# Microeconomics

***A notes of Intermediate Microeconomics course***

First Edition

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*"Economics is not a set of answers,
but a way of thinking."*

--- Paul Samuelson

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# Preface

Each price a whisper, every choice a vow,
The hand unseen adjusts the trembling scale;
From chaos born, an equilibrium now,
So fragile that one breath might tip the veil.
Yet reason builds its temples out of thought,
Not marble, but theorems shaped by light;
Each symbol holds the battles reason fought,
Each graph a dawn that conquered human night.
Thus study, patient soul, this art of cause and will—
Where freedom bends, yet harmony lies still.

This book of notes is primarily based on Hal R. Varian's *Intermediate Microeconomics*, and was compiled during distinguished Professor **Zhang Haiyang**'s course at the University of International Business and Economics. The author's understanding is limited, and omissions or errors are entirely his own responsibility.

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# Chapter 1

# The Market

## 1.1 Introduction

In microeconomics, we study the behavior of individual economic agents—primarily consumers and firms—and how they interact in markets. We typically build simplified models to understand complex economic phenomena. A key principle in our modeling is the **optimization principle**, which states that people try to choose the best patterns of consumption that they can afford. Another is the **equilibrium principle**, where prices adjust until the amount that people demand of something is equal to the amount that is supplied.

In this introductory chapter, we will look at a fundamental concept for evaluating economic outcomes: efficiency. This will provide a criterion to judge how well an economic system performs.

## 1.2 Pareto Efficiency

When we want to evaluate the desirability of different economic allocations, we need a standard. One of the most widely used concepts is named after the Italian economist Vilfredo Pareto (1848–1923).

**Definition 1.1** (Pareto Improvement and Efficiency)**.**

- *A **Pareto improvement** is a change to a different allocation that makes at least one individual better off without making any other individual worse off.*
- *If an allocation allows for a Pareto improvement, it is called **Pareto inefficient**.*
- *If an allocation is such that no Pareto improvements are possible, it is called **Pareto efficient**.*

A Pareto efficient outcome can be thought of as one with "no wasted welfare". That is, in a Pareto efficient allocation, the only way to improve one person's welfare is to lower another person's welfare.

*Remark* 1.1.

- A Pareto inefficient outcome implies that there are still unrealized mutual gains-to-trade. It is possible to rearrange the allocation of goods to make someone happier without hurting anyone else.
- Any market outcome that achieves all possible gains-to-trade must be Pareto efficient. The concept of Pareto efficiency is a cornerstone of welfare economics.

# Chapter 2

# Budget Constraint

## 2.1 Consumption Choice Sets

Consumers choose to consume bundles of goods and services. A **consumption bundle** is a vector of quantities of each good, e.g., $(x_1, x_2, \ldots, x_n)$, where $x_i$ is the quantity of good $i$. The **consumption choice set** is the collection of all consumption bundles available to the consumer.

The choices a consumer can make are constrained by various factors, with the most prominent being the budget constraint. Other constraints can include time, resource limitations, and legal restrictions. For now, we will focus on the budgetary constraint.

## 2.2 The Budget Constraint

Let's consider a consumer with a disposable income of $m$ who can consume $n$ goods. The prices of these goods are given by the price vector $(p_1, p_2, \ldots, p_n)$.

**Definition 2.1** (Affordable Consumption Bundles)**.** *A consumption bundle $(x_1, \ldots, x_n)$ is affordable if its total cost is no more than the consumer's income. That is:*

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq m \tag{2.1}$$

*This inequality is the consumer's **budget constraint**.*

**Definition 2.2** (Budget Set)**.** *The **budget set** is the set of all affordable consumption bundles. Assuming non-negative consumption, the budget set $B$ is:*

$$B(p_1, \ldots, p_n, m) = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \textit{ for all } i, \textit{ and } \sum_{i=1}^{n} p_i x_i \leq m\} \tag{2.2}$$

*The upper boundary of the budget set, where the consumer spends all their income, is called the **budget line**:*

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = m \tag{2.3}$$

### 2.2.1 The Two-Good Case

For simplicity, we often analyze the case with only two goods. The budget line is then given by $p_1x_1 + p_2x_2 = m$. We can rearrange this to express $x_2$ as a function of $x_1$:

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{m}{p_2} \tag{2.4}$$

This is a linear equation.

- The vertical intercept is $m/p_2$, which is the maximum amount of good 2 the consumer can buy.
- The horizontal intercept is $m/p_1$, which is the maximum amount of good 1 the consumer can buy.
- The slope is $-p_1/p_2$. The slope measures the rate at which the market is willing to "substitute" good 2 for good 1. It is the **opportunity cost** of consuming good 1: to consume one more unit of good 1, the consumer must give up $p_1/p_2$ units of good 2.

*Remark* 2.1 (Graphical Representation). The budget set for two goods is a right-angled triangle in the $(x_1, x_2)$ plane, with vertices at $(0, 0)$, $(m/p_1, 0)$, and $(0, m/p_2)$. The hypotenuse of this triangle is the budget line. Bundles inside the triangle are affordable but do not exhaust income. Bundles on the budget line are just affordable. Bundles outside the triangle are unaffordable.

## 2.3 Changes in the Budget Line

The budget set depends on prices and income. When these parameters change, the set of affordable choices changes as well.

### 2.3.1 Income Changes

An increase in income $m$ to $m' > m$ leads to a new budget line:

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{m'}{p_2}$$

The slope $(-p_1/p_2)$ remains unchanged, but the vertical intercept increases. This means the budget line makes a **parallel shift outwards**. An income increase expands the budget set, meaning no original choices are lost and new choices are added. Thus, a higher income cannot make a consumer worse off. Conversely, a decrease in income shifts the budget line inward, shrinking the choice set.

### 2.3.2 Price Changes

Suppose the price of good 1 decreases from $p_1$ to $p_1' < p_1$. The new budget line is:

$$x_2 = -\frac{p_1'}{p_2}x_1 + \frac{m}{p_2}$$

The vertical intercept $(m/p_2)$ is unchanged. The horizontal intercept increases to $m/p_1'$. The slope becomes flatter (less negative). The budget line **pivots outward** around the vertical intercept. Reducing the price of one commodity also expands the budget set, and thus cannot make the consumer worse off.

## 2.4 Taxes, Subsidies, and Rationing

### 2.4.1 Uniform Ad Valorem Sales Tax

An *ad valorem* (from the value) sales tax is levied as a percentage of the price. If a uniform tax rate $t$ is applied to all goods, the new prices become $p_1(1+t)$ and $p_2(1+t)$. The budget constraint becomes:

$$p_1(1+t)x_1 + p_2(1+t)x_2 = m$$

This can be rewritten as:

$$p_1x_1 + p_2x_2 = \frac{m}{1+t}$$

A uniform sales tax is thus equivalent to a decrease in income from $m$ to $m/(1+t)$. This causes a parallel inward shift of the budget line.

**The Food Stamp Program**

Food stamps are coupons that can be legally exchanged only for food. Let $F$ be food and $G$ be all other goods. Assume $p_F = p_G = \$1$ and income is $m = \$100$. The initial budget line is $F + G = 100$.
Suppose the consumer receives $40 worth of food stamps.

