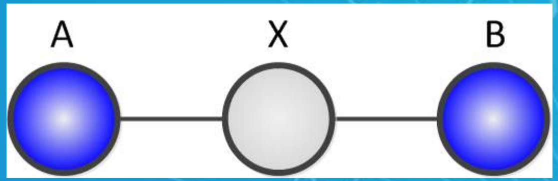
Probabilistic Graphical Models

Spectral Learning for Graphical Models

Eric Xing

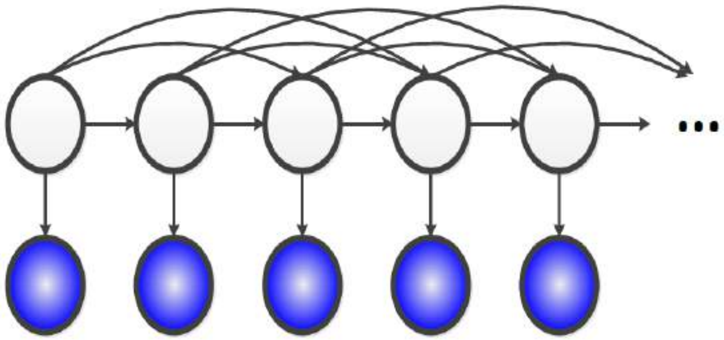
Lecture 25, April 20, 2020



Reading: see class homepage

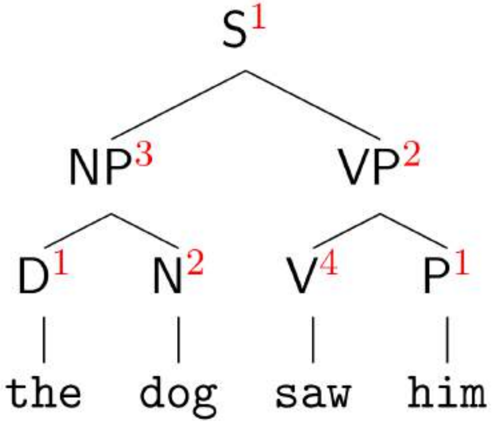
Acknowledgement: slides drafted by Ankur Parikh

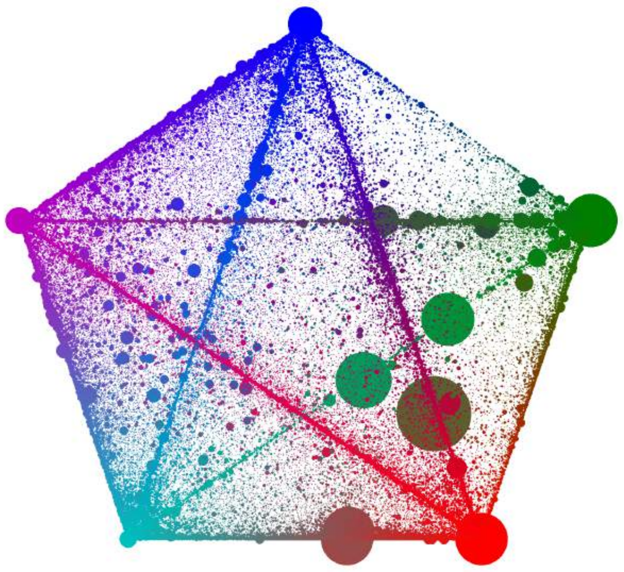
Latent Variable Models



Sequence models

Parsing

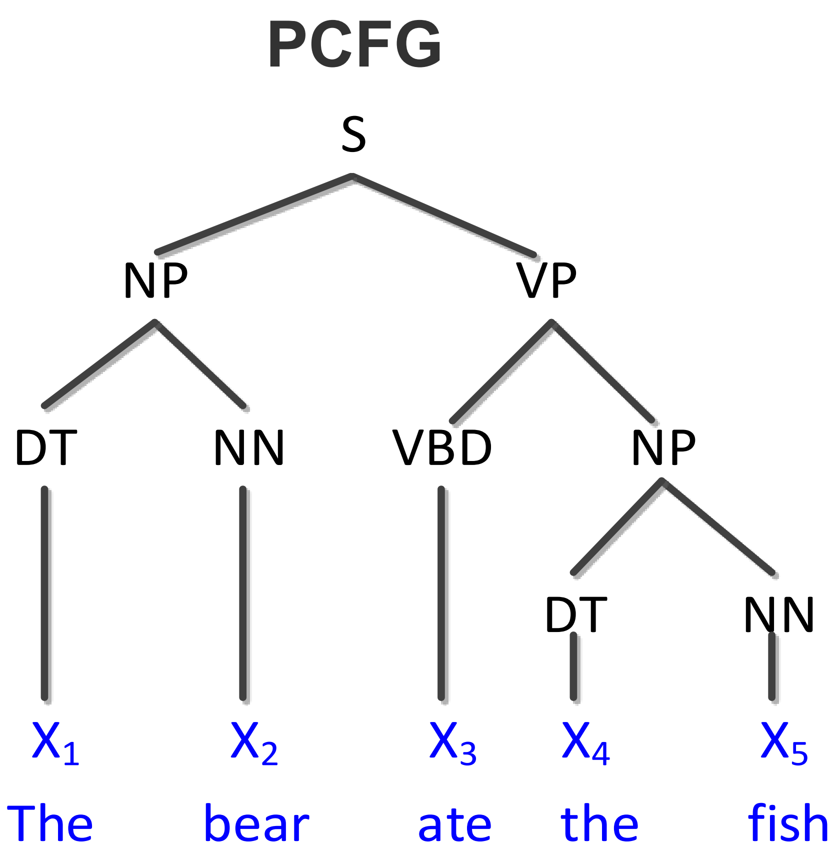


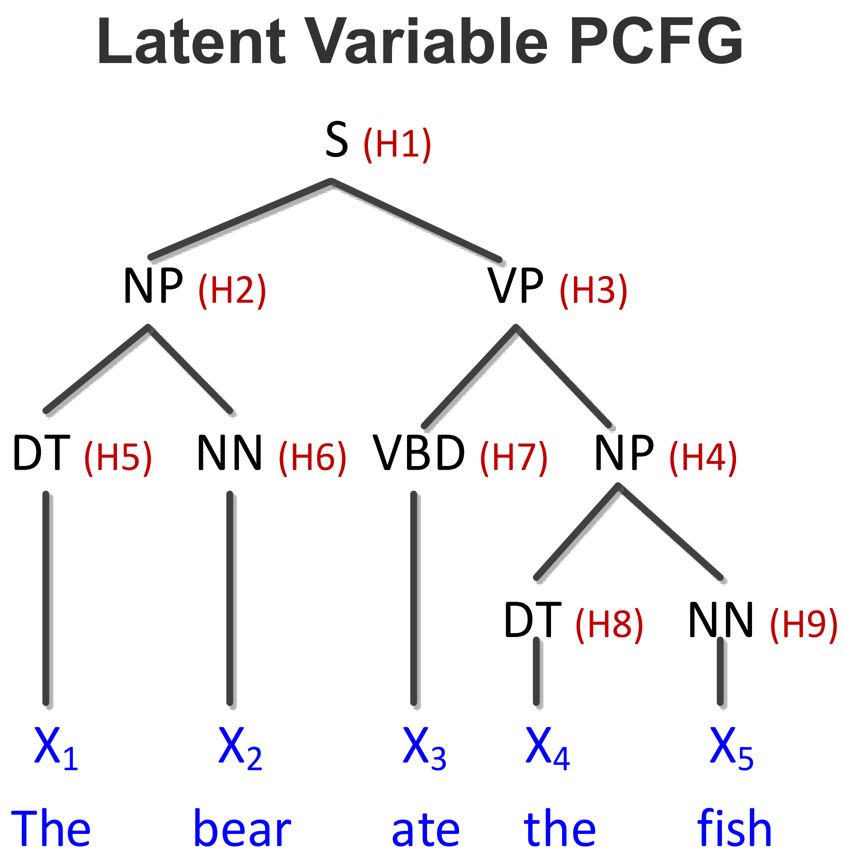


Ho. et al. 2012

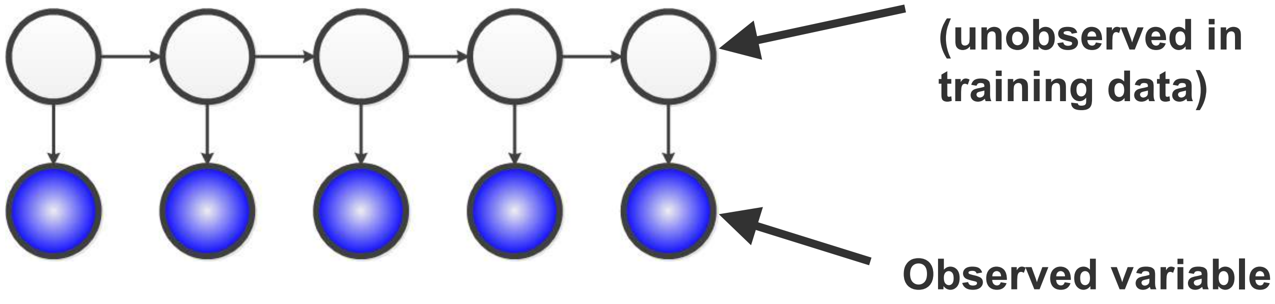
Mixed membership models

Latent Variable PCFG [Matsuzaki et al., 2005, Petrov et al. 2006]





