CHAPTER 6

INTRODUCTION TO OPTIMAL INVESTMENT

In this chapter we provide an introduction to the theory of optimal investment in finite one-period models. We assume throughout that a finite one-period model with k basic risky assets and interest rate $r \geq 0$ is given. The reference probability measure $\mathbb{P}$ will now play a central role. As usual, the sample space will be denoted by Ω.

Although the reference probabilities are not used directly to compute arbitrage-free prices of derivative securities, they *are* used directly to determine how desirable each security is to include in a portfolio. **Throughout this chapter the term "expected return" refers to the expected value of the return with respect the actual probability measure $\mathbb{P}$.** Recall that the return of a portfolio with strictly positive initial capital is defined by

$$\text{return} = \frac{\text{terminal capital} - \text{initial capital}}{\text{initial capital}}.$$

6.1 Risk Aversion and Expected Returns Under the Reference Measure

Webster's Finance and Investment Dictionary defines risk as "the possibility that a negative event will occur such as the value of investments declining below what was paid for them $\cdots$". The no-arbitrage principle tells us that any investment (with initial capital $X_0 > 0$) that has the possibility of doing better than the bank account (i.e., $X_1 > X_0(1+r)$) must also have the possibility of doing worse than the bank account (i.e., $X_1 < X_0(1+r)$). An investor who hopes to outperform the bank account must therefore be willing to take some risk. There is an investment principle known as the *risk-reward trade-off* which says that investments must offer higher expected returns (under the reference measure) as compensation for higher risk. Before attempting to quantify the risk-reward trade-off, let us look at a simple example.

Example 6.1: Consider a one-period binomial model with $u = 2$, $d = \frac{1}{2}$, $r = .25$ and $S_0 = 100$. The risk-neutral probabilities are

$$\tilde{p} = \tilde{\mathbb{P}}(H) = \frac{1}{2}, \quad \tilde{q} = \tilde{\mathbb{P}}(T) = \frac{1}{2}.$$

Let us denote the reference (or actual) probabilities by

$$p = \mathbb{P}(H), \quad q = \mathbb{P}(T).$$

Recall that the return of the stock is defined by

$$\rho^S(\omega) = \frac{S_1(\omega) - 100}{100} \quad \text{for all } \omega \in \{H, T\},$$

and the return of the bank account is $r = .25$ (for sure). Under the risk-neutral measure, the stock has the same expected return as the bank account, i.e.

$$\tilde{\mathbb{E}}(\rho^S) = .25.$$

In order for the stock to be an attractive investment to most rational investors we must have

$$\mathbb{E}(\rho^S) > .25 \tag{1}$$

because the bank account offers a guaranteed return of .25 and purchase of the stock involves risk. (Indeed investors who purchases the stock might lose half of their money.) Most investors will want to feel confident that (1) holds in order for them to buy any stock.

It is straightforward to show that (1) holds if and only if $p > \frac{1}{2}$. For concreteness, let us assume that

$$p = \frac{2}{3}, \quad q = \frac{1}{3},$$

in which case

$$\mathbb{E}(\rho^S) = \frac{2}{3}2 + \frac{1}{3}\left(\frac{1}{2}\right) - 1 = .5.$$

A *risk-averse* investor is an investor who if presented with two alternatives having the same expected return will always choose the one which has lower risk. A *risk-seeking* investor will do just the opposite. In other words, if a risk-seeking investor is presented with two investment alternatives having the same expected return, he will always choose the one with higher risk. A *risk-neutral* investor does not take risk into account at all when making investment decisions. He is indifferent between alternatives with the same expected return; presented with alternatives having different expected returns, he will choose the one with the highest expected return, regardless of the risk associated with it.

Under normal circumstances most investors are risk averse. However, if the potential loss is small, rational individuals will sometimes behave as risk-seeking investors. A common example of this behavior is the purchase of lottery tickets. Typically, only about 50% of the money collected from the sale of lottery tickets is returned to the players. This means that typically, the purchase of a lottery ticket has a negative expected return. However, for many people the potential loss of $1 is outweighed by the (very small, but strictly positive) chance of winning, say $1,000,000.

Consider the following three attractive (but hypothetical) possibilities.

1. You receive $10,000,000 for sure.
2. A fair coin is tossed: If it shows heads you receive $20,000,000; if it shows tails you receive nothing.
3. A fair coin is tossed. If it shows heads you receive $30,000,000; if it shows tails you receive nothing.

Notice that alternative 1 is riskless - the payoff is certain. Alternatives 2 and 3 are risky- you may receive a very large payoff, but you may receive nothing.

A risk-averse investor would prefer alternative 1 to alternative 2 because they have the same expected payoff, namely $10,000,000 and alternative 1 is less risky. A risk-seeking investor would prefer alternative 2 to alternative 1. A risk-neutral investor would regard alternatives 1 and 2 as equally attractive, but would prefer alternative 3 because it has the highest expected return. A risk-averse investor would probably prefer alternative 1 to alternative 3, but we cannot say for sure without additional information about the investor's risk preferences. (See Exercise 2 of this chapter.) The notion of risk aversion can be quantified by means of a utility function.

6.2 An Explanation of the Term Risk-Neutral Measure

Consider a risk-neutral investor with initial capital X_0 to invest. She would try to construct her portfolio in such a way that $\mathbb{E}(X_1)$ is maximized; in other words she wants the highest possible expected return – regardless of the risk involved. Let us denote the basic risky assets by $S^1, S^2, \cdots, S^k$. Suppose there exists $j \in \{1, 2, \ldots k\}$ such that

$$\mathbb{E}(S_1^j) > (1+r)S_0^j. \tag{2}$$

The investor can then construct a portfolio with $\Delta_0^j = \alpha$ and $\Delta_0^i = 0$ for $i \neq j$. The terminal capital of this portfolio is

$$X_1(\omega) = (X_0 - \alpha S_0^j)(1+r) + \alpha S_1^j(\omega),$$

so that

$$\mathbb{E}(X_1) = X_0(1+r) + \alpha((\mathbb{E}(S_1^j)) - (1+r)S_0^j).$$

It follows that $\mathbb{E}(X_1) \to +\infty$ as $\alpha \to +\infty$. In other words, the risk-neutral investor can get an arbitrarily large expected return by borrowing a lot of money and investing it in the j^{th} stock.

On the other hand, suppose there exists $m \in \{1, 2, \ldots m\}$ such that

$$\mathbb{E}(S_1^m) < (1+r)S_0^m. \tag{3}$$

The investor can construct a portfolio with $\Delta_0^m = \beta$ and $\Delta_0^i = 0$ for $i \neq m$. For this portfolio

$$\mathbb{E}(X_1) = X_0(1+r) + \beta((\mathbb{E}(S_1^m)) - (1+r)S_0^m)$$

so that $\mathbb{E}(X_1) \to +\infty$ as $\beta \to -\infty$. The risk-neutral investor can create a portfolio with an arbitrarily large expected return by selling short many shares of S^m and investing the money in the bank. If there exists j such that (2) holds or m such that (3) holds then a risk-neutral investor cannot “exist in equilibrium”; in other words, no matter how matter how her money is invested, there will always be another portfolio that she finds preferable. The only way that a risk-neutral investor could be in equilibrium is if

$$\mathbb{E}(S_1^i) = (1+r)S_0^i \quad \text{for all } i = 1, 2, \ldots, k. \tag{4}$$

It is very unnatural to expect that (4) would hold. Indeed, if (4) holds most investors would never buy any stock, because putting money in the bank would have the same expected return, but less risk, than purchasing stock. Most rational investors want a higher expected return in exchange for taking risk. Although we cannot expect the actual probability measure to satisfy the “equilibrium condition for risk-neutral investors”, every pricing measure must satisfy this condition. For this reason, pricing measures are often called risk-neutral measures. In fact, the vast majority of practitioners use the term “risk-neutral measure”.

6.3 Utility Functions

The theory of optimal investment attempts to quantify the "desirability" of securities relative to one another. Throughout this chapter, we assume that a finite one-period model with k basic risky securities and interest rate $r \geq 0$ is given. The reference probability measure will be denoted by $\mathbb{P}$ and Ω will denote the sample space.

We consider a single economic agent, or investor, who has initial capital $X_0 > 0$ to invest and wishes to construct an "optimal portfolio" using the k risky assets and the bank account. Of course, we need to quantify what makes a portfolio optimal.

