0^i + \mu \Delta_0^i. \tag{5.15}$$

Combining (5.13), (5.14), and (5.15) we find that

$$W_1(\omega) = \left(0 - \sum_{i=1}^{k} \Lambda_0^i S_0^i\right)(1+r) + \sum_{i=1}^{k} \Lambda_0^i S_0^i(\omega) \text{ for all } \omega \in \Omega. \tag{5.16}$$

Since $\mathbb{P}[W_1 \geq 0] = 1$ and $\mathbb{P}[W_1 > 0] > 0$, we conclude that the strategy in the original model having 0 initial capital and holding Λ_0^i shares of S_i for $i = 1, 2, \cdots, k$ is an arbitrage strategy. (Indeed, the terminal capital of this strategy is the same as the terminal capital of W.) This is not possible since the original model was assumed to be arbitrage free. $\square$

Remark 5.15. Let V be a non-replicable derivative security in an arbitrage-free one-period finite model. Then there will be an interval of arbitrage-free prices for V. This phenomenon will be discussed in Section 5.6.

Definition 5.16. A finite one-period model is said to be *complete* if it is arbitrage-free and every derivative security is replicable.

In a complete model, every derivative security has a unique arbitrage-free price. An elegant and very useful characterization of complete models is provided by the Second Fundamental Theorem of Asset Pricing, which will be introduced in the next section.

Example 5.17 (An Incomplete Model). Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1,\ k = 1$, $S_0^1 = 20$, and $S_1^1(\omega_1) = 24$, $S_1^1(\omega_2) = 18$, $S_1^1(\omega_3) = 16$. We shall show that this model is incomplete by explicitly exhibiting a derivative security that cannot be replicated. Later, we shall use the Second Fundamental Theorem of Asset Pricing to show that this model is incomplete.

Consider a put option V on S^1 with $K = 20$. Notice that $V_1(\omega_1) = 0,\ V_1(\omega_2) = 2$, $V_1(\omega_3) = 4$. To construct a replicating strategy we must find X_0 and Δ_0^1 if possible so that

$$\begin{aligned}
X_1(\omega_1) &= (X_0 - 20\Delta_0^1) \times 1.1 + 24\Delta_0^1 = V_1(\omega_1)\\
X_1(\omega_2) &= (X_0 - 20\Delta_0^1) \times 1.1 + 18\Delta_0^1 = V_1(\omega_2)\\
X_1(\omega_3) &= (X_0 - 20\Delta_0^1) \times 1.1 + 16\Delta_0^1 = V_1(\omega_3).
\end{aligned}$$

This leads to the system of equations

$$(X_0 - 20\Delta_0^1) \times 1.1 + 24\Delta_0^1 = 0 \tag{5.17}$$
$$(X_0 - 20\Delta_0^1) \times 1.1 + 18\Delta_0^1 = 2 \tag{5.18}$$
$$(X_0 - 20\Delta_0^1) \times 1.1 + 16\Delta_0^1 = 4. \tag{5.19}$$

Subtracting (5.18) from (5.17) we find that $\Delta_0^1 = -\frac{1}{3}$. On the other hand, subtracting (5.19) from (5.17) yields $\Delta_0^1 = -\frac{1}{2}$, which is a contradiction. It follows that V is not replicable and the model is not complete.

5.3 Pricing Measures and the Fundamental Theorems of Asset Pricing

The First and Second Fundamental Theorems of Asset Pricing are based on the notion of "magical" probability measures on Ω that can be used to relate the initial capital to the terminal capital for all strategies, via a formula analogous to (5.12). Such a measure is called a *pricing measure* or *risk-neutral measure*.

Definition 5.18. By a *pricing measure* or *risk-neutral measure*, we mean a probability measure $\tilde{\mathbb{P}}$ on Ω such that

(i) $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$

(ii) $X_0 = \frac{1}{1+r}\mathbb{E}^{\tilde{\mathbb{P}}}(X_1)$ for every strategy X.

Here $\mathbb{E}^{\tilde{\mathbb{P}}}$ denotes the expected value with respect to $\tilde{\mathbb{P}}$, i.e.

$$\mathbb{E}^{\tilde{\mathbb{P}}}(Y) = \sum_{\omega\in\Omega} \tilde{\mathbb{P}}(\omega)Y(\omega).$$

Remark 5.19. The term "risk-neutral measure" seems to lead to quite a bit of confusion. In our opinion, the term "pricing measure" is preferable. However, in practice the term "risk-neutral measure" is used by most people. We shall discuss the meaning of the term "risk-neutral" in Chapter 6.

The condition (ii) in the above definition is not very useful for direct computation of risk-neutral probabilities. Below we give a more convenient characterization.

Lemma 5.20. *A probability measure $\tilde{\mathbb{P}}$ on Ω is a pricing measure if and only if*

(i) $\tilde{\mathbb{P}}(\omega) > 0$ *for all* $\omega \in \Omega$,

(ii) $S_0^i = \frac{1}{1+r}\mathbb{E}^{\tilde{\mathbb{P}}}(S_1^i)$ *for all* $i = 1, 2, \ldots, k$.

Proof. For a pricing measure $\tilde{\mathbb{P}}$ the conditions of the lemma hold trivially. Indeed, the second item in the lemma is just a restriction of the corresponding item in Definition (5.18) to the case of buy and hold strategies for the individual stocks.

To prove the reverse implication consider a strategy with initial capital X_0 and the initial number of stocks $\Delta_0 = (\Delta_0^i)_{1\le i\le k}$. The terminal capital of the strategy is given by

$$\begin{aligned} X_1(\omega) &= (X_0 - \langle\Delta_0, S_0\rangle)(1+r) + \langle\Delta_0, S_1(\omega)\rangle \\ &= X_0(1+r) + \sum_{i=1}^{k} \Delta_0^i(S_1^i(\omega) - S_0^i(1+r)) \text{ for all } \omega \in \Omega. \end{aligned}$$

By taking the expected value under a probability measure $\tilde{\mathbb{P}}$ satisfying the conditions of the lemma, we obtain

$$\mathbb{E}^{\tilde{\mathbb{P}}}(X_1) = X_0(1+r).$$

Hence, $\tilde{\mathbb{P}}$ is a pricing measure. □

Risk-neutral probabilities are artificial probabilities in the sense that they are introduced as a device to compute arbitrage-free prices and do not represent the probabilities of real-world events. They are extremely powerful theoretical and practical tools as the fundamental theorems of asset pricing show.

Theorem 5.21 (First Fundamental Theorem of Asset Pricing). *In the general one-period finite model, the following two statements are equivalent:*

(i) The model is arbitrage-free.

(ii) There is at least one pricing measure $\tilde{\mathbb{P}}$.

Theorem 5.22 (Second Fundamental Theorem of Asset Pricing). *In the general one-period finite model the following two statements are equivalent:*

(i) The model is complete.

(ii) There is exactly one pricing measure $\tilde{\mathbb{P}}$.

Remark 5.23. In a complete model, where there is exactly one pricing measure $\tilde{\mathbb{P}}$, we use the notation $\tilde{\mathbb{E}}$ in place of the more cumbersome $\mathbb{E}^{\tilde{\mathbb{P}}}$.

In models that are arbitrage-free, but incomplete, it is important to know whether or not a given security is replicable. The following remark addresses this issue in terms of pricing measures.

Remark 5.24. Consider a general one-period model that is arbitrage free and let V be a derivative security with payment function V_1. Then V is replicable if and only if

$$\mathbb{E}^{\tilde{\mathbb{P}}}(V_1) = \mathbb{E}^{\tilde{\mathbb{Q}}}(V_1)$$

for all pricing measures $\tilde{\mathbb{P}}, \tilde{\mathbb{Q}}$.

