Chapter 3

Forward Contracts, European and American Put and Call Options

Recall that a *forward contract* is an agreement between two parties made at some time τ concerning the sale of an asset at a future time T, called the *delivery time*, *delivery date*, or *maturity*. The party taking the *long position* agrees to buy a prescribed amount of the asset on the delivery date from the counterparty, who takes the *short position*, at a price $\mathcal{F}$, called the *forward price*. The party taking the short position agrees to sell the asset at time T at the price $\mathcal{F}$. The forward price is agreed upon at time τ and neither party pays anything to enter into the agreement. It is customary to scale $\mathcal{F}$ so that it represents the forward price per unit of the asset.

Although the forward price is chosen so that the value of both positions is zero at time τ, both positions will generally have nonzero values at times t with $\tau < t \leq T$. The values of the two positions will always have the same magnitude, but opposite signs. In particular, at time T, the value (per unit) of the long position is

$$S_T - \mathcal{F}$$

and the value (per unit) of the short position is

$$\mathcal{F} - S_T,$$

where S_T is the price (per unit) of the underlying asset at time T.

At times after the contract is initiated, $\mathcal{F}$ is often referred to as the *delivery price* in order to avoid confusion with the forward price in contracts being issued currently. In situations when there could be ambiguity concerning the time at which the contract was initiated, or the delivery time, we shall use the notation $\mathcal{F}_{\tau,T}$ in place of $\mathcal{F}$. It is quite common to use the phrase "buy a forward contract" to mean "take the long position on a forward contract" and the phrase "sell a forward contract" to mean "take the short position on a forward contract". In these notes, we generally use the more formal terminology "take a long position" and "take a short postion".

Forward contracts have existed since ancient times. They can potentially remove risk for both parties. However, there are some practical difficulties that limit their use in real-world financial markets. For example, it may not be easy for an individual wanting to take a position on a forward contract to find a party who wants the counterposition for the same quantity and delivery date of the asset. Another serious concern is that one of that parties may be unable to fulfill their obligation on the delivery date. In practice, it is much more common to use a closely related kind of contract called a *futures contract*. The main difference between futures and forwards is the manner in which settlement is made. Futures contracts are more complicated to analyze mathematically; they will be discussed briefly in Section 3.6 and treated treated in detail in 21-370 and 21-378. Although forward contracts are used only sparingly in the real world, it is extremely useful to study them. In particular, forward contracts will be extremely helpful in analyzing put and call options. Moreover, if interest rates are deterministic, then futures prices are the same as forward prices.

Throughout this chapter, we assume that the present time is $t = 0$ and that there is an ideal bank with effective spot rate function R_*. We denote the spot rate for investments initiated at time t and settled with a lump-sum payment at time T by $\mathcal{R}_{t,T}$. Recall that $R_*(T) = \mathcal{R}_{0,T}$.

3.1 Forward Contract for a Zero-Coupon Bond

Consider a zero-coupon bond with maturity T_b and face value F. The arbitrage-free price of the bond at $t = 0$ is

$$\mathcal{P} = Fd(T_b).$$

We want to determine the arbitrage-free forward price $\mathcal{F}$ (at time 0) for delivery of the bond at time T_d with $0 < T_d < T_b$. Let us replicate the long position. To do so, we need to set ourselves up so that at time T_d we pay $\mathcal{F}$ and receive the bond. (Of course, we do not need to physically receive the bond at time T_d; what we need is a self-financing strategy having terminal capital at time T_d equal to the price of the bond minus $\mathcal{F}$.) We do not know what the price of the bond will be at time T_d, so it is impossible to know how much money we should deposit in the bank at $t = 0$ in order to have precisely the correct amount available to purchase the bond at time T_d. However, since the bond does not pay coupons, (and since it will cost us nothing to store the bond) we can buy the bond at $t = 0$, "put it away in a drawer", and then simply take it out of the drawer at time T_d. The net financial effect to us will be the same as if someone delivers the bond to us at time T_d.

We can set ourselves to pay (or owe) $\mathcal{F}$ at time T_d by simply borrowing $\mathcal{F}d(T_d)$ between $t = 0$ and $t = T_d$. The initial capital of this strategy is

$$\mathcal{P} - \mathcal{F}d(T_d) = Fd(T_b) - \mathcal{F}d(T_d).$$

Since the initial capital should be zero, we must have

$$\mathcal{F} = \frac{Fd(T_b)}{d(T_d)} = \frac{F(1 + R_*(T_d))^{T_d}}{(1 + R_*(T_b))^{T_b}}. \tag{3.1}$$

Notice that as T_d approaches T_b, $\mathcal{F}$ approaches F and as T_d approaches 0, $\mathcal{F}$ approaches $\mathcal{P}$, as we should expect.

We can also arrive at (3.1) by a slightly different approach. Consider the strategy X in which we borrow $\mathcal{F}d(T_d)$ between $t = 0$ and $t = T_d$, buy the bond at $t = 0$, and take the short position at $t = 0$ on a forward contract for delivery of the bond at time T_d. If we denote by B_{T_d} the value of the bond at time T_d then the terminal capital of this strategy is

$$X_{T_d} = -\mathcal{F} + B_{T_d} + (\mathcal{F} - B_{T_d}) = 0.$$

Notice that the initial capital is given by

$$X_0 = Fd(T_b) - \mathcal{F}d(T_d).$$

Since the strategy is self-financing and has zero terminal capital, the initial capital must also be zero and this leads to (3.1).

Remark 3.1. There is, of course, a very basic relationship between forward interest rates and forward contracts for zero coupon bonds. The reader should verify that if it is agreed at time 0 to invest the amount $\mathcal{F}$ given by (3.1) between times T_d and T_b at the effective forward rate $\mathcal{R}^{for}_{0,T_d,T_b}$ then the value of the investment at time T_b will be precisely F.

Example 3.2. If $R_*(3.5) = .05$ and $R_*(5) = .0525$ then the forward price (at time 0) for delivery at $T_d = 3.5$ years of a zero coupon bond with face value $F = \$2,500$ and maturity $T_b = 5$ years is given by

$$\mathcal{F} = \frac{\$2,500d(5)}{d(3.5)} = \frac{\$2,500(1 + R_*(3.5))^{3.5}}{(1 + R_*(5))^5} = \frac{\$2,500(1.05)^{3.5}}{(1.0525)^5} = \$2,296.11.$$

3.2 Forward Contract for a General Security with Fixed Payments

Consider a general security with fixed payments $(F_i, T_i)_{1 \le i \le N}$ and a forward contract issued at time 0 for delivery of the security at time T, with $0 < T < T_N$, at the forward price $\mathcal{F}$. In order to avoid worrying about distinctions between clean and dirty prices, we assume that $T = T_j$ for some given $j \in \{1, 2, \cdots N-1\}$. We make the convention that delivery will take place just after the payment F_j is made. In other words, the agent with the long forward position will pay $\mathcal{F}$ at time T and receive the payments F_i at each of the times T_i, $i = j+1, j+2, \cdots, N$.

Let us replicate the long position. We need to modify the approach used for the zero-coupon bond. Indeed, if we purchase the security at time 0, we will receive payments that a holder of a long forward position is not entitled to (or make payments that we are not obligated to make if some of the relevant F_i are negative). In describing our strategy, we shall use language that is appropriate when the F_i are positive.

(Everything works out in the general case with the interpretation that borrowing a negative amount A means investing $|A|$.)

We can replicate the long position by purchasing the security at $t = 0$ and borrowing from the bank to create debts that will be balanced out by the payments $F_1, F_2, \ldots F_j$ made by the security at times $T_1, T_2, \ldots T_j$. We can also replicate the long position by simply setting ourselves up to receive the payments made after time T_j, i.e. we can purchase the security with fixed payments $(F_i, T_i)_{j+1 \leq i \leq N}$. We shall follow the former approach here. It is recommended that you try the latter approach on your own and verify that it gives the same result.

Consider the following strategy. At $t = 0$, we purchase the security $(F_i, T_i)_{1 \leq i \leq N}$, borrow $\mathcal{F}d(T_j)$ over the interval $[0, T_j]$, and for each $i = 1, 2, \cdots, j$, we borrow $F_i d(T_i)$ over the interval $[0, T_i]$. The payments F_i received at T_i for $i = 1, 2, \cdots, j$ (from holding the security) are used to repay the loans. At time T_j, the net position of this strategy is to hold the desired security and owe $\mathcal{F}$ to the bank. Moreover, this strategy is self-financing over the interval $[0, T_j]$, so it replicates the long forward position. Recall that the arbitrage-free price at time 0 of the security with fixed payments $(F_i, T_i)_{1 \leq 1 \leq N}$ is

$$\mathcal{P} = \sum_{i=1}^{N} F_i d(T_i).$$

Consequently, the initial capital of the replicating strategy is

$$-\mathcal{F}d(T_j) + \sum_{i=1}^{N} F_i d(T_i) - \sum_{i=1}^{j} F_i d(T_i) = -\mathcal{F}d(T_j) + \sum_{i=j+1}^{N} F_i d(T_i) = 0.$$

It follows that

$$\mathcal{F} = \sum_{i=j+1}^{N} \frac{F_i d(T_i)}{d(T_j)}. \tag{3.2}$$

When using (3.2) it is often convenient to make use of the observation

$$\sum_{i=j+1}^{N} F_i d(T_i) = \mathcal{P} - \sum_{i=1}^{j} F_i d(T_i).$$

Example 3.3. Consider a coupon bond with face value $F = \$10,000$ and maturity 5 years that pays coupons twice per year at the nominal coupon rate $q[2] = 4.5\%$. Given that

$$d(3.5) = .84866, \quad d(4) = .82900,$$

$$d(4.5) = .81153, \quad d(5) = .78353,$$

find the forward price (at time 0) for delivery of this bond at time 3.5, just after the coupon has been paid.

For each $i = 1, 2, \cdots, 10$, we put $T_i = \frac{i}{2}$. The coupon payments are given by

$$C = \$10,000 \times \left(\frac{.045}{2}\right) = \$225,$$

so we have $F_i = 225$ for $i = 1, 2, \cdots, 9$ and $F_{10} = 10,225$. The delivery time of the bond is $3.5 = T_7$ and the maturity of the bond is $5 = T_{10}$, so that we should use $j = 7$ and $N = 10$ in (3.2). Substituting in the numbers, we find that

$$\mathcal{F} = \frac{1}{.84866}\left((.829)225 + (.81153)225 + (.78353)10,225\right) = \$9,875.23.$$

3.3 Forward Currency Exchange

Corporations frequently enter into forward contracts for foreign currency. We have already discussed such contracts briefly in Chapter 1. We give an example here involving actual data from February 2005.

