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Pareto-optimal reinsurance policies with maximal synergy

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ABSTRACT

Optimal reinsurance policies have been studied extensively in the economics and insurance literature. Two types of optimality criteria, expected utility (EU) maximization and risk minimization, are commonly used. Understandably, applying the two types of criteria usually will result in different "optimal" policies. To reconcile the conflicting interests of the insurer and reinsurer and strike a balance between EU maximization and risk minimization, we follow the approach in Borch (1960b) but assume that the involved two parties both apply distortion risk measures instead of variance. We first identify a set of reinsurance policies that minimize the total risk borne by the two parties, then we take this set of policies as admissible and derive the Pareto-optimal policies that maximize the weighted EU of these two parties. A Nash bargaining model is applied to identify the "best" weights allocated to the two parties. In addition, we extend our results to a situation where the insurer's decision making is dictated by the rank-dependent expected utility (RDEU) theory. Finally, we present numerical examples to show the applicability of our methodology and some implications of the obtained results.

1. Introduction

A reinsurance contract is a mechanism for redistributing risks between an insurer and a reinsurer. It is characterized by a pair $(I(X), P)$, where $I(X)$ is the ceded function specifying the amount that the insurer is indemnified by the reinsurer when it suffers a loss of size X and P is the reinsurance premium.

Extensive results exist for "optimal" reinsurance policies in the economics and insurance literature. Classical results on policies that maximize the insurer's EU can be found in Borch (1962), Arrow (1963, 1974) and Raviv (1979). Results on policies that minimize the insurer's risks, measured by variance, Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) and more general distortion risk measures (DRM), are available in Borch (1960a), Aase (2002), Cai and Tan (2007), Assa (2015), (Zhuang et al., 2016), and the references therein. Policies that maximize EU under VaR constraint were studied by Huang (2006), Zhou and Wu (2009) and Bernard and Tian (2010).

When negotiating a reinsurance contract, the interests of both the insurer and reinsurer should be considered simultaneously. Historically, the Pareto-optimal policies are usually sought to reconcile the conflicting interests of the insurer and reinsurer. With the Pareto-optimal policies, one party's expected utility (risk) cannot be increased (reduced) further without reducing (increasing) that of the other party. Results on EU-maximizing Pareto-optimal policies can be found in, for example, Borch (1962), Raviv (1979), Gerber and Pafumi (1998), Golubin (2006b) and Aase (2009). Results on risk-minimizing Pareto-optimal policies can be found in, for example, Cai and Tan (2007), Cai et al. (2017), Jiang et al. (2017, 2018), Asimit and Boonen (2018) and the references therein.

To identify a unique policy from the set of Pareto-optimal policies, one could consider the competitive equilibrium or some bargaining solution in the context of game theory. The optimal policy corresponding to the competitive equilibrium is a policy in the Pareto-optimal set, where the price is determined by the market such that the demand and supply of reinsurance are equal (market is clear). For results about optimal reinsurance in the competitive equilibrium within the framework of EU maximization, see, for example, Borch (1962) and Gerber and Pafumi (1998). Results within the risk minimization framework can be found in, for example, Embrechts et al. (2018).

The optimal policy corresponding to a bargaining solution is a policy in the Pareto-optimal set, where the benefits of cooperation are distributed among the negotiating parties in accordance with some rationality axioms. For example, Borch (1960c) first identified the set of Pareto-optimal reinsurance policies that maximize the EU of both parties, then a unique policy was determined by making use of Nash's solution for bargaining games. Kihlstrom and Roth (1982) studied the effects of insurance buyer's risk

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aversion on the bargaining outcomes. Much more recently, Boonen (2016) studied the Nash bargaining solution for insurance risk redistribution by assuming that the set of admissible policies are regulated so that both parties benefit from the reinsurance transaction. For results about optimal reinsurance policies corresponding to bargaining solutions within the framework of risk minimization, see for example, Asimit and Boonen (2018). An insightful and comprehensive review of reinsurance policies in the competitive equilibrium versus those corresponding to bargaining solutions was given by Aase (2009).

The optimal reinsurance policies that maximize EU and that minimize risk are in general different. In practice, insurance companies are likely to consider increasing EU and controlling risk at the same time when negotiating reinsurance policies. One approach to consider both is maximizing EU under some risk constraints, as was done in Huang (2006), Zhou and Wu (2009) and Bernard and Tian (2010). An alternative approach was introduced by Borch (1960b), which identified the optimal policy by solving a multilevel optimization problem. On the first level, the quota-share policy was proved optimal in minimizing the total variance of insurer and reinsurer. On the second level, the quota-share policies that meet Pareto optimality were picked out through maximizing the weighted EU of the two parties. Lastly, the Pareto-optimal policy corresponding to the Nash bargaining solution was identified. We note that the criterion of minimizing total variance was further discussed in Hürlimann (2011).

In this paper, following the approach in Borch (1960b), we assume that the two parties apply distortion risk measures instead of variance. This is because nowadays the distortion risk measures, such as VaR and TVaR, are commonly used in determining companies' solvency status and capital requirement under the international solvency standard such as Basel III and Solvency II. We first identify a set of reinsurance policies that minimize the total risk shared by the two parties, then we take this set of policies as admissible and determine the Pareto-optimal policies that maximize the weighted EU of the two parties. In contrast to the results in Borch (1960b), we show that applying risk measures such as VaR and TVaR results in multi-layered policies.

The approach we adopted is also related to the maximal-synergy-based risk sharing (Section 9 in Gerber and Pafumi, 1998), in the sense that we only consider policies that minimize the total risk (maximize synergy) when maximizing EU. It is also somewhat similar to that in Boonen (2016), where the Nash bargaining solutions are determined in some priorly determined set of feasible policies. From now on, we refer to the reinsurance policies that minimize the total risk of insurer and reinsurer as the synergy-maximizing reinsurance policies.

We remark that minimizing the total risk in the system is essential in designing "internal reinsurance" (captive reinsurance), where the parent company aims at minimizing the total capital requirement of fronting company and reinsurance captive. It is also very important from a societal point of view, where the "total risk" of the society is minimized.

In addition to examining the Pareto-optimal policies in the traditional EU framework, we also extend our results in the framework of rank dependent expected utility (RDEU) theory (Quiggin, 1982) to take into consideration of psychological effects on human decision-making process. The extension from EU to non-EU framework in optimal (re)insurance policy design has led to numerous results in the past few years. See, for example, Bernard et al. (2015), Xu et al. (2019) and Ghossoub (2019) and the references therein. In this paper, we contribute by deriving the synergy-maximizing Pareto-optimal reinsurance policies by assuming that the insurer is a RDEU maximizer and the reinsurer is risk neutral.

The remainder of this paper proceeds as follows. Section 2 reviews the basic concepts of distortion risk measure and describes the objective function of this paper. Section 3 determines the set of synergy-maximizing reinsurance policies. Section 4 identifies the set of Pareto-optimal policies on the basis of synergy-maximizing policies. In addition, the policy corresponding to the Nash bargaining solution is determined. Section 5 refines the optimal policies when additional risk constraints are imposed. Section 6 studies the optimal policies in the RDEU framework. Section 7 provides numerical examples. Section 8 concludes.

2. Background and model formulation

Let the aggregate loss of the insurer in a period be represented by a random variable X with support $[0, M]$ where $M \leq \infty$. Assume that X is non-atomic and has cumulative distribution function F_X, probability density function f_X and survival function S_X. The distortion risk measure of X is defined as $\rho_g(X) = \int_0^\infty g(S_X(x))\,dx$, where the distortion function $g : [0,1] \to [0,1]$ is non-decreasing and satisfies $g(0) = 0$ and $g(1) = 1$ (Denuit et al., 2006). Distortion risk measures satisfy the properties of translation invariance, positive homogeneity, monotonicity and comonotonic additivity (Wang et al., 1997). For detailed discussions of comonotonic random variables and distortion risk measures, see, for example, Dhaene et al. (2002a,b), Balbás et al. (2009) and the references therein.

The following two results (Ludkovski and Young, 2009; Cheung and Lo, 2017) play vital roles in identifying the synergy-maximizing policies: for any distortion function g and non-decreasing 1-Lipschitz continuous function $I(x)$,

$$\rho_g(X) = \rho_g(I(X)) + \rho_g(X - I(X)), \tag{2.1}$$

$$\rho_g(I(X)) = \int_0^\infty g(S_X(x))dI(x). \tag{2.2}$$

Let the insurer and reinsurer have initial wealth w_1 and w_2 and adopt distortion risk measures with distortion functions g_1 and g_2 respectively. The insurer is considering to share its aggregate loss X with the reinsurer through a reinsurance policy $(I(X), \pi(I(X)))$, where $\pi(\cdot)$ is a premium principle. Optimizing $\pi(\cdot)$ is not the focus of this paper, we simply assume that it is determined by some actuarial premium principles. As such, the policy relies on $I(x)$ only and thus we do not distinguish between the ceded function and reinsurance policy in the rest of this paper.

We assume that the set of admissible ceded functions is given by

$$\mathcal{C} = \left\{ I : [0, M] \to [0, M] \;\middle|\; \begin{array}{l} 0 \leq I(x) \leq x \text{ for all } x \geq 0, \\ 0 \leq I(x_1) - I(x_2) \leq x_1 - x_2 \text{ if } 0 \leq x_2 \leq x_1 \end{array} \right\}.$$

With $I \in \mathcal{C}$, the ceded loss $I(X)$ and retained loss $X - I(X)$ are comonotonic as they are both non-decreasing with respect to X. The functions belonging to set $\mathcal{C}$ are said to satisfy the *no-sabotage* property.[1] (Carlier and Dana, 2003; Boonen and Ghossoub, 2019) and have been widely applied in the literature. With the no-sabotage property, the increment of indemnity does not exceed the increment of loss, which to some extent rules out the motivations for *ex post* moral hazards.

With a reinsurance policy, the insurer's total loss is $X - I(X) + \pi(I(X))$ and the reinsurer's total loss is $I(X) - \pi(I(X))$. The total risk in the system is given by

$$\rho_{g_1}(X - I(X) + \pi(I(X))) + \rho_{g_2}(I(X) - \pi(I(X))).$$

We then define the set of synergy-maximizing ceded functions as

$$\mathcal{C}_g = \underset{I \in \mathcal{C}}{\arg\min}\; \rho_{g_1}(X - I(X) + \pi(I(X))) + \rho_{g_2}(I(X) - \pi(I(X))). \tag{2.3}$$

[1] This is also called incentive compatibility (Huberman et al., 1983) See Xu et al. (2019) and Chi and Zhuang (2020) for more recent discussions.

The special feature of this paper is that we take $\mathcal{C}_g$ as the admissible set of ceded functions when decision makers' EU (or RDEU) are maximized. Therefore, the synergy for risk sharing is maximized. We emphasize again this approach is inspired by Borch (1960b).

To this end, we denote the utility function of the insurer and reinsurer by u and v respectively. These two utility functions are assumed to be non-decreasing and concave. In addition, we assume that

$$\lim_{x\to\infty} u'(x) = \lim_{x\to\infty} v'(x) = 0 \tag{2.4}$$

and

$$\lim_{x\to-\infty} u'(x) = \lim_{x\to-\infty} v'(x) = \infty. \tag{2.5}$$

Since a reinsurance treaty can be reached only if both parties in the transaction are better off from it, the rationality constraints should be considered when discussing the optimal policy. Therefore, we confine ourselves to the following set

$$\mathcal{C}_r = \left\{ I : [0, M] \to [0, M] \,\middle|\, \begin{array}{l} \mathbf{E}[u(w_1 - X + I(X) - \pi(I(X)))] \geq \mathbf{E}[u(w_1 - X)] \\ \mathbf{E}[v(w_2 - I(X) + \pi(I(X)))] \geq v(w_2) \end{array} \right\}.$$

Note that the point $(\mathbf{E}[u(w_1 - X)], v(w_2))$ corresponds to the two parties' utilities without reinsurance contract and it is called the disagreement point in the game theory literature (Nash, 1953; Lemaire, 1991).