To this end, we assume that our economic agent uses a *utility function* $U : (0, \infty) \to \mathbb{R}$ to rank portfolios. If $\hat{X}$ and X are portfolios with terminal capitals $\hat{X}_1$ and X_1, our investor will prefer $\hat{X}$ to X if and only if

$$\mathbb{E}(U(\hat{X}_1)) > \mathbb{E}(U(X_1));$$

in other words, the investor wants the portfolio such that the utility of the terminal capital has the highest possible expected value (with respect to the reference probability measure).

We assume throughout that U is twice differentiable on $(0, \infty)$. We also assume that

$$U'(x) > 0 \quad \text{for all } x > 0, \tag{5}$$

so that U is strictly increasing (i.e., more is always better). Most of the time, we shall assume that

$$U''(x) < 0 \quad \text{for all } x > 0. \tag{6}$$

Notice that (6) expresses the law of diminishing returns: an additional \$1 is less important to a wealthy investor than it would be to a monetarily-challenged investor.

Condition (6) also says something important about risk aversion. In order to elaborate on this issue, we need an important result known as Jensen's Inequality.

Proposition 6.2 (Jensen's Inequality): Let Y be a random variable on Ω and let J be an open interval such that $Y(\omega) \in J$ for all $\omega \in \Omega$. Assume that $\varphi : J \to \mathbb{R}$ is twice differentiable on J.

(i) If $\varphi''(x) \geq 0$ for all $x \in J$, then

$$\varphi(\mathbb{E}(Y)) \leq \mathbb{E}(\varphi(Y)).$$

(ii) If $\varphi''(x) \leq 0$ for all $x \in J$, then

$$\varphi(\mathbb{E}(Y)) \geq \mathbb{E}(\varphi(Y)).$$

Proof: (i) Assume that $\varphi''(x) \geq 0$ for all $x \in J$. Let $\alpha = \mathbb{E}(Y)$ and notice that $\alpha \in J$. For each $x \in J$, there exists c_x between α and x such that

$$\varphi(x) = \varphi(\alpha) + \varphi'(\alpha)(x-\alpha) + \frac{1}{2}\varphi''(c_x)(x-\alpha)^2 \quad \text{for all } x \in J,$$

by virtue of Taylor's Theorem. Since $\varphi''(c_x) \geq 0$ we have

$$\varphi(x) \geq \varphi(\alpha) + \varphi'(\alpha)(x-\alpha) \quad \text{for all } x \in J.$$

It follows that

$$\varphi(Y(\omega)) \geq \varphi(\alpha) + \varphi'(\alpha)(Y(\omega)-\alpha) \quad \text{for all } \omega \in \Omega. \tag{7}$$

Taking expected values in (7) we find that

$$\mathbb{E}(\varphi(Y)) \geq \mathbb{E}(\varphi(\alpha)) + \varphi'(\alpha)\mathbb{E}(Y-\alpha). \tag{8}$$

Notice that $\mathbb{E}(\varphi(\alpha)) = \varphi(\alpha)$ and $\mathbb{E}(Y-\alpha) = 0$. Since $\alpha = \mathbb{E}(Y)$, it follows from (8) that

$$\mathbb{E}(\varphi(Y)) \geq \varphi(\mathbb{E}(Y)),$$

which is the desired conclusion. To obtain (ii), we simply apply (i) to the function $-\varphi$. $\square$

Remark 6.3: Assume that $U''(x) < 0$ for all $x > 0$ and let $\alpha > 0$ be given. Let $\hat{X}$ denote the portfolio obtained by investing $\frac{\alpha}{1+r}$ in the bank at $t = 0$ (so that $\hat{X}_0 = \frac{\alpha}{1+r}$ and $\hat{X}_1(\omega) = \alpha$ for all $\omega \in \mathbb{R}$). Observe that $\mathbb{E}(\hat{X}_1) = \alpha$. Let X be any other portfolio with $\mathbb{E}(X_1) = \alpha$. Then, by Jensen's Inequality (ii), we have

$$U(\alpha) \geq \mathbb{E}(U(X_1)). \tag{9}$$

Since $U(\alpha) = \mathbb{E}(U(\alpha))$ and $\hat{X}_1 = \alpha$, it follows from (9) that

$$\mathbb{E}(U(\hat{X}_1)) \geq \mathbb{E}(U(X_1)).$$

This tells us that the investor prefers the portfolio having a certain (or sure) payoff of α to all other portfolios having expected payoff α. For this reason, utility functions with $U'' < 0$ are called *risk-averse*.

Remark 6.4: A utility function having $U'' > 0$ is called *risk-seeking* because in this case the portfolio with sure payoff α is the least desirable of all portfolios with expected payoff α.

Remark 6.5: A utility function with $U''(x) = 0$ for all $x > 0$ is called *risk-neutral* because all portfolios having expected payoff α are equally desirable.

An example of a commonly-employed utility function is the natural logarithm.

$$U(x) = lnx, \quad x > 0. \tag{10}$$

This function is used not because it accurately reflects real-world investor behavior, but because it leads to clean analytical expressions in many cases and because it exhibits appropriate qualitative behavior. (In fact, portfolios constructed by using a logarithmic utility function generally take much more risk than typical investors would feel comfortable with.)

Other examples of popular utility functions are

$$U(x) = x^\alpha, \quad x > 0 \quad (0 < \alpha < 1), \tag{11}$$

$$U(x) = \frac{-1}{x^\beta}, \quad x > 0 \quad (\beta > 0), \tag{12}$$

$$U(x) = \alpha x - \beta x^2, \quad 0 < x < \frac{\alpha}{2\beta} \quad (\alpha, \beta > 0), \tag{13}$$

$$U(x) = -e^{-\alpha x}, \quad x > 0 \quad (\alpha > 0). \tag{14}$$

Remark 6.6: Observe that if U is a utility function and A, B are constants with $A > 0$ then the utility function $\hat{U}$ defined by

$$\hat{U}(x) = AU(x) + B \quad \text{for all } x > 0,$$

leads to exactly the same rankings of portfolios as the utility function U. It is common practice to combine utility functions of types (11) and (12) into one family

$$U(x) = \frac{x^\gamma}{\gamma}, \quad x > 0 \quad (\gamma < 1, \quad \gamma \neq 0). \tag{15}$$

The level of risk aversion of a utility function can be measured by the *Arrow-Pratt risk-aversion coefficient*

$$a(x) = \frac{-U''(x)}{U'(x)}. \tag{16}$$

Higher values of $a(x)$ indicate higher levels of risk aversion. For utility functions of the form (10), (11), (12), the level of risk aversion decreases as the level of wealth increases. Exponential utility functions (14) have constant risk-aversion coefficient

$$a(x) = \alpha.$$

Utility functions of the form (11) correspond to more risk than utility functions of the form (10) and utility functions of the form (10) correspond to more risk than utility functions of the form (12). Within the class (11), higher values of α correspond to more risk. Within the class (12), higher values of β correspond to less risk. (The reader is urged to verify these claims by computing the risk-aversion coefficients for the various utility functions.) It should be noted that since utility functions are used to compare portfolios with one another, the actual value of the utility of a portfolio (on some absolute scale) is irrelevant. In particular, it is perfectly reasonable to have utility functions whose values are always negative.

Before attempting to develop a general theory, let us look at a simple example.

Example 6.7: Consider a one-period binomial model with $u = 2$, $d = \frac{1}{2}$, $r = .25$, $S_0 = 16$, $\mathbb{P}(H) = \frac{2}{3}$, $\mathbb{P}(T) = \frac{1}{3}$. Suppose that an investor has initial capital $X_0 = \$100$ and employs the utility function $U(x) = lnx, \; x > 0$.

Consider a portfolio with initial capital $X_0 = \$100$ and denote by y the number of shares of stock purchased at $t = 0$. Observe that

$$\begin{aligned} X_1(H) &= (100 - 16y)(1.25) + 32y = 125 + 12y, \\ X_1(T) &= (100 - 16y)(1.25) + 8y = 125 - 12y. \end{aligned}$$

It follows that

$$\mathbb{E}(U(X_1)) = \frac{2}{3}\ ln(125 + 12y) + \frac{1}{3}\ ln(125 - 12y).$$

Let us put

$$f(y) = \frac{2}{3}\ ln(125 + 12y) + \frac{1}{3}\ ln(125 - 12y)$$

and use calculus to find a value of y that maximizes f. Notice that

$$f'(y) = \left(\frac{2}{3}\right)\ \frac{12}{125 + 12y} - \left(\frac{1}{3}\right)\ \frac{12}{125 - 12y}.$$

Setting $f'(y) = 0$ and solving for y we find that

$$\frac{2}{125 + 12y} = \frac{1}{125 - 12y},$$

$$250 - 24y = 125 + 12y,$$

$$125 = 36y,$$

$$y = \frac{125}{36}.$$

It is straightforward to check that this value of y does indeed correspond to a maximum. It follows that the initial capital should be invested as follows at $t = 0$:

$55.56 in stock

$44.44 in the bank.