- The consumer can now consume up to $140 of food if they spend all their income on it.
- However, the food stamps cannot be used for other goods, so the maximum amount of other goods is still $100.
- The budget constraint becomes kinked. It is $F + G = 100$ for $F > 40$, but for $F \leq 40$, the consumer can still spend their full $100 on other goods. The new budget set is larger. The budget line is:

$$G = \begin{cases} 100 & \text{if } 0 \leq F \leq 40 \\ 140 - F & \text{if } 40 < F \leq 140 \end{cases}$$

What if the food stamps can be traded on a black market, e.g., for \$0.50 each?

- The \$40 in food stamps could be sold for $40 \times \$0.50 = \$20$.
- This effectively increases the consumer's cash income to $\$100 + \$20 = \$120$.
- The budget line would then be $F + G = 120$, further expanding the budget set.

## 2.5 Shapes of Budget Constraints

If prices are not constant, the budget constraint may not be a straight line.

- **Quantity Discounts:** Suppose $p_1 = \$2$ for the first 20 units and $p_1 = \$1$ for any unit thereafter, with $p_2 = \$1$ and $m = \$100$. The budget line will have a slope of $-2$ for $x_1 \leq 20$ and a slope of $-1$ for $x_1 > 20$. The budget line becomes flatter after $x_1 = 20$, creating a kink.
- **Quantity Penalties (Taxes):** The opposite occurs, and the budget line becomes steeper after a certain quantity.
- **One Price Negative:** If good 1 is a "bad" like garbage, you might be paid to accept it, i.e., $p_1 < 0$. If $p_1 = -\$2$, $p_2 = \$1$ and $m = \$10$, the budget line is $-2x_1 + x_2 = 10$, or $x_2 = 2x_1 + 10$. The slope is positive, and the budget set is unbounded on the $x_1$ side.

## 2.6 More General Choice Sets

Choices are often constrained by more than just a budget. For example, there might be a time constraint, or a minimum consumption requirement for survival. A bundle is available only if it meets *every* constraint. The final choice set is the **intersection** of all the individual constraint sets.

# Chapter 3

# Preferences

After establishing what a consumer can afford (the budget set), we now turn to modeling what they want to consume. We do this using the concept of preferences.

## 3.1 Consumer Preferences

We will consider two consumption bundles, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. The consumer can rank these bundles according to their desirability. There are three possibilities:

- **Strict Preference:** The consumer strictly prefers bundle $\mathbf{x}$ to bundle $\mathbf{y}$. We write this as $\mathbf{x} \succ \mathbf{y}$.
- **Indifference:** The consumer is exactly as satisfied with bundle $\mathbf{x}$ as with bundle $\mathbf{y}$. We write this as $\mathbf{x} \sim \mathbf{y}$.
- **Weak Preference:** The consumer prefers or is indifferent between bundle $\mathbf{x}$ and bundle $\mathbf{y}$. We write this as $\mathbf{x} \succeq \mathbf{y}$.

These relations are linked. For example, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{x}$, then we can conclude that $\mathbf{x} \sim \mathbf{y}$. If $\mathbf{x} \succeq \mathbf{y}$ but it is *not* the case that $\mathbf{y} \succeq \mathbf{x}$, then we can conclude $\mathbf{x} \succ \mathbf{y}$.

## 3.2 Assumptions about Preferences

To have a sensible theory of consumer choice, we need to impose some assumptions on preferences. These are often called axioms of rational choice.

- **Completeness:** For any two bundles $\mathbf{x}$ and $\mathbf{y}$, the consumer can make a comparison. That is, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ (or both).
- **Reflexivity:** Any bundle $\mathbf{x}$ is at least as good as itself: $\mathbf{x} \succeq \mathbf{x}$. This is a trivial assumption.

- **Transitivity:** If a consumer thinks that $\mathbf{x}$ is at least as good as $\mathbf{y}$, and that $\mathbf{y}$ is at least as good as $\mathbf{z}$, then they must think that $\mathbf{x}$ is at least as good as $\mathbf{z}$. Formally: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$.

A consumer with preferences that satisfy these three axioms is said to be **rational**.

## 3.3 Indifference Curves

Preferences can be represented graphically using **indifference curves**. An indifference curve is a set of all consumption bundles among which a consumer is indifferent.

- The **weakly preferred set** for a bundle $\mathbf{x}'$ is the set of all bundles $\mathbf{y}$ such that $\mathbf{y} \succeq \mathbf{x}'$.
- The **strictly preferred set** for $\mathbf{x}'$ is the set of all bundles $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{x}'$.

A key property, derived from transitivity and the "more is better" assumption (monotonicity, see below), is that **indifference curves cannot intersect**. If they did, a point on the intersection would be indifferent to bundles on both curves. By transitivity, all bundles on both curves would have to be indifferent to each other, which contradicts the idea that one curve represents a higher level of preference than the other.

## 3.4 Examples of Preferences

- **Perfect Substitutes:** If a consumer always regards units of commodities 1 and 2 as equivalent (e.g., at a one-to-one ratio), the commodities are perfect substitutes. The consumer's preference depends only on the total sum of the goods. The indifference curves are parallel straight lines.
- **Perfect Complements:** If a consumer always consumes commodities 1 and 2 in a fixed proportion (e.g., one-to-one, like left shoes and right shoes), the commodities are perfect complements. The indifference curves are L-shaped.
- **Bads:** If less of a commodity is always preferred, it is a **bad**. If good 1 is a bad and good 2 is a good, the indifference curves will be positively sloped.
- **Satiation:** A consumer may have a most preferred bundle, called a **satiation point** or **bliss point**. Bundles further away from this point are less preferred. The indifference curves are circles or ellipses centered on the bliss point.

## 3.5 Well-Behaved Preferences

We often make two further assumptions about preferences, which lead to "well-behaved" indifference curves.

**Definition 3.1** (Monotonicity)**.** *Preferences are **monotonic** if more of any commodity is always preferred to less (holding the other commodity constant). This implies that commodities are **goods**, not bads, and that there is no satiation. Monotonicity ensures that indifference curves are negatively sloped.*

**Definition 3.2** (Convexity)**.** *Preferences are **convex** if mixtures of bundles are (at least weakly) preferred to the bundles themselves. If $\mathbf{x} \sim \mathbf{y}$, then for any $t \in (0,1)$, the mixture bundle $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ is at least as good as $\mathbf{x}$ or $\mathbf{y}$ (i.e., $\mathbf{z} \succeq \mathbf{x}$).*

- ***Strict convexity** holds if the mixture is always strictly preferred ($\mathbf{z} \succ \mathbf{x}$).*
- *Convexity implies that consumers prefer balanced bundles over extreme bundles. Graphically, the set of bundles weakly preferred to $\mathbf{x}$ is a convex set.*

## 3.6 The Marginal Rate of Substitution (MRS)

The slope of an indifference curve at a particular point is called the **marginal rate-of-substitution (MRS)**.

**Definition 3.3** (MRS)**.** *The MRS measures the rate at which the consumer is just willing to substitute a small amount of good 2 for good 1, while remaining on the same indifference curve. Mathematically, it is the derivative of the indifference curve:*

$$MRS = \frac{dx_2}{dx_1}$$

- For monotonic preferences (goods), the MRS is negative, as the consumer must give up some of one good to get more of the other.
- For strictly convex preferences, the MRS becomes less negative as $x_1$ increases. This is known as a **diminishing marginal rate of substitution**. The indifference curve becomes flatter as we move to the right. This means the consumer is willing to give up less of good 2 to get an additional unit of good 1 as they consume more and more of good 1.

# Chapter 4

# Utility

In the previous chapter, we used indifference curves to describe preferences. While graphical, this can be cumbersome. A more convenient way to describe preferences is through a **utility function**.