As per the literature (Raviv, 1979; Golubin, 2006b; Jiang et al., 2019), the Pareto-optimal reinsurance policies are derived through maximizing the weighted EU of insurer and reinsurer. Therefore, the main problem to be solved in this paper is

Problem 1 (*Main Problem*)**.**

$$\max_{I\in\mathcal{C}_g\cap\mathcal{C}_r} \mathbf{E}\left[u(w_1 - X + I(X) - \pi(I(X)))\right] + k\mathbf{E}\left[v(w_2 - I(X) + \pi(I(X)))\right], \quad k \geq 0 \tag{2.6}$$

Note that the parameter k in the objective function can be interpreted as the relative negotiation power of reinsurer. When $k \searrow 0 (\nearrow \infty)$, only the EU of insurer (reinsurer) is considered.

We remark that Problem 1 differs from those studied in Raviv (1979) and Golubin (2006b) in that it restricts the admissible ceded functions to a set such that the synergy between the insurer and reinsurer is maximized. Jiang et al. (2019) also studied the Pareto-optimal reinsurance policies. However, under the heterogeneous beliefs, their obtained policies cannot prevent *ex post* moral hazard issues.

Problem 1 is solved in the following two sections. In Section 3, we identify the synergy-maximizing policies for general distortion risk measures as well as for some important special cases. In Section 4, we derive the Pareto-optimal EU-maximizing policies based on the set of synergy-maximizing policies.

3. The set of synergy-maximizing policies

Before determining set $\mathcal{C}_g$ of synergy-maximizing policies, we point out that the optimization problem embedded in (2.3) is in fact a special case of a more general problem

$$\min_{I\in\mathcal{C}} \beta\rho_{g_1}(X - I(X) + \pi(I(X))) + (1-\beta)\rho_{g_2}(I(X) - \pi(I(X))), \quad 0 \leq \beta \leq 1, \tag{3.1}$$

which was studied in Cai et al. (2016, 2017) and Jiang et al. (2018) in the context of determining the Pareto optimal reinsurance policies so that one party's risk cannot be reduced further without increasing that of the other party.

However, because of the peculiarity of (2.3), instead of applying the general results in the literature, we use a simple argument to provide the intuition.

Applying properties (2.1), (2.2) and the translation invariance property of distortion risk measures, we have

$$\begin{aligned} \mathcal{C}_g &= \arg\min_{I\in\mathcal{C}} \ \rho_{g_1}(X - I(X) + \pi(I(X))) + \rho_{g_2}(I(X) - \pi(I(X))) \\ &= \arg\min_{I\in\mathcal{C}} \ \rho_{g_2}(I(X)) - \rho_{g_1}(I(X)), \\ &= \arg\min_{I\in\mathcal{C}} \ \int_0^M (g_2(S_X(x)) - g_1(S_X(x)))I'(x)dx, \end{aligned} \tag{3.2}$$

from which it is easy to derive

$$I'^{*}(x) = \mathbb{1}_{\{x:\ g_2(S_X(x))<g_1(S_X(x))\}}(x) + \eta(x)\cdot\mathbb{1}_{\{x:\ g_2(S_X(x))=g_1(S_X(x))\}}(x), \tag{3.3}$$

where $\mathbb{1}_A(x)$ is an indicator function which equals to 1 if $x \in A$ and 0 otherwise, $\eta(x)$ is any function such that $I^*(x) \in \mathcal{C}$. In the literature, $I'(\cdot)$ is called the marginal indemnification function (MIF). The concepts and applications of MIF can be found in, for example, Assa (2015), Zhuang et al. (2016), Cheung and Lo (2017) and Lo (2017).

Essentially, (3.3) states that the synergy is maximized if the party who is less "risk averse" for the losses in the layer $(x, x+dx)$ bears those losses. Apparently, when only the "synergy" (collective interest) is considered, premium transfer between the parties has no effect.

We next discuss in detail the set of synergy-maximizing ceded functions when the risk measures used by the two parties are VaR or TVaR.

3.1. VaR

The VaR of X at confidence level α is defined as

$$VaR_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x : F_X(x) \geq \alpha\}, \tag{3.4}$$

of which the distortion function is

$$g_{V,\alpha}(x) = \mathbb{1}_{[1-\alpha,1]}(x). \tag{3.5}$$

Suppose that the insurer and reinsurer adopt VaR, with confidence levels α_c and α_r respectively, to measure their risks. Denote $a_c = VaR_{\alpha_c}(X)$ and $a_r = VaR_{\alpha_r}(X)$, then applying (3.5) to Eq. (3.3) yields the set of synergy-maximizing ceded functions. The results depend on the order of α_c and α_r, which are stated in detail in the following:

Case 1: $\alpha_c > \alpha_r$

Let $\mathcal{C}_{V_1}$ denote the set of synergy-maximizing ceded functions for this case. Applying (3.5) to Eq. (3.3), the derived ceded function, denoted by $I_{V_1}(x)$, is of form

$$I'_{V_1}(x) = \eta(x)\cdot\mathbb{1}_{[0,a_r)\cup(a_c,M]}(x) + \mathbb{1}_{[a_r,a_c]}(x), \tag{3.6}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{V_1}(x) \in \mathcal{C}$. This indicates that the synergy-maximizing ceded functions have slope one in $[a_r, a_c]$. Intuitively, when $\alpha_c \geq \alpha_r$, the losses in layer $[a_r, a_c]$ contribute to the insurer's VaR, but not to that of the reinsurer. So these losses should be ceded.

Further, we denote by $\mathcal{C}_{V_1,b_1}$ the set of functions $I_{V_1}(x)$ which satisfy $I_{V_1}(a_r) = b_1$ for some constant $b_1 \in [0, a_r]$. Then the upper and lower bounds of $\mathcal{C}_{V_1,b_1}$, denoted by $\bar{I}_{V_1,b_1}(x)$ and $\underline{I}_{V_1,b_1}(x)$ respectively, are given by

- $\bar{I}_{V_1,b_1}(x) = \{x \wedge b_1\} + (x - a_r)_+$,
- $\underline{I}_{V_1,b_1}(x) = (x - (a_r - b_1))_+ \wedge (b_1 + a_c - a_r)$.

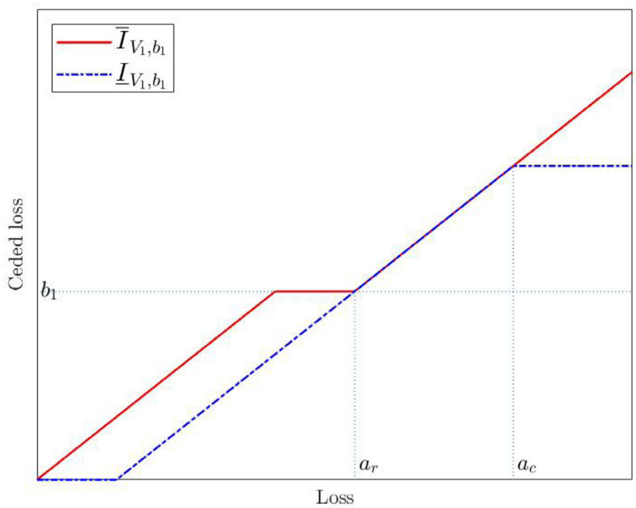

Fig. 1. An illustration of the upper and lower bounds of $\mathcal{C}_{V_1,b_1}$.

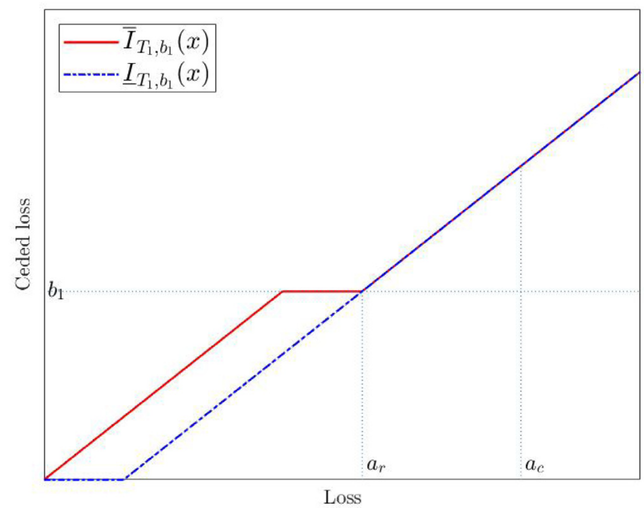

Fig. 2. An illustration of the upper and lower bounds of $\mathcal{C}_{T_1,b_1}$.

These bounds will be used in Section 4 in determining the EU-maximizing polices. A graphical illustration is given in Fig. 1.

Case 2: $\alpha_c < \alpha_r$

In this case, one easily obtain that a ceded function is synergy-maximizing if and only if it is flat in $[a_c, a_r]$. Intuitively, when $\alpha_c \le \alpha_r$, the losses in the layer $[a_c, a_r]$ contribute to the reinsurer's VaR, but not to the insurer's. Therefore, losses in this layer should be retained.

Case 3: $\alpha_c = \alpha_r$

In this case, for all ceded functions in $\mathcal{C}$,

$$VaR_{\alpha_c}(X - I(X) + \pi(I(X))) + VaR_{\alpha_c}(I(X) - \pi(I(X))) = VaR(X). \tag{3.7}$$

The reduction in total risk due to the reinsurance policy is always zero for every ceded function in $\mathcal{C}$. Thus the set of synergy-maximizing ceded functions is just $\mathcal{C}$. This is true whenever the two companies apply exactly the same distorting risk measures.

Remark 3.1. VaR is often used to set required capital reserve by regulators. Since reinsurance companies are usually less regulated than the primary insurance companies, for the same underlying risk, the required capital is higher for insurers than for reinsurers. In other words, the insurer is more risk averse than the reinsurer and has a natural demand for reinsurance protection. Please also see Blazenko (1986) for related discussions. Therefore, we assume $\alpha_c > \alpha_r$ in the remainder of this paper.

3.2. TVaR

The TVaR of X at confidence level α is defined by

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_t(X)dt, \tag{3.8}$$

for which the distortion function is

$$g_{T,\alpha}(x) = \frac{x}{1-\alpha}\cdot \mathbb{1}_{[0,1-\alpha)}(x) + \mathbb{1}_{[1-\alpha,1]}(x). \tag{3.9}$$

Here we suppose that the insurer and reinsurer adopt TVaR, with confidence levels α_c and α_r respectively, to measure their risks. In line with Remark 3.1, it is assumed that $\alpha_c > \alpha_r$ and the insurer is more risk averse.

Let $\mathcal{C}_{T_1}$ denote the set of synergy-maximizing ceded functions. By applying (3.9) to Eq. (3.3), the derived ceded function, denoted by $I_{T_1}(x)$, is of form

$$I'_{T_1}(x) = \eta(x)\cdot \mathbb{1}_{[0,a_r]}(x) + \mathbb{1}_{(a_r,M]}(x), \tag{3.10}$$

where $\eta(x) \in [0, 1]$ is any function such that $I_{T_1}(x) \in \mathcal{C}$. In this case, the synergy between the two parties is maximized when the reinsurer covers the losses greater than a_r.

Further, we denote by $\mathcal{C}_{T_1,b_1}$ the set of functions $I_{T_1}(x)$ that satisfy $I_{T_1}(a_r) = b_1$ for some constant $b_1 \in [0, a_r]$. The upper and lower bounds of $\mathcal{C}_{T_1,b_1}$, denoted by $\bar{I}_{T_1,b_1}(x)$ and $\underline{I}_{T_1,b_1}(x)$ respectively, are given by

- Upper bound: $\bar{I}_{T_1,b_1}(x) = (x \wedge b_1) + (x - a_r)_+$,
- Lower bound: $\underline{I}_{T_1,b_1}(x) = (x - (a_r - b_1))_+$.