It is interesting to ask what would happen if a different utility function is employed, but everything else stays the same in the example. The results for several utility functions are given below.

Utility Function	Initial Capital in Stock	Initial Capital in Bank
$U(x) = x^{3/4}$	\$147.06	$-$\$47.06
$U(x) = x^{1/2}$	\$100	0
$U(x) = lnx$	\$55.56	\$44.44
$U(x) = -\frac{1}{x^2}$	\$19.17	\$80.83

The reader is urged to verify these results.

Notice that if the utility function $U(x) = x^{3/4}$ is used then the optimal portfolio has the investor borrowing money to buy stock. Observe also that the initial amount invested in stock decreases as we go down the table. This is consistent with the remarks made earlier about the level of risk aversion of various types of utility functions. □

6.5 Utility Maximization in Complete One-Period Models

The approach used in the Example 6.7 works very nicely when there is only one basic risky asset, but leads to very complicated equations when there are many risky assets. We now develop a general approach in which we seek the terminal capital of the optimal portfolio rather than the numbers of shares of stocks. We assume that the model is complete and we denote by $\tilde{\mathbb{P}}$ the unique risk-neutral measure. We assume that our investor has initial capital $X_0 = z$ and uses a general utility function U with

$$U''(x) < 0 \quad \text{for all } x > 0.$$

Let $\mathcal{X}$ denote the set of all terminal capitals X_1 of strategies having initial capital $X_0 = z$ and satisfying $X_1(\omega) > 0$ for all $\omega \in \Omega$. Notice that $\mathcal{X}$ is simply the set of all random variables X_1 on Ω such that

$$\tilde{\mathbb{E}}(X_1) = (1 + r)z \tag{17}$$

and

$$X_1(\omega) > 0 \quad \text{for all } x > 0. \tag{18}$$

Let $\hat{X}_1 \in \mathcal{X}$ be given and assume that

$$\mathbb{E}(U(X_1)) \le \mathbb{E}(U(\hat{X}_1)) \quad \text{for all } X_1 \in \mathcal{X}. \tag{19}$$

We want to see what can be said about $\hat{X}_1$. For this purpose it is convenient to have a way to generate terminal capitals $X_1 \in \mathcal{X}$ that are related to $\hat{X}_1$ in a simple way. It is therefore natural to let $\mathcal{Y}$ denote the set of all random variables Y_1 on Ω such that

$$\tilde{\mathbb{E}}(Y_1) = 0.$$

(Notice that $\mathcal{Y}$ is simply the set of all terminal capitals of strategies having zero initial capital.)

Let $Y_1 \in \mathcal{Y}$ be given and observe that

$$\begin{aligned} \tilde{\mathbb{E}}(\hat{X}_1 + yY_1) &= \tilde{\mathbb{E}}(\hat{X}_1) + \tilde{\mathbb{E}}(yY_1) \\ &= \tilde{\mathbb{E}}(\hat{X}_1) + y\tilde{\mathbb{E}}(Y_1) \\ &= z(1+r) + y \cdot 0 = z(1+r) \quad \text{for all } y \in \mathbb{R}. \end{aligned}$$

By virtue of (18) and the fact that Ω is finite, we may choose an open interval I (depending on Y_1) such that $0 \in I$ and

$$\hat{X}_1(\omega) + yY_1(\omega) > 0 \quad \text{for all } y \in I.$$

It follows that

$$\hat{X}_1 + yY_1 \in \mathcal{X} \quad \text{for all } y \in I$$

and consequently

$$\mathbb{E}(U(\hat{X}_1 + yY_1)) \le \mathbb{E}(U(\hat{X}_1)) \quad \text{for all } y \in I. \tag{20}$$

Now, define $g : I \to \mathbb{R}$ by

$$g(y) = \mathbb{E}(U(\hat{X}_1 + yY_1)) \quad \text{for all } y \in I. \tag{21}$$

Since $g(0) = \mathbb{E}(U(\hat{X}_1))$, it follows from (20) that

$$g(y) \le g(0) \quad \text{for all } y \in I,$$

i.e., g attains a maximum at 0.

It is straightforward to check that g is differentiable and

$$g'(y) = \mathbb{E}(U'(\hat{X}_1 + yY_1)Y_1) \quad \text{for all } y \in I. \tag{22}$$

Since g attains a maximum at 0, we must have

$$g'(0) = \mathbb{E}(U'(\hat{X}_1)Y_1) = 0. \tag{23}$$

Using the fact that Y_1 was an arbitrary element of $\mathcal{Y}$ we conclude that

$$\mathbb{E}(U'(\hat{X}_1)Y_1) = 0 \quad \text{for all } Y_1 \in \mathcal{Y}. \tag{24}$$

In order to understand what (24) is telling us, we make the following observations: (24) is the statement that a certain expected value (with respect to the reference probability measure) is zero for all random variables having risk-neutral expected value zero. It therefore seems like a good idea to express the left-hand side of (24) as a risk-neutral expected value. To this end, observe that

$$\begin{aligned}
\mathbb{E}(U'(\hat{X}_1)Y_1) &= \sum_{\omega\in\Omega} \mathbb{P}(\omega)U'(\hat{X}_1(\omega))Y_1(\omega) \\
&= \sum_{\omega\in\Omega} \frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)}U'(\hat{X}_1(\omega))Y_1(\omega)\tilde{\mathbb{P}}(\omega) \\
&= \tilde{\mathbb{E}}\left(\frac{\mathbb{P}}{\tilde{\mathbb{P}}}U'(\hat{X}_1)Y_1\right).
\end{aligned} \tag{25}$$

At this point it is convenient to define the random variable G on Ω by

$$G(\omega) = \frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)}U'(\hat{X}_1(\omega)) \quad \text{for all } \omega \in \Omega. \tag{26}$$

Combining (24), (25), and (26) we arrive at

$$\tilde{\mathbb{E}}(GY_1) = 0 \quad \text{for all } Y_1 \in \mathcal{Y}. \tag{27}$$

It is easy to see that if G is constant, then (27) holds. We shall prove that the converse is also true. In other words, it follows from (27) that G is constant.

Let $\lambda = \tilde{\mathbb{E}}(G)$ and observe that

$$\tilde{\mathbb{E}}((G-\lambda)Y_1) = \tilde{\mathbb{E}}(GY_1) - \lambda\tilde{\mathbb{E}}(Y_1) = 0 \quad \text{for all } Y_1 \in \mathcal{Y} \tag{28}$$

by virtue of (27). Observe further that

$$\tilde{\mathbb{E}}(G-\lambda) = 0 \tag{29}$$

since $\lambda = \tilde{\mathbb{E}}(G)$. Therefore $G - \lambda \in \mathcal{Y}$, so we may put $Y_1 = G - \lambda$ in (28) to deduce that

$$\tilde{\mathbb{E}}((G-\lambda)^2) = 0. \tag{30}$$

It follows easily that

$$G(\omega) = \lambda \quad \text{for all } \omega \in \Omega. \tag{31}$$

(Indeed, (30) shows that G has risk-neutral variance equal to zero.) We conclude that if (27) holds, then there is a constant λ such that

$$\frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)} U'(\hat{X}_1(\omega)) = \lambda \quad \text{for all } \omega \in \Omega. \tag{32}$$

We shall now show that if (32) holds, then the strategy $\hat{X}_1$ is optimal, i.e. (19) holds. Let $\hat{X}_1 \in \mathcal{X}$ be given and assume that there exists $\lambda \in \mathbb{R}$ such that (32) holds. Let $X_1 \in \mathcal{X}$ be given and put

$$Y_1 = X_1 - \hat{X}_1. \tag{33}$$

Observe that $Y_1 \in \mathcal{Y}$. We may choose an open interval I such that $[0,1] \subset I$ and

$$\hat{X}_1(\omega) + yY_1(\omega) \in \mathcal{X} \quad \text{for all } y \in I.$$

(The reader is urged to verify this claim.)

