Microeconomics

A notes of Intermediate Microeconomics course

First Edition

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*"Economics is not a set of answers,
but a way of thinking."*

--- Paul Samuelson

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Preface

Beneath the curve where want and wisdom meet,
Desire bends softly toward constraint's embrace;
The market hums—a silent, measured beat,
Where countless hearts converge in ordered grace.
Each price a whisper, every choice a vow,
The hand unseen adjusts the trembling scale;
From chaos born, an equilibrium now,
So fragile that one breath might tip the veil.
Yet reason builds its temples out of thought,
Not marble, but theorems shaped by light;
Each symbol holds the battles reason fought,
Each graph a dawn that conquered human night.
Thus study, patient soul, this art of cause and will—
Where freedom bends, yet harmony lies still.

This book of notes is primarily based on Hal R. Varian's *Intermediate Microeconomics*, and was compiled during distinguished Professor **Zhang Haiyang**'s course at the University of International Business and Economics. The author's understanding is limited, and omissions or errors are entirely his own responsibility.

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Table of Contents

Chapter 1

The Market

1.1 Introduction

In microeconomics, we study the behavior of individual economic agents—primarily consumers and firms—and how they interact in markets. We typically build simplified models to understand complex economic phenomena. A key principle in our modeling is the **optimization principle**, which states that people try to choose the best patterns of consumption that they can afford. Another is the **equilibrium principle**, where prices adjust until the amount that people demand of something is equal to the amount that is supplied.

In this introductory chapter, we will look at a fundamental concept for evaluating economic outcomes: efficiency. This will provide a criterion to judge how well an economic system performs.

1.2 Pareto Efficiency

When we want to evaluate the desirability of different economic allocations, we need a standard. One of the most widely used concepts is named after the Italian economist Vilfredo Pareto (1848–1923).

Definition 1.1 (Pareto Improvement and Efficiency)**.**

- *A **Pareto improvement** is a change to a different allocation that makes at least one individual better off without making any other individual worse off.*
- *If an allocation allows for a Pareto improvement, it is called **Pareto inefficient**.*
- *If an allocation is such that no Pareto improvements are possible, it is called **Pareto efficient**.*

A Pareto efficient outcome can be thought of as one with "no wasted welfare". That is, in a Pareto efficient allocation, the only way to improve one person's welfare is to lower another person's welfare.

Remark 1.1.

- A Pareto inefficient outcome implies that there are still unrealized mutual gains-to-trade. It is possible to rearrange the allocation of goods to make someone happier without hurting anyone else.
- Any market outcome that achieves all possible gains-to-trade must be Pareto efficient. The concept of Pareto efficiency is a cornerstone of welfare economics.

Chapter 2

Budget Constraint

2.1 Consumption Choice Sets

Consumers choose to consume bundles of goods and services. A **consumption bundle** is a vector of quantities of each good, e.g., $(x_1, x_2, \dots, x_n)$, where x_i is the quantity of good i. The **consumption choice set** is the collection of all consumption bundles available to the consumer.

The choices a consumer can make are constrained by various factors, with the most prominent being the budget constraint. Other constraints can include time, resource limitations, and legal restrictions. For now, we will focus on the budgetary constraint.

2.2 The Budget Constraint

Let's consider a consumer with a disposable income of m who can consume n goods. The prices of these goods are given by the price vector $(p_1, p_2, \dots, p_n)$.

Definition 2.1 (Affordable Consumption Bundles)**.** *A consumption bundle* $(x_1, \dots, x_n)$ *is affordable if its total cost is no more than the consumer's income. That is:*

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m \tag{2.1}$$

*This inequality is the consumer's **budget constraint**.*

Definition 2.2 (Budget Set)**.** *The **budget set** is the set of all affordable consumption bundles. Assuming non-negative consumption, the budget set B is:*

$$B(p_1, \dots, p_n, m) = \{(x_1, \dots, x_n) \mid x_i \geq 0 \textit{ for all } i, \textit{ and } \sum_{i=1}^{n} p_ix_i \leq m\} \tag{2.2}$$

*The upper boundary of the budget set, where the consumer spends all their income, is called the **budget line**:*

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n = m \tag{2.3}$$

2.2.1 The Two-Good Case

For simplicity, we often analyze the case with only two goods. The budget line is then given by $p_1x_1 + p_2x_2 = m$. We can rearrange this to express x_2 as a function of x_1:

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{m}{p_2} \tag{2.4}$$

This is a linear equation.

- The vertical intercept is m/p_2, which is the maximum amount of good 2 the consumer can buy.
- The horizontal intercept is m/p_1, which is the maximum amount of good 1 the consumer can buy.
- The slope is $-p_1/p_2$. The slope measures the rate at which the market is willing to "substitute" good 2 for good 1. It is the **opportunity cost** of consuming good 1: to consume one more unit of good 1, the consumer must give up p_1/p_2 units of good 2.

Remark 2.1 (Graphical Representation). The budget set for two goods is a right-angled triangle in the (x_1, x_2) plane, with vertices at $(0, 0)$, $(m/p_1, 0)$, and $(0, m/p_2)$. The hypotenuse of this triangle is the budget line. Bundles inside the triangle are affordable but do not exhaust income. Bundles on the budget line are just affordable. Bundles outside the triangle are unaffordable.

2.3 Changes in the Budget Line

The budget set depends on prices and income. When these parameters change, the set of affordable choices changes as well.

2.3.1 Income Changes

An increase in income m to $m' > m$ leads to a new budget line:

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{m'}{p_2}$$

The slope $(-p_1/p_2)$ remains unchanged, but the vertical intercept increases. This means the budget line makes a **parallel shift outwards**. An income increase expands the budget set, meaning no original choices are lost and new choices are added. Thus, a higher income cannot make a consumer worse off. Conversely, a decrease in income shifts the budget line inward, shrinking the choice set.

2.3.2 Price Changes

Suppose the price of good 1 decreases from p_1 to $p_1' < p_1$. The new budget line is:

$$x_2 = -\frac{p_1'}{p_2}x_1 + \frac{m}{p_2}$$

The vertical intercept (m/p_2) is unchanged. The horizontal intercept increases to m/p_1'. The slope becomes flatter (less negative). The budget line **pivots outward** around the vertical intercept. Reducing the price of one commodity also expands the budget set, and thus cannot make the consumer worse off.

2.4 Taxes, Subsidies, and Rationing

2.4.1 Uniform Ad Valorem Sales Tax

An *ad valorem* (from the value) sales tax is levied as a percentage of the price. If a uniform tax rate t is applied to all goods, the new prices become $p_1(1+t)$ and $p_2(1+t)$. The budget constraint becomes:

$$p_1(1+t)x_1 + p_2(1+t)x_2 = m$$

This can be rewritten as:

$$p_1x_1 + p_2x_2 = \frac{m}{1+t}$$

A uniform sales tax is thus equivalent to a decrease in income from m to $m/(1+t)$. This causes a parallel inward shift of the budget line.

The Food Stamp Program

Food stamps are coupons that can be legally exchanged only for food. Let F be food and G be all other goods. Assume $p_F = p_G = \$1$ and income is $m = \$100$. The initial budget line is $F + G = 100$.
Suppose the consumer receives $40 worth of food stamps.

- The consumer can now consume up to $140 of food if they spend all their income on it.
- However, the food stamps cannot be used for other goods, so the maximum amount of other goods is still $100.
- The budget constraint becomes kinked. It is $F + G = 100$ for $F > 40$, but for $F \leq 40$, the consumer can still spend their full $100 on other goods. The new budget set is larger. The budget line is:

$$G = \begin{cases} 100 & \text{if } 0 \leq F \leq 40 \\ 140 - F & \text{if } 40 < F \leq 140 \end{cases}$$

What if the food stamps can be traded on a black market, e.g., for \$0.50 each?

- The \$40 in food stamps could be sold for $40 \times \$0.50 = \$20$.
- This effectively increases the consumer's cash income to $\$100 + \$20 = \$120$.
- The budget line would then be $F + G = 120$, further expanding the budget set.

2.5 Shapes of Budget Constraints

If prices are not constant, the budget constraint may not be a straight line.

- **Quantity Discounts:** Suppose $p_1 = \$2$ for the first 20 units and $p_1 = \$1$ for any unit thereafter, with $p_2 = \$1$ and $m = \$100$. The budget line will have a slope of -2 for $x_1 \leq 20$ and a slope of -1 for $x_1 > 20$. The budget line becomes flatter after $x_1 = 20$, creating a kink.
- **Quantity Penalties (Taxes):** The opposite occurs, and the budget line becomes steeper after a certain quantity.
- **One Price Negative:** If good 1 is a "bad" like garbage, you might be paid to accept it, i.e., $p_1 < 0$. If $p_1 = -\$2$, $p_2 = \$1$ and $m = \$10$, the budget line is $-2x_1 + x_2 = 10$, or $x_2 = 2x_1 + 10$. The slope is positive, and the budget set is unbounded on the x_1 side.

2.6 More General Choice Sets

Choices are often constrained by more than just a budget. For example, there might be a time constraint, or a minimum consumption requirement for survival. A bundle is available only if it meets *every* constraint. The final choice set is the **intersection** of all the individual constraint sets.

Chapter 3

Preferences

After establishing what a consumer can afford (the budget set), we now turn to modeling what they want to consume. We do this using the concept of preferences.

3.1 Consumer Preferences

We will consider two consumption bundles, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. The consumer can rank these bundles according to their desirability. There are three possibilities:

- **Strict Preference:** The consumer strictly prefers bundle $\mathbf{x}$ to bundle $\mathbf{y}$. We write this as $\mathbf{x} \succ \mathbf{y}$.
- **Indifference:** The consumer is exactly as satisfied with bundle $\mathbf{x}$ as with bundle $\mathbf{y}$. We write this as $\mathbf{x} \sim \mathbf{y}$.
- **Weak Preference:** The consumer prefers or is indifferent between bundle $\mathbf{x}$ and bundle $\mathbf{y}$. We write this as $\mathbf{x} \succeq \mathbf{y}$.

These relations are linked. For example, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$, then we can conclude that $\mathbf{x} \sim \mathbf{y}$. If $\mathbf{x} \succeq \mathbf{y}$ but it is *not* the case that $\mathbf{y} \succeq \mathbf{x}$, then we can conclude $\mathbf{x} \succ \mathbf{y}$.

3.2 Assumptions about Preferences

To have a sensible theory of consumer choice, we need to impose some assumptions on preferences. These are often called axioms of rational choice.

- **Completeness:** For any two bundles $\mathbf{x}$ and $\mathbf{y}$, the consumer can make a comparison. That is, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ (or both).
- **Reflexivity:** Any bundle $\mathbf{x}$ is at least as good as itself: $\mathbf{x} \succeq \mathbf{x}$. This is a trivial assumption.

- **Transitivity:** If a consumer thinks that $\mathbf{x}$ is at least as good as $\mathbf{y}$, and that $\mathbf{y}$ is at least as good as $\mathbf{z}$, then they must think that $\mathbf{x}$ is at least as good as $\mathbf{z}$. Formally: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$.

A consumer with preferences that satisfy these three axioms is said to be **rational**.

3.3 Indifference Curves

Preferences can be represented graphically using **indifference curves**. An indifference curve is a set of all consumption bundles among which a consumer is indifferent.

- The **weakly preferred set** for a bundle $\mathbf{x}'$ is the set of all bundles $\mathbf{y}$ such that $\mathbf{y} \succeq \mathbf{x}'$.
- The **strictly preferred set** for $\mathbf{x}'$ is the set of all bundles $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{x}'$.

A key property, derived from transitivity and the "more is better" assumption (monotonicity, see below), is that **indifference curves cannot intersect**. If they did, a point on the intersection would be indifferent to bundles on both curves. By transitivity, all bundles on both curves would have to be indifferent to each other, which contradicts the idea that one curve represents a higher level of preference than the other.

3.4 Examples of Preferences

- **Perfect Substitutes:** If a consumer always regards units of commodities 1 and 2 as equivalent (e.g., at a one-to-one ratio), the commodities are perfect substitutes. The consumer's preference depends only on the total sum of the goods. The indifference curves are parallel straight lines.
- **Perfect Complements:** If a consumer always consumes commodities 1 and 2 in a fixed proportion (e.g., one-to-one, like left shoes and right shoes), the commodities are perfect complements. The indifference curves are L-shaped.
- **Bads:** If less of a commodity is always preferred, it is a **bad**. If good 1 is a bad and good 2 is a good, the indifference curves will be positively sloped.
- **Satiation:** A consumer may have a most preferred bundle, called a **satiation point** or **bliss point**. Bundles further away from this point are less preferred. The indifference curves are circles or ellipses centered on the bliss point.