A graphical illustration is given in Fig. 2

Remark 3.2. It can be easily verified that $\mathcal{C}_{T_1} \subset \mathcal{C}_{V_1}$. It means that the TVaR synergy-maximizing policies are also VaR synergy-maximizing. In other words, the TVaR constraint is more stringent than the VaR constraint, which is quite intuitive.

Remark 3.3. A more general risk measure that could be used here is Range Value-at-Risk (RVaR) (Cont et al., 2010), which connects VaR and TVaR as limiting cases. Its distortion function is given by

$$g_R(x) = \frac{x - 1 + \omega}{\omega - \alpha}\cdot \mathbb{1}_{[1-\omega,1-\alpha]}(x) + \mathbb{1}_{(1-\alpha,1]}(x).$$

The set of synergy-maximizing ceded functions can be easily derived in a similar fashion as the VaR and TVaR cases, expect that the form of the derived ceded function will have more layers. The details are omitted here.

Overall, when the insurer and reinsurer apply VaR, TVaR or RVaR as their risk measures, synergy maximization does not completely determine the ceded function, as $g_2(S_X(x)) = g_1(S_X(x))$ for x belonging to a set whose Lebesgue measure is non-zero. Therefore, the set $\mathcal{C}_g$ contains infinitely many ceded functions. In the next section, we focus on the case when the two parties apply VaR as their risk measures and propose a step-wise optimization procedure to uniquely determine the optimal ceded function.

The situation is analogous to that in Borch (1960b), which first verified the optimality of proportional reinsurance in minimizing the system's total variance and then determined the optimal proportion through EU maximization.

4. EU-based Pareto-optimal policies

In this section we apply a step-wise optimization procedure and derive the optimal reinsurance policy that solves Problem 1. As this procedure is applicable to all the distortion risk measures, we present the details for the VaR case only.

As shown in Sections 3.1 and 3.2 , when $\alpha_c > \alpha_r$, synergy-maximization requires that the insurer cedes certain part of the losses to the reinsurer, regardless of the premium. Apparently, if the premium is too high or too low, the insurer's or reinsurer's rationality constraint will be violated. To simplify discussions, we assume that the premium is given by

$$\pi(I(X)) = (1+\theta)\mathbf{E}[I(X)], \tag{4.1}$$

where $\theta > 0$ is the safety loading. With the expectation premium principle, the reinsurer's rationality constraint is always satisfied by a simple application of Jensen's inequality. To make the insurer's rationality constraint also satisfied, we assume that $\theta < \theta^{max}$. As proved by Arrow (1973), the stop-loss function $I_d(x) = (x-d)_+$ is optimal in maximizing the insurer's EU subject to $(1+\theta)\mathbf{E}[I(X)] = P$ where P is a given premium. Suppose there exists a policy $(I_d(X), P)$ such that the insurer's rationality constraint is satisfied (otherwise no policy will satisfy the insurer's rationality constraint). If θ is not unreasonably large, the deductible point d could be smaller than a_r such that $I_d \in \mathcal{C}_{V_1}$, which indicates that the feasible set $\mathcal{C}_{V_1} \cap \mathcal{C}_r$ to Problem 1 is not empty. The exact value of θ^{max} will of course depend on the form of utility function and is not pursued here.

It should also be noted that, without adding any complexity, the premium principle (4.1) could be generalized to any actuarial-premium-based principle $P = H(\mathbf{E}[I(X)])$ where $H(\cdot)$ is a continuous function on $[0, \mathbf{E}[X]]$ such that $H(x) \geq x$ (Golubin, 2006b; Bernard and Tian, 2010).

Our strategy to solve Problem 1 includes three steps: first, fix the premium $\pi(I(X)) = P$ and $I(a_r) = b$ and derive the parametric form of ceded function in $[0, a_r]$ and $[a_c, \infty]$; second, we search for the optimal values for P and b; finally, we check whether or not the solution satisfies the rationality constraints. Specifically, we first derive the solution $I^*_{b,P}$ to

Problem 1a (*VaR Synergy-Maximizing Policy with Fixed b and P*).

$$\max_{I\in\mathcal{C}_{V_1,b}} \ \mathbf{E}\left[u(w_1 - X + I(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I(X) + P)\right], \tag{4.2}$$
$$\text{s.t.} \quad P = (1+\theta)\mathbf{E}[I(X)].$$

Then we seek (b^*, P^*) that solves

Problem 1b (*Optimal Parameters*).

$$\max_{P,b} \ \mathbf{E}\left[u(w_1 - X + I^*_{b,P}(X) - P)\right] + k\mathbf{E}\left[v(w_2 - I^*_{b,P}(X) + P)\right],$$

where $b \in [0, a_r]$ and $P \in [\underline{P}_{V_1,b}, \overline{P}_{V_1,b}]$ with

$$\underline{P}_{V_1,b} = (1+\theta)\mathbf{E}[\underline{I}_{V_1,b}(X)]$$

and

$$\overline{P}_{V_1,b} = (1+\theta)\mathbf{E}[\bar{I}_{V_1,b}(X)]$$

being the minimum and maximum possible premiums in this scenario.

Theorem 4.1. *The solution to Problem 1a is given by*

$$I^*_{b,P}(x) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}, \tag{4.3}$$

where $y(x,\lambda)$ is the solution to

$$u'(w_1 - x + y(x,\lambda) - P) = k \cdot v'(w_2 - y(x,\lambda) + P) - \lambda(1+\theta), \tag{4.4}$$

and the Lagrange multiplier coefficient λ is selected such that $(1+\theta)\mathbf{E}\left[I^*_{b,P}(x)\right] = P$.

The proof of Theorem 4.1 is given in the Appendix A.

With the optimal form of ceded function given by Theorem 4.1, one can next seek the optimal parameter values P^* and b^* by solving Problem 1b. Since the functional form of the ceded function implicitly depends on P and b, analytical results are impossible. However, numerical solution could be obtained because it is a maximization problem over two real parameters. We will illustrate this in the numerical examples provided in Section 7.

Remark 4.1. If TVaR is the risk measure, the solution can be obtained by modifying Theorem 4.1 slightly by replacing $\bar{I}_{V_1,b}$ and $\underline{I}_{V_1,b}$ with $\bar{I}_{T_1,b}$ and $\underline{I}_{T_1,b}$ respectively. The search range for Problem 1b changes accordingly to $[\underline{P}_{T_1,b}, \overline{P}_{T_1,b}]$, where

$$\underline{P}_{T_1,b} = (1+\theta)\mathbf{E}[\underline{I}_{T_1,b}(X)]$$

and

$$\overline{P}_{T_1,b} = (1+\theta)\mathbf{E}[\bar{I}_{T_1,b}(X)].$$

Remark 4.2. Taking derivative with respect to x on both sides of Eq. (4.4), one gets

$$y'(x,\lambda) = \frac{u''(w_1 - x + y(x,\lambda) - P)}{u''(w_1 - x + y(x,\lambda) - P) + kv''(w_2 - y(x,\lambda) + P)}, \tag{4.5}$$

which is the same as equation (6) in Golubin (2006a). In addition, from (4.5), it is seen that the slope of ceded function (4.3) is between 0 and 1.

Without the synergy-maximizing requirement, or equivalently when the admissible set of ceded function is $\mathcal{C}$ (which has the upper bound $\bar{I}(x) = x$ and the lower bound $\underline{I}(x) = 0$), the optimal reinsurance takes the form

$$I^*(x) = x \wedge \{0 \vee y(x,\lambda)\}, \tag{4.6}$$

where $y(x,\lambda)$ is the solution to Eq. (4.4) and λ is selected such that $(1+\theta)\mathbf{E}[I^*(x)] = P$.

Remark 4.3. For $k = 0$,

$$y(x,\lambda) = x - w_1 + P + [u']^{-1}(\lambda(1+\theta)),$$

where $[u']^{-1}(\cdot)$ is the inverse function of $u'(\cdot)$. Therefore,

$$I^*_{b,P}(x) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x), x - w_1 + P + [u']^{-1}(\lambda(1+\theta))\right\}.$$

One can see that the optimal ceded function in this case is piecewise linear.

After deriving the Pareto-optimal reinsurance policy $I^*_{b^*,P^*}(x)$ corresponding to a negotiation weight parameter k, we can obtain the whole Pareto efficient frontier of the reinsurance policies by varying the weight parameter k from 0 to ∞ (or some very large value in implementation). The policies on the frontier that belong to $\mathcal{C}_r$ are solutions to Problem 1. Note that whether a Pareto-optimal policy belongs to $\mathcal{C}_r$ can be easily checked by a direct substitution.

To shed more light on insurance practice, we hereby briefly discuss the influence of parameters involved in Theorem 4.1 on the optimal reinsurance policy. Intuitively, the larger the difference between α_c and α_r, the larger the synergy potential for total risk reduction. However, as per Eq. (4.3), to make maximal use of the potential synergy for total risk reduction, the admissible set of functions for EU maximization become smaller.

On the other hand, from the EU maximization perspective, as per equation (4.4), the more capital the insurer holds (everything else the same), the less coverage it needs. Therefore, ceding out the losses in layer (a_r, a_c) as required by risk-reduction synergy maximization may not be to the advantage of EU maximization.

Furthermore, with the synergy constraint, the weighted EU at maximum given by (4.2) is surely less than that without the constraint. However, it remains unclear whether such constraint is disadvantageous to both parties or only one of them. In an extreme case such as $k \to \infty$, the optimal ceded function for Problem 1a without the synergy constraint is given by $I^d(x) = x \wedge d$. If $d \geq a_c$, then $I^d \in \mathcal{C}_r$. Consequently, adding the synergy constraint does not affect the solution and thus does not affect the two parties' utility gain. However, if $d < a_c$ and there exists a $b \in [0, a_r]$ such that $(1+\theta)\mathbf{E}[I_b^d(X)] = P$ where

$$I_b^d(x) = \bar{I}_{V_1,b}(x) \wedge \left\{ \underline{I}_{V_1,b}(x) \vee I^d(x) \right\},$$

then there exists a point $c \in [a_r, a_c]$ such that $I^d(x) \geq I_b^d(x)$ when $x \in [0, c]$ and $I^d(x) \leq I_b^d(X)$ when $x \in [c, M]$. According to the cross-point condition (Osuna, 2012) and the definition of second order stochastic dominance (SSD), we have

$$w_1 - X + I_b^d(X) - P \succ_{SSD} w_1 - X + I^d(X) - P,$$
$$w_2 - I^d(X) + P \succ_{SSD} w_2 - I_b^d(X) + P,$$

which means for fixed P the insurer's EU increases while the reinsurer's EU decreases after imposing the synergy constraint. Generally, the effects of synergy maximization constraint on the two parties' utility gain depend on the model parameters and the optimal values of P and b, which are difficult to calculate explicitly.

Methodologically, step-wise optimization can be used to efficiently identify the parametric form of the optimal insurance, which translates the original infinite-dimensional optimization problem to a finite-dimensional one. The method has been widely used in the optimal (re)insurance literature. See for e.g. Cheung (2010), Asimit et al. (2013) and Jiang et al. (2017).

Remark 4.4. In our model, if the premium is negotiable instead of determined by actuarial premium principle, then the optimal policy $(I(X), P)$ is searched in the space $\mathcal{C}_g \bigotimes [0, \infty)$. This setting was analyzed by Raviv (1979) and later by Asimit and Boonen (2018).

4.1. Optimal policies as the Nash bargaining solutions

In many situations, it is desirable to identify an "optimal" reinsurance policy from a set of Pareto-optimal policies. Therefore, we next apply the Nash bargaining model (Nash, 1950) to identify the policy such that the benefits of cooperation are "fairly" shared by the two parties.