We shall give a detailed proof of the second fundamental theorem, and a sketch of the proof of the first fundamental theorem. Before discussing the proofs, we shall consider several examples.

Example 5.25. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1$, $k = 2, S_0 = (20, 20)$, $S_1(\omega_1) = (24, 24)$, $S_1(\omega_2) = (18, 24)$, $S_1(\omega_3) = (16, 8)$.

Let us try to find a risk-neutral measure $\tilde{\mathbb{P}}$. For this purpose it is convenient to put $\tilde{p}_1 = \tilde{\mathbb{P}}(\omega_1)$, $\tilde{p}_2 = \tilde{\mathbb{P}}(\omega_2)$, $\tilde{p}_3 = \tilde{\mathbb{P}}(\omega_3)$. Notice that

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}}(S_1^1) &= 24\tilde{p}_1 + 18\tilde{p}_2 + 16\tilde{p}_3 \\ \mathbb{E}^{\tilde{\mathbb{P}}}(S_1^2) &= 24\tilde{p}_1 + 24\tilde{p}_2 + 8\tilde{p}_3. \end{aligned}$$

We must have $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1$ since $\tilde{\mathbb{P}}$ is a probability measure. We are thus led to the system of equations

$$24\tilde{p}_1 + 18\tilde{p}_2 + 16\tilde{p}_3 = 20 \times (1.1) = 22 \tag{5.20}$$

$$24\tilde{p}_1 + 24\tilde{p}_2 + 8\tilde{p}_3 = 20 \times (1.1) = 22 \tag{5.21}$$

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1. \tag{5.22}$$

It follows from (5.22) that

$$\tilde{p}_3 = 1 - \tilde{p}_1 - \tilde{p}_2. \tag{5.23}$$

Substituting from (5.23) into (5.20) and (5.21) we obtain

$$8\tilde{p}_1 + 2\tilde{p}_2 = 6 \tag{5.24}$$

$$16\tilde{p}_1 + 16\tilde{p}_2 = 14. \tag{5.25}$$

Subtracting 2 times equation (5.24) from (5.25) we find that

$$12\tilde{p}_2 = 2,$$

so that

$$\tilde{p}_2 = \frac{1}{6}. \tag{5.26}$$

Substituting (5.26) into (5.24) we find that

$$\tilde{p}_1 = \frac{17}{24}. \tag{5.27}$$

Using (5.23) we find that

$$\tilde{p}_3 = \frac{1}{8}. \tag{5.28}$$

It follows that there is exactly one risk neutral measure $\tilde{\mathbb{P}}$, so the model is complete.

We now use $\tilde{\mathbb{P}}$ to price the following derivative securities U, V, W.

(a) Put Option U on S^1 with $K = 20$.

(b) Basket Option V with payoff

$$V_1(\omega) = \max\{S_1^1(\omega),\ S_1^2(\omega)\} \text{ for all } \omega \in \Omega.$$

(c) The derivative security W with payoff

$$W_1(\omega) = \left|S_1^1(\omega) - S_1^2(\omega)\right| \text{ for all } \omega \in \Omega.$$

(A) Notice that $U_1(\omega_1) = 0,\ U_1(\omega_2) = 2,\ U_1(\omega_3) = 4.$ It follows that

$$\tilde{\mathbb{E}}(U_1) = \left(\left(\frac{17}{24}\right) \times 0 + \left(\frac{1}{6}\right) \times 2 + \left(\frac{1}{8}\right) \times 4\right) = .833$$

We conclude that

$$U_0 = \left(\frac{1}{1.1}\right) \times (.833) = .76.$$

(B) Since $V_1(\omega_1) = 24,\ V_1(\omega_2) = 24$, and $V_1(\omega_3) = 16$, we conclude that

$$V_0 = \frac{1}{1.1}\left(\left(\frac{17}{24}\right) \times 24 + \left(\frac{1}{6}\right) \times 24 + \left(\frac{1}{8}\right) \times 16\right) = 20.91.$$

(C) Since $W_1(\omega_1) = 0$, $W_1(\omega_2) = 6$, and $W_1(\omega_3) = 8$, we conclude that

$$W_0 = \frac{1}{1.1}\left(\left(\frac{17}{24}\right) \times 0 + \left(\frac{1}{6}\right) \times 6 + \left(\frac{1}{8}\right) \times 8\right) = 1.82.$$

Example 5.26. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1$, $k = 1$, $S_0^1 = 20$, $S_1^1(\omega_1) = 24$, $S_1^1(\omega_2) = 18$, $S_1^1(\omega_3) = 16$. We have already seen that this model is incomplete by producing a derivative security that is not replicable. We shall now apply the fundamental theorems to this model.

We seek a pricing measure $\tilde{\mathbb{P}}$. Let $\tilde{p}_1 = \tilde{\mathbb{P}}(\omega_1)$, $\tilde{p}_2 = \tilde{\mathbb{P}}(\omega_2)$, and $\tilde{p}_3 = \tilde{\mathbb{P}}(\omega_3)$. We must solve the pair of equations

$$24\tilde{p}_1 + 18\tilde{p}_2 + 16\tilde{p}_3 = 22 \tag{5.29}$$

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1. \tag{5.30}$$

There are more unknowns than there are equations, so we should not expect a unique solution. Let us set $\tilde{p}_3 = \tau$, for some parameter τ with $0 < \tau < 1$, so that (5.29) and (5.30) become

$$24\tilde{p}_1 + 18\tilde{p}_2 = 22 - 16\tau \tag{5.31}$$

$$\tilde{p}_1 + \tilde{p}_2 = 1 - \tau. \tag{5.32}$$

Solving (5.31) and (5.32) for $\tilde{p}_1$ and $\tilde{p}_2$ we find that

$$\tilde{p}_1 = \frac{2}{3} + \frac{1}{3}\tau$$
$$\tilde{p}_2 = \frac{1}{3} - \frac{4}{3}\tau.$$

In order to have $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$ it is necessary and sufficient to have

$$0 < \tau < \frac{1}{4}.$$

There are infinitely many risk-neutral measures, so the model is arbitrage-free, but not complete.

Example 5.27. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .25$, and $k = 2$. One of the risky assets is a stock S with $S_0 = 8$, $S_1(\omega_1) = 4$, $S_1(\omega_2) = 8$, and $S_1(\omega_3) = 12$. The other risky asset is a call C on S with $K = 10$ and initial price $C_0 = .80$. Notice that $C_1(\omega_1) = 0$, $C_1(\omega_2) = 0$, and $C_1(\omega_3) = 2$. Let us try to find a risk-neutral measure $\tilde{\mathbb{P}}$. We put $\tilde{p}_1 = \tilde{\mathbb{P}}(\omega_1)$, $\tilde{p}_2 = \tilde{\mathbb{P}}(\omega_2)$, $\tilde{p}_3 = \tilde{\mathbb{P}}(\omega_3)$. This leads to the system of equations

$$4\tilde{p}_1 + 8\tilde{p}_2 + 12\tilde{p}_3 = 8 \times (1.25) = 10 \tag{5.33}$$

$$0 \cdot \tilde{p}_1 + 0 \cdot \tilde{p}_2 + 2\tilde{p}_3 = .80 \times (1.25) = 1 \tag{5.34}$$

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1. \tag{5.35}$$

It follows from (5.34) thet $\tilde{p}_3 = \frac{1}{2}$. Substituting this value for $\tilde{p}_3$ into (5.33) and (5.35) we obtain

$$4\tilde{p}_1 + 8\tilde{p}_2 = 4 \tag{5.36}$$
$$\tilde{p}_1 + \tilde{p}_2 = \frac{1}{2}. \tag{5.37}$$

Subtracting 4 times (5.37) from (5.36) gives $4\tilde{p}_2 = 2$, so that $\tilde{p}_2 = \frac{1}{2}$. Putting $\tilde{p}_2 = \frac{1}{2}$ in (5.37) gives $\tilde{p}_1 = 0$. This solution does **not** correspond to a risk-neutral measure because $\tilde{p}_1$ is not strictly positive.