Now, define $g : I \to \mathbb{R}$ by (21). Notice that g is differentiable on I and that g' is given by (22). It follows that g' is differentiable on I and that

$$g''(y) = \mathbb{E}(U''(\hat{X}_1 + yY_1)Y_1^2) \leq 0 \quad \text{for all } y \in I. \tag{34}$$

From computations done previously, we conclude that

$$g'(0) = \tilde{\mathbb{E}}\left(\frac{\mathbb{P}}{\tilde{\mathbb{P}}}U'(\hat{X}_1)Y_1\right). \tag{35}$$

By virtue of (32) and the fact that $\tilde{\mathbb{E}}(Y_1) = 0$, we conclude that

$$g'(0) = 0.$$

Since $g'(0) = 0$ and $g''(y) \le 0$ for all $y \in I$, it follows that g attains a maximum at 0, i.e.

$$g(y) \le g(0) \quad \text{for all } y \in I.$$

In particular, we have

$$g(1) \le g(0).$$

Notice that

$$g(1) = \mathbb{E}(U(\hat{X}_1 + X_1 - \hat{X}_1)) = \mathbb{E}(U(X_1)).$$

We conclude that

$$\mathbb{E}(U(X_1)) \le \mathbb{E}(U\hat{X}_1)).$$

Since $X_1 \in \mathcal{X}$ was arbitrary this shows that $\hat{X}_1$ is optimal.

We have just proved the following theorem.

Theorem 6.8: Assume that U is twice differentiable on $(0, \infty)$ and that $U''(x) < 0$ for all $x > 0$. Let $\hat{X}_1$ be the terminal capital of a strategy with initial capital $\hat{X}_0 = z > 0$ and $\hat{X}_1(\omega) > 0$ for all $\omega \in \Omega$. Then

$$\mathbb{E}(U(X_1)) \le \mathbb{E}(U(\hat{X}_1))$$

for all strategies X with $X_0 = z$ and $X_1(\omega) > 0$ for all $\omega \in \Omega$ if and only if there exists a constant $\lambda \in \mathbb{R}$ such that

$$U'(\hat{X}_1(\omega)) = \lambda \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad \text{for all } \omega \in \Omega. \tag{36}$$

Remark 6.9: It is convenient to introduce a random variable Z, called the *Radon-Nikodym derivative* of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ via the formula

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad \text{for all } \omega \in \Omega. \tag{37}$$

Notice that (36) can be expressed as

$$U'(\hat{X}_1(\omega)) = \lambda Z(\omega) \quad \text{for all } \omega \in \Omega, \tag{38}$$

which says "the marginal utility of the terminal capital of an optimal portfolio is proportional to the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$".

Remark 6.10: An important property of the Radon-Nikodym derivative is the following "change of variables" formula

$$\mathbb{E}(X_1) = \tilde{\mathbb{E}}\left(\frac{X_1}{Z}\right),$$

or equivalently

$$\tilde{\mathbb{E}}(X_1) = \mathbb{E}(ZX_1)$$

for all random variables X_1 on Ω. In fact, it was such a change of variables that motivated the definition of Z.

We shall now apply the theorem to an example considered previously.

Example 6.11: Consider the one-period binomial model with $\mathbb{P}(H) = \frac{2}{3}$, $\mathbb{P}(T) = \frac{1}{3}$, $u = 2$, $d = \frac{1}{2}$, $r = .25$, and $S_0 = 16$. Suppose that an investor has utility function $U(x) = lnx,\ x > 0$, and has initial capital $X_0 = 100$.

For this model, we have already seen that

$$\tilde{p} = \tilde{\mathbb{P}}(H) = \frac{1}{2},\ \tilde{q} = \tilde{\mathbb{P}}(T) = \frac{1}{2}.$$

It follows that

$$Z(H) = \frac{1/2}{2/3} = \frac{3}{4}, \ Z(T) = \frac{1/2}{1/3} = \frac{3}{2}.$$

Let us look for the terminal capital $\hat{X}_1$ of an optimal strategy. Since $U'(x) = \frac{1}{x}$ for all $x > 0$, we want to find $\lambda \in \mathbb{R}$ such that

$$\frac{1}{\hat{X}_1(\omega)} = \lambda Z(\omega) \quad \text{for all } \omega \in \{H, T\}.$$

This leads to the pair of equations

$$\frac{1}{\hat{X}_1(H)} = \frac{3}{4}\lambda$$

$$\frac{1}{\hat{X}_1(T)} = \frac{3}{2}\lambda.$$

Solving for $\hat{X}_1(H), \ \hat{X}_1(T)$ we find that

$$\hat{X}_1(H) = \frac{4}{3\lambda}, \ \hat{X}_1(T) = \frac{2}{3\lambda}.$$

To determine the appropriate value of λ, we use the fact that $\hat{X}_0 = 100$, i.e.

$$100 = \frac{1}{1.25}\tilde{\mathbb{E}}(\hat{X}_1) = \frac{1}{1.25}\left(\frac{1}{2}\hat{X}_1(H) + \frac{1}{2}\hat{X}_1(T)\right).$$

This leads us to the condition

$$125 = \frac{2}{3\lambda} + \frac{1}{3\lambda} = \frac{1}{\lambda}$$

which is equivalent to

$$\lambda = \frac{1}{125}.$$

It follows that

$$\hat{X}_1(H) = \frac{500}{3}, \ \hat{X}_1(T) = \frac{250}{3}.$$

Suppose now that we wish to find out how the initial capital should be invested at $t = 0$. Let y denote the number of shares of stock purchased at $t = 0$. Then we have

$$32y + (100 - 16y)(1.25) = \frac{500}{3}$$

$$8y + (100 - 16y)(1.25) = \frac{250}{3}.$$

Solving this system we find that

$$y = \frac{125}{36}.$$

This means that at $t = 0$ we should invest $\frac{500}{9}$ in stock and put $\frac{400}{9}$ in the bank. □

Example 6.12: Assume that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1$, $\mathbb{P}(\omega_1) = \frac{1}{4}$, $\mathbb{P}(\omega_2) = \frac{1}{2}$, $\mathbb{P}(\omega_3) = \frac{1}{4}$, $\tilde{\mathbb{P}}(\omega_1) = \tilde{\mathbb{P}}(\omega_2) = \tilde{\mathbb{P}}(\omega_3) = \frac{1}{3}$. Consider an investor with initial capital z and utility function $U(x) = -x^{-1}$. Let us find the terminal capital $\hat{X}_1$ of an optimal portfolio.

Notice that

$$\begin{aligned}
Z(\omega_1) &= \frac{1/3}{1/4} = \frac{4}{3}, \\
Z(\omega_2) &= \frac{1/3}{1/2} = \frac{2}{3}, \\
Z(\omega_3) &= \frac{1/3}{1/4} = \frac{4}{3}.
\end{aligned}$$

Since $U'(x) = x^{-2}$, we seek $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}
\frac{1}{\hat{X}_1(\omega_1)^2} &= \frac{4\lambda}{3}, \\
\frac{1}{\hat{X}_1(\omega_2)^2} &= \frac{2\lambda}{3}, \\
\frac{1}{\hat{X}_1(\omega_3)^2} &= \frac{4\lambda}{3}.
\end{aligned}$$

These conditions are equivalent to

$$\begin{aligned}
\hat{X}_1(\omega_1) &= \frac{\sqrt{3}}{2\sqrt{\lambda}}, \\
\hat{X}_1(\omega_2) &= \frac{\sqrt{3}\sqrt{2}}{2\sqrt{\lambda}}, \\
\hat{X}_1(\omega_3) &= \frac{\sqrt{3}}{2\sqrt{\lambda}}.
\end{aligned}$$

Using the fact that

$$\hat{X}_0(1+r) = \tilde{\mathbb{E}}(\hat{X}_1),$$

we find that

$$1.1z = \frac{1}{3}\hat{X}_1(\omega_1) + \frac{1}{3}\hat{X}_1(\omega_2) + \frac{1}{3}\hat{X}_1(\omega_3)$$

which yields

$$3.3z = \frac{\sqrt{3}}{2\sqrt{\lambda}}\,(2+\sqrt{2}).$$

We conclude that

$$\frac{\sqrt{3}}{2\sqrt{\lambda}} = \frac{3.3z}{2+\sqrt{2}}$$

and consequently

$$\hat{X}_1(\omega_1) = \hat{X}_1(\omega_3) = \frac{3.3z}{2+\sqrt{2}}, \quad \hat{X}_1(\omega_2) = \frac{3.3\sqrt{2}z}{2+\sqrt{2}}. \ \square$$

6.6 Mean-Variance Analysis

In this section we briefly explore optimal portfolio design via mean-variance analysis. This procedure was developed by Harry Markowitz as part of his Ph.D. thesis (in the 1950's). Markowitz shared the Nobel prize in 1990 for pioneering work in financial economics, based on his fundamental contributions to optimal investment.