Example 3.4. Suppose that today's exchange rate for British pounds to US dollars is $E_d^p = 1.8861$ (i.e. it costs \$1.8861 to buy one pound). Suppose also that the 5-year effective spot rate in the United States for investments in dollars is $R_*^d(5) = 3.71\%$ and that the 5-year effective spot rate in Great Britain for investments in pounds is $R_*^p(5) = 4.55\%$. Find the arbitrage-free forward price F_d^p to purchase 1 pound for F_d^p dollars in 5 years.

It is convenient to put

$$d^d(T) = \frac{1}{(1 + R_*^d(T))^T}, \quad d^p(T) = \frac{1}{(1 + R_*^p(T))^T}.$$

To replicate the long position we need to set ourselves up to pay F_d^p dollars and receive 1 pound at $t = 5$. This can be accomplished by the following strategy: At $t = 0$, we borrow $F_d^p d^d(5)$ dollars until $t = 5$ and we invest $d^p(5)$ pounds (which will cost us $E_d^p d^p(5)$ dollars) until $t = 5$. The initial capital of this strategy in dollars is

$$-F_d^p d^d(5) + E_d^p d^p(5) = 0.$$

We find that

$$\begin{aligned} F_d^p &= \frac{E_d^p d^p(5)}{d^d(5)} \\ &= E_d^p \left(\frac{1 + R_*^d(5)}{1 + R_*^p(5)}\right)^5 \\ &= 1.8861 \left(\frac{1.0371}{1.0455}\right)^5 = 1.8115. \quad \square \end{aligned}$$

3.4 Forward Prices for Stocks

An important issue concerning the forward price of a stock is whether or not the stock pays dividends. Recall that shares of stock give their holders a claim to ownership of a fixed portion of a company. Some companies distribute some (or all of) their profits to the stockholders through cash payments called dividends. The amounts of dividend payments are generally not known very far in advance of the dates at which the payments will be made. We assume that when a cash dividend is paid, the price of a share of stock drops by the amount of the dividend payment. More precisely, suppose that the stock will pay a cash dividend of amount δ per share at time τ. Then,

$$S_{\tau_+} = S_{\tau_-} - \delta \tag{3.3}$$

where S_{τ_-} and S_{τ_+} are the stock prices just before and just after the dividend is paid, respectively.

Remark 3.5. Sometimes companies pay dividends in the form of additional shares of stock or property of other kinds (such as shares of stock in a subsidiary). There are important legal differences between dividends paid in cash and dividends paid as additional shares. Unfortunately, the language can be very confusing because in ordinary parlance it is common practice to refer to any kind of a dividend payment resulting from ownership of a stock as a “stock dividend”. In investment circles, the term “stock dividend” is reserved for dividends paid as additional shares. The term “dividend” without any qualification usually means a cash dividend. Throughout these notes, all dividends are assumed to be cash dividends.

Remark 3.6. Empirical evidence shows that in practice, stock prices typically only drop by approximately 80% of the amount of the dividend payment. Market frictions (in particular, taxation and transaction costs) play a crucial role in explaining this phenomenon. In some situations involving stock dividends, we will not need to know what happens to the stock price when the dividend is paid. However, in those situations when we do need an assumption about the behavior of the stock price when a dividend is paid we shall always assume that (3.3) holds.

3.4.1 Stocks that Do Not Pay Dividends

Consider a stock S that pays no dividends. We want to determine the arbitrage free forward price $\mathcal{F}$ for delivery of one share of stock at time $T > 0$. We can replicate the long forward position by purchasing one share of stock at $t = 0$ (and holding the stock until $t = T$) and borrowing $\mathcal{F}d(T)$ between $t = 0$ and $t = T$. The initial capital of this strategy is $S_0 - \mathcal{F}d(T)$. Since the initial capital must be zero, we have

$$\mathcal{F} = \frac{S_0}{d(T)} = S_0(1 + R_*(T))^T. \tag{3.4}$$

As with a zero-coupon bond, we can also arrive at (3.4) by considering a strategy X in which we borrow $\mathcal{F}d(T)$ until time T, buy one share of stock, and take a short forward position for delivery of the stock at time T. The strategy X is self-financing and has terminal capital

$$X_T = -\mathcal{F} + S_T + (\mathcal{F} - S_T) = 0.$$

Notice that the initial capital of this strategy is

$$X_0 = -\mathcal{F}d(T) + S_0.$$

Since the strategy is self-financing and has zero terminal capital, the initial capital must also be zero and this leads to (3.4).

Example 3.7. Consider a stock S that pays no dividends and has initial price $S_0 =$ \$60 per share. If $R_*(1) = .05$ then the forward price (at time 0) for delivery of one share of stock at $T = 1$ is

$$\mathcal{F} = \$60(1.05) = \$63.$$

Remark 3.8. Equation (3.4) actually holds for any asset that can be shorted as well as purchased, is not subject to spoilage, can be stored at no cost, and for which there are no benefits associated with holding the asset that are not available to an agent holding a long position on a forward contract.

Remark 3.9. The strategy used to derive (3.4) does not apply directly to situations when the stock pays dividends at times between 0 and T. The reason for this is that if we purchase the stock at $t = 0$ and hold it until $t = T$, we will receive dividend payments that the holder of a long forward position will not receive. There are two special situations in which it is possible to overcome this difficulty with simple modifications of the replicating strategy:
(i) When the dividend payment times and amounts are known at $t = 0$; and
(ii) When the number of dividend payments between $t = 0$ and $t = T$ is known at $t = 0$ and the amount of each dividend payment is a known fraction of the price of the stock just before the dividend payment.

3.4.2 Stocks that Pay Known Dividends

If the dividend payment times and (dollar) amounts are known at time 0, then it is straightforward to replicate forward contracts. Let us assume that τ_i, $i = 1, 2, \cdots, N$ and T are specified at time 0 and satisfy

$$0 < \tau_1 < \tau_2 < \cdots < \tau_N < T.$$

Consider a stock S, with initial price S_0, that will pay a dividend of amount δ_i per share at each of the times τ_i, $i = 1, 2, \cdots, N$, where the amounts δ_i are specified at time 0. (It is assumed that no other dividends will be paid during the time interval $[0, T]$.) Let $\mathcal{F}$ denote the forward price (at time 0) for delivery of one share of stock at time T. An agent who purchases the stock at time 0 and holds it until time T will receive dividend payments that the holder of a long forward position will not receive.

We can replicate the long forward position by borrowing $\mathcal{F}d(T)$, purchasing the stock, and shorting a security having the same cash flows as the dividend payments. More precisely, let X be the strategy in which $\mathcal{F}d(T)$ is borrowed between time 0 and time T, one share stock is purchased at time 0 and held until time T and one security with fixed payments $(\delta_i, \tau_i)_{1 \le i \le N}$ is sold short at time 0. (Notice that the dividend payments received by the implementer of this strategy match exactly the payments that this agent is obliged to make by virtue of the short sale.) It is clear that X is a replicating strategy for the long forward position. Observe that

$$X_0 = -\mathcal{F}d(T) + S_0 - \sum_{i=1}^{N} d(\tau_i)\delta_i.$$

Since we must have $X_0 = 0$, it follows that

$$\mathcal{F} = \frac{1}{d(T)}\left(S_0 - \sum_{i=1}^{N} d(\tau_i)\delta_i\right) = \left(S_0 - \sum_{i=1}^{N} d(\tau_i)\delta_i\right)(1 + R_*(T))^T. \tag{3.5}$$

Notice that this is exactly the same as the forward price for one share of a stock $\hat{S}$ that does not pay dividends and has initial price $\hat{S}_0$ that is lower than S_0 by precisely the net present value of all of the dividend payments made by S between times 0 and T.

Example 3.10. Today's date is $t = 0$. Consider a stock S with initial price $S_0 = \$60$ per share. Suppose that the stock will pay a dividend of \$.85 per share in three months and in nine months (and no other dividends will be paid during the year). Find the forward price (at $t = 0$) for delivery of one share of stock at $T = 1$ assuming that $R_*(.25) = R_*(.75) = R_*(1) = .05$.

Observe first that

$$d(.25) = \frac{1}{(1.05)^{.25}} = .98788, \quad d(.75) = \frac{1}{(1.05)^{.75}} = .96407.$$

Substituting the numbers into (3.5), we find that

$$\mathcal{F} = [60 - (.98788)(.85) - (.96407)(.85)]\,(1.05) = \$61.26.$$

The reader should compare this result with Example 3.7.

3.4.3 Stocks with a Known Dividend Yield

We now consider a stock S with initial price S_0 that will make dividend payments that are a known fraction of the stock price just prior to the dividend payment. We begin by considering the case when a single dividend payment δ will be made at some time $\tau > 0$ and the amount of this payment will be

$$\delta = \alpha S_{\tau_-},$$

where α is known at time 0 and satisfies $0 < \alpha < 1$. We are interested in determining the forward price (at time 0) for delivery of one share of the stock at a specified time $T > \tau$. An important warning is in order here: One cannot cancel out the dividend payment in a replicating strategy for the long position by borrowing $\alpha S_{\tau_-} d(\tau)$ between $t = 0$ and $t = \tau$ because S_{τ_-} is not known prior to time τ and strategies are not allowed to look into the future! The appropriate way to deal with the dividend in this case is to reinvest it.

Suppose that at time 0 we purchase Δ shares of stock, and as soon as we receive the dividend payment at time τ we use the precise amount of the payment to purchase more stock. Let us see how many shares of stock we will have at time τ_+. The cash amount of the dividend payment will be $\alpha \Delta S_{\tau_-}$. We want to use this amount of money to purchase more stock at the adjusted share price

$$S_{\tau_+} = S_{\tau_-} - \delta = S_{\tau_-} - \alpha S_{\tau_-} = (1-\alpha) S_{\tau_-}.$$

Let x be the number of additional shares purchased. Since x is equal to the dividend payment divided by the the adjusted share price, we have

$$x = \frac{\Delta \delta}{S_{\tau_+}} = \frac{\Delta \alpha S_{\tau_-}}{S_{\tau_+}} = \frac{\Delta \alpha S_{\tau_-}}{(1-\alpha) S_{\tau_-}} = \frac{\Delta \alpha}{1-\alpha}.$$

The dividend payment is therefore enough to purchase precisely $x = \Delta\alpha/(1-\alpha)$ additional shares of stock. The total number of shares of stock in our portfolio after reinvestment of the dividend will be

$$\Delta + x = \Delta + \frac{\Delta \alpha}{1-\alpha} = \frac{\Delta}{1-\alpha}.$$

To replicate the long forward position we need to wind up with one share of stock, i.e. we need

$$\frac{\Delta}{1-\alpha} = 1,$$

so we should choose $\Delta = 1 - \alpha$.