There are no risk-neutral measures, so this model must admit arbitrage. Let's produce an example of an arbitrage strategy.

Although the proofs of the fundamental theorems are quite complicated, one very important part has a very simple proof, which we give below.

Proposition 5.28. *Consider a general one-period finite model and assume that there is at least one pricing measure $\tilde{\mathbb{P}}$. Then the model is arbitrage free.*

Proof. Let us write $\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$. Consider a strategy X having $X_1(\omega) \geq 0$ for all $\omega \in \Omega$ and $\mathbb{P}[X_1(\omega) > 0] > 0$. We may choose $j \in \{1, 2, \cdots, N\}$ such that $X_1(\omega_j) > 0$. Then, we have

$$\tilde{\mathbb{E}}(X_1) \geq \tilde{\mathbb{P}}(\omega_j) X_1(\omega_j) > 0,$$

which implies that $X_0 > 0$. It follows that there can be no arbitrage strategies. □

5.4 Arrow-Debreu Securities

In this section we describe some very simple, but extremely useful securities, that were first introduced by Kenneth Arrow and Gerard Debreu [AD] in 1954. (Use of these securities in a finite one-period model is analogous to the use of a carefully chosen basis for a finite-dimensional vector space in Linear Algebra.)

Let N be the number of elements of Ω, so that we may write

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

For each $i = 1, 2, \cdots, N$ we let V^i denote the security with payment function

$$V_1^i(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_i \\ 0 & \text{if } \omega \neq \omega_i. \end{cases} \tag{5.38}$$

We refer to V^i as the i^{th} *Arrow-Debreu* security. These securities will play an important role in our proof of the second fundamental theorem. We shall illustrate their use first in a simple numerical example.

Example 5.29. Consider a finite one-period model in which Ω contains 3 elements: $\Omega = \{\omega_1, \omega_2, \omega_3\}$. We record the payoffs of the Arrow-Debreu securities in the table below.

ω	$V_1^1(\omega)$	$V_1^2(\omega)$	$V_1^3(\omega)$
ω_1	1	0	0
ω_2	0	1	0
ω_3	0	0	1

Observe that a portfolio that holds one of each of the Arrow-Debreu securities has a sure payoff of 1, i.e.

$$V_1^1(\omega) + V_1^2(\omega) + V_1^3(\omega) = 1 \text{ for all } \omega \in \Omega.$$

The portfolio obtained by investing $1/(1+r)$ in the bank at time zero has the same payoff. Consequently, if there is no arbitrage and if the Arrow-Debreu securities are traded at the initial prices V_0^1, V_0^2, V_0^3 then we must have

$$V_0^1 + V_0^2 + V_0^3 = \frac{1}{1+r}. \tag{5.39}$$

Another important property of Arrow-Debreu securities concerns their use in replication arguments. Consider, for example, the derivative security U having payoff

$$U_1(\omega_1) = 3, \quad U_1(\omega_2) = 11, \quad U_1(\omega_3) = 5.$$

Observe that

$$U_1(\omega) = 3V_1^1(\omega) + 11V_1^2(\omega) + 5V_1^3(\omega_3) \text{ for all } \omega \in \Omega.$$

In fact, for any derivative security W we can write

$$W_1(\omega) = W_1(\omega_1)V_1^1(\omega) + W_1(\omega_2)V_1^2(\omega) + W(\omega_3)V_1^3(\omega) \text{ for all } \omega \in \Omega. \tag{5.40}$$

In other words, an arbitrary derivative security can always be replicated in terms of the Arrow-Debreu secuirties—without solving any equations.

Example 5.30. Equations (5.39) and (5.40) generalize easily. Consider a general finite one-period model in which Ω contains N elements: $\Omega = \{\omega_1, \cdots, \omega_N\}$. Notice that

$$V_1^1(\omega) + V_1^2(\omega) + \ldots + V_1^N(\omega) = 1 \text{ for all } \omega \in \Omega.$$

In other words, the terminal capital of a strategy that holds exactly one of each of the Arrow-Debreu Securities is 1. The strategy in which $1/(1+r)$ is invested in the bank at $t = 0$ also has terminal capital 1. If there is no arbitrage and the Arrow-Debreu securities are traded at the initial prices $V_0^1, V_0^2, \cdots, V_0^N$ then we must have

$$\sum_{i=1}^{N} V_0^i = \frac{1}{1+r}. \tag{5.41}$$

Let W be an arbitrary derivative security with payment function W_1. It is straightforward to check that

$$W_1(\omega) = \sum_{j=1}^{N} V_1^j(\omega) W_1(\omega_j) \text{ for all } \omega \in \Omega. \tag{5.42}$$

Indeed, for each $i = 1, 2, \ldots, N$, substitution of ω_i into (5.42) gives

$$W_1(\omega_i) = \sum_{j=1}^{N} V_1^j(\omega_i) W_1(\omega_j).$$

Since $V_1^j(\omega_i) = 0$ except when $j = i$, we have

$$W_1(\omega_i) = V^i(\omega_i) W_1(\omega_i) = W_1(\omega_i) \text{ for all } i = 1, 2, \ldots, N,$$

which validates (5.42).

5.5 Proof of the Second Fundamental Theorem

Let N be the number of elements in Ω, so that we may write

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}.$$

We begin by showing that (i) $\Rightarrow$ (ii) in Theorem 5.22, i.e. that in a complete model there is exactly one pricing measure.

Assume that the model is complete. We need to show that there is exactly one risk neutral measure $\tilde{\mathbb{P}}$. For this purpose, it is convenient to use the Arrow-Debreu securities $V^1, V^2, \cdots, V^N$ defined by (5.38).

Since the model is complete, each V^i is replicable. Let x^i denote the initial capital of a replicating strategy for $V^i, i = 1, 2, \ldots, N$. Notice that for each $i = 1, 2, \ldots, N$, we have

$$V_1^i(\omega) \geq 0 \text{ for all } \omega \in \Omega$$

and

$$V_1^i(\omega_i) > 0,$$

so that

$$\mathbb{P}[V_1^i \geq 0] = 1,$$

and

$$\mathbb{P}[V_1^i > 0] = \mathbb{P}(\omega_i) > 0.$$

It follows easily that

$$x^i > 0 \text{ for all } i = 1, 2, \ldots, N \tag{5.43}$$

since the model is arbitrage-free. (Indeed, if $x^i = 0$, then a replicating strategy for V^i must be an arbitrage strategy. If $x^i < 0$ we could create an arbitrage-strategy by replicating V^i and investing the surplus capital in the bank.) Recall that

$$V_1^1(\omega) + V_1^2(\omega) + \ldots + V_1^N(\omega) = 1 \text{ for all } \omega \in \Omega.$$

In other words, the terminal capital of a strategy that holds exactly one of each of the Arrow-Debreu Securities is 1. The strategy in which $1/(1+r)$ is invested in the bank at $t=0$ also has terminal capital 1. Since there is no arbitrage, the initial capitals of these two strategies must coincide, i.e.