3.5 Well-Behaved Preferences

We often make two further assumptions about preferences, which lead to "well-behaved" indifference curves.

Definition 3.1 (Monotonicity)**.** *Preferences are **monotonic** if more of any commodity is always preferred to less (holding the other commodity constant). This implies that commodities are **goods**, not bads, and that there is no satiation. Monotonicity ensures that indifference curves are negatively sloped.*

Definition 3.2 (Convexity)**.** *Preferences are **convex** if mixtures of bundles are (at least weakly) preferred to the bundles themselves. If $\mathbf{x} \sim \mathbf{y}$, then for any $t \in (0, 1)$, the mixture bundle $\mathbf{z} = t\mathbf{x} + (1 - t)\mathbf{y}$ is at least as good as $\mathbf{x}$ or $\mathbf{y}$ (i.e., $\mathbf{z} \succeq \mathbf{x}$).*

- ***Strict convexity** holds if the mixture is always strictly preferred ($\mathbf{z} \succ \mathbf{x}$).*
- *Convexity implies that consumers prefer balanced bundles over extreme bundles. Graphically, the set of bundles weakly preferred to $\mathbf{x}$ is a convex set.*

3.6 The Marginal Rate of Substitution (MRS)

The slope of an indifference curve at a particular point is called the **marginal rate-of-substitution (MRS)**.

Definition 3.3 (MRS)**.** *The MRS measures the rate at which the consumer is just willing to substitute a small amount of good 2 for good 1, while remaining on the same indifference curve. Mathematically, it is the derivative of the indifference curve:*

$$MRS = \frac{dx_2}{dx_1}$$

- For monotonic preferences (goods), the MRS is negative, as the consumer must give up some of one good to get more of the other.
- For strictly convex preferences, the MRS becomes less negative as x_1 increases. This is known as a **diminishing marginal rate of substitution**. The indifference curve becomes flatter as we move to the right. This means the consumer is willing to give up less of good 2 to get an additional unit of good 1 as they consume more and more of good 1.

Chapter 4

Utility

In the previous chapter, we used indifference curves to describe preferences. While graphical, this can be cumbersome. A more convenient way to describe preferences is through a **utility function**.

4.1 Utility Functions

A utility function is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers than less-preferred bundles.

Definition 4.1 (Utility Function)**.** *A function $U(\mathbf{x})$ represents a preference relation $\succeq$ if for any two bundles $\mathbf{x}'$ and $\mathbf{x}''$:*

- $\mathbf{x}' \succ \mathbf{x}'' \iff U(\mathbf{x}') > U(\mathbf{x}'')$
- $\mathbf{x}' \prec \mathbf{x}'' \iff U(\mathbf{x}') < U(\mathbf{x}'')$
- $\mathbf{x}' \sim \mathbf{x}'' \iff U(\mathbf{x}') = U(\mathbf{x}'')$

A preference relation can be represented by a continuous utility function if it is complete, reflexive, transitive, and continuous.

An indifference curve consists of all bundles that have the same level of utility. The equation for an indifference curve is therefore $U(x_1, x_2) = k$ for some constant k.

4.1.1 Ordinal Utility

Utility is an **ordinal** concept. This means that the magnitude of the utility function is only important for ranking bundles. The absolute values, or the differences between them, do not have a specific meaning. If $U(\mathbf{x}) = 6$ and $U(\mathbf{y}) = 2$, we know $\mathbf{x}$ is preferred to $\mathbf{y}$, but we cannot say that $\mathbf{x}$ is "three times as good" as $\mathbf{y}$.

4.1.2 Monotonic Transformations

Because utility is ordinal, any strictly increasing transformation of a utility function will represent the same preferences. If $U(x_1, x_2)$ is a utility function representing some preferences, and f is a strictly increasing function (i.e., $f'(z) > 0$), then $V(x_1, x_2) = f(U(x_1, x_2))$ is a new utility function that represents the exact same preferences.

For example, if $U = x_1 x_2$, then $V = (x_1 x_2)^2 = x_1^2 x_2^2$ and $W = \ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$ both represent the same preferences as U, because squaring (for positive numbers) and taking the natural log are both strictly increasing transformations.

4.2 Examples of Utility Functions

- **Perfect Substitutes:** Preferences can be represented by a linear utility function of the form $U(x_1, x_2) = ax_1 + bx_2$. The indifference curves are lines with slope $-a/b$.

- **Perfect Complements:** Preferences for goods consumed in a fixed proportion (e.g., one-to-one) can be represented by a function like $U(x_1, x_2) = \min\{ax_1, bx_2\}$.

- **Quasi-linear Preferences:** A utility function that is linear in one good, e.g., $U(x_1, x_2) = v(x_1) + x_2$. The indifference curves are vertically shifted copies of each other.

- **Cobb-Douglas Preferences:** A commonly used functional form is the Cobb-Douglas utility function, $U(x_1, x_2) = x_1^a x_2^b$, where $a, b > 0$. These preferences exhibit well-behaved, convex indifference curves.

4.3 Marginal Utility and MRS

Definition 4.2 (Marginal Utility)**.** *The **marginal utility** (MU) of a good i is the rate of change in total utility from consuming an infinitesimally small additional amount of good i, holding all other goods constant. It is the partial derivative of the utility function with respect to x_i:*

$$MU_i = \frac{\partial U}{\partial x_i}$$

The magnitude of marginal utility depends on the specific utility function chosen and is not meaningful on its own (due to the ordinal nature of utility). However, it is useful for calculating the MRS.

4.3.1 Deriving MRS from Utility

Consider a small change in consumption (dx_1, dx_2) that keeps the consumer on the same indifference curve. The total change in utility, dU, must be zero. The total differential of the utility function is:

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

$$MU_1 dx_1 + MU_2 dx_2 = 0$$

Rearranging this equation gives us the slope of the indifference curve:

$$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -\frac{MU_1}{MU_2} \tag{4.1}$$

The MRS is the ratio of the marginal utilities. This ratio is independent of any monotonic transformation of the utility function, which is a desirable property since the MRS has a real economic meaning (a psychological rate of trade-off) while the MU values do not.

Chapter 5

Choice

5.1 Introduction

In the previous chapters, we modeled the consumer's constraints (the budget set) and their preferences (indifference curves and utility functions). We now put these two pieces together to analyze the consumer's choice. The fundamental assumption is that the consumer will choose the most preferred bundle from their budget set.

This is a problem of **constrained maximization**. Mathematically, the consumer's problem is to:

$$\begin{aligned} \max_{x_1, x_2} \quad & U(x_1, x_2) \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \leq m \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

In economics, this is known as the rational choice problem. The solution to this problem, the optimal consumption bundle (x_1^*, x_2^*), is the consumer's **demanded bundle**.

Definition 5.1 (Marshallian Demand)**.** *The consumer's optimal choice (x_1^*, x_2^*) at a given set of prices (p_1, p_2) and income m is called the consumer's demanded bundle. The functions that give the optimal amount of each good as a function of prices and income, $x_1^*(p_1, p_2, m)$ and $x_2^*(p_1, p_2, m)$, are called the* ***Marshallian demand functions****.*

5.2 Finding the Optimal Choice

Graphically, the optimal bundle is the point in the budget set that lies on the highest possible indifference curve.

5.2.1 Interior Solutions with Well-Behaved Preferences

For well-behaved preferences (monotonic and strictly convex), the optimal choice will typically be an **interior solution**, where the consumer purchases positive amounts of both goods ($x_1^* > 0$ and $x_2^* > 0$). In this case, the optimal bundle (x_1^*, x_2^*) is characterized by two conditions:

1. **The budget is exhausted.** The optimal point must lie on the budget line, not inside it. Because preferences are monotonic (more is better), any bundle inside the budget set has a more-preferred bundle to its northeast that is also affordable.

$$p_1 x_1^* + p_2 x_2^* = m$$

2. **Tangency condition.** At the optimal point, the indifference curve is tangent to the budget line. This means their slopes are equal.

$$\text{Slope of Indifference Curve} = \text{Slope of Budget Line}$$

$$MRS = -\frac{p_1}{p_2}$$

Since we know $MRS = -MU_1/MU_2$, the tangency condition can be rewritten as:

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

This second condition has a powerful economic intuition: at the optimal choice, the rate at which the consumer is *willing* to trade one good for another (the MRS) is equal to the rate at which the market *allows* them to trade (the price ratio).

Computing Demand with Cobb-Douglas Preferences

Let the consumer's utility function be of the Cobb-Douglas form:

$$U(x_1, x_2) = x_1^a x_2^b$$

First, we find the marginal utilities and the MRS:

$$MU_1 = \frac{\partial U}{\partial x_1} = a x_1^{a-1} x_2^b$$

$$MU_2 = \frac{\partial U}{\partial x_2} = b x_1^a x_2^{b-1}$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}$$

Now we apply the two conditions for an interior optimum:

1. **Tangency:** $MRS = -p_1/p_2$

$$-\frac{a x_2^*}{b x_1^*} = -\frac{p_1}{p_2} \implies \frac{a x_2^*}{b x_1^*} = \frac{p_1}{p_2} \implies p_2 x_2^* = \frac{b}{a} p_1 x_1^*$$

2. **Budget Exhaustion:** $p_1 x_1^* + p_2 x_2^* = m$

Substitute the result from the tangency condition into the budget constraint:

$$p_1 x_1^* + \left(\frac{b}{a} p_1 x_1^*\right) = m$$

$$p_1 x_1^* \left(1 + \frac{b}{a}\right) = m \implies p_1 x_1^* \left(\frac{a+b}{a}\right) = m$$

Solving for x_1^* gives the demand function for good 1:

$$x_1^*(p_1, p_2, m) = \frac{a}{a+b} \frac{m}{p_1}$$

Substituting this back into the expression for $p_2 x_2^*$ from the tangency condition gives the demand for good 2:

$$p_2 x_2^* = \frac{b}{a} p_1 \left(\frac{a}{a+b} \frac{m}{p_1}\right) = \frac{b}{a+b} m \implies x_2^*(p_1, p_2, m) = \frac{b}{a+b} \frac{m}{p_2}$$

These are the Marshallian demand functions for a consumer with Cobb-Douglas preferences. They show that the consumer spends a fixed fraction of income ($a/(a+b)$ on good 1 and $b/(a+b)$ on good 2).

5.3 Non-Tangency Solutions

The tangency condition only holds for interior solutions with smooth indifference curves. In other cases, the optimal choice may occur where the tangency condition is not met.

5.3.1 Corner Solutions

A **corner solution** occurs when the optimal quantity of one of the goods is zero. This often happens with perfect substitutes or non-convex preferences.

Optimal Choice with Perfect Substitutes

Consider the utility function $U(x_1, x_2) = x_1 + x_2$. Here, the MRS is constant at -1. The indifference curves are straight lines with a slope of -1. The consumer will compare their personal trade-off rate (1-for-1) with the market's trade-off rate (p_1/p_2).

- If $p_1 < p_2$, the slope of the budget line ($-p_1/p_2$) is flatter than the slope of the indifference curves (-1). The consumer gets more utility per dollar from good 1. The optimal choice is to spend all income on good 1. This is a corner solution at $(m/p_1, 0)$.
- If $p_1 > p_2$, the budget line is steeper than the indifference curves. The

consumer gets more utility per dollar from good 2 and will spend all income on it. The corner solution is at $(0, m/p_2)$.

- If $p_1 = p_2$, the budget line and indifference curves have the same slope. Any affordable bundle on the budget line is an optimal choice.

So, the demand for good 1 is:

$$x_1^*(p_1, p_2, m) = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any amount in } [0, m/p_1] & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

5.3.2 Kinky Solutions

If indifference curves have kinks, such as with perfect complements, the MRS is not well-defined at the optimal point, and the tangency condition cannot be used.

Optimal Choice with Perfect Complements

Consider the utility function $U(x_1, x_2) = \min\{x_1, x_2\}$. The consumer always wants to consume the goods in a one-to-one ratio. The optimal choice will always be at the corner of an L-shaped indifference curve, where $x_1 = x_2$.

1. **Consumption in fixed proportion:** $x_1^* = x_2^*$
2. **Budget Exhaustion:** $p_1 x_1^* + p_2 x_2^* = m$

Substituting the first condition into the second gives:

$$p_1 x_1^* + p_2 x_1^* = m \implies (p_1 + p_2) x_1^* = m$$

Solving gives the demand functions:

$$x_1^*(p_1, p_2, m) = x_2^*(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

Chapter 6

Demand

6.1 Introduction

The demand functions derived in the previous chapter, $x_i^*(p_1, p_2, m)$, tell us the optimal quantities of each good as a function of prices and income. In this chapter, we explore how the quantity demanded changes as these variables change. This is known as **comparative statics**.