Remark 3.11. The argument above shows that if a stock pays a dividend that is α times the pre-dividend share price and we reinvest this dividend in the stock, we will wind up with $1/(1-\alpha)$ times the number of shares we had before the dividend payment. This is a handy fact to know.

We can now give a replicating strategy for the long forward position. At $t = 0$, we purchase $1 - \alpha$ shares to stock and borrow $\mathcal{F} d(T)$ from the bank until time T; when the dividend payment is received at time τ, we use this money to purchase additional stock. At time T we will hold one share of stock and owe $\mathcal{F}$ to the bank.

The initial capital of this strategy is

$$X_0 = (1-\alpha) S_0 - \mathcal{F} d(T).$$

Since $X_0 = 0$ we conclude that

$$\mathcal{F} = \frac{(1-\alpha) S_0}{d(T)} = (1-\alpha) S_0 (1 + R_*(T))^T.$$

It is straightforward to modify this strategy to account for multiple dividend payments, each of which is a known fraction of the stock price. For simplicity we treat the case when all dividends are exactly the same fraction of the stock price. More precisely, let $T > 0$ be a given delivery time and assume that there are times τ_i, $i = 1, 2, \cdots, N$ satisfying

$$0 < \tau_1 < \tau_2 < \cdots < \tau_N < T,$$

and a prescribed constant α, with $0 < \alpha < 1$, such that the stock pays a cash dividend αS_{τ_-} at each of the times $\tau = \tau_i$, $i = 1, 2, \cdots, N$. (We assume that these represent the only dividend payments between times 0 and T.) Notice that if we buy and hold stock, and reinvest each dividend payment, then the number of shares of stock we have will be multiplied by $(1 - \alpha)^{-1}$ each time that a dividend is reinvested. We can replicate a long forward position by implementing the following strategy:

(i) At $t = 0$, we purchase $(1-\alpha)^N$ shares of stock and borrow $\mathcal{F}d(T)$ between $t = 0$ and $t = T$.

(ii) At each of the times τ_i, the dividend payment is reinvested to buy additional stock. (Notice that just after the i^{th} dividend payment is reinvested, we will hold precisely $(1 - \alpha)^{N-i}$ shares of stock.)

The initial capital of this strategy is $X_0 = (1 - \alpha)^N S_0 - \mathcal{F}d(T)$. Since we must have $X_0 = 0$, we conclude that

$$\mathcal{F} = \frac{(1-\alpha)^N S_0}{d(T)} = (1-\alpha)^N S_0 (1 + R_*(T))^T. \tag{3.6}$$

Notice that we do not need to know the discount factors corresponding to the dividend payment times τ_i. Observe also that the times τ_i do not need to be known with certainty; we only need to know the precise number of dividend payments that will be made between $t = 0$ and $t = T$.

Example 3.12. Today's date is $t = 0$. Consider a stock S with initial price $S_0 = \$60$ per share. Suppose that the stock will pay a dividend in three months and in nine months (and no other dividends will be paid during the year). Each dividend payment will be .015 times the price of the stock just prior to the payment. Find the forward price (at $t = 0$) for delivery of one share of stock at $T = 1$ assuming that $R_*(1) = .05$.

Substituting the numbers into (3.6) we find that

$$\mathcal{F} = \$60(1 - .015)^2(1.05) = \$61.12.$$

You should compare this result with Examples 3.7 and 3.10.

3.5 Forward Prices for Commodities

It is problematic to compute theoretical forward prices for commodities. A number of difficulties materialize if we try to apply the replicating strategies used above. In particular, many commodities are subject to spoilage or degradation of quality. For such commodities, the buy-and-hold strategy obviously runs into difficulties. Moreover, there are storage costs associated with many commodities. Even if the commodity is not subject to spoilage and we know the storage costs exactly, there are additional difficulties. In particular, there are advantages associated with holding some commodities (such as the ability to keep a production process running) that are not available to an agent with a long forward position. The benefits associated with having a physical commodity on hand are generally referred to as the *convenience yield* of the commodity. It is extremely difficult to determine a numerical value of the convience yield *a priori*. Finally, most commodities cannot be sold short in real-world markets. Although the impossibility of short sales may not seem to present an obvious difficulty, it does indeed present an extremely serious difficulty. The result that "if two different self-financing strategies have the same terminal capital then they must also have the same initial capital" is based on the assumption that short sales are possible.

The current market price (for purchase "on the spot") of a commodity is called the *spot price*. There are mathematical treatments that explain the relationships between spot prices and forward prices of commodities by accounting for storage fees and convenience yields. We shall not attempt to give such an analysis here. However, we shall consider two examples that illustrate these points. The interested reader is referred to [H] and [L] for much more information.

Example 3.13 (Forward Price of Platinum)**.** This example is based on actual financial data from November 2001.

Suppose that the spot price of platinum is $S_0 = \$456$ per ounce, the effective spot interest rate for maturity 6 months is $R_*(.5) = 1.8\%$ and that the forward price for delivery of platinum at $T = .5$ is $\mathcal{F} = \$440$ per ounce. It looks like we may be able to make an arbitrage profit by selling platinum short and taking the long position on a forward contract for delivery of platinum at $T = .5$. Let's see how this might work for $N = 10,000$ ounces of platinum.

Consider the following strategy:

(i) At $t = 0$, we sell short N ounces of platinum for $NS_0 = \$4,560,000$, invest the money for 6 months at the effective rate $R_*(.5) = 1.8\%$, and take the long position on a forward contract for delivery of N ounces of platinum in 6 months.

(ii) At $t = .5$, we withdraw $NS_0(1 + R_*(.5))^{\frac{1}{2}} = \$4,600,856.96$ from the bank, buy N ounces of platinum for $N\mathcal{F} = \$4,400,000$, and return the borrowed platinum.

The initial capital of this strategy is zero and the terminal capital is

$$\$4,600,856.96 - \$4,400,000 = \$200,856.96,$$

an apparent arbitrage. The catch here is that it is practically impossible to find anyone who will lend platinum at no charge (i.e., we cannot sell platinum short). The arbitrage opportunity will disappear if the cost to borrow the platinum (payable at maturity) is greater than or equal to \$200,856.96.

It is interesting to observe that someone who owns 10,000 ounces of platinum at $t = 0$, and has no need for the platinum between $t = 0$ and $t = .5$ could implement the strategy above and make a profit of \$200,856.96. There are, however, advantages associated with holding platinum that may dissuade someone who currently owns it from implementing the strategy. Indeed, one out of five goods maufactured today either contains platinum or is produced using platinum. The fact that such a "mis-match" in pricing occurs in practice indicates that holders of platinum must feel that there are advantages to holding this commodity that are not realized by holding a long forward position; otherwise holders of platinum would flock to sell it and take long forward positions. Such activity would cause prices in the platinum market to adjust so that this profit opportunity would disappear.

Example 3.14 (Forward Price of Heating Oil)**.** This example is based on actual financial data[1] from February 2002.

Suppose that the spot price of heating oil is $S_0 = \$.52$ per gallon, the effective spot interest rate for maturity one year is $R_*(1) = 2.3\%$ and the forward price for delivery of heating oil at $T = 1$ is $\mathcal{F} = \$.59$ per gallon. It looks like we may be able to make an arbitrage profit by buying oil now and taking the short position on a forward contract for delivery of oil in one year. Let's see how this might work for $N = 10,000,000$ gallons of oil.

Consider the following strategy:

(i) At time $t = 0$, we borrow $NS_0 = \$5,200,000$ for one year at the effective rate $R_*(1) = 2.3\%$, buy N gallons of oil for $NS_0 = \$5,200,000$ and take the short position on a forward contract for delivery of N gallons of oil in one year.

(ii) At time $T = 1$, we deliver the oil and receive $NF = \$5,900,000$. Then we give the bank $NS_0(1 + R_*(1)) = \$5,319,600$ to repay the loan.

This strategy has zero initial capital and the terminal capital is

$$\$5,900,000 - \$5,319,600 = \$580,400,$$

an apparent arbitrage. The "catch" here is that we would need to store the oil between $t = 0$ and $t = T$. In practice, it is almost impossible to arrange free storage of oil. The arbitrage opportunity will disappear if the time 1 value of the storage cost is greater than or equal to \$580,400. We should expect the storage cost to exceed this number because heating oil has a significant convenience yield. (Indeed, it is much more comforting to have a full oil tank in the winter than to have a long forward position for delivery of oil in the spring.) See Remark 3.15 below.

[1] Because there are significant transportation and storage costs associated with heating oil, prices are regional. The prices in this example are from the Northeast region of the United States.

Remark 3.15. Consider a commodity that can be stored safely between times 0 and T for a total time 0 cost per unit of $\mathcal{C}^s_{0,T}$. (In other words, $\mathcal{C}^s_{0,T}$ is the net present value at time 0 of the total cost to store one unit of the commodity between times 0 and T.) We should expect the spot price S_0 and the forward price $\mathcal{F}_{0,T}$ to obey

$$S_0 + \mathcal{C}^s_{0,T} \geq d(T)\mathcal{F}_{0,T} \tag{3.7}$$

where $d(T)$ is the discount factor for time T. The reason is that having the commodity in storage should be worth at least much as a long forward position. Indeed, if (3.7) were voilated an agent could buy and store the commodity and take a short forward position and be guaranteed to make an arbitrage profit. In order to establish the opposite inequality by arbitrage considerations, it would be necessary that the commodity could be shorted in such a way that the agent who shorts it would receive the storage costs (or we would need to know that there are many people who are holding the commodity solely for investment purposes).

When we encounter forward contracts for commodities, we shall assume that the forward prices are determined by a market (rather than try to give a theoretical justification for the forward prices.) The following example illustrates the fact that at times after initiation, a position on a forward contract will generally have a nonzero value.

Example 3.16. Suppose that today's date is March 1, 2017 and that we consider this date to be $t = 0$. Suppose that some time ago you took the long position on a forward contract for 50,000 pounds of cotton to be delivered on March 1, 2018 at the forward price of \$.70 per pound. Forward contracts being issued today for delivery of cotton on March 1, 2018 are being made with a forward price of \$.75 per pound. Given that the effective spot rate for maturity 1 year is $R_*(1) = 5\%$, determine the arbitrage-free value today of your position on the old contract.