$$\sum_{i=1}^{N} x^i = \frac{1}{1+r}. \tag{5.44}$$

Let W be an arbitrary derivative security with payment function W_1. Recall from Section 5.4 that

$$W_1(\omega) = \sum_{j=1}^{N} V_1^j(\omega) W_1(\omega_j) \text{ for all } \omega \in \Omega. \tag{5.45}$$

It follows from (5.45) that the initial capital W_0 in any replicating strategy for W must satisfy

$$W_0 = \sum_{j=1}^{N} x^j W_1(\omega_j). \tag{5.46}$$

Let us define $\tilde{\mathbb{P}} : \Omega \to \mathbb{R}$ by

$$\tilde{\mathbb{P}}(\omega_j) = x^j(1+r) \text{ for all } j = 1, 2, \ldots, N. \tag{5.47}$$

It follows from (5.43), (5.44), and (5.47) that

$$\tilde{\mathbb{P}}(\omega_j) > 0 \text{ for all } j = 1, 2, \ldots N \tag{5.48}$$

and

$$\sum_{j=1}^{N} \tilde{\mathbb{P}}(\omega_j) = 1. \tag{5.49}$$

Combining (5.46) and (5.47) we find that

$$W_0 = \frac{1}{1+r} \sum_{j=1}^{N} \tilde{\mathbb{P}}(\omega_j) W_1(\omega_j) = \frac{1}{1+r} \mathbb{E}^{\tilde{\mathbb{P}}}(W_1). \tag{5.50}$$

It follows easily from (5.48), (5.49), and (5.50) that $\tilde{\mathbb{P}}$ is a risk-neutral measure.

Suppose that $\hat{\mathbb{P}}$ is another risk neutral measure. Then for each $i = 1, 2, \ldots, N$, we must have

$$x^i = \frac{1}{1+r} \mathbb{E}^{\hat{\mathbb{P}}}(V_1^i) = \frac{1}{1+r} \sum_{j=1}^{N} \hat{\mathbb{P}}(\omega_j) V_1^i(\omega) = \frac{1}{1+r} \hat{\mathbb{P}}(\omega_i).$$

It follows that

$$\hat{\mathbb{P}}(\omega_i) = (1+r)x^i = \tilde{\mathbb{P}}(\omega_i) \text{ for all } i = 1, 2, \ldots, N,$$

so that $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$, and there is exactly one risk-neutral measure. This completes the proof of (i) $\Rightarrow$ (ii).

To prove that (ii) $\Rightarrow$ (i), we assume that there is exactly one risk-neutral measure $\tilde{\mathbb{P}}$. It follows from Proposition 5.28 that the model is arbitrage free.

It remains to show that every derivative security is replicable. To this end, let W be an arbitrary derivative security with payment function W_1. We want to find a replicating strategy for W. Let $\mathcal{X}$ denote the set of terminal capitals of strategies. Notice that the elements of $\mathcal{X}$ are random variables on Ω. In other words, if we denote by $\mathcal{Y}$ the set of all random variables on Ω, then $\mathcal{X} \subset \mathcal{Y}$. We need to show that $W_1 \in \mathcal{X}$. In order to accomplish this, we shall find $\hat{X}_1 \in \mathcal{X}$ that best approximates W_1 in a sense described below. We shall then show that the approximation is exact, i.e. $W_1 = \hat{X}_1$. Since $\hat{X}_1 \in \mathcal{X}$, this will show that $W_1 \in \mathcal{X}$ and the proof will be complete.

Consider the problem of minimizing

$$\tilde{\mathbb{E}}((X_1 - W_1)^2)$$

over all $X_1 \in \mathcal{X}$. It can be shown (using methods of Linear Algebra, for example) that there is a unique minimizer $\hat{X}_1$, i.e. there is exactly one $\hat{X}_1 \in \mathcal{X}$ such that

$$\tilde{\mathbb{E}}((\hat{X}_1 - W_1)^2) \leq \tilde{\mathbb{E}}((X_1 - W_1)^2) \text{ for all } X_1 \in \mathcal{X}. \tag{5.51}$$

Let $Z_1 \in \mathcal{X}$ be given. Then we have

$$yZ_1 + (1-y)\hat{X}_1 \in \mathcal{X} \text{ for all } y \in \mathbb{R}. \tag{5.52}$$

It follows from (5.51) and (5.52) that

$$\tilde{\mathbb{E}}((\hat{X}_1 - W_1)^2) \leq \tilde{\mathbb{E}}((yZ_1 + (1-y)\hat{X}_1 - W_1)^2) \text{ for all } y \in \mathbb{R}. \tag{5.53}$$

This suggests that we define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) = \tilde{\mathbb{E}}((yZ_1 + (1-y)\hat{X}_1 - W_1)^2) \text{ for all } y \in \mathbb{R}. \tag{5.54}$$

Using (5.53) we can rewrite (5.54) as

$$g(0) \leq g(y) \text{ for all } \in \mathbb{R}. \tag{5.55}$$

It is straightforward to check that g is differentiable on $\mathbb{R}$ and

$$g'(y) = 2\tilde{\mathbb{E}}((yZ_1 + (1-y)\hat{X}_1 - W_1)(Z_1 - \hat{X}_1)) \text{ for all } y \in \mathbb{R}. \tag{5.56}$$

(To obtain (5.56) from (5.55), we can simply express the right-hand side of (5.55) as a finite sum and use the basic rules of differentiation.)

It follows from (5.55) that g attains a minimum at 0 so we must have $g'(0) = 0$. Using (5.56) we arrive at

$$\tilde{\mathbb{E}}((\hat{X}_1 - W_1)(Z_1 - \hat{X}_1)) = 0. \tag{5.57}$$

It is convenient to define $G : \Omega \to \mathbb{R}$ by

$$G(\omega) = \hat{X}_1(\omega) - W_1(\omega) \text{ for all } \omega \in \Omega. \tag{5.58}$$

Since $Z_1 \in \mathcal{X}$ was arbitrary, it follows from (5.57) and (5.58) that

$$\tilde{\mathbb{E}}(G \times (Z_1 - \hat{X}_1)) = 0 \text{ for all } Z_1 \in \mathcal{X}. \tag{5.59}$$

We deduce immediately from (5.59) that

$$\tilde{\mathbb{E}}(GZ_1) = \tilde{\mathbb{E}}(G\hat{X}_1) \text{ for all } Z_1 \in \mathcal{X}. \tag{5.60}$$

Since $0 \in \mathcal{X}$, we deduce from (5.60) that

$$\tilde{\mathbb{E}}(GZ_1) = \tilde{\mathbb{E}}(G\hat{X}_1) = 0 \text{ for all } Z_1 \in \mathcal{X}. \tag{5.61}$$

If we invest $\frac{1}{1+r}$ in bank at $t = 0$, then we will have capital 1 at $t = 1$, so the constant random variable with value 1 belongs to $\mathcal{X}$. Therefore we may put $Z_1 = 1$ in (5.61) to deduce that

$$\tilde{\mathbb{E}}(G) = 0.$$

Since Ω has only a finite number of elements, we may choose $\epsilon > 0$ (but possibly very small) so that

$$1 + \epsilon G(\omega) > 0 \text{ for all } \omega \in \Omega. \tag{5.62}$$

now define $\hat{\mathbb{P}} : \Omega \to \mathbb{R}$ by

$$\hat{\mathbb{P}}(\omega) = (1 + \epsilon G(\omega))\tilde{\mathbb{P}}(\omega) \text{ for all } \omega \in \Omega. \tag{5.63}$$

Our goal is to show that $\hat{\mathbb{P}}$ is a risk-neutral measure. Since there is only one risk-neutral measure, this will tell us that $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$, and allow us to conclude from (5.63) that $G = 0$.