6.2 Own-Price Changes

We first analyze how the demand for a good, say good 1, changes as its own price, p_1, changes, while holding p_2 and income m constant.

Definition 6.1 (Price Offer Curve and Demand Curve)**.**

- *The set of optimal bundles traced out as p_1 changes is called the p_1**-price offer curve** or simply the **price offer curve**.*

- *A plot of the optimal quantity demanded of good 1, x_1^*, against its own price, p_1, is the **demand curve** for good 1.*

Definition 6.2 (Ordinary and Giffen Goods)**.**

- *A good is called an **ordinary good** if the quantity demanded for it always increases as its own price decreases. Ordinary goods have downward-sloping demand curves.*

- *A good is called a **Giffen good** if, for some range of prices, the quantity demanded rises as its own-price increases. Giffen goods have a segment of their demand curve that is upward-sloping. They are a theoretical curiosity and are rarely observed in reality.*

6.2.1 The Inverse Demand Function

The standard demand curve plots quantity as a function of price, $x_1 = x_1^*(p_1)$. Sometimes it is useful to ask the inverse question: at what price would a given quantity be demanded? This gives the **inverse demand function**, which plots price as a function of quantity, $p_1 = p_1(x_1)$.

Cobb-Douglas Inverse Demand

The ordinary demand function for good 1 is $x_1^* = \frac{a}{a+b}\frac{m}{p_1}$. To find the inverse demand function, we solve for p_1:

$$p_1(x_1^*) = \frac{a}{a+b}\frac{m}{x_1^*}$$

6.3 Income Changes

We now analyze how the demand for a good changes as income m changes, holding prices (p_1, p_2) constant.

Definition 6.3 (Income Offer Curve and Engel Curve)**.**

- *The set of optimal bundles traced out as income m changes is called the* ***income offer curve*** *or* ***income expansion path****.*
- *A plot of the quantity demanded of a good against income is called an* ***Engel curve****.*

Definition 6.4 (Normal and Inferior Goods)**.**

- *A good is a* ***normal good*** *if the quantity demanded rises with income ($\partial x_i^*/\partial m > 0$). A normal good has a positively sloped Engel curve.*
- *A good is an* ***inferior good*** *if the quantity demanded falls as income increases ($\partial x_i^*/\partial m < 0$). An inferior good has a negatively sloped Engel curve.*

6.3.1 Homothetic Preferences

An important class of preferences are those that are **homothetic**. For these preferences, the MRS depends only on the ratio of the goods, not on the scale.

Proposition 6.1. *If a consumer has homothetic preferences, then their income offer curve is a straight line through the origin, and their Engel curves are straight lines.*

This implies that the consumer will always spend a fixed proportion of their income on each good. Cobb-Douglas, perfect substitutes, and perfect complements are all examples of homothetic preferences.

6.3.2 Non-homothetic Preferences: Quasi-linear Utility

Quasi-linear preferences of the form $U(x_1, x_2) = v(x_1) + x_2$ are a key example of non-homothetic preferences. For these preferences, an increase in income (once it is sufficiently high) does not change the demand for good 1 at all; all extra income is spent on good 2. This results in a vertical income offer curve and a vertical Engel curve for good 1 above a certain income level.

6.4 Cross-Price Effects

Finally, we consider how the demand for good 1 changes when the price of *another* good, p_2, changes.

Definition 6.5 (Gross Substitutes and Complements)**.**

- *Good 1 is a* ***gross substitute*** *for good 2 if an increase in p_2 increases the demand for good 1 ($\partial x_1^*/\partial p_2 > 0$).*
- *Good 1 is a* ***gross complement*** *for good 2 if an increase in p_2 reduces the demand for good 1 ($\partial x_1^*/\partial p_2 < 0$).*

Cross-Price Effects Examples

- **Perfect Complements:** $x_1^* = m/(p_1 + p_2)$. Taking the derivative:

$$\frac{\partial x_1^*}{\partial p_2} = -\frac{m}{(p_1 + p_2)^2} < 0$$

As expected, the goods are gross complements.

- **Cobb-Douglas:** $x_1^* = \frac{a}{a+b}\frac{m}{p_1}$. Taking the derivative:

$$\frac{\partial x_1^*}{\partial p_2} = 0$$

In the Cobb-Douglas case, the demand for one good is independent of the other good's price. The goods are neither gross substitutes nor gross complements.

Chapter 7

Revealed Preference

In previous chapters, we started with a consumer's preferences (or utility function) and used this information to derive their optimal choices. This chapter reverses the process. We ask: can we start by observing a consumer's choices at different prices and income levels, and from these observations, deduce their underlying preferences? This is the central question of revealed preference theory.

7.1 The Concept of Revealed Preference

The basic idea is simple yet powerful: a consumer's choice *reveals* information about their preferences. If a consumer chooses a bundle of goods x when another bundle y was also affordable, we can infer that the consumer prefers x to y.

7.1.1 Maintained Assumptions

To make this inference robust, we rely on a few standard assumptions about consumer behavior:

- **Stable Preferences:** The consumer's preferences do not change during the period of observation.
- **Well-Behaved Preferences:** We assume preferences are monotonic (more is better) and strictly convex. A key implication of strict convexity is that for any given budget, there is a *unique* most-preferred affordable bundle.
- **Rationality:** The consumer always chooses the most preferred bundle they can afford.

7.1.2 Direct and Indirect Revelation

Definition 7.1 (Directly Revealed Preferred)**.** *Let* $\mathbf{x} = (x_1, x_2)$ *be the bundle chosen at prices* $\mathbf{p} = (p_1, p_2)$*, and let* $\mathbf{y} = (y_1, y_2)$ *be another bundle. If* $\mathbf{y}$ *was affordable at* *prices* $\mathbf{p}$ *when* $\mathbf{x}$ *was chosen, i.e.,* $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$*, and* $\mathbf{x} \neq \mathbf{y}$*, then we say that* $\mathbf{x}$ ***is directly revealed preferred to*** $\mathbf{y}$*. We denote this by* $\mathbf{x} \succ_D \mathbf{y}$*.*

The logic is that since the consumer could have chosen $\mathbf{y}$ but instead chose $\mathbf{x}$, they must prefer $\mathbf{x}$.

We can extend this idea through transitivity.

Definition 7.2 (Indirectly Revealed Preferred)**.** *If we have a chain of direct revelations, such as* $\mathbf{x} \succ_D \mathbf{y}$ *and* $\mathbf{y} \succ_D \mathbf{z}$*, then we say that* $\mathbf{x}$ ***is indirectly revealed preferred to*** $\mathbf{z}$*. We denote this by* $\mathbf{x} \succ_I \mathbf{z}$*.*

This allows us to compare bundles that were not available in the same budget set, by linking them through intermediate choices.

7.2 Axioms of Revealed Preference

For observed choices to be consistent with our model of a rational, utility-maximizing consumer, they must satisfy certain consistency conditions. These are known as the axioms of revealed preference.

7.2.1 The Weak Axiom of Revealed Preference (WARP)

WARP is the most basic consistency check. It ensures that if a consumer reveals a preference for one bundle over another, they don't subsequently reveal the opposite preference.

Definition 7.3 (Weak Axiom of Revealed Preference (WARP))**.** *If a bundle* $\mathbf{x}$ *is directly revealed preferred to a bundle* $\mathbf{y}$ *(*$\mathbf{x} \succ_D \mathbf{y}$*), then it can never be the case that* $\mathbf{y}$ *is directly revealed preferred to* $\mathbf{x}$ *(*$\mathbf{y} \succ_D \mathbf{x}$*).*

$$\textit{If } \mathbf{x} \succ_D \mathbf{y}, \textit{ then it cannot be that } \mathbf{y} \succ_D \mathbf{x}.$$

Choice data that violate WARP are inconsistent with the model of economic rationality. WARP is a *necessary* condition for choices to be rationalized by a utility function.

Checking for WARP Violations

A consumer makes the following choices:

- At prices $\mathbf{p}^A = (\$2, \$2)$, the choice is $\mathbf{x}^A = (10, 1)$.
- At prices $\mathbf{p}^B = (\$2, \$1)$, the choice is $\mathbf{x}^B = (5, 5)$.
- At prices $\mathbf{p}^C = (\$1, \$2)$, the choice is $\mathbf{x}^C = (5, 4)$.

To check for WARP violations, we calculate the cost of each bundle at each set of prices. The chosen bundle's cost represents the consumer's income in that situation.

Prices	**Cost of $\mathbf{x}^A$**	**Cost of $\mathbf{x}^B$**	**Cost of $\mathbf{x}^C$**
$\mathbf{p}^A = (\$2, \$2)$	**$22**	$20	$18
$\mathbf{p}^B = (\$2, \$1)$	$21	**$15**	$14
$\mathbf{p}^C = (\$1, \$2)$	$12	$15	**$13**

Let's analyze the choices:

- At prices $\mathbf{p}^A$, income is \$22. Both $\mathbf{x}^B$ (\$20) and $\mathbf{x}^C$ (\$18) were affordable. Thus, $\mathbf{x}^A \succ_D \mathbf{x}^B$ and $\mathbf{x}^A \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^B$, income is \$15. $\mathbf{x}^C$ (\$14) was affordable. Thus, $\mathbf{x}^B \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^C$, income is \$13. $\mathbf{x}^A$ (\$12) was affordable. Thus, $\mathbf{x}^C \succ_D \mathbf{x}^A$.

We have found that $\mathbf{x}^A \succ_D \mathbf{x}^C$ and $\mathbf{x}^C \succ_D \mathbf{x}^A$. This is a direct violation of WARP. These data are not consistent with a rational consumer.

7.2.2 The Strong Axiom of Revealed Preference (SARP)

WARP is not sufficient to guarantee that choices can be described by a well-behaved utility function. We need a stronger condition that also rules out cycles in *indirect* revealed preferences.

Definition 7.4 (Strong Axiom of Revealed Preference (SARP))**.** *If a bundle* $\mathbf{x}$ *is revealed preferred (directly or indirectly) to a bundle* $\mathbf{y}$ *(*$\mathbf{x} \succ_D \mathbf{y}$ *or* $\mathbf{x} \succ_I \mathbf{y}$*), and* $\mathbf{x} \neq \mathbf{y}$*, then it can never be the case that* $\mathbf{y}$ *is revealed preferred (directly or indirectly) to* $\mathbf{x}$.

SARP is the key condition. It has been proven that if a finite set of observed choices satisfies SARP, then there exists a well-behaved (monotonic, convex) preference relation that "rationalizes" these choices. Thus, SARP is both a *necessary and sufficient* condition for the data to be consistent with the standard economic model of consumer choice.

A SARP Violation (with no WARP violation)

Consider the following data:

- Prices $\mathbf{p}^A = (1, 3, 10)$, Choice $\mathbf{x}^A = (3, 1, 4)$.
- Prices $\mathbf{p}^B = (4, 3, 6)$, Choice $\mathbf{x}^B = (2, 5, 3)$.
- Prices $\mathbf{p}^C = (1, 1, 5)$, Choice $\mathbf{x}^C = (4, 4, 3)$.

Prices	**Cost of $\mathbf{x}^A$**	**Cost of $\mathbf{x}^B$**	**Cost of $\mathbf{x}^C$**
$\mathbf{p}^A = (1, 3, 10)$	**$46**	$47	$46
$\mathbf{p}^B = (4, 3, 6)$	$39	**$41**	$46
$\mathbf{p}^C = (1, 1, 5)$	$24	$22	**$23**

Let's analyze the direct revelations:

- At prices $\mathbf{p}^A$, $\mathbf{x}^C$ was affordable (\$46 $\leq$ \$46). So, $\mathbf{x}^A \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^B$, $\mathbf{x}^A$ was affordable (\$39 $\leq$ \$41). So, $\mathbf{x}^B \succ_D \mathbf{x}^A$.
- At prices $\mathbf{p}^C$, $\mathbf{x}^B$ was affordable (\$22 $\leq$ \$23). So, $\mathbf{x}^C \succ_D \mathbf{x}^B$.