Based on a set of simple and reasonable axioms: scale invariance, symmetry, Pareto efficiency, and independence of irrelevant alternatives, Nash (1950) proposed that the solution to a two-person bargaining game could be obtained by maximizing the product of their utility gains. In our context, the optimal reinsurance policy in the sense of Nash bargaining solution can be obtained by solving

$$\max_{I \in \mathcal{C}_s} \left\{ \mathbf{E}\left[u(w_1 - X + I(X) - \pi(I(X)))\right] - \mathbf{E}\left[u(w_1 - X)\right] \right\}$$
$$\times \left\{ \mathbf{E}\left[v(w_2 - I(X) + \pi(I(X)))\right] - v(w_2) \right\}, \tag{4.7}$$

where $\mathcal{C}_s$ contains all the solutions to Problem 1 with different values of k ($k \in [0, \infty]$).

It is known that the Nash bargaining solution locates on the Pareto efficient frontier. Therefore, it may be identified by checking which Pareto-optimal policy solves (4.7). A numerical example is presented in Section 7.

5. Optimal policies with additional risk constraints

In this section, we study the optimal reinsurance policies when the following additional risk constraints are imposed by the two parties:

$$\rho_{g_1}(X - I(X) + \pi(I(X))) \leq L_1, \quad \rho_{g_2}(I(X) - \pi(I(X))) \leq L_2, \tag{5.1}$$

where L_1 and L_2 are the risk tolerance levels of the insurer and reinsurer respectively. We derive the results for the cases when VaR and TVaR are risk measures in the following.

5.1. VaR

Again, we fix $\pi(I(X)) = P$ in the first stage. Because the ceded function $I \in \mathcal{C}$ is 1-Lipschitz continuous and VaR has the properties of translation invariance and comonotonic additivity, we have

$$VaR_{\alpha_c}(X - I(X) + P) \leq L_1 \iff P \leq L_1 + I(a_c) - a_c,$$

and

$$VaR_{\alpha_r}(I(X) - P) \leq L_2 \iff P \geq I(a_r) - L_2.$$

Recall that for $I \in \mathcal{C}_{V_1,b}$, $I(a_r) = b$ and $I(a_c) - I(a_r) = a_c - a_r$. The above two inequalities are combined into

$$P \in [b - L_2, b + L_1 - a_r]. \tag{5.2}$$

Consequently, the optimal ceded function with the new VaR constraints can be obtained by using the results in Section 4 with a very slight modification, where we change the search range for P (in Problem 1b) from $[\underline{P}_{V_1,b}, \bar{P}_{V_1,b}]$ to

$$[\underline{P}_{V_1,b}, \bar{P}_{V_1,b}] \cap [b - L_2, b + L_1 - a_r].$$

Note that, if the above set is empty, then the problem has no feasible solution.

5.2. TVaR

In this case, let $t_c = TVaR_{\alpha_c}(X)$, $t_r = TVaR_{\alpha_r}(X)$ and $a_s = VaR_s(X)$ for $0 \leq s \leq 1$. Then

$$TVaR_{\alpha_c}(X - I(X) + P) \leq L_1 \iff P \leq L_1 + \frac{1}{1-\alpha_c}\int_{\alpha_c}^1 I(a_s)ds - t_c,$$

and

$$TVaR_{\alpha_r}(I(X) - P) \leq L_2 \iff P \geq \frac{1}{1-\alpha_r}\int_{\alpha_r}^1 I(a_s)ds - L_2.$$

Recall that for $I \in \mathcal{C}_{T_1,b}$, $I(a_r) = b$ and $I(x) = b + x - a_r$ for $x > a_r$. Therefore, the above two inequalities are combined into

$$P \in [b - L_2 + t_r - a_r, b + L_1 - a_r]. \tag{5.3}$$

Analogous to the VaR case, the optimal ceded function with the additional TVaR risk constraints can be obtained by changing the search range for P (in Problem 1b) from $[\underline{P}_{T_1,b}, \bar{P}_{T_1,b}]$ to

$$[\underline{P}_{T_1,b}, \bar{P}_{T_1,b}] \cap [b - L_2 + t_r - a_r, b + L_1 - a_r].$$

Remark 5.1. Comparing (5.2) and (5.3), we observe that the allowable range of P is narrower for the TVaR case than for the VaR case. This is consistent with that the TVaR constraints are more stringent than the VaR constraints.

6. Optimal policies under rank dependent expected utility theory

In this section, we study the synergy-maximizing Pareto-optimal policies when the insurer is a RDEU maximizer whereas the reinsurer is risk neutral. For simplicity, we present results for the VaR type synergy-maximizing policies when $\alpha_c > \alpha_r$.

To start, we let $T : [0, 1] \to [0, 1]$ be a strictly increasing and continuously differentiable probability weighting function that satisfies $T(0) = 0$ and $T(1) = 1$. Then the preference value of random wealth, X, in RDEU framework is defined as

$$U^{RDEU}(X) = \int_{-\infty}^{\infty} u(x)d[1 - T(1 - F_X(x))]. \tag{6.1}$$

With this setup, our optimization problem is formulated as

Problem 2 (*RDEU-based Problem*)**.**

$$\max_{I\in\mathcal{C}_{V_1,b}} U^{RDEU}(w_1 - X + I(X) - P) + k\mathbf{E}[w_2 - I(X) + P],$$
$$\text{s.t. } P = (1+\theta)\mathbf{E}[I(X)].$$

Note that under the premium constraint $P = (1+\theta)E[I(X)]$, the second term in the objective function becomes

$$k\mathbf{E}[w_2 - I(X) + P] = k \cdot \left(w_2 + \frac{\theta}{1+\theta}P\right).$$

Consequently, Problem 2 reduces to a unilateral problem. Furthermore, optimizing $I(x)$ is equivalent to optimizing the retention function $R(x) = x - I(x)$. Thus, applying the quantile formulation introduced in Bernard et al. (2015) and Xu et al. (2019), the first term of objective function could be rewritten as

$$\begin{aligned} U^{RDEU}(w_1 - X + I(X) - P) &= \int_0^1 u(w_1 - P - F^{-1}_{R(X)}(z))T'(z)dz \\ &= \int_0^1 u(w_1 - P - R(F_X^{-1}(z)))T'(z)dz, \end{aligned}$$

where the first equality is derived in the same way as (Xu et al., 2019) and the second equality is due to the fact that the function R also belongs to the class $\mathcal{C}$. Moreover, the premium constraint could also be written in terms of R:

$$\mathbf{E}[R(X)] = \int_0^1 R(F_X^{-1}(z))dz = \mathbf{E}[X] - \mathbf{E}[I(X)] = \mathbf{E}[X] - \frac{P}{1+\theta}.$$

In Section 4, we find the solution to Problem 1a by bounding the solution for the case without considering the synergy maximization constraint by the upper and lower bounds of $\mathcal{C}_{V_1,b}$. Similarly, in this section, we first find the point-wise maximizer to Problem 2 without the synergy maximization constraint, or in the larger class $\mathcal{C}$, and then bound it using the upper and lower bounds of $\mathcal{C}_{V_1,b}$, i.e. $\bar{I}_{V_1,b}$ and $\underline{I}_{V_1,b}$. The bounded solution solves Problem 2.

In the following, to simplify the notations, we let $G(z) = R(F_X^{-1}(z))$. Noting that $R(F_X^{-1}(z)) = F^{-1}_{R(X)}(z)$ (see Xu et al., 2019), we define the following set

$$\tilde{\mathcal{C}} = \left\{F^{-1}_{R(X)}(z) \;\middle|\; R \in \mathcal{C}\right\}.$$

Then every function G satisfies $0 \le G(z) \le F_X^{-1}(z)$ and $0 \le G'(z) \le \left(F_X^{-1}(z)\right)'$. In addition, we let $W_\Delta = w_1 - P$, whose range is $[w_1 - (1+\theta)\mathbf{E}[X], w_1]$. With these notations, we consider the following problem.

Problem 3 (*Problem 2 Without Synergy Maximization Constraint*)**.**

$$\max_{G\in\tilde{\mathcal{C}}} \int_0^1 u(W_\Delta - G(z))T'(z)dz,$$
$$\text{s.t.} \int_0^1 G(z)dz = \mathbf{E}[X] - \frac{P}{1+\theta}.$$

We remark that variants of Problem 3 have been studied extensively in the literature. For example, Bernard et al. (2015) solved Problem 3 without considering the no-sabotage property and thus their optimal ceded function may lead to moral hazards. Ghossoub (2019) also solved Problem 3 without taking into account the no-sabotage property, but introduced the cost of verification to avoid possible moral hazards. So far, to the best of our knowledge, only (Xu et al., 2019) solved Problem 3 by imposing the no-sabotage property on the optimal policy. However, their results have to rely on two additional assumptions on the utility and probability weighting functions. Until now, how to incorporate the no-sabotage property into the optimal insurance problem within the framework of RDEU is still an open question.

In this paper, we also need to rely on two important assumptions similarly to those in Xu et al. (2019) to handle the no-sabotage property. Different from (Xu et al., 2019), as per our strategy to determine the synergy-maximizing optimal policy, the solution to Problem 3 needs to be characterized by point-wise maximization, which in turn requires the technique of constructing convex or concave "envelope" applied in He et al. (2017), Ghossoub (2019) and Boonen and Ghossoub (2020). We will show that our proposed assumptions could be perfectly combined with the constructed concave "envelope" to derive the solution to Problem 2.

First, for any twice differentiable function η, define its Arrow–Pratt measure of absolute risk aversion to be $A_\eta(x) = -\frac{\eta''(x)}{\eta'(x)}$. Our assumptions are

(1). The function $A_u(x)$ is increasing on its domain.
(2). $A_T(F_X(x))f_X(x) \le A_u(w_1 - (1+\theta)\mathbf{E}[X] - x)$.

These two assumptions in fact complement assumptions 5.2 (i) and (ii) of Xu et al. (2019). Assumption (1) is not very restrictive because there are many utility functions with increasing absolute risk aversion, including for example, the quadratic utility function. Under an inverse S-shaped function T, Assumption (2) supposes that the degree of the insurer's concern for small losses is small relative to the absolute risk aversion of the utility function.

Remark 6.1. If T is convex, then assumption (2) is automatically satisfied for all the $x \in [0, M]$.

If T has an inverse S shape, then assumption (2) only needs to hold for $x \in [0, F_X^{-1}(a)]$ where a is defined as the unique solution to $T'(a) = \frac{1-T(a)}{1-a}$ (see Xu et al., 2019).

However, if T is concave on $[0, 1]$, then assumption (2) needs to be assumed to hold for all the $x \in [0, M]$ (see the proof for Theorem 6.1 for the details). This can be achieved by choosing a large w_1, e.g.

$$w_1 = \max\left\{A_u^{-1}\left(A_T(F_X(x))f_X(x)\right) + (1+\theta)\mathbf{E}[X] + x \;\middle|\; x \in [0, M]\right\}.$$

Now we are ready to present the main result of this section. The proof is provided in the Appendix.

Theorem 6.1. *Under assumptions (1) and (2), let δ be the concave envelope of the function T on $[0, 1]$, the following results hold:*

(1). If $\lim_{z\to 0+} T'(z) = T'(0+) = \infty$, the solution to Problem 2 is given by

$$I^{**}_{b,P}(x) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee (x - R^*(x))\right\}, \tag{6.2}$$

where $R^*(x) = 0 \vee \left(W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(F_X(x))}\right)\right)$ is the solution to Problem 3 and λ is chosen such that $\mathbf{E}[I^{**}_{b,P}(x)] = \frac{P}{1+\theta}$.