Observe first that

$$\hat{\mathbb{P}}(\omega) > 0 \text{ for all } \omega \in \Omega \tag{5.64}$$

by virtue of (5.62), (5.63), and the fact that $\tilde{\mathbb{P}}(\omega) > 0$. Next, we observe that

$$\begin{aligned}\sum_{\omega\in\Omega}\hat{\mathbb{P}}(\omega) &= \sum_{\omega\in\Omega}(1 + \epsilon G(\omega))\tilde{\mathbb{P}}(\omega) \\ &= \tilde{\mathbb{E}}(1 + \epsilon G) \\ &= \tilde{\mathbb{E}}(1) + \epsilon\tilde{\mathbb{E}}G \\ &= 1 + 0 = 1.\end{aligned} \tag{5.65}$$

To show that $\hat{\mathbb{P}}$ is a risk-neutral measure, it remains only to verify the appropriate relationship between the initial and terminal capitals of replicable strategies. Let $Z_1 \in \mathcal{X}$ be given with initial capital Z_0. Then we have

$$\begin{aligned}\hat{\mathbb{E}}(Z_1) &= \tilde{\mathbb{E}}((1 + \epsilon G)Z_1) \\ &= \tilde{\mathbb{E}}(Z_1) + \epsilon\tilde{\mathbb{E}}(GZ_1) \\ &= Z_0(1 + r) + 0 \\ &= Z_0(1 + r),\end{aligned} \tag{5.66}$$

by virtue of (5.61) and the fact that $\tilde{\mathbb{P}}$ is a risk-neutral measure.

It follows that $\hat{\mathbb{P}}$ is a risk-neutral measure. Since there is only one risk-neutral measure, we conclude that

$$\hat{\mathbb{P}}(\omega) = \tilde{\mathbb{P}}(\omega) \text{ for all } \omega \in \Omega. \tag{5.67}$$

Since $\epsilon > 0$, it follows from (5.62) that

$$G(\omega) = 0 \text{ for all } \omega \in \Omega. \tag{5.68}$$

We conclude from (5.58) that $\hat{X}_1 = W_1$ which tells us that W is replicable. This completes the proof of the second fundamental theorem.

5.6 Sketch of a Proof of the First Fundamental Theorem

The idea of the proof is to extend the model so that it becomes complete and invoke the second fundamental theorem to obtain a pricing measure for the extended model. A pricing measure in the extended model is clearly a pricing measure in the original model.

If a model contains all of the Arrow-Debreu securities as basic securities, then any derivative security can be replicated. Lemma 5.32 below tells us that in an arbitrage-free model, every derivative security has at least one arbitrage-free price. Consequently, if we start with an arbitrage-free model, we can add Arrow-Debreu securities to the model without introducing arbitrage. Before stating Lemma 5.32, we give a simple lemma that will be used in the proof of Lemma 5.32.

Lemma 5.31. *Consider a general one-period finite model that is arbitrage free and let V be a derivative security. If there is an arbitrage strategy in the extended model in which V is traded at the initial price η, then there is an arbirtage strategy in the extended model that either is long exactly one share of V or is short exactly one share of V.*

Proof. Let X be an arbitrage strategy in the extended model. Let Δ^i denote the number of shares of S^i held in the strategy, and let μ denote the number of shares of V held in the strategy. We know that $\mu \neq 0$ since the original model is arbitrage free. Observe that the terminal capital X_1 is given by

$$X_1(\omega) = (0 - \mu\eta - \sum_{i=1}^{k} \Delta_0^i S_0^i)(1+r) + \mu V_1(\omega) + \sum_{i=1}^{k} \Delta_0^i S_0^i(\omega) \text{ for all } \omega \in \Omega.$$

Suppose that $\mu < 0$ and put

$$\hat{\mu} = \frac{\mu}{|\mu|} = -1, \quad \hat{\Delta}_0^i = \frac{\Delta_0^i}{|\mu|}, \quad i = 1, 2, \cdots, k.$$

Let $\hat{X}$ be the strategy in the extended model that has initial capital $\hat{X}_0 = 0$, is short 1 share of V and holds $\hat{\Delta}_0^i$ shares of S^i for $i = 1, 2, \cdots, k$. The terminal capital $\hat{X}_1$ of this strategy satisfies

$$\hat{X}_1(\omega) = \frac{1}{|\mu|} X_1(\omega) \quad \text{for all } \omega \in \Omega.$$

It follows that $\hat{X}$ is an arbitrage strategy in the extended model. A similar argument can be used to construct an arbitrage strategy that is long exactly one share of V if $\mu > 0$. $\square$

Lemma 5.32. *Consider a general one-period finite model that is arbitrage free and let V be a derivative security. Then there is at least one arbitrage-free price for V.*

A sketch of the proof is given below.

Proof. Let $\mathcal{X}^*$ denote the set of all initial capitals of strategies X (in the original model) such that

$$X_1(\omega) \geq V_1(\omega) \quad \text{for all } \omega \in \Omega,$$

and let $\mathcal{X}_*$ denote the set of all initial capitals of strategies Y (in the original model) such that

$$Y_1(\omega) \leq V_1(\omega) \quad \text{for all } \omega \in \Omega.$$

It is not difficult to show that $\mathcal{X}^*$ and $\mathcal{X}_*$ are semi-infinite intervals, with $\mathcal{X}^*$ being unbounded to the right, and $\mathcal{X}_*$ being unbounded to the left. It can also be shown (although this is a bit tricky and we omit the proof) that both of these intervals are closed (i.e., contain their endpoints). In other words, we may choose $\alpha, \beta \in \mathbb{R}$ such that

$$\mathcal{X}^* = [\beta, \infty)$$

and

$$\mathcal{X}_* = (-\infty, \alpha].$$

Since the original model is arbitrage free, it follows that $\alpha \leq \beta$. (Indeed, if $\alpha > \beta$, then we may choose strategies X and Y (in the original model) with $Y_1 \leq V_1 \leq X_1$ and $\beta = Y_0 < X_0 = \alpha$. The strategy Z in which one purchases X, sells Y, and invests $Y_0 - X_0$ in the bank is an arbitrage in the original model.)

Suppose that $\alpha = \beta$. Then we may choose strategies X and Y in the original model with

$$Y_0 = \beta = \alpha = X_0$$

and

$$Y_1(\omega) \leq V_1(\omega) \leq X_1(\omega) \quad \text{for all } \omega \in \Omega. \tag{5.69}$$

Since $X_1 \geq Y_1$ and $X_0 = Y_0$, the absence of arbitrage in the original model implies that

$$Y_1(\omega) = X_1(\omega) \quad \text{for all } \omega \in \Omega. \tag{5.70}$$

In view of (5.69), we conclude from (5.70) that

$$Y_1(\omega) = V_1(\omega) = X_1(\omega) \text{ for all } \omega \in \Omega,$$

i.e. V is replicable (in the original model). In this case, Proposition 5.13 implies the existence of a unique arbitrage-free price, namely the common value of α, β.

Suppose now that $\alpha < \beta$ and let ξ be any real number satisfying

$$\alpha < \xi < \beta.$$

We claim that ξ is an arbitrage-free price for V. To verify this claim, we shall show that the existence of an arbitrage in the extended model (with V being traded at the initial price ξ) implies the existence of an arbitrage in the original model.