A quick analysis of the situation shows that with the old contract you will be better off in one year by $(50,000) \times (\$.05) = \$2,500$ than a party who takes the long position on a forward contract today for delivery of 50,000 pounds of cotton in one year. The arbitrage-free value today of a long position in the new contract is zero. Therefore, the arbitrage-free value today of your position on the old contract is

$$\begin{aligned} \$2,500d(1) &= \frac{\$2,500}{1.05} \\ &= \$2,380.95. \end{aligned}$$

For practice in replication, let us replicate the long position in the old contract and check that the initial capital agrees with the amount above. Consider the following strategy:

At $t = 0$, we take the long position on a forward contract for 50,000 pounds of cotton to be delivered at $T = 1$ at the forward price \$.75 per pound. (This will set us up to receive 50,000 pounds of cotton, but we will need to pay too much at $t = 1$). At $t = 0$ we should also invest $(50,000) \times (\$.05)d(1)$ to make up for the difference between the two delivery prices. The initial capital of this strategy is

$$\frac{(50,000) \times (\$.05)}{1.05} = \$2,380.95$$

Remark 3.17. The example above can be generalized easily. Suppose that $\tau < 0 < T$. The arbitrage-free price at $t = 0$ of a long position on a previously issued forward contract for delivery of an asset at time T at the delivery price $\mathcal{F}_{\tau,T}$ is

$$(\mathcal{F}_{0,T} - \mathcal{F}_{\tau,T})d(T),$$

where $\mathcal{F}_{0,T}$ is the forward price at time 0 for delivery of the same asset at time T.

3.6 Some Remarks on Futures Contracts

The idea of "locking in" a price now for a sale at a future date can be extremely useful for both parties. However, there are certain practical difficulties associated with forward contracts. As mentioned previously, it may be difficult for an individual wanting to take a position on a forward contract to find a party who wants the counterposition for the same quantity and delivery date of the asset. Another serious concern is that one of the parties may be unable to fulfill their obligation on the delivery date. Instead of forward contracts, it is much more common in practice to use *futures contracts*, which also concern delivery of a prescribed amount of a particular asset at a specified future date.

Futures contracts are issued by an organized exchange, which helps to standardize many features of contracts and to define universal prices. A futures contract is an agreement to buy or sell a specified amount of an asset at a prescribed future date. A party wanting to take a position on such a contract does so with the exchange serving as the counterparty. Nothing is paid initially to take a position on a futures contract.

Before entering a futures contract an investor must open a cash account called a *margin account* with the exchange. The balance of this account will be updated on a daily basis, depending on the change in the futures price for delivery of the same asset at the same delivery time. To illustrate this, let us consider futures contracts for delivery of gold in December 2017. (In contrast with forward contracts, futures contracts often specify a delivery month rather than a particular day.) Suppose that on March 1, 2017, the futures price for delivery of gold in December is \$1,250 per ounce. Interested parties can take a long or short position on a contract for delivery in December at \$1,250 per ounce, with the exchange serving as the counterparty. Suppose that on March 2, the futures price for delivery of gold in December goes up to \$1,253 per ounce. Then the accounts of investors who already hold long positions will be credited with \$3 per ounce and the accounts of investors holding short positions will be charged \$3 per ounce. All contracts will then have their delivery prices revised to \$1,253 per ounce. If, on March 3, the futures price for delivery of gold in December drops to \$1,252 per ounce then the accounts of investors who already hold a long position will be charged \$1 per ounce and the accounts of investors who already hold a short position will be credited with \$1 per ounce; all contracts will then have their delivery prices adjusted to \$1,252 per ounce $\cdots$ and so on. This removes the risk of default and it also makes the book-keeping much easier - all contracts for December delivery have the same (unit) futures price. In particular, there is no need to keep track of the initiation dates of futures contracts.

Investors can close out their positions – at no cost – before the delivery time if desired. (Since it costs nothing to take a futures position, an existing position can be closed out by simply taking the opposite position.) In fact, this is what happens in practice in the majority of cases. Most investors who actually need to buy or sell a commodity close out their accounts early and simply deal with their usual suppliers or customers. If the balance of an investor's account becomes too low at some point during the life of the contract, the investor will receive a *margin call* requiring her to either deposit more money or close out the futures position.

Remark 3.18. Let $\mathcal{F}^{for}_{\tau,T}$ and $\mathcal{F}^{fut}_{\tau,T}$ denote the forward and futures prices at time τ for delivery of one unit of the same asset at time T. If, at time τ, the interest rates that will prevail at all times between τ and T are known with certainty then we can conclude that $\mathcal{F}^{for}_{\tau,T} = \mathcal{F}^{fut}_{\tau,T}$. If the interest rates that will prevail between times τ and T are unknown, but it is assumed that interest rate fluctuations will be small then $\mathcal{F}^{for}_{\tau,T} \approx \mathcal{F}^{fut}_{\tau,T}$.

3.7 Put and Call Basics

Investors who want to buy or sell an asset at a future date T may decide to enter into a forward contract. They pay nothing to take a position on the contract, but they are obligated to go through with the sale at time T even if the current market price of the asset is such that they will lose money in the transaction. Options give their holders the right, but not the obligation, to make a transaction at a future date. We recall some terminology pertaining to options.

Definition 3.19. (i) A *European put option* on an asset gives its holder the right to sell a specified amount of the asset for a specified price K at a specified future date T.

(ii) A *European call option* on an asset gives its holder the right to purchase a specified amount of the asset for a specified price K at a specified future date T.

(iii) For both types of options the specified price K is called the *strike price*.

(iv) For both types of options the specified date T is called the *expiration date*, *exercise date*, or *maturity* of the option.

As mentioned in Chapter 1, options are generally cash settled. In other words, the holder does not actually buy or sell the asset at maturity. If the transaction allowed by the option is favorable to the option holder, then the seller (or writer) of the option simply pays the option holder the difference between the asset price and the strike price in cash. Since the holder of an option will not pay any money at time T, and has the possibility of receiving money at that time, the holder will have to pay for the option. The initial price of an option is generally referred to as the *premium*. In practice, one can purchase options on a given asset with a variety of expiration dates and strike prices. In particular, in contrast with forward contracts (where there is only one forward price for a given maturity), options having the same maturity are available with a number of different strike prices.

Remark 3.20. *American options* (puts and calls) allow their holders to exercise them (i.e. sell or buy the asset at the strike price K) at any time up to and including the expiration date T. American options must cost at least as much as their European counterparts (having the same K and T) because they give their holders all of the advantages of a European option as well as additional flexibility.

Remark 3.21. For both American and European options, it is clear that if the expiration date is held fixed and the strike price is increased, then the price of a put option (on a given asset) must increase and the price of a call option (on a given asset) must decrease. This can be seen by examining the effect of the strike price on the payoff. (Securities with higher payoffs must have higher prices.) For American puts and calls (on a given asset), the price must increase if the strike is held fixed and the maturity is increased. This issue will be discussed in Section 3.9. In general, there is no simple rule regarding the effect that increased maturity has on prices of European options. We can however, make a definite statement for European calls on stocks that do not pay dividends. (See Remark 3.33)

Remark 3.22. The vast majority of exchange-traded options are American. On the other hand, most OTC[2] options are European. European options are generally much simpler to treat mathematically and we will focus on them first. Some of the results that we obtain about European options will help us analyze American options. In particular, assuming that interest rates are positive, we shall show that for stocks that do not pay dividends the price of an American call is the same as the price of a European call having the same expiration date and strike price. (In general, an American put is worth strictly more than a European put having the same expiration date and strike price.)

Remark 3.23. Years ago, it was common practice with OTC stock options to adjust the strike price each time the underlying stock paid a cash dividend. This practice is no longer followed. Unless stated explicitly otherwise, you should assume that the strike price of an option remains constant throughout the life of the option. Of course, if a stock split[3] occurs during the life of an option the strike price and the number of

[2]Recall from Chapter 1 that "OTC" is an abbreviation for "over the counter". In the OTC market, the parties interact directly with one another to create transactions with terms that are tailored to the individual needs of the client. Exchange-traded options are offered with a limited variety of expiration dates and strike prices. As one would expect, transaction costs are generally much lower for exchange-traded options.

[3]A stock split occurs when the company feels that price per share is sufficiently high that individual investors may shy away from buying the stock. In a 2-for-1 split (most common), the price per share is cut in half and the number of outstanding shares is doubled. In a 3-for 2 split (also common) the price per share is multiplied by 2/3 and the number of outstanding shares is multiplied by 3/2.

shares must be adjusted accordingly. Throughout these notes, we assume that there are no stock splits.

Many different types of payoffs can be achieved by combining different options and holding (or shorting) the underlying asset. Some of the more common possibilities are illustrated in the exercises. An agent who writes (or issues) a call option without holding the underlying asset is said to have written a *naked call*. Of course, writing a naked call creates a very significant potential liability because the price of the underlying asset may increase dramatically during the life of the option. An agent who writes a call option and also holds one unit of the underlying asset is said to have written a *covered call*. In such a case, the writer can be sure to cover the payoff of the call by selling the asset[4]. The liability created by writing a put option can always be covered by having a cash amount equal to the strike price available. As discussed in the beginning of Chapter 1, put options can be used to provide a kind of insurance to someone who holds an asset. A put option that is held together with one unit of the underlying asset is called a *protective put*; the value of this combination is greater than or equal to the strike price at times when the put can be exercised.

In contrast with forward contracts, put and call options on a stock cannot be replicated withou having a specific model for the evolution of the stock price. However, even without a specific model for the evolution of the stock price, we can establish some useful inequalities for put and call prices and a very powerful expression called *put-call parity* that relates the prices of European puts and calls to one another and the forward price. The remainder of this chapter gives a brief development of option-pricing results that are independent of a model for the evolution of the price of the underlying asset. The fundamental paper of Merton [M] contains a wealth of information on this topic. Another excellent source for this material is the treatise of Hull [H]. As we develop models for stock prices, we will revisit the topic of option prices. We begin by deriving the put-call parity formula.

3.8 Put-Call Parity

We assume that the present time is $t = 0$. Let $T > 0$ and $K > 0$ be given. Consider a European put option P and a European call option C on an asset S. We assume that the strike price is K and that the expiration date is T for both options. (We also assume that the puts and calls can be purchased and sold short in any amounts.) Let P_T, C_T, and S_T denote the values at time T of the put, call, and the asset, respectively. Observe that

$$C_T = \max\{S_T - K, 0\}$$

$$P_T = \max\{K - S_T, 0\}.$$

[4]Naked and covered calls represent extremes. It is often desirable to hedge the short position on a call option by holding some fractional number of units of the underlying asset for each option. However, to set up this type of hedge properly, we need a model for the evolution of the price of the underlying asset.

It is convenient to introduce the notation $\alpha^+ = \max\{\alpha, 0\}$, so that we can rewrite the equations above as

$$C_T = (S_T - K)^+$$
$$P_T = (K - S_T)^+.$$

Suppose that the forward price for delivery of the asset at time T is $\mathcal{F}$. Recall that the value at time T of the long position on the forward contract is

$$S_T - \mathcal{F}.$$

Consider the following self-financing strategy: At $t = 0$, we

(a) Take the long position on a forward contract for delivery of the asset at time T.