The basic idea is that the potential reward of a portfolio will be measured by its expected terminal capital and the risk of a portfolio will be measured by the variance of its terminal capital. (Here, expected values and variances are computed using the reference probability measure $\mathbb{P}$.) This leads us naturally to two essentially equivalent optimization problems.

I. Of all portfolios having the same given initial capital and the same given expected terminal capital, find the one whose terminal capital has the smallest variance.

II. Of all portfolios having the same given initial capital and whose terminal capitals have the same given variance, find the one whose terminal capital has the largest expected value.

We shall consider problems of the type I because they are more convenient to treat mathematically.

Remark 6.13: Although we shall implement this procedure only within the context of a finite one-period model, it can be used in much more general contexts.

Remark 6.14: The mean-variance approach to portfolio optimization is closely related to maximizing the expected utility with a quadratic utility function.

Remark 6.15: Mean-Variance analysis can be applied to portfolios built with a finite number of assets on an infinite probability space, assuming that all of the relevant expected values, variances, and covariances are meaningful. If the stock returns are normal random variables then any portfolio that is optimal in the sense of maximum expected utility, with a risk-averse utility function U, is also optimal in the sense of mean-variance analysis. (See, for example, [L].)

Two very important consequences of mean-variance analysis will be:

1. "Don't put all your eggs in one basket." Diversification of a portfolio can significantly lower risk, without lowering the expected return.
2. Assuming that risk-free investing and borrowing is possible at the same interest rate, then there exists a single fund M (made up of the basic risky assets in certain proportions) such that every portfolio that is optimal in the sense of mean-variance analysis can be expressed completely in terms of M and the bank account. In other words, individual investors will always want to buy the stocks in the same proportions relative to one another – regardless of the risk preferences of the individual investor. Conservative investors will put a higher percentage of their initial capital into the bank (and less into the fund M), but all investors will like the mix of stocks used to create M. This result is known as the one-fund theorem and helps give credence to the concept of a mutual fund.

For the remainder of the chapter we assume that a finite one-period model with interest rate $r \geq 0$ and basic risky assets $S^1, S^2, \ldots, S^k$ is given. We assume that the model is arbitrage free (although we do not require the model to be complete.) In some sense, the initial stock price S_0^i can be regarded as a choice of unit used to measure the amount of capital invested in the i^{th} stock. It can be misleading to compare variances of stock prices if different units are being used. For example, when a two-for-one stock split occurs, the share price is cut in half and investors who already hold the stock receive two "new shares" for each share already held. Suppose that a two-for-one split occurs at time 0 and let us denote the pre-split price by S_0 and the post-split price by S_0^* so that $S_0 = 2S_0^*$. We denote the price of one new share at time 1 by S_1^*. The value at time 1 of one old share will be $S_1 = 2S_1^*$. Notice that

$$\mathrm{Var}(S_1^*) = \mathrm{Var}(\frac{1}{2}S_1) = \frac{1}{4}\mathrm{Var}(S_1).$$

The stock split cuts the variance of the stock price at time 1 by a factor of 4; however it should be clear that the stock split does not reduce the actual risk of investing in this particular stock. If we look at variance per unit of capital invested, then we can compare different stocks without worrying about the initial stock prices. For a stock S, we will look mostly at $\mathrm{Var}(S_1/S_0)$ rather than $\mathrm{Var}(S_1)$. It is also useful to be able to compare expected stock performances directly to the interest rate in the bank account. Consequently, it is convenient to introduce the stock returns ρ^i defined by

$$\rho^i(\omega) = \frac{S_1^i(\omega) - S_0^i}{S_0^i}.$$

Notice that each ρ^i is "dimensionless". We define the numbers $\mu^i, \sigma_{ij},\ i, j = 1, 2, \ldots, k$ by

$$\begin{aligned} \mu^i &= \mathbb{E}(\rho^i), \\ \sigma_{ij} &= \mathrm{Cov}(\rho^i, \rho^j). \end{aligned}$$

Recall that $\sigma_{ii} = \mathrm{Var}(\rho^i)$ and that $\sigma_{ij} = \sigma_{ji}$ for all $i, j = 1, 2, \ldots, k$.

Notice that

$$\mathbb{E}(S_1^i) = (1 + \mu^i)S_0^i,$$

and that

$$\mathrm{Cov}(S_1^i, S_1^j) = \sigma_{ij} S_0^i S_0^j \quad \text{for all } i, j = 1, 2, \ldots, k.$$

In order to be able to express portfolios in a simple manner, we define the random variables $\bar{S}_1^1,\ \bar{S}_1^2, \ldots \bar{S}_1^k$ on Ω by

$$\bar{S}_1^i(\omega) = \frac{S_1^i(\omega)}{S_0^i}.$$

We refer to $\bar{S}_1^i$ as the *scaled* price of the i^{th} stock at time 1. Observe that

$$\mathbb{E}(\bar{S}_1^i) = 1 + \mu^i$$

and

$$\mathrm{Cov}(\bar{S}_1^i,\ \bar{S}_1^j) = \sigma_{ij} \quad \text{for all } i, j = 1, 2, \ldots, k.$$

Example 6.16 Before getting into the mathematical formulation of the mean-variance problem, it seems worthwhile to look at a simple example (with some numbers) that illustrates the value of diversification. Suppose that there are N stocks $S^1, S^2, \cdots, S^N$ satisfying

$$\mu^i = .1, \quad \sigma_{ii} = .16, \quad i = 1, 2, \cdots k,$$

$$\sigma_{ij} = 0, \quad i \neq j.$$

Consider an investor who wants to invest \$100 in stock at $t = 0$. Since the stocks are indistinguishable to us on the basis of the information given, and since every portfolio that invests only in these N stocks has expected return .1, a naive argument suggests that it does not matter how the \$100 investment is split between the stocks. However, although the expected value of the terminal capital will be \$110 in all cases, the variance of the terminal capital is impacted on very significantly by the manner in which the initial capital is split between the stocks. We shall illustrate this by looking at two extreme cases:

Case 1: Suppose that the entire \$100 is invested in one stock S^j. Then the variance of the terminal capital is given by

$$\mathrm{Var}(X_1) = \mathrm{Var}(100\bar{S}_1^j) = (100)^2(.16) = 1,600.$$

Case 2: Suppose that \$100/N is invested in each of the N stocks. Recall that for uncorrelated random variables, the variance of a sum is the sum of the variances. Therefore, we have

$$\mathrm{Var}(X_1) = \mathrm{Var}(\sum_{i=1}^{N} \frac{100\bar{S}_1^i}{N}) = N(\frac{100}{N})^2(.16) = \frac{1,600}{N}.$$

Notice that variance has been divided by N. $\square$

Consider an economic agent with initial capital X_0. For each $i = 1, 2, \ldots, k$, let x^i denote the amount of capital that the agent invests in S^i at $t = 0$. The amount of capital invested in the bank account at time 0 is therefore

$$X_0 - \sum_{j=1}^{k} x^j,$$

and consequently the terminal capital of the agent's portfolio is

$$X_1 = \left(X_0 - \sum_{j=1}^{k} x^j \right) (1 + r) + \sum_{j=1}^{k} x^j \bar{S}_1^j. \tag{39}$$

The variance of X_1 is given by

$$\mathrm{Var}(X_1) = \sum_{i,j=1}^{k} \sigma_{ij} x^i x^j. \tag{40}$$

We assume that

$$\sum_{i,j=1}^{k} \sigma_{ij} x^i x^j = 0$$

if and only if

$$x^i = 0 \quad \text{for all } i = 1, 2, \ldots, k.$$

This means that the only portfolio that is risk-free is the portfolio in which all capital is invested in the bank account. It is convenient to introduce the return ρ of the portfolio defined by

$$\rho = \frac{X_1 - X_0}{X_0}. \tag{41}$$

Our investor chooses a desired value $\hat{r}$ for $\mathbb{E}(\rho)$ and tries to determine $x^1, x^2, \ldots x^k$ so that

$$\mathbb{E}(\rho) = \hat{r} \tag{42}$$

and

$$\operatorname{Var}(X_1) \text{ is as small as possible.}$$

In view of (39), (41), it is easy to see that (42) is equivalent to

$$\sum_{j=1}^{k} (\mu^j - r) x^j = (\hat{r} - r) X_0. \tag{43}$$

We assume that

$$\hat{r} > r;$$

indeed a rational investor who would be satisfied with an expected return of r (or below) would simply invest all of the initial capital in the bank. We also assume that

$$\mu^j \neq r \qquad \text{for at least one } j = 1, 2, \ldots, k. \tag{44}$$

Indeed, if $\mu^j = r$ for all $j = 1, 2, \ldots, k$ then every portfolio will have expected return r and a risk-averse investor will simply put all of the initial capital in the bank.