Learning Parameters (EM)



$$\mathbb{P}[X_1, \ldots, X_5, H_1, \ldots, H_5] = \mathbb{P}[H_1] \prod_{i=2}^{5} \mathbb{P}[H_i | H_{i-1}] \prod_{i=1}^{5} \mathbb{P}[X_i | H_i]$$

Since latent variables are not observed in the data, we have to use Expectation Maximization (EM) to learn parameters

- **Slow**
- **Local Minima**

Spectral Learning

- Different paradigm of learning in latent variable models based on linear algebra

- Theoretically,
  - Provably consistent
  - Can offer deeper insight into the identifiability

- Practically,
  - Local minima free
  - As of now, performs comparably to EM with 10-100x speed-up
  - Can also model non-Gaussian continuous data using kernels (usually performs much better than EM in this case)

Related References

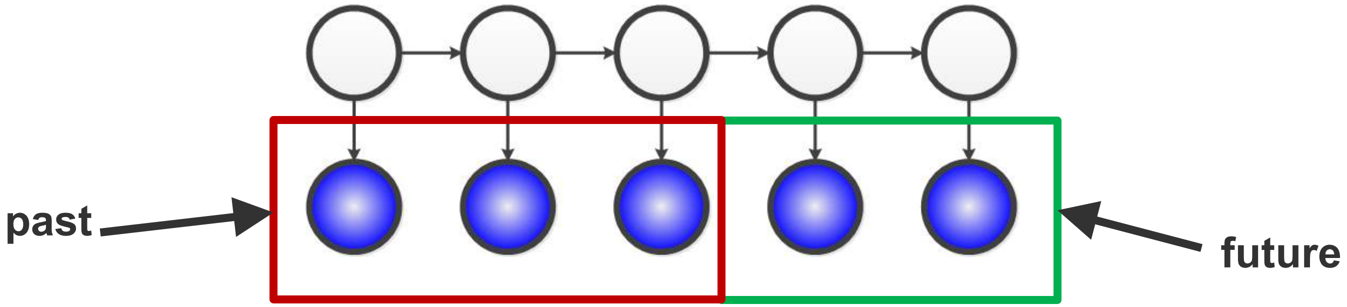
- Relevant works
  - **Hsu et al. 2009** – Spectral HMMs (also Bailly 2009)
  - **Siddiqi et al. 2009** – Features in Spectral Learning
  - **Parikh et al. 2011/2012** –Tensors to Generalize to Trees/Low Treewidth Graphs
  - **Cohen et al. 2012 / 2013** – Spectral Learning of latent PCFGs

- Will present it from "matrix factorization" view:
  - **Balle et al. 2012** – Connection between Spectral Learning / Hankel Matrix Factorization
  - **Song et al. 2013** – Spectral Learning as Hierarchical Tensor Decomposition

Focusing on Prediction

- In many applications that use latent variable models, the end task is not to recover the latent states, but rather to use the model for prediction among observed variables.

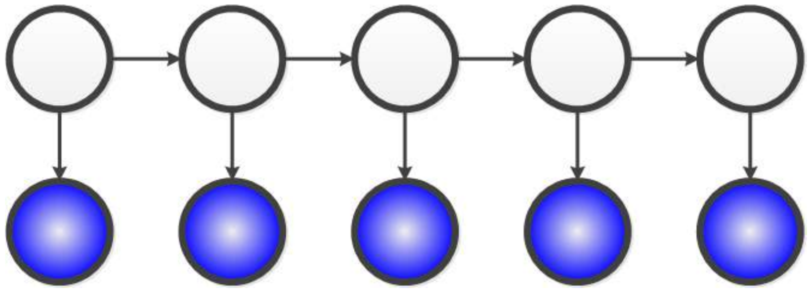
- Dynamical Systems – Predict future given past



Focusing on Prediction

- We will only be concerned with quantities related to the observed variables: $\mathbb{P}[X_1, X_2, X_3, X_4, X_5]$

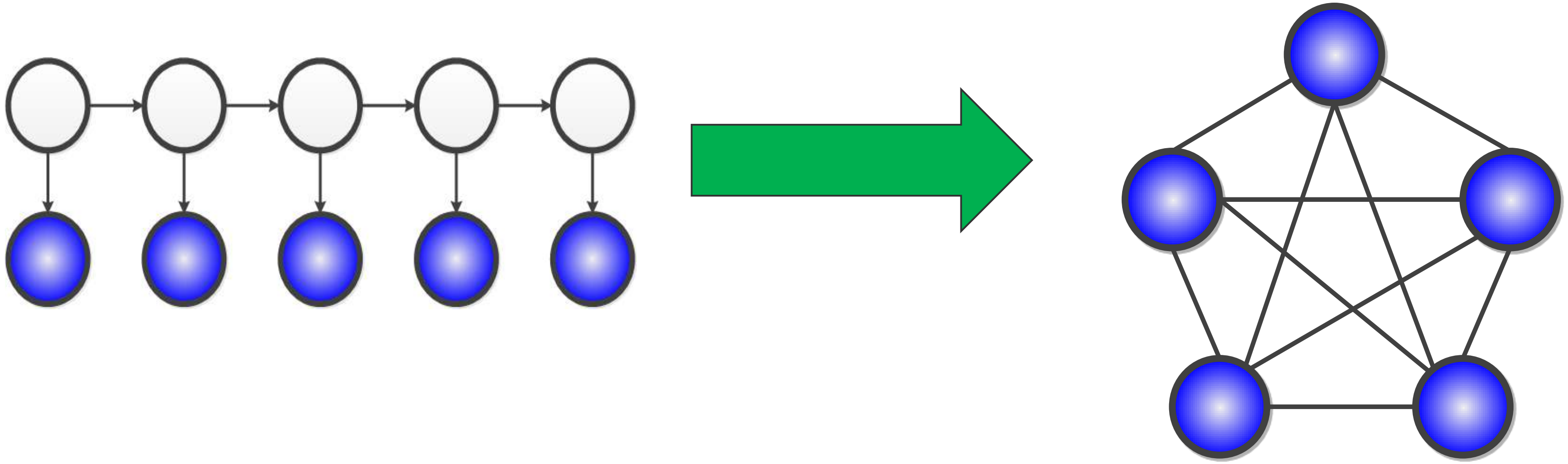
- We do not care about the latent variables explicitly.



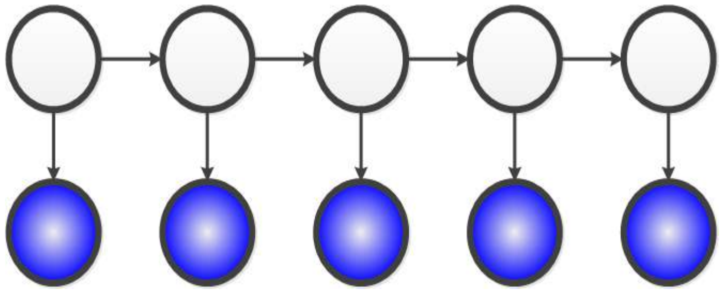
- **Do we still need EM to learn the parameters?**

But if we don’t care about the latent variables....

- Why don’t we just integrate them out?
- Because integrating them out results in a clique ☹



Marginal Does Not Factorize

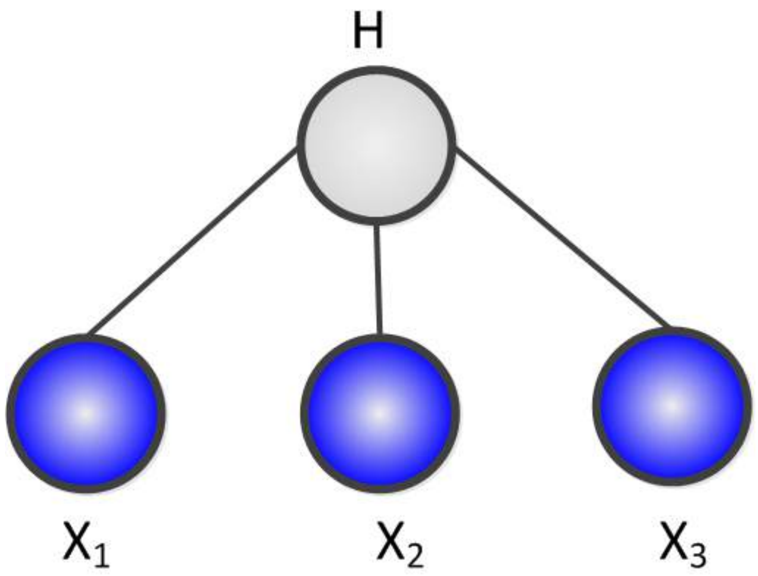


$$\mathbb{P}[X_1, X_2, X_3, X_4, X_5] = \sum_{H_1,\ldots,H_5} \mathbb{P}[H_1]\mathbb{P}[H_1] \prod_{i=2}^{5} \mathbb{P}[H_i|H_{i-1}] \prod_{i=1}^{5} \mathbb{P}[X_i|H_i]$$

Does not factorize due to the outer sum (Can somewhat distribute the sum, but doesn’t solve problem)