□

Using the lemma, we can start with any arbitrage-free one-period finite model and add the Arrow-Debreu securities one-by-one to the list of traded securities (at some arbitrage-free price). If any of the Arrow-Debreu securities are on original list of traded securities, it does not hurt to include them again. Once all of the Arrow-Debreu securities are on the list of traded securities, the model will be complete and the second fundamental theorem ensures the existence of a pricing measure.

5.7 Additional Examples

We close this chapter with three additional examples.

Example 5.33. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $r = .05$. There are two basic risky assets:

(i) A stock S^1 with $S^1_0 = 100$, $S^1_1(\omega_1) = 80$, $S^1_1(\omega_2) = 100$, and $S^1_1(\omega_3) = 120$.

(ii) A call option C on the stock with strike price $K_c = 100$ and initial price $C_0 = 9$.

We want to check and see if this model is arbitrage-free and if it is complete. Let us look for a risk-neutral measure $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{P}}(\omega_1) = \tilde{p}_1$, $\tilde{\mathbb{P}}(\omega_2) = \tilde{p}_2$, and $\tilde{\mathbb{P}}(\omega_3) = \tilde{p}_3$. Notice that

$$C_1(\omega_1) = 0, \; C_1(\omega_2) = 0, \; C_1(\omega_3) = 20. \tag{5.71}$$

We therefore consider the system of equations

$$\begin{aligned} 80\tilde{p}_1 + 100\tilde{p}_2 + 120\tilde{p}_3 &= 100(1.05) \\ 20\tilde{p}_3 &= 9(1.05) \\ \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 &= 1. \end{aligned} \tag{5.72}$$

This system has the unique solution

$$\tilde{p}_1 = .2225, \; \tilde{p}_2 = .305, \; \tilde{p}_3 = .4725. \tag{5.73}$$

We conclude that the model is complete.

Let us determine the arbitrage-free price P_0 of a put option P on S^1 with $K_P = 95$. Notice that

$$P_1(\omega_1) = 15,\ P_1(\omega_2) = 0,\ P_1(\omega_3) = 0. \tag{5.74}$$

It follows that

$$P_0 = \frac{1}{1.05}\tilde{\mathbb{E}}(P_1) = \frac{1}{1.05}(15 \times (.2225)) = 3.1786.$$

Example 5.34. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $r = .1$. There is a single risky asset, namely a stock S^1 with $S^1_0 = 20$, and $S^1_1(\omega_1) = 24$, $S^1_1(\omega_2) = 18$, $S^1_1(\omega_3) = 16$. We investigated this model previously in Example 5.26 and found a one-parameter family of risk-neutral measures given by

$$\tilde{p}_1 = \frac{2}{3} + \frac{1}{3}\tau,\ \tilde{p}_2 = \frac{1}{3} - \frac{4}{3}\tau,\ \tilde{p}_3 = \tau,\ 0 < \tau < \frac{1}{4}.$$

The model is arbitrage-free, but incomplete. Let us attempt to complete the model by adding a put option P on S^1 with strike price $K = 20$. Notice that

$$P_1(\omega_1) = 0,\ P_1(\omega_2) = 2,\ P_1(\omega_3) = 4.$$

In order for the extended model to be complete it is necessary and sufficient that there is exactly one $\tau \in (0, \frac{1}{4})$ such that

$$P_0 = \frac{1}{1.1}\left(0 + 2 \times \left(\frac{1}{3} - \frac{4}{3}\tau\right) + 4\tau\right) = \frac{1}{1.1}\left(\frac{2}{3} + \frac{4}{3}\tau\right).$$

It is clear there can be at most one such τ. In order for there to exist such a $\tau \in (0, \frac{1}{4})$, it is necessary and sufficient to have

$$\frac{1}{1.1}\left(\frac{2}{3}\right) < P_0 < \frac{1}{1.1}.$$

In view of Example 5.25, it is interesting to consider the choice

$$P_0 = \frac{1}{1.1}\left(\frac{2}{3} + \frac{1}{6}\right) \approx .76 \tag{5.75}$$

(which corresponds to $\tau = \frac{1}{8}$). Assume that P_0 is given by (5.75) and consider the derivative security V with payment function

$$V_1(\omega_1) = 24,\ V_1(\omega_2) = 24,\ V_1(\omega_3) = 8.$$

Then the arbitrage-free price V_0 is given by

$$V_0 = \frac{1}{1.1}\left(24 \times \left(\frac{17}{24}\right) + 24 \times \left(\frac{1}{6}\right) + 24 \times \left(\frac{1}{8}\right)\right) = 20.$$

Example 5.35. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $r = 0$. There are two basic risky assets:

(i) A stock S^1 with $S_0^1 = 15$, $S_1^1(\omega_1) = 16$, $S_1^1(\omega_2) = 12$, and $S_1^1(\omega_3) = 8$.

(ii) A put option P on the stock with strike price $K = 12$ and initial price $P_0 = 3$.

We want to check and see if this model is arbitrage-free and if it is complete. Let us look for a risk-neutral measure $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{P}}(\omega_1) = \tilde{p}_1$, $\tilde{\mathbb{P}}(\omega_2) = \tilde{p}_2$, and $\tilde{\mathbb{P}}(\omega_3) = \tilde{p}_3$. Notice that

$$P_1(\omega_1) = 0,\ P_1(\omega_2) = 0,\ P_1(\omega_3) = 4. \tag{5.76}$$

We therefore consider the system of equations

$$\begin{aligned} 16\tilde{p}_1 + 12\tilde{p}_2 + 8\tilde{p}_3 &= 15 \\ 4\tilde{p}_3 &= 3 \\ \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 &= 1. \end{aligned} \tag{5.77}$$

This system has the unique solution

$$\tilde{p}_1 = \frac{3}{2},\ \tilde{p}_2 = \frac{-5}{4},\ \tilde{p}_3 = \frac{3}{4}. \tag{5.78}$$

This solution does not correspond to a pricing measure since $\tilde{p}_2 \leq 0$. There are no pricing measures. The model admits arbitrage and is therefore not complete.

Let us look for an arbitrage startegy. We seek a strategy X with $X_0 = 0$, $X_1(\omega) \geq 0$ for all $\omega \in \Omega$, and $X_1(\omega) > 0$ for at least one value of $\omega \in \Omega$. Let us denote the number of shares of stock and the number of put options in the portfolio by x and y, respectively. Then we have

$$\begin{aligned} X_1(\omega_1) &= -15x - 3y + 16x = x - 3y, \\ X_1(\omega_2) &= -15x - 3y + 12x = -3x - 3y, \\ X_1(\omega_3) &= -15x - 3y + 8x + 4y = -7x + y. \end{aligned}$$

We need to satisfy the three inequalities

$$x - 3y \geq 0, \quad -3x - 3y \geq 0, \quad -7x + y \geq 0,$$

with at least one of the inequalities being strict. It is straightforward to check that $x = y = -1$ satisfies these conditions. (Of course, there are other solutions. The reader is invited to find them all as an exercise.) Therefore, an arbitrage startegy is to sell short one share of stock, sell short one put option, and invest the money obtained from the short sales in the bank.

5.8 Exercises for Chapter 5

Exercise 5.1. Consider a one period binomial model with $S_0 = 100$, $r = .05$, $u = 1.4$, and $d = .8$.

(a) Find the arbitrage-free price C_0 of a standard call option on the stock with strike price $K_c = 90$.

(b) Find the arbitrage-free price P_0 of a standard put option on the stock with strike price $K_p = 105$.