This dataset does not violate WARP (check: no pairs like $X \succ_D Y$ and $Y \succ_D X$). However, let's look at the indirect revelations. We have a cycle:

$$\mathbf{x}^B \succ_D \mathbf{x}^A \quad \text{and} \quad \mathbf{x}^A \succ_D \mathbf{x}^C \implies \mathbf{x}^B \succ_I \mathbf{x}^C$$

But we also found that $\mathbf{x}^C \succ_D \mathbf{x}^B$. This is a violation of SARP. These choices cannot be rationalized by a well-behaved preference relation.

7.3 Applications of Revealed Preference

The theory of revealed preference provides the foundation for practical economic tools, such as index numbers, which are used to measure changes in welfare and the cost of living.

7.3.1 Index Numbers and Welfare

Index numbers compare expenditures in a base period (b) and a current period (t). Let $\mathbf{p}^b, \mathbf{x}^b$ be the prices and chosen bundle in the base period, and $\mathbf{p}^t, \mathbf{x}^t$ be for the current period.

Quantity Indices

A quantity index measures the change in consumption levels, holding prices constant.

- The **Laspeyres Quantity Index** uses base period prices as weights:

$$L_q = \frac{\mathbf{p}^b \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^b} = \frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b}$$

  If $L_q < 1$, it means $\mathbf{p}^b \cdot \mathbf{x}^b > \mathbf{p}^b \cdot \mathbf{x}^t$. At base period prices, the consumer chose $\mathbf{x}^b$ when $\mathbf{x}^t$ was affordable. By revealed preference, the consumer was better off in the base period b.

- The **Paasche Quantity Index** uses current period prices as weights:

$$P_q = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^t \cdot \mathbf{x}^b} = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b}$$

  If $P_q > 1$, it means $\mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{x}^b$. At current period prices, the consumer chose $\mathbf{x}^t$ when $\mathbf{x}^b$ was affordable. The consumer is better off in the current period t.

Price Indices

A price index measures the change in prices, holding quantities constant.

- The **Laspeyres Price Index** uses the base period bundle as weights:

$$L_p = \frac{\mathbf{p}^t \cdot \mathbf{x}^b}{\mathbf{p}^b \cdot \mathbf{x}^b} = \frac{p_1^t x_1^b + p_2^t x_2^b}{p_1^b x_1^b + p_2^b x_2^b}$$

- The **Paasche Price Index** uses the current period bundle as weights:

$$P_p = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^t} = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^t + p_2^b x_2^t}$$

Let $M = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^b}$ be the ratio of total expenditure. If $L_p < M$, it can be shown this implies $\mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{x}^b$, meaning the consumer is better off in the current period.

7.3.2 Application: Indexation

Price indices like the Consumer Price Index (CPI), which is a type of Laspeyres price index, are often used to adjust wages or benefits for inflation. This is called **indexation**. If an individual's income is adjusted by the full amount of the Laspeyres price index, they are typically made *strictly better off*.

Why? Full indexation gives the consumer enough income ($m' = \mathbf{p}^t \cdot \mathbf{x}^b$) to buy their *old* base-period bundle $\mathbf{x}^b$ at the *new* current-period prices $\mathbf{p}^t$. However, since relative prices have likely changed, the consumer can usually improve their welfare by substituting away from goods that have become relatively more expensive. They will choose a new bundle $\mathbf{x}^t$ on their new budget line. Since the old bundle $\mathbf{x}^b$ is still affordable, but a different bundle $\mathbf{x}^t$ is chosen, it must be that $\mathbf{x}^t$ is revealed preferred to $\mathbf{x}^b$. Thus, full indexation tends to overcompensate for price changes.

Chapter 8

Slutsky Equation

When the price of a good changes, the consumer's optimal choice is affected. This overall change in demand can be decomposed into two distinct effects. Understanding this decomposition is crucial for analyzing consumer behavior. The Slutsky equation provides a formal framework for this analysis.

8.1 Effects of a Price Change

What happens when a commodity's price decreases? The total change in the quantity demanded arises from two separate phenomena:

- **Substitution Effect:** The commodity becomes relatively cheaper compared to other goods. A rational consumer will substitute towards the cheaper good, away from the now relatively more expensive ones. This effect captures the change in demand due to the change in the rate of exchange between two goods.

- **Income Effect:** The consumer's purchasing power increases. With the same nominal income, say $m, the consumer can now afford to buy more goods than before. It is as if the consumer's real income has risen. This change in purchasing power leads to a change in quantity demanded, which we call the income effect.

Let's visualize this. Consider a consumer with income m facing prices p_1 and p_2. The budget line is $p_1x_1 + p_2x_2 = m$. If the price of good 1 falls to $p_1' < p_1$, the budget line pivots outwards around the vertical intercept (m/p_2). The consumer can now reach a higher indifference curve, representing a new optimal bundle.

The core idea behind the Slutsky equation is to separate this pivot into two conceptual steps: a pivot and a shift. To isolate the substitution effect, we need to hold the consumer's purchasing power constant.

8.1.1 Slutsky Compensation

How do we hold purchasing power constant? Slutsky's clever idea was to define it as the ability to purchase the *original bundle* of goods.

Definition 8.1 (Slutsky Compensation)**.** *The* ***Slutsky compensation*** *is the hypothetical adjustment to a consumer's income such that, at the new prices, they can just afford their original consumption bundle. This adjusted income level keeps the consumer's purchasing power constant in the sense that the original choice remains affordable.*

If the original bundle was (x_1^*, x_2^*) at prices (p_1, p_2), and the price of good 1 changes to p_1', the compensated income m' would be:

$$m' = p_1' x_1^* + p_2 x_2^*$$

The change in income required for this compensation is:

$$\Delta m = m' - m = (p_1' x_1^* + p_2 x_2^*) - (p_1 x_1^* + p_2 x_2^*) = (p_1' - p_1) x_1^* = \Delta p_1 x_1^*$$

8.2 Decomposing the Total Effect

Let's trace the full decomposition graphically. Let the initial optimal choice be bundle A = (x_1^*, x_2^*) at prices (p_1, p_2) and income m. Now, let the price of good 1 fall to p_1'. The final optimal choice is bundle C = (x_1'', x_2'') at prices (p_1', p_2) and income m. The total change in demand for good 1 is $\Delta x_1 = x_1'' - x_1^*$.

8.2.1 The Substitution Effect

To isolate the substitution effect, we give the consumer the compensated income $m' = p_1' x_1^* + p_2 x_2^*$. The new (compensated) budget line is $p_1' x_1 + p_2 x_2 = m'$. Notice that this line has the slope of the *new* price ratio but passes through the *original* bundle A.

The consumer chooses a new optimal bundle on this compensated budget line, let's call it B = (x_1^s, x_2^s). The **Slutsky substitution effect** is the change in demand from A to B:

$$\Delta x_1^s = x_1^s - x_1^*$$

Proposition 8.1 (Sign of the Substitution Effect)**.** *The Slutsky substitution effect is always negative (or non-positive). That is, the change in quantity demanded due to the substitution effect is always in the opposite direction to the change in price. If price falls (*$\Delta p_1 < 0$*), the substitution effect on demand will be positive (*$\Delta x_1^s \geq 0$*).*

Justification via WARP. At the original prices (p_1, p_2), bundle A was chosen over bundle B (since B was not affordable, assuming B is not A). At the compensated budget with prices (p_1', p_2), bundle B is chosen. Bundle A is also affordable on this line by construction. Therefore, B is revealed preferred to A. If we consider a price increase from p_1' to p_1, B is chosen at p_1' and A is chosen at p_1. Since A was affordable at the prices where B was chosen, WARP is not violated. Crucially, the optimal bundle B on the compensated budget line must lie to the right of A if $p_1' < p_1$, meaning $x_1^s \geq x_1^*$. □

8.2.2 The Income Effect

The substitution effect is a hypothetical construct. The consumer's actual income is still m, not m'. The second part of the decomposition is to restore the consumer's original income, which means moving from the compensated budget line back to the final budget line. This is a parallel shift, representing a change in income from m' to m.

The **income effect** is the change in demand from the intermediate bundle B to the final bundle C:

$$\Delta x_1^n = x_1'' - x_1^s$$

This change is purely due to the change in purchasing power, as prices are held constant at (p_1', p_2) during this step.

The overall change in demand is the sum of these two effects:

$$\text{Total Effect} = \text{Substitution Effect} + \text{Income Effect}$$

$$(x_1'' - x_1^*) = (x_1^s - x_1^*) + (x_1'' - x_1^s)$$

8.3 Slutsky's Effects for Different Types of Goods

The sign of the income effect depends on whether the good is normal or inferior. This determines how the two effects combine. Let's assume a price **decrease** for good 1 ($p_1 \downarrow$).

8.3.1 Normal Goods

A good is **normal** if demand increases as income increases ($\frac{\partial x_1}{\partial m} > 0$).

- **Substitution Effect:** Price falls, so demand increases ($\Delta x_1^s > 0$).
- **Income Effect:** Price falls, real income rises. Since the good is normal, demand increases ($\Delta x_1^n > 0$).

For a normal good, the substitution and income effects **reinforce** each other. A price decrease unambiguously leads to an increase in quantity demanded. Therefore, the Law of Demand always holds for normal goods.

8.3.2 Income-Inferior Goods

A good is **inferior** if demand decreases as income increases ($\frac{\partial x_1}{\partial m} < 0$).

- **Substitution Effect:** Price falls, so demand increases ($\Delta x_1^s > 0$).
- **Income Effect:** Price falls, real income rises. Since the good is inferior, demand decreases ($\Delta x_1^n < 0$).

For an income-inferior good, the substitution and income effects **oppose** each other. The total effect depends on which effect is stronger. In most cases, the substitution effect outweighs the income effect, so a price decrease still leads to an increase in total demand.

8.3.3 Giffen Goods

In rare cases of extreme income-inferiority, the income effect can be stronger than the substitution effect.

Definition 8.2 (Giffen Good)**.** *A* ***Giffen good*** *is a good for which a decrease in price causes the quantity demanded to fall. This happens when the good is so strongly inferior that the negative income effect outweighs the positive substitution effect.*

For a Giffen good, the demand curve is upward-sloping. It is a violation of the Law of Demand. Slutsky's decomposition provides the theoretical explanation for this phenomenon.

8.4 The Slutsky Identity

We can express the decomposition mathematically. The total change in demand for good 1, Δx_1, when its price changes from p_1 to p_1' is:

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m)$$

We can add and subtract the term $x_1(p_1', m')$, where $m' = p_1' x_1(p_1, m) + p_2 x_2(p_1, m)$:

$$\Delta x_1 = \underbrace{[x_1(p_1', m') - x_1(p_1, m)]}_{\text{Substitution Effect, } \Delta x_1^s} + \underbrace{[x_1(p_1', m) - x_1(p_1', m')]}_{\text{Income Effect, } \Delta x_1^n}$$

This is the **Slutsky identity**.

To express this in terms of rates of change, we divide by Δp_1:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + \frac{\Delta x_1^n}{\Delta p_1}$$

Let's analyze the income effect term. We know $\Delta m = \Delta p_1 x_1$. Also, $\Delta x_1^n = x_1(p_1', m) - x_1(p_1', m')$. Notice that income changes from m' to m, so $\Delta m = m - m'$. Therefore, we can write:

$$\frac{\Delta x_1^n}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1', m')}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1', m')}{m - m'} \frac{m - m'}{\Delta p_1} = -\frac{\Delta x_1^n}{\Delta m} x_1$$

(The minus sign appears because Δx_1^n as defined is for an income change from m' to m, while Δm from our compensation formula was $m' - m$). Substituting this back gives the **Slutsky Equation**:

The Slutsky Equation

The relationship between the total effect, substitution effect, and income effect is given by:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1}{\Delta m} x_1$$

Or in calculus terms:

$$\frac{\partial x_1(p_1, m)}{\partial p_1} = \frac{\partial x_1^s(p_1, \bar{u})}{\partial p_1} - \frac{\partial x_1(p_1, m)}{\partial m} x_1(p_1, m)$$

- $\frac{\partial x_1}{\partial p_1}$: Total effect — the slope of the ordinary (Marshallian) demand curve.
- $\frac{\partial x_1^s}{\partial p_1}$: Substitution effect — the slope of the compensated (Hicksian) demand curve. This term is always negative.
- $-\frac{\partial x_1}{\partial m} x_1$: Income effect — captures how demand changes with income, scaled by the amount of the good being consumed.

8.5 Examples of Slutsky Decomposition

Perfect Complements

For perfect complements (e.g., left shoes and right shoes), the indifference curves are L-shaped. The consumer always consumes at the corner of the indifference curve. When we perform the Slutsky pivot around the original bundle, the optimal choice on the compensated budget line remains the same as the original bundle. Therefore, the **substitution effect is zero**. The entire change in demand is due to the income effect.