(b) Sell one call option with strike price K and exercise date T.

(c) Buy one put option with strike price K and exercise date T.

We hold our position until time T. The terminal capital X_T of this strategy is

$$X_T = S_T - \mathcal{F} - (S_T - K)^+ + (K - S_T)^+.$$

Observe that if $S_T \geq K$ then $(S_T - K)^+ = S_T - K$ and $(K - S_T)^+ = 0$. On the other hand if $S_T < K$ then $(S_T - K)^+ = 0$ and $(K - S_T)^+ = K - S_T$. It follows that the value of our position at time T is simply

$$X_T = K - \mathcal{F}.$$

Another self-financing strategy that has the same terminal capital is to invest $d(T)(K - \mathcal{F})$ between $t = 0$ and $t = T$, where

$$d(T) = \frac{1}{(1 + R_*(T))^T}$$

is the discount factor for maturity T. Assuming that there is no arbitrage, these two strategies must have the same initial capital, i.e.

$$P_0 - C_0 = d(T)(K - \mathcal{F}), \tag{3.8}$$

where P_0 and C_0 are the prices of the put and the call respectively. Equation (3.8) is known as put-call parity. If the asset is a stock that does not pay dividends we can rearrange (3.8) in a useful way by substitutting (3.4) into (3.8). More precisely, we have the following remark.

Remark 3.24. If S is a stock that does not pay dividends then put-call parity takes the form

$$P_0 - C_0 + S_0 = d(T)K. \tag{3.9}$$

Remark 3.25. In textbooks, put-call parity is often stated in the form (3.9), together with a caveat that this relation holds only for assets satisfying the conditions described in Remark 3.8. (In particular, (3.9) is not valid for stocks paying dividends between times 0 and T.) Equation (3.8) holds in much greater generality. Our derivation of (3.8) requires only that long and short forward positions are possible, and that puts and calls can be shorted as well as purchased. (It does not matter whether or not the underlying asset itself can be shorted, pays dividends, or has storage costs, etc. For such assets, even though it may not be possible to compute theoretical forward prices, or option prices, these prices must obey (3.8).)

Example 3.26. Suppose that the forward price for delivery of gold in 2 months is $\mathcal{F} = \$1,227.64$ per ounce and that European call options on one ounce of gold with strike price \$1,170 and exercise date 2 months are trading at $C_0 = \$78.66$. Compute the arbitrage-free price of a European put option on one ounce of gold with the same strike price and exercise date, assuming that $R_*(\frac{1}{6}) = 5.12\%$.

We first compute

$$d\left(\frac{1}{6}\right) = \frac{1}{(1.0512)^{1/6}} = .99171.$$

Using (3.8) we find that

$$\begin{aligned} P_0 &= C_0 + d(T)(K - \mathcal{F}) \\ &= \$78.66 + .99171(\$1,170 - \$1,227.64) \\ &\approx \$21.50 \end{aligned}$$

Example 3.27. Today's date is $t = 0$. Consider a stock S with initial price $S_0 = \$60$ per share. Suppose that the stock will pay a dividend in three months and in nine months (and no other dividends will be paid during the year). Each dividend payment will be .015 times the price of the stock just prior to the payment. European put options on the stock with expiration $T = 1$ and strike price $K = \$65$ are currently trading at $P_0 = \$6.64$. Assuming that $R_*(1) = .05$, find the price of a European call option with expiration $T = 1$ and strike price $K = \$65$.

The forward price for delivery of one share at $T = 1$ was found to be $\mathcal{F} = \$61.12$ in Example 3.12. Using (3.8) we find that

$$\begin{aligned} C_0 &= P_0 - d(T)(K - \mathcal{F}) \\ &= \$6.64 - \tfrac{1}{1.05}(\$65 - \$61.12) \\ &\approx \$2.94 \end{aligned}$$

It is very important to understand the derivation of put-call parity. The same ideas can be used to obtain many other important pricing relations. We record below some basic inequalities for European put and call prices for stocks that do not pay dividends.

Remark 3.28. Consider a single stock that pays no dividends and has current price S_0 per share. Let $T > 0$ and $K > 0$ be given and let P and C denote European put and call options on the stock with exercise date T and strike price K. Then we must have

$$(S_0 - d(T)K)^+ \leq C_0 \leq S_0, \tag{3.10}$$

$$(d(T)K - S_0)^+ \leq P_0 \leq Kd(T). \tag{3.11}$$

The proof of this remark and extensions to dividend-paying stocks are left to Exercise 11 of this chapter.

It is possible to extend put-call parity and Remark 3.23 to prices at times $t \in (0, T)$. Such extensions will, of course, involve the spot rate $\mathcal{R}_{t,T}$. For simplicity, we record the relevant results below for the case of an ideal money market with constant effective rate R and a stock that pays no dividends. The only new observation needed for this remark is the fact that in the presence of an ideal money market with constant effective rate R, the appropriate factor for discounting from time T back to time t is $(1 + R)^{t-T}$.

Remark 3.29. Assume that there is an ideal money market with constant effective rate R. Consider a single stock that pays no dividends and has current price S_0. Let $T > 0$ and $K > 0$ be given and let P and C denote European put and call options on the stock with exercise date T and strike price K. Let S_t, P_t, and C_t denote the prices of the stock, put, and call at time t. Then for all $t \in [0, T]$ we have

$$P_t - C_t + S_t = (1 + R)^{t-T}K, \tag{3.12}$$

$$(S_t - (1 + r)^{t-T}K)^+ \leq C_t \leq S_t, \tag{3.13}$$

$$((1 + R)^{t-T}K - S_t)^+ \leq P_t \leq K(1 + R)^{t-T}. \tag{3.14}$$

Example 3.30 (Chooser Options)**.** A (European-style) *chooser option* V on a stock S is described by two given times τ and T with $0 < \tau < T$ and a strike price $K > 0$. At time τ the holder of the option chooses between a European call C and a European put P on the stock. The put and the call both have exercise date T and strike price K. We assume that the holder of V will choose whichever option is more valuable at time τ. We also assume that the stock does not pay dividends and that there is an ideal money market with constant effective rate R.

Using Remark 3.29, we know that

$$P_\tau = C_\tau - S_\tau + (1 + R)^{\tau-T}K. \tag{3.15}$$

It follows that the value of V at time τ is given by

$$V_\tau = \max\left\{C_\tau, C_\tau - S_\tau + (1 + R)^{\tau-T}K\right\} = C_\tau + \max\left\{0, (1 + R)^{\tau-T}K - S_\tau\right\} \tag{3.16}$$

If we let $\hat{P}$ denote a European put option on S with exercise date τ and strike price $K(1+R)^{\tau-T}$, then (3.16) becomes

$$V_\tau = C_\tau + \hat{P}_\tau,$$

and it follows immediately that

$$V_0 = C_0 + \hat{P}_0.$$

Remark 3.31. Chooser options are also called *as-you-like-it* options. There are much more complicated types of chooser options. For example the expiration dates and/or strikes can be different for the put and the call. Also the underlying put and call could be American rather than European. In Example 3.30, it is not necessary that interest rates be constant; however, it is crucial that the interest rate $\mathcal{R}_{\tau,T}$ be known at time 0, because the strike price for $\hat{P}$ depends on this interest rate.

3.9 American Options

Consider an asset S, whose value at each time t will be denoted by S_t. Assume that the present time is $t = 0$. An American option on S is described by an expiration date (or maturity) $T > 0$ and a nonnegative function g called the *intrinsic value function* of the option. At any time τ with $0 \le \tau \le T$ the holder can decide to *exercise* the option and receive $g(S_\tau)$ at that time. (The holder can only exercise the option once.) American options are much more difficult to analyze mathematically than options of European type, because the exercise time τ is not know when the option is sold and because different holders may choose different exercise times. In this section we give a brief discussion of American puts and calls. We refer to $g(S_t)$ as the intrinsic value of the option at time t.

For an American call option with strike price K the intrinsic value function is given by

$$g(x) = (x-K)^+,$$

whereas for an American put option we have

$$g(x) = (K-x)^+.$$

For a given intrinsic value function g, the price of an American option on S increases as the maturity increases. More precisely, for each $T > 0$, let $V^{A,T}$ denote the American option on S with intrinsic value function g and maturity T. The price of this option at time t will be denoted by $V_t^{A,T}$. Then for all t, T_1 and T_2 we have

$$V_t^{A,T_1} \le V_t^{A,T_2}. \tag{3.17}$$

To prove (3.17) we consider a portfolio in which one option V^{A,T_2} is purchased and one option V^{A,T_1} is sold short at time t. If the option V^{A,T_1} is exercised by its holder at some time $\tau \in [t, T_1]$, then the agent implementing the strategy can exercise the option V^{A,T_2} to make the required payment. It follows that the initial capital of this strategy must be nonnegative, and this yields (3.17).

Remark 3.32. Consider an American option with intrinsic value function g and maturity T and let $t \in [0, T)$ be given. It is important to understand that in general the intrinsic value $g(S_t)$ will be different from the price V_t. We must have

$$V_t \geq g(S_t).$$

In other words, the price must be at least as high as the value of immediate exercise. If $V_t > g(S_t)$, then the option should not be exercised at t. (The option holder could sell the option for more than she would get by exercising it.)

3.9.1 Prices of American Versus European Options

It seems clear that an American option should be worth at least as much as the corresponding European option. It is not difficult to prove this statement using replication, but we must be careful in analyzing portfolios in which American options have been sold short, because we cannot control when they will be exercised. Let g be a given nonnegative function and let $T > 0$ be given. Let V^A denote the American option with intrinsic value g and expiration date T and let V^E denote the corresponding European option. We treat V^E as a security that makes a single payment of $V_T^E = g(S_T)$ to its holder at time T. (Of course, it could happen that $g(S_T) = 0$ so holders of V^E receive no money.)

Let $T > 0$ and a nonnegative function g be given. Let us denote by V^E the European option that pays its holder $g(S_T)$ and V^A denote the American option with instrinsic value $g(S_t)$ and expiration date T. For each $t \in [0, T]$ let V_t^E and V_t^A denote the prices of these options at time t. (V_t^A is the price of the American option at time t, assuming that the option has not been exercised yet; once the option has been exercised, it no longer has any value.) We shall show that

$$V_t^A \geq V_t^E \tag{3.18}$$

for all $t \in [0, T]$.