Exercise 5.2. Consider a one-period binomial model with $r = .1,\ u = 1.2,\ d = .9$, and $S_0 = \$50$.

(a) Find the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$.

(b) Find the initial price C_0 and the number of shares of stock Δ_0 in the replicating portfolio for a standard call option C on the stock with strike price $K = \$55$.

Exercise 5.3. Consider a one-period binomial model with $r = .1,\ u = 1.2$, and $d = .9$. Alice thinks the stock will go up and Andy thinks that the stock will go down, so they make a bet. If $S_1 > S_0$, Andy must pay Alice \$100. If $S_1 < S_0$, Alice must pay Andy \$100. Does either person have an advantage with this bet? Explain.

Exercise 5.4. Consider a one-period binomial model with $r = .1,\ u = 1.4,\ d = .8$, and $S_0 = 60$.

(a) Find the risk-neutral probabilities $\tilde{p}, \tilde{q}$.

(b) Find the arbitrage-free price C_0 of a standard call option C on the stock with strike price $K_c = 50$.

(c) Find the arbitrage-free price P_0 of a standard put option P on the stock with $K_p = 60$.

(d) Find the number of shares Δ_0 in the replicating portfolio for the derivative security V with payoff $V_1(H) = 20$, $V_1(T) = 40$.

Exercise 5.5. Consider a one-period binomial model with $u = 1.5,\ d = .7,\ r = .2$ and $S_0 = \$60$.

(a) Find the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$.

(b) Let V be a standard straddle option on the stock with strike price $K = \$50$. (Note that $V_1(\omega) = |S_1(\omega) - 50|$ for all $\omega \in \{H, T\}$.) Find the initial price V_0 and the number of shares Δ_0 of stock in the replicating portfolio for V.

Exercise 5.6. Consider a one-period binomial model with $u = 1.3$, $d = .9$, $r = .2$, and $S_0 = 100$.

(a) Find $\tilde{p}$ and $\tilde{q}$.

(b) Find the arbitrage-free price C_0 of a standard call option C on S with strike price $K_c = 105$.

(c) A standard straddle option V on the stock is selling at time 0 at the arbitrage-free price $V_0 = 16\frac{2}{3}$. Find all possible strike prices K. (Recall that $V_1(\omega) = |S_1(\omega) - K|$.)

Exercise 5.7. Consider a one-period binomial model with $r = .1,\ u = 1.3,\ d = .9$, and $S_0 = \$100$. Suppose that you are working for an investment bank. A client comes to you and says, "I don't want to put money in the bank, because I think that the stock is going to go up. On the other hand, I am afraid to buy the stock because it might go down, and I don't want to lose money. Can you create a security for me that has a guaranteed return of at least 5% and will have a return of greater than 10% if the stock does well?"

You tell the client that you can create a security with initial price $V_0 = \$100$ and payoff

$$V_1(\omega) = \max\{\$105,\ \alpha S_1(\omega)\}$$

with $1.3\alpha > 1.1$. The client likes the idea, but wants to know how big α will be. Find the arbitrage-free value of α.

Exercise 5.8. Consider a one-period binomial model with $r = .25,\ u = 2,\ d = \frac{1}{2}$ and $S_0 = 100$. Let V be a derivative security with $V_0 = 100$ and payment function V_1 given by $V_1(H) = \alpha S_1(H)$, $V_1(T) = 100$, where α is some positive real number. Assume that there is no arbitrage.

(a) Find α.

(b) Find the number of shares of stock Δ_0 in the replicating portfolio for V.

Exercise 5.9. Consider a one-period binomial model with $u = 2,\ d = \frac{1}{2}$, and $r = .1$. The arbitrage-free initial price of a standard call option C on the stock with strike price $K = \$50$ is $C_0 = \$10$. What can you say about S_0?

Exercise 5.10. Consider a financial model, where

(i) There is an ideal money market with constant effective (annual) interest rate R.

(ii) One can take long or short positions on forward contracts for delivery of a stock S at time $T = 0.5$ at the forward price $\mathcal{F} = \$110$ per share.

(iii) One can buy or sell standard put options on S with the strike $K = \$120$ and exercise date $T = 0.5$. The current price of the option is $P_0 = \$17.50$

(iv) The price of the stock at the time $t = 0.5$ is one of the two possible values: \$90 or \$140, each with strictly positive probability.

Given that the model is arbitrage-free, compute R.

Exercise 5.11. Consider a one-period binomial model in which the initial price of the stock is $S_0 = 100$. Let C denote a standard call option on the stock with $K = 110$ and P denote a standard put option on the stock with $K = 110$. The call and the put are trading at $t = 0$ at the initial prices $C_0 = P_0 = 10$. Given that there is no arbitrage, determine as much as you can about the possible values of u, d, and r.

Exercise 5.12. Consider the one-period model described in Example 1.25 and let C be a standard call option on S with strike price $K = 10$ and let V be the derivative security with payment function V_1 given by $V_1(\omega_1) = 0, \quad V_1(\omega_2) = V_1(\omega_3) = 1$.

(a) Show (by explicitly finding it) that there is exactly one risk-neutral measure.

(b) Use the measure found in part (a) to compute the arbitrage-free price C_0 of C and verify that it agrees with the price determined in Example 1.25.

(c) Use the measure found in part (a) to compute the arbitrage-free price V_0 of V and verify that it agrees with the price determined in Example 1.25.

Exercise 5.13. For each of the two one-period models below, use the first and second fundamental theorems of asset pricing to answer the following two questions.

(a) Is the model arbitrage free?

(b) Is the model complete?

Model (i): $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .05, S_0^1 = 15, S_0^2 = 10, S_1^1(\omega_1) = 25$, $S_1^1(\omega_2) = 20$, $S_1^1(\omega_3) = 5$, $S_1^2(\omega_1) = 16$, $S_1^2(\omega_2) = 13$, $S_1^2(\omega_3) = 3$.

Model (ii): $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .2$, $S_0^1 = 20$, $S_0^2 = 25$, $S_1^1(\omega_1) = 16$, $S_1^1(\omega_2) = 40$, $S_1^1(\omega_3) = 24$, $S_1^2(\omega_1) = 30$, $S_1^2(\omega_2) = 32$, $S_1^2(\omega_3) = 28$.

Exercise 5.14. Consider a one-period model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $r = .05$. There is a single stock S that cannot be purchased directly at $t = 0$. It is known that $S_1(\omega_1) = 260$, $S_1(\omega_2) = 280$, and $S_1(\omega_3) = 300$. There are two traded securities, other than the bank account, in this model:

(i) A forward contract for delivery of the stock at $T = 1$ at the forward price $\mathcal{F} = 280$. (It costs nothing to enter into the forward contract. You may take either a long or a short position for as many shares as you wish.)

(ii) A standard call option on the stock with strike price $K = 280$ and maturity $T = 1$. You can buy or sell as many options as you want at the initial price $C_0 = 3.5$.

(a) Is this model arbitrage-free?

(b) Is this model complete?

(c) If the model is complete, compute the arbitrage-free prices of a standard put option on the stock with strike price $K_p = 290$ and a standard call option on the stock with strike price $K_c = 250$.

Exercise 5.15. Consider a one-period finite model with $r = .2$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. There are two basic risky securities.

(i) A stock S^1 with initial price $S_0^1 = 12$, and $S_1^1(\omega_1) = 18$, $S_1^1(\omega_2) = 4$, $S_1^1(\omega_3) = 15$.

(ii) A standard call option C on S^1 with strike price $K = 15$ and initial price $C_0 = .60$.