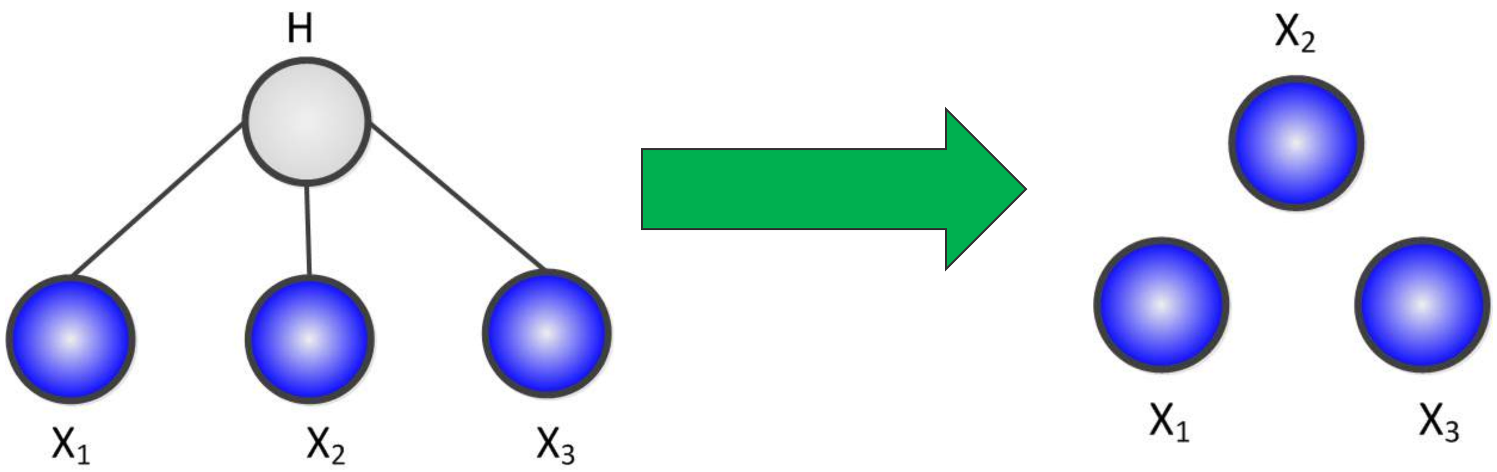
But isn't an HMM different from a clique?

- It depends on the number of latent states.
- Consider the following model.



If H has only one state.....

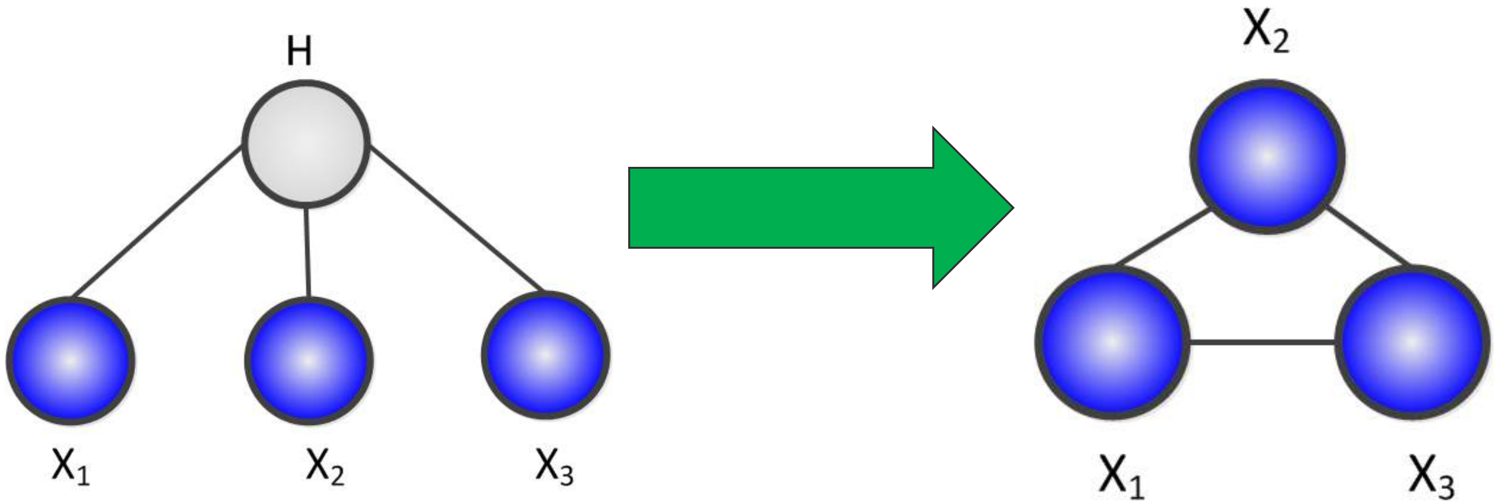
- Then the observed variables are independent!



What if H has many states?

- Let us say the observed variables each have m states.

- Then if H has m^3 states then the latent model can be exactly equivalent to a clique (depending on how parameters are set).



- But what about all the other cases?

The Question

- Under existing methods, latent models all require EM to learn regardless of the number of hidden states.
- However, is there a formulation of latent variable models where the difficulty of learning is a function of the number of latent states?
- This is the question that the *spectral view* will answer.

Sum Rule (Matrix Form)

- Sum Rule

$$\mathbb{P}[X] = \sum_{Y} \mathbb{P}[X|Y]\mathbb{P}[Y]$$

- Equivalent view using Matrix Algebra

$$\mathcal{P}[X] = \mathcal{P}[X|Y] \times \mathcal{P}[Y]$$

$$\begin{pmatrix} \mathbb{P}[X=0] \\ \mathbb{P}[X=1] \end{pmatrix} = \begin{pmatrix} \mathbb{P}[X=0|Y=0] & \mathbb{P}[X=0|Y=1] \\ \mathbb{P}[X=1|Y=0] & \mathbb{P}[X=1|Y=1] \end{pmatrix} \times \begin{pmatrix} \mathbb{P}[Y=0] \\ \mathbb{P}[Y=1] \end{pmatrix}$$

Chain Rule (Matrix Form)

- Chain Rule

$$\mathbb{P}[X, Y] = \mathbb{P}[X|Y]\mathbb{P}[Y] = \mathbb{P}[Y|X]\mathbb{P}[Y]$$

- Equivalent view using Matrix Algebra

Means on diagonal

$$\mathcal{P}[X, Y] = \quad \mathcal{P}[X|Y] \quad \times \mathcal{P}[\oslash Y]$$

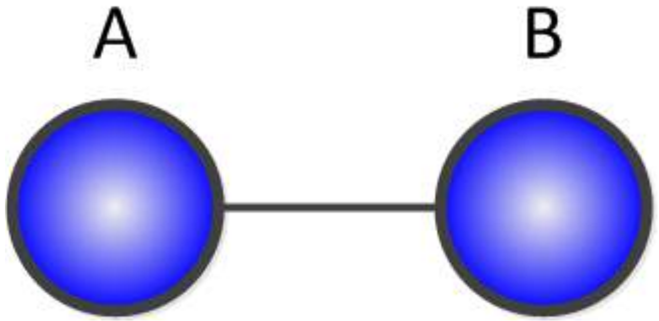
$$\begin{pmatrix} \mathbb{P}[X=0, Y=0] & \mathbb{P}[X=0, Y=1] \\ \mathbb{P}[X=1, Y=0] & \mathbb{P}[X=1, Y=1] \end{pmatrix} =$$

$$\begin{pmatrix} \mathbb{P}[X=0|Y=0] & \mathbb{P}[X=0|Y=1] \\ \mathbb{P}[X=1|Y=0] & \mathbb{P}[X=1|Y=1] \end{pmatrix} \times \begin{pmatrix} \mathbb{P}[Y=0] & 0 \\ 0 & \mathbb{P}[Y=1] \end{pmatrix}$$

- Note how diagonal is used to keep ***Y*** from being marginalized out.

Graphical Models: The Linear Algebra View

- In general, nothing we can say about the nature of this matrix.

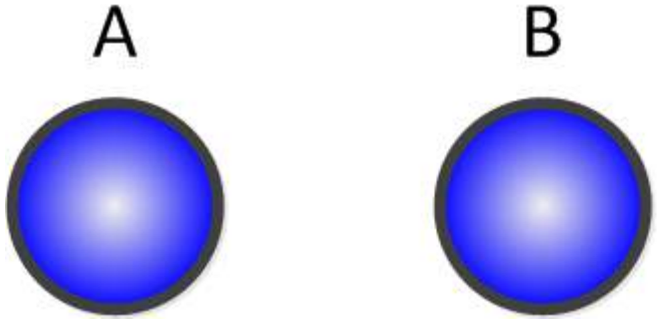


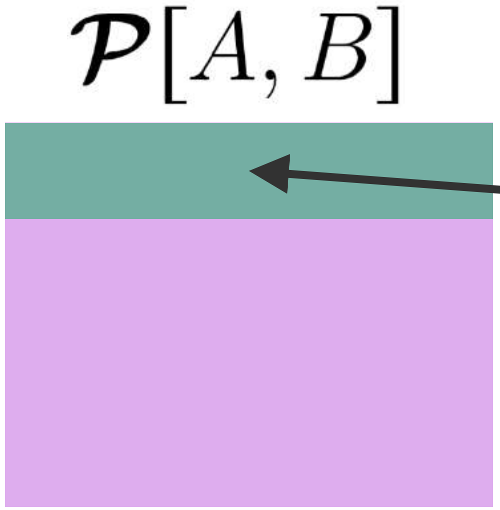
$$\mathcal{P}[A, B]$$

A and B have m states each.