Fix $t \in [0, T]$ and suppose that $V_t^A < V_t^E$. Let $\alpha = V_t^E - V_t^A$ and notice that $\alpha > 0$. Consider an agent who at time t sells V^E, buys V^A and invests α in the bank until time T. Suppose that the agent decides to store the American option in a drawer and exercise it at time T. The agent's strategy is self-financing and has zero initial capital, i.e. $X_t = 0$. The terminal capital of this strategy is

$$X_T = \alpha(1 + \mathcal{R}_{t,T})^{T-t} > 0,$$

and this is an arbitrage. Therefore it is impossible to have $V_t^A < V_t^E$. We conclude that (3.18) must hold. Notice that we cannot use this argument to obtain the reverse inequality, because an agent who sells the American option cannot control when the holder of the option will exercise it.

3.9.2 Early Exercise and American Calls

There are indeed situations when exercise of an American option prior to maturity is useful and for such options one actually has $V_t^A > V_t^E$. However, when interest rates are strictly positive and the underlying asset is a stock that does not pay dividends, it is never optimal to exercise an American call option early and we can conclude that American and European calls with the same expiration date and strike price must always have the same market price. We now establish this result; for simplicity, we assume that interest rates are constant.

Suppose that there is an ideal money market with constant effective rate R and let S be a stock that does not pay dividends. Let $T > 0$ and $K > 0$ be given. We denote by C^A an American call on the stock with expiration date T and strike price K and we denote by C^E the corresponding European option. Let $t \in [0, T)$ be given and let C_t^A and C_t^E denote the prices of the American and European calls at time t. We know that

$$C_t^A \geq C_t^E, \tag{3.19}$$

by virtue of (3.18). We can obtain a useful lower bound for C_t^E from (3.13). For the sake of completeness, let's derive this lower bound here.

Let V^E be a European-style derivative security that makes a single payment of amount

$$V_T^E = S_T - K$$

at time T. It is immediate that $C_T^E \geq V_T^E$ (because we always have $x^+ \geq x$.) This allows us to conclude that $C_t^E \geq V_t^E$. (Otherwise we could create an arbitrage by selling V^E and buying C^E at time t.) We can compute the price V_t^E of V^E at time t by replication: If, at time t, we buy one share of S (and hold it until time T) and borrow $(1+R)^{t-T}K$ from the bank until time T, the value of the portfolio at time T will be $S_T - K$. The capital required at time t to create this portfolio is $S_t - (1+R)^{t-T}K$. It follows that

$$V_t^E = S_t - (1+R)^{t-T}K.$$

(Notice that this argument makes use of the fact that S does not pay dividends.) Since $C_t^E \geq V_t^E$, we conclude that

$$C_t^E \geq S_t - K(1+R)^{t-T}. \tag{3.20}$$

Observe that (3.20) is valid without any sign restrictions on R.

Suppose now that $R > 0$ and $t < T$. Then we have $K(1+R)^{t-T} < K$ so we deduce from (3.19) and (3.20) that

$$C_t^A \geq C_t^E > S_t - K. \tag{3.21}$$

The holder of the American call can receive money by exercising at time t only if $S_t > K$; however, in this case (3.21) shows that the price of the option is strictly larger than the amount of money that would be received by exercising. In other words, the option holder could sell the option for more money than would be received by exercising it. This indicates that that the option should not be exercised prior to maturity and suggests that

$$C_t^A = C_t^E. \tag{3.22}$$

We can show that (3.22) actually holds for $R \geq 0$. We begin by observing that if $R \geq 0$, we can conclude from (3.20)

$$C_t^E \geq S_t - K. \tag{3.23}$$

Assume that $R \geq 0$ and let $t \in [0, T)$ be given. To see that (3.23) holds, suppose that $C_t^A > C_t^E$. Then, at time t, we could sell C^A and buy C^E and invest $C_t^A - C_t^E$ in the money market. If C^A is exercised at some time $\tau < T$ (and we owe any money because of this exercise) then we would have $S_\tau > K$ and we can sell C^E for at least

$$S_\tau - K$$

(by virtue of (3.23)). This amount of money is sufficient to pay the holder of C^A. We would still have a strictly positive amount of money in the bank – so this strategy would be an arbitrage.

It is very important to observe that in order for this argument to work we need to know that interest rates are nonnegative and that the stock will not make any dividend payments between time 0 and time T. For stocks that pay dividends it is sometimes advantageous to exercise an American call early.

Remark 3.33. If there is an ideal money market with constant effective rate $R \geq 0$ and S is a stock that does not pay dividends, then the price of a European call option on S must increase if the strike price is held fixed and the maturity is increased. This follows from the observations that American call options always have this property and, under the assumptions made above, European and American calls have the same prices.

3.9.3 Relationships Between American Put and Call Prices

The put-call parity relation is not valid for American options. However, it is possible to obtain a very useful pair of inequalities for the difference in price of an American put and American call having the same expiration date and strike price. For simplicity, we assume that there is an ideal money market with constant effective rate $R > 0$ and we consider a stock that does not pay dividends.

Let $T > 0$ and $K > 0$ be given and let P^A, P^E, C^A, and C^E denote American and European put and call options on the stock with expiration date T and strike price K. For simplicity, we shall write the inequalities in question for the prices of these options at time 0; similar inequalities hold for each $t \in [0, T)$.

In Exercise 14 of this chapter you are asked to show that

$$P_0^A + S_0 \leq C_0^E + K. \tag{3.24}$$

Using the fact that $C_0^A = C_0^E$ together with (3.24) we obtain

$$P_0^A - C_0^A = P_0^A - C_0^E \leq K - S_0. \tag{3.25}$$

We can obtain a lower bound for $P_0^A - C_0^A$ by using the facts that $P_0^A \geq P_0^E$ and $C_0^A = C_0^E$ together with the put-call parity relation (3.9) for stocks that do not pay dividends. We find that

$$P_0^A - C_0^A \geq P_0^E - C_0^E = K(1+R)^{-T} - S_0. \tag{3.26}$$

Combining (3.25) and (3.26) we arrive at

$$K(1+R)^{-T} - S_0 \leq P_0^A - C_0^A \leq K - S_0. \tag{3.27}$$

If $(1+R)^{-T}$ is very close to 1, then (3.27) provides relatively tight bounds for $P_0^A - C_0^A$.

Remark 3.34. We can obtain an inequality very similar to (3.27) without assuming that interest rates are constant. In particular, if we know at time 0 that $\mathcal{R}_{0,t} \geq 0$ for all $t \in [0,T)$, then we can replace (3.27) with

$$d(T)K - S_0 \leq P_0^A - C_0^A \leq K - S_0. \tag{3.28}$$

The following example is based on actual data from February 2007.

Example 3.35. The current price of Microsoft stock is \$28.21 per share and put options with $T = 2$ months and $K = \$27.50$ are trading at $P_0^A = \$.48$ Assuming that no dividends will be paid within the next two months and that $R_*(\frac{1}{6}) = 5.25\%$, let us estimate C_0^A, the price of a call option on Microsoft with $T = 2$ months and $K = \$27.50$

Substituting the numbers into (3.28) we find that

$$-\$.9435 \leq \$.48 - C_0^A \leq -\$.71,$$

which is equivalent to

$$\$1.19 \leq C_0^A \leq \$1.4235.$$

The actual market price quoted at the time for the American calls was $C_0^A = \$1.35$, which falls within the calculated range. An important word of caution is in order, however. One must be very careful with this type of calculation because the stock and the options are traded on different exchanges and it is difficult to know whether or not the option prices and stock price were recorded at the same time.

3.10 Exercises for Chapter 3

Exercise 3.1. The current exchange rate for currencies A and B is $E_{A,0}^B = 5$ (i.e., it costs 5 units of A to buy 1 unit of B). The effective 3-year spot rates for investments in currencies A and B are $R_*^A(3) = 6\%$ and $R_*^B(3) = 4\%$. Consider a forward contract in which 1 unit of B will be sold at time $T = 3$ for F_A^B units of A. Find the aribtrage-free value of F_A^B.

Exercise 3.2. Consider a coupon bond issued today with face value $F = \$1,000$ and maturity 10 years. The bond pays coupons twice per year at the nominal rate $q[2] = 5\%$. The current price of the bond is $\mathcal{P} = \$1,000$. Given that $R_*(.5) = 4.2\%$ and $R_*(1) = 4.4\%$ find the arbitrage-free forward price $\mathcal{F}$ for delivery of the bond at time 1, just after the coupon payment has been made. (The holder of the long position on the forward contract will receive her first coupon payment 6 months after delivery of the bond.)

Exercise 3.3. A trader has an opportunity to buy or sell a limited number of call options on a stock at the price $\hat{C}_0 = \$15$. The strike price of the option is $K = \$90$ and its exercise date is $T = 1$ year. The current price of the stock is $S_0 = \$100$ and the stock will pay a dividend $\delta = \$10$ per share in six months. (No other dividend payments will be made in the time interval $[0, 1]$.) Assume that the discount factors for six months and one year are given by $d(0.5) = 0.9$ and $d(1) = 0.8$, respectively.

Will the trader buy the options, sell the options, or decide that there is no advantage in trading this security? If he does trade the options, discuss the profit (per option) from this transaction.

Exercise 3.4. In this problem we use the symbol "\$" for US dollars. The current exchange rate for the Canadian Dollar is $E^C_\$ = 0.6298$. In other words, to buy 100 Canadian Dollars we need to pay \$62.98. The effective spot rate for maturity 3 months in the United States is $R^\$_*(.25) = 3\%$ and the effective spot rate for maturity 3 months in Canada is $R^C_*(.25) = 2\%$. The price of a European put option on 100,000 Canadian Dollars with exercise date $T = .25$ and strike price \$64,120 (\$0.6412 per Canadian Dollar) is \$1,100 (\$0.011 per Canadian Dollar). The put option gives its holder the right to sell 100,000 Canadian Dollars at the strike price at time $T = .25$.

Compute the arbitrage-free price of a European call option on the same amount of Canadian Dollars and with the same exercise date and strike price.

Exercise 3.5. Some time ago, Susan took the short position on a forward contract for delivery of 100 ounces of gold one year from today at the forward price of \$1,285 per ounce. Forward contracts being issued today for delivery of gold one year from today have a forward price of \$1,272 per ounce. If the arbitrage-free value today of Susan's position in the old contract is \$1,247, find the effective spot rate $R_*(1)$.

Exercise 3.6. Suppose that the forward price for delivery of gold at $T = 1$ is \$1,184 per ounce. Let P and C be European put and call options on one ounce of gold with exercise date $T = 1$ and the same strike price K. Given that there is an ideal money market with constant effective rate $R = .1$ and that the arbitrage-free prices of these options today are $P_0 = \$36.42$ and $C_0 = \$21.87$, find the strike price K of the options.