(a) Show that the model is complete. (Explain briefly.)

(b) Find the arbitrage-free price of the derivative security V with payment function $V_1(\omega_1) = 30$, $V_1(\omega_2) = 2$, $V_1(\omega_3) = 6$.

Exercise 5.16. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $r = .1$. There are two basic securities, other than the bank account.

(i) A stock S^1 with $S_0^1 = 100$ and $S_1^1(\omega_1) = 70$, $S_1^1(\omega_2) = 90$, $S_1^1(\omega_3) = 110$, and $S_1^1(\omega_4) = 130$.

(ii) A standard call option C on S^1 with strike price $K = 110$ and initial price $C_0 = 6$.

(a) Show that the model is arbitrage-free, but incomplete.

(b) Let P be a standard put option on S^1 with strike price $K = 110$. Determine whether or not P is replicable.

(c) If the answer to (b) is yes, find the arbitrage-free price P_0.

Exercise 5.17. Consider a complete one-period model with $r = .21$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and

$$\tilde{\mathbb{P}}(\omega_1) = \frac{1}{4},\ \tilde{\mathbb{P}}(\omega_2) = \frac{1}{2},\ \tilde{\mathbb{P}}(\omega_3) = \frac{1}{4}.$$

The bank decides to become an ideal money market with constant effective rate $R = .21$. (You can borrow or invest any amount between any two times t_1, t_2 with $t_1 < t_2$. Capital in the bank account evolves according to the rule $V(t) = V(t_1)(1+R)^{t-t_1}$ for $t_1 \leq t \leq t_2$.) A company then decides to issue a stock S with $S_1(\omega_1) = 12$, $S_1(\omega_2) = 6$, $S_1(\omega_3) = 18$. The stock will pay a dividend $d = 2$ at time $t = .5$. (No other dividends will be paid.) Assuming that the assets other than the bank account can only be traded at times $t = 0$ and $t = 1$, and that there is no arbitrage, find the initial price S_0 of stock.

Exercise 5.18. Consider a one-period model with $r = .05$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and a single stock S^1. The stock cannot be purchased directly at $t = 0$, but certain options on the stock can be purchased at $t = 0$. It is known that $S_1^1(\omega_1) = 95$, $S_1^1(\omega_2) = 120$, $S_1^1(\omega_3) = 145$. The following two securities can be bought or sold short in any amount at $t = 0$.

(i) A standard call option C on the stock with $K_c = 115$. The price at time 0 of this security is $C_0 = 12$.

(ii) A standard put option P on the stock with $K_p = 105$. The price at time 0 of this security is $P_0 = 5$.

(a) Is the model arbitrage free?

(b) Is the model complete?

(c) Consider a straddle option V with strike price $K_s = 110$. The payoff of this option is

$$V_1(\omega) = \left|S_1^1(\omega) - K_s\right| \quad \text{for all } \omega \in \Omega.$$

Is this option replicable? If so, and if the model is arbitrage free, find the arbitrage-free price V_0.

Exercise 5.19. Consider a finite one-period model with $k = 2, r = .25$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Assume that

$$\begin{aligned} S_1^1(\omega_1) = \$60,\ S_1^1(\omega_2) = \$40,\ S_1^1(\omega_3) = \$20 \\ S_1^2(\omega_1) = \$20,\ S_1^2(\omega_2) = \$30,\ S_1^2(\omega_3) = \$60 \end{aligned}$$

and that $S_0^1 = \$32$. Find the values of S_0^2 (if any) such that the model is arbitrage-free.

Exercise 5.20. Consider a one-period finite model with $r = .1$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. There are two basic risky securities: a stock with $S_0 = 10$, $S_1(\omega_1) = 10$, $S_1(\omega_2) = 6$, $S_1(\omega_3) = 20$ and a European put option P on the stock with strike price $K = 8$, expiration time $T = 1$, and initial price $P_0 = \frac{2}{1.1}$. Answer the following questions.

(a) Is the model arbitrage-free?

(b) Is the model complete?

(c) (i) If the model admits arbitrage, give an example of an arbitrage strategy.

(ii) If the model is arbitrage-free but incomplete, give an example of a non-replicable derivative security.

(iii) If the model is complete find the arbitrage-free price of a European call option on the stock with strike price $K = 9$ and expiration time $T = 1$.

Exercise 5.21. Consider a complete one-period model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and let V^1, V^2, V^3, V^4 denote the Arrow-Debreu securities with payment functions

$$V_1^i(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_i \\ 0 & \text{if } \omega \neq \omega_i. \end{cases}$$

Assume that the initial prices of these securities are

$$V_0^1 = \frac{1}{4},\ V_0^2 = \frac{1}{3},\ V_0^3 = \frac{1}{8},\ V_0^4 = \frac{1}{6}.$$

(a) Find the arbitrage-free price W_0 of the derivative security with payment function W_1 given by

$$W_1(\omega_1) = 12,\ W_1(\omega_2) = 0,\ W_1(\omega_3) = 5,\ W_1(\omega_4) = 8.$$

(b) Find the interest rate r.

Exercise 5.22. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $r = .25$. Let V^1, V^2, V^3, V^4 denote the Arrow-Debreu securities, i.e.

$$V_1^i(\omega_j) = \begin{cases} 1 & \text{if} \quad i = j \\ \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Assume that the model is complete and that

$$V_0^1 = .4,\ V_0^2 = .1,\ V_0^3 = .2.$$

Find the arbitrage-free price of the derivative security W with payoff $W_1(\omega_1) = 8$, $W_1(\omega_2) = 24$, $W_1(\omega_3) = 16$, $W_1(\omega_4) = 8$.

Exercise 5.23 (Double binomial Model). Consider a one-period model with two stocks S^1 and S^2, and a bank account with one-period interest rate $r \geq 0$. The initial prices of the stocks are S_0^1, S_0^2. The prices of the stocks at $t = 1$ will be determined by tossing two fair coins independently, so that an appropriate probability space $(\Omega, \mathbb{P})$ is given by $\Omega = \{(H,H), (H,T),\ (T,H),\ (T,T)\}$ with $\mathbb{P}(\omega) = \frac{1}{4}$ for all $\omega \in \Omega$. The stock prices at $t = 1$ are given by

$$\begin{aligned} S_1^1(H,H) &= u_1 S_0^1 \quad & S_1^2(H,H) &= u_2 S_0^2 \\ S_1^1(H,T) &= u_1 S_0^1 \quad & S_1^2(H,T) &= d_2 S_0^2 \\ S_1^1(T,H) &= d_1 S_0^1 \quad & S_1^2(T,H) &= u_2 S_0^2 \\ S_1^1(T,T) &= d_1 S_0^1 \quad & S_1^2(T,T) &= d_2 S_0^2. \end{aligned}$$

Here u_1, u_2, d_1, d_2 are given numbers satisfying

$$u_1 > d_1 > 0 \quad , \quad u_2 > d_2 > 0.$$

(a) Under what conditions is the model arbitrage free?

(b) Under what conditions is the model complete?

Exercise 5.24. Consider a general one-period finite model and assume that $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$ are pricing measures. Let $\alpha \in (0,1)$ be given and define $\hat{\mathbb{P}} : \Omega \to \mathbb{R}$ by

$$\hat{\mathbb{P}}(\omega) = \alpha\tilde{\mathbb{P}} + (1-\alpha)\tilde{\mathbb{Q}}(\omega) \text{ for all } \omega \in \Omega.$$

Show that $\hat{\mathbb{P}}$ is a pricing measure.