Perfect Substitutes

For perfect substitutes, the indifference curves are straight lines. The consumer typically consumes only one of the goods (a corner solution). A change in price can cause the consumer to switch entirely from one good to the other. In this case, the shift from the original corner to the new corner can often be entirely attributed to the substitution effect. The compensated bundle is the same as the final bundle, making the **income effect zero**.

Quasilinear Preferences

For quasilinear preferences of the form $u(x_1, x_2) = v(x_1) + x_2$, there is no income effect for good 1 (assuming an interior solution). The demand for x_1

depends only on the price ratio, not on income. Therefore, the change in demand due to a price change is entirely composed of the **substitution effect**. The income effect is zero.

8.6 The Hicks Substitution Effect

An alternative way to define the substitution effect was proposed by John Hicks.

Definition 8.3 (Hicks Substitution Effect)**.** *The **Hicks substitution effect** is the change in demand when prices change, while adjusting income to keep the consumer on the original indifference curve. This holds utility constant, rather than just the affordability of the original bundle.*

Graphically, instead of pivoting the new budget line around the old bundle (A), we roll it along the original indifference curve until it is tangent at the new price ratio. Like the Slutsky effect, the Hicks substitution effect is also always negative. Let bundle $X = (x_1, x_2)$ be chosen at prices $P = (p_1, p_2)$ and bundle $Y = (y_1, y_2)$ be chosen at prices $Q = (q_1, q_2)$. If the consumer is indifferent between X and Y, then by the logic of revealed preference:

$$p_1x_1 + p_2x_2 \leq p_1y_1 + p_2y_2$$

$$q_1y_1 + q_2y_2 \leq q_1x_1 + q_2x_2$$

Subtracting the two inequalities yields:

$$(p_1 - q_1)x_1 + (p_2 - q_2)x_2 - (p_1 - q_1)y_1 - (p_2 - q_2)y_2 \geq 0$$

Rearranging gives:

$$(p_1 - q_1)(x_1 - y_1) + (p_2 - q_2)(x_2 - y_2) \leq 0$$

If only the price of good 1 changes, so $p_2 = q_2$, then we have:

$$(p_1 - q_1)(x_1 - y_1) \leq 0$$

This shows that the change in price $(p_1 - q_1)$ and the change in quantity demanded from the substitution effect $(x_1 - y_1)$ must have opposite signs. For practical purposes, the Slutsky and Hicks decompositions are very similar for small price changes.

Chapter 9

Buying and Selling

9.1 Introduction

In the preceding chapters, we assumed that a consumer's income was exogenously fixed. We now expand our basic consumer choice framework to a more realistic scenario where consumers are endowed with a bundle of goods, which they can either consume or sell to generate income. This allows us to analyze a broader range of economic decisions, such as labor supply.

The theoretical framework remains fundamentally the same: consumers choose the most preferred bundle from their budget set. The key difference is that the budget constraint is now determined by the market value of the consumer's initial **endowment**.

9.2 The Budget Constraint

9.2.1 Endowments

We begin by defining the concept of an endowment.

Definition 9.1 (Endowment)**.** *The list of resource units with which a consumer starts is called their* ***endowment****. It is denoted by the vector* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$.

Endowment Bundle

Consider a consumer in a two-good world. An endowment vector $\boldsymbol{\omega} = (\omega_1, \omega_2) = (10, 2)$ means that the consumer starts with 10 units of good 1 and 2 units of good 2 before entering the market.

Given this endowment, the consumer can trade these goods at market prices. The fundamental questions are: what is the total value of this endowment, and what consumption bundles can the consumer afford?

9.2.2 Constructing the Budget Line

The consumer's income is no longer a fixed amount of money, but is instead determined by the market value of their endowed goods. If the market prices are (p_1, p_2), the value of the endowment $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is $p_1\omega_1 + p_2\omega_2$.

The consumer can sell their endowment to purchase any other consumption bundle (x_1, x_2) as long as its total cost does not exceed the value of their endowment. The budget constraint is therefore:

$$p_1x_1 + p_2x_2 = p_1\omega_1 + p_2\omega_2$$

The left-hand side represents the consumer's expenditure, and the right-hand side represents their income from selling the endowment.

The consumer's budget set consists of all affordable bundles (x_1, x_2):

$$\text{Budget Set} = \{(x_1, x_2) \mid p_1x_1 + p_2x_2 \leq p_1\omega_1 + p_2\omega_2, \text{ and } x_1 \geq 0, x_2 \geq 0\}$$

A crucial feature of this budget line is that it must always pass through the endowment point (ω_1, ω_2). This is because if the consumer chooses to consume their endowment bundle, i.e., $(x_1, x_2) = (\omega_1, \omega_2)$, the budget constraint is satisfied: $p_1\omega_1 + p_2\omega_2 = p_1\omega_1 + p_2\omega_2$.

Remark 9.1. Because the endowment point is always on the budget line, any change in prices will cause the budget line to **pivot** around the endowment point. This is different from the parallel shift we saw when money income was fixed.

9.3 Net and Gross Demands

We can rearrange the budget constraint to provide a different perspective on the consumer's decision.

$$\begin{aligned} p_1x_1 + p_2x_2 &= p_1\omega_1 + p_2\omega_2 \\ p_1x_1 - p_1\omega_1 + p_2x_2 - p_2\omega_2 &= 0 \\ p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) &= 0 \end{aligned}$$

This formulation is insightful. It states that the net value of a consumer's purchases and sales must be zero. To understand this, we define two types of demands.

Definition 9.2 (Gross and Net Demands)**.** *The consumer's final consumption bundle* (x_1, x_2) *is their* ***gross demand****. The quantities* $(x_1 - \omega_1)$ *and* $(x_2 - \omega_2)$ *are the consumer's* ***net demands****.*

- If $x_i - \omega_i > 0$, the consumer is a **net buyer** or **net demander** of good i.
- If $x_i - \omega_i < 0$, the consumer is a **net seller** or **net supplier** of good i.

The equation $p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0$ simply means that the value of what the consumer buys must equal the value of what they sell. For example, if a consumer is a net seller of good 1 ($x_1 - \omega_1 < 0$), they must be a net buyer of good 2 ($x_2 - \omega_2 > 0$) for the equation to hold.

9.4 Slutsky's Equation with Endowments

When prices change, not only do relative prices change (affecting substitution), but the value of the consumer's endowment also changes. This alters their income, introducing an additional income effect.

Recall that for a consumer with fixed money income m, the Slutsky decomposition was:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - x_1 \frac{\Delta x_1^m}{\Delta m}$$

Here, the total effect of a price change on demand is the sum of a substitution effect and an ordinary income effect.

Now, income is not fixed but is given by $m = p_1\omega_1 + p_2\omega_2$. A change in p_1 affects m. The change in money income when p_1 changes is:

$$\frac{\Delta m}{\Delta p_1} = \omega_1$$

This change in income generates an additional income effect, which we call the **endowment income effect**.

The endowment income effect is the change in demand due to the change in the value of the endowment:

$$\text{Endowment Income Effect} = \frac{\Delta m}{\Delta p_1} \times \frac{\Delta x_1^m}{\Delta m} = \omega_1 \frac{\Delta x_1^m}{\Delta m}$$

The overall change in demand is now the sum of three components:

1. **Pure Substitution Effect:** The effect from the change in relative prices, holding purchasing power constant.

2. **Ordinary Income Effect:** The effect from the change in purchasing power, as the original bundle is now more or less expensive.

3. **Endowment Income Effect:** The effect from the change in the value of the endowment.

Combining these, the full Slutsky equation with endowments is:

$$\frac{\Delta x_1}{\Delta p_1} = \underbrace{\frac{\Delta x_1^s}{\Delta p_1}}_{\text{Substitution Effect}} \underbrace{-x_1 \frac{\Delta x_1^m}{\Delta m}}_{\text{Ordinary Income Effect}} + \underbrace{\omega_1 \frac{\Delta x_1^m}{\Delta m}}_{\text{Endowment Income Effect}}$$

We can combine the two income effects into a single term:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1) \frac{\Delta x_1^m}{\Delta m} \tag{9.1}$$

Remark 9.2. The term $(\omega_1 - x_1)$ represents the consumer's net supply of good 1.

- If the consumer is a net seller of good 1 ($\omega_1 > x_1$) and good 1 is a normal good, an increase in p_1 has a positive income effect, encouraging more consumption of good 1. This can potentially outweigh the negative substitution effect.

- If the consumer is a net buyer of good 1 ($\omega_1 < x_1$), the term $(\omega_1 - x_1)$ is negative. The total income effect is reinforced, making the demand curve for a normal good unambiguously downward-sloping.

9.5 Application: Labor Supply

The model of buying and selling is perfectly suited to analyze a worker's decision on how many hours to work.

9.5.1 The Labor-Leisure Choice

Consider a worker who chooses between two goods: a composite consumption good, C, and leisure, R.

- The price of the consumption good is p.

- The "price" of leisure is the wage rate, w, as it represents the opportunity cost of not working.

The worker is endowed with some non-labor income, M, and a total amount of time, $\bar{R}$ (e.g., 24 hours a day). The endowment bundle is $(\bar{R}, M)$. The worker sells some of their time endowment as labor to fund consumption. Labor supplied, L, is the total time endowment minus the amount of leisure consumed: $L = \bar{R} - R$.

The budget constraint can be expressed as:

$$\text{Expenditure} = \text{Income}$$

$$pC = w(\bar{R} - R) + M$$

Rearranging this gives us the standard endowment budget line:

$$pC + wR = w\bar{R} + M$$

The left side is the total expenditure on consumption and leisure. The right side is the total potential income, or the value of the endowment of time and non-labor money. The slope of this budget line is $-w/p$, which is the **real wage**.

The worker chooses the optimal combination (C^*, R^*) that maximizes utility, which in turn determines the amount of labor supplied, $L^* = \bar{R} - R^*$.

9.5.2 Backward-Bending Labor Supply Curve

How does the amount of labor supplied change when the wage rate, w, increases? We can use the Slutsky equation to analyze the effect of a change in w on the demand for leisure, R. Here, leisure is good 1, and its price is w. The endowment of leisure is $\bar{R}$.

From Equation (9.1), we have:

$$\frac{\Delta R}{\Delta w} = \underbrace{\frac{\Delta R^s}{\Delta w}}_{\text{(-)}} + \underbrace{(\bar{R} - R)}_{\text{(+)}} \underbrace{\frac{\Delta R^m}{\Delta m}}_{\text{(?)}} \tag{9.2}$$

Let's analyze the signs:

- **Substitution Effect ($\frac{\Delta R^s}{\Delta w}$):** An increase in the wage rate w makes leisure more expensive. The consumer will substitute away from leisure towards consumption. Thus, the substitution effect on the demand for leisure is negative. This effect implies that the worker supplies more labor.
- **Income Effect ($(\bar{R} - R)\frac{\Delta R^m}{\Delta m}$):**
  - $(\bar{R} - R)$ is the amount of labor supplied, which is typically positive.
  - If leisure is a normal good (which is a reasonable assumption), then $\frac{\Delta R^m}{\Delta m}$ is positive. An increase in income leads to a desire for more leisure.
  - Therefore, the entire income effect term is positive. A higher wage makes the worker richer, so they choose to "purchase" more leisure (i.e., work less).

The substitution and income effects work in opposite directions.

- At low wage rates, the substitution effect often dominates. An increase in the wage leads to an increase in labor supply.
- At high wage rates, the income effect may dominate. The worker is already earning a high income, and a further wage increase may lead them to value leisure more, thus reducing their labor supply.

This can lead to a **backward-bending labor supply curve**, where the quantity of labor supplied first increases with the wage and then decreases.

9.5.3 Overtime

An overtime wage creates a kink in the budget constraint. Suppose a worker is paid a wage w for the first L_0 hours of work, and an overtime wage $w' > w$ for any hours beyond L_0. The budget line becomes steeper after the worker has supplied L_0 hours of labor.

This non-linear wage structure can induce an employee to work more hours than a simple increase in the flat wage rate. The overtime offer effectively isolates the substitution effect over a certain range, encouraging more work without the large offsetting income effect that a high flat wage for all hours would create.

Chapter 10

Intertemporal Choice

10.1 Introduction

In previous chapters, we analyzed consumer choices among different goods at a single point in time. However, many important economic decisions, such as saving for retirement, taking out a loan for education, or investing in a new project, involve trade-offs over time. **Intertemporal choice** is the study of how individuals allocate their consumption and resources across different time periods.