Exercise 3.7. Assume that there is an ideal money market with constant effective rate $R = 12\%$. A stock S is trading today at the initial price $S_0 = \$100$. The stock will pay a dividend 5 months from today. The amount of the dividend payment will be $\alpha S_{\frac{5}{12}-}$, and just after the dividend is paid the stock price will drop by the exact amount of the dividend payment. (Here $S_{\frac{5}{12}-}$ denotes the price of the stock just before the dividend is paid.) Let C and P be European call options on the stock with exercise date $T = .5$ and $K = 100$ for both options. The number α is known to investors at $t = 0$ (but is currently unknown to you). Given that the arbitrage-free prices of the options at $t = 0$ are $C_0 = \$5.76$ and $P_0 = \$10.25$, determine α.

Exercise 3.8. Assume that there is an ideal money market with constant effective annual rate $R = 2\%$. The spot price of gold is \$1,128 per ounce and the forward price for delivery of gold in 6 months is \$1,145 per ounce. Suppoes that a bank offers you the opportunity to store 1,000 ounces of gold in their vault for the next 6 months for \$4 per ounce due as soon as the gold is put in storage. Can you make an aribtrage profit? If so, what is the time-0 value of the profit?

Exercise 3.9. A textile company in North Carolina will need 500,000 pounds of cotton two months from today. The company has two choices to obtain the cotton:

(i) Purchase the cotton today at the spot price $S_0 = 45$ cents per pound and store it for two months. In order to store the cotton, the company will have to pay $\frac{1}{6}$ cents per pound at the end of each month that the cotton is in storage.

(ii) Take a long position on a forward contract for delivery of the cotton in two months at the forward price of 46 cents per pound.

Assuming that there is an ideal money market with effective annual rate $R = (1.01)^{12} - 1$, which choice should the company make? Discuss the assumptions needed to reach your conclusion.

Exercise 3.10. Assume that there is an ideal money market with constant effective rate $R = 6\%$. A stock S is trading today at the initial price $S_0 = \$100$. The stock will pay a dividend $\delta = \$2$ four months from today. A European call option C on the stock with exercise date $T = 6$ months and strike price $K = \$105$ is trading today at the initial price $C_0 = \$6$. Assuming that there is no arbitrage, find the initial price P_0 of a European put option P on the stock with exercise date $T = 6$ months and strike price $K = \$105$.

Exercise 3.11. (a) Prove Remark 3.28.

(b) How should (3.10) and (3.11) be modified if it is known that the stock pays a dividend of $\$\delta$ per share at time $\tau \in (0, T)$? Assume that δ and τ are known at time 0 and that there will be no other dividend payments made between time 0 and time T.

(c) How should (3.10) and (3.11) be modified if the stock pays a dividend of amount $\delta = \alpha S_{\tau_-}$ at time $\tau \in (0, T)$? Here $\alpha \in (0, 1)$ is a known constant. Assume that there will be no other dividend payments between time 0 and time T and that as soon as the dividend is paid, the stock price will drop by precisely the amount of the dividend payment.

Exercise 3.12. Consider the chooser option of Example 3.30 for a stock S that does not pay dividends.

(a) Show that the initial price can also be expressed as $V_0 = \hat{C}_0 + P_0$, where $\hat{C}$ is a call option with expiration date τ and strike price $K(1+R)^{\tau-T}$.

(b) Suppose that you are a broker and that you sell one chooser option to a client at the price $V_0 = C_0 + \hat{P}_0$, plus a small commission. You want to invest the amount V_0 in such a way that you will be able to pay your client at time T, no matter what the stock price is. You construct a portfolio by purchasing one call option C and one put option $\hat{P}$ at time 0 and holding your position until time τ. How should you adjust your portfolio at time τ? (Your position between time τ and time T will depend on the choice made by the client.) Explain how you can make a profit at time τ (and still be sure that you can cover your liability at time T) if the client makes the wrong choice.

Exercise 3.13. Consider the chooser option of Example 3.29 with $S_0 = \$100$, $\tau = 6$ months, $T = 1$ year, $K = \$105$, and $R = .05$. Let P and C denote European put and call options on S with expiration date $T = 1$ year and strike price $K = \$105$. Let $\hat{P}$ denote a European put on S with expiration date 6 months and strike price $\$105(1.05)^{-1/2}$. Assume that

$$C_0 = \$8.02, \quad \hat{P}_0 = \$5.61,$$

and that the stock does not pay dividends, so that $V_0 = \$13.63$. At $t = 0$ a client purchases 10,000 of the chooser options described above from a broker for $10,000V_0$ plus a commission. In order to hedge her short position, the broker purchases 10,000 each of C and $\hat{P}$ and holds them for 6 months. At time $t = 6$ months the prices of S and C are $S_{.5} = \$110$ and $C_{.5} = \$10.49$. Since $S_{.5} > \$105(1.05)^{-1/2}$, we know that $C_{.5} > P_{.5}$, so that the client should decide to choose the call for all of the shares of V. However the client has insider information and knows that, although it has not been announced yet, a regulatory agency has made a decision that will have a significant and negative impact on the profits of the company that issued the stock. Once the decision is made public, the stock price will almost certainly make a dramatic drop. The client is convinced that stock price will not recover in six months and decides to choose the puts for all 10,000 options, even though the calls are currently more valuable. In order to keep her short position hedged, the broker sells all 10,000 calls C and purchases 10,000 puts P.

(a) The broker will have cash left over at $t = 6$ months that can safely be consumed (or invested). How much extra cash will the broker have at $t = 6$ months?

(b) Suppose that $S_1 = \$69.78$. What will be the value of the client's portfolio at time 1?

(c) Assuming that $S_1 = \$69.78$, both the broker and the client have made a lot of money on this deal. Where did this money come from? In particular, do you think that there is a victim here? (There are currently strict regulations in the United States governing securities trades based on insider information. See, for example, [BKM] for a discussion. It is worth noting that experts have differing views on insider trading.)

Exercise 3.14. Assume that there is an ideal money market with constant effective interest rate $R > 0$. Let $T > 0$ and $K > 0$ be given. Consider a stock S that pays no dividends and let C^E and P^A denote a European call and an American put on the stock, both having expiration date T and strike price K. Show that

$$P_0^A + S_0 \leq C_0^E + K.$$

(Suggestion: Show that, no matter when the put option is exercised, an investor who purchases one European call and invests K in the bank at $t = 0$ is always at least as well off as an investor who purchases an American put and one share of stock at $t = 0$.)

Exercise 3.15. Assume that there is an ideal money market with constant effective rate $R = 0$ and let S be a stock that does not pay dividends. Let $T > 0$ and $K > 0$ be given and let P^A and P^E be American and European put options on S both having expiration date T and strike price K. Show that

$$P_0^A = P_0^E.$$

Exercise 3.16. Let $T > 0$, $K^{(1)}$, $K^{(2)}$ with $0 < K^{(1)} < K^{(2)}$ and $\alpha \in (0, 1)$ be given. Set

$$K = \alpha K^{(1)} + (1 - \alpha)K^{(2)}$$

and notice that $K^{(1)} < K < K^{(2)}$. Let $C^{(1)}$, $C^{(2)}$, and C be European call options on S all with expiration date T and having strike prices $K^{(1)}$, $K^{(2)}$, and K respectively.

(a) Show that

$$C_0 \leq \alpha C_0^{(1)} + (1 - \alpha)C_0^{(2)}.$$

(This shows that for a fixed expiration date, the price of a European call option is a convex function of the strike price.) Suggestion: Consider a portfolio that is long α calls $C^{(1)}$, long $(1 - \alpha)$ calls $C^{(2)}$ and short one call C.

(b) What is the analogous result for European put options?

Exercise 3.17. A commodity swap (for N units) is an agreement made (at time 0) between two parties A and B under which A pays a fixed amount of cash to B at prescribed future dates, and B pays A a variable amount of cash equal to N times the spot price of the designated commodity on each of those dates. Neither party pays anything to enter into the agreement. Let us assume that the swap dates are $T_i = \frac{i}{m}$, $i = 1, 2, \cdots, mn$ where m and n are given positive integers. At each of the times T_i, A pays the amount NF to B and B pays the amount NS_{T_i} to A, where S_t denotes the price of the commodity at time t. The swap price F, as well as the number of units N, must be specified at time 0.

(a) Find a formula for F in terms of in terms of the effective spot rates $R_*(T_i)$ and the forward prices $\mathcal{F}_{0,T_i}$ for delivery of one unit of the commodity at T_i.

(b) A manufacturer of electrical equipment wants to enter a swap agreement (as party A) for copper with $N = 20,000$ pounds and $m = n = 2$. Assuming that there is an ideal money market with constant effective rate $R = 5\%$ and that the forward prices (per pound) of copper are as follows:

$$\mathcal{F}_{0,\frac{1}{2}} = \$2.795, \quad \mathcal{F}_{0,1} = \$2.74, \quad \mathcal{F}_{0,\frac{3}{2}} = \$2.685, \quad \mathcal{F}_{0,2} = \$2.64,$$

determine the swap price F for one pound of copper.

(c) Assume that the spot price of copper (at $t = 0$) is \$2.845 per pound. Even if the company in part (b) can store 80,000 pounds of copper at no cost, it will save money by entering the swap agreement as opposed to purchasing 80,000 pounds of copper at $t = 0$. Calculate the net present value (at $t = 0$) of the savings, assuming that the copper can be stored for free.

Exercise 3.18 (Currency Swap). The current exchange rate for dollars and Japanese yen is $E^{\$}_{y,0} = 120$, i.e. it costs 120 yen to purchase one dollar. The effective spot rates for investments in dollars and yen are

$$\begin{aligned}
&R^{\$}_*(.25) = 3.00\%, \quad R^{\$}_*(.5) = 3.12\%, \quad R^{\$}_*(.75) = 3.15\%, \quad R^{\$}_*(1) = 3.25\%, \\
&R^{y}_*(.25) = 1.00\%, \quad R^{y}_*(.5) = 1.05\%, \quad R^{y}_*(.75) = 1.25\%, \quad R^{y}_*(1) = 1.30\%.
\end{aligned}$$

A US company and a Japanese company are making an agreement today to exchange dollars and yen at each of the dates $T_i = \frac{i}{4}$, $i = 1, 2, 3, 4$. At each T_i, the Japanese company will receive \$10,000,000 from the US company in exchange for a payment of $10,000,000\mathcal{F}$ yen. Assuming that nothing is paid by either company to enter the agreement, find the appropriate value of $\mathcal{F}$.

Exercise 3.19. Although gold has industrial uses, sufficiently many people hold gold solely for investment purposes that it is reasonable to assume that the forward prices and spot prices are related by

$$\mathcal{F}_{0,T} = (S_0 + \mathcal{C}^s_{0,T})(1 + R_*(T))^T,$$

where S_0 is the spot price of gold and $\mathcal{C}^s_{0,T}$ is the net present value at $t = 0$ of the cost per ounce to store gold between $t = 0$ and $t = T$. Suppose that the spot price of gold is $\$1,396.60$ per ounce and that the forward price for delivery of one ounce of gold in 10 months is $\$1,457.53$. Assuming that the effective spot rate for maturity 10 months is $R_*(\frac{5}{6}) = 4.87\%$, determine (approximately) the net present value of the total cost to store 1,000 ounces of gold for the next 10 months.