Independence: The Linear Algebra View

- What if we know A and B are independent?





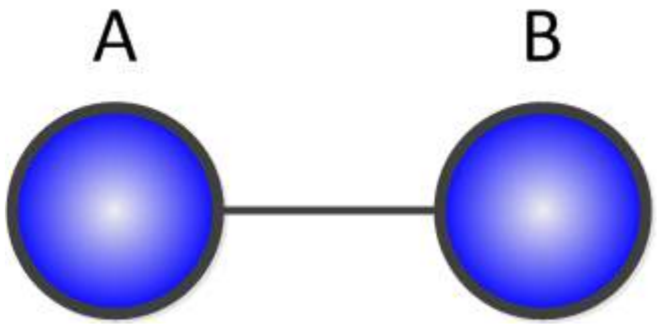
$$\left(\mathbb{P}[A=1, B=1], \ldots, \mathbb{P}[A=1, B=m] \right)$$

$$= \left(\mathbb{P}[A=1] (\mathbb{P}[B=1], \ldots, \mathbb{P}[B=m]) \right)$$

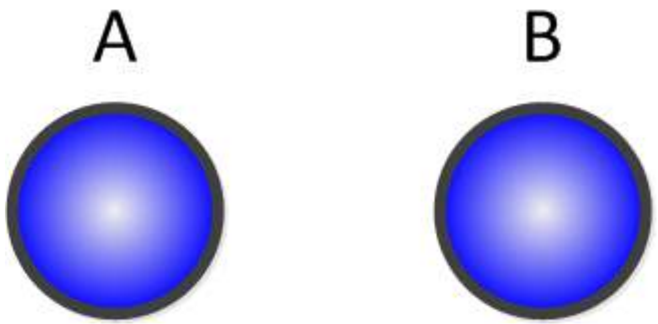
- Joint probability matrix is rank one, since all rows are multiples of one another!!

Independence and Rank

- What about rank in between 1 and m?



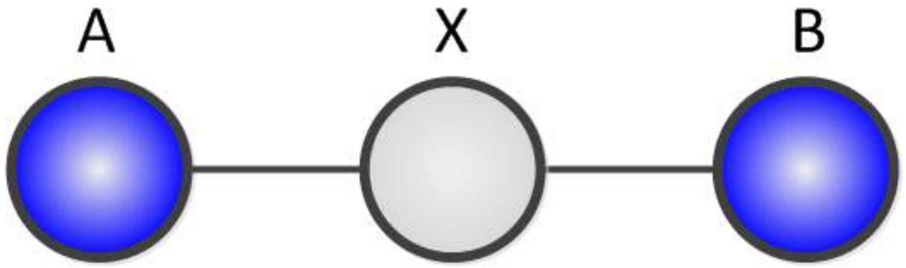
$\mathcal{P}[A, B]$ **has rank m (at most)**



$\mathcal{P}[A, B]$ **has rank 1**

Low Rank Structure

- ***A*** and ***B*** are not marginally independent (They are only conditionally independent given ***X***).



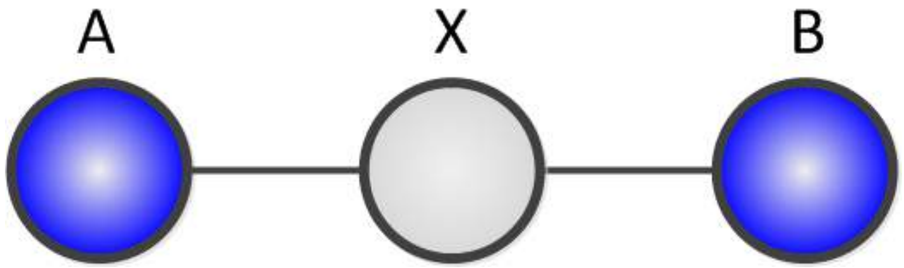
- Assume ***X*** has ***k*** states (while ***A*** and ***B*** have ***m*** states).

- Then,

$$rank(\mathcal{P}[A, B]) \leqslant k$$

- Why?

Low Rank Structure



$$\mathcal{P}[A, B] = \mathcal{P}[A|X] \quad \mathcal{P}(\oslash X) \quad \mathcal{P}[B|X]^\top$$

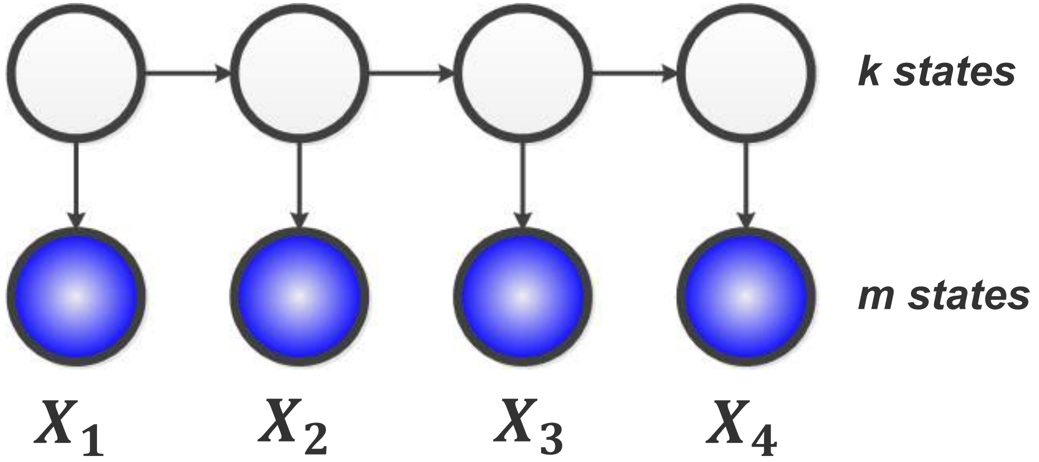
$rank \leq k$ $\qquad$ $rank \leq k$ $\qquad$ $\cancel{rank \leq k}$ $\qquad$ $rank \leq k$

The Spectral View

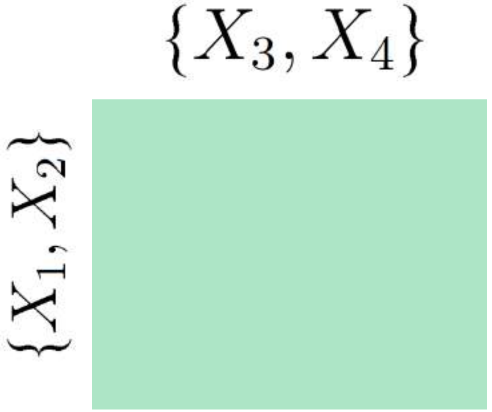
- Latent variable models encode low rank dependencies among variables *(both marginal and conditional)*

- Use tools from linear algebra to exploit this structure.
  - Rank
  - Eigenvalues
  - SVD
  - Tensors

A More Interesting Example



$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$



has rank k

Low Rank Matrices "Factorize"

$$M = LR$$

m by n   m by k   k by n

If M has rank k

We already know one factorization!!!

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}} | H_2] \mathcal{P}[\oslash H_2] \mathcal{P}[X_{\{3,4\}} | H_2]^\top$$

Factor of 4 variables   Factor of 3 variables   Factor of 3 variables

Factor of 1 variable

Alternate Factorizations

- The key insight is that this factorization is not unique.

- Consider Matrix Factorization. Can add any invertible transformation:

$$M = LR$$
$$M = LSS^{-1}R$$

- **The magic of spectral learning is that there exists an alternative factorization that only depends on observed variables!**

An Alternate Factorization

- Let us say we only want to factorize this matrix of 4 variables

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$

such that it is product of matrices that contain at most three ***observed*** variables e.g.

$$\mathcal{P}[X_{\{1,2\}}, X_3]$$

$$\mathcal{P}[X_2, X_{\{3,4\}}]$$

An Alternate Factorization

- Note that

$$\mathcal{P}[X_{\{1,2\}}, X_3] = \underline{\mathcal{P}[X_{\{1,2\}}|H_2]}\,\underline{\mathcal{P}[\oslash H_2]}\,\underline{\mathcal{P}[X_3|H_2]^\top}$$

$$\mathcal{P}[X_2, X_{\{3,4\}}] = \underline{\mathcal{P}[X_2|H_2]}\,\underline{\mathcal{P}[\oslash H_2]}\,\underline{\mathcal{P}[X_{\{3,4\}}|H_2]^\top}$$

- Product of green terms (in some order) is

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]$$

- Product of red terms (in some order) is

$$\mathcal{P}[X_2, X_3]$$

An Alternate Factorization

$$\underbrace{\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]}_{\text{factor of 4 variables}} = \underbrace{\mathcal{P}[X_{\{1,2\}}, X_3]}_{\text{factor of 3 variables}} \underbrace{\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4\}}]}_{\text{factor of 3 variables}}$$

