# Course Notes for Introduction to Mathematical Finance (21-270)

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# Part I

# Introduction to Financial Markets, Replication and No-Arbitrage Pricing

# Chapter 1

# Introduction to Financial Markets, Replication and No-Arbitrage Pricing

The aim of this course is to provide an introduction to the mathematical modeling of financial markets with particular emphasis on the pricing of derivative securities, optimal investment, and the management of risk. Topics covered will include: an introduction to financial instruments and markets; fixed income securities, interest rates, and yields; simple models for random variation in prices; utility functions and optimal investment, the fundamental concepts of arbitrage, replication, and completeness; and the use of arbitrage-free models for the valuation of securities and for the management of risk.

We begin with a discussion of financial instruments, markets, and models. This introductory chapter contains a lot of terminology that may be unfamiliar to students who have not had previous exposure to finance or investing, and it is not expected that such students will feel completely at home with all of the material presented here immediately after the first reading. However this material *will* become "second nature" as you work through the subsequent chapters. It should prove useful to revisit Chapter 1 periodically.

Before attempting to explain the meanings of terms such as *replication*, *derivative security*, and *arbitrage*, it seems instructive to consider some examples.

## 1.1 A First Look at Stock Options

For purposes of illustration, let us pretend for a moment that it is now 1:00 pm on January 11, 2019. Suppose that you have agreed to have a construction project done that will be completed 3 months from now (in April 2019) at a cost of $25,000. You have also agreed to pay the contractor the full amount of $25,000 3 months from today. The only asset that you have available to pay for the project is 200 shares of Apple stock (abbreviated AAPL) that you currently own. The price of AAPL now is $152.43 per share, making the total value of your AAPL stock $\$152.34 \times 200 = \$30,486$. What should you do?

You could simply do nothing and hope (or just assume) that the price of AAPL will be above \$125 per share when you need the money – but this is risky and is probably not a very wise strategy. One very sensible possibility is to sell approximately \$25,000 of AAPL today and put the money in the bank to pay for the project. On the other hand, what if you think that the price of AAPL may go way up, to say \$200 (or even \$250) per share over the next 3 months? It would then be tempting to hold the stock until April. However, this strategy involves significant risk because the price of the stock might go way down, to say \$100 (or even \$75) per share by April. If this happens you would not be able to pay the contractor without borrowing money. This is a typical scenario: Alternatives that have the potential for a higher payoff generally involve greater risk.

There is an interesting kind of security, called a *put option*, that can be very useful for this type of situation.

**Definition 1.1.** A *put option* on a stock gives the holder the right, but not the obligation, to sell one share of the stock on or before a specified date $T$ (called the *expiration date* or *maturity*) for a specified price $K$ (called the *strike price*). If and when the holder of a put option sells the stock under the terms of the option, we say that the option has been *exercised*.

*Remark* 1.2. Some put options (*European options*) allow the holder to exercise them only on the expiration date $T$, while other put options (*American options*) can be exercised at any time up to and including $T$. We will discuss this distinction in detail later.

Of course, you should expect to pay something for a put option, because it gives a significant financial advantage to the holder and exposes the counterparty (or *writer*) to financial risk. In your situation, you may wish to consider purchasing 200 put options on AAPL with $K = \$125$ and $T = 3$ months (i.e., with expiration in April 2019). Such options are traded on an organized exchange. The current price for put options on AAPL with $K = \$125$ and $T = 3$ months is \$1.47 per share. This looks potentially attractive: By investing an additional \$294 (\$1.47 $\times$ 200) now, you can be sure that you can liquidate your stock for at least \$25,000 in April. If the price of AAPL in April is above \$125 per share, then you would not exercise the options and you would simply sell some (or all) of your stock on the open market. On the other hand, if the price of AAPL drops to \$75 per share in April, you can exercise the options and sell your stock for \$125 per share, even though the market price is only \$75.

In the scenario involving the construction project, it seems natural to purchase options that expire in April, but you may want to consider some other strike prices. (It might be nice to have some money left over after you pay the contractor.) The prices of put options on AAPL, expiring in April, are given in the table below for several different strike prices.

| Strike Price | Put Price |
|---|---|
| 125 | 1.47 |
| 130 | 2.13 |
| 135 | 3.05 |
| 140 | 4.25 |
| 145 | 5.80 |
| 150 | 7.75 |
| 155 | 10.20 |
| 160 | 13.00 |

If you purchase 200 puts on AAPL with $K = 150$ and $T = 3$ months then you can be sure that you can liquidate your stock for at least $30,000 in April. However, this higher level insurance comes at the expense of a greater premium – you would be required to pay $\$7.75 \times 200 = \$1550$ now rather than $294.

The prices of securities such as stocks and put options fluctuate throughout the day, based on demand. For securities that are being actively traded, there will be two prices quoted at the exchange: an *ask price* and a *bid price*. The ask price is the price at which brokers will sell the security to investors and the bid price is the price at which brokers will buy the security from investors. Of course, the bid price is always less than or equal to the ask price; the difference between these two prices is known as the *bid-ask spread*. The prices given above for AAPL stock and put options are the ask prices that were being quoted at 1:00 pm on January 11 for American put options expiring on April 18, 2019.

We shall now stop pretending that it is January 11, 2019.

You may be interested to look on *Yahoo Finance* (or a similar source) to check the current prices of AAPL stock and of put options on AAPL, and to explore how changing the strike price (or the expiration date) effects the price of a put option.

There is an analogous kind of security, known as a *call option*, involving the potential purchase of stock.

**Definition 1.3.** A *call option* on a stock gives the holder the right, but not the obligation, to purchase one share of stock at a specified date $T$ (called the *expiration date* or *maturity*) for a specified price $K$ (called the *strike price*). If and when the holder of a call option buys the stock under the terms of the option, we say that the option has been *exercised*.

*Remark* 1.4. *European call options* can be exercised only on the expiration date $T$, whereas *American call options* can be exercised at any time up to and including $T$.

It is recommended that you choose several stocks of interest and look on *Yahoo Finance* (or a similar source) to get a feel for the prices of puts and calls in relation to one another and to the current stock price for various values of the strike price and expiration date. You will see that even if two different stocks have nearly the same current price, their option prices may be very different even when all of the option parameters are identical. We note that puts and calls are usually sold in batches of hundreds or thousands. Sometimes put and call options are settled without any stock actually being bought or sold, i.e. when the investor exercises an option the writer of the option simply pays the investor the difference between the current stock price and the strike price. This is referred to as *cash settlement*. However, in many situations settlement of options requires that stock be transacted. When you actually trade options, it is important to understand all of the details of the settlement procedure. In this class, we will always treat options as though they are cash settled.

Stock prices are listed according to an abbreviation called a *ticker symbol* that identifies the company. For example, the ticker symbol for Apple is AAPL, the ticker symbol for Microsoft is MSFT, and the ticker symbol for General Motors is GM. Ticker symbols always contain between one and 5 characters.

Here is how you can look up prices on *Yahoo Finance*. First, go to *www.yahoo.com* and click on the link marked "finance" near the top left (or go directly to *finance.yahoo.com*). At the top of the finance page, there is a "search box". You can enter either a ticker symbol or a company name. When you click on *search* you will arrive at a page that contains a great deal of financial information (including stock prices) for the company in question. Information about options (including prices) can be found by clicking on the link marked "options". When you get to the options page, there will be a box on the left (a little bit down from the top) where you can select the expiration date. As you scroll down the page, you will find prices of calls first and then puts for a variety of strike prices. You can also get to the page of company information by entering the company name or the ticker symbol in the box labelled *Quote Lookup* that appears on the right of the finance page, a little bit down from the top.

There are numerous other websites such as *www.bloomberg.com* (and smart phone apps) where you can access prices of various securities and a great deal of other useful financial information.

As the discussion above illustrates, stock options can be used to *reduce risk*. They can also be used for *speculation*. An investor who has a strong belief that the stock price will drop in the future may wish to buy put options in order to capitalize on this belief. On the other hand, an investor who believes that the stock price will rise in the near future may wish to buy call options. It is possible to make a much larger profit (for a given initial investment) by purchasing call options than by purchasing the stock itself. Indeed, call options with strike price approximately equal to the current stock price typically sell for a small fraction of the stock price. If the stock price goes way up, an investor who purchases such options will have nearly the same gain per share as an investor who purchases the stock itself, but the investor who purchases the calls will hold many more shares. There is some serious risk involved with purchasing call options to try to realize a larger profit. To see why, consider the case of call options with strike price equal to the initial stock price. If, at the exercise date, the stock price is equal to the strike price, then the investor who has purchased the stock will lose nothing, whereas the investor who purchased the call options will lose his or her entire investment. We shall illustrate these points with a simple numerical example. In this example (as will typically be the case), we ignore the bid-ask spread on both the stock and the options, and we ignore any transaction costs.

**Example 1.5.** Suppose that today's date is $t = 0$ and consider a stock whose current price per share is $S_0 = \$10$. (We assume that the stock does not pay dividends.) European call options on the stock with $K = \$10$ and $T = 1$ year are trading at the current price of $C_0 = \$.80$ per share. There are two investors, "A" and "B", who will each invest \$1,000 at $t = 0$. Investor "A" uses the entire \$1,000 to purchase $1,250$ of the call options described above and holds them until $t = 1$. (Since the options are European, they cannot be exercised before $t = 1$, but investor "A" could sell them before $t = 1$ if he or she were so inclined.) Investor "B" uses the entire \$1,000 to purchase 100 shares of stock and holds those shares until time $t = 1$. Let us denote the price per share of the stock at time 1 by $S_1$. Of course, $S_1$ is not known at $t = 0$, so at the time the investments are made it is impossible to know which investor will be better off at time $t = 1$. Let's check what happens in several different situations.

(a) Suppose that $S_1 = \$25$ (a very dramatic increase in the stock price). The value of one call option at $t = 1$ is $C_1 = \$25 - \$10 = \$15$. (Indeed, an option holder can purchase a share of stock for \$10 and sell it immediately for \$25, yielding an instant profit of \$15 per share. As noted previously, we assume that the calls will be cash settled so there is no need for the investor to actually purchase the stock. He or she can collect \$15 per option from the agent who sold the options.) Therefore, the value of investor "A"'s portfolio at $t = 1$ is $1,250 \times (\$15) = \$18,750$. The value of investor "B"'s portfolio at $t = 1$ is $100 \times (\$25) = \$2,500$.

(b) Suppose that the stock price stays the same, so that $S_1 = \$10$. In this case, the call options are worthless at $t = 1$ and investor "A" has lost his or her entire initial investment. On the other hand, the value of Investor "B"'s portfolio at $t = 1$ is \$1,000 (the same as its initial value).

(c) Suppose that $S_1 = \frac{\$250}{23} \approx \$10.87$ The reader is asked to verify that in this case the two portfolios have the same value at $t = 1$, namely \$1,086.96 (rounded to the nearest cent).

Prior to 1973, stock options were not traded on exchanges in the US. Investors who wanted to trade puts and calls advertised in publications such as *The Wall Street Journal* to find other investors, or institutions, wanting to take a counterposition. At the time, there was no mathematically sound theory of determining option prices.

The landmark paper of Fisher Black & Myron Scholes [BS], together with related fundamental contributions by Robert Merton [Me], laid the foundation of the modern mathematical theory of derivative securities by providing a completely rational, and mathematically sound, theory for determining option prices. (The term *derivative security* will be defined later. For now, it suffices to know that put and call options are special cases of derivative securities.) Merton and Scholes shared the Nobel Prize in Economics in 1997 for their fundamental contributions to option pricing. (Fisher Black died in 1995 and was therefore not eligible to share the prize.)

Two crucial concepts underlying the Black-Scholes-Merton theory are *replication* and the assumption that *arbitrage* should not be possible. Roughly speaking, the idea of replication is that it may be possible to create the payment structure of a complicated security by using a portfolio (called a *replicating portfolio* or *replicating strategy*) of other, more primitive securities. In this case, the capital required to create the replicating portfolio should be the same as the price of the given security; otherwise the mismatch in pricing could be used to create wealth out of nothing – with no risk. A trading strategy that requires no input of capital, has no risk of loss, and has a strictly positive probability of creating wealth is called an *arbitrage strategy.* Consequently, the assumption that there is no arbitrage means that *if a trading strategy has the possibility of turning nothing into something of strictly positive value, then this strategy must also have the possibility of incurring a loss.* The no-arbitrage assumption goes along with the old proverb "There is no such thing as a free lunch.".

To help develop a feel for the concepts of replication and arbitrage, we shall consider a scenario involving currency exchange.

## 1.2 A Currency Exchange Scenario

Suppose that there are two currency exchange booths at an airport - Booth 1 and Booth 2. At Booth 1, you can exchange dollars for British pounds or pounds for dollars at the rate of \$2.00 per pound. At Booth 2, you can exchange dollars for euros or euros for dollars at the rate of \$1.25 per euro. (We assume that no fees or commissions are charged to make an exchange and the same exchange rate is used for both buying and selling.) The airport announces that another exchange booth, Booth 3, is about to open. At Booth 3, you will be able to exchange euros for pounds or pounds for euros. What can we say about the exchange rate at Booth 3?

A bit of thought shows that the rate at Booth 3 must be 1.60 euros per pound because we can actually exchange euros for pounds (and vice versa) at the rate of 1.60 euros per pound by making transactions at Booth 1 and Booth 2. (Indeed, we can take 1.60 euros to Booth 2 and receive \$2.00 which we could then take to Booth 1 and receive one pound. This strategy can be used in reverse to convert one pound to 1.60 euros.)

If an exchange rate other than 1.60 euros per pound were to be used at Booth 3, there would be an opportunity to make a very large profit (virtually instantly) without any risk just by trading among the different booths. To see how this might work, let us suppose that a lower rate, say 1.50 euros per pound were being used. Then we could start with \$75, purchase 60 euros at Booth 2, take the 60 euros to Booth 3 and purchase 40 pounds, then take the 40 pounds to Booth 1 and receive \$80, leaving us with a \$5 riskless profit. We could make a much larger profit by starting with an amount much larger than \$75, or by simply repeating the process many times. A similar strategy could be employed if an exchange rate greater than 1.60 euros per dollar were used. It seems clear that an opportunity such as this would not exist – at least not for very long – in the real world.

The crucial point here is that we can *replicate* the transactions available at Booth 3, i.e., we can in effect exchange euros for pounds or pounds for euros, by making transactions at Booths 1 and 2. For a fixed amount of euros, the number of pounds obtained by making an exchange at Booth 3 must be the same as the number of pounds obtained by exchanging the euros for dollars at Booth 2 and then exchanging the dollars for pounds at Booth 1 - otherwise there will be an arbitrage opportunity. (The trading strategy discussed above – when the exchange rate at Booth 3 is 1.50 euros per pound – requires some initial capital, and therefore does not satisfy our definition of arbitrage. We can fit this situation within the definition of arbitrage by assuming that it is possible to borrow money for short periods of time without paying any significant interest charges. This is a very reasonable assumption.)

## 1.3 Financial Markets and Models

Real-world financial markets are extremely complex and it is not possible to capture all of their features with a simple mathematical model. Mathematical models that are extremely complicated may be of limited use because they are too difficult to analyze. On the other hand, models that are too simplistic may also be of limited use. In an overly simplistic model, it may be possible to prove powerful conclusions with relative ease, but such results may bear little relation to anything in the real world. The financial models that we consider in this course are relatively simple, yet rich enough to provide insights into real financial markets. These models will lead to conclusions that are qualitatively reasonable, but, for the most part, the models will not be sophisticated enough to make quantitative decisions about complicated real-world financial situations. If you take the follow-up courses 21-370 (Discrete-Time Finance) and 21-420 (Continuous-Time Finance), you will learn about much more sophisticated financial models, including ones (such as the Black-Scholes model) that *are* used to make important quantitative decisions on Wall Street.

Many types of assets are traded in financial markets. An especially important type for us will be *securities*. Roughly speaking, a security[1] is a document that gives its holder the right to a financial claim. Some examples of common securities are *shares of stock* (which give their holders a claim to ownership of a fixed portion of a company) and *zero-coupon bonds* (which give their holders the right to receive a fixed payment at a prescribed future date). There are, of course, much more complicated kinds of securities. Another very important type of asset for us will be *commodities* such as gold, silver, oil, cotton, corn, wheat, etc.

Some companies distribute a portion of their profits to the stockholders by means of periodic cash payments called *dividends*. The precise amounts of such payments are usually not known very far in advance of the payment dates. Profits that are not paid to stockholders as dividends are generally reinvested in the firm (and presumably this helps to increase the stock price). Some companies reinvest all of their profits and do not pay dividends to the stockholders.

In real-world financial markets there are securities that are actively traded on an exchange and their prices are determined by the market. We shall refer to such securities as *traded securities*. (More generally, we use the term *traded assets* to refer to

[1]The term "security" was given a precise legal meaning by the Securities Act of 1933 and the Securities Exchange Act of 1934. The latter act created the Securities and Exchange Commission (SEC), an agency that oversees the sale of securities in the United States.

assets whose prices are determined by an active market.) Individuals and companies may have good reasons to want "custom-made" securities that are not actively traded so that there is no notion of market price. Such securities can often be purchased *over the counter* (OTC) from investment banks (or other financial institutions), but their prices must be computed. We shall refer to securities that are not actively traded on an exchange as *non-traded securities*, although this term may be somewhat misleading because such securities can be bought and sold outside of the exchange. In practice, it can happen that a security changes status during its lifetime and switches from being traded to being non-traded (or vice versa) as time evolves. For example, as the price of a stock changes, the strike prices at which puts and calls are actively traded also change.

Another very important notion is that of a *derivative security*, i.e., a security whose payoff is derived from, or computed from, the values of more primitive securities (or other types of assets). The securities (or assets) used to determine the value of a derivative are called the *underlying securities* (or *underlying assets*). Some derivatives are actively traded on exchanges; others are not. The term *contingent claim* is frequently used as a synonym for derivative security.

Put options on AAPL are examples of derivative securities. In this case, the underlying security is the AAPL stock. On January 11, 2019, when the stock price was \$152.43 per share, put options on AAPL with $T = 3$ months and $K = \$125$ were traded, whereas put options on AAPL with $T = 3$ months and $K = \$250$ were non-traded because there is little demand for options whose strike price is so far from the current stock price.

There are a number of exchanges in the United States (and throughout the world) at which various kinds of securities are traded. A brief discussion of some of the more important exchanges in the US is given in Appendix I. There are literally thousands of securities traded at these exchanges. It would be an impossible task to follow the price movements of all of them. There are a number of market *indices* that try to measure the performance of broad sectors of financial markets through a single number. The oldest index in the United States is the *Dow Jones Industrial Average* (abbreviated DJIA or simply called the *Dow*). It is computed from the stock prices of 30 major companies. Another very important index is the *Standard and Poor's 500* (abbreviated S&P 500) which is based on a weighted average of the stock prices of 500 of the largest companies. The correlation between the DJIA and the S&P 500 is very high. An important difference between the Dow and the S&P 500 is the way in which each of the individual companies influences the index. The DJIA is simply a multiple of the average price per share of the individual stocks – the sizes of the companies are not taken into account. Companies having higher share prices have a greater influence on the DJIA. The S&P 500 is based on a *weighted average* of the stock prices – the influence of each company is proportional to the company's market capitalization (i.e., the price per share times the number of outstanding shares). More information on market indices is contained in Appendix I.

In real-world financial markets there are effects known as *market frictions* that are difficult to model and analyze mathematically. Examples of market frictions include transaction costs (including bid-ask spreads), taxes, and the fact that an order to buy or sell a very large amount of a given asset can have an immediate impact on the unit price of the asset. Unless stated otherwise, we shall not try to account for market frictions in our mathematical models for financial markets. In certain situations, it is quite reasonable to ignore market frictions; in other situations it may be appropriate to perform our analysis assuming that the market is frictionless and then make some kind of adjustment to the final answer in order to account for important frictions that were neglected in the basic analysis.

Roughly speaking by a *financial model* we mean a collection of traded securities (and possibly other assets), together with a set $\mathcal{T}$ of possible trading times, rules on how securities can be traded, and some mathematical assumptions concerning the manner in which the prices of the traded securities change with time. We assume that the set $\mathcal{T}$ contains an initial time $t_0$ such that $t_0 \leq t$ for all $t \in \mathcal{T}$. In order to avoid trivialities we assume that there is at least one time $t \in \mathcal{T}$ with $t > t_0$. We usually take $t_0 = 0$. In situations where the set of trading times is clear from context, or is unimportant, we shall not mention $\mathcal{T}$ explicitly.

The prices of the traded assets at future times are generally unknown at the initial time, and therefore must be modeled as *random variables* on some *probability space.* (These terms will be explained in Chapter 4.) Sometimes, however, very powerful conclusions can be drawn without making a specific assumption about the precise manner in which the prices of the traded assets will evolve with time. In other words, we may be able to draw important conclusions based only on initial prices, some simple assumptions regarding asset sales, and our belief that prices will evolve forward in time in such a way that arbitrage is not possible. In other situations, the conclusions that we draw will depend crucially on the choice of a mathematical model that describes the random evolution of prices. It is very important to distinguish carefully between those formulas that do not depend on the choice of a particular pricing model for the basic securities and those that do. Much of the material in Chapters 1 through 3 applies independently of particular random pricing models. After a brief discussion of finite probability spaces in Chapter 4, we shall introduce some simple models for the random variation of prices. In order to help keep track of whether or not specific assumptions regarding the evolution of asset prices are being used in our analysis, we shall use the term *financial market* to indicate a framework that is like a financial model, but without any specific assumptions about the evolution of asset prices.

One can purchase US Treasury securities that promise to make a single payment of a specific amount at a prescribed future maturity date (zero coupon bonds). Such a security is generally considered to be risk free if it is held to maturity; in other words, the payoff amount at maturity is assumed to be certain. It is considered extremely unlikely that the US government would default on bonds. (Indeed, the government can always raise taxes or simply print more money[2].) However, even if one is certain that a zero-coupon bond will pay the full amount promised at maturity, purchase of such a security involves risk in the sense that the price of the security at times prior

[2] If the government prints too much money, then the purchasing power of a dollar may decline significantly so that bond investors who show a profit on paper may actually suffer a loss in terms of spending power of their money because of inflation. Throughout these notes, we ignore the effects of inflation.

to maturity is not known in advance and the value of such a security might decline unexpectedly. Federally insured bank deposits are also generally modeled as being free of default risk.

Throughout these notes, all of our financial markets and models will have a bank account that is free of default risk. The prices, or values, of bank investments such as certificates of deposits are usually characterized by interest rates. (There are several popular conventions for computing account values from interest rates; this issue will be treated in detail in Chapter 2.) We shall assume that customers at a bank can borrow or invest as much money as they like. The interest rate for borrowing will be the same as the rate for investing. Although we frequently assume that interest rate is constant, we will also study situations in which it will be allowed to depend on the initiation time as well as the duration of the deposit or loan. Since the interest rate for borrowing is the same as for investing, a loan can be treated as a negative deposit. For this reason, we shall often use the term "deposit" in place of "deposit or loan". Moreover, We shall use the terms "bank account" and "money market account" interchangeably. Although interest rates that will prevail for deposits made at future dates need not be known at the present time, we assume that promises made by a bank at the time of a deposit will always be honored.

Risky assets, such as stocks, should have higher "expected returns" than bank deposits; otherwise most investors simply would not buy them. Historically, stocks have outperformed bank accounts (on average) over time. According to the *Investor's Business Daily Guide to the Markets* [IBD], stocks in the US posted an average annual return of 8.8% during the period from 1871 to 1992, whereas the average return on bank deposits in that period was approximately 4.2%. Of course, some stocks will do much better than average, but some other stocks will do much worse than average. In fact, stockholders can lose their entire investment if a company fails. Investors cannot be certain in advance which stocks will perform better than others. Also, there have been periods of time during which stocks have lost money on average. Investors cannot be certain in advance whether or not stocks will outperform bank accounts in the future. Understanding and quantifying the tradeoff between risk and potentially higher rewards will be an important goal of this course.

Another major goal of this course will be to understand the theory behind pricing non-traded securities. A central component in this theory is the notion of *arbitrage*. In order to give a reasonable definition of arbitrage, we must explain what is meant by a "trading strategy" and by "the capital of a strategy". A *trading strategy* or *portfolio* in a financial model with initial time $t_0$ is described by a number $T > t_0$, called the *terminal time* of the strategy, and procedures to decide how many shares of each traded security to hold and how much money to have in the bank at each time $t$ with $t_0 \leq t < T$. The decisions regarding how to allocate capital at each time $t$ must be based solely on the information available at time $t$; in other words, strategies cannot "look into the future". In particular, decisions about how to allocate capital at time $t$ can take into account prices of all of the basic securities (and prevailing interest rates) at times up to and including $t$, but cannot be based on prices at future times. Indeed, we cannot have tomorrow's *Wall Street Journal* today.

It is, of course, crucial to know if and when capital can be added to or removed from a strategy. Most of the time we shall require strategies to be *self-financing*. This means capital can be added or removed only at the initial time. However, capital can be reallocated at any time $t < T$ at which trading is allowed. In other words, assets can be sold during the middle of the strategy, but the full amount of money obtained must be reinvested in other assets within our model; we cannot simply put money "under the mattress", nor can we introduce new capital to purchase assets. A self-financing strategy can hold assets that make payments at multiple times, provided that the payments received are invested in assets that are part of the strategy.

It is also important to study situations in which capital is introduced or removed after the initial time, but, in such cases, we shall always spell out that we are not dealing with a self-financing strategy and we shall give the rules by which capital can be introduced or removed. We shall use the terms "strategy" and "portfolio" synonymously. However, it is sometimes helpful to think of the portfolio as the actual collection of assets and the strategy as the procedure used to build the portfolio.

By the initial capital $X_{t_0}$ of a strategy, we mean the net capital invested at the initial time $t_0$. Capital received at the initial time by an agent implementing a strategy is considered to be negative. For each $t \in \mathcal{T}$ with $t_0 < t \leq T$, the capital $X_t$ at time $t$ is the net value of all assets in the portfolio at time $t$. For strategies that are not self-financing, if capital is introduced (or removed) at time $t$, we must be careful to explain whether $X_t$ represents the value of the portfolio *before* or *after* the new capital has been added. (In general, payments made by the implementer of a strategy make a positive contribution to the capital of the strategy, because they increases the value of the portfolio, and payments received by an agent implementing a strategy contribute negatively to the capital of the strategy because they decrease the value of the portfolio.) If $T$ is the terminal time of a strategy, we refer to $X_T$ as the terminal capital of the strategy. In a self-financing strategy, there may be some securities that make payments at the terminal time $T$. We can lump these payments together with the money in the bank account. The terminal capital in a self-financing strategy reflects the total value of all of the assets plus any payments received at the terminal time.

It is crucial to distinguish carefully between the capital of a strategy and the payment stream, or *cash flow*, associated with implementing the strategy. Of course, the capital of a strategy is generally nonzero even at times when there is no cash flow. One must be very careful about sign conventions when dealing with cash flows and capitals of strategies. This will be reinforced as we do examples. For self-financing strategies, the only time at which there can be a nonzero cash flow to the strategy is the initial time. (Of course, securities in the portfolio can make payments, but these payments must be immediately invested in the portfolio, if the strategy is to be self financing)

**Definition 1.6.** By an *arbitrage strategy* we mean a self-financing strategy having zero initial capital, nonnegative terminal capital for sure (i.e. with probability one), and strictly positive probability of having strictly positive terminal capital.

Some authors use slightly different definitions of arbitrage. What really matters is not whether a particular strategy is or is not an arbitrage, but whether or not a given model is free of arbitrage strategies. Our financial models will always have a "risk-free" bank account. For such models, there are alternative definitions of arbitrage that lead to exactly the same class of arbitrage-free models. (See Exercises 1.13 and 1.14.)

*Remark* 1.7. Some authors introduce a concept known as *strong arbitrage*; the essential point is that a strong arbitrage strategy is certain to yield a profit, rather than simply have a strictly positive probability of making a profit. The absence of strong arbitrage is a weaker assumption than the absence of arbitrage. (See Exercise 5.x.)

In order for our definition of arbitrage to be meaningful, there must be a way for an investor with no initial capital to generate a nontrivial portfolio. We shall therefore always assume that it is possible for an investor to borrow money and we shall usually assume that transactions known as short sales are possible.

A *short sale* occurs when someone sells an asset that they do not currently own. The way this works in practice is that the party who sells the asset short borrows it from someone who currently owns it and sells it to a third party. When the party who lent the asset wants it back, the party who borrowed it must return it either by borrowing it from someone else or by purchasing it on the open market. Short sales will play a central role in our mathematical treatment of financial markets. However, it is important to note that they can be very risky in practice. If you buy stock, your potential loss is limited to the purchase price of the stock. On the other hand, if you sell stock short, your potential loss is theoretically unlimited because if the price of the stock goes up drastically, you may have to purchase the stock at an extremely high price in order to return it to the party from whom you borrowed it. In practice, there are strict rules governing short sales of stock. Also, it should be noted that if a security that pays dividends (or other income) is sold short, then the party who sells the security short is responsible for paying the dividends to the party from whom the security is borrowed. Indeed, a party who lends an asset for purposes of a short sale should not lose any financial benefits associated with holding the asset. The act of purchasing the borrowed asset and returning it is referred to as *covering the short position* or *closing out the short position*.

In real-world financial markets, certain assets (such as most stocks) can be sold short, but others (such as most commodities) cannot. As we shall see later in the course, the fact that commodities cannot be sold short plays a crucial role in understanding prices in certain kinds of contracts. We now give two simple numerical examples involving short sales. As before, we ignore bid-ask spreads as well as any transaction costs. We also assume that the stocks do not pay dividends.

**Example 1.8.** Today's date is $t = 0$. There are two stocks, $S^{(1)}$ and $S^{(2)}$ with initial prices $S_0^{(1)} = \$150$ and $S_0^{(2)} = \$200$ per share, respectively. An investor builds a portfolio by buying 6 shares of $S^{(1)}$ and selling short 4 shares of $S^{(2)}$ at $t = 0$ and holding this position until $t = 1.5$.

(a) Let us find the initial capital. The amounf of money required at $t = 0$ to purchase 6 shares of $S^{(1)}$ is \$900. The amount of money received at $t = 0$ by selling 4 shares of $S^{(2)}$ is \$800. The initial capital is

$$X_0 = \$900 - \$800 = \$100.$$

Indeed, the money that is needed from external sources to get the portfolio started is \$100. (The proceeds of the short sale *reduce* the amount of money needed by the agent implementing the strategy, and this accounts for the minus sign that goes along with the \$800 associated with selling $S^{(2)}$.)

(b) Suppose that the stock prices at $t = 1.5$ are given by $S^{(1)}_{1.5} = 180$ and $S^{(2)}_{1.5} = 190$. In this case, the terminal capital is geven by

$$X_{1.5} = 6 \times (\$180) - 4 \times (\$190) = \$320.$$

(The minus sign that goes with the contribution from $S^{(2)}$ is there because the portfolio *owes* 4 shares of $S^{(2)}$.)

(c) Suppose that the stock prices at $t = 1.5$ are given by $S^{(1)}_{1.5} = \$140$ and $S^{(2)}_{1.5} = \$225$. In this case the terminal capital is given by

$$X_{1.5} = 6 \times (\$140) - 4 \times (\$225) = -\$60,$$

i.e. the strategy is in debt by \$60 at time 1.5. (Indeed the portfolio owns \$840 of $S^{(1)}$, but it owes 4 shares of $S^{(2)}$ and the value of 4 shares of $S^{(1)}$ at time 1.5 is \$900, so the portfolio owes \$60 more than the value of the shares of $S^{(1)}$.)

**Example 1.9.** Today's date is $t = 0$. Stock of the ABC company is currently trading at $S_0 = \$20$ per share. An investor who feels strongly that the stock price will drop over the next year decides to create a portfolio, with initial capital $X_0 = 0$, that will make money if the stock price goes way down. The investor notices that European put options on the stock with $T = 1$ year and $K = \$22.50$ are currently trading at $P_0 = \$5$ per share. The investor decides to sell short 100 shares of stock and use the proceeds of the short sale to buy as many of the put options described above as possible. Observe that by selling short 100 shares of stock at $t = 0$, the investor will receive \$2,000 which will allow him or her to purchase 400 put options. Let $S_1$ denote the price per share of the stock at $t = 1$. Assuming that the investor makes no transactions between $t = 0$ and $t = 1$, we shall calculate the capital $X_1$ of his or her portfolio at $t = 1$ for some possible values of $S_1$.

(a) Suppose that $S_1 = \$10$ (a dramatic decrease in the stock price). In the case, the value per share at $t = 1$ of the put options is $P_1 = \$12.50$. (Indeed, the holder of a put can purchase a share of stock for \$10 and sell it immediately for \$22.50.) The total value of the investor's put options is therefore $400 \times (\$12.50) = \$5,000$. However, the investor also has a debt of 100 shares of stock; at $t = 1$, it would cost a total of $100 \times (\$10) = \$1,000$ to purchase the borrowed stock. There the net value of the investor's portfolio at $t = 1$ is $X_1 = \$5,000 - \$1,000 = \$4,000$. It is very important to understand that even though this strategy required no initial capital, and yielded a profit in this case, it is *not* an arbitrage strategy because at the time that strategy was initiated the investor had no way of being *certain* that the stock price would drop. (See Remark 1.9 below.) As we shall see, if the stock price goes up (or drops by less than a certain amount) this strategy will leave the investor in debt at $t = 1$.

(b) Suppose that the stock price stays the same, so that $S_1 = \$20$. In this case the value per share at $t = 1$ of the put options is $P_1 = \$2.50$ and the total cost of the borrowed stock is \$2,000. Therefore the value of the portfolio at $t = 1$ is

$$X_1 = 400 \times (\$2.50) - \$2,000 = -\$1,000.$$

In other words, the investor is in debt by \$1,000 at $t = 1$.

(c) Suppose that $S_1 = \$25$. In this case the put options are worthless at $t = 1$ and the cost of the borrowed stock is \$2,500. The capital of the portfolio at $t = 1$ is therefore $X_1 = -\$2,500$ and the investor is in debt by \$2,500.

(d) The reader is asked to show as an exercise that if $S_1 < \$18$ then the value of the portfolio at $t = 1$ will be strictly positive and if $S_1 > \$18$ then the value of the portfolio at $t = 1$ will be strictly negative. (Since, at $t = 0$, there is no way of knowing for sure whether or not $S_1$ will be below \$18, there is no arbitrage opportunity here.)

*Remark* 1.10. We note that in practice traders sometimes use the term "arbitrage" for a strategy (with zero initial capital) in which there is a positive probability of making money and only a small (but nonzero) probability of incurring a loss. Such a strategy does not have a guarantee of avoiding a loss, and therefore does not fit our definition of arbitrage. Throughout these notes we reserve the use of the term "arbitrage" only for situations in which the probability of a loss is zero.

An obvious example of an arbitrage opportunity would occur if Bank 1 is offering one-year loans at 4% annual interest, while Bank 2 is offering one-year CDs (certificates of deposit) paying 6% annual interest; I would simply borrow as much money as I could from Bank 1 and use it to purchase CDs at Bank 2. At the end of one year, I could use the money in Bank 2 to repay the loan at Bank 1 and have money left over - with no risk. It seems that such an opportunity would not exist in the real world. (Why would Bank 1 lend money at 4% when it could simply deposit the money in Bank 2 and receive 6%?) If such an opportunity did actually exist, then presumably many investors would try to take advantage of it and the interest rates at the two banks would adjust until the arbitrage opportunity disappeared.

A more involved, but still rather transparent, example of an arbitrage can be seen in the currency exchange scenario of Section 1.2 if Booth 3 is using the exchange rate of 1.50 euros per pound. In order to fit this situation within our definition of arbitrage we assume that it is possible to borrow currency for short periods of time without paying interest (or other fees) of any significance.

In financial models with many assets, there can be rather complicated arbitrage strategies that cannot be discovered simply by inspection. For this reason, it is essential to develop some machinery that can be used to determine whether or not a given model is arbitrage free.

Arbitrage opportunities in the real world are generally very short-lived and/or have other "catches" that make it extremely difficult to make any significant amount of money. For example, several years ago I received an offer from a bank to lend me money for 6 months at an interest rate lower than the rate offered for 6 month CDs at my own bank. However, there was a limit on how much money I could borrow and this made me decide that it would not be worthwhile to try to take advantage of the arbitrage opportunity. (My maximum profit would have been about \$50, and if I failed to make the loan payment on time I would have had to pay a penalty larger than \$50.) In practice, arbitrage opportunities that exist on paper frequently disappear when transaction costs are taken into account. Moreover, if a "genuine" arbitrage opportunity is discovered, then as investors attempt to take advantage of the opportunity, the prices of the assets involved will adjust so that the arbitrage disappears.

## 1.4 Basic Ideas Behind No-Arbitrage Pricing

The simple idea that financial markets and models should not permit arbitrage has far-reaching consequences. Before applying this idea in any specific situations, it seems worthwhile to discuss the basic mechanics of no-arbitrage pricing. However, some caution must be exercised in interpreting the remarks below because terms such as "trading strategy" and "security" do not have clear cut meanings that apply in all conceivable situations. In particular, it seems quite difficult to give a simple and reasonable definition of replicating strategy here that will apply to all situations of interest. However, within the context of a specific model, a list of traded (or basic) securities as well as trading rules will be part of the prescription of the model and this will allow us to give precise formulations (and mathematical proofs) of these remarks.

For now, let us restrict our attention to non-traded securities that make a single payment of amount $V_T$ to their holders at a known terminal time $T$. The payment amount $V_T$ is generally not known at the initial time. (We shall model $V_T$ as a *random variable* on a *probability space*.) By a replicating strategy for such a security, we mean a self-financing strategy involving only traded securities and the bank account such that the terminal capital $X_T$ of the strategy will be equal to the payment amount $V_T$ in all possible "states of the world" at time $T$; in other words, no matter how the prices of the traded securities evolve, we can be sure to have $X_T = V_T$. We say that a non-traded security is *replicable* if there exists a replicating strategy.

Unless stated otherwise, we assume that all securities can be purchased and sold short in any amounts we choose (including fractional numbers of shares of stock) and we ignore transaction costs, including the bid-ask spread. Moreover, we assume the interest rate for investing is the same as the rate for borrowing (although we do allow the interest rate to depend on the duration of the loan or investment as well as on the inition date). We shall also assume that the price per share of a security is independent of the number of shares involved in the transaction. We shall also ignore the effects of taxation[3]. A discussion of the validity of these assumptions from a practical point of view is given in Appendix II. For assets other than securities, we shall always indicate whether or not short sales are allowed.

The following three remarks summarize several key ideas ideas that underlie the theory of no-arbitrage pricing.

*Remark* 1.11. If a model is arbitrage free, then the initial capitals of all replicating strategies for a given security must be equal.

*Remark* 1.12. If a model is arbitrage free, and a non-traded replicable security is added to the model, then the extended model will be arbitrage free if and only if the initial price of the additional security is equal to the initial capital of a replicating strategy.

*Remark* 1.13. Suppose that we are working in an arbitrage-free model and that we have two self-financing strategies, with the same terminal time $T$, whose terminal capitals are the same for all possible "states" of the world at time $T$. Then the initial capitals of the strategies must be the same. (On the other hand it is easy to give examples of two self-financing strategies with the same initial capital such that for each possible state at time $T$, the capitals of the two strategies will be different.)

Remarks 1.11 and 1.12 motivate the following definition.

**Definition 1.14.** In an arbitrage-free model, the *arbitrage-free price* of a non-traded replicable security is defined to be the initial capital of a replicating strategy.

For securities that make payments at more than one time, we obviously cannot require replicating strategies to be self-financing. It is important to note that although the validity of Remark 1.13 requires the strategies to be self-financing, Remarks 1.11 and 1.12 can be generalized to cases where the replicating strategies are not self-financing.

Remarks 1.11, 1.12, and 1.13 are very closely related. The idea behind a proof of the first part of Remark 1.13 goes as follows. Consider two self-financing strategies having the property that at the terminal time $T$ the values of the portfolios generated by these two strategies are the same for all possible states. Suppose that the initial capitals of the two strategies are different. Then, at the initial time $t = t_0$, we can sell the portfolio corresponding to the more expensive strategy and buy the portfolio corresponding to the less expensive strategy. This will give us some money that we can put in the bank. At time $T$, our positions on the two portfolios will cancel out exactly (no matter what outcome has occurred) and we will have money in the bank. In other words, we started with nothing, and wound up with money left at the terminal time. This is an arbitrage. Therefore the supposition that the initial capitals were different is not possible.

[3]Taxation policies can be very important in practice. Most investors are more concerned with the after-tax values of their portfolios than they are with generating tax revenue for the government. In particular, a security with a lower return, but non-taxable gains may be preferable to some securities with higher returns, but for which taxes must be paid on the gains.

A major theme of this course will be that

**PRICING = REPLICATION**.

In addition to studying derivative securities, we shall also be interested in analyzing *financial contracts* in which two parties agree to exchange assets (cash, securities, commodities, etc.) at future dates according to some prearranged rules. (A contract simply links the parties through an agreement; there is no intrinsic notion of holder.) In such situations we shall replicate the position of one of the parties. The initial capital of a replicating strategy for a particular position on the contract must equal the price paid (by the party holding that position) to enter into the contract.

Although there are important practical differences between contracts and securities, it is often possible mathematically to treat a contract as a security by choosing one of the parties to be considered the holder; in cases where the party designated as holder has an obligation to deliver an asset to the counterparty, we say that the holder receives a negative amount of that asset. It is also possible to view the sale of a security as a contract linking the buyer and the seller of the security. For these reasons, we shall usually be very casual about the distinction between a security and a contract. However, when we construct replicating strategies, we shall always be very careful to indicate the position being replicated.

## 1.5 Some Examples of Replicating Strategies

Our first example concerns an agreement made at the present time for an exchange of currency at a prescribed future date. Before beginning the example it will be useful to define a very basic type of contract and to make a few remarks about currency exchanges.

**Definition 1.15.** A *forward contract* is an agreement made between two parties at some initiation date $\tau$. One of parties agrees to purchase a specified amount of an asset (called the *underlying asset*) from the other party for a specified price $\mathcal{F}_{\tau,T}$ (called the *forward price*) at a prescribed future date $T$ (called the *delivery date* or *maturity*). The other party agrees to sell the asset for the specified price on the specified date. Nothing is paid by either party at time $\tau$ to enter into this agreement. The party who agrees to buy the asset is said to hold the *long position* and the party who agrees to sell the asset is said to hold the *short position.*

In contrast with put and call options, the parties in a forward contract are contractually obligated to go through with the sale. Since the initial price to enter into a forward contract is zero (by definition), it is not the initial price that needs to be computed, but rather the forward price. At times after initiation of the contract, it is standard practice to refer to $\mathcal{F}_{\tau,T}$ as the *delivery price* in order to avoid potential confusion with the forward price on contracts being made at the present time. When there is no danger of ambiguity concerning the dates $\tau$ and $T$, we write $\mathcal{F}$ in place of $\mathcal{F}_{\tau,T}$.

*Remark* 1.16. Forward contracts are generally not traded on exchanges. They are, however, commonly used between two financial institutions or between a financial institution and one of its major customers. Foreign currency is often the underlying asset.

*Remark* 1.17. There is a closely related (but somewhat more complicated) kind of contract called a *futures contract* concerning the sale of an asset at a prescribed future date and price. Futures contracts are actively traded on exchanges.

*Remark* 1.18 (Currency Exchange). Unless stated otherwise, we always assume that currency is bought and sold at the same exchange rate and that no fees (or commissions) are charged to make an exchange. If there are two currencies, say $A$ and $B$, it will be convenient to use the notation $E_A^B$ to denote the value of one unit of $B$ expressed in units of $A$ (i.e. it costs $E_A^B$ units of $A$ to purchase one unit of $B$). It is clear that

$$E_B^A = 1/E_A^B.$$

If there is also a third currency $C$, then a simple generalization of the earlier example on currency exchange shows that in order to avoid arbitrage we must have

$$E_C^A = E_B^A E_C^B.$$

Exchange rates, like prices of securities, evolve with time. We shall write $E_{A,t}^B$ to denote the value of $E_A^B$ at time $t$.

**Example 1.19** (Forward Exchange Rate)**.** Consider a simple financial market in which there are two times, $t = 0$ and $t = 1$, and there is a domestic currency $A$ and a foreign currency $B$. In this market, it is assumed that we can

(a) Exchange any amounts of currencies $A$ and $B$ at $t = 0$ at the initial exchange rate $E_{A,0}^B = 4$.

(b) Borrow or invest any amount of $A$ at the domestic interest rate $r_A = .1$ between $t = 0$ and $t = 1$. (If we borrow or invest an amount $\alpha$ at $t = 0$, then the amount owed or held at $t = 1$ will be $\alpha(1 + r_A)$.)

(c) Borrow or invest any amount of $B$ at the foreign interest rate $r_B = .2$ between $t = 0$ and $t = 1$.

The exchange rate $E_{A,1}^B$ is, of course, unknown at $t = 0$. Consider a forward contract in which it is agreed at $t = 0$ that one unit of $B$ will be sold at time $t = 1$ for the forward price $F_A^B$ units of A. We want to determine what, if anything, the absence of arbitrage implies regarding the value of $F_B^A$. To this end, we shall try to find a replicating strategy using loans and investments in the two currencies. For definiteness, let us agree to replicate the long position. We must create an obligation to pay $F_A^B$ units of $A$ at time $t = 1$. This can be accomplished by borrowing $F_A^B/(1 + r_A)$ units of $A$ at time $t = 0$ (at the domestic interest rate $r_A$). We also need to set ourselves up to receive 1 unit of $B$ at time $t = 1$. This can be achieved by investing

$1/(1+r_B)$ units of $B$ at time 0 (at the foreign interest rate $r_B$). We can now write an expression for the initial capital $X_0$ of this strategy, i.e., the capital required to initiate the strategy. For this purpose, it is crucial that the amounts borrowed and invested be expressed in units of the same currency. Let us agree to express $X_0$ in units of $A$. Notice that the amount of $B$ invested at $t = 0$ is equal to $E_A^B/(1+r_B)$ when expressed in units of $A$. Since the initial capital is the amount of capital required to initiate the strategy, money invested carries a plus sign and money borrowed carries a minus sign. (This is consistent with our convention that money received at the initial time by an agent implementing a strategy carries a minus sign.) We find that

$$X_0 = \frac{E_A^B}{1 + r_B} - \frac{F_A^B}{1 + r_A}.$$

Since nothing is paid by either party to enter into the contract, we must have $X_0 = 0$. This yields

$$F_A^B = E_A^B \frac{(1 + r_A)}{(1 + r_B)} = \frac{4(1.1)}{1.2} \approx 3.67.$$

*Remark* 1.20. The short position can be replicated by borrowing $1/(1 + r_B)$ units of $B$ and investing $F_A^B/(1 + r_A)$ units of $A$ at $t = 0$. A natural question arises: If both positions can be replicated, why would anyone want to enter into the contract? To answer this, let us assume that the party taking the long position is a company that has ordered some expensive equipment from a foreign supplier and that the party taking the short position is a financial institution. The equipment will be delivered at $t = 1$ and must be paid for upon delivery in foreign currency. The company may not have easy access to a foreign bank account, whereas a financial institution presumably does. It might be much more efficient for the company to enter into a forward contract than it would be to open a foreign bank account at $t = 0$. Of course, the financial institution would not want to enter the contract unless it will receive some compensation. In practice, financial institutions charge a transaction fee to the counterparty in this type of contract.

*Remark* 1.21. One of the basic principles of mathematical finance is that (in an arbitrage-free model) the initial price of a position on a contract is given by the initial capital of a replicating strategy. Since initial capitals can turn out to be negative, this principle can yield a negative price. It seems worthwhile to elaborate a bit on this possibility. A positive price means that the party holding the position being replicated should pay the counterparty. A negative price indicates that the holder of the position being replicated should receive money from the counterparty. Note that in the forward contract on the exchange discussed above, the forward price $F_A^B$ is set so that the prices of both the long and short forward contract positions are zero.

We now look at a simple example in which a replicating strategy is staring us in the face.

**Example 1.22.** Consider a financial market in which there are two times, $t = 0$ and $t = 1$, and three stocks: $S^{(1)}$, $S^{(2)}$, and $S^{(3)}$. The initial prices of the stocks are $S_0^{(1)} =$

\$100, $S_0^{(2)} = \$50$, and $S_0^{(3)} = \$75$, and we assume that there is no arbitrage. The stocks do not pay dividends. Their prices $S_1^{(1)}$, $S_1^{(2)}$, $S_1^{(3)}$ at $t = 1$ are, of course, unknown at $t = 0$. Consider a non-traded security $V$ whose value at $t = 1$ is determined by

$$V_1 = S_1^{(1)} - S_1^{(2)} + 2S_1^{(3)}.$$

We wish to find the arbitrage-free price $V_0$. We can replicate the long position (i.e., the position of holding the security) by purchasing one share of $S^{(1)}$, two shares of $S^{(3)}$ and selling short one share of $S^{(2)}$ at $t = 0$. The value at $t = 1$ of this portfolio is $S_1^{(1)} - S_1^{(2)} + 2S_1^{(3)}$. (Indeed, we hold one share of $S^{(1)}$ and 2 shares of $S^{(3)}$, but we owe one share of $S^{(2)}$. To close out our position at $t = 1$ we would sell one share of $S^{(1)}$ and receive the market price $S_1^{(1)}$, sell two shares of $S^{(3)}$ and receive $2S_1^{(3)}$, but the proceeds from the sale of $S^{(1)}$ and $S^{(3)}$ would be reduced by the amount $S_1^{(2)}$ required to purchase and return the borrowed share of $S^{(2)}$.) The initial capital of this strategy is

$$X_0 = \$100 - \$50 + 2 \times (\$75) = \$200,$$

which tells us that the arbitrage-free price is $V_0 = \$200$. (Notice that when the share of $S^{(2)}$ is sold short, we receive \$50, and this reduces by \$50 the amount of capital required to purchase the shares of $S^{(1)}$ and $S^{(2)}$.)

Next, we shall consider an important example in which a replicating strategy is not obvious at all. The observations made here are extremely important and they are very surprising to many people when encountering this type of example for the first time. In contrast with the previous two examples, we shall now employ a mathematical model for the evolution of prices.

**Example 1.23** (Put Options)**.** Consider a financial model with two times, $t = 0$ and $t = 1$, and a single stock that pays no dividends. The initial price of the stock is $S_0 = \$50$. The price $S_1$ of the stock at $t = 1$ will be either \$100 or \$25. The probability of $S_1 = \$100$ is .8, while the probability of $S_1 = \$25$ is .2. We are allowed to buy or sell any number of shares of stock (including fractional shares) at $t = 0$. There is also a bank that will allow us to borrow or invest any amount of money at $t = 0$ for the (one period) interest rate $r = .25$. It is not difficult to verify that this model is arbitrage free. We shall do so later in the course. This model is, of course, too simplistic to accurately model real-world stock behavior. Nevertheless, a number of important ideas are illustrated very nicely by this type of model.

Consider a put option $V$ on the stock with $K = \$40$ and $T = 1$. What is the arbitrage free price $V_0$ of the option?

As a first step, let us determine the value of the option at $t = 1$. If the stock goes up (to \$100) then the option is worthless. (Who would sell the stock for \$40 when they are free to sell it on the open market for \$100?) In this case the option would not be exercised. On the other hand, if the stock goes down to \$25, then the option is worth \$15 to the holder (because he or she can buy a share of stock for \$25 and sell it immediately for \$40).

What does this tell us about $V_0$? A natural (but incorrect) guess is that $V_0 = \$0(.8) + \$15(.2) = \$3$ since $V_1 = \$0$ with probability .8 and $V_1 = \$15$ with probability .2. A slightly more sophisticated (but still incorrect) guess is that $V_0 = \frac{\$3}{1.25} = \$2.40$ (with the factor $\frac{1}{1.25}$ inserted because \$1 invested in the bank at $t = 0$ will be worth \$1.25 at $t = 1$, so that a payment of \$3 at $t = 1$ can be created by investing \$2.40 in the bank at $t = 0$).

Let us try to find a replicating strategy for the long position (i.e., the position of holding the put). Suppose that at $t = 0$ we buy $\alpha$ shares of stock and invest $\beta$ in the bank. We want to determine $\alpha$ and $\beta$ (if possible) so that if the stock goes up to \$100 the value of our portfolio will be zero and if the stock goes down to \$25 then the value of our portfolio will be \$15. Notice that the value of the portfolio at $t = 1$ is

$$\alpha S_1 + 1.25\beta.$$

This leads to the pair of equations

$$100\alpha + 1.25\beta = 0$$

$$25\alpha + 1.25\beta = 15,$$

which has the unique solution $\alpha = -\frac{1}{5}$, $\beta = \$16$. The fact that $\alpha$ is negative indicates that we should sell short $\frac{1}{5}$ share of stock at $t = 0$. (This means that at $t = 0$, we borrow $\frac{1}{5}$ share of stock from someone and sell it to a third party. At time $t = 1$, we would have to return the borrowed stock by purchasing it on the open market.) The initial capital $X_0$ of the strategy is simply

$$X_0 = 50\alpha + \beta = -10 + 16 = 6.$$

We have found a simple investment strategy whose terminal capital matches the option payoff exactly – whether the stock price goes up or down. In order to avoid arbitrage, the price of the option at $t = 0$ must match the initial capital of the replicating strategy. Therefore $V_0 = \$6$. In order to elaborate on this point a bit, suppose that the option is trading at an initial price $V_0^* < 6$ at $t = 0$. In this case the option is *underpriced* and we could make an arbitrage profit by purchasing some puts and selling the replicating strategy. Let's see how this would work for 5 options (in order to avoid fractional shares of stock.) At time 0, we borrow \$80 from the bank, and purchase one share of stock for \$50, and purchase 5 put options for $5V_0^*$. This would leave us with $\$30 - 5V_0^*$ which we could deposit in the bank. (Actually, since the interest rate for borrowing and investing is the same, the loan and deposit can be combined into a loan of $\$50 - 5V_0^*$, but for our present purpose we shall think of them as being separate transactions with the bank.) Notice that the initial capital of this strategy is $X_0 = -80 + 50 + 5V_0^* + (30 - 5V_0^*) = 0$.

(i) If the stock goes up to \$100 at $t = 1$, the puts will be worthless, our share of stock will be worth \$100, and we will owe the bank $\$80 \times (1.25) = \$100$ to pay off the loan. The stock can be used to pay off the loan. However, our deposit will be worth $(\$30 - 5V_0^*) \times (1.25) > 0$.

(ii) If the stock goes down to \$25 at $t = 1$, then the puts will be worth $\$15 \times (5) = \$75$, so the combined value of the stock and the puts will be \$100 which can be used to pay off the loan at the bank. Once again, our deposit will be worth $(\$30 - 5V_0^*) > 0$.

If the option is trading at an initial price $V_0^* < 6$ then there is a strategy with zero initial capital that has strictly positive terminal capital for sure. Such a strategy is certainly an arbitrage. A similar argument shows that there will also be arbitrage if the option is trading at an initial price $V_0^{**} > 6$. The reader is invited to give the details. The reader is also invited to verify that if the option is trading at the initial price $V_0 = 6$, then no strategy holding stock, put options, and money in the bank can be an arbitrage strategy. It is interesting (and perhaps a bit surprising) that the probabilities of the stock going up or down do not play a direct role in determining the option price. This issue will be discussed in more detail in Remark 1.24 below and be revisited at several stages of the course.

In conclusion, we have shown that a replicating strategy for the put is to sell short $\frac{1}{5}$ share of stock and invest \$16 in the bank. Consequently, a replicating strategy for the short position (i.e. the position of the party selling the put) is to buy $\frac{1}{5}$ share of stock and borrow \$16 at $t = 0$.

Notice that the party holding the short position has exposed herself to some risk. She might have to pay \$15 at $t = 1$, but she only collects $V_0 = \$6$ at $t = 0$. (Even if she invests the \$6 in the bank at $t = 0$, the value of that investment at $t = 1$ will only be \$7.50.) The party holding the short position presumably would not want to replicate that position because this would create additional risk. Instead, she may wish to *hedge* the position by selling short $\frac{1}{5}$ share of stock and investing \$16. In other words, she may wish to replicate the long position because this cancels the risk of holding the short position.

According to the dictionary, the meaning of the word "hedge" is to "protect". In Mathematical Finance it is difficult to give a formal definition of "hedging" that applies to all conceivable situations, but we should think of a "hedging strategy" as a strategy employed by an economic agent to help ensure that the agent will be able to meet all possible future financial obligations. For now, we should observe that by using the hedging strategy described above, the party taking the short position has set herself up so that she will not owe anything at $t = 1$. Of course, she has no chance of making any money either. In practice brokers buy and sell securities at two different prices. The difference between these two prices is called the *bid-ask spread*. We will revisit this example later with a bid-ask spread on the option. In practice, brokers also generally charge a commission when they buy or sell securities.

**Warning:** The process of hedging options in the real world is MUCH more complicated than this because the mathematical assumptions used above to model the evolution of the stock price are too simplistic to accurately model real-world stock prices.

*Remark* 1.24. It is interesting, and perhaps somewhat surprising, that the probability $p$ of the stock going up does not play a direct role in determining the price of the option - so long as this probability is neither 0 nor 1. (Of course, the fact that the initial price is \$50 and that the two possible values at $t = 1$ are \$100 and \$25 does matter; changing these values would have an impact on the option price.) More precisely, if the probability of $S_1 = \$100$ is $p$ and the probability of $S_1 = \$25$ is $1 - p$ with $0 < p < 1$, and all other parameters in the example stay the same, then the argument above shows that we still have $V_0 = \$6$. It may seem that the option should be more valuable if $p$ is closer to 0 and less valuable if $p$ is closer to 1, but the mathematics shows that the option price should not change. In particular, even if $p$ is only .01, the broker in the above example needs no extra cash to set up her short position hedge – she is still sure to cover her liabilities by charging \$6 per put (and to make money by charging anything above \$6 per put). On the other hand, investors will likely feel that the put is more attractive to them if $p$ is close to 0. This issue is related to *risk preferences* and *optimal investment* and will be discussed in detail later. [4]

**Example 1.25.** Consider a financial model with two times $t = 0$ and $t = 1$. There is a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one-period interest rate $r = .2$. There are two basic risky securities: a stock $S$ that pays no dividends and a European put option $P$ on the stock with expiration date $T = 1$ and strike price $K = \$10$. The initial stock price is $S_0 = \$10$, and the initial price of the put is $P_0 = \$2.50$. There are three possible outcomes $\omega_1, \omega_2, \omega_3$ concerning the stock price at $t = 1$; each of these three outcomes has strictly positive probability of occurring. If $\omega_1$ occurs the stock price at $t = 1$ will be \$4 per share; if $\omega_2$ occurs then the stock price at $t = 1$ will be \$21 per share; if $\omega_3$ occurs the stock price at $t = 1$ will be \$18 per share. We write this as

$$S_1(\omega_1) = 4, \quad S_1(\omega_2) = 21, \quad S_1(\omega_3) = 18. \tag{1.1}$$

(Using the results of Chapter 5, it can be shown that this model is arbitrage free.)

(a) Let $C$ denote a European call option on the stock with expiration date $T = 1$ and strike price $K = \$10$ (the same strike price as the put). Find the arbitrage-free price $C_0$ at $t = 0$ of the call.

Let us look for a replicating strategy for the long position (i.e., the position of holding the call). Suppose at $t = 0$ we purchase $\alpha$ shares of stock, $\beta$ put options, and invest $\$\gamma$ in the bank. The capital $X_1$ at time 1 of our portfolio will be

$$X_1 = \alpha S_1 + \beta P_1 + 1.2\gamma. \tag{1.2}$$

Observe that

$$P_1(\omega_1) = 6, \quad P_1(\omega_2) = P_1(\omega_3) = 0, \tag{1.3}$$

[4] In real-world markets, investor's beliefs about the probability of the stock going up are reflected in the initial stock price $S_0$. Therefore, it is perhaps unrealistic to imagine that the value of $p$ can change without having an impact on $S_0$. However, it is not reasonable to expect that the initial stock price should satisfy $S_0 = (1 + r)^{-1}(100p + 25(1 - p))$.

and

$$C_1(\omega_1) = 0, \quad C_1(\omega_2) = 11, \quad C_1(\omega_3) = 8. \tag{1.4}$$

We need to choose $\alpha$, $\beta$, and $\gamma$, if possible, so that

$$X_1(\omega_i) = C_1(\omega_i) \text{ for all } i = 1, 2, 3. \tag{1.5}$$

Substituting (1.1) and (1.3) into (1.2) and using (1.4) and (1.5) we obtain the following system of equations:

$$\begin{array}{rcrcrcl} 4\alpha & + & 6\beta & + & 1.2\gamma & = & 0 \\ 21\alpha & & & + & 1.2\gamma & = & 11 \\ 18\alpha & & & + & 1.2\gamma & = & 8. \end{array} \tag{1.6}$$

It is straightforward to solve the system (1.6). Subtracting the third equation from the second we find that $3\alpha = 3$, i.e. $\alpha = 1$. Substituting $\alpha = 1$ into the third equation yields $\gamma = -\frac{25}{3}$. Substituting these values for $\alpha$ and $\gamma$ into the first equation gives $\beta = 1$. The initial capital of the corresponding strategy is given by

$$X_0 = \alpha S_0 + \beta P_0 + \gamma = 10 + 2.50 - \frac{25}{3} = \frac{25}{6}.$$

Consequently, we have $C_0 = \$\frac{25}{6} \approx \$4.17$. The fact that $\alpha$ and $\beta$ both turned out to be equal to 1 was not an accident. This happened because the strike price for the put is the same as the strike price for the call. There is a simple relationship called *put-call parity* that applies in this situation. This topic will be discussed in Chapter 3. We will revisit this example there.

(b) Let $V$ be a derivative security that pays \$1 at time 1 if $S_1 > 10$ and pays nothing if $S_1 \leq 10$. We wish to determine the arbitrage-free price $V_0$ of $V$ at time 0. In place of (1.4) we write

$$V_1(\omega_1) = 0, \quad V_1(\omega_2) = 1, \quad V_1(\omega_3) = 1. \tag{1.7}$$

We need to determine $\alpha$, $\beta$, and $\gamma$, if possible, so that

$$\begin{array}{rcrcrcl} 4\alpha & + & 6\beta & + & 1.2\gamma & = & 0 \\ 21\alpha & & & + & 1.2\gamma & = & 1 \\ 18\alpha & & & + & 1.2\gamma & = & 1. \end{array} \tag{1.8}$$

To solve the system (1.8), we first subtract the third equation from the second to obtain $\alpha = 0$. Substitution of $\alpha = 0$ into the second (or third) equation yields $\gamma = \frac{1}{1.2}$. Substituting these values for $\alpha$ and $\beta$ into the first equation, we find that $\beta = -\frac{1}{6}$. The initial capital of the corresponding strategy is given by

$$X_0 = \alpha S_0 + \beta P_0 + \gamma = 0 - \frac{2.5}{6} + \frac{1}{1.2} = \frac{5}{12}.$$

Consequently, we have $V_0 = \$\frac{5}{12} \approx \$.42$.

## 1.6 Exercises for Chapter 1

Throughout these exercises, you should assume that the stocks do not pay dividends. Many of the exercises include bank accounts. Since we have not discussed interest-rate mechanics yet, we shall usually limit our attention to the situation when there are only two trading times, namely 0 and 1, and a one-period interest rate $r$ is specified. In this situation an investment (or loan) of an amount $A$ at $t = 0$ will grow to a value of $A(1 + r)$ at $t = 1$. In all exercises, the present time is $t = 0$.

**Exercise 1.1.** Let $T > 0$ be given. Let $C$ be a European call option (on a stock $S$) with maturity $T$ and strike price $K_c = \$50.00$. Let $P$ be a European put option on the same stock with maturity $T$ and strike price $K_p = \$47.50$. Let $S_T$ denote the price per share of the stock at time $T$ and let $C_T$ and $P_T$ denote the value per share of the call and put option (respectively) at time $T$.

Find $C_T$ and $P_T$ if

(a) $S_T = \$53.47$;

(b) $S_T = \$48.52$;

(c) $S_T = \$42.71$.

**Exercise 1.2.** Stock of the Acme Company is currently trading at the price of $S_0 = \$46.00$ per share. European call options on the stock with maturity $T = 1$ and strike price $K_c = \$55.00$ are currently trading at the price of $C_0 = \$1.00$ per option and European put options on the stock with maturity $T = 1$ and strike price $K_p = \$50.00$ are currently trading at the price of $P_0 = \$5.00$ per option. At $t = 0$, an investor (with no initial capital to invest) sells short 1,000 of the put options described above and uses the proceeds of the short sale to purchase as many of the call options described above as possible. Let $S_1$ denote the price per share of the stock at $t = 1$. Assume that the investor does not make any trades between $t = 0$ and $t = 1$. Find the value $X_1$ of the investor's portfolio at $t = 1$ if

(a) $S_1 = \$60.00$;

(b) $S_1 = \$35.00$.

**Exercise 1.3.** Stock of the QRS company is currently trading at the price $S_0 = \$50$ per share. European put options on the stock with maturity $T = 1$ and strike price $K = \$50$ are currently trading at the price $P_0 = \$4$ per option. An investor with no initial capital believes that the stock price is going to drop, so she constructs a portfolio by selling short 400 shares of stock and using all of the proceeds of the short sale to purchase put options of the type described above. She makes no trades between $t = 0$ and $t = 1$. Let $S_1$ denote the price per share of the stock at $t = 1$. Find the value $X_1$ of her portfolio at $t = 1$ if

(a) $S_1 = \$25$;

(b) $S_1 = \$75$.