We wish to minimize $\operatorname{Var}(X_1)$ subject to the constraint (43). It will clean up the formulas a bit if we minimize $\frac{1}{2}\operatorname{Var}(X_1)$. Therefore, we define

$$g(x^1, x^2, \ldots x^k) = \frac{1}{2} \sum_{i,j=1}^{k} \sigma_{ij} x^i x^j. \tag{45}$$

Using a minor modification of the proof of Theorem 3.6, we can obtain the following result.

Theorem 6.17: Let the list $(\hat{x}^i)_{1 \le i \le k}$ of real numbers satisfying

$$\sum_{j=1}^{k} (\mu^j - r) \hat{x}^j = (\hat{r} - r) X_0$$

be given. Then

$$g(x^1, x^2, \dots x^k) \geq g(\hat{x}^1, \hat{x}^2, \dots \hat{x}^k)$$

for all lists $(x^i)_{1\leq i\leq k}$ satisfying (43) if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\sum_{j=1}^{k} \sigma_{ij}\hat{x}^j = \lambda(\mu^i - r) \quad \text{for all } i = 1, 2, \dots, k. \tag{46}$$

Remark 6.18: It can be shown that there is exactly one list $(\hat{y}^i)_{1\leq i\leq k}$ satisfying

$$\sum_{j=1}^{k}\sigma_{ij}\hat{y}^j = \mu^i - r. \tag{47}$$

(Indeed, if we denote by A the $k\times k$ matrix whose ij entry is σ_{ij}, then our assumption that the only risk-free portfolio is the one in which all of the initial capital is invested in the bank account implies that the null space of A is trivial and consequently A is invertible. This ensures us that there is exactly one list $(\hat{y}^i)_{1\leq i\leq k}$ satisfying (46). The matrix A is called the covariance matrix for the stock returns. It is symmetric and positive definite.)

The list $(\hat{y}^i)_{1\leq i\leq k}$ can be used to construct the optimal portfolio as follows: Put

$$\hat{x}^i = \lambda\hat{y}^i \quad \text{for } i = 1, 2, \dots, k$$

and use (43) to solve for λ. This leads to the expression

$$\lambda\sum_{j=1}^{k}(\mu^j - r)\hat{y}^j = (\hat{r} - r)X_0.$$

It can be shown that

$$\sum_{j=1}^{k}(\mu^j - r)\hat{y}^j > 0 \tag{48}$$

and consequently

$$\lambda = \frac{(\hat{r} - r)X_0}{\sum_{j=1}^{k} (\mu^j - r)\hat{y}^j} > 0. \tag{49}$$

(The inequality (48) follows from the fact the covariance matrix A is positive definite.)

It is very interesting to observe that the list $(\hat{y}^i)_{1 \le i \le k}$ depends only on properties of the stocks and on the interest rate in the bank account; in particular this list is the same for every investor – regardless of their initial capital and their choice of $\hat{r}$. The characteristics of an individual investor are measured by the constant λ. All portfolios that are optimal in the sense of mean-variance analysis will hold the stocks in the same proportion to one another. Investors who want a higher expected return will put a higher percentage of their initial capital in the stocks, but they will purchase the stocks in the same proportion to one another as an investor whose desired expected return is only a little bit above the risk-free interest rate. In other words, we can construct a portfolio M (which holds only stocks and does not invest in the bank account) having the property that all portfolios that are optimal in the sense of mean-variance analysis can be expressed in terms of M and the bank account. We refer to M as an *optimal mutual fund*. Of course, there is a trivial nonuniqueness in constructing optimal mutual funds, because any constant multiple of an optimal mutual fund will also be an optimal mutual fund. One way to arrive at a unique optimal mutual fund is to specify the initial share price (initial capital). For simplicity, we shall describe a share of the optimal mutual fund here by simply investing the amount of capital $\hat{y}^i$ in S^i at $t = 0$ for each $i = 1, 2, \cdots, k$, where the $\hat{y}^i$ are determined by (47). The number of shares of S^i in M is therefore given by

$$\Delta_0^i = \frac{\hat{y}^i}{S_0^i}.$$

The argument above establishes the following important result.

Theorem 6.19 (One-Fund Theorem) There is a portfolio M having the property that every portfolio that is optimal in the sense of mean variance analysis can be expressed in terms of M and an investment in the bank account.

6.7 Mean-Variance Analysis with Uncorrelated Stock Returns:

In this section we look at a special case in which the optimal portfolio can be written down explicitly. Assume that

$$\sigma_{ii} > 0 \quad \text{for all } i = 1, 2, \ldots, k \tag{50}$$

and

$$\sigma_{ij} = 0 \quad \text{whenever } i \neq j. \tag{51}$$

When (51) holds, we say that the returns $\rho^1, \rho^2, \ldots, \rho^k$ are uncorrelated. (This condition will certainly hold if $S_1^1, S_1^2, \ldots, S_1^k$ are independent.)

In this special case, the condition for optimality reduces to

$$\sigma_{ii}\hat{x}^i = \lambda(\mu^i - r), \quad i = 1, 2, \ldots, k \tag{52}$$

so that

$$\hat{x}^i = \frac{\lambda(\mu^i - r)}{\sigma_{ii}}. \tag{53}$$

Using (49), we find that

$$\lambda = \frac{(\hat{r} - r)X_0}{\displaystyle\sum_{j=1}^{k} \frac{(\mu^j - r)^2}{\sigma_{jj}}} > 0. \tag{54}$$

We see that

1. If $\mu^j > r$, then the j^{th} stock has been purchased in the optimal portfolio.
2. If $\mu^j < r$, then the j^{th} stock has been sold short in the optimal portfolio.
3. If $\mu^j = r$, then the optimal portfolio avoids the j^{th} stock.

We also see that if two stocks have the same variance, the one with the higher expected return is more desirable. Moreover, if $\mu^i = \mu^j > r$, then the stock with smaller variance is more desirable to hold in the portfolio.

Notice that the initial capitals $\hat{y}^i$ describing the optimal mutual fund M are given by

$$\hat{y}^i = \frac{\mu^i - r}{\sigma_{ii}}, \tag{55}$$

and the number of shares of S^i in M is given by

$$\Delta_0^i = \frac{\mu^i - r}{\sigma_{ii} S_0^i}. \tag{56}$$

Example 6.20: Suppose that $r = .1$ and that there are three stocks S^1, S^2, and S^3. Assume that $S_0^1 = 10$, $S_0^2 = 12$, $S_0^3 = 8$, $\mathbb{E}(S_1^1) = 11.2$, $\mathbb{E}(S_1^2) = 13.68$, $\mathbb{E}(S_1^3) = 9.04$, $\mathrm{Var}(S_1^1) = 20$, $\mathrm{Var}(S_1^2) = 21.6$, $\mathrm{Var}(S_1^3) = 6.4$, and that $\mathrm{Cov}(S_1^i, S_1^j) = 0$ if $i \neq j$. Suppose that an investor has initial capital 1,000 and wishes to construct a portfolio with expected return $\hat{r} = .12$ that is optimal in the sense of mean-variance analysis.