## 4.1 Utility Functions

A utility function is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers than less-preferred bundles.

**Definition 4.1** (Utility Function)**.** *A function $U(\mathbf{x})$ represents a preference relation $\succeq$ if for any two bundles $\mathbf{x}'$ and $\mathbf{x}''$:*

- $\mathbf{x}' \succ \mathbf{x}'' \iff U(\mathbf{x}') > U(\mathbf{x}'')$
- $\mathbf{x}' \prec \mathbf{x}'' \iff U(\mathbf{x}') < U(\mathbf{x}'')$
- $\mathbf{x}' \sim \mathbf{x}'' \iff U(\mathbf{x}') = U(\mathbf{x}'')$

*A preference relation can be represented by a continuous utility function if it is complete, reflexive, transitive, and continuous.*

An indifference curve consists of all bundles that have the same level of utility. The equation for an indifference curve is therefore $U(x_1, x_2) = k$ for some constant $k$.

### 4.1.1 Ordinal Utility

Utility is an **ordinal** concept. This means that the magnitude of the utility function is only important for ranking bundles. The absolute values, or the differences between them, do not have a specific meaning. If $U(\mathbf{x}) = 6$ and $U(\mathbf{y}) = 2$, we know $\mathbf{x}$ is preferred to $\mathbf{y}$, but we cannot say that $\mathbf{x}$ is "three times as good" as $\mathbf{y}$.

### 4.1.2 Monotonic Transformations

Because utility is ordinal, any strictly increasing transformation of a utility function will represent the same preferences. If $U(x_1, x_2)$ is a utility function representing some preferences, and $f$ is a strictly increasing function (i.e., $f'(z) > 0$), then $V(x_1, x_2) = f(U(x_1, x_2))$ is a new utility function that represents the exact same preferences.

For example, if $U = x_1x_2$, then $V = (x_1x_2)^2 = x_1^2x_2^2$ and $W = \ln(x_1x_2) = \ln(x_1)+\ln(x_2)$ both represent the same preferences as $U$, because squaring (for positive numbers) and taking the natural log are both strictly increasing transformations.

## 4.2 Examples of Utility Functions

- **Perfect Substitutes:** Preferences can be represented by a linear utility function of the form $U(x_1, x_2) = ax_1 + bx_2$. The indifference curves are lines with slope $-a/b$.

- **Perfect Complements:** Preferences for goods consumed in a fixed proportion (e.g., one-to-one) can be represented by a function like $U(x_1, x_2) = \min\{ax_1, bx_2\}$.

- **Quasi-linear Preferences:** A utility function that is linear in one good, e.g., $U(x_1, x_2) = v(x_1) + x_2$. The indifference curves are vertically shifted copies of each other.

- **Cobb-Douglas Preferences:** A commonly used functional form is the Cobb-Douglas utility function, $U(x_1, x_2) = x_1^a x_2^b$, where $a, b > 0$. These preferences exhibit well-behaved, convex indifference curves.

## 4.3 Marginal Utility and MRS

**Definition 4.2** (Marginal Utility)**.** *The* ***marginal utility*** *(MU) of a good $i$ is the rate of change in total utility from consuming an infinitesimally small additional amount of good $i$, holding all other goods constant. It is the partial derivative of the utility function with respect to $x_i$:*

$$MU_i = \frac{\partial U}{\partial x_i}$$

The magnitude of marginal utility depends on the specific utility function chosen and is not meaningful on its own (due to the ordinal nature of utility). However, it is useful for calculating the MRS.

### 4.3.1 Deriving MRS from Utility

Consider a small change in consumption $(dx_1, dx_2)$ that keeps the consumer on the same indifference curve. The total change in utility, $dU$, must be zero. The total differential of the utility function is:

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

$$MU_1 dx_1 + MU_2 dx_2 = 0$$

Rearranging this equation gives us the slope of the indifference curve:

$$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -\frac{MU_1}{MU_2} \tag{4.1}$$

The MRS is the ratio of the marginal utilities. This ratio is independent of any monotonic transformation of the utility function, which is a desirable property since the MRS has a real economic meaning (a psychological rate of trade-off) while the MU values do not.

# Chapter 5

# Choice

## 5.1 Introduction

In the previous chapters, we modeled the consumer's constraints (the budget set) and their preferences (indifference curves and utility functions). We now put these two pieces together to analyze the consumer's choice. The fundamental assumption is that the consumer will choose the most preferred bundle from their budget set.

This is a problem of **constrained maximization**. Mathematically, the consumer's problem is to:

$$\begin{aligned} \max_{x_1, x_2} \quad & U(x_1, x_2) \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \leq m \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

In economics, this is known as the rational choice problem. The solution to this problem, the optimal consumption bundle $(x_1^*, x_2^*)$, is the consumer's **demanded bundle**.

**Definition 5.1** (Marshallian Demand)**.** *The consumer's optimal choice $(x_1^*, x_2^*)$ at a given set of prices $(p_1, p_2)$ and income $m$ is called the consumer's demanded bundle. The functions that give the optimal amount of each good as a function of prices and income, $x_1^*(p_1, p_2, m)$ and $x_2^*(p_1, p_2, m)$, are called the* ***Marshallian demand functions****.*

## 5.2 Finding the Optimal Choice

Graphically, the optimal bundle is the point in the budget set that lies on the highest possible indifference curve.

### 5.2.1 Interior Solutions with Well-Behaved Preferences

For well-behaved preferences (monotonic and strictly convex), the optimal choice will typically be an **interior solution**, where the consumer purchases positive amounts of both goods ($x_1^* > 0$ and $x_2^* > 0$). In this case, the optimal bundle $(x_1^*, x_2^*)$ is characterized by two conditions:

1. **The budget is exhausted.** The optimal point must lie on the budget line, not inside it. Because preferences are monotonic (more is better), any bundle inside the budget set has a more-preferred bundle to its northeast that is also affordable.

$$p_1 x_1^* + p_2 x_2^* = m$$

2. **Tangency condition.** At the optimal point, the indifference curve is tangent to the budget line. This means their slopes are equal.

$$\text{Slope of Indifference Curve} = \text{Slope of Budget Line}$$

$$MRS = -\frac{p_1}{p_2}$$

Since we know $MRS = -MU_1/MU_2$, the tangency condition can be rewritten as:

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

This second condition has a powerful economic intuition: at the optimal choice, the rate at which the consumer is *willing* to trade one good for another (the MRS) is equal to the rate at which the market *allows* them to trade (the price ratio).

**Computing Demand with Cobb-Douglas Preferences**

Let the consumer's utility function be of the Cobb-Douglas form:

$$U(x_1, x_2) = x_1^a x_2^b$$

First, we find the marginal utilities and the MRS:

$$MU_1 = \frac{\partial U}{\partial x_1} = a x_1^{a-1} x_2^b$$

$$MU_2 = \frac{\partial U}{\partial x_2} = b x_1^a x_2^{b-1}$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}$$

Now we apply the two conditions for an interior optimum:

1. **Tangency:** $MRS = -p_1/p_2$

$$-\frac{a x_2^*}{b x_1^*} = -\frac{p_1}{p_2} \implies \frac{a x_2^*}{b x_1^*} = \frac{p_1}{p_2} \implies p_2 x_2^* = \frac{b}{a} p_1 x_1^*$$

2. **Budget Exhaustion:** $p_1 x_1^* + p_2 x_2^* = m$

Substitute the result from the tangency condition into the budget constraint:

$$p_1 x_1^* + \left(\frac{b}{a} p_1 x_1^*\right) = m$$

$$p_1 x_1^* \left(1 + \frac{b}{a}\right) = m \implies p_1 x_1^* \left(\frac{a+b}{a}\right) = m$$

Solving for $x_1^*$ gives the demand function for good 1:

$$x_1^*(p_1, p_2, m) = \frac{a}{a+b} \frac{m}{p_1}$$

Substituting this back into the expression for $p_2 x_2^*$ from the tangency condition gives the demand for good 2:

$$p_2 x_2^* = \frac{b}{a} p_1 \left(\frac{a}{a+b} \frac{m}{p_1}\right) = \frac{b}{a+b} m \implies x_2^*(p_1, p_2, m) = \frac{b}{a+b} \frac{m}{p_2}$$

These are the Marshallian demand functions for a consumer with Cobb-Douglas preferences. They show that the consumer spends a fixed fraction of income ($a/(a+b)$ on good 1 and $b/(a+b)$ on good 2).

## 5.3 Non-Tangency Solutions

The tangency condition only holds for interior solutions with smooth indifference curves. In other cases, the optimal choice may occur where the tangency condition is not met.

### 5.3.1 Corner Solutions

A **corner solution** occurs when the optimal quantity of one of the goods is zero. This often happens with perfect substitutes or non-convex preferences.

**Optimal Choice with Perfect Substitutes**

Consider the utility function $U(x_1, x_2) = x_1 + x_2$. Here, the MRS is constant at $-1$. The indifference curves are straight lines with a slope of $-1$. The consumer will compare their personal trade-off rate (1-for-1) with the market's trade-off rate ($p_1/p_2$).

- If $p_1 < p_2$, the slope of the budget line ($-p_1/p_2$) is flatter than the slope of the indifference curves ($-1$). The consumer gets more utility per dollar from good 1. The optimal choice is to spend all income on good 1. This is a corner solution at $(m/p_1, 0)$.
- If $p_1 > p_2$, the budget line is steeper than the indifference curves. The

consumer gets more utility per dollar from good 2 and will spend all income on it. The corner solution is at $(0, m/p_2)$.

- If $p_1 = p_2$, the budget line and indifference curves have the same slope. Any affordable bundle on the budget line is an optimal choice.

So, the demand for good 1 is:

$$x_1^*(p_1, p_2, m) = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any amount in } [0, m/p_1] & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

### 5.3.2 Kinky Solutions

If indifference curves have kinks, such as with perfect complements, the MRS is not well-defined at the optimal point, and the tangency condition cannot be used.

**Optimal Choice with Perfect Complements**

Consider the utility function $U(x_1, x_2) = \min\{x_1, x_2\}$. The consumer always wants to consume the goods in a one-to-one ratio. The optimal choice will always be at the corner of an L-shaped indifference curve, where $x_1 = x_2$.

1. **Consumption in fixed proportion:** $x_1^* = x_2^*$
2. **Budget Exhaustion:** $p_1 x_1^* + p_2 x_2^* = m$

Substituting the first condition into the second gives:

$$p_1 x_1^* + p_2 x_1^* = m \implies (p_1 + p_2) x_1^* = m$$

Solving gives the demand functions:

$$x_1^*(p_1, p_2, m) = x_2^*(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

# Chapter 6

# Demand

## 6.1 Introduction

The demand functions derived in the previous chapter, $x_i^*(p_1, p_2, m)$, tell us the optimal quantities of each good as a function of prices and income. In this chapter, we explore how the quantity demanded changes as these variables change. This is known as **comparative statics**.

## 6.2 Own-Price Changes

We first analyze how the demand for a good, say good 1, changes as its own price, $p_1$, changes, while holding $p_2$ and income $m$ constant.

**Definition 6.1** (Price Offer Curve and Demand Curve)**.**

- *The set of optimal bundles traced out as $p_1$ changes is called the $p_1$-**price offer curve** or simply the **price offer curve**.*
- *A plot of the optimal quantity demanded of good 1, $x_1^*$, against its own price, $p_1$, is the **demand curve** for good 1.*

**Definition 6.2** (Ordinary and Giffen Goods)**.**

- *A good is called an **ordinary good** if the quantity demanded for it always increases as its own price decreases. Ordinary goods have downward-sloping demand curves.*
- *A good is called a **Giffen good** if, for some range of prices, the quantity demanded rises as its own-price increases. Giffen goods have a segment of their demand curve that is upward-sloping. They are a theoretical curiosity and are rarely observed in reality.*

### 6.2.1 The Inverse Demand Function

The standard demand curve plots quantity as a function of price, $x_1 = x_1^*(p_1)$. Sometimes it is useful to ask the inverse question: at what price would a given quantity be demanded? This gives the **inverse demand function**, which plots price as a function of quantity, $p_1 = p_1(x_1)$.

**Cobb-Douglas Inverse Demand**

The ordinary demand function for good 1 is $x_1^* = \frac{a}{a+b}\frac{m}{p_1}$. To find the inverse demand function, we solve for $p_1$:

$$p_1(x_1^*) = \frac{a}{a+b}\frac{m}{x_1^*}$$

## 6.3 Income Changes

We now analyze how the demand for a good changes as income $m$ changes, holding prices $(p_1, p_2)$ constant.

**Definition 6.3** (Income Offer Curve and Engel Curve)**.**

- *The set of optimal bundles traced out as income $m$ changes is called the **income offer curve** or **income expansion path**.*
- *A plot of the quantity demanded of a good against income is called an **Engel curve**.*

**Definition 6.4** (Normal and Inferior Goods)**.**

- *A good is a **normal good** if the quantity demanded rises with income ($\partial x_i^*/\partial m > 0$). A normal good has a positively sloped Engel curve.*
- *A good is an **inferior good** if the quantity demanded falls as income increases ($\partial x_i^*/\partial m < 0$). An inferior good has a negatively sloped Engel curve.*

### 6.3.1 Homothetic Preferences

An important class of preferences are those that are **homothetic**. For these preferences, the MRS depends only on the ratio of the goods, not on the scale.

**Proposition 6.1.** *If a consumer has homothetic preferences, then their income offer curve is a straight line through the origin, and their Engel curves are straight lines.*

This implies that the consumer will always spend a fixed proportion of their income on each good. Cobb-Douglas, perfect substitutes, and perfect complements are all examples of homothetic preferences.

### 6.3.2 Non-homothetic Preferences: Quasi-linear Utility

Quasi-linear preferences of the form $U(x_1, x_2) = v(x_1) + x_2$ are a key example of non-homothetic preferences. For these preferences, an increase in income (once it is sufficiently high) does not change the demand for good 1 at all; all extra income is spent on good 2. This results in a vertical income offer curve and a vertical Engel curve for good 1 above a certain income level.

## 6.4 Cross-Price Effects

Finally, we consider how the demand for good 1 changes when the price of *another* good, $p_2$, changes.

**Definition 6.5** (Gross Substitutes and Complements)**.**

- *Good 1 is a* ***gross substitute*** *for good 2 if an increase in* $p_2$ *increases the demand for good 1 (*$\partial x_1^*/\partial p_2 > 0$*).*
- *Good 1 is a* ***gross complement*** *for good 2 if an increase in* $p_2$ *reduces the demand for good 1 (*$\partial x_1^*/\partial p_2 < 0$*).*

**Cross-Price Effects Examples**

- **Perfect Complements:** $x_1^* = m/(p_1 + p_2)$. Taking the derivative:

$$\frac{\partial x_1^*}{\partial p_2} = -\frac{m}{(p_1 + p_2)^2} < 0$$

  As expected, the goods are gross complements.

- **Cobb-Douglas:** $x_1^* = \frac{a}{a+b}\frac{m}{p_1}$. Taking the derivative:

$$\frac{\partial x_1^*}{\partial p_2} = 0$$

  In the Cobb-Douglas case, the demand for one good is independent of the other good's price. The goods are neither gross substitutes nor gross complements.

# Chapter 7

# Revealed Preference

In previous chapters, we started with a consumer's preferences (or utility function) and used this information to derive their optimal choices. This chapter reverses the process. We ask: can we start by observing a consumer's choices at different prices and income levels, and from these observations, deduce their underlying preferences? This is the central question of revealed preference theory.

## 7.1 The Concept of Revealed Preference

The basic idea is simple yet powerful: a consumer's choice *reveals* information about their preferences. If a consumer chooses a bundle of goods $x$ when another bundle $y$ was also affordable, we can infer that the consumer prefers $x$ to $y$.

### 7.1.1 Maintained Assumptions

To make this inference robust, we rely on a few standard assumptions about consumer behavior:

- **Stable Preferences:** The consumer's preferences do not change during the period of observation.
- **Well-Behaved Preferences:** We assume preferences are monotonic (more is better) and strictly convex. A key implication of strict convexity is that for any given budget, there is a *unique* most-preferred affordable bundle.
- **Rationality:** The consumer always chooses the most preferred bundle they can afford.

### 7.1.2 Direct and Indirect Revelation

**Definition 7.1** (Directly Revealed Preferred). *Let* $\mathbf{x} = (x_1, x_2)$ *be the bundle chosen at prices* $\mathbf{p} = (p_1, p_2)$*, and let* $\mathbf{y} = (y_1, y_2)$ *be another bundle. If* $\mathbf{y}$ *was affordable at* *prices* $\mathbf{p}$ *when* $\mathbf{x}$ *was chosen, i.e.,* $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$*, and* $\mathbf{x} \neq \mathbf{y}$*, then we say that* $\mathbf{x}$ ***is directly revealed preferred to*** $\mathbf{y}$*. We denote this by* $\mathbf{x} \succ_D \mathbf{y}$*.*

The logic is that since the consumer could have chosen $\mathbf{y}$ but instead chose $\mathbf{x}$, they must prefer $\mathbf{x}$.

We can extend this idea through transitivity.

**Definition 7.2** (Indirectly Revealed Preferred)**.** *If we have a chain of direct revelations, such as* $\mathbf{x} \succ_D \mathbf{y}$ *and* $\mathbf{y} \succ_D \mathbf{z}$*, then we say that* $\mathbf{x}$ ***is indirectly revealed preferred to*** $\mathbf{z}$*. We denote this by* $\mathbf{x} \succ_I \mathbf{z}$*.*

This allows us to compare bundles that were not available in the same budget set, by linking them through intermediate choices.

## 7.2 Axioms of Revealed Preference

For observed choices to be consistent with our model of a rational, utility-maximizing consumer, they must satisfy certain consistency conditions. These are known as the axioms of revealed preference.

### 7.2.1 The Weak Axiom of Revealed Preference (WARP)

WARP is the most basic consistency check. It ensures that if a consumer reveals a preference for one bundle over another, they don't subsequently reveal the opposite preference.

**Definition 7.3** (Weak Axiom of Revealed Preference (WARP))**.** *If a bundle* $\mathbf{x}$ *is directly revealed preferred to a bundle* $\mathbf{y}$ *(*$\mathbf{x} \succ_D \mathbf{y}$*), then it can never be the case that* $\mathbf{y}$ *is directly revealed preferred to* $\mathbf{x}$ *(*$\mathbf{y} \succ_D \mathbf{x}$*).*

$$\textit{If } \mathbf{x} \succ_D \mathbf{y}, \textit{ then it cannot be that } \mathbf{y} \succ_D \mathbf{x}.$$

Choice data that violate WARP are inconsistent with the model of economic rationality. WARP is a *necessary* condition for choices to be rationalized by a utility function.

**Checking for WARP Violations**

A consumer makes the following choices:

- At prices $\mathbf{p}^A = (\$2, \$2)$, the choice is $\mathbf{x}^A = (10, 1)$.
- At prices $\mathbf{p}^B = (\$2, \$1)$, the choice is $\mathbf{x}^B = (5, 5)$.
- At prices $\mathbf{p}^C = (\$1, \$2)$, the choice is $\mathbf{x}^C = (5, 4)$.

To check for WARP violations, we calculate the cost of each bundle at each set of prices. The chosen bundle's cost represents the consumer's income in that situation.

| **Prices** | **Cost of $\mathbf{x}^A$** | **Cost of $\mathbf{x}^B$** | **Cost of $\mathbf{x}^C$** |
|---|---|---|---|
| $\mathbf{p}^A = (\$2, \$2)$ | **$22** | $20 | $18 |
| $\mathbf{p}^B = (\$2, \$1)$ | $21 | **$15** | $14 |
| $\mathbf{p}^C = (\$1, \$2)$ | $12 | $15 | **$13** |

Let's analyze the choices:

- At prices $\mathbf{p}^A$, income is $22. Both $\mathbf{x}^B$ ($20) and $\mathbf{x}^C$ ($18) were affordable. Thus, $\mathbf{x}^A \succ_D \mathbf{x}^B$ and $\mathbf{x}^A \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^B$, income is $15. $\mathbf{x}^C$ ($14) was affordable. Thus, $\mathbf{x}^B \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^C$, income is $13. $\mathbf{x}^A$ ($12) was affordable. Thus, $\mathbf{x}^C \succ_D \mathbf{x}^A$.

We have found that $\mathbf{x}^A \succ_D \mathbf{x}^C$ and $\mathbf{x}^C \succ_D \mathbf{x}^A$. This is a direct violation of WARP. These data are not consistent with a rational consumer.

### 7.2.2 The Strong Axiom of Revealed Preference (SARP)

WARP is not sufficient to guarantee that choices can be described by a well-behaved utility function. We need a stronger condition that also rules out cycles in *indirect* revealed preferences.

**Definition 7.4** (Strong Axiom of Revealed Preference (SARP))**.** *If a bundle* $\mathbf{x}$ *is revealed preferred (directly or indirectly) to a bundle* $\mathbf{y}$ *(*$\mathbf{x} \succ_D \mathbf{y}$ *or* $\mathbf{x} \succ_I \mathbf{y}$*), and* $\mathbf{x} \neq \mathbf{y}$*, then it can never be the case that* $\mathbf{y}$ *is revealed preferred (directly or indirectly) to* $\mathbf{x}$.

SARP is the key condition. It has been proven that if a finite set of observed choices satisfies SARP, then there exists a well-behaved (monotonic, convex) preference relation that "rationalizes" these choices. Thus, SARP is both a *necessary and sufficient* condition for the data to be consistent with the standard economic model of consumer choice.

**A SARP Violation (with no WARP violation)**

Consider the following data:

- Prices $\mathbf{p}^A = (1, 3, 10)$, Choice $\mathbf{x}^A = (3, 1, 4)$.
- Prices $\mathbf{p}^B = (4, 3, 6)$, Choice $\mathbf{x}^B = (2, 5, 3)$.
- Prices $\mathbf{p}^C = (1, 1, 5)$, Choice $\mathbf{x}^C = (4, 4, 3)$.

| **Prices** | **Cost of $\mathbf{x}^A$** | **Cost of $\mathbf{x}^B$** | **Cost of $\mathbf{x}^C$** |
|---|---|---|---|
| $\mathbf{p}^A = (1, 3, 10)$ | **$46** | $47 | $46 |
| $\mathbf{p}^B = (4, 3, 6)$ | $39 | **$41** | $46 |
| $\mathbf{p}^C = (1, 1, 5)$ | $24 | $22 | **$23** |

Let's analyze the direct revelations:

- At prices $\mathbf{p}^A$, $\mathbf{x}^C$ was affordable ($\$46 \leq \$46$). So, $\mathbf{x}^A \succ_D \mathbf{x}^C$.
- At prices $\mathbf{p}^B$, $\mathbf{x}^A$ was affordable ($\$39 \leq \$41$). So, $\mathbf{x}^B \succ_D \mathbf{x}^A$.
- At prices $\mathbf{p}^C$, $\mathbf{x}^B$ was affordable ($\$22 \leq \$23$). So, $\mathbf{x}^C \succ_D \mathbf{x}^B$.

This dataset does not violate WARP (check: no pairs like $X \succ_D Y$ and $Y \succ_D X$). However, let's look at the indirect revelations. We have a cycle:

$$\mathbf{x}^B \succ_D \mathbf{x}^A \quad \text{and} \quad \mathbf{x}^A \succ_D \mathbf{x}^C \implies \mathbf{x}^B \succ_I \mathbf{x}^C$$

But we also found that $\mathbf{x}^C \succ_D \mathbf{x}^B$. This is a violation of SARP. These choices cannot be rationalized by a well-behaved preference relation.

## 7.3 Applications of Revealed Preference

The theory of revealed preference provides the foundation for practical economic tools, such as index numbers, which are used to measure changes in welfare and the cost of living.

### 7.3.1 Index Numbers and Welfare

Index numbers compare expenditures in a base period (b) and a current period (t). Let $\mathbf{p}^b, \mathbf{x}^b$ be the prices and chosen bundle in the base period, and $\mathbf{p}^t, \mathbf{x}^t$ be for the current period.

#### Quantity Indices

A quantity index measures the change in consumption levels, holding prices constant.

- The **Laspeyres Quantity Index** uses base period prices as weights:

$$L_q = \frac{\mathbf{p}^b \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^b} = \frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b}$$

  If $L_q < 1$, it means $\mathbf{p}^b \cdot \mathbf{x}^b > \mathbf{p}^b \cdot \mathbf{x}^t$. At base period prices, the consumer chose $\mathbf{x}^b$ when $\mathbf{x}^t$ was affordable. By revealed preference, the consumer was better off in the base period b.

- The **Paasche Quantity Index** uses current period prices as weights:

$$P_q = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^t \cdot \mathbf{x}^b} = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b}$$

  If $P_q > 1$, it means $\mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{x}^b$. At current period prices, the consumer chose $\mathbf{x}^t$ when $\mathbf{x}^b$ was affordable. The consumer is better off in the current period t.

**Price Indices**

A price index measures the change in prices, holding quantities constant.

- The **Laspeyres Price Index** uses the base period bundle as weights:

$$L_p = \frac{\mathbf{p}^t \cdot \mathbf{x}^b}{\mathbf{p}^b \cdot \mathbf{x}^b} = \frac{p_1^t x_1^b + p_2^t x_2^b}{p_1^b x_1^b + p_2^b x_2^b}$$

- The **Paasche Price Index** uses the current period bundle as weights:

$$P_p = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^t} = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^t + p_2^b x_2^t}$$

Let $M = \frac{\mathbf{p}^t \cdot \mathbf{x}^t}{\mathbf{p}^b \cdot \mathbf{x}^b}$ be the ratio of total expenditure. If $L_p < M$, it can be shown this implies $\mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{x}^b$, meaning the consumer is better off in the current period.

### 7.3.2 Application: Indexation

Price indices like the Consumer Price Index (CPI), which is a type of Laspeyres price index, are often used to adjust wages or benefits for inflation. This is called **indexation**. If an individual's income is adjusted by the full amount of the Laspeyres price index, they are typically made *strictly better off*.

Why? Full indexation gives the consumer enough income ($m' = \mathbf{p}^t \cdot \mathbf{x}^b$) to buy their *old* base-period bundle $\mathbf{x}^b$ at the *new* current-period prices $\mathbf{p}^t$. However, since relative prices have likely changed, the consumer can usually improve their welfare by substituting away from goods that have become relatively more expensive. They will choose a new bundle $\mathbf{x}^t$ on their new budget line. Since the old bundle $\mathbf{x}^b$ is still affordable, but a different bundle $\mathbf{x}^t$ is chosen, it must be that $\mathbf{x}^t$ is revealed preferred to $\mathbf{x}^b$. Thus, full indexation tends to overcompensate for price changes.

# Chapter 8

# Slutsky Equation

When the price of a good changes, the consumer's optimal choice is affected. This overall change in demand can be decomposed into two distinct effects. Understanding this decomposition is crucial for analyzing consumer behavior. The Slutsky equation provides a formal framework for this analysis.

## 8.1 Effects of a Price Change

What happens when a commodity's price decreases? The total change in the quantity demanded arises from two separate phenomena:

- **Substitution Effect:** The commodity becomes relatively cheaper compared to other goods. A rational consumer will substitute towards the cheaper good, away from the now relatively more expensive ones. This effect captures the change in demand due to the change in the rate of exchange between two goods.

- **Income Effect:** The consumer's purchasing power increases. With the same nominal income, say $m, the consumer can now afford to buy more goods than before. It is as if the consumer's real income has risen. This change in purchasing power leads to a change in quantity demanded, which we call the income effect.

Let's visualize this. Consider a consumer with income $m$ facing prices $p_1$ and $p_2$. The budget line is $p_1 x_1 + p_2 x_2 = m$. If the price of good 1 falls to $p_1' < p_1$, the budget line pivots outwards around the vertical intercept ($m/p_2$). The consumer can now reach a higher indifference curve, representing a new optimal bundle.

The core idea behind the Slutsky equation is to separate this pivot into two conceptual steps: a pivot and a shift. To isolate the substitution effect, we need to hold the consumer's purchasing power constant.

### 8.1.1 Slutsky Compensation

How do we hold purchasing power constant? Slutsky's clever idea was to define it as the ability to purchase the *original bundle* of goods.

**Definition 8.1** (Slutsky Compensation)**.** *The **Slutsky compensation** is the hypothetical adjustment to a consumer's income such that, at the new prices, they can just afford their original consumption bundle. This adjusted income level keeps the consumer's purchasing power constant in the sense that the original choice remains affordable.*

If the original bundle was $(x_1^*, x_2^*)$ at prices $(p_1, p_2)$, and the price of good 1 changes to $p_1'$, the compensated income $m'$ would be:

$$m' = p_1' x_1^* + p_2 x_2^*$$

The change in income required for this compensation is:

$$\Delta m = m' - m = (p_1' x_1^* + p_2 x_2^*) - (p_1 x_1^* + p_2 x_2^*) = (p_1' - p_1) x_1^* = \Delta p_1 x_1^*$$

## 8.2 Decomposing the Total Effect

Let's trace the full decomposition graphically. Let the initial optimal choice be bundle A = $(x_1^*, x_2^*)$ at prices $(p_1, p_2)$ and income $m$. Now, let the price of good 1 fall to $p_1'$. The final optimal choice is bundle C = $(x_1'', x_2'')$ at prices $(p_1', p_2)$ and income $m$. The total change in demand for good 1 is $\Delta x_1 = x_1'' - x_1^*$.

### 8.2.1 The Substitution Effect

To isolate the substitution effect, we give the consumer the compensated income $m' = p_1' x_1^* + p_2 x_2^*$. The new (compensated) budget line is $p_1' x_1 + p_2 x_2 = m'$. Notice that this line has the slope of the *new* price ratio but passes through the *original* bundle A.

The consumer chooses a new optimal bundle on this compensated budget line, let's call it B = $(x_1^s, x_2^s)$. The **Slutsky substitution effect** is the change in demand from A to B:

$$\Delta x_1^s = x_1^s - x_1^*$$

**Proposition 8.1** (Sign of the Substitution Effect)**.** *The Slutsky substitution effect is always negative (or non-positive). That is, the change in quantity demanded due to the substitution effect is always in the opposite direction to the change in price. If price falls (*$\Delta p_1 < 0$*), the substitution effect on demand will be positive (*$\Delta x_1^s \geq 0$*).*

*Justification via WARP.* At the original prices $(p_1, p_2)$, bundle A was chosen over bundle B (since B was not affordable, assuming B is not A). At the compensated budget with prices $(p_1', p_2)$, bundle B is chosen. Bundle A is also affordable on this line by construction. Therefore, B is revealed preferred to A. If we consider a price increase from $p_1'$ to $p_1$, B is chosen at $p_1'$ and A is chosen at $p_1$. Since A was affordable at the prices where B was chosen, WARP is not violated. Crucially, the optimal bundle B on the compensated budget line must lie to the right of A if $p_1' < p_1$, meaning $x_1^s \geq x_1^*$. □

### 8.2.2 The Income Effect

The substitution effect is a hypothetical construct. The consumer's actual income is still $m$, not $m'$. The second part of the decomposition is to restore the consumer's original income, which means moving from the compensated budget line back to the final budget line. This is a parallel shift, representing a change in income from $m'$ to $m$.

The **income effect** is the change in demand from the intermediate bundle B to the final bundle C:

$$\Delta x_1^n = x_1'' - x_1^s$$

This change is purely due to the change in purchasing power, as prices are held constant at $(p_1', p_2)$ during this step.

The overall change in demand is the sum of these two effects:

$$\text{Total Effect} = \text{Substitution Effect} + \text{Income Effect}$$

$$(x_1'' - x_1^*) = (x_1^s - x_1^*) + (x_1'' - x_1^s)$$

## 8.3 Slutsky's Effects for Different Types of Goods

The sign of the income effect depends on whether the good is normal or inferior. This determines how the two effects combine. Let's assume a price **decrease** for good 1 ($p_1 \downarrow$).

### 8.3.1 Normal Goods

A good is **normal** if demand increases as income increases ($\frac{\partial x_1}{\partial m} > 0$).

- **Substitution Effect:** Price falls, so demand increases ($\Delta x_1^s > 0$).
- **Income Effect:** Price falls, real income rises. Since the good is normal, demand increases ($\Delta x_1^n > 0$).

For a normal good, the substitution and income effects **reinforce** each other. A price decrease unambiguously leads to an increase in quantity demanded. Therefore, the Law of Demand always holds for normal goods.

### 8.3.2 Income-Inferior Goods

A good is **inferior** if demand decreases as income increases ($\frac{\partial x_1}{\partial m} < 0$).

- **Substitution Effect:** Price falls, so demand increases ($\Delta x_1^s > 0$).
- **Income Effect:** Price falls, real income rises. Since the good is inferior, demand decreases ($\Delta x_1^n < 0$).

For an income-inferior good, the substitution and income effects **oppose** each other. The total effect depends on which effect is stronger. In most cases, the substitution effect outweighs the income effect, so a price decrease still leads to an increase in total demand.

### 8.3.3 Giffen Goods

In rare cases of extreme income-inferiority, the income effect can be stronger than the substitution effect.

**Definition 8.2** (Giffen Good)**.** *A **Giffen good** is a good for which a decrease in price causes the quantity demanded to fall. This happens when the good is so strongly inferior that the negative income effect outweighs the positive substitution effect.*

For a Giffen good, the demand curve is upward-sloping. It is a violation of the Law of Demand. Slutsky's decomposition provides the theoretical explanation for this phenomenon.

## 8.4 The Slutsky Identity

We can express the decomposition mathematically. The total change in demand for good 1, $\Delta x_1$, when its price changes from $p_1$ to $p_1'$ is:

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m)$$

We can add and subtract the term $x_1(p_1', m')$, where $m' = p_1' x_1(p_1, m) + p_2 x_2(p_1, m)$:

$$\Delta x_1 = \underbrace{[x_1(p_1', m') - x_1(p_1, m)]}_{\text{Substitution Effect, } \Delta x_1^s} + \underbrace{[x_1(p_1', m) - x_1(p_1', m')]}_{\text{Income Effect, } \Delta x_1^n}$$

This is the **Slutsky identity**.

To express this in terms of rates of change, we divide by $\Delta p_1$:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + \frac{\Delta x_1^n}{\Delta p_1}$$

Let's analyze the income effect term. We know $\Delta m = \Delta p_1 x_1$. Also, $\Delta x_1^n = x_1(p_1', m) - x_1(p_1', m')$. Notice that income changes from $m'$ to $m$, so $\Delta m = m - m'$. Therefore, we can write:

$$\frac{\Delta x_1^n}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1', m')}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1', m')}{m - m'} \frac{m - m'}{\Delta p_1} = -\frac{\Delta x_1^n}{\Delta m} x_1$$

(The minus sign appears because $\Delta x_1^n$ as defined is for an income change from $m'$ to $m$, while $\Delta m$ from our compensation formula was $m' - m$). Substituting this back gives the **Slutsky Equation**:

**The Slutsky Equation**

The relationship between the total effect, substitution effect, and income effect is given by:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1}{\Delta m} x_1$$

Or in calculus terms:

$$\frac{\partial x_1(p_1, m)}{\partial p_1} = \frac{\partial x_1^s(p_1, \bar{u})}{\partial p_1} - \frac{\partial x_1(p_1, m)}{\partial m} x_1(p_1, m)$$

- $\frac{\partial x_1}{\partial p_1}$: Total effect — the slope of the ordinary (Marshallian) demand curve.
- $\frac{\partial x_1^s}{\partial p_1}$: Substitution effect — the slope of the compensated (Hicksian) demand curve. This term is always negative.
- $-\frac{\partial x_1}{\partial m} x_1$: Income effect — captures how demand changes with income, scaled by the amount of the good being consumed.

## 8.5 Examples of Slutsky Decomposition

**Perfect Complements**

For perfect complements (e.g., left shoes and right shoes), the indifference curves are L-shaped. The consumer always consumes at the corner of the indifference curve. When we perform the Slutsky pivot around the original bundle, the optimal choice on the compensated budget line remains the same as the original bundle. Therefore, the **substitution effect is zero**. The entire change in demand is due to the income effect.

**Perfect Substitutes**

For perfect substitutes, the indifference curves are straight lines. The consumer typically consumes only one of the goods (a corner solution). A change in price can cause the consumer to switch entirely from one good to the other. In this case, the shift from the original corner to the new corner can often be entirely attributed to the substitution effect. The compensated bundle is the same as the final bundle, making the **income effect zero**.

**Quasilinear Preferences**

For quasilinear preferences of the form $u(x_1, x_2) = v(x_1) + x_2$, there is no income effect for good 1 (assuming an interior solution). The demand for $x_1$

depends only on the price ratio, not on income. Therefore, the change in demand due to a price change is entirely composed of the **substitution effect**. The income effect is zero.

## 8.6 The Hicks Substitution Effect

An alternative way to define the substitution effect was proposed by John Hicks.

**Definition 8.3** (Hicks Substitution Effect)**.** *The* ***Hicks substitution effect*** *is the change in demand when prices change, while adjusting income to keep the consumer on the original indifference curve. This holds utility constant, rather than just the affordability of the original bundle.*

Graphically, instead of pivoting the new budget line around the old bundle (A), we roll it along the original indifference curve until it is tangent at the new price ratio. Like the Slutsky effect, the Hicks substitution effect is also always negative. Let bundle $X = (x_1, x_2)$ be chosen at prices $P = (p_1, p_2)$ and bundle $Y = (y_1, y_2)$ be chosen at prices $Q = (q_1, q_2)$. If the consumer is indifferent between X and Y, then by the logic of revealed preference:

$$p_1x_1 + p_2x_2 \leq p_1y_1 + p_2y_2$$

$$q_1y_1 + q_2y_2 \leq q_1x_1 + q_2x_2$$

Subtracting the two inequalities yields:

$$(p_1 - q_1)x_1 + (p_2 - q_2)x_2 - (p_1 - q_1)y_1 - (p_2 - q_2)y_2 \geq 0$$

Rearranging gives:

$$(p_1 - q_1)(x_1 - y_1) + (p_2 - q_2)(x_2 - y_2) \leq 0$$

If only the price of good 1 changes, so $p_2 = q_2$, then we have:

$$(p_1 - q_1)(x_1 - y_1) \leq 0$$

This shows that the change in price $(p_1 - q_1)$ and the change in quantity demanded from the substitution effect $(x_1 - y_1)$ must have opposite signs. For practical purposes, the Slutsky and Hicks decompositions are very similar for small price changes.

# Chapter 9

# Buying and Selling

## 9.1 Introduction

In the preceding chapters, we assumed that a consumer's income was exogenously fixed. We now expand our basic consumer choice framework to a more realistic scenario where consumers are endowed with a bundle of goods, which they can either consume or sell to generate income. This allows us to analyze a broader range of economic decisions, such as labor supply.

The theoretical framework remains fundamentally the same: consumers choose the most preferred bundle from their budget set. The key difference is that the budget constraint is now determined by the market value of the consumer's initial **endowment**.

## 9.2 The Budget Constraint

### 9.2.1 Endowments

We begin by defining the concept of an endowment.

**Definition 9.1** (Endowment)**.** *The list of resource units with which a consumer starts is called their* ***endowment****. It is denoted by the vector* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$.

**Endowment Bundle**

Consider a consumer in a two-good world. An endowment vector $\boldsymbol{\omega} = (\omega_1, \omega_2) = (10, 2)$ means that the consumer starts with 10 units of good 1 and 2 units of good 2 before entering the market.

Given this endowment, the consumer can trade these goods at market prices. The fundamental questions are: what is the total value of this endowment, and what consumption bundles can the consumer afford?

### 9.2.2 Constructing the Budget Line

The consumer's income is no longer a fixed amount of money, but is instead determined by the market value of their endowed goods. If the market prices are $(p_1, p_2)$, the value of the endowment $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is $p_1\omega_1 + p_2\omega_2$.

The consumer can sell their endowment to purchase any other consumption bundle $(x_1, x_2)$ as long as its total cost does not exceed the value of their endowment. The budget constraint is therefore:

$$p_1x_1 + p_2x_2 = p_1\omega_1 + p_2\omega_2$$

The left-hand side represents the consumer's expenditure, and the right-hand side represents their income from selling the endowment.

The consumer's budget set consists of all affordable bundles $(x_1, x_2)$:

$$\text{Budget Set} = \{(x_1, x_2) \mid p_1x_1 + p_2x_2 \leq p_1\omega_1 + p_2\omega_2, \text{ and } x_1 \geq 0, x_2 \geq 0\}$$

A crucial feature of this budget line is that it must always pass through the endowment point $(\omega_1, \omega_2)$. This is because if the consumer chooses to consume their endowment bundle, i.e., $(x_1, x_2) = (\omega_1, \omega_2)$, the budget constraint is satisfied: $p_1\omega_1 + p_2\omega_2 = p_1\omega_1 + p_2\omega_2$.

*Remark* 9.1. Because the endowment point is always on the budget line, any change in prices will cause the budget line to **pivot** around the endowment point. This is different from the parallel shift we saw when money income was fixed.

## 9.3 Net and Gross Demands

We can rearrange the budget constraint to provide a different perspective on the consumer's decision.

$$\begin{aligned} p_1x_1 + p_2x_2 &= p_1\omega_1 + p_2\omega_2 \\ p_1x_1 - p_1\omega_1 + p_2x_2 - p_2\omega_2 &= 0 \\ p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) &= 0 \end{aligned}$$

This formulation is insightful. It states that the net value of a consumer's purchases and sales must be zero. To understand this, we define two types of demands.

**Definition 9.2** (Gross and Net Demands)**.** *The consumer's final consumption bundle $(x_1, x_2)$ is their **gross demand**. The quantities $(x_1 - \omega_1)$ and $(x_2 - \omega_2)$ are the consumer's **net demands**.*

- If $x_i - \omega_i > 0$, the consumer is a **net buyer** or **net demander** of good $i$.
- If $x_i - \omega_i < 0$, the consumer is a **net seller** or **net supplier** of good $i$.

The equation $p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0$ simply means that the value of what the consumer buys must equal the value of what they sell. For example, if a consumer is a net seller of good 1 ($x_1 - \omega_1 < 0$), they must be a net buyer of good 2 ($x_2 - \omega_2 > 0$) for the equation to hold.

## 9.4 Slutsky's Equation with Endowments

When prices change, not only do relative prices change (affecting substitution), but the value of the consumer's endowment also changes. This alters their income, introducing an additional income effect.

Recall that for a consumer with fixed money income $m$, the Slutsky decomposition was:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - x_1 \frac{\Delta x_1^m}{\Delta m}$$

Here, the total effect of a price change on demand is the sum of a substitution effect and an ordinary income effect.

Now, income is not fixed but is given by $m = p_1\omega_1 + p_2\omega_2$. A change in $p_1$ affects $m$. The change in money income when $p_1$ changes is:

$$\frac{\Delta m}{\Delta p_1} = \omega_1$$

This change in income generates an additional income effect, which we call the **endowment income effect**.

The endowment income effect is the change in demand due to the change in the value of the endowment:

$$\text{Endowment Income Effect} = \frac{\Delta m}{\Delta p_1} \times \frac{\Delta x_1^m}{\Delta m} = \omega_1 \frac{\Delta x_1^m}{\Delta m}$$

The overall change in demand is now the sum of three components:

1. **Pure Substitution Effect:** The effect from the change in relative prices, holding purchasing power constant.

2. **Ordinary Income Effect:** The effect from the change in purchasing power, as the original bundle is now more or less expensive.

3. **Endowment Income Effect:** The effect from the change in the value of the endowment.

Combining these, the full Slutsky equation with endowments is:

$$\frac{\Delta x_1}{\Delta p_1} = \underbrace{\frac{\Delta x_1^s}{\Delta p_1}}_{\text{Substitution Effect}} