10-701 Introduction to Machine Learning (PhD)
Lecture 5: Naive Bayes

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Slides based on Tom Mitchell's
10-701 Spring 2016 material

Announcements

- HW2 was released, due on February 15th
- Projects
  - Teams + project due February 11th (let us know if you have problems!)
  - Midway Report due March 6th
  - Poster Session on April 17th, 9am-2pm
  - Final Report due May 1st

You want to build a classifier for new customers by learning P(Y|X)

O: Is older than 35 years	I: Has Personal Income	S: Is a Student	J: Birthday before July 1st	Y: Buys computer
0	1	0	0	0
0	1	0	1	0
1	1	0	1	1
1	1	0	1	1
1	0	1	0	1
1	0	1	1	0
0	0	1	0	1
0	1	0	0	0
0	0	1	1	1
0	1	1	0	1
0	0	1	1	1
1	1	0	1	1
0	1	1	0	1
1	1	0	0	0

New customer i:

0	1	1	1	?

Want to find P(Yi = 0| O=0, I = 0, S = 1, J = 1)

How many parameters must we estimate?

Suppose $X = (X_1, X_2)$
where X_i and Y are boolean RV's

To estimate $P(Y| X_1, X_2)$

X_1	X_2	$P(Y = 1 \mid X_1, X_2)$	$P(Y = 0 \mid X_1, X_2)$
0	0	0.1	0.9
1	0	0.24	0.76
0	1	0.54	0.46
1	1	0.23	0.77

How many parameters must we estimate?

Suppose X =(X_1,X_2)
where X_i and Y are boolean RV's

4 parameters:
P(Y=0|X) = 1-P(Y=1|X)

To estimate P(Y| X_1, X_2)

X_1	X_2	P(Y = 1 | X_1,X_2)	P(Y = 0 | X_1,X_2)
0	0	0.1	0.9
1	0	0.24	0.76
0	1	0.54	0.46
1	1	0.23	0.77

How many parameters must we estimate?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

To estimate P(Y| X_1, X_2, … X_n)

X_1	X_2	X_3	…	X_N	P(Y = 1 | X)	P(Y = 0 | X)
0	0	0	…	0	0.1	0.9
1	0	0	…	…	0.24	0.76
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…
1	1	1	1	1	0.52	0.48

How many parameters must we estimate?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

To estimate P(Y| X_1, X_2, … X_n)

X_1	X_2	X_3	…	X_N	P(Y = 1 | X)	P(Y = 0 | X)
0	0	0	…	0	0.1	0.9
1	0	0	…	…	0.24	0.76
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…
1	1	1	1	1	0.52	0.48

2^n rows!

How many parameters must we estimate?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

To estimate P(Y| X_1, X_2, … X_n)

X_1	X_2	X_3	…	X_N	P(Y = 1 | X)	P(Y = 0 | X)
0	0	0	…	0	0.1	0.9
1	0	0	…	…	0.24	0.76
…	…	…	…	…	…	…
…	…	…	…	…	…	…
…	…	…	…	…	…	…
1	1	1	1	1	0.52	0.48

2^n rows!

If we have 30 boolean X_i's: P(Y | X_1, X_2, … X_{30}) ~ 1Billion

Bayes Rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Which is shorthand for:

$$(\forall i,j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{P(X = x_j)}$$

Equivalently:

$$(\forall i,j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{\sum_k P(X = x_j | Y = y_k) P(Y = y_k)}$$

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

Y	X_1	X_2	…	X_N	P(X| Y)
0	0	0	…	0	0.1
0	1	0	…	…	0.02
…	…	…	…	…	…
…	…	…	…	…	…
…	…	…	…	…	…
0	1	1	1	1	0.16

2^n rows!

-1 because all the probabilities sum to 1

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

Y	X_1	X_2	…	X_N	P(X| Y)
0	0	0	…	0	0.1
0	1	0	…	…	0.02
…	…	…	…	…	…
…	…	…	…	…	…
…	…	…	…	…	…
0	1	1	1	1	0.16

2^n -1

Y	X_1	X_2	…	X_N	P(X| Y)
1	0	0	…	0	
1	1	0	…	…	
…	…	…	…	…	
…	…	…	…	…	
…	…	…	…	…	
1	1	1	1	1	

Should I compute these as well?

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

Y	X_1	X_2	…	X_N	P(X|Y)
0	0	0	…	0	0.1
0	1	0	…	…	0.02
…	…	…	…	…	…
…	…	…	…	…	…
…	…	…	…	…	…
0	1	1	1	1	0.16

2^n -1

Y	X_1	X_2	…	X_N	P(X|Y)
1	0	0	…	0	
1	1	0	…	…	
…	…	…	…	…	
…	…	…	…	…	
…	…	…	…	…	
1	1	1	1	1	

2^n -1

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

$2^*(2^n - 1)$

If n = 30, 2 billion

How many parameters to define P(Y)?

Can we reduce params using Bayes Rule?

Suppose X =(X_1,… X_n)
where X_i and Y are boolean RV's

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How many parameters to define P(X_1,… X_n | Y)?

$2^*(2^n - 1)$

If n = 30, 2 billion

How many parameters to define P(Y)?

1

Naïve Bayes

Naïve Bayes assumes

$$P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$$

i.e., that X_i and X_j are conditionally independent given Y, for all i≠j

Conditional Independence

Definition: X is conditionally independent of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

$$(\forall i,j,k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write

$$P(X|Y,Z) = P(X|Z)$$

E.g.,

P(thunder|raining,lightning) = P(thunder|lightning)

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$P(X_1, X_2|Y) =$$

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$P(X_1, X_2|Y) = P(X_1|X_2,Y)P(X_2|Y)$$

Chain rule

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$\begin{aligned} P(X_1, X_2|Y) &= P(X_1|X_2,Y)P(X_2|Y) \\ &= P(X_1|Y)P(X_2|Y) \end{aligned}$$

Conditional independence

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$P(X_1,X_2|Y) = P(X_1|X_2,Y)P(X_2|Y) = P(X_1|Y)P(X_2|Y)$$

in general: $P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$P(X_1,X_2|Y) = P(X_1|X_2,Y)P(X_2|Y) = P(X_1|Y)P(X_2|Y)$$

in general: $P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$

How many parameters to describe $P(X_1 \ldots X_n|Y)$? $P(Y)$?

- Without conditional indep assumption? $2(2^n - 1)$ and 1
- With conditional indep assumption?

Naïve Bayes uses assumption that the X_i are conditionally independent, given Y. E.g., $P(X_1|X_2,Y) = P(X_1|Y)$

Given this assumption, then:

$$P(X_1,X_2|Y) = P(X_1|X_2,Y)P(X_2|Y) = P(X_1|Y)P(X_2|Y)$$

in general: $P(X_1 \ldots X_n|Y) = \prod_i P(X_i|Y)$

How many parameters to describe $P(X_1 \ldots X_n|Y)$? $P(Y)$?

- Without conditional indep assumption? $2(2^n - 1)$ and 1
- With conditional indep assumption? 2n and 1

Naïve Bayes in a Nutshell

Bayes rule:

$$P(Y = y_k|X_1 \ldots X_n) = \frac{P(Y = y_k)P(X_1 \ldots X_n|Y = y_k)}{\sum_j P(Y = y_j)P(X_1 \ldots X_n|Y = y_j)}$$

Assuming conditional independence among X_i's:

$$P(Y = y_k|X_1 \ldots X_n) = \frac{P(Y = y_k)\prod_i P(X_i|Y = y_k)}{\sum_j P(Y = y_j)\prod_i P(X_i|Y = y_j)}$$ (estimate in training)

So, to pick most probable Y for $X^{new} = (X_1, \ldots, X_n)$

$$Y^{new} \leftarrow \arg\max_{y_k} P(Y = y_k)\prod_i P(X_i^{new}|Y = y_k)$$ (testing)

Naïve Bayes Algorithm – discrete X_i

- Train Naïve Bayes (examples)
  for each* value y_k
    estimate $\pi_k \equiv P(Y = y_k)$
    for each* value x_{ij} of each attribute X_i
      estimate $\theta_{ijk} \equiv P(X_i = x_{ij} | Y = y_k)$
- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \theta_{ijk}$$

* probabilities must sum to 1, so need estimate only v-1 of these, where v is the number of values, which is 2 in the binary case

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

O: Is older than 35 years	S: Is a Student	J: Birthday before July 1st	Y: Buys computer
0	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
0	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
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- Y= buys computer

What probability parameters must we estimate?

P(Y=1) :
P(O=1 | Y=1) :
P(O=1 | Y=0) :
P(S=1 | Y=1) :
P(S=1 | Y=0) :
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) :
P(O=0 | Y=1) :
P(O=0 | Y=0) :
P(S=0 | Y=1) :
P(S=0 | Y=0) :
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

P(Y=1) : 9/14
P(O=1 | Y=1) :
P(O=1 | Y=0) :
P(S=1 | Y=1) :
P(S=1 | Y=0) :
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) : 5/14
P(O=0 | Y=1) :
P(O=0 | Y=0) :
P(S=0 | Y=1) :
P(S=0 | Y=0) :
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) :
P(S=1 | Y=1) :
P(S=1 | Y=0) :
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) : **5/14**
P(O=0 | Y=1) :
P(O=0 | Y=0) :
P(S=0 | Y=1) :
P(S=0 | Y=0) :
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
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What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) :
P(S=1 | Y=1) :
P(S=1 | Y=0) :
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) :
P(S=0 | Y=1) :
P(S=0 | Y=0) :
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

- O= Is older than 35
- S= Is a student
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- Y= buys computer

What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) : **2/5**
P(S=1 | Y=1) :
P(S=1 | Y=0) :
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **3/5**
P(S=0 | Y=1) :
P(S=0 | Y=0) :
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(Y|O,S,J)

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What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) : **4/5**
P(S=1 | Y=1) : **6/9**
P(S=1 | Y=0) : **1/5**
P(J=1 | Y=1) :
P(J=1 | Y=0) :

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) :
P(J=0 | Y=0) :

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's train! Buy computer? P(YIO,S,J)

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What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) : **4/5**
P(S=1 | Y=1) : **6/9**
P(S=1 | Y=0) : **1/5**
P(J=1 | Y=1) : **5/9**
P(J=1 | Y=0) : **2/5**

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) : **4/9**
P(J=0 | Y=0) : **3/5**

O	S	J	Y
1	0	0	0
0	0	1	0
1	0	1	1
1	0	1	1
1	1	0	1
1	1	1	0
0	1	0	1
1	0	0	0
0	1	1	1
0	1	0	1
0	1	1	1
1	0	1	1
0	1	0	1
1	0	0	0

Let's test! Buy computer? P(YIO,S,J)

- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) : **4/5**
P(S=1 | Y=1) : **6/9**
P(S=1 | Y=0) : **1/5**
P(J=1 | Y=1) : **5/9**
P(J=1 | Y=0) : **2/5**

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) : **4/9**
P(J=0 | Y=0) : **3/5**

0	1	1	?

Let's test! Buy computer? P(YIO,S,J)

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What probability parameters must we estimate?

P(Y=1) : **9/14**
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P(J=1 | Y=0) : **2/5**

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) : **4/9**
P(J=0 | Y=0) : **3/5**

0	1	1	?

P(O=0,S=1,J=1|Y=0) * P(Y=0) = 1/5 * 1/5 * 2/5 * 5/14 = 0.0057

Let's test! Buy computer? P(YIO,S,J)

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P(O=1 | Y=0) : **4/5**
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P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) : **4/9**
P(J=0 | Y=0) : **3/5**

0	1	1	?

P(O=0,S=1,J=1|Y=0) * P(Y=0) = 1/5 * 1/5 * 2/5 * 5/14 = 0.0057

P(O=0,S=1,J=1|Y=1) * P(Y=1) = 5/9 * 6/9 * 5/9 * 9/14 = 0.13

Let's test! Buy computer? P(YIO,S,J)

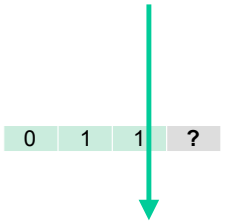
- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

Can already pick label 1

P(Y=1) : **9/14**
P(O=1 | Y=1) : **4/9**
P(O=1 | Y=0) : **4/5**
P(S=1 | Y=1) : **6/9**
P(S=1 | Y=0) : **1/5**
P(J=1 | Y=1) : **5/9**
P(J=1 | Y=0) : **2/5**

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
P(S=0 | Y=1) : **3/9**
P(S=0 | Y=0) : **4/5**
P(J=0 | Y=1) : **4/9**
P(J=0 | Y=0) : **3/5**



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Let's test! Buy computer? P(YIO,S,J)

- O= Is older than 35
- S= Is a student
- J = Birthday before July 1
- Y= buys computer

What probability parameters must we estimate?

$$P(X) = \sum_i P(X|Y=i)P(Y=i)$$

P(Y=1) : **9/14**
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P(O=1 | Y=0) : **4/5**
P(S=1 | Y=1) : **6/9**
P(S=1 | Y=0) : **1/5**
P(J=1 | Y=1) : **5/9**
P(J=1 | Y=0) : **2/5**

P(Y=0) : **5/14**
P(O=0 | Y=1) : **5/9**
P(O=0 | Y=0) : **1/5**
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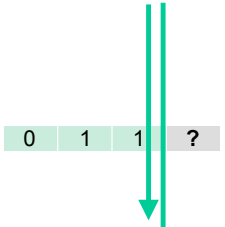

P(O=0,S=1,J=1|Y=0) * P(Y=0) = 1/5 * 1/5 * 2/5 * 5/14 = 0.0057

P(O=0,S=1,J=1|Y=1) * P(Y=1) = 5/9 * 6/9 * 5/9 * 9/14 = 0.13

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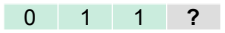

P(Y=0|O=0,S=1,J=1) = 0.04

P(Y=1|O=0,S=1,J=1) = 0.96 ← Choice is Label 1

Assume we have a lot of data

- We estimate these probabilities
- We obtain 68% test accuracy.
  - What does this mean?
- Our estimated P(Y=1) was 64.5%, let's say this is actually the truth.
- Is 68% impressive?

- What is chance performance?
- What happens if I flip an unbiased coin?
- If P(Y=1)>0.5, then we can just predict 1 all the time!
  - What will be the accuracy?
- What happens if you predict Y=1 with probability 0.645 ==> this is called probability matching in cognitive science

Naïve Bayes: Point #1

Often the X_i are not really conditionally independent

- We use Naïve Bayes in many cases anyway, and it often works pretty well
  - often the right classification, even when not the right probability (see [Domingos&Pazzani, 1996])
- What is effect on estimated P(Y|X)?
  - Extreme case: what if we add two copies: $X_i = X_k$

Extreme case: what if we add two copies: $X_i = X_k$

$$\frac{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1)}{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1) + P(Y=0)\,P(O=1|Y=0)\,P(S=0|Y=0)\,P(J=0|Y=0)}$$

Naïve Bayes: Point #2

Irrelevant variables, do they affect you?

$$\frac{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1)}{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1) + P(Y=0)\,P(O=1|Y=0)\,P(S=0|Y=0)\,P(J=0|Y=0)}$$

Naïve Bayes: Point #2

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If J is not relevant then P(J|Y=0) = P(J|Y=1) = P(J)

Does it hurt classification?

Naïve Bayes: Point #2

Irrelevant variables, do they affect you?

$$\frac{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1)}{P(Y=1)\,P(O=1|Y=1)\,P(S=0|Y=1)\,P(J=0|Y=1) + P(Y=0)\,P(O=1|Y=0)\,P(S=0|Y=0)\,P(J=0|Y=0)}$$

If J is not relevant then P(J|Y=0) = P(J|Y=1) = P(J)

Does it hurt classification?

If we had the correct estimate, performance is not affected!
If we have noisy estimates ==> performance is affected

Naïve Bayes: Point #3

Another way to view Naïve Bayes (Boolean Y, X_i's):

Decision rule: is this quantity greater or less than 1?

$$\frac{P(Y=1|X_1 \ldots X_n)}{P(Y=0|X_1 \ldots X_n)} = \frac{P(Y=1)\prod_i P(X_i|Y=1)}{P(Y=0)\prod_i P(X_i|Y=0)}$$

(is q_1 larger than q_0?)

Naïve Bayes: Point #3

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Decision rule: is this quantity greater or less than 1?

$$\frac{P(Y=1|X_1 \ldots X_n)}{P(Y=0|X_1 \ldots X_n)} = \frac{P(Y=1)\prod_i P(X_i|Y=1)}{P(Y=0)\prod_i P(X_i|Y=0)}$$ **> or < 1 ?**

$$log\frac{P(Y=1|X_1 \ldots X_n)}{P(Y=0|X_1 \ldots X_n)} = log\frac{P(Y=1)}{P(Y=0)} + \sum_i log\frac{P(X_i|Y=1)}{P(X_i|Y=0)}$$

> or < 0 ?

$$= log\frac{P(Y=1)}{P(Y=0)} + \sum_i X_i log\frac{\theta_{i1}}{\theta_{i0}} + (1-X_i)\frac{1-\theta_{i1}}{1-\theta_{i0}}$$

$$\theta_{ik} = \hat{P}(X_i = 1|Y=k)$$

$$1-\theta_{ik} = \hat{P}(X_i = 0|Y=k)$$

Naïve Bayes: Point #3

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What happens when **X** has a large number of dimensions?

Naïve Bayes: Point #3

Another way to view Naïve Bayes (Boolean Y, X_i's):

Decision rule: is this quantity greater or less than 1?

$$\frac{P(Y=1|X_1 \ldots X_n)}{P(Y=0|X_1 \ldots X_n)} = \frac{P(Y=1)\prod_i P(X_i|Y=1)}{P(Y=0)\prod_i P(X_i|Y=0)}$$

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$$= log\frac{P(Y=1)}{P(Y=0)} + \sum_i x_i log\frac{\theta_{i1}}{\theta_{i0}} + (1-x_i)\frac{1-\theta_{i1}}{1-\theta_{i0}}$$

$$\theta_{ik} = \hat{P}(x_i = 1|Y=k)$$

$$1-\theta_{ik} = \hat{P}(x_i = 0|Y=k)$$

Prevents underflow

Naïve Bayes: Point #4

If unlucky, our MLE estimate for $P(X_i \mid Y)$ might be zero.
(for example, X_i = *birthdate*. X_i = *Jan_25_1992*)

- Why worry about just one parameter out of many?

- What can be done to address this?

Naïve Bayes: Point #4

If unlucky, our MLE estimate for $P(X_i \mid Y)$ might be zero.
(for example, X_i = *birthdate*. X_i = *Jan_25_1992*)

- Why worry about just one parameter out of many?

$$P(Y=i)P(X_1|Y=i)P(X_2|Y=i)\ldots P(X_n|Y=i)$$

- What can be done to address this?

Estimating Parameters

- Maximum Likelihood Estimate (MLE): choose θ that maximizes probability of observed data $\mathcal{D}$

$$\hat{\theta} = \arg\max_{\theta} P(\mathcal{D} \mid \theta)$$

- Maximum a Posteriori (MAP) estimate: choose θ that is most probable given prior probability and the data

$$\hat{\theta} = \arg\max_{\theta} P(\theta \mid \mathcal{D})$$

$$= \arg\max_{\theta} = \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}$$

Estimating Parameters: Y, X_i discrete-valued

Maximum likelihood estimates:

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$$

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_j | Y = y_k) = \frac{\#D\{X_i = x_j \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

MAP estimates (Beta, Dirichlet priors):

$$\hat{\pi}_k = \hat{P}(Y = y_k) = \frac{\#D\{Y = y_k\} + (\beta_k - 1)}{|D| + \sum_m (\beta_m - 1)}$$

$$\hat{\theta}_{ijk} = \hat{P}(X_i = x_j | Y = y_k) = \frac{\#D\{X_i = x_j \wedge Y = y_k\} + (\beta_k - 1)}{\#D\{Y = y_k\} + \sum_m (\beta_m - 1)}$$

Only difference: "imaginary" examples

How many counts should we add?

Learning to classify text documents

- Classify which emails are spam?
- Classify which emails promise an attachment?
- Classify which web pages are student home pages?

How shall we represent text documents for Naïve Bayes?

Baseline: Bag of Words Approach

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

How can we express X?

- Y discrete valued. e.g., Spam or not
- $X = (X_1, X_2, \ldots X_n)$ with n the number of words in English.

  - What are the problems with this representation?

How can we express X?

- Y discrete valued. e.g., Spam or not
- $X = (X_1, X_2, \ldots X_n)$ with n the number of words in English.

  - What are the problems with this representation?
    - Some words always present
    - Some words very infrequent
    - Doesn't count how often a word appears
    - Conditional independence assumption is false…

Learning to classify document: P(YIX) the "Bag of Words" model

- Y discrete valued. e.g., Spam or not
- $X = (X_1, X_2, \ldots X_n)$ = document

- X_i is a random variable describing the word at position i in the document
- possible values for X_i : any word w_k in English
- X_i represents the *i^{th} word position* in document
- X_1 = "I", X_2 = "am", X_3 = "pleased"
- and, let's assume the X_i are iid (indep, identically distributed)

$$P(X_i|Y) = P(X_j|Y) \quad (\forall i, j)$$

Learning to classify document: P(YIX) the "Bag of Words" model

- Y discrete valued. e.g., Spam or not
- $X = (X_1, X_2, \ldots X_n)$ = document

- X_i is a random variable describing the word at position i in the document
- possible values for X_i : any word w_k in English

Multinomial distribution

$$P(D|\theta) \propto \theta_1^{\alpha_1}\theta_2^{\alpha_2}\ldots\theta_n^{\alpha_n}$$

All X_i have the same distribution

Learning to classify document: P(YIX) the "Bag of Words" model

- Y discrete valued. e.g., Spam or not
- X = (X_1, X_2, … X_n) = document
- X_i is a random variable describing the word at position i in the document
- possible values for X_i : any word w_k in English
- Document = bag of words: the vector of counts for all w_k's
  - like #heads, #tails, but we have many more than 2 values
  - assume word probabilities are position independent (i.i.d. rolls of a 50,000-sided die)

Naïve Bayes Algorithm – discrete X_i

- Train Naïve Bayes (examples)
  for each value y_k
    estimate $\pi_k \equiv P(Y = y_k)$
    for each value x_j of each attribute X_i
      estimate $\theta_{ijk} \equiv P(X_i = x_j | Y = y_k)$

prob that word xj appears in position i, given Y=y_k

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \theta_{ijk}$$

* Additional assumption: word probabilities are position independent

$$\theta_{ijk} = \theta_{mjk} \quad \text{for all } i, m$$

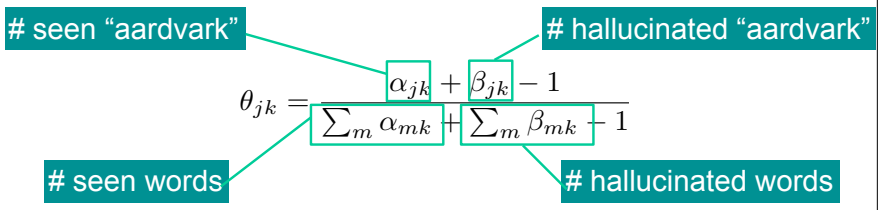
MAP estimates for bag of words

Map estimate for multinomial

$$\theta_{jk} = \frac{\alpha_{jk} + \beta_{jk} - 1}{\sum_m \alpha_{mk} + \sum_m \beta_{mk} - 1}$$

MAP estimates for bag of words

Map estimate for multinomial



What β's should we choose?

MAP estimates for bag of words

Map estimate for multinomial

seen "aardvark"

hallucinated "aardvark"

$$\theta_{jk} = \frac{\alpha_{jk} + \beta_{jk} - 1}{\sum_m \alpha_{mk} + \sum_m \beta_{mk} - 1}$$

seen words

hallucinated words

What β's should we choose?

Probabilities over all classes?

Constant per word?

MAP estimates for bag of words

$$P(\theta_k) = \frac{\theta_{1k}^{\beta_{1k}}, \theta_{2k}^{\beta_{2k}}, \ldots, \theta_{mk}^{\beta_{mk}}}{Beta(\beta_{1k}, \beta_{2k}, \ldots, \beta_{mk})}$$

Map estimate for multinomial

Prior is Dirichlet

$$\theta_{jk} = \frac{\alpha_{jk} + \beta_{jk} - 1}{\sum_m \alpha_{mk} + \sum_m \beta_{mk} - 1}$$

Posterior is also Dirichlet

What β's should we choose?

Probabilities over all classes?

Constant per word?

(Dirichlet is the conjugate prior for a multinomial likelihood function)

Twenty NewsGroups

For code and data, see www.cs.cmu.edu/~tom/mlbook.html click on "Software and Data"

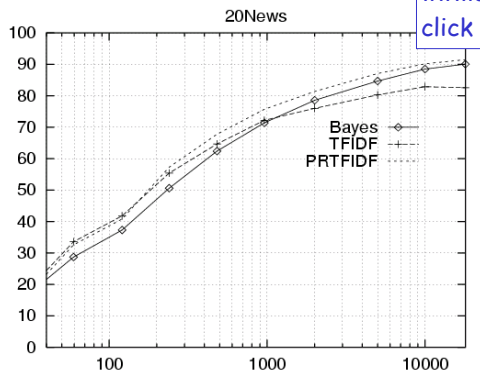
Given 1000 training documents from each group
Learn to classify new documents according to
which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naive Bayes: 89% classification accuracy

Learning Curve for 20 Newsgroups

For code and data, see www.cs.cmu.edu/~tom/mlbook.html click on "Software and Data"



19 Prior probabilities

Accuracy vs. Training set size (1/3 withheld for test)

Learning Curve for 20 Newsgroups

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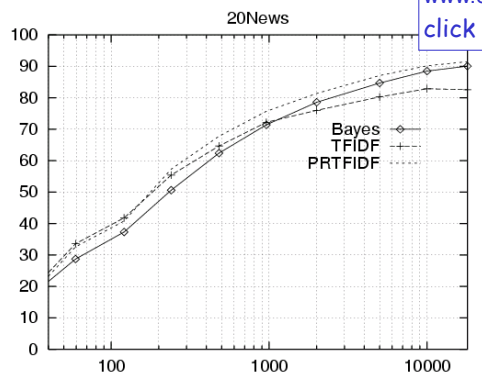


Accuracy vs. Training set size (1/3 withheld for test)

19 Prior probabilities

Can group over sub-classes to generate prior counts

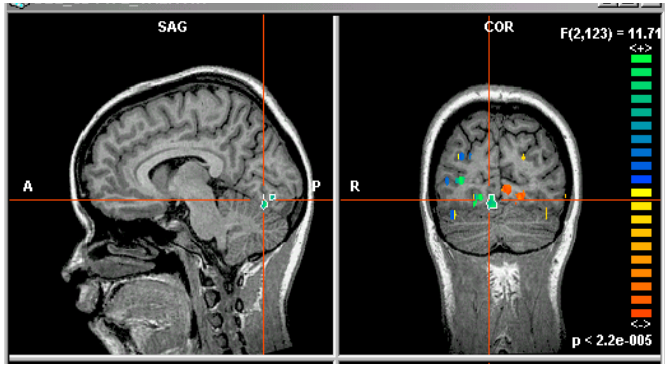
(hallucinate training examples)

Performance can be very good

- Even when taking half of the email
  - Assumption doesn't hurt the particular problem?
  - Redundancy?
  - Leads less examples to train? Converges faster to asymptotic performance? (Ng and Jordan)

What if we have continuous X_i ?

Eg., image classification: X_i is real-valued ith pixel



What if we have continuous X_i ?

Eg., image classification: X_i is real-valued ith pixel

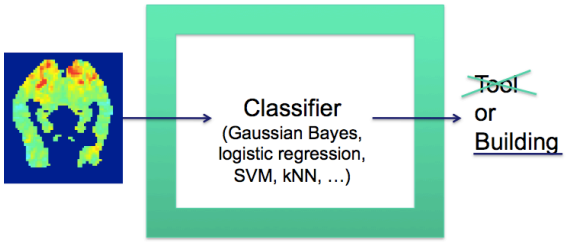
Naïve Bayes requires $P(X_i \mid Y{=}y_k)$, but X_i is real (continuous)

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

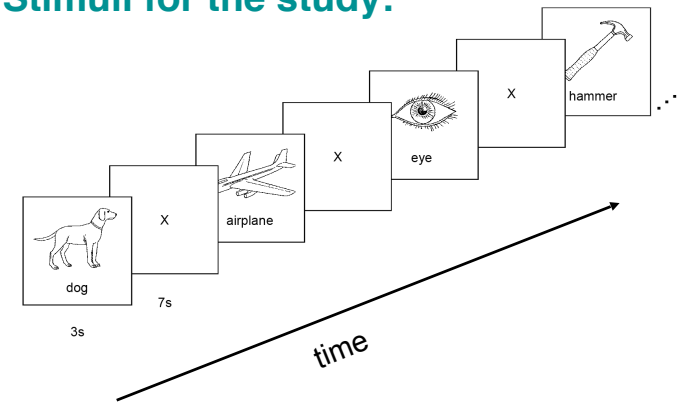
Common approach: assume $P(X_i \mid Y{=}y_k)$ follows a Normal (Gaussian) distribution

GNB Example: Classify a person's cognitive state, based on brain image

- reading a sentence or viewing a picture?
- reading the word describing a "Tool" or "Building"?
- answering the question, or getting confused?



Stimuli for the study:



60 distinct exemplars, presented 6 times each

What if we have continuous X_i ?

Gaussian Naïve Bayes (GNB): assume

$$p(X_i = x | Y = y_k) = \frac{1}{\sqrt{2\pi\sigma_{ik}^2}} \; e^{-\frac{1}{2}(\frac{x-\mu_{ik}}{\sigma_{ik}})^2}$$

Sometimes assume variance

- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)

Gaussian Naïve Bayes Algorithm – continuous X_i (but still discrete Y)

- Train Naïve Bayes (examples)
  for each value y_k
    estimate* $\pi_k \equiv P(Y = y_k)$
    for each attribute X_i estimate $P(X_i | Y = y_k)$
    - class conditional mean μ_{ik}, variance σ_{ik}

- Classify (X^{new})

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i^{new} | Y = y_k)$$

$$Y^{new} \leftarrow \arg\max_{y_k} \; \pi_k \prod_i \mathcal{N}(X_i^{new}; \mu_{ik}, \sigma_{ik})$$

* probabilities must sum to 1, so need estimate only n-1 parameters...

Estimating Parameters: Y discrete, X_i continuous

Maximum likelihood estimates:

$$\hat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

jth training example

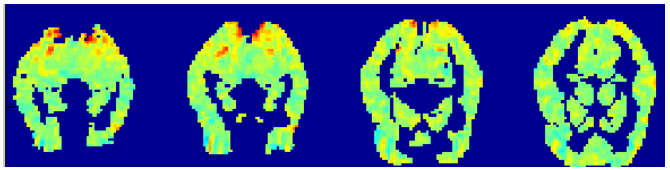
ith feature

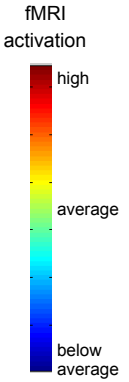
kth class

δ()=1 if (Yj=yk) else 0

$$\hat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j (X_i^j - \hat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

Mean activations over all training examples for Y="bottle"

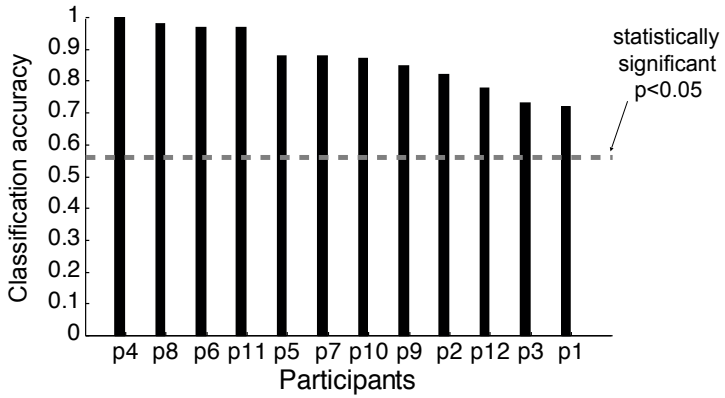




Y is the mental state (reading "house" or "bottle")
X_i are the voxel activities,

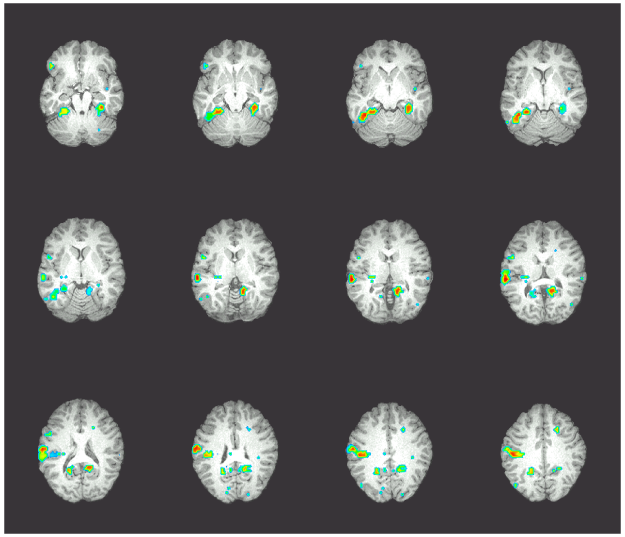
this is a plot of the μ's defining P(X_i | Y="bottle")

Classification task: is person viewing a "tool" or "building"?



Where is information encoded in the brain?

Accuracies of cubical 27-voxel classifiers centered at each significant voxel [0.7-0.8]



Questions to think about:

- How can we extend Naïve Bayes if just 2 of the X_i's are dependent?
- What error will the classifier achieve if Naïve Bayes assumption is satisfied and we have infinite training data?
- What does the decision surface of a Naïve Bayes classifier look like?
- Can you use Naïve Bayes for a combination of discrete and real-valued X_i?

We covered:

- Bayes classifiers to learn P(Y|X)
- MLE and MAP estimates for parameters of P
- Conditional independence
- Naïve Bayes → make Bayesian learning practical
- Text classification
- Naïve Bayes and continuous variables X_i:
  - Gaussian Naïve Bayes classifier

Next:

- Learn P(Y|X) directly
  - Logistic regression, Regularization, Gradient ascent
- Naïve Bayes or Logistic Regression?
  - Generative vs. Discriminative classifiers

Gaussian Naïve Bayes – Big Picture

$$Y^{new} \leftarrow \arg\max_{y \in \{0,1\}} P(Y=y) \prod_i P(X_i^{new} | Y=y)$$ assume P(Y=1) = 0.5

Example: Y= PlayBasketball (boolean), X1=Height, X2=MLgrade

Logistic Regression

Idea:

- Naïve Bayes allows computing P(Y|X) by learning P(Y) and P(X|Y)
- Why not learn P(Y|X) directly?