**Exercise 1.4.** Let $S$ be a stock that is currently trading at $S_0 = 50$ per share. Let $C$ be a European call on the stock with strike price $K = 48$ and maturity $T = 1$. Assume that the initial price of the call is $C_0 = 6$. At $t = 0$ an investor with no initial capital sells short 30 shares of stock and uses all of the proceeds from the short sale to buy call options. Assuming that the investro makes no trades between $t = 0$ and $t = 1$, find all values of $S_1$ (the stock price at time 1) such that $X_1 > 0$. Here $X_1$ is the investor's capital at time 1.

**Exercise 1.5.** Let $T > 0$ be given. Let $C$ be a European call option (on a stock $S$) with maturity $T = .6$ and strike price $K_c = 28$. Let $P$ be a European put on $S$ with maturity $T = .6$ and strike price $Kp = 30$. An investor buys 500 calls and 200 puts at time 0 and holds this position until time .6. What will be the investor's terminal capital $X_{.6}$ if

(a) $S_{.6} = \$31$?

(b) $S_{.6} = \$29.75$?

(b) $S_{.6} = \$25.00$?

Here $S_{.6}$ is the stock price at time .6.

**Exercise 1.6.** Let $S$ be a stock with initial price $S_0 = 50$ per share. European put options on $S$ with maturity $T = 1$ year and strike price $K = 50$ are trading at the initial price $P_0 = 1.75$ per option. An investor buys 10 shares of stock and 200 put options at time 0 and holds the position for one year.

(a) What is the initial capital of this strategy?

(b) Find all values of $S_1$ (the stock price at time 1) for which the $X_1 \geq 600$ (where $X_1$ is the capital of the strategy at time 1).

**Exercise 1.7.** A *European straddle option* on a stock $S$ with maturity $T$ and strike price $K$ is a security that pays its holder the amount $|S_T - K|$ at time $T$, where $S_T$ is the stock price at time $T$.

(a) What is the relationship between European straddles, calls, and puts?

(b) Let $S$ be a stock that has current price $S_0 = \$120$ per share. European calls on $S$ with maturity $T = 1$ and strike price $K = 120$ are currently trading at the price $C_0 = \$10.50$ and European puts on $S$ with $T =$ and $K = 120$ are trading at the price $P_0 = \$9.50$. An investor with initial capital $X_0 = \$10,000$ buys 140 European straddle options on $S$ with $T = 1$ and $K = 120$. She uses the rest of her initial capital to purchase stock. What will be the value of her portfolio at time 1

  (i) if $S_1 = 120$?

  (ii) if $S_1 = 130$?

  (iii) if $S_1 = 110$?

**Exercise 1.8.** A stock is currently trading at the initial price $S_0 = 100$. European put options on the stock with strike price $K = 95$ and maturity $T = 1$ are trading at the initial price $P_0 = 2$ per option. At time $t = 0$ an investor buys 2 shares of stock and also buys 150 put options. The investor holds the position until $t = 1$.

(a) What is the initial capital $X_0$ of this strategy?

(b) Find the terminal capital $X_1$ of this strategy for each of the following scenarios: (i) $S_1 = 110$, (ii) $S_1 = 100$, (iii) $S_1 = 90$, (iv) $S_1 = 85$.

**Exercise 1.9.** Let $S$ be a stock that has initial price $S_0 = 100$. Let $C$ be a European call on the stock with strike price $K = 105$ and maturity $T = 1$. Assume that the initial price of the call is $C_0 = 1.25$. At $t = 0$, an investor with no initial capital sells short 10 shares of the stock and uses the proceeds of the short sale to buy call options. Find all values of $S_1$ (the stock price at time 1) such that $X_1 > 0$. Here $X_1$ is the investor's capital at time 1.

**Exercise 1.10.** Stock of the XYZ company is currently trading at the price of $S_0 = \$40.00$ per share. European call options on the stock with maturity $T = .5$ years and $K = \$45.00$ are trading at the current price of $C_0 = \$.40$ per option. At $t = 0$ an investor purchases 500 of the call options described above and takes a short position on a forward contract for delivery of 500 shares of stock with delivery date $T = .5$ and forward price $\mathcal{F} = \$41.00$ per share. Assume that the investor does not make any trades between $t = 0$ and $t = .5$. Let $S_{.5}$ denote the price per share of the stock at $t = .5$.

(a) What is the initial capital $X_0$ of this strategy?

(b) Find the value of the investor's portfolio at $t = .5$ (i.e., find $X_{.5}$) if $S_{.5} = \$25.00$.

(c) Find the value of the investor's portfolio at $t = .5$ (i.e., find $X_{.5}$) if $S_{.5} = \$48.00$.

**Exercise 1.11.**

Let $K, T > 0$ be given. Let $C$ and $P$ be European calls and puts, respectively, on the same stock $S$ (that pays no dividends). Assume that $C$ and $P$ have the same maturity date $T$ and the same strike price $K$. At $t = 0$ an investor buys one put, buys one share of stock, and sells short one call option. Find a formula (simplified as much as possible) for the terminal capital $X_T$ of this strategy. Sketch the graph of $X_T$ as a function of $S_T$ (the terminal stock price).

**Exercise 1.12.**

Suppose that you can exchange Japanese yen for US dollars and dollars for yen at the rate of

- 120 yen per dollar,

exchange US dollars for British pounds and pounds for dollars at the rate of

- \$1.85 per pound,

and also exchange British pounds for euros and euros for pounds at the rate of

- 1.55 euros per pound.

No fees are charged to make exchanges. Assuming that there is no arbitrage, what should the exchange rate be for converting euros to yen. (Express your answer in terms of yen per euro.)

**Exercise 1.13** (Alternative Definition of Arbitrage)**.** Consider a financial model with two times, namely 0 and $T > 0$. Assume that one of the traded securities has initial price $B_0 > 0$ and makes a single payment $B_T = F$ to its holder at time $T$, where $F > 0$ is a constant that is specified (or known) at time 0. (Such a security is called a *zero coupon bond* with *maturity* $T$ and *face value* $F$.) Let us agree to say that a strategy is of type (A) provided that it is self-financing, has initial capital $X_0 \leq 0$, its terminal capital $X_T$ is nonnegative for sure, and there is a strictly positive probability that $X_T > 0$. Show that the model is arbitrage-free if and only if there are no strategies of type (A).

**Exercise 1.14** (Another Alternative Definition of Arbitrage)**.** Consider a financial model with two times: $t = 0$ and $t = 1$. Assume that there is a bank at which one can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one-period interest rate $r \geq 0$, where $r$ is a constant that is known at time 0. Let us agree to say that a strategy is of type (Ar) provided that it is self-financing and the initial capital $X_0$ and terminal capital $X_1$ satisfy (i) and (ii) below:

(i) $X_1 \geq (1 + r)X_0$ for sure;

(ii) There is a strictly positive probability that $X_1 > (1 + r)X_0$.

Show that the model is arbitrage-free if and only if there are no strategies of type (Ar).

**Exercise 1.15.**

Consider a simple financial model with two times $t = 0$ and $t = 1$. There is a domestic currency $A$ and a foreign currency $B$. The initial exchange rate is $E_{A,0}^B = 2.5$, i.e. one can exchange any amount of currency $A$ for $B$ (or $B$ for $A$) at time 0 at the rate of 2.5 units of $A$ for 1 unit of $B$. (No fees are charged to make an exchange.) There is a bank at which one can borrow or invest any amount of $A$ between $t = 0$ and $t = 1$ at the domestic one-period interest rate $r_A = .06$. There is also a bank at which one can borrow or invest any amount of $B$ between $t = 0$ and $t = 1$ at the foreign one-period rate $r_B$. The forward exchange rate for delivery date 1 is $F_A^B = 2.36$, i.e. it costs nothing at time 0 to enter into an agreement to purchase 1 unit of $B$ at time 1 for $F_A^B$ units of $A$. (An agent who enters this agreement is obligated to make the purchase at time 1.) Assuming that there is no arbitrage, find the foreign interest rate $r_B$.

**Exercise 1.16.** Consider a simple financial market with two times $t = 0$ and $t = 1$. There is a single stock $S$ with initial price $S_0 = \$98$ per share. There is also a bank at which one can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one-period interest rate $r = .06$. Consider a forward contract initiated at time 0 for delivery of one share of stock at time 1. Assuming that there is no arbitrage, find forward price $\mathcal{F}$.

**Exercise 1.17.** Consider a financial model with two times, $t = 0$ and $t = 1$, and a single stock $S$. We can buy or sell any number of shares of stock at $t = 0$ for the initial price of $S_0 = \$50$ per share. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .1$. At $t = 1$ the price of the stock will be either \$80 or \$40. The probability of $S_1 = \$80$ is $\frac{2}{3}$, while the probability of $S_1 = \$40$ is $\frac{1}{3}$. (You may take it for granted that this model is free of arbitrage.) Consider a call option $V$ on the stock with strike price $K = \$60$ and expiration date $T = 1$. Find the arbitrage-free price $V_0$ of the option at $t = 0$.

**Exercise 1.18.** Consider a financial model with two times, $t = 0$ and $t = 1$, and a single stock $S$. We can buy or sell any number of shares of stock at $t = 0$ for the initial price of $S_0 = \$50$ per share. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .25$. At $t = 1$ the price per share $S_1$ of the stock will be either \$100 or \$25. The probability of $S_1 = \$100$ is .7, while the probability of $S_1 = \$25$ is .3. (You may take it for granted that this model is free of arbitrage.) An investor who has \$10,000 to invest believes strongly that the stock will go up. However, the investor must be certain that the value of his portfolio at $t = 1$ is at least \$8,000, so he is reluctant to put his entire initial capital into stock. He goes to a broker for advice. The broker says that she can create a derivative security $V$ with initial price $V_0 = \$10,000$ that will pay as follows at $t = 1$:

$$V_1 = \begin{cases} A & \text{if} \quad S_1 = \$100 \\ \$8,000 & \text{if} \quad S_1 = \$25, \end{cases}$$

where $A > \$12,500$. Find the arbitrage-free value of $A$.

**Exercise 1.19.** Consider a financial model with two times, $t = 0$ and $t = 1$, and a single stock $S$. We can buy or sell any number of shares of stock at $t = 0$ for the initial price $S_0 = \$16$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .25$. At $t = 1$, the price of the stock will be either $S_1 = \$32$ or $S_1 = \$8$. The probability of $S_1 = \$32$ is $\frac{2}{3}$, while the probability of $S_1 = \$8$ is $\frac{1}{3}$. (You may take it for granted that the model is arbitrage free.)

(a) Find the arbitrage-free forward price $\mathcal{F}$ for delivery of one share of stock at time 1.

(b) Consider a put option $P$ on the stock with strike price $K = \$24$ and exercise date $T = 1$. Find the arbitrage-free price $P_0$ of the option at $t = 0$.