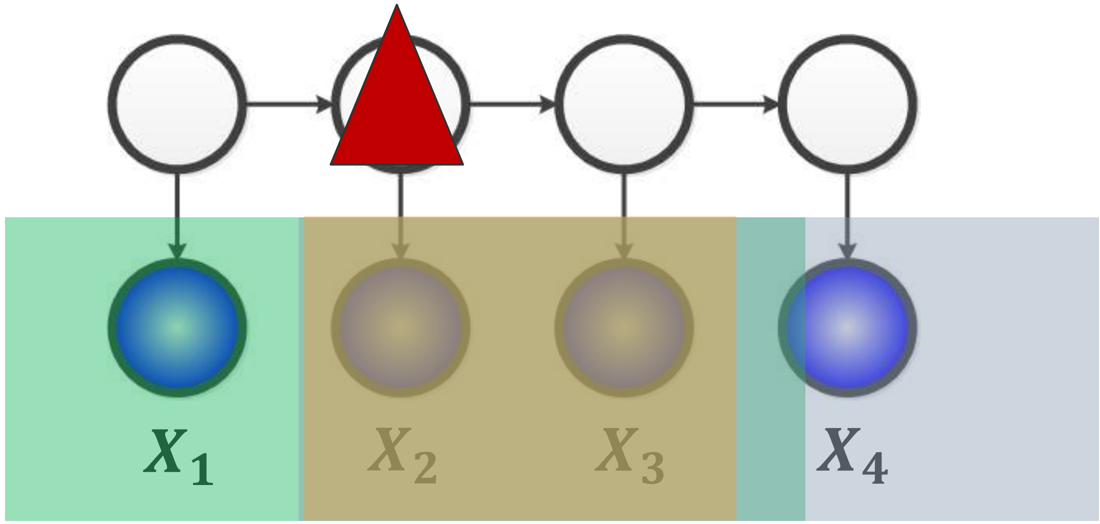
Advantage: Factors are only functions of observed variables! Can be directly computed from data without EM!!!!

Caveat: some factors are no longer probability tables (do not have to be non-negative)

We will call this factorization the observable factorization.

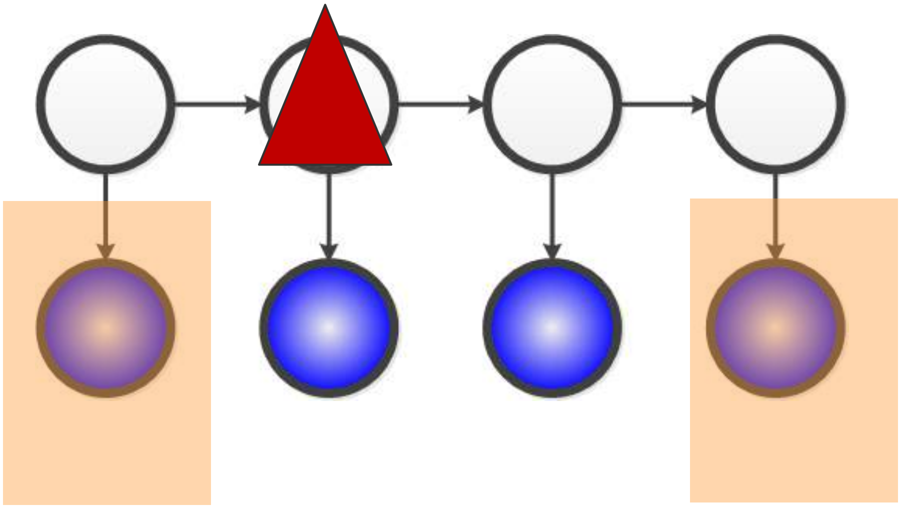
Graphical Relationship

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \underline{\mathcal{P}[X_{\{1,2\}}, X_3]}\,\underline{\mathcal{P}[X_2, X_3]^{-1}}\,\underline{\mathcal{P}[X_2, X_{\{3,4\}}]}$$



Another Factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_4]\mathcal{P}[X_1, X_4]^{-1}\mathcal{P}[X_1, X_{\{3,4\}}]$$



- Seems we would do better empirically if you could “combine” both factorizations. Will come back to this later.

Relationship to Original Factorization

- What is the relationship between the original factorization and the new factorization?

$$\underbrace{\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]}_{\boldsymbol{M}} = \underbrace{\mathcal{P}[X_{\{1,2\}}|H_2]\mathcal{P}[\oslash H_2]}_{\boldsymbol{L}}\underbrace{\mathcal{P}[X_{\{3,4\}}|H_2]}_{\boldsymbol{R}}^{\top}$$

$$\boldsymbol{M} = \boldsymbol{L}\boldsymbol{R}$$

$$\boldsymbol{M} = \boldsymbol{L}\boldsymbol{S}\boldsymbol{S}^{-1}\boldsymbol{R}$$

Can I choose S to get the observable factorization?

Relationship to Original Factorization

- Let

$$\boldsymbol{S} := \mathcal{P}[X_3|H_2]$$

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \underbrace{\mathcal{P}[X_{\{1,2\}}, X_3]}_{= \boldsymbol{L}\boldsymbol{S}}\underbrace{\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4\}}]}_{= \boldsymbol{S}^{-1}\boldsymbol{R}}$$

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}|H_2]\mathcal{P}[\oslash H_2]\mathcal{P}[X_{\{3,4\}}|H_2]^\top$$

Our Alternative Factorization

$$\underbrace{\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}]}_{\text{factor of 4 variables}} = \underbrace{\mathcal{P}[X_{\{1,2\}}, X_3]}_{\text{factor of 3 variables}} \underbrace{\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4\}}]}_{\text{factor of 3 variables}}$$

- It may not seem very amazing at the moment (we have only reduced the size of the factor by 1)
- What is cool is that every latent tree of ***V*** variables has such a factorization where:
  - All factors are of size 3
  - All factors are only functions of observed variables

Generalizing To More Variables

- Consider HMM with 5 observations. Using similar arguments as before we will get that:

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4,5\}}] = \mathcal{P}[X_{\{1,2\}}, X_3]\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4,5\}}]$$

reshape and decompose recursively

$$\mathcal{P}[X_{\{2,3\}}, X_{\{4,5\}}] = \mathcal{P}[X_{\{2,3\}}, X_4]\mathcal{P}[X_3, X_4]^{-1}\mathcal{P}[X_3, X_{\{4,5\}}]$$

Training / Testing with Spectral Learning

- We have that

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3]\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4\}}]$$

- In training, we compute estimates:

$$\mathcal{P}_{MLE}[X_{\{1,2\}}, X_3] \quad \mathcal{P}_{MLE}[X_2, X_3]^{-1} \quad \mathcal{P}_{MLE}[X_2, X_{\{3,4\}}]$$

- In test time, we can compute probability estimates (let lowercase letters denote fixed evidence values):

$$\widehat{\mathbb{P}}_{spec}[x_1, x_2, x_3, x_4] = \mathcal{P}_{MLE}[x_{\{1,2\}}, X_3]\mathcal{P}_{MLE}[X_2, X_3]^{-1}\mathcal{P}_{MLE}[X_2, x_{\{3,4\}}]^{\top}$$

Consistency

- A trivial consistent estimator is to simply attempt to estimate the "big" probability table from the data without making any conditional independence assumptions

$$\mathcal{P}_{MLE}[X_1, X_2; X_3, X_4] \rightarrow \mathcal{P}[X_1, X_2; X_3, X_4]$$

as number of samples increases

- While this is consistent, it is not very statistically efficient

Unsupervised Parsing

Training Set – Given sentences and part-of-speech tags

DT	NN	VB	NN	
The	*bear*	*likes*	*fish*	
DT	NN	VB	DT	NN
The	*llama*	*eats*	*the*	*grass*

Test Set – Find (unlabeled) parse tree for each sentence

NN	ADV	VB	NN	CONJ	NN
Lions	*quickly*	*chase*	*deer*	*and*	*antelope*

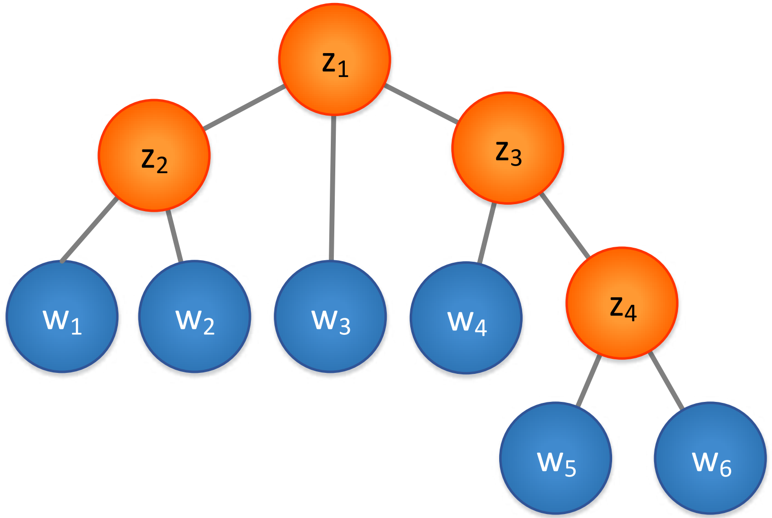
Parse tree structure is a *latent* variable

Conditional Latent Tree Model

- Each tag sequence $\boldsymbol{x}$ associated with a latent tree

$\boldsymbol{x_2} = (DT, NN, VBD, DT, ADJ, NN)$

$$p(\boldsymbol{w}, \boldsymbol{z} \mid \boldsymbol{x}) = \prod_{i=1}^{H} p(z_i \mid \pi_{\boldsymbol{x}}(z_i)) \times \prod_{i=1}^{\ell(x)} p(w_i \mid \pi_{\boldsymbol{x}}(w_i))$$



The bear ate the big fish

The moose ran the tiring race

- Traditional Approach

Training

(Given the latent tree) Estimate parameters using *nonconvex optimization*:

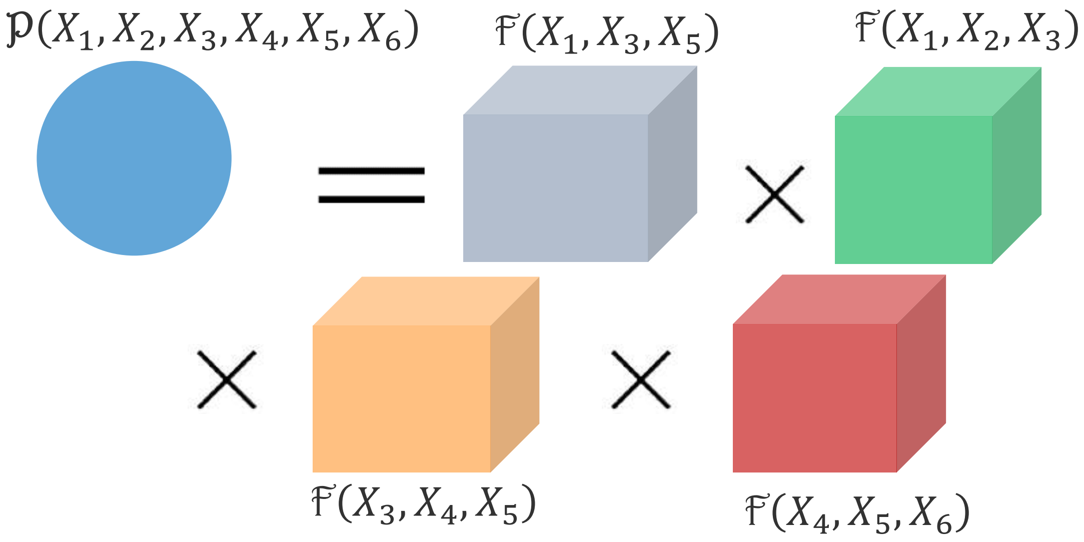
$\hat{P}(H_1)$ $\hat{P}(X_1|H_2)$ $\hat{P}(X_5|H_4)$ …..

Test

To query probabilities: $\mathrm{P}(X_1 = 0, \dots X_6 = 1)$
multiply learned parameters

The Spectral Approach

Latent Tree *Observable* Factorization



Training

Estimate alternate parameters:

$\mathcal{F}(X_1, X_2, X_3)$ $\mathcal{F}(X_1, X_3, X_5)$ $\mathcal{F}(X_3, X_4, X_5)$ $\mathcal{F}(X_4, X_5, X_6)$

Test

To query probabilities: $\mathcal{P}(X_1 = 0, \dots X_6 = 1)$

tensor multiply parameters

Consistency

- A better estimate is to compute likelihood estimates of the factorization:

$$\mathcal{P}_{MLE}[X_{\{1,2\}}|H_2]\mathcal{P}_{MLE}[\oslash H_2]\mathcal{P}_{MLE}[X_{\{3,4\}}|H_2]^\top \\ \rightarrow \mathcal{P}[X_1, X_2; X_3, X_4]$$

- But this requires running EM, which will get stuck in local optima and is not guaranteed to obtain the MLE of the factorized model

Consistency

- In spectral learning, we estimate the alternate factorization from the data

$$\mathcal{P}_{MLE}[X_{\{1,2\}}, X_3]\mathcal{P}_{MLE}[X_2, X_3]^{-1}\mathcal{P}_{MLE}[X_2, X_{\{3,4\}}]$$
$$\rightarrow \mathcal{P}[X_1, X_2; X_3, X_4]$$

- This is consistent and computationally tractable (at some loss of statistical efficiency due to the dependence on the inverse)

Where's the Catch?

- Before we said that if the number of latent states was very large then the model was equivalent to a clique.

- Where does that scenario enter in our factorization?

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3]\mathcal{P}[X_2, X_3]^{-1}\mathcal{P}[X_2, X_{\{3,4\}}]$$

When does this inverse exist?

When Does the Inverse Exist

$$\mathcal{P}[X_2, X_3] = \mathcal{P}[X_2|H_2]\mathcal{P}[\oslash H_2]\mathcal{P}[X_3|H_2]^\top$$

- All the matrices on the right hand side must have full rank. (This is in general a requirement of spectral learning, although it can be somewhat relaxed)

When m > k

- The inverse cannot exist, but this situation is easily fixable (project onto lower dimensional space)

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3]\boldsymbol{V}\left(\boldsymbol{U}^\top \mathcal{P}[X_2, X_3]\boldsymbol{V}\right)^{-1}\boldsymbol{U}^\top \mathcal{P}[X_2, X_{\{3,4\}}]$$

- Where ***U***, ***V*** are the top left/right **k** singular vectors of $\mathcal{P}[X_2, X_3]$

When k > m

- The inverse does exist. But it no longer satisfies the following property, which we used to derive the factorization

$$\mathcal{P}[X_2, X_3]^{-1} = (\mathcal{P}[X_3|H_2]^\top)^{-1}\mathcal{P}[\oslash H_2]^{-1}\mathcal{P}[X_2|H_2]^{-1}$$

- This is much more difficult to fix, and intuitively corresponds to how the problem becomes intractable if ***k >> m***.

What does k>m mean?

- Intuitively, large ***k***, small ***m*** means long range dependencies

- Consider following generative process:
  (1) With probability 0.5, let ***S= X***, and with probability 0.5 let ***S=Y***.
  (2) Print ***A n*** times.
  (3) Print ***S***
  (4) Go back to step (2)

With ***n=1*** we either generate:
AXAXAXA…… or AYAYAYA…..

m=3
k=2

With ***n=2*** we either generate:
AAXAAXAA….. or AAYAAYAA…….

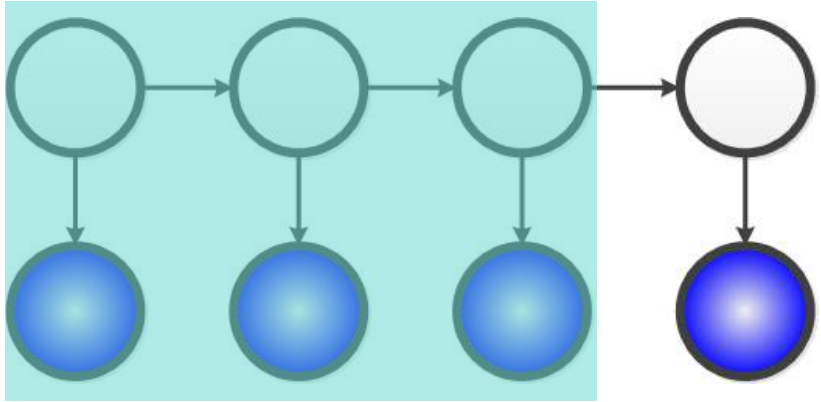
How many hidden states does HMM need?

- HMM needs **2n** states.
- Needs to remember count as well as whether we picked $S=X$ or $S=Y$
- However, number of observed states m does not change, so our previous spectral algorithm will break for $n > 2$.
- How to deal with this in spectral framework?

Making Spectral Learning Work In Practice

- We are only using marginals of pairs/triples of variables to construct the full marginal among the observed variables.

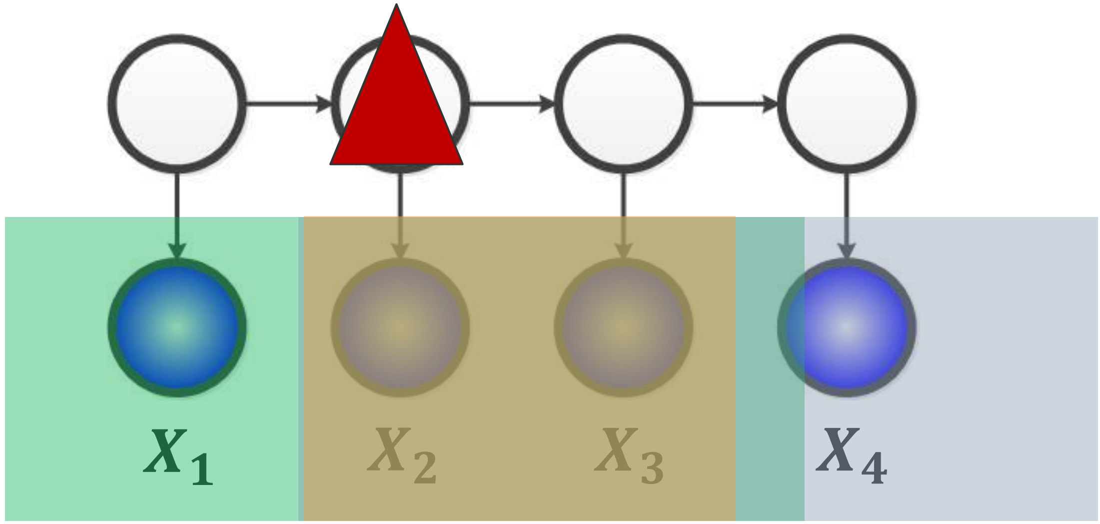
- Only works when $k < m$.