Exercise 3.20. A zero coupon bond with maturity 1 year and face value \$10,000 is trading at the current price $\mathcal{P} = \$9,558.40$. Assuming that $R_*(.75) = 4.57\%$, find the arbitrage-free forward price for delivery of the bond in 9 months.

Exercise 3.21. Assume that there is an ideal money market with constant effective rate $R = .06$. Consider a stock S with current price per share $S_0 = \$48$. Find the arbitrage-free forward price for delivery of one share of stock in 10 months assuming that

(a) the stock will not pay any dividends during the next 10 months;

(b) the stock will pay dividends twice during the next 10 months: a payment of \$.55 per share in 2 months and a payment of \$.60 per share in 8 months;

(c) the stock will pay dividends twice during the next 10 months: each payment will be exactly .012 times the share price just prior to the dividend payment.

Exercise 3.22. An annuity is being issued today. It will make payments of \$1,000 quarterly for the next 10 years. The effective yield to maturity of this annuity is $R_I = .05$. Assuming that $d(.25) = .99$ and $d(.5) = .98$, find the arbitrage-free forward price $\mathcal{F}_{0,\frac{1}{2}}$ for delivery of the annuity immediately after the second payment.

Exercise 3.23. Consider a stock S that pays no dividends and has initial price $S_0 = \$50$ per share. Assume that there is an ideal money market with constant effective rate $R = 6\%$. Let $T = 4$ months. Assume that the price of a European call $C^{(1)}$ on S with expiration date T and strike price $K^{(1)} = \$50$ is given by $C_0^{(1)} = \$4.50$ and that the price of a European put $P^{(2)}$ with expiration date T and strike price $K^{(2)} = \$60$ is given by $P_0^{(2)} = \$10.15$. Determine the arbitrage-free price V_0 of a (European-style) derivative security that pays its holder the amount V_T at time T if

(a) $V_T = \max\{50, S_T\}$;

(b) $V_T = \min\{50, S_T\}$;

(c) $V_T = \max\{10, S_T - 50\}$;

(d) $V_T = \min\{10, (S_T - 50)^+\}$.

(It is a good idea to sketch the graph of each payoff as a function of S_T.)

Exercise 3.24 (This is an adaptation of Exercise 10 from Chapter 12 of [L).] Let $T > 0$, $K > 0$, and $\alpha \in (0, 1)$ be given and let S be a stock. Consider a (European-style) derivative security V that pays its holder the amount

$$V_T = \max\{\alpha S_T, S_T - K\}$$

at time T. Determine constants β, γ, and $\hat{K}$ such that the arbitrage-free price V_0 of this security is given by

$$V_0 = \beta S_0 + \gamma \hat{C}_0,$$

where $\hat{C}$ is a European call option on S with expiration T and strike price $\hat{K}$. (The constants that you find may depend on α and K.)

Exercise 3.25 (Bull Spreads). In this exercise all options are understood to be European options with the same exercise date $T > 0$. Let $K^{(1)}$ and $K^{(2)}$ with $0 < K^{(1)} < K^{(2)}$ be given. A *bull spread* on a stock S is created by purchasing one call option $C^{(1)}$ with strike price $K^{(1)}$ and selling one call option $C^{(2)}$ with strike price $K^{(2)}$. The payoff of a bull spread at time T is therefore

$$V_T^{bull} = C_T^{(1)} - C_T^{(2)} = (S_T - K^{(1)})^+ - (S_T - K^{(2)})^+.$$

(a) Sketch the graph V_T^{bull} versus S_T.

(b) Show that the same payoff can be achieved by using two put options and a zero coupon bond with maturity T and face value $F = K^{(2)} - K^{(1)}$. (The portfolio consisting of the put options, without the bond is also referred to as a bull spread; spreads of this type result in cash received at time 0 and a potential liability at time T.)

(c) Suppose that $K^{(1)} = \$55$, $C_0^{(1)} = \$8.22$, $K^{(2)} = \$65$, and $C_0^{(2)} = \$3.19$. For which values of S_T will it be true that $V_T^{bull} \geq V_0^{bull}$?

Exercise 3.26. (Bear Spreads) In this exercise all options are understood to be European options with the same exercise date $T > 0$. Let $K^{(1)}$ and $K^{(2)}$ with $0 < K^{(1)} < K^{(2)}$ be given. A *bear spread* on a stock S is created by by purchasing one call option $C^{(2)}$ with strike price $K^{(1)}$ and selling one call option $C^{(1)}$ with strike price $K^{(1)}$. Notice that an agent who creates a bear spread in this way receives cash initially and has a potential liability at time T. The payoff of a bear spread at time T is given by

$$V_T^{bear} = C_T^{(2)} - C_T^{(1)} = (S_T - K^{(2)})^+ - (S_T - K^{(1)})^+.$$

(a) Sketch the graph of V_T^{bear} versus S_T.

(b) Show that the same payoff can be achieved by using put options and a zero-coupon bond with maturity T and face value $F = K^{(2)} - K^{(1)}$. (The portfolio consisting of the put options without the bond is also called a bear spread; an agent who creates a spread in this way must pay money at $t = 0$ and has the possibility of receiving money at time T.)

(c) Suppose that $K^{(1)} = \$50$, $C_0^{(1)} = \$11.96$, $K^{(2)} = \$60$, and $C_0^{(2)} = \$5.29$. For which values of S_T will it be true that $V_T^{bear} \geq V_0^{bear}$?

Exercise 3.27 (Butterfly Spreads). In this exercise, all options are European options with the same exercise date $T > 0$. Let $K^{(1)}$ and $K^{(2)}$ with $0 < K^{(1)} < K^{(2)}$ be given and let $K = \frac{1}{2}(K^{(1)} + K^{(2)})$. A *butterfly spread* on a stock S is created by purchasing one call $C^{(1)}$ with strike $K^{(1)}$, one call $C^{(2)}$ with strike $K^{(2)}$ and selling 2 calls C with strike K, i.e. the holder will receive

$$V_T^{but} = C_T^{(1)} + C_T^{(2)} - 2C_T = (S_T - K^{(1)})^+ + (S_T - K^{(2)})^+ - (2S_T - K^{(1)} - K^{(2)})^+$$

at time T. Typically, the strike prices are chosen so that $K \approx S_0$.

(a) Sketch the graph of V_T^{but} versus S_T.

(b) Suppose that $T = 6$ months, $S_0 = \$45$, $K^{(1)} = \$40$, $K^{(2)} = \$50$ (so that $K = \$45$), $C_0^{(1)} = \$6.45$, $C_0^{(2)} = \$1.17$, $C_0 = \$3.09$, and that $R_*(.5) = .05$. For which values of $S_{.5}$ will it be true that $V_{.5}^{but} \geq V_0^{but}$?

(c) Consider the situation of part (b) and assume that $R_*(.5) = .05$. For which values of $S_{.5}$ will an agent who purchases V^{but} be at least well off at $t = .5$ as an agent who invests V_0^{but} in the bank between $t = 0$ and $t = .5$?

Exercise 3.28 (Straddles, Strips, and Straps). In this exercise, all options are understood to be European. Let $T > 0$ and $K > 0$ be given and let P and C denote put and call options on a stock S with expiration T and strike price K. Recall that a straddle option on S with expiration date T and strike price K pays its holder the amount

$$V_T^{strad} = |S_T - K| = C_T + P_T$$

at time T. The holder of a straddle benefits when the price of the stock at expiration deviates significantly from the strike price – deviations above or below the strike lead to the same benefit. It is sometimes desirable to assign greater weight to deviations to one side or the other. This is accomplished by *strips* and *straps*. A strip pays its holder the amount

$$V_T^{strip} = C_T + 2P_T = (S_T - K)^+ + 2(K - S_T)^+,$$

at time T and a strap pays its holder the amount

$$V_T^{strap} = P_T + 2C_T = (K - S_T)^+ + 2(S_T - K)^+,$$

at time T.

(a) Sketch the graphs of V^{strad}, V^{strip}, and V^{strap} versus S_T.

(b) Suppose that $S_0 = \$53$, $K = \$55$, $T = 6$ months and $R_*(.5) = .06$. Assume that S does not pay dividends and that $P_0 = \$3.94$. Compute V_0^{strad}, V_0^{strip}, and V_0^{strap}.

Exercise 3.29. (Strangles) In this exercise all options are understood to be European. Let $T > 0$ be given. A *strangle* option is similar to a straddle, except that there are two strike prices, rather than one. Let $K^{(1)}$ and $K^{(2)}$ with $0 < K^{(1)} < K^{(2)}$ be given. A strangle with expiration date T and strike prices $K^{(1)}, K^{(2)}$ pays its holder the amount

$$V_T^{stran} = (S_T - K^{(2)})^+ + (K^{(1)} - S_T)^+ = C_T^{(2)} + P_T^{(1)},$$

at time T, where $C^{(2)}$ is a call with expiration date T and strike price $K^{(2)}$ and $P^{(1)}$ is a put with expiration T and strike price $K^{(1)}$.

(a) Sketch the graph of V^{stran} versus S_T.

(b) Assume that $K^{(1)} = \$42.50$, $P_0^{(1)} = \$1.00$, $K^{(2)} = \$47.50$, and $C_0^{(2)} = \$2.05$. For which values of S_T will it be true that $V_T^{stran} \geq V_0^{stran}$?

Exercise 3.30. Assume that $d(.5) = .98$ and $d(1) = .95$. In this problem there are three bonds U, V, and W. All of them have face value $1,000.00, maturity 10 years, and pay coupons twice per year at the nominal coupon rate $q[2] = .06$. The bond U is a plain vanilla bond (i.e., an ordinary coupon bond). The time-0 price of U is $U_0 = 1,000$. However, V and W have optionality features as described below. V is *callable* at time 1; this means that at time 1 (just after the coupon is paid) the issuer of the bond can decide to pay face value ($1,000) to the bond holder and not have to make any more payments. (In other words, the issuer has the right, but not the obligation to buy back the bond from the bond holder for $1,000 just after the coupon is paid at time 1.) W is *putable* at time 1; this means that at time 1 (just after the coupon is paid) the bond holder has the right to collect face value ($1,000) from the issuer and give up any claim to payments after time 1. (In other words, the bond holder has the right, but not the obligation, to sell the bond back to the issuer for $1,000 just after the coupon is paid at time 1.) The time-0 price of V is $V_0 = 995$. Determine the time-0 price W_0 of W.