**Exercise 1.20.** Consider a financial model with two times, $t = 0$ and $t = 1$, and two stocks $S^{(1)}$ and $S^{(2)}$. We can buy or sell any number of shares of each of the stocks at $t = 0$ at the initial prices $S_0^{(1)} = S_0^{(2)} = \$20$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .1$. There are three possible outcomes $\omega_1, \omega_2$, and $\omega_3$ regarding the stock prices, each having probability $\frac{1}{3}$. If $\omega_1$ occurs then $S_1^{(1)} = S_1^{(2)} = \$24$. If $\omega_2$ occurs then $S_1^{(1)} = \$18$ and $S_1^{(2)} = \$24$. If $\omega_3$ occurs then $S_1^{(1)} = \$16$ and $S_1^{(2)} = \$8$. Here $S_1^{(i)}$ is the price per share of $S^{(i)}$ at time 1. (You may take it for granted that this model is arbitrage free.) Consider a European put option $V$ on $S^{(1)}$ with $K = \$20$ and $T = 1$. Find the arbitrage-free price $V_0$ of the option at $t = 0$. (Suggestion: Look for a replicating strategy that involves both stocks and the bank account.)

**Exercise 1.21.** Consider a financial model with two times $t = 0$ and $t = 1$, a single stock $S$, and a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .25$. There are two derivatives of $S$ that are being traded, namely a call option $V^{(1)}$ on $S$ with maturity $T = 1$ and strike price $K^{(1)} = \$64$ and a put option $V^{(2)}$ on $S$ with maturity $T = 1$ and strike price $K^{(2)} = \$24$. There are four possible outcomes $\omega_1, \omega_2, \omega_3, \omega_4$ regarding the stock price $S_1$ at $t = 1$. Each outcome has probability $\frac{1}{4}$. It is known that $S_1(\omega_1) = \$80$, $S_1(\omega_2) = \$60$, $S_1(\omega_3) = \$40$, and $S_1(\omega_4) = \$20$. (Here $S_1(\omega_i)$ represents the stock price at $t = 1$ corresponding to the outcome $\omega_i$.) The securities $S, V^{(1)}$, and $V^{(2)}$ can be bought or sold at $t = 0$ (in any amounts) at the prices $S_0 = \$36$, $V_0^{(1)} = \$1.60$, and $V_0^{(2)} = \$.80$. (You may take it for granted that this model is arbitrage free.) Let $V^{(3)}$ denote a put option on $S$ with maturity $T = 1$ and strike price $K^{(3)} = \$48$. Find the arbitrage-free price $V_0^{(3)}$ of $V^{(3)}$ at $t = 0$. (Suggestion: Look for a replicating strategy involving $S, V^{(1)}, V^{(2)}$, and the bank account.)

**Exercise 1.22.** Consider a financial model with two times, $t = 0$ and $t = 1$, and two stocks $S^{(1)}$ and $S^{(2)}$. We can buy or sell any number of shares of each of the stocks at $t = 0$ at the initial prices $S_0^{(1)} = S_0^{(2)} = \$20$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .1$. There are three possible outcomes $\omega_1, \omega_2$, and $\omega_3$ regarding the stock prices, each having probability $\frac{1}{3}$. The possible stock prices at $t = 1$ are given by

$$S_1^{(1)}(\omega_1) = \$24, \quad S_1^{(1)}(\omega_2) = \$18, \quad S_1^{(1)}(\omega_3) = \$16,$$
$$S_1^{(2)}(\omega_1) = \$24, \quad S_1^{(2)}(\omega_2) = \$20, \quad S_1^{(2)}(\omega_3) = \$12.$$

(You may take it for granted that this model is arbitrage free.)

Consider a derivative security $V$ with payoff at $t = 1$ given by

$$V_1(\omega_i) = \max\{S_1^{(1)}(\omega_i),\ S_1^{(2)}(\omega_i)\}, \quad i = 1, 2, 3,$$

i.e. if outcome $\omega_i$ occurs, the holder of the security receives the larger of $S_1^{(1)}(\omega_i)$, $S_1^{(2)}(\omega_i)$ at $t = 1$. (This is an example of a *basket option*.) Let $V_0$ be the arbitrage-free price of $V$ at $t = 0$.

(a) Explain why we know that $\$20 < V_0 < \frac{\$24}{1.1}$ without finding a replicating strategy.

(b) Find a replicating strategy for $V$ (in terms of $S^{(1)}$, $S^{(2)}$, and the bank account) and use it determine $V_0$.

**Exercise 1.23.** Consider a financial model with two times $t = 0$ and $t = 1$, and two stocks $S^{(1)}$ and $S^{(1)}$. We can buy or sell any number of shares of these stocks at $t = 0$ at the initial prices $S_0^{(1)} = \$32$, $S_0^{(2)} = \$28$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest $r = .25$. There are three possible outcomes $\omega_1, \omega_2, \omega_3$ regarding the stock prices at $t = 1$; each of these outcomes has strictly positive probability of occurring. The possible stock prices at $t = 1$ are given by

$$S_1^{(1)}(\omega_1) = \$60, \quad S_1^{(1)}(\omega_2) = \$40, \quad S_1^{(1)}(\omega_3) = \$20,$$

$$S_1^{(2)}(\omega_1) = \$20, \quad S_1^{(2)}(\omega_2) = \$30, \quad S_1^{(2)}(\omega_3) = \$60.$$

(You may take it for granted that this model is free of arbitrage.)

Consider a derivative security $V$ with payoff at $t = 1$ given by

$$V_1(\omega_i) = \left|S_1^{(1)}(\omega_i) - 40\right|, \quad i = 1, 2, 3.$$

(This security is called a *straddle option* on $S^{(1)}$ with strike price \$40 and maturity $T = 1$.) Find the arbitrage-free price $V_0$ of $V$ at $t = 0$.

**Exercise 1.24.** Consider a simple financial model in which there are two times, $t = 0$ and $t = 1$, and a stock $S$. There is also a European put option $P$ on the stock with $T = 1$ and $K = \$48$, and there is a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one-period interest rate $r = .25$. There are three possible outcomes $\omega_1, \omega_2, \omega_3$ regarding the stock price, each having strictly positive probability. It is known that the stock price $S_1$ at time 1 satisfies $S_1(\omega_1) = \$80$, $S_1(\omega_2) = \$60$, $S_1(\omega_3) = \$40$. At $t = 0$, one can buy or sell any number of shares of stock and put options at the initial prices $S_0 = \$44$ and $P_0 = \$2.40$. (You may take it for granted that this model is arbitrage free.) Let $C$ be a European call option on the stock with $T = 1$ and $K = \$64$.

(a) Find a replicating strategy for $C$ in terms of $S, P$, and the bank account.

(b) Find the arbitrage-free price $C_0$ of $C$ at time 0.

**Exercise 1.25.** Consider a simple financial market with three times $t = 0, 1, 2$ and a domestic currency, say dollars, and a foreign currency, say British pounds. In this model, we can

(i) Exchange any amount of dollars and pounds at $t = 0$ at the exchange rate $E_{\$}^{P} = 2$, i.e., it costs \$2 to purchase one pound at time 0.

(ii) Borrow or invest any amount of dollars between $t = 0$ and $t = 1$ at the one period interest rate $r_0^{\$} = .1$ and borrow or invest any amount of dollars between $t = 1$ and $t = 2$ at the one-period interest rate $r_1^{\$} = .12$. An amount $\alpha$ invested at $t = i$ will grow to the amount $\alpha(1 + r_i^{\$})$ at $t = i + 1$. Similarly for loans. (In particular, a dollar-deposit of amount $\alpha$ made at $t = 0$ and left in the bank until $t = 2$ will grow to $\alpha(1.1)(1.12)$ at $t = 2$.)

(iii) Borrow or invest any amount of pounds between $t = 0$ and $t = 1$ at the one-period interest rate $r_0^{p} = .2$ and borrow or invest any amount of pounds between $t = 1$ and $t = 2$ at the one-period interest rate $r_1^{p} = .15$. An amount $\beta$ invested at time $i$ will grow to $\beta(1 + r_i^{p}$ at time $i + 1$. (In particular a pound-deposit of amount $\beta$ made at $t = 0$ and left in the bank until $t = 2$ will grow to $\beta(1.2)(1.15)$ at $t = 2$.)

Consider a contract made between two investors A and B at $t = 0$ in which it is agreed that investor A will pay Investor B \$2 at each of the times $t = 1$ and $t = 2$ and Investor B will pay Investor A 2 pounds at each of the times $t = 1$ and $t = 2$. Assuming that there is no arbitrage, find the value, in pounds, of investor B's position at $t = 0$.

**Exercise 1.26.**

Consider a financial model in which there are 4 times, $t = 0, 1, 2, 3$ and two currencies: US dollars and British pounds. The current exchange rate between dollars and pounds is $E_{\$,0}^{p} = 2$, (i.e. it costs \$2 to purchase 1 pound at time 0).There is a bank buying and selling zero-coupon bonds that make a single payment of \$1 at maturity. These bonds are available with maturities 1, 2, and 3. For each $i = 1, 2, 3$ let $B_i^{\$}$ be the price in dollars at time zero of the bond that pays \$1 at time i. There is a also bank buying and selling zero-coupon bonds that make a single payment of 1 pound at maturity. These bonds are available with maturities 1, 2, and 3. For each $i = 1, 2, 3$

let $B_i^p$ be the price in pounds at time zero of the bond that pays 1 pound at time i. Assume that

$$B_1^\$ = .9615, \quad B_2^\$ = .9070, \quad B_3^\$ = .8396,$$

$$B_1^p = .9259, \quad B_2^p = .8573, \quad B_3^p = .7938.$$

A US company and a British company want to enter into a currency swap at time 0. The British company will pay the US company 1,000,000 pounds at each of the times $t = 1, 2, 3$ and the US company will pay the British company A dollars at each of the times $t = 1, 2, 3$. Nothing is paid by either party to enter the agreement and the (constant) amount $A$ is agreed to at time 0. Assuming that there is no arbitrage, determine the value of $A$.

**Exercise 1.27.** (Chooser Put Option) This exercise should be compared with Exercise 1.20. Consider a financial model with two times $t = 0$ and $t = 1$, and two stocks $S^{(1)}$ and $S^{(1)}$. We can buy or sell any number of shares of these stocks at $t = 0$ at the initial prices $S_0^{(1)} = S_0^{(2)} = \$20$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest $r = .1$. There are three possible outcomes $\omega_1, \omega_2, \omega_3$ regarding the stock prices at $t = 1$; each of these outcomes has strictly positive probability of occurring. The possible stock prices at $t = 1$ are given by

$$S_1^{(1)}(\omega_1) = \$24, \quad S_1^{(1)}(\omega_2) = \$18, \quad S_1^{(1)}(\omega_3) = \$16,$$

$$S_1^{(2)}(\omega_1) = \$24, \quad S_1^{(2)}(\omega_2) = \$24, \quad S_1^{(2)}(\omega_3) = \$8.$$

(You may take it for granted that this model is free of arbitrage.) Let $W$ be a derivative security that gives the holder the right to choose, at time 1, either a put option on $S^{(1)}$ with $K = 20$ and $T = 1$ or a put option on $S^{(2)}$ with $K = 20$ and $T = 1$. Determine the the arbitrage-free price $W_0$ of $W$ at time 0.