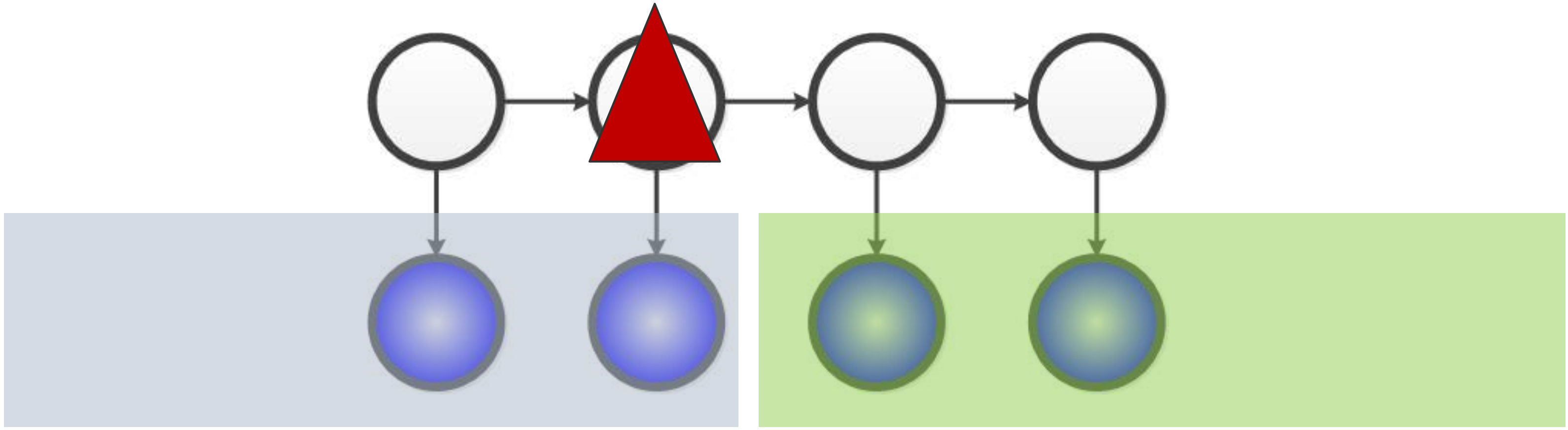
- However, in real problems we need to capture longer range dependencies.

Recall our factorization

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \underline{\mathcal{P}[X_{\{1,2\}}, X_3]}\,\underline{\mathcal{P}[X_2, X_3]^{-1}}\,\underline{\mathcal{P}[X_2, X_{\{3,4\}}]}$$



Key Idea: Use Long-Range Features



Construct feature vector of left side

$$\phi_L$$

Construct feature vector of right side

$$\phi_R$$

Spectral Learning With Features

Delta thing is just an indicator for the value X_j takes on

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\boldsymbol{\delta}_2 \otimes \boldsymbol{\delta}_3] := \mathbb{E}[\boldsymbol{\delta}_2 \boldsymbol{\delta}_3^\top]$$

Use more complex feature instead:

$$\mathbb{E}[\phi_L \otimes \phi_R]$$

$$\begin{aligned}\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] &= \mathbb{E}[\boldsymbol{\delta}_{1\otimes 2}, \boldsymbol{\delta}_{3\otimes 4}] \\ &= \mathbb{E}[\boldsymbol{\delta}_{1\otimes 2}, \phi_R]\boldsymbol{V}(\boldsymbol{U}^\top \mathbb{E}[\phi_L \otimes \phi_R]\boldsymbol{V})^{-1}\boldsymbol{U}^\top \mathcal{P}[\phi_L, X_{\{3,4\}}]\end{aligned}$$

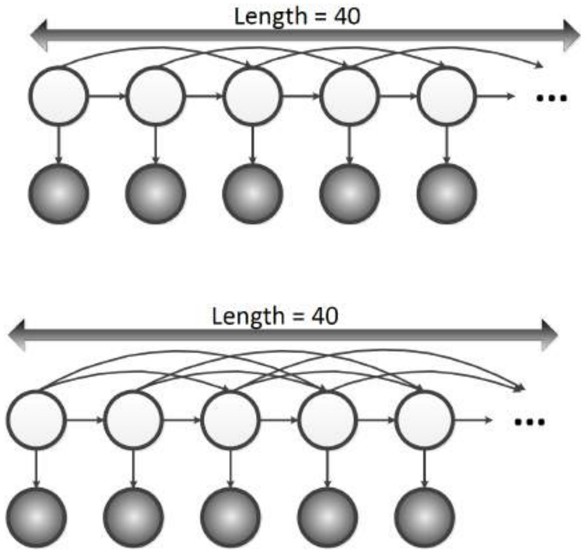
Experimentally,

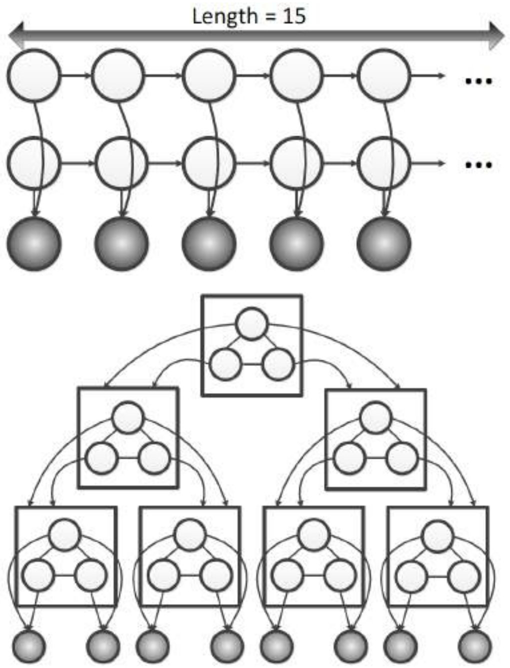
- Has been shown by many authors that (with some work) spectral methods achieve comparable results to EM but are 10-50x faster
  - Parikh et al. 2011 / 2012
  - Balle et al. 2012
  - Cohen et al. 2012 / 2013

- The following are some synthetic and real data results demonstrating the comparison between EM and spectral methods.

Synthetic Data [Parikh et al. 2012]

- Different latent variable models

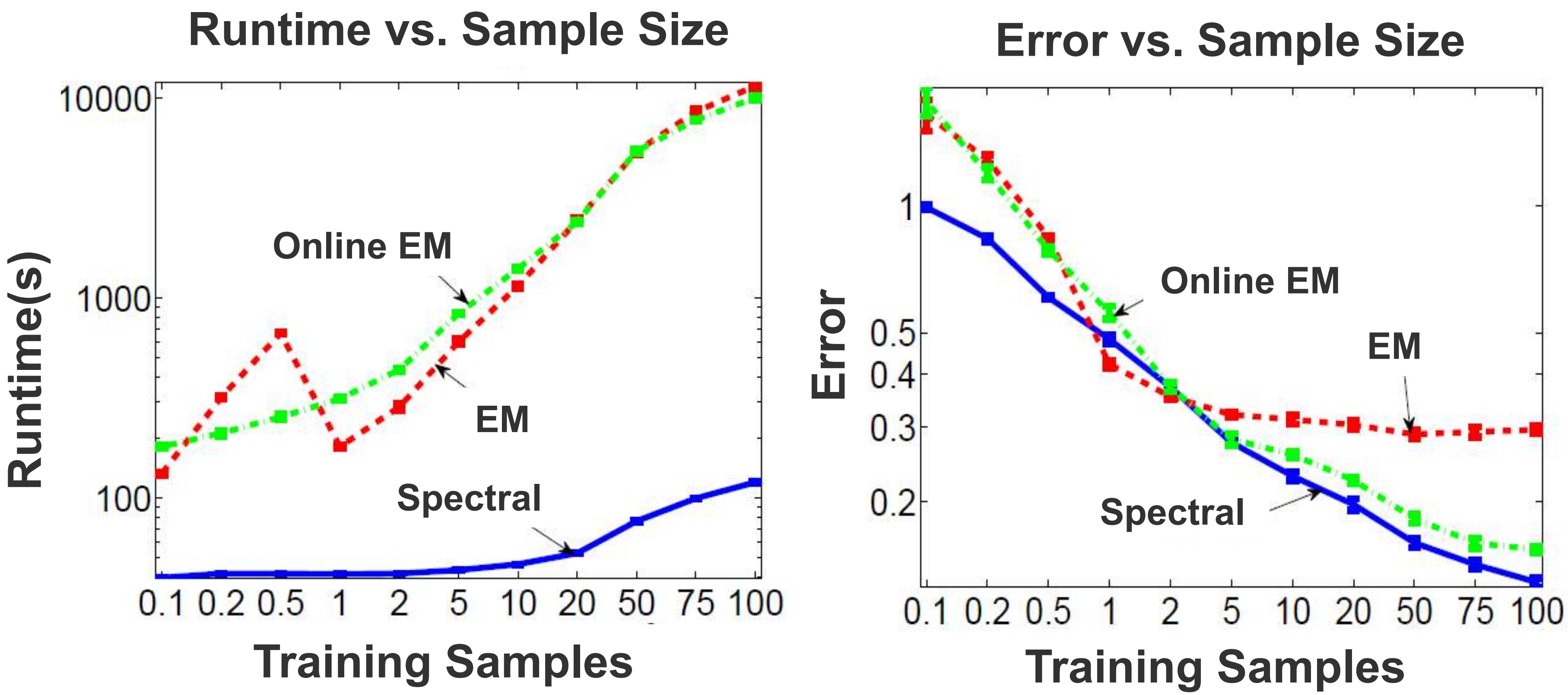




- Train: Learn parameters for a given model given samples of observed variables
- Test: Evaluate likelihood of random samples drawn from model and compare to the true likelihood

Synthetic Data [Parikh et al. 2012]

- Synthetic 3rd order HMM Example (Spectral/EM/Online EM):

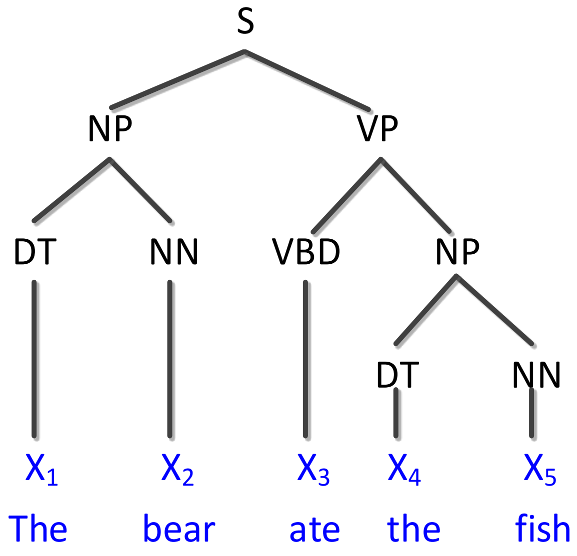


- Results for other structures look similar

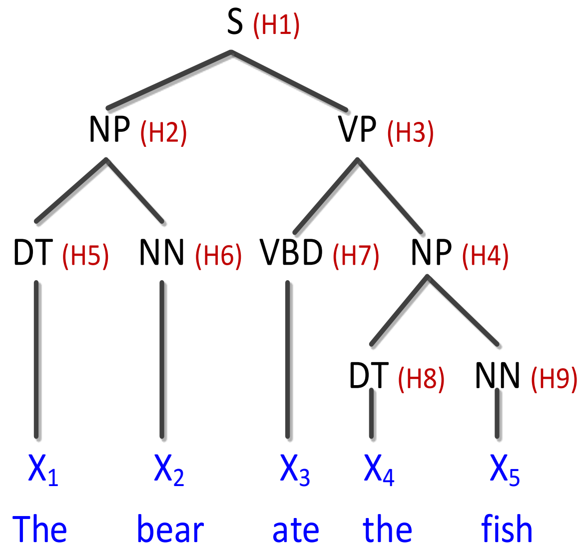
Supervised Parsing [Cohen et al. 2012/2013]

- Learn a latent variable Probabilistic Context Free Grammar model (latent PCFG) which is a PCFG augmented with additional latent states

PCFG



Latent Variable PCFG



- Train: Learn parameters given parse trees on training examples.
- Test: Estimate most likely parse structure on test sentences

Empirical Results for Latent PCFGs [Cohen et al. 2013]

	section 22		section 23	
	EM	spectral	EM	spectral
$m = 8$	86.87	85.60	—	—
$m = 16$	88.32	87.77	—	—
$m = 24$	88.35	88.53	—	—
$m = 32$	88.56	88.82	87.76	88.05

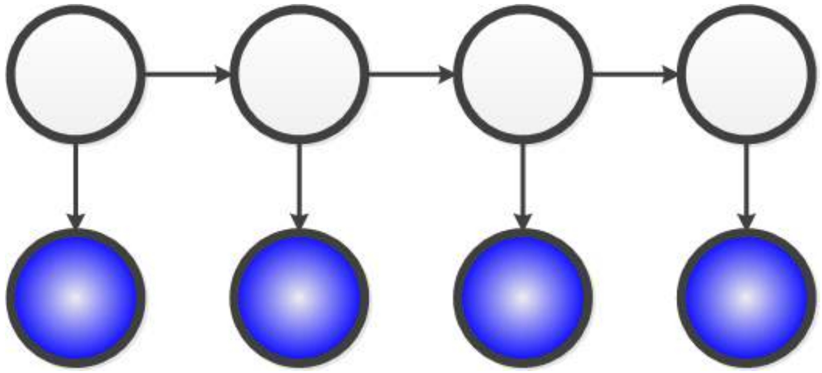
Evaluation Measure: *F1 bracketing score*

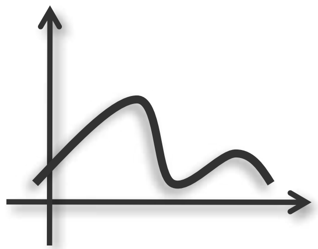
Timing Results on Latent PCFGs [Cohen et al. 2013]

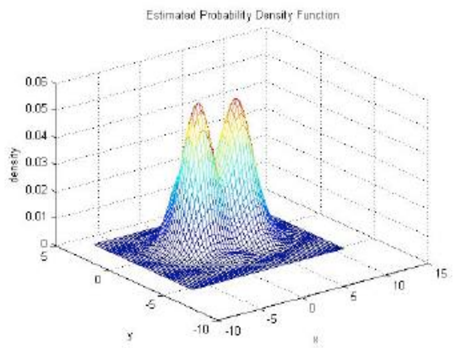
	single EM iter.	EM best model	spectral algorithm total	feature	transfer + scaling	SVD	$a \to b\ c$	$a \to x$
$m = 8$	6m	3h	3h32m	22m	49m	36m	1h34m	10m
$m = 16$	52m	26h6m	5h19m			34m	3h13m	19m
$m = 24$	3h7m	93h36m	7h15m			36m	4h54m	28m
$m = 32$	9h21m	187h12m	9h52m			35m	7h16m	41m

Dealing with Nonparametric, Continuous Variables

- It is difficult to run EM if the conditional/marginal distributions are continuous and do not easily fit into a parametric family.







- However, we will see that Hilbert Space Embeddings can easily be combined with spectral methods for learning nonparametric latent models.

Connection to Hilbert Space Embeddings

- Recall that we could substitute features for variables

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\boldsymbol{\delta}_2 \otimes \boldsymbol{\delta}_3] := \mathbb{E}[\boldsymbol{\delta}_2 \boldsymbol{\delta}_3^\top]$$

Use more complex feature instead:

$$\mathbb{E}[\phi_L \otimes \phi_R]$$

Can Also Use Infinite Dimensional Features

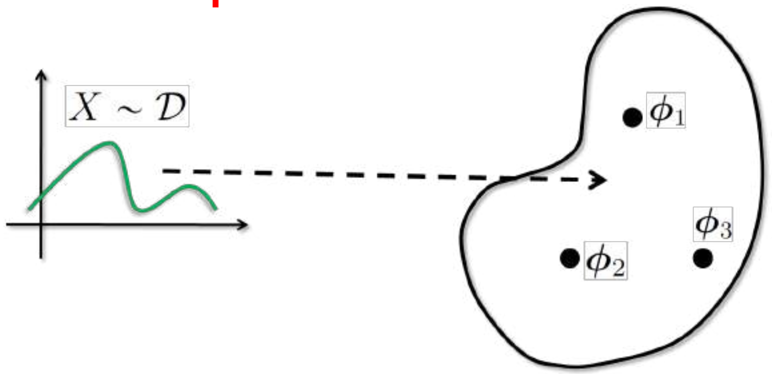
- Replace

$$\mathcal{P}[X_2, X_3] = \mathbb{E}[\boldsymbol{\delta}_2 \otimes \boldsymbol{\delta}_3] := \mathbb{E}[\boldsymbol{\delta}_2 \boldsymbol{\delta}_3^\top]$$

- with

$$\mathcal{C}[X_2, X_3] = \mathbb{E}[\phi_{X_2} \otimes \phi_{X_3}]$$

covariance operator



- (and similarly for other quantities)

Connection to Hilbert Space Embeddings

Discrete case:

$$\mathcal{P}[X_{\{1,2\}}, X_{\{3,4\}}] = \mathcal{P}[X_{\{1,2\}}, X_3]\boldsymbol{V}\left(\boldsymbol{U}^\top \mathcal{P}[X_2, X_3]\boldsymbol{V}\right)^{-1}\boldsymbol{U}^\top \mathcal{P}[X_2, X_{\{3,4\}}]$$

Continuous case:

$$\mathcal{C}[X_{\{1,2\}}; X_{\{3,4\}}] = \mathcal{C}[X_{\{1,2\}}; X_3]\boldsymbol{V}\left(\boldsymbol{U}^\top \mathcal{C}[X_2, X_3]\boldsymbol{V}\right)^{-1}\boldsymbol{U}^\top \mathcal{C}[X_2; X_{\{3,4\}}]$$

Summary - EM & Spectral (Part I)

EM

- Aims to Find MLE so more "statistically" efficient
- Can get stuck in local-optima
- Lack of theoretical guarantees
- Slow
- Easy to derive for new models

Spectral

- Does not aim to find MLE so less statistically efficient.
- Local-optima-free
- Provably consistent
- Very fast
- Challenging to derive for new models (Unknown whether it can generalize to arbitrary loopy models)

Summary - EM & Spectral (Part II)

EM

- No issues with negative numbers
- Allows for easy modelling with conditional distributions
- Difficult to incorporate long-range features (since it increases treewidth).
- Generalizes poorly to non-Gaussian continuous variables.

Spectral

- Problems with negative numbers. Requires explicit normalization to compute likelihood.
- Allows for easy modelling with marginal distributions
- Easy to incorporate long-range features.
- Easy to generalize to non-Gaussian continuous variables via Hilbert Space Embeddings