In this chapter, we will adapt our existing framework of consumer theory to this new problem. We will treat consumption at different times as different goods and analyze the consumer's decision-making process. The central questions we seek to answer are:

- How does a consumer choose between consumption today and consumption in the future? - How do interest rates affect the decision to save or borrow? - How can we place a value on streams of payments that occur over time?

10.2 The Budget Constraint for Intertemporal Choice

To analyze the choice problem, we first need to understand the consumer's budget constraint. We will start with a simple two-period model. Let's denote consumption in period 1 as c_1 and consumption in period 2 as c_2. Similarly, the consumer receives income m_1 in period 1 and m_2 in period 2. This income bundle, (m_1, m_2), is called the consumer's **endowment**.

For now, let's assume the price of consumption is normalized to 1 in both periods, so $p_1 = p_2 = 1$. The key variable that connects the two periods is the interest rate.

10.2.1 Present and Future Value

Before constructing the budget constraint, we must understand the concepts of present and future value, which are essential tools for comparing money across time. Let r be the interest rate per period.

Definition 10.1 (Future Value)**.** *The **Future Value (FV)** of an amount of money is its worth at a specified future date. If you save an amount* $\$m$ *today at an interest rate* r*, its value one period from now will be:*

$$FV = m(1 + r) \tag{10.1}$$

Definition 10.2 (Present Value)**.** *The **Present Value (PV)** of a future amount of money is its equivalent value today. To find the present value of an amount* $\$m$ *to be received one period from now, we ask: how much money would you need to save today to have* $\$m$ *in the next period? The answer is:*

$$PV = \frac{m}{1 + r} \tag{10.2}$$

This is because if you save $m/(1 + r)$ *today, it will grow to* $(m/(1 + r))(1 + r) = m$ *in the next period.*

Paying \$1 today for a promise of \$1 tomorrow is a bad deal, as you could save the \$1 and have $\$(1 + r)$ tomorrow. The most you should be willing to pay today for \$1 tomorrow is its present value, $\$1/(1 + r)$.

10.2.2 Constructing the Intertemporal Budget Constraint

A consumer can choose to consume their endowment (m_1, m_2) in each period. This is known as the "Polonius Point," from Shakespeare's *Hamlet* ("Neither a borrower, nor a lender be."). However, consumers can typically transfer resources between periods by saving or borrowing at the interest rate r.

Case 1: The Lender (Saver) If a consumer's first-period consumption c_1 is less than their income m_1, they are a saver. The amount saved is $s_1 = m_1 - c_1$. This saving earns interest and can be used for consumption in the second period. The consumption in period 2 will be their income in that period plus the principal and interest from their savings:

$$c_2 = m_2 + (m_1 - c_1)(1 + r) \tag{10.3}$$

Case 2: The Borrower If a consumer's first-period consumption c_1 is greater than their income m_1, they are a borrower. The amount borrowed is $c_1 - m_1$. This loan must be repaid with interest in the second period. Their consumption in period 2 will be their income in that period minus the loan repayment:

$$c_2 = m_2 - (c_1 - m_1)(1 + r) \tag{10.4}$$

Notice that rearranging this equation yields $c_2 = m_2 + (m_1 - c_1)(1 + r)$, which is identical to the saver's equation.

Thus, a single equation describes the trade-off between consumption in the two periods for both savers and borrowers. We can rearrange this equation in two standard forms:

1. **Future Value Form:** By moving all consumption terms to the left and income terms to the right, we get:

$$(1+r)c_1 + c_2 = (1+r)m_1 + m_2 \tag{10.5}$$

This equation states that the future value of the consumption stream must equal the future value of the income stream.

2. **Present Value Form:** By dividing the future value form by $(1+r)$, we get:

$$c_1 + \frac{c_2}{1+r} = m_1 + \frac{m_2}{1+r} \tag{10.6}$$

This equation states that the present value of the consumption stream must equal the present value of the income stream.

This is the consumer's intertemporal budget constraint. It shows all affordable combinations of (c_1, c_2). Graphically, it is a straight line that passes through the endowment point (m_1, m_2) with a slope of $-(1+r)$. The slope represents the opportunity cost: to consume one more unit today, the consumer must give up $(1+r)$ units of consumption tomorrow.

10.2.3 Inflation and the Real Interest Rate

So far, we assumed the price of consumption was 1 in both periods. Let's introduce inflation. Let $p_1 = 1$, but allow the price level in period 2, p_2, to be different. The rate of inflation, π, is defined by the relation:

$$p_2 = p_1(1+\pi) = 1+\pi \tag{10.7}$$

For example, $\pi = 0.2$ means 20% inflation.

The budget constraint must now be written in terms of monetary values. The present value of expenditure must equal the present value of income:

$$p_1c_1 + \frac{p_2c_2}{1+r} = p_1m_1 + \frac{p_2m_2}{1+r} \tag{10.8}$$

Substituting $p_1 = 1$ and $p_2 = 1+\pi$, we get:

$$c_1 + \frac{1+\pi}{1+r}c_2 = m_1 + \frac{1+\pi}{1+r}m_2 \tag{10.9}$$

The slope of this budget constraint is now $-\frac{1+r}{1+\pi}$. The slope represents the trade-off in terms of *real goods*, not just dollars.

We can define a new variable, the **real interest rate** ρ, to simplify this expression.

Definition 10.3 (Real Interest Rate)**.** *The real interest rate ρ is defined such that:*

$$1+\rho = \frac{1+r}{1+\pi} \tag{10.10}$$

Rearranging gives the ***Fisher Equation****:*

$$\rho = \frac{r - \pi}{1 + \pi} \tag{10.11}$$

When inflation π is small, this is often approximated as $\rho \approx r - \pi$. The real interest rate measures the return on saving in terms of how many additional goods you can buy in the future.

Using the real interest rate, the budget constraint takes on a familiar and intuitive form:

$$c_1 + \frac{c_2}{1+\rho} = m_1 + \frac{m_2}{1+\rho} \tag{10.12}$$

The slope of the budget line is simply $-(1 + \rho)$.

10.3 Optimal Choice and Comparative Statics

The consumer's problem is to choose the bundle (c_1, c_2) on the budget line that gives the highest utility. Assuming well-behaved preferences, the optimal choice occurs where an indifference curve is tangent to the budget line. At this point, the marginal rate of substitution (MRS) between consumption today and consumption tomorrow equals the slope of the budget line:

$$MRS = -(1 + \rho) \tag{10.13}$$

How does the optimal choice change when the real interest rate ρ changes? A change in ρ pivots the budget line around the endowment point (m_1, m_2).

Case 1: The Lender (Saver) Suppose the consumer is initially a lender ($c_1^* < m_1$). If the real interest rate ρ *decreases*, the budget line becomes flatter. The lender is now worse off, as the reward for saving has fallen. They will unambiguously reach a lower indifference curve.

Case 2: The Borrower Suppose the consumer is initially a borrower ($c_1^* > m_1$). If the real interest rate ρ *decreases*, the budget line becomes flatter. The borrower is now better off, as the cost of borrowing has fallen. They will unambiguously reach a higher indifference curve.

10.3.1 The Slutsky Equation and Intertemporal Choice

We can analyze the effect of an interest rate increase on current consumption, c_1, using the Slutsky equation. An increase in the interest rate is like an increase in the price of current consumption (since its opportunity cost rises). The total effect is:

$$\frac{\Delta c_1}{\Delta r} = \underbrace{\frac{\Delta c_1^s}{\Delta r}}_{\text{Substitution Effect}} + \underbrace{(m_1 - c_1)\frac{\Delta c_1^m}{\Delta m}}_{\text{Income Effect}}$$

- **Substitution Effect:** An increase in r makes future consumption cheaper relative to current consumption. Therefore, the substitution effect is always negative: $\frac{\Delta c_1^s}{\Delta r} < 0$.
- **Income Effect:** The sign depends on whether the consumer is a borrower or a lender.
  - For a **lender**, $m_1 - c_1 > 0$. An increase in r is like an increase in their income. Assuming consumption is a normal good, this effect is positive. The total effect is ambiguous as the two effects work in opposite directions.
  - For a **borrower**, $m_1 - c_1 < 0$. An increase in r is like a decrease in their income. This effect is negative. The total effect is unambiguously negative, as both effects push in the same direction. A borrower will always reduce current consumption when the interest rate rises.

10.4 Valuing Streams of Payments

The concept of present value can be extended to value any stream of payments over multiple periods. The present value of an income stream $(m_1, m_2, \ldots, m_T)$ is:

$$PV = m_1 + \frac{m_2}{1+r} + \frac{m_3}{(1+r)^2} + \cdots + \frac{m_T}{(1+r)^{T-1}} \tag{10.14}$$

If interest rates are not constant, with r_t being the rate between period t and $t+1$, the formula becomes:

$$PV = m_1 + \frac{m_2}{1+r_1} + \frac{m_3}{(1+r_1)(1+r_2)} + \ldots \tag{10.15}$$

Proposition 10.1. *If a consumer can freely borrow and lend at a constant interest rate, they will always prefer an income stream with a higher present value to one with a lower present value.*

Proof. A higher present value of income shifts the intertemporal budget constraint outward in a parallel fashion, allowing the consumer to afford more consumption in all periods and thus reach a higher indifference curve. □

This principle is the foundation for investment analysis. To decide whether to undertake a project, one calculates its **Net Present Value (NPV)**—the present value of its future returns minus the initial cost. Projects with a positive NPV should be undertaken.

10.5 Applications: Valuing Assets

Financial assets like stocks and bonds are essentially claims to a future stream of payments. Their price in a competitive market should therefore be equal to the present value of that payment stream.

10.5.1 Bonds

A **bond** is a security that promises to pay a fixed amount, the **coupon** (x), each period for a set number of periods, until its **maturity date** (T). At maturity, the bond also pays back its **face value** (F). The payment stream is $(x, x, \ldots, x, F)$, where the final payment is typically coupon plus face value, but for simplicity we follow the common textbook formula where the T-th payment is F. The present value (and thus the price) of such a bond is:

$$PV_{\text{bond}} = \frac{x}{1+r} + \frac{x}{(1+r)^2} + \cdots + \frac{x}{(1+r)^{T-1}} + \frac{F}{(1+r)^T} \tag{10.16}$$

10.5.2 Perpetuities (Consols)

A **perpetuity** or **consol** is a special type of bond that promises to pay a coupon x forever. Its payment stream is $(x, x, x, \ldots)$.

Proposition 10.2. *The present value of a perpetuity paying x per period is given by:*

$$PV_{perpetuity} = \frac{x}{r} \tag{10.17}$$

Proof. The present value is the sum of an infinite geometric series:

$$PV = \frac{x}{1+r} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \ldots$$

Let $v = 1/(1+r)$. Then $PV = x(v + v^2 + v^3 + \ldots)$. The sum of this series is $x\left(\frac{v}{1-v}\right)$. Substituting back $v = 1/(1+r)$:

$$PV = x\left(\frac{\frac{1}{1+r}}{1-\frac{1}{1+r}}\right) = x\left(\frac{\frac{1}{1+r}}{\frac{r}{1+r}}\right) = \frac{x}{r}$$

□

This simple formula is one of the most useful in finance.

Valuing an Installment Loan

Suppose you borrow \$1000 and agree to pay it back in 12 monthly installments of \$100. What interest rate are you paying?
The stream of payments from the lender's perspective is $(-1000, 100, 100, \ldots, 100)$. The value of this stream must be zero at the true interest rate. We need to find the monthly interest rate r_m that solves:

$$1000 = \frac{100}{1+r_m} + \frac{100}{(1+r_m)^2} + \cdots + \frac{100}{(1+r_m)^{12}}$$

This equation must be solved numerically. The solution is approximately $r_m =$

2.92%. The annual percentage rate (APR) is often quoted as $12 \times 2.92\% = 35.04\%$.

10.6 Conclusion: The Right Interest Rate

In the real world, there is no single interest rate. Rates vary by duration (short-term vs. long-term), risk (government bonds vs. corporate debt), and tax treatment. When using present value for analysis, it is crucial to select an interest rate that accurately reflects the opportunity cost of the funds for the specific time horizon and risk profile of the payments being valued. The core principle remains: present value is the correct and indispensable tool for converting streams of future payments into today's dollars.

Chapter 11

Asset Markets

11.1 Introduction to Assets

We begin by defining what an asset is in the context of economics.

Definition 11.1 (Asset)**.** *An* ***asset*** *is a commodity that provides a flow of services or payments over time.*

- Physical assets, like a house or a computer, provide a flow of services (shelter, computing power).
- A **financial asset**, often called a security, provides a flow of money over time. Examples include stocks and bonds.

Typically, the future services or payments from an asset are uncertain. However, incorporating uncertainty into economic models is complex. For this introductory chapter, we will make a strong simplifying assumption.

Remark 11.1 (Perfect Certainty). Throughout this chapter, we will assume that the future is known with **perfect certainty**. This means all future prices, payments, and interest rates are known today.

11.2 Arbitrage

Under the assumption of perfect certainty, a powerful principle emerges that governs the pricing of assets: the principle of no arbitrage.

Definition 11.2 (Arbitrage)**.** ***Arbitrage*** *is the practice of buying and selling equivalent assets to profit from a difference in their price. It involves trading for profit in commodities that are not intended for immediate consumption.*

With no uncertainty, any and all opportunities for risk-free profit will be immediately exploited by traders. This activity, in turn, eliminates the profit opportunities that gave rise to it. The implication is that in a well-functioning market, there can be no opportunities for arbitrage. This is often called the **no-arbitrage principle**.

11.2.1 The No-Arbitrage Condition and Rates of Return

Let's consider a simple asset. Let its price today be p_0 and its price tomorrow be p_1. The **rate of return**, R, from holding this asset for one period is the change in its price as a fraction of its initial price:

$$R = \frac{p_1 - p_0}{p_0} \tag{11.1}$$

Rearranging this equation gives us the relationship between today's price, tomorrow's price, and the rate of return:

$$p_1 = p_0 + Rp_0 = (1 + R)p_0 \tag{11.2}$$

Now, consider an alternative investment. You could sell the asset today for p_0 and deposit the money in a bank to earn a risk-free interest rate, r. Tomorrow, you would have $(1 + r)p_0$.

The principle of arbitrage helps us determine the equilibrium relationship between R and r.

- If $R > r$, the return from holding the asset is greater than the return from the bank. Everyone would want to buy the asset, driving its price p_0 up. This increase in p_0 would lower the rate of return $R = (p_1/p_0) - 1$ until the inequality no longer holds.

- If $R < r$, the return from the bank is higher. Everyone would want to sell the asset and deposit the money in the bank. This would drive the price p_0 down, increasing the rate of return R until the inequality is eliminated.

Therefore, in equilibrium, all assets must earn the same risk-free rate of return:

$$R = r \tag{11.3}$$

This is the fundamental **no-arbitrage condition**. It implies that for any asset:

$$p_1 = (1 + r)p_0 \tag{11.4}$$

This equation states that the price of the asset tomorrow is simply the future value of its price today. Equivalently, we can express today's price in terms of tomorrow's price:

$$p_0 = \frac{p_1}{1 + r}$$

This means that today's price must be the **present value** of tomorrow's price.

11.2.2 Applications of the No-Arbitrage Condition

Arbitrage in Bonds

Bonds are financial assets that promise a fixed stream of payments. A common question is: why do the market prices of existing bonds fall when the interest rate paid by banks rises?
A bond's fixed payments mean that its rate of return, R, is inversely related to its market price, p_0. Suppose initially the market is in equilibrium, so the bond's return equals the bank interest rate, $R = r'$.
Now, suppose the bank interest rate rises to $r'' > r'$. At the current market price of the bond, its return R is now less than the new bank rate ($R < r''$). Arbitrageurs will sell the bond to move their money to the bank. This massive selling pressure on bonds causes their market prices to fall. The price of bonds will continue to fall until their rate of return, R, rises to meet the new, higher interest rate r''.

Taxation of Asset Returns

The no-arbitrage principle implies that the *after-tax* rates of return of all assets must be equal. Let r_b be the before-tax rate of return on a taxable asset (like a corporate bond) and r_e be the rate of return on a tax-exempt asset (like a municipal bond). Let the tax rate on interest income be t.
The after-tax return on the taxable asset is $(1 - t)r_b$. For an investor to be indifferent between the two assets, their after-tax returns must be equal:

$$(1 - t)r_b = r_e \tag{11.5}$$

Assets with Consumption Returns

Some assets, like a house, provide returns in two forms: a monetary return (appreciation) and a consumption return (the value of living in it, or the implicit rent). The total rate of return must equal the market interest rate in equilibrium. Let:

- P = Initial price (investment)
- A = Appreciation (capital gain over one year)
- T = Implicit rental value for one year

The total rate of return, h, is the sum of the financial and consumption returns, divided by the initial investment:

$$h = \frac{A + T}{P} \tag{11.6}$$

In equilibrium, this total return must equal the opportunity cost of the funds, i.e., the market interest rate:

$$\frac{A+T}{P} = r \tag{11.7}$$

11.3 Optimal Holding Time: When to Sell an Asset

The no-arbitrage principle can also tell us the optimal time to sell an asset whose value changes over time.

When to Sell an Asset

Suppose the value of an asset (e.g., a painting, a plot of land) at time t is given by the function $V(t)$. The owner's alternative is to sell the asset and invest the proceeds in a bank at a constant interest rate r. When should they sell?
The asset itself is like a savings account, but its rate of return changes over time. The instantaneous rate of return from holding the asset at time t is the rate of growth of its value, which is $\frac{V'(t)}{V(t)}$.
The optimal rule is to hold the asset as long as its rate of return is greater than the market interest rate and sell it at the exact moment its rate of return falls to the market interest rate. The optimal time to sell, t^*, is when:

$$\frac{V'(t^*)}{V(t^*)} = r \tag{11.8}$$

Let's consider a specific function: $V(t) = -1000 + 1000t - 10t^2$. The derivative is $V'(t) = 1000 - 20t$. Suppose the market interest rate is $r = 0.10$ (or 10%). We need to solve:

$$\frac{1000 - 20t}{-1000 + 1000t - 10t^2} = 0.1$$

Solving this equation yields $t = 10$.
At $t = 10$, the value of the asset is $V(10) = \$8,000$. Note that the asset's value is maximized at $t = 50$, where $V(50) = \$24,000$. Why sell so early?
Because at $t = 10$, the asset's value is growing at exactly 10%. After this point, its growth rate will be less than 10%. If you sell at $t = 10$ for \$8,000 and invest this amount at 10% for the next 40 years, at $t = 50$ you will have:

$$\$8,000 \times (1 + 0.1)^{40} \approx \$362,074$$

This amount is vastly greater than the \$24,000 you would have by holding the asset until its peak value. The optimal strategy is to harvest the high returns from the asset early and then switch to the market investment once the asset's return diminishes.

11.3.1 Application: When to Cut a Forest

This same logic applies to renewable resources. Let $F(t)$ be the volume (and thus value, assuming a constant price) of lumber in a forest. The rate of growth of the forest is $\frac{F'(t)}{F(t)}$. The optimal time to harvest the forest, T, is when its biological rate of growth equals the financial rate of return (the interest rate).

$$\frac{F'(T)}{F(T)} = r \tag{11.9}$$

11.3.2 Continuous Time and Optimal Harvesting

The analysis is more formally done in continuous time. The present value of harvesting a forest of value $F(T)$ at time T is given by $V(T) = F(T)e^{-rT}$, where r is the continuously compounded interest rate. To find the time T that maximizes this present value, we set the derivative with respect to T equal to zero:

$$\frac{dV(T)}{dT} = F'(T)e^{-rT} - rF(T)e^{-rT} = 0 \tag{11.10}$$

$$F'(T) - rF(T) = 0 \tag{11.11}$$

$$r = \frac{F'(T)}{F(T)} \tag{11.12}$$

This confirms our earlier result: the optimal harvest time is when the resource's growth rate equals the interest rate.

11.4 Asset Bubbles

The no-arbitrage condition assumes that prices are based on the fundamental value of an asset (the present value of its future payments or services). Sometimes, however, prices seem to disconnect from these fundamentals.

Definition 11.3 (Asset Bubble)**.** *An* ***asset bubble*** *occurs when the price of an asset is pushed to an unreasonably high level, driven by expectations of future price increases rather than its fundamental value.*

In a bubble, an initial price increase leads people to expect further increases. This expectation boosts demand, which pushes the price up even more rapidly, creating a self-fulfilling cycle for a period.

- **Economic Fundamentals**: The key to identifying a potential bubble is to compare the market price to the asset's fundamental value. For a house, the fundamental value is the present value of the stream of rental services it provides.

- **Key Indicators**:

1. **Price-to-Rent Ratio**: The ratio of a house's price to its annual rental rate. A very high ratio suggests prices are detached from the fundamental rental value. If the interest rate is r and annual rent is R_{annual}, the fundamental price is $P_{fundamental} = R_{annual}/r$. The fundamental price-to-rent ratio is simply $1/r$. If the market price-to-rent ratio is significantly higher than $1/r$, it could signal a bubble.
2. **Price-to-Income Ratio**: The ratio of median house prices to median income. This is a measure of affordability and can indicate when prices have become unsustainable for the general population.

All bubbles eventually burst. When they do, prices fall rapidly, and those who bought at the peak are left with assets worth much less than they paid. The belief that "this time is different" is a common and hazardous feature of bubble psychology.

11.5 Depletable Resources

How should the price of a finite,depletable resource (like oil) change over time? We can think of oil in the ground as an asset. The owner of the oil must decide whether to extract and sell it today or leave it in the ground for the future.

By the no-arbitrage principle, the owner must be indifferent between these two options.

- Option 1: Extract and sell one unit today at price p_t. Invest the proceeds at rate r. Next year, this will be worth $p_t(1 + r)$.
- Option 2: Leave the unit in the ground and sell it next year at price p_{t+1}.

For equilibrium, these must be equal: $p_{t+1} = p_t(1 + r)$. This result is known as **Hotelling's rule**. It states that the price of a depletable resource must grow at the rate of interest.

This helps explain why news that affects the *future* supply of a resource can cause its *current* price to jump immediately. If news (e.g., a war or natural disaster) suggests that oil will be scarcer in the future, the entire expected future price path must be higher. For the no-arbitrage condition to hold along this new, higher path, today's price must also jump up immediately.

11.6 Financial Intermediaries

In a modern economy, financial institutions exist to help people reallocate their consumption and wealth over time.

- **Banks**: Banks act as intermediaries between savers and borrowers. For example, they can take the savings of an older individual who desires a steady stream of income and lend it as a lump sum to a younger individual who wants to buy a house. Both parties are better off.

- **Stock Market**: The stock market allows successful entrepreneurs to convert their ownership of a firm (a claim on a future stream of profits) into a lump-sum payment by selling shares. In turn, it allows investors with lump sums of money to purchase a share of that future profit stream. In this way, both sides of the market can reallocate their wealth across time to better suit their needs.

Chapter 12

Uncertainty

12.1 Introduction

In the preceding chapters, we analyzed consumer choice in a world of certainty. Consumers knew the prices of goods, their income, and the quality of the products they were purchasing. However, many, if not most, economic decisions are made under conditions of uncertainty. For example, when you invest in the stock market, you are uncertain about the future returns. When you buy a car, you are uncertain about its reliability and potential repair costs.

This chapter extends our model of consumer choice to handle uncertainty. We will follow a familiar structure:

- First, we will describe the budget constraint of a consumer facing uncertain outcomes. This involves introducing the concept of **state-contingent consumption**.

- Second, we will model preferences over these uncertain outcomes using the **Expected Utility Theorem**.

- Finally, we will combine the budget constraint and preferences to determine the consumer's optimal choice, with a particular focus on the market for insurance.

Uncertainty is a pervasive feature of economic life, affecting decisions about savings, career paths, and even simple purchases. Rational responses to uncertainty include actions like buying insurance or diversifying investments, which we will explore in detail.

12.2 State-Contingent Consumption

To analyze choice under uncertainty, we first need a way to describe the different possible outcomes an individual might face.

Definition 12.1 (States of Nature)**.** *A **state of nature** is a complete and mutually exclusive description of a possible outcome of an uncertain event. The set of all possible states of nature exhausts all possibilities.*

For example, if we are considering the risk of a car accident, there are two relevant states of nature: "accident occurs" and "no accident occurs". Let's denote the "accident" state as state a and the "no accident" state as state na.

Let π_a be the probability that an accident occurs, and π_{na} be the probability that no accident occurs. Since these are the only two possibilities, we must have $\pi_a + \pi_{na} = 1$. Suppose an accident results in a monetary loss of $\$L$.

12.2.1 Contingent Consumption Plans

A consumer's consumption may depend on which state of nature occurs. A consumption plan that specifies consumption levels for each possible state of nature is called a **state-contingent consumption plan**.

Definition 12.2 (State-Contingent Consumption Good)**.** *A **state-contingent consumption good** is a good that is delivered only if a specific state of nature occurs. For example, "\$1 if an accident occurs" is a different commodity from "\$1 if no accident occurs".*

We can think of the consumer choosing a bundle of state-contingent consumption goods. Let C_a be the consumption value in the accident state and C_{na} be the consumption value in the no-accident state. A consumption plan is then a bundle (C_a, C_{na}).

12.2.2 The Budget Constraint with Insurance

Insurance is a common tool for managing risk. An insurance contract is a prime example of a state-contingent plan: the payout occurs only if the specified adverse event (like an accident) happens.

Let's construct the consumer's budget constraint. Assume:

- The consumer has initial wealth of $\$m$.
- An accident causes a loss of $\$L$.
- The consumer can buy insurance. The price of \$1 of insurance coverage is γ. Let's say the consumer buys $\$K$ of coverage. The total cost of this insurance is γK.

Without any insurance, the consumer's state-contingent consumption bundle is their **endowment point**:

$$C_a = m - L$$
$$C_{na} = m$$

Now, suppose the consumer buys $\$K$ of insurance. The premium γK must be paid regardless of the state of nature. If an accident occurs, the consumer receives a payout of $\$K$ from the insurance company. The consumption in each state is now:

$$C_{na} = m - \gamma K \tag{12.1}$$
$$C_a = m - L - \gamma K + K = m - L + (1-\gamma)K \tag{12.2}$$

To derive the budget constraint in terms of C_a and C_{na}, we can eliminate K from these two equations. From (12.2), we can solve for K:

$$K = \frac{C_a - m + L}{1-\gamma}$$

Substituting this expression for K into (12.1) gives:

$$C_{na} = m - \gamma\left(\frac{C_a - m + L}{1-\gamma}\right)$$

Rearranging this equation to express C_{na} as a function of C_a yields the budget line:

$$C_{na} = \frac{m(1-\gamma) + \gamma(m-L)}{1-\gamma} - \frac{\gamma}{1-\gamma}C_a = \frac{m-\gamma L}{1-\gamma} - \frac{\gamma}{1-\gamma}C_a \tag{12.3}$$

This is a linear equation in the (C_a, C_{na}) space. The slope of the state-contingent budget constraint is:

$$\text{Slope} = -\frac{\gamma}{1-\gamma}$$

The budget line passes through the consumer's initial endowment point $(m-L, m)$, as purchasing zero insurance ($K = 0$) is always an option.

12.3 Preferences Under Uncertainty

How do consumers evaluate different state-contingent consumption plans? The standard model for this is the **von Neumann-Morgenstern expected utility theory**.

Definition 12.3 (Expected Utility)**.** *Suppose a consumer faces a set of outcomes* $\{c_1, c_2, \ldots, c_n\}$ *with corresponding probabilities* $\{\pi_1, \pi_2, \ldots, \pi_n\}$*, where* $\sum \pi_i = 1$*. The consumer has a utility function over certain outcomes,* $U(c)$*. The* ***expected utility*** *(EU) of this uncertain prospect is the probability-weighted average of the utilities of each outcome:*

$$EU = \sum_{i=1}^{n} \pi_i U(c_i) \tag{12.4}$$

The function $U(\cdot)$ is called the von Neumann-Morgenstern utility function, or Bernoulli utility function. It measures the utility of wealth in a given state, while EU measures the utility of the uncertain prospect as a whole.

12.3.1 Risk Attitudes

We can classify individuals based on their preferences toward risk by comparing the utility of the expected value of a gamble with the expected utility of the gamble itself.

Consider a simple lottery that pays $90 with probability 1/2 and $0 with probability 1/2.

- The **expected monetary value** (EMV) is $EM = \frac{1}{2}(\$90) + \frac{1}{2}(\$0) = \$45$.
- The **expected utility** is $EU = \frac{1}{2}U(\$90) + \frac{1}{2}U(\$0)$.

We can compare $U(\$45)$, the utility of getting the expected value for sure, with EU.

Definition 12.4 (Risk Attitudes)**.** *An individual's attitude toward risk is characterized as follows:*

- ***Risk Aversion****: If $U(EM) > EU$. The individual prefers the certain expected value to the gamble. This occurs when the utility function $U(w)$ is strictly concave ($U''(w) < 0$), which implies diminishing marginal utility of wealth.*
- ***Risk Loving****: If $U(EM) < EU$. The individual prefers the gamble to the certain expected value. This occurs when the utility function $U(w)$ is strictly convex ($U''(w) > 0$), implying increasing marginal utility of wealth.*
- ***Risk Neutrality****: If $U(EM) = EU$. The individual is indifferent between the gamble and the certain expected value. This occurs when the utility function $U(w)$ is linear ($U''(w) = 0$), implying constant marginal utility of wealth.*

Most economic analysis assumes that individuals are risk-averse, which aligns with behaviors like buying insurance.

12.3.2 Indifference Curves for State-Contingent Plans

An indifference curve represents all bundles (C_a, C_{na}) that provide the same level of expected utility. For our two-state example, the expected utility is:

$$EU(C_a, C_{na}) = \pi_a U(C_a) + \pi_{na} U(C_{na})$$

To find the slope of an indifference curve, we take the total differential and set it to zero ($dEU = 0$):

$$dEU = \pi_a U'(C_a) dC_a + \pi_{na} U'(C_{na}) dC_{na} = 0$$

Rearranging this gives the slope, which is the Marginal Rate of Substitution (MRS) between consumption in the two states:

$$MRS = \frac{dC_{na}}{dC_a} = -\frac{\pi_a U'(C_a)}{\pi_{na} U'(C_{na})} = -\frac{\pi_a MU(C_a)}{\pi_{na} MU(C_{na})} \tag{12.5}$$

The MRS depends on both the probabilities of the states and the marginal utility of consumption in each state. For a risk-averse individual, the indifference curves are convex to the origin.

12.4 Optimal Choice and Insurance

A rational consumer will choose the state-contingent consumption plan that maximizes their expected utility, given their budget constraint. The optimal choice is found at the tangency point between an indifference curve and the state-contingent budget line.

At the optimal bundle (C_a^*, C_{na}^*), the slope of the indifference curve equals the slope of the budget line:

$$MRS = \text{Slope of Budget Line}$$

$$-\frac{\pi_a MU(C_a^*)}{\pi_{na} MU(C_{na}^*)} = -\frac{\gamma}{1-\gamma} \tag{12.6}$$

This simplifies to the fundamental optimality condition for choice under uncertainty:

$$\frac{\pi_a MU(C_a^*)}{\pi_{na} MU(C_{na}^*)} = \frac{\gamma}{1-\gamma} \tag{12.7}$$

12.4.1 Fair Insurance

In a perfectly competitive insurance market, entry is free, and firms are expected to make zero economic profit in the long run. The expected profit for an insurance company from selling a policy with coverage $\$K$ at price γK is:

$$\text{Expected Profit} = \gamma K - (\pi_a \cdot K + \pi_{na} \cdot 0) = (\gamma - \pi_a)K$$

For expected profit to be zero, the price of \$1 of coverage must equal the probability of the loss: $\gamma = \pi_a$. An insurance policy with this pricing is called **actuarially fair insurance**.

If insurance is fair ($\gamma = \pi_a$), then $1 - \gamma = 1 - \pi_a = \pi_{na}$. The optimality condition (12.7) becomes:

$$\frac{\pi_a MU(C_a^*)}{\pi_{na} MU(C_{na}^*)} = \frac{\pi_a}{\pi_{na}}$$

This simplifies to:

$$MU(C_a^*) = MU(C_{na}^*)$$

For a risk-averse individual, marginal utility is diminishing. Therefore, the only way for the marginal utilities to be equal is if the consumption levels are equal:

$$C_a^* = C_{na}^*$$

Proposition 12.1. *A risk-averse consumer facing fair insurance will always choose to* ***fully insure****, equalizing their consumption across all possible states of nature.*

Full insurance means buying coverage equal to the potential loss, $K = L$. We can verify this by setting $C_a = C_{na}$:

$$\begin{aligned} m - L + (1-\gamma)K &= m - \gamma K \\ -L + K - \gamma K &= -\gamma K \\ K &= L \end{aligned}$$

12.4.2 Unfair Insurance

In reality, insurance companies have administrative costs and aim to make a profit. This means the price of insurance is typically "unfair" in the sense that $\gamma > \pi_a$. In this case, the insurance company makes a positive expected profit.

If $\gamma > \pi_a$, then it can be shown that $\frac{\gamma}{1-\gamma} > \frac{\pi_a}{\pi_{na}}$. The optimality condition (12.7) now implies:

$$\frac{\pi_a MU(C_a^*)}{\pi_{na} MU(C_{na}^*)} > \frac{\pi_a}{\pi_{na}} \implies \frac{MU(C_a^*)}{MU(C_{na}^*)} > 1$$

So, we must have $MU(C_a^*) > MU(C_{na}^*)$. For a risk-averse individual, this means consumption in the accident state is lower than in the no-accident state:

$$C_a^* < C_{na}^*$$

Proposition 12.2. *A risk-averse consumer facing unfair insurance ($\gamma > \pi_a$) will purchase some insurance but will not fully insure. They will choose a point where consumption is higher in the "good" state than in the "bad" state.*

12.5 Risk Management Strategies

Besides insurance, individuals and firms use other strategies to manage risk, most notably diversification and risk spreading.

12.5.1 Diversification

The principle of **diversification** states that one should not "put all their eggs in one basket." By spreading investment across multiple assets whose returns are not perfectly positively correlated, an investor can reduce the overall risk of their portfolio.

Example: Diversification

Consider two firms, A and B. A share in either costs \$10. You have \$100 to invest. There are two states of nature, each with probability 1/2.

- State 1: A's profit is \$100, B's profit is \$20.
- State 2: A's profit is \$20, B's profit is \$100.

If you buy 10 shares of A, your earnings are \$1000 in State 1 and \$200 in State 2. The expected earning is \$600. If you buy 5 shares of A and 5 shares of B, your portfolio is diversified.

- In State 1, your earning is $5 \times (\$100/10) + 5 \times (\$20/10) = \$500 + \$100 = \$600$.
- In State 2, your earning is $5 \times (\$20/10) + 5 \times (\$100/10) = \$100 + \$500 = \$600$.

With diversification, you earn \$600 for sure. You have maintained the same expected earning while completely eliminating the risk. This powerful result occurs because the returns of the assets are perfectly negatively correlated. More generally, diversification reduces risk as long as asset returns are not perfectly positively correlated.

12.5.2 Risk Spreading and the Role of the Stock Market

Risk spreading involves sharing a risk among a large number of people, so that the potential impact on any single individual is small. Mutual insurance is a form of risk spreading. If 100 people each face a 1% chance of a \$10,000 loss, the expected loss per person is \$100. By each contributing \$100 to a common fund, they can cover the loss of any member who is unlucky. Each person exchanges a small probability of a large loss for a certain small loss (the \$100 premium), which is a desirable trade for a risk-averse individual.

The **stock market** is a crucial institution for both diversification and risk spreading. It allows:

1. Entrepreneurs to spread the risk of their enterprise by selling shares to a multitude of investors.

2. Investors to diversify their wealth by holding shares in many different companies.

3. The transfer of risk from individuals who are more risk-averse to those who are more willing to bear it (risk-tolerant investors or speculators).

12.6 Measures of Risk Aversion

While the concavity of the utility function indicates risk aversion, it is useful to have more precise measures of *how* risk-averse an individual is.

Definition 12.5 (Certainty Equivalent and Risk Premium)**.** *The **certainty equivalent** (CE) of a gamble is the amount of certain wealth that provides the same utility as the expected utility of the gamble. It is defined by the equation:*

$$U(CE) = E[U(g)]$$

*The **risk premium** (P) is the difference between the expected monetary value of the gamble and its certainty equivalent. It represents the maximum amount an individual would pay to avoid the gamble.*

$$P = E[g] - CE$$

For a risk-averse individual, $P > 0$.

Definition 12.6 (Absolute Risk Aversion)**.** *The **Arrow-Pratt measure of absolute risk aversion** is defined as:*

$$R_a(w) = -\frac{U''(w)}{U'(w)} \tag{12.8}$$

$R_a(w)$ is a measure of the curvature of the utility function at wealth level w. A higher value indicates greater risk aversion.

- *For a risk-averse person, $U'' < 0$, so $R_a(w) > 0$.*
- *For a risk-loving person, $U'' > 0$, so $R_a(w) < 0$.*
- *For a risk-neutral person, $U'' = 0$, so $R_a(w) = 0$.*

This measure is useful because it is invariant to positive linear transformations of the utility function.

Index