We begin by observing that

$$\begin{aligned}
\mu^1 &= \frac{\mathbb{E}(S_1^1)}{S_0^1} - 1 = \frac{11.2}{10} - 1 = .12, \\
\mu^2 &= \frac{\mathbb{E}(S_1^2)}{S_0^2} - 1 = \frac{13.68}{12} - 1 = .14, \\
\mu^3 &= \frac{\mathbb{E}(S_1^3)}{S_0^3} - 1 = \frac{9.04}{8} - 1 = .13, \\
\sigma_{11} &= \frac{\mathrm{Var}(S_1^1)}{(S_0^1)^2} = \frac{20}{100} = .2, \\
\sigma_{22} &= \frac{\mathrm{Var}(S_1^2)}{(S_0^2)^2} = \frac{21.6}{144} = .15, \\
\sigma_{33} &= \frac{\mathrm{Var}(S_1^3)}{(S_0^3)^2} = \frac{6.4}{64} = .1,
\end{aligned}$$

and that $\sigma_{ij} = 0$ whenever $i \neq j$. The optimal portfolio is therefore described by

$$\begin{aligned}
\hat{x}^1 &= \frac{(.12 - .1)}{.2}\lambda = .1\lambda \\
\hat{x}^2 &= \frac{(.14 - .1)}{.15}\lambda = .2666\lambda \\
\hat{x}^3 &= \frac{(.13 - .1)}{.1}\lambda = .3\lambda,
\end{aligned}$$

where λ is given by

$$\lambda = \frac{X_0(\hat{r} - r)}{\displaystyle\sum_{j=1}^{k} \frac{(\mu^j - r)^2}{\sigma_{jj}}} = 923.08.$$

It follows that

$$\hat{x}^1 = 92.31, \ \hat{x}^2 = 246.15, \ \hat{x}^3 = 276.92,$$

and that the amount of initial capital invested in the bank should be $1000 - \hat{x}^1 - \hat{x}^2 - \hat{x}_3 = 384.62$. In terms of shares of stock, the optimal portfolio will hold

$$\begin{aligned}
\frac{\hat{x}^1}{S_0^1} &= \frac{92.31}{10} = 9.231 \text{ shares of } S^1, \\
\frac{\hat{x}^2}{S_0^2} &= \frac{246.15}{12} = 20.51 \text{ shares of } S^2, \\
\frac{\hat{x}^3}{S_0^3} &= \frac{276.92}{8} = 34.62 \text{ shares of } S^3. \ \square
\end{aligned}$$

Exercises for Chapter 6

1. Consider a one-period binomial model with $u = 1.3,\ d = .94,\ r = .06, \mathbb{P}(H) = \frac{1}{2},\ \mathbb{P}(T) = \frac{1}{2}$. (The characteristics of this model are roughly consistent with annual behavior of the $S\&P$ 500, based on 10-year averages.)

   (a) Suppose that $U(x) = lnx$, $x > 0$, and that $X_0 = 100$. Find the amount of money invested in stock at $t = 0$ and the amount of money in the bank at $t = 0$ in the portfolio having the largest expected utility. (Use $S_0 = 100$ in your computations.)

   (b) Choose a percentage α, with $.2 \le \alpha \le .8$, that you feel would be a reasonable percentage of an initial investment X_0 to be put into a stock fund with a one-year investment horizon. (In other words, you will invest αX_0 in stocks and $(1 - \alpha)X_0$ in the bank at $t = 0$ with the goal of maximizing your expected utility at time 1.) Find a value $\beta > 0$ such that if the utility function

   $$U(x) = \frac{-1}{x^\beta}, \quad x > 0$$

   is employed, then the optimal portfolio will have αX_0 invested in stock at $t = 0$ and $(1 - \alpha)X_0$ in the bank at $t = 0$.

2. Consider an economic agent having utility function

   $$U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma < 1, \quad \gamma \neq 0.$$

   The agent is a contestant on a game show and can choose either alternative (I) or (II) below:

   (I) The agent will receive \$10,000,000 for sure.

   (II) A fair coin will be tossed. If it shows a head, the agent will receive \$30,000,000; if it shows a tail, the agent will receive nothing.

   Which alternative will the agent choose. (The answer may depend on the value of γ.)

3. Consider a one-period binomial model with $u = 2,\ d = \frac{1}{2},\ r = \frac{1}{4},\ S_0 = 100,\ \mathbb{P}(H) = \frac{3}{4}$, and $\mathbb{P}(T) = \frac{1}{4}$. An investor with utility function $U(x) = lnx$ has initial capital \$100. Find the terminal capitals $\hat{X}_1(H),\ \hat{X}_1(T)$ for the portfolio that is optimal in the sense of maximum expected utility.

4. Consider a one-period binomial model with $u = 1.1,\ d = .9,\ r = .05$, and $S_0 = 100$. An investor has utility function $U(x) = lnx$ and initial capital 100. Assume that the optimal strategy (in the sense of maximum expected utility) is to buy 1 share of stock (and invest nothing in the bank) at time 0. Determine $\mathbb{P}(H)$.

5. Consider a complete one-period model with $r = .2$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and $\mathbb{P}(\omega_1) = \frac{1}{3}$, $\mathbb{P}(\omega_2) = \frac{1}{2}$, $\mathbb{P}(\omega_3) = \frac{1}{6}$, $\tilde{\mathbb{P}}(\omega_1) = \frac{1}{4}$, $\tilde{\mathbb{P}}(\omega_2) = \frac{3}{8}$, $\tilde{\mathbb{P}}(\omega_3) = \frac{3}{8}$. Suppose that an investor has initial capital $X_0 = 100$ and that the investor's utility function is

$$U(x) = lnx, \quad x > 0.$$

Find the terminal capital $\hat{X}_1$ of the portfolio $\hat{X}$ that maximizes $\mathbb{E}(U(X_1))$ over all portfolios X having initial capital 100.

6. Consider a one-period complete model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .1$, $\mathbb{P}(\omega_1) = \frac{1}{3}$, $\mathbb{P}(\omega_2) = \frac{5}{12}$, $\mathbb{P}(\omega_3) = \frac{1}{4}$, $\tilde{\mathbb{P}}(\omega_1) = \frac{1}{2}$, $\tilde{\mathbb{P}}(\omega_2) = \frac{1}{4}$, $\tilde{\mathbb{P}}(\omega_3) = \frac{1}{4}$. An investor has utility function $U(x) = lnx$ and initial capital \$500. Find the terminal capital of the strategy that is optimal in the sense of maximum expected utility.

7. A CMU student is going to play a dice game in a casino. The game is based on one roll of a single die and has the following rules.

    1. One can bet on any number from 1 to 6.
    2. The price of one bet equals 1/6.
    3. The casino pays a gambler \$1 if the outcome of the die coincides with the gambler's bet.
    4. One can make "fractional" bets. For example, one can make half a bet on the number 1, by paying \$1/12 and receiving \$0.5 if the outcome equals 1.

    After observing the game for a while, the student discovers that the die is not perfect, i.e. the probabilities of different outcomes are not equal. The student's estimate of the probabilities is given in the following table:

1	2	3	4	5	6
20%	18%	16%	13%	18%	15%

    The student is going to play only one time with initial capital $z = \$1$. The utility function of the student is $U(x) = -e^{-x}$. The student can borrow cash for a short interval of time without any interest. Find the optimal betting strategy for the student, i.e. the quantities of bets on the different numbers from 1 to 6 he needs to make in order to maximize the expected utility of the terminal wealth.

8. Consider a complete one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $r = .1$ and reference (or actual) probabilities $\mathbb{P}(\omega_1) = .2$, $\mathbb{P}(\omega_2) = .3$ $\mathbb{P}(\omega_3) = .3$, and $\mathbb{P}(\omega_4) = .2$. An investor with utility function $U(x) = lnx$, $x > 0$ constructs an optimal portfolio $\hat{X}$. We know that $\hat{X}_1(\omega_1) = 100$, $\hat{X}_1(\omega_2) = 150$, $\hat{X}_1(\omega_3) = 300$, and $\hat{X}_1(\omega_4) = 200$. Find the initial capital of the investor.

9. Consider a one-period finite model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $r = .07$, and reference probabilities $\mathbb{P}(\omega_1) = .25$, $\mathbb{P}(\omega_2) = .45$, and $\mathbb{P}(\omega_3) = .3$. There are two basic risky assets: A stock S^1 with $S_0^1 = 100/1.07$ and $S_1^1(\omega_1) = 80$, $S_1^1(\omega_2) = 100$ and $S_1^1(\omega_3) = 120$, and a standard put option P on the stock with strike price $K = 90$ and initial price $P_0 = 4$. Consider an investor with initial capital $z = 1,000$ and utility function $U(x) = \sqrt{x}$, $x > 0$. Determine the number of shares of stock and the number of put options in the optimal portfolio.

10. Consider a one-period binomial model with $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$, $u = 1.3$, $d = .94$ and $S_0 = 100$. (The interest rate is to be determined later.) An investor with utility function $U(x) = lnx$ and initial capital $X_0 = 100$ constructs a portfolio that is optimal in the sense of expected utility. The optimal strategy is to buy one share of stock (and invest nothing in the bank) at $t = 0$. find the interest rate r.

11. Suppose you are a representative for a financial company that has access to a financial market in which

    (i) one can borrow or invest money for any period of time at the effective annual interest rate $R = 6\%$

    (ii) one can take the long or short position on a forward contract for delivery of a stock S at time $t = .5$ at the forward price $F = \$150$ per share

    (iii) the price of the stock at $t = .5$ takes only two possible values \$130 and \$180 with

$$\begin{aligned} \mathbb{P}[S_{.5} = 130] &= .3 \text{ and} \\ \mathbb{P}[S_{.5} = 180] &= .7 \end{aligned}$$

    A client of the company wants to invest $X_0 = \$100,000$ up to $t = .5$. The client's utility function is $U(x) = lnx$. The client will buy from you any derivative security having the same expected utility as investing the initial price of the security into a savings account with effective annual rate $R = 6\%$. You naturally want to maximize the profit of your company. Describe the payment function of the derivative security that you will sell the client. What will the time 0 value of your profit be?

12. Consider a finite one-period model with $r = 0$ and $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The model is complete with risk-neutral probabilities $\tilde{p}_1 = \tilde{\mathbb{P}}(\omega_1) = .2$, $\tilde{p}_2 = \tilde{\mathbb{P}}(\omega_2) = .5$, $\tilde{p}_3 = \tilde{\mathbb{P}}(\omega_3) = .3$. There are two basic securities: a stock S and a put option P on S with maturity $T = 1$ and strike price $K = \$100$. We know that $S_1(\omega_1) = 50$, $S_1(\omega_2) = 100$, $S_1(\omega_3) = 150$, $S_0 = 105$, $P_0 = 10$. An investor with utility function $U(x) = lnx$ and initial capital $X_0 = \$1,000$ designs a portfolio that is optimal in the sense of maximum expected utility. The optimal strategy is to buy 2 shares of stock and 5 put options at $t = 0$ (and invest the remaining capital in the bank). Let $p_1 = \mathbb{P}(\omega_1)$, $p_2 = \mathbb{P}(\omega_2)$, $p_3 = \mathbb{P}(\omega_3)$. Find p_1, p_2, and p_3.

13. Consider a one-period finite model with $r = .08$ and two stocks S^1 and S^2. Assume that $S_0^1 = 50$, $S_0^2 = 100$, $\mathbb{E}(S_1^1) = 55$, $\mathbb{E}(S_1^2) = 120$, $\mathrm{Var}(S_1^1) = 600$, $\mathrm{Var}(S_1^2) = 1800$, $\mathrm{Cov}(S_1^1, S_1^2) = 450$. Let X be a portfolio that is constructed by investing \$1,000 in the bank, \$1,000 in S^1 and \$1,000 in S^2 at $t = 0$. Find the expected return of the portfolio (i.e., $\mathbb{E}(\rho)$) and also the variance of the terminal capital (i.e., $\mathrm{Var}(X_1)$).

14. Consider a one-period finite model with $r = .1$ and two stocks S^1 and S^2. Assume that $S_0^1 = S_0^2 = 100$ and that $\mathbb{E}(S_1^1) = 115, \mathbb{E}(S_1^2) = 120$, $\mathrm{Var}(S_1^1) = 1000$, $\mathrm{Var}(S_1^2) = 2000$, and $\mathrm{Cov}(S_1^1, S_1^2) = -1000$. An investor with initial capital $X_0 = 1000$ wants to construct a portfolio with expected return $\hat{r} = .12$ (i.e., $\mathbb{E}(X_1) = X_0(1 + \hat{r}) = 1120$) having the smallest possible variance of all portfolios with this expected return and initial capital 1000 (i.e., the portfolio should be optimal in the sense of mean-variance analysis). Find the amounts $\hat{x}^1$ and $\hat{x}^2$ of money that should be invested in S^1 and S^2 at $t = 0$.

15. Consider a one-period finite model with $r = .1$ and three stocks S^1, S^2, S^3 satisfying

$$S_0^1 = S_0^2 = S_0^3 = 10,$$

$$\mathbb{E}(S_1^1) = 12, \quad \mathbb{E}(S_1^2) = 11, \quad \mathbb{E}(S_1^3) = 13,$$

$$\mathrm{Var}(S_1^1) = 15, \quad \mathrm{Var}(S_1^2) = 10, \quad \mathrm{Var}(S_1^3) = 25, \text{ and}$$

$$\mathrm{Cov}(S_1^i, S_1^j) = 0 \text{ for } i \neq j.$$

An investor with initial capital \$1,000 and desired expected return $\hat{r} = .14$ constructs a portfolio that is optimal in the sense of mean-variance analysis. Find the amounts $\hat{x}^1, \hat{x}^2, \hat{x}^3$ of capital in S^1, S^2, and S^3 at $t = 0$.

16. Consider a one-period model with $r = .05$ and three stocks S^1, S^2, S^3. Assume that $S_0^1 = S_0^2 = S_0^3 = 100$ and that $\mathbb{E}(S_1^1) = 108$, $\mathbb{E}(S_1^2) = 104$, $\mathbb{E}(S_1^3) = 109$, $\mathrm{Var}(S_1^1) = 500$, $\mathrm{Var}(S_1^2) = 200$, $\mathrm{Var}(S_1^3) = 400$, $\mathrm{Cov}(S_1^i, S_1^j) = 0$ whenever $i \neq j$. An investor with initial capital \$10,000 desires an expected return $\hat{r} = .07$. Find the portfolio that is optimal in the sense of mean-variance analysis.

17. Consider a one-period finite model with $r = .1$ and three stocks S^1, S_1^2, S^3. Assume that $S_0^1 = S_0^2 = S_0^3 = 100$, that $\mathbb{E}(S_1^1) = 115$, $\mathbb{E}(S_1^2) = 120$. $\mathbb{E}(S_1^3) = 112$, $\mathrm{Var}(S_1^1) = 1000$, $\mathrm{Var}(S_1^2) = 1000$, $\mathrm{Var}(S_1^3) = 800$, $\mathrm{Cov}(S_1^1, S_1^2) = \mathrm{Cov}(S_1^1, S_1^3) = \mathrm{Cov}(S_1^2, S_1^3) = 0$.

    (a) An investor with initial capital \$10,000 wants to construct a portfolio that is optimal in the sense of mean-variance analysis and has an expected return of $\hat{r} = .12$. Find the amounts $\hat{x}^1, \hat{x}^2, \hat{x}^3$ of capital that should be invested in S^1, S^2, and S^3 at time 0.

(b) Let M denote the optimal mutual fund. Find the number of shares of S^1, S^2, and S^3 needed to construct one share of M.

18. Consider a one-period finite model with three stocks S^1, S^2, S^3, and interest rate $r \geq 0$. Alice and Betty are mean-variance investors, each having initial capital \$1,000. Alice's desired expected return is $\hat{r}_A = .15$ and Betty's desired expected return is $\hat{r}_B = .12$. Let $\hat{x}_A^i$ and $\hat{x}_B^i$ denote the amounts of initial capital invested in S^i at time 0 in Alice's and Betty's optimal portfolios, respectively. It is known that

$$\hat{x}_A^1 = 300,\ \hat{x}_A^2 = 150,\ \hat{x}_A^3 = 225,$$

$$\hat{x}_B^1 = 200,\ \hat{x}_B^2 = 100,\ \hat{x}_B^3 = 150.$$

Find the interest rate r.

19. Consider a one-period finite model with $r = .08$ and 3 stocks S^1, S^2, S^3. Al and Bill are mean-variance investors each having initial capital \$1,000. The desired expected returns of Al and Bill are $\hat{r}_A = .15$ and $\hat{r}_B = .1$, respectively. Al's optimal portfolio invests \$400 in S^1, \$300 in S^2, and \$600 in S^3 at $t = 0$. How much capital should Bill invest in each of the stocks at $t = 0$?