(2). If T is concave, the solution to Problem 2 is given by

$$I^{**}_{b,P}(x) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee (x - R^*(x))\right\}, \tag{6.3}$$

where $R^*(x) = x \wedge \left\{0 \vee \left(W_\Delta - [u']^{-1}\left(\frac{\lambda}{T'(F_X(x))}\right)\right)\right\}$ is the solution to Problem 3 and λ is chosen such that $\mathbf{E}[I^{**}_{b,P}(x)] = \frac{P}{1+\theta}$.

(3). If T is convex, the solution to Problem 2 is given by

$$I^{**}_{b,P}(x) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee (x - d^*)_+\right\}, \tag{6.4}$$

where d^* is determined by $\mathbf{E}[I^{**}_{b,P}(x)] = \frac{P}{1+\theta}$.

Remark 6.2. Our obtained solution $R^*(x)$ to Problem 3 agrees with that in Ghossoub (2019), in which the author applied the convex envelope of T^{-1}. Here we apply the concave envelope of T. Note that, in the first case of Theorem 6.1, we assume that the probability weighting function T satisfies $\lim_{z\to 0+} T'(z) = \infty$, instead of being a general inverse S-shaped probability weighting function. This can be interpreted as that the insurer distorts the probability for small losses greatly. Technically, this eliminates the restriction on λ as was the case in Theorem 3.5(i) of Ghossoub (2019).

Remark 6.3. Compare with (Ghossoub, 2019), our obtained state-contingent indemnification function satisfies the no-sabotage property. It is understandable that more strict assumptions are needed to achieve this.

Remark 6.4. Problem 3 is quite similar to the problem studied in Xu et al. (2019), except that our assumptions (1) and (2) are different from theirs. Our assumptions were introduced to ensure that the solution derived from the point-wise maximization automatically satisfies the no-sabotage condition. This facilitates the imposition of the upper and lower bounds ($\bar{I}_{V_1,b}(x)$ and $\underline{I}_{V_1,b}(x)$), which are derived from the synergy-maximizing constraint.

In Xu et al. (2019), the calculus of variation was applied to determine the optimal marginal indemnity function ($I'(x)$). Although it seems to be a more natural way to deal with the no-sabotage condition, it gives rise to some technical difficulties since the optimal $I'(x)$ is determined by the tail integral of a complex function. As such, their assumptions 5.2(i) and (ii) were introduced in order to get an explicit solution.

7. Numerical examples

The focus of this paper is to derive the Pareto-optimal policy and Nash bargaining solution when the synergy gained from the insurer–reinsurer cooperation gets maximized. In this section, we present an example of step-to-step derivation of the solution to our main Problem 1 when VaR or TVaR is used as the risk measure.

Suppose that the utility functions of the insurer and the reinsurer are given by

$$u(x) = -\frac{1}{2}\beta_1 x^2 + x, \quad x \le \frac{1}{\beta_1},$$

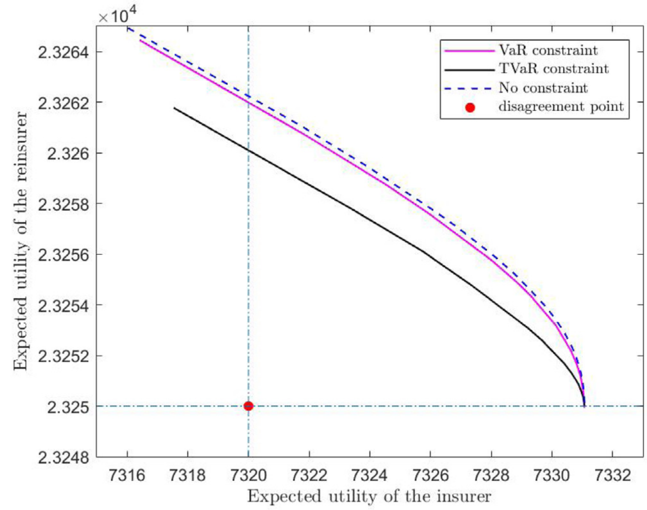

Fig. 3. EU-based Pareto efficient frontiers.

and

$$v(x) = -\frac{1}{2}\beta_2 x^2 + x, \quad x \le \frac{1}{\beta_2}.$$

Then solving equation (4.4) yields

$$y(x, \lambda) = \frac{\beta_1 x + k\beta_2(w_2 + P) - k + \lambda(1+\theta) + 1 - \beta_1(w_1 - P)}{\beta_1 + k\beta_2},$$

and the optimal reinsurance policy is given by (4.3). One can see that the optimal ceded function is piecewise linear, with slope being either one, zero, or $\frac{\beta_1}{\beta_1 + k\beta_2}$.

More specifically, we next provide numerical solutions to the problem with the following hypothetical parameters.

- The insurer and reinsurer have initial wealth $w_1 = \$10000$ and $w_2 = \$30000$.
- The parameters for quadratic utility functions are $\beta_1 = 0.00002$ and $\beta_2 = 0.000015$, so that the insurer is more risk averse than the reinsurer.
- The insurer and reinsurer apply VaR (TVaR) as risk measures with confidence levels $\alpha_c = 0.95$ and $\alpha_r = 0.9$ respectively.
- The aggregate loss X follows an exponential distribution with mean \$2000. In this case $a_c = VaR_{\alpha_c}(X) = \$5991.5$ and $a_r = VaR_{\alpha_r}(X) = \$4605.2$.
- The safety loading θ is set to be 0.05.

7.1. Pareto efficient frontier

We solve Problem 1 with the admissible ceded functions given by the sets $\mathcal{C}$, $\mathcal{C}_{V_1}$ and $\mathcal{C}_{T_1}$ respectively. The obtained optimal policies are denoted by I^*, I^*_V and I^*_T. By varying the negotiation weight parameter k, we obtain the Pareto efficient frontiers, which are shown in Fig. 3. The following observations are noted:

- The EU-based Pareto efficient frontier becomes lower and lower in the order of none, VaR and TVaR synergy constraint. This is not surprising because the admissible sets have the relationship $\mathcal{C} \supset \mathcal{C}_{V_1} \supset \mathcal{C}_{T_1}$. The TVaR synergy-maximization constraint restricts the form of ceded function significantly and thus sacrifices the two parties' expected utilities. Understandably, increasing EU and minimizing risk (VaR or TVaR) can be contradictory objectives and one has to strike a balance.
- The range of k which could make the rationality constraints satisfied is approximately $[0, 1.4]$ in all three cases. The

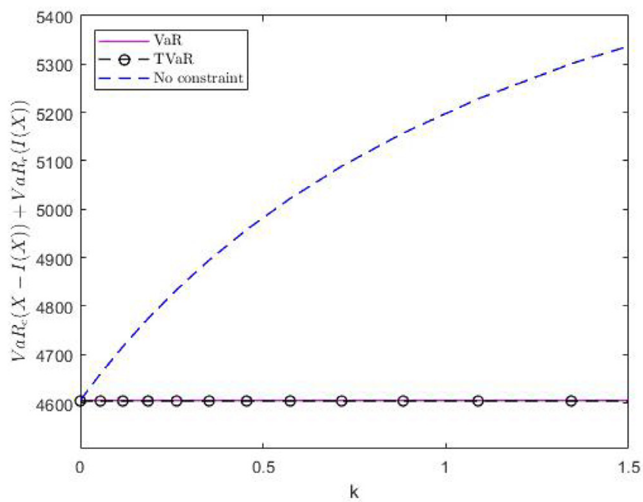

Fig. 4. The total VaR of the insurer and reinsurer.

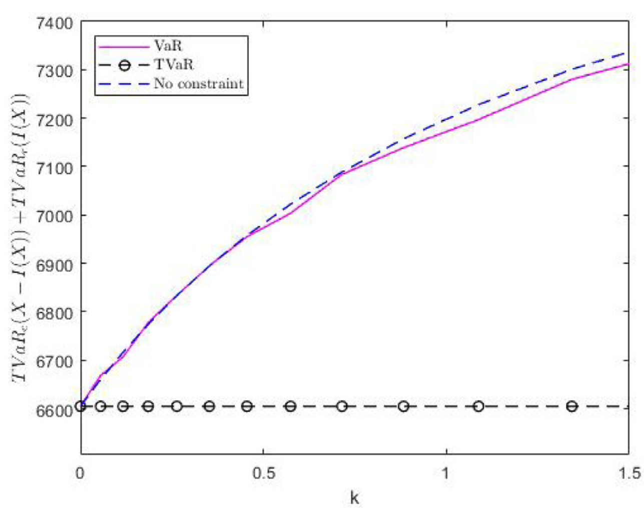

Fig. 5. The total TVaR of the insurer and reinsurer.

frontiers coincide at the lower right corner point, which is due to the optimality of stop-loss function in all three cases when $k = 0$.

To illustrate the effects of synergy constraints on the level of risks, the total risk (in terms of VaR and TVaR respectively) of the two parties corresponding to the optimal policies I^*, I_V^*, I_T^* are shown in Figs. 4 and 5. Two observations are noted:

- TVaR synergy-maximizing policies are also VaR synergy-maximizing. This is reasonable because $\mathcal{C}_{V_1} \supset \mathcal{C}_{T_1}$.
- Without the synergy-maximizing constraint, the total risk increases with the reinsurer's relative negotiation power k. This is because we have assumed the reinsurer is less risk averse ($\alpha_r < \alpha_c$), maximizing its EU will increase the total risk in the system.

7.2. The Nash bargaining solution

Because the Nash solution is on the efficient frontier, it can be numerically identified by seeking the best parameter k so that (4.7) is maximized.

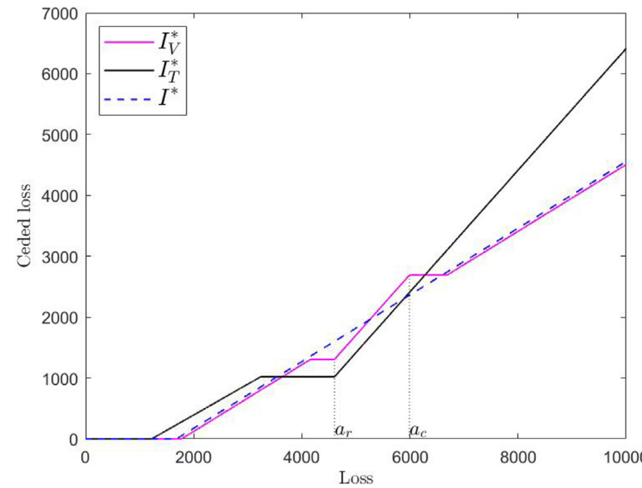

Fig. 6. Pareto-optimal reinsurance policies corresponding to the Nash bargaining solutions.

Specifically, the Nash bargaining solutions are as follows:

- No constraint: $k = 1.1$, $P^* = \$496.08$, $I^*(x) = 0.55(x - 1682.9)_+$.

$$I_V^*(x) = \begin{cases} 0.55(x - 1770.9)_+ \wedge 1305.9, & x \in [0, 4605.2], \\ x - 3299.3, & x \in [4605.2, 5991.5], \\ 0.55(x - 1770.9)_+ \vee 2692.2, & x \in [5991.5, \infty). \end{cases} \tag{7.1}$$

- TVaR constraint: $k = 1.3$, $P^* = \$575.83$,

$$I_T^*(x) = \begin{cases} 0.51(x - 1216.1)_+ \wedge 1012, & x \in [0, 4605.2], \\ x - 3585.2, & x \in [4605.2, \infty). \end{cases} \tag{7.2}$$

The ceded functions are plotted in Fig. 6. We have the following observations:

- Under the VaR synergy constraint, the obtained optimal reinsurance policies are quite close to the one obtained without constraint. This happens when the values of α_c and α_r are close. In fact, as commented at the end of Section 3.1, with $\alpha_c = \alpha_r$, no reinsurance policy in the set $\mathcal{C}$ can reduce the total risk in the system, in which case $\mathcal{C}_{V_1} = \mathcal{C}$.
- By comparing I^* and I_T^*, it is seen that the former requires only 55% coverage for the losses after a deductible amount \$1682.9 while the later requires that 100% of tail risk (larger than $VaR_{\alpha_r}(X) = 4605$) to be ceded, which may result in a big difference in premiums.

7.3. Optimal policies under additional risk constraints

7.3.1. Additional VaR constraint

We assume that an additional risk constraint is imposed by the insurer such that

$$VaR_{\alpha_c}(X - I(X) + P) \le L_1,$$

where L_1 is $0.6 \times VaR_{\alpha_c}(X) = \$3594.9$ for illustrative purpose. The optimal policy Eq. (7.1) derived in Section 7.2 results in $VaR_{\alpha_c}(X - I_V^*(X) + P) = \$3774.3$, which violates the imposed risk constraint. Therefore, we adjust the search range for premium as

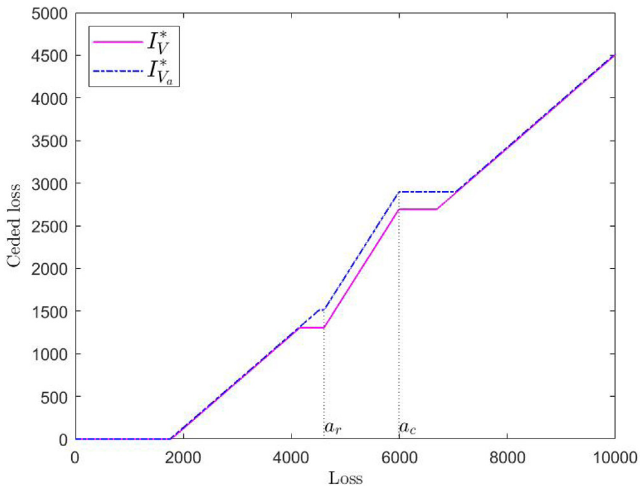

Fig. 7. Pareto-optimal reinsurance policy corresponding to the Nash bargaining solution with additional VaR constraint of the insurer.

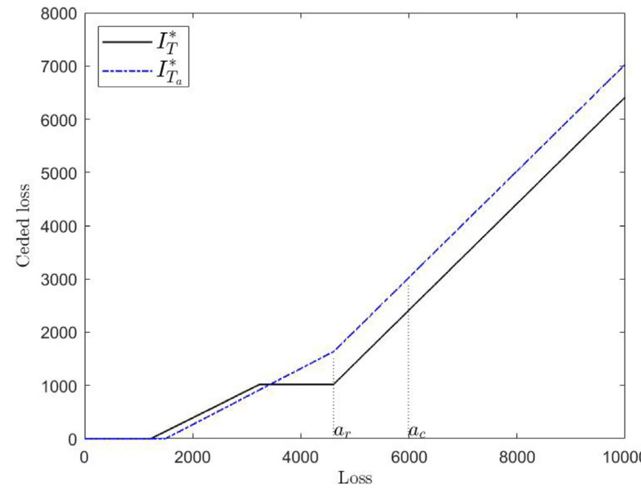

Fig. 8. Pareto-optimal reinsurance policy corresponding to the Nash bargaining solution with additional TVaR constraint of the insurer.

discussed in Section 5 and obtain the optimal policy as follows:

$$I_{V_a}^*(x) = \begin{cases} 0.55(x-1770.9)_+ \wedge 1512.4, & x \in [0, 4605.2], \\ x - 3092.8, & x \in [4605.2, 5991.5], \\ 0.55(x-1770.9)_+ \vee 2898.7, & x \in [5991.5, \infty). \end{cases} \tag{7.3}$$

The premium for this policy is \$497.21. The ceded function $I_{V_a}^*(x)$ is compared with $I_V^*(x)$ in (7.1) in Fig. 7. It is seen that $I_{V_a}^*(x)$ provides more coverage for losses around the layer $[a_r, a_c]$, resulting in a lower VaR of the insurer. Of course, the premium increases from \$475 for $I_V^*(x)$ to \$497 for $I_{V_a}^*(x)$.

7.3.2. Additional TVaR constraint

Now we assume that an additional risk constraint is imposed by the insurer such that $TVaR_{\alpha_c}(X - I(X) + P) \le L_1$, where L_1 is again set to be $0.6 \times VaR_{\alpha_c}(X) = \$3594.9$. The optimal policy I_T^* in (7.2) results in $TVaR_{\alpha_c}(X - I(X) + P) = \$4161.1$. Thus the additional risk constraint is violated. Therefore, we adjust the search range for premium as discussed in Section 5 and obtain the optimal policy as follows:

$$I_{T_a}^*(x) = \begin{cases} 0.53(x-1484.8)_+ \wedge 1632.7, & x \in [0, 4605.2], \\ x - 2972.5, & x \in [4605.2, \infty). \end{cases} \tag{7.4}$$

The premium for this policy is \$622.39. These two policies are shown in Fig. 8. As expected, the policy $I_{T_a}^*$ in (7.4) covers more losses in the right tail than I_T^* in (7.2).

8. Concluding remark and future research

In this paper, we study the Pareto-optimal reinsurance design considering two optimality criteria: EU maximization and risk minimization. We first identify a set of reinsurance policies that minimize the total risk shared by the two parties, then we take this set of policies as admissible and determine the Pareto-optimal policies that maximize the weighted EU of the two parties. Then we show how to obtain the policy corresponding to the Nash bargaining solution numerically. In addition, we characterize the optimal policy when additional risk constraints are imposed by the two parties. A step-wise optimization is applied in this paper, which provides an implementable way to deal with the multiple-objective optimization problem. At last, we go beyond the EU framework and extend our results to the case where the insurer is a RDEU maximizer and the reinsurer is risk neutral.

There are many ways to advance the results of this paper. For example, one could consider a risk averse reinsurer in the non-EU framework, which is an important topic and will definitely introduce new technical difficulties. Another direction of extension is to investigate a more general multiple-objective risk-sharing problem which involves, e.g. multiple insurers or reinsurers. Some related works could be found in Asimit and Boonen (2018) and Boonen and Ghossoub (2019).

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Appendix A. Proof of Theorem 4.1

Denote terms under the expectation operators in the objective function of Problem 1a by

$$M(x, I(x), P) := u(w_1 - x + I(x) - P) + k \cdot v(w_2 - I(x) + P), \tag{A.1}$$

and let

$$N(x, I(x), P, \lambda) := M(x, I(x), P) + \lambda\left[(1+\theta)I(x) - P\right] \tag{A.2}$$

be the Lagrange augmented function. We first have the following verification Lemma.

Lemma A.1. *A ceded function $I^*(x, \lambda^*) \in \mathcal{C}_{V_1,b}$ solves Problem 1a if there exists a constant $\lambda^* \in \mathbf{R}$ such that the following two conditions are satisfied:*

Condition 1: *For all $I \in \mathcal{C}_{V_1,b}$ that satisfies $(1+\theta)\mathbf{E}[I(X)] = P$,*

$$N\left(x, I^*(x, \lambda^*), P, \lambda^*\right) \ge N\left(x, I(x), P, \lambda^*\right), \quad x \in [0, M].$$

Condition 2:

$$(1+\theta)\,\mathbf{E}\left[I^*(X, \lambda^*)\right] = P, \tag{A.3}$$

where the ceded function I^* includes a second argument λ to emphasize its dependence on the Lagrange coefficient λ.

Proof. Based on Condition 1, we have for all $x \geq 0$,

$$M\left(x, I^*(x,\lambda^*), P\right) - M\left(x, I(x), P\right) \geq \lambda(1+\theta)\left(I^*(x,\lambda^*) - I(x)\right).$$

Thus

$$\int_0^M \left(M\left(x, I^*(x,\lambda^*), P\right) - M\left(x, I(x), P\right)\right) dF(x)$$
$$\geq \lambda(1+\theta)\Big[\mathbf{E}[I^*(X,\lambda^*)] - \mathbf{E}[I(X)]\Big] = 0,$$

which completes the proof. □

Lemma A.1 in fact states that Problem 1a can be solved pointwisely by identifying $I^*(x,\lambda^*)$ that maximizes the Lagrange augmented function.

We next consider the problem

$$\max_{y\in[\underline{I}_{V_1,b}(x),\bar{I}_{V_1,b}(x)]} N\left(x, y, P, \lambda\right), \tag{A.4}$$

for fixed x, b and λ. We will use the notation $N_1(\cdot,\cdot,\cdot,\cdot)$ for the first partial derivative of N with respective to the first argument and $N_{11}(\cdot,\cdot,\cdot,\cdot)$ for the second derivative with respective to the first argument and so on.

Due to the concavities of $u(\cdot)$ and $v(\cdot)$, we have

$$N_{22}\left(x, y, P, \lambda\right) = u''(w_1 - x + y - P) + kv''(w_2 - y + P) < 0.$$

Thus $N\left(x, y, P, \lambda\right)$ is strictly concave in y and there must exist a solution to (A.4), which we denote as $I^*(x;\lambda)$ and determine its form in the next Lemma.

Lemma A.2. *The solution to problem (A.4) is given by*

$$I^*(x;\lambda) := \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}, \tag{A.5}$$

where $y(x,\lambda)$ be the solution to the first-order condition

$$N_2\left(x, y, P, \lambda\right) = 0. \tag{A.6}$$

Proof. Firstly, notice that

$$N_2\left(x, y, P, \lambda\right) = u'(w_1 - x + y - P) - kv'(w_2 - y + P) + \lambda(1+\theta)$$

is continuous and non-increasing in y. Second, because of the assumptions (2.4) and (2.5), $N_2\left(x, -\infty, P, \lambda\right) > 0$ and $N_2\left(x, \infty, P, \lambda\right) < 0$. Therefore, a solution, $y(x,\lambda)$, to (A.6) always exists in $[-\infty, \infty]$.

If

$$N_2\left(x, y, P, \lambda\right)|_{y=\underline{I}_{V_1,b}(x)} = u'(w_1 - x + \underline{I}_{V_1,b}(x) - P) - k\cdot v'(w_2 - \underline{I}_{V_1,b}(x) + P) + \lambda(1+\theta) < 0,$$

then $y(x,\lambda) < \underline{I}_{V_1,b}(x)$ and the solution to problem (A.4) is $I^*(x,\lambda) = \underline{I}_{V_1,b}(x)$.

On the other hand, if

$$N_2\left(x, y, P, \lambda\right)|_{y=\bar{I}_{V_1,b}(x)} = u'(w_1 - x + \bar{I}_{V_1,b}(x) - P) - k\cdot v'(w_2 - \bar{I}_{V_1,b}(x) + P) + \lambda(1+\theta) > 0$$

then $y(x,\lambda) > \bar{I}_{V_1,b}(x)$ and $I^*(x,\lambda) = \bar{I}_{V_1,b}(x)$.

Finally, if $y(x,\lambda) \in [\underline{I}_{V_1,b}(x), \bar{I}_{V_1,b}(x)]$ then $I^*(x,\lambda) = y(x,\lambda)$. This ends the proof. □

Lemma A.3. *For any $P \in \left(\underline{P}_{V_1,b}, \bar{P}_{V_1,b}\right)$, there exists a λ^* such that*

$$(1+\theta)\,\mathbf{E}\left[I^*(X,\lambda^*)\right] = P.$$

Proof. Let

$$\phi(\lambda) := (1+\theta)\mathbf{E}\left[I^*(X;\lambda)\right]$$

where $I^*(x;\lambda)$ is given by Lemma A.2. The result is true if we can show that $\phi(\lambda)$ is continuous and non-decreasing with respect to λ and that

$$\lim_{\lambda\to-\infty}\phi(\lambda) = (1+\theta)\mathbf{E}[\underline{I}_{V_1,b}(X)] = \underline{P}_{V_1,b},$$
$$\lim_{\lambda\to\infty}\phi(\lambda) = (1+\theta)\mathbf{E}\left[\bar{I}_{V_1,b}(X)\right] = \bar{P}_{V_1,b}.$$

For this purpose, we first note that $y(x,\lambda)$ is the unique solution to Eq. (A.6), taking derivative on both sides of Eq. (A.6) with respect to λ gives

$$\frac{\partial y(x,\lambda)}{\partial\lambda} = -\frac{(1+\theta)}{u''(w_1 - x + y(x,\lambda) - P) + k\cdot v''(w_2 - y(x,\lambda) + P)} \geq 0.$$

Thus $y(x,\lambda)$ is non-decreasing in λ. It follows that $I^*(x;\lambda) = \bar{I}_{V_1,b}(x) \wedge \left\{\underline{I}_{V_1,b}(x) \vee y(x,\lambda)\right\}$ is also non-decreasing in λ.

For each value of x, $I^*(x;\lambda)$ is continuous in λ. In addition, we have by construction that

$$|I^*(x;\lambda)| \leq \bar{I}_{V_1,b}(x).$$

Then by applying Lebesgue Dominated Convergence Theorem we conclude that $\phi(\lambda) = \mathbf{E}[I^*(x;\lambda)]$ is continuous with respect to λ.

It is easy to see that when $\lambda \to -\infty$, $N_2\left(x, y, P, \lambda\right) < 0$ for all y. Then $I^*(x,-\infty) = \underline{I}_{V_1,b}(x)$, which implies $(1+\theta)\phi(-\infty) = \underline{P}_{V_1,b}$. On the other hand, when $\lambda \to \infty$, $N_2\left(x, y, P, \lambda\right) > 0$ for all y. Consequently, $I^*(x,\infty) = \bar{I}_{V_1,b}(x)$ and $(1+\theta)\phi(\infty) = \bar{P}_{V_1,b}$. This ends the proof. □

Combining Lemmas A.1–A.3, Theorem 4.1 is proved.

Appendix B. Proof of Theorem 6.1

We only prove the result for Theorem 6.1(1) here. The results for Theorem 6.1(2) and (3) could be derived using the same manner as Theorem 3.5(ii) and (iii) in Ghossoub (2019), which are thus omitted.

Analogous to the proof of Theorem 4.1, we need to solve the Lagrangian form of Problem 3

$$\max_{G\in\tilde{\mathcal{C}}}\left\{\int_0^1 u(W_\Delta - G(z))T'(z) + \lambda G(z)\right\} dz. \tag{B.1}$$

We need some Lemmas to proceed. First we modify Lemma B.1 and B.2 in He et al. (2017) to fit our concave envelope construction. The proof is almost as same as theirs and thus omitted here.

Lemma B.1. *Let f be a continuous real-valued function on a nonempty convex subset of* $\mathbf{R}$ *containing* $[0,1]$, *and let g be its concave envelope on* $[0,1]$. *Then*

(1). g is continuous and concave on $[0,1]$*;*
(2). $g(0) = f(0)$ and $g(1) = f(1)$;
(3). for all $x \in [0,1]$, $g(x) \geq f(x)$;
(4). g is affine on $\{x \in [0,1] : g(x) > f(x)\}$.

Moreover, if f is increasing, then so is g. If f is continuously differentiable on $(0,1)$, then so is g.

For our purpose, we let δ be the concave envelope of the function T on $[0,1]$ and define the following two sets:

$$\mathcal{D} := \{z\in[0,1]:\ \delta(z) = T(z)\}, \quad \mathcal{E} := \{z\in[0,1]:\ \delta(z) > T(z)\}.$$

Our next Lemma is analogous to Lemma F.11 of Boonen and Ghossoub (2020). It plays a vital role in characterizing the solution to Problem 3.

Lemma B.2. *For any $g \in \tilde{\mathcal{C}}$, we have*

$$\int_0^1 u(W_\Delta - g(z))T'(z)dz \le \int_0^1 u(W_\Delta - g(z))\delta'(z)dz.$$

Proof. Denote $\tilde{u}(x) = u(W_\Delta + x)$, then by Fubini's Theorem, we have

$$\begin{aligned}
0 &\ge \int_0^1 [(T(1)-\delta(1)) - (T(y)-\delta(y))]\, d\tilde{u}(-g(y)) \\
&= \int_0^1 \left\{ \int_y^1 [T'(z)-\delta'(z)]dz \right\} d\tilde{u}(-g(y)) \\
&= \int_0^1 \left[\int_0^z d\tilde{u}(-g(y)) \right] [T'(z)-\delta'(z)]dz \\
&= \int_0^1 [\tilde{u}(-g(z)) - \tilde{u}(-g(0))][T'(z)-\delta'(z)]dz \\
&= \int_0^1 \tilde{u}(-g(z))T'(z)dz - \int_0^1 \tilde{u}(-g(z))\delta'(z)dz.
\end{aligned}$$

Therefore we get $\int_0^1 \tilde{u}(-g(z))T'(z)dz \le \int_0^1 \tilde{u}(-g(z))\delta'(z)dz$. This ends the proof. □

In light of Lemma B.2, we have

$$\begin{aligned}
&\int_0^1 \left\{ u(W_\Delta - G(z))T'(z) + \lambda G(z) \right\} dz \\
&\le \int_0^1 \left\{ u(W_\Delta - G(z))\delta'(z) + \lambda G(z) \right\} dz.
\end{aligned}$$

Our next Lemma gives the optimal policy when the probability weighting function T is replaced by its concave envelop δ.

Lemma B.3. *The solution to problem*

$$\max_{G \in \tilde{\mathcal{C}}} \left\{ \int_0^1 u(W_\Delta - G(z))\delta'(z) + \lambda G(z) \right\} dz. \tag{B.2}$$

is given by

$$G^*(z) = F_X^{-1}(z) \wedge \left\{ 0 \vee \left(W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(z)} \right) \right) \right\}. \tag{B.3}$$

Proof. Similar to the proof for Theorem 4.1, let

$$K(y, \lambda) = u(W_\Delta - y)\delta'(z) + \lambda \cdot y.$$

Applying the first order condition yields

$$K_1(y, \lambda) = 0 \implies y(z) = W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(z)} \right). \tag{B.4}$$

It is easy to verify that $K_{11}(y, \lambda) < 0$, which indicates that $K(y, \lambda)$ is concave w.r.t y. Then, by recognizing that $G(z) \in [0, F_X^{-1}(z)]$, the solution to problem (B.2) is given by (B.3). □

We next show how to derive $R^*(x)$ in Theorem 6.1 (1) when $\lim_{z \to 0+} T'(z) = T'(0+) = \infty$.

First, under assumptions (1) and (2) we have

$$\begin{aligned}
& A_T(F_X(x))f_X(x) \le A_u(w_1 - (1+\theta)\mathbf{E}[X] - x) \\
\Longrightarrow\ & A_T(F_X(x)) \le A_u(w_1 - (1+\theta)\mathbf{E}[X] - x) \cdot \frac{1}{f_X(x)} \\
\Longrightarrow\ & A_T(F_X(x)) \le A_u(W_\Delta - x) \cdot \frac{1}{f_X(x)} \\
\xRightarrow{x = F_X^{-1}(z)}\ & A_T(z) \le A_u(W_\Delta - F_X^{-1}(z)) \cdot \frac{1}{f_X(F_X^{-1}(z))} \\
\Longrightarrow\ & A_T(z) \le A_u(W_\Delta - F_X^{-1}(z)) \cdot (F_X^{-1}(z))'.
\end{aligned} \tag{B.5}$$

Second, from Eq. (B.4), $K_1(y, \lambda) = 0$ indicates that $u'(W_\Delta - y(z))\delta'(z) = \lambda$. Then taking derivative on both sides of the former w.r.t z leads to

$$y'(z) = \frac{u'(W_\Delta - y(z)) \cdot \delta''(z)}{u''(W_\Delta - y(z)) \cdot \delta'(z)} = \begin{cases} \dfrac{A_T(z)}{A_u(W_\Delta - y(z))}, & z \in \mathcal{D}, \\ 0, & z \in \mathcal{E}. \end{cases}$$

If $y(z) \le F_X^{-1}(z)$, according to assumption (1) and the inequality (B.5), the above equation indicates that

$$y'(z) = \frac{A_T(z)}{A_u(W_\Delta - y(z))} \le \frac{A_T(z)}{A_u(W_\Delta - F_X^{-1}(z))} \le (F_X^{-1}(z))'.$$

Thus, defining $H(z) = F_X^{-1}(z) - y(z)$, we have

$$H'(z) \ge 0 \quad \text{if} \quad H(z) \ge 0. \tag{B.6}$$

Now let us look at the value of $y(z)$ at point $z = 0$. Since $\delta \ge T$ and $\delta(0) = T(0)$, we have

$$\lim_{\Delta z \to 0+} \frac{\delta(\Delta z) - \delta(0)}{\Delta z} \ge \lim_{\Delta z \to 0+} \frac{T(\Delta z) - T(0)}{\Delta z} = T'(0+) = \infty.$$

Then $u'(W_\Delta - y(0)) \cdot \delta'(0) > \lambda$, which indicates that $y(0) \le 0 = F_X^{-1}(0)$. Hence, $H(0) \ge 0$. Combining with (B.6), we get $H(z) \ge 0$ for all the $z \in [0, 1]$.

As such, (B.3) is reduced to

$$G^*(z) = 0 \vee \left(W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(z)} \right) \right). \tag{B.7}$$

Equivalently,

$$R^*(x) = 0 \vee \left(W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(F_X(x))} \right) \right).$$

Next, we have:

Lemma B.4. *The function G^* also solves problem (B.1).*

Proof. Note that with (B.7), we have

$$dG^*(z) = \begin{cases} 0, \\ \quad \text{if } 0 \ge W_\Delta - [u']^{-1}\left(\dfrac{\lambda}{\delta'(z)} \right), \\ \lambda \cdot \left([u']^{-1} \right)' \left(\dfrac{\lambda}{\delta'(z)} \right) \cdot \dfrac{\delta''(z)}{(\delta'(z))^2}, \\ \quad \text{if } 0 < W_\Delta - [u']^{-1}\left(\dfrac{\lambda}{\delta'(z)} \right). \end{cases}$$

Furthermore, we have

$$\begin{aligned}
&\int_0^1 u'(W_\Delta - G^*(z))d(\delta(z) - T(z)) \\
=& u'(W_\Delta - G^*(1))(\delta(1) - T(1)) - u'(W_\Delta - G^*(0))(\delta(0) - T(0)) \\
&- \int_0^1 (\delta(z) - T(z))du'(W_\Delta - G^*(z)) \\
=& \int_0^1 (\delta(z) - T(z))u''(W_\Delta - G^*(z))dG^*(z) \\
=& \int_{\mathcal{E}} (\delta(z) - T(z))u''(W_\Delta - G^*(z))dG^*(z) = 0,
\end{aligned}$$

where the last equation is due to $\delta''(z) = 0$ on $\mathcal{E}$.

This ends the proof. □

Finally, we prove that the derived function $R^*(x)\ (= G^*(F_X(x)))$ satisfies the no-sabotage property.

Lemma B.5. *Under assumptions (1) and (2), we have that $0 \le R^{*\prime}(x) \le 1$.*

Proof. Let $Y(x) = W_\Delta - [u']^{-1}\left(\frac{\lambda}{\delta'(F_X(x))}\right)$, then $R^*(x) = x \wedge (0 \vee Y(x))$. We get

$$R^{*\prime}(x) = \begin{cases} 1, & \text{if } x < Y(x), \\ Y'(x), & \text{if } 0 < Y(x) \le x, \\ 0, & \text{if } Y(x) \le 0. \end{cases}$$

Therefore we need to prove $Y'(x) \in [0, 1]$ when $0 < Y(x) \le x$. Note that $Y(x)$ actually solves

$$u'(W_\Delta - Y(x)) \cdot \delta'(F_X(x)) = \lambda$$

at point x. Taking the derivative on both sides of the above equation w.r.t x gives

$$\begin{aligned} Y'(x) &= \frac{u'(W_\Delta - Y(x)) \cdot \delta''(F_X(x)) \cdot f_X(x)}{u''(W_\Delta - Y(x)) \cdot \delta'(F_X(x))} \\ &= \begin{cases} \dfrac{u'(W_\Delta - Y(x)) \cdot T''(F_X(x)) \cdot f_X(x)}{u''(W_\Delta - Y(x)) \cdot T'(F_X(x))}, & \text{if } F_X(x) \in \mathcal{D}, \\ 0, & \text{if } F_X(x) \in \mathcal{E}. \end{cases} \end{aligned}$$

As both u and δ are increasing and concave, $Y'(x) \ge 0$. Moreover, under assumptions (1) and (2), for $x \in \{x : F_X(x) \in \mathcal{D}\}$, we have

$$\begin{aligned} Y'(x) &= \frac{u'(W_\Delta - Y(x)) \cdot T''(F_X(x)) \cdot f_X(x)}{u''(W_\Delta - Y(x)) \cdot T'(F_X(x))} = \frac{A_T(F_X(x))f_X(x)}{A_u(W_\Delta - Y(x))} \\ &\le \frac{A_T(F_X(x))f_X(x)}{A_u(w_1 - (1+\theta)\mathbf{E}[X] - x)} \le 1. \end{aligned}$$

This ends the proof. □

With all of above, we have verified that $R^*(x)$ solves Problem 3 (without synergy requirement) point-wisely and that $R^*(x)$ satisfies the no-sabotage property. Therefore, imposing the upper and lower bounds required by synergy maximization, we get the $I^{**}_{b,P}$ in Theorem 6.1.

We close the proof by showing the existence of λ. We are only interested in the case $\underline{P}_{V_1,b} < P < \bar{P}_{V_1,b}$ as otherwise we get trivial result $I^{**}_{b,P} = \underline{I}_{V_1,b}$ or $I^{**}_{b,P} = \bar{I}_{V_1,b}$. To emphasize the dependence of $I^{**}_{b,P}$ on λ, we write it as $I^{**}_{b,P}(x; \lambda)$ in the following lemma.

Lemma B.6. *For $P \in (\underline{P}_{V_1,b}, \bar{P}_{V_1,b})$, there exists a λ such that $\mathbf{E}[I^{**}_{b,P}] = \frac{P}{1+\theta}$.*

Proof. The proof follows exactly the same way as that for Lemma A.3. First, let

$$\phi(\lambda) = (1+\theta)\mathbf{E}[I^{**}_{b,P}(X; \lambda)].$$

Note that $I^{**}_{b,P}(x; \lambda)$ is decreasing w.r.t λ and

$$\lim_{\lambda\to\infty} I^{**}_{b,P}(x; \lambda) = \underline{I}_{V_1,b}(x), \quad \lim_{\lambda\to 0+} I^{**}_{b,P}(x; \lambda) = \bar{I}_{V_1,b}(x).$$

By applying Lebesgue Dominated Convergence Theorem we get that $\phi(\lambda)$ is continuous w.r.t λ and

$$\lim_{\lambda\to\infty} \phi(\lambda) = \underline{P}_{V_1,b}, \quad \lim_{\lambda\to 0+} \phi(\lambda) = \bar{P}_{V_1,b}.$$

Therefore, there exists a λ such that $\phi(\lambda) = P$ if $P \in (\underline{P}_{V_1,b}, \bar{P}_{V_1,b})$. This ends the proof. □

References

Aase, K., 2002. Perspectives of risk sharing. Scand. Actuar. J. 2002 (2), 73–128.
Aase, K., 2009. The Nash bargaining solution vs. equilibrium in a reinsurance syndicate. Scand. Actuar. J. 2009 (3), 219–238.
Arrow, K., 1963. Uncertainty and the welfare economics of medical care. Am. Econ. Rev. 53 (5), 941–973.
Arrow, K., 1973. Optimal Insurance and Generalized Deductibles. Rand Corporation. Technical Report, R-1108-OEO.
Arrow, K., 1974. Optimal insurance and generalized deductibles. Scand. Actuar. J. 1974 (1), 1–42.
Asimit, A., Badescu, A., Verdonck, T., 2013. Optimal risk transfer under quantile-based risk measurers. Insurance Math. Econom. 53 (1), 252–265.
Asimit, V., Boonen, T., 2018. Insurance with multiple insurers: A game-theoretic approach. European J. Oper. Res. 267 (2), 778–790.
Assa, H., 2015. On optimal reinsurance policy with distortion risk measures and premiums. Insurance Math. Econom. 61 (1), 70–75.
Balbás, A., Garrido, J., Mayoral, S., 2009. Properties of distortion risk measures. Methodol. Comput. Appl. Probab. 11 (3), 385–399.
Bernard, C., He, X., Yan, J.-A., Zhou, X.Y., 2015. Optimal insurance design under rank-dependent expected utility. Math. Finance 25 (1), 154–186.
Bernard, C., Tian, W., 2010. Insurance market effects of risk management metrics. Geneva Risk Insur. Rev. 35 (1), 47–80.
Blazenko, G., 1986. The economics of reinsurance. J. Risk Insurance 53 (2), 258–277.
Boonen, T., 2016. Nash equilibria of Over-The-Counter bargaining for insurance risk redistributions: The role of a regulator. European J. Oper. Res. 250 (3), 955–965.
Boonen, T., Ghossoub, M., 2019. On the existence of a representative reinsurer under heterogeneous beliefs. Insurance Math. Econom. 88 (1), 209–225.
Boonen, T.J., Ghossoub, M., 2020. Bilateral risk sharing with heterogeneous beliefs and exposure constraints. ASTIN Bull. 50 (1), 293–323.
Borch, K., 1960a. An attempt to determine the optimum amount of stop loss reinsurance. In: Transactions of the XVI International Congress of Actuaries, Vol. 1, pp. 597–610.
Borch, K., 1960b. Reciprocal reinsurance treaties seen as a two-person co-operative game. Scand. Actuar. J. 1960 (2), 29–58.
Borch, K., 1960c. Reciprocal reinsurance treaties. ASTIN Bull. 1 (4), 170–191.
Borch, K., 1962. Equilibrium in a reinsurance market. Econometrica 424–444.
Cai, J., Lemieux, C., Liu, F., 2016. Optimal reinsurance from the perspectives of both an insurer and a reinsurer. ASTIN Bull. 46 (3), 815–849.
Cai, J., Liu, H., Wang, R., 2017. Pareto-optimal reinsurance arrangements under general model settings. Insurance Math. Econom. 77 (1), 24–37.
Cai, J., Tan, K., 2007. Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. ASTIN Bull. 37 (1), 93–112.
Carlier, G., Dana, R., 2003. Pareto efficient insurance contracts when the insurer's cost function is discontinuous. Econom. Theory 21 (4), 871–893.
Cheung, K., 2010. Optimal reinsurance revisited–a geometric approach. ASTIN Bull. 40 (01), 221–239.
Cheung, K., Lo, A., 2017. Characterizations of optimal reinsurance treaties: A cost-benefit approach. Scand. Actuar. J. 2017 (1), 1–28.
Chi, Y., Zhuang, S.C., 2020. Optimal insurance with belief heterogeneity and incentive compatibility. Insurance Math. Econom. 92, 104–114.
Cont, R., Deguest, R., Scandolo, G., 2010. Robustness and sensitivity analysis of risk measurement procedures. Quant. Finance 10 (6), 593–606.
Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., 2006. Actuarial Theory for Dependent Risks: Measures, Orders and Models. John Wiley & Sons.
Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R., Vyncke, D., 2002a. The concept of comonotonicity in actuarial science and finance: applications. Insurance Math. Econom. 31 (2), 133–161.
Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R., Vyncke, D., 2002b. The concept of comonotonicity in actuarial science and finance: theory. Insurance Math. Econom. 31 (1), 3–33.
Embrechts, P., Liu, H., Wang, R., 2018. Quantile-based risk sharing. Oper. Res. 66 (4), 936–949.
Gerber, H., Pafumi, G., 1998. Utility functions: from risk theory to finance. N. Am. Actuar. J. 2 (3), 74–91.
Ghossoub, M., 2019. Optimal insurance under rank-dependent expected utility. Insurance Math. Econom. 87, 51–66.
Golubin, A., 2006a. An optimal insurance policy in the individual risk model seen as a bargaining game. Game Theory Appl. 11, 60–71.
Golubin, A., 2006b. Pareto-optimal insurance policies in the models with a premium based on the actuarial value. J. Risk Insurance 73 (3), 469–487.
He, X.D., Kouwenberg, R., Zhou, X.Y., 2017. Rank-dependent utility and risk taking in complete markets. SIAM J. Financial Math. 8 (1), 214–239.
Huang, H., 2006. Optimal insurance contract under a Value-at-Risk constraint. Geneva Risk Insur. Rev. 31 (2), 91–110.
Huberman, G., Mayers, D., Smith, Jr., C.W., 1983. Optimal insurance policy indemnity schedules. Bell J. Econ. 415–426.
Hürlimann, W., 2011. Optimal reinsurance revisited–point of view of cedent and reinsurer. ASTIN Bull. 41 (2), 547–574.
Jiang, W., Hong, H., Ren, J., 2018. On Pareto-optimal reinsurance with constraints under distortion risk measures. Eur. Actuar. J. 8 (1), 215–243.
Jiang, W., Ren, J., Yang, C., Hong, H., 2019. On optimal reinsurance treaties in cooperative game under heterogeneous beliefs. Insurance Math. Econom. 85 (1), 173–184.
Jiang, W., Ren, J., Zitikis, R., 2017. Optimal reinsurance policies under the VaR risk measure when the interests of both the cedent and the reinsurer are taken into account. Risks 5 (1), 11–32.

Kihlstrom, R., Roth, A., 1982. Risk aversion and the negotiation of insurance contracts. J. Risk Insurance 49 (3), 372–387.
Lemaire, J., 1991. Cooperative game theory and its insurance applications. ASTIN Bull. 21 (1), 17–40.
Lo, A., 2017. A Neyman–Pearson perspective on optimal reinsurance with constraints. ASTIN Bull. 47 (2), 467–499.
Ludkovski, M., Young, V., 2009. Optimal risk sharing under distorted probabilities. Math. Financ. Econ. 2 (2), 87–105.
Nash, J., 1950. The bargaining problem. Econometrica 155–162.
Nash, J., 1953. Two-person cooperative games. Econometrica 128–140.
Osuna, E., 2012. Crossing points of distributions and a theorem that relates them to second order stochastic dominance. Statist. Probab. Lett. 82 (4), 758–764.
Quiggin, J., 1982. A theory of anticipated utility. J. Econ. Behav. Organ. 3 (4), 323–343.
Raviv, A., 1979. The design of an optimal insurance policy. Am. Econ. Rev. 69 (1), 84–96.
Wang, S., Young, V., Panjer, H., 1997. Axiomatic characterization of insurance prices. Insurance Math. Econom. 21 (2), 173–183.
Xu, Z.Q., Zhou, X.Y., Zhuang, S.C., 2019. Optimal insurance under rank-dependent utility and incentive compatibility. Math. Finance 29 (2), 659–692.
Zhou, C., Wu, C., 2009. Optimal insurance under the insurer's VaR constraint. Geneva Risk Insur. Rev. 34 (2), 140–154.
Zhuang, S., Weng, C., Tan, K., Assa, H., 2016. Marginal indemnification function formulation for optimal reinsurance. Insurance Math. Econom. 67 (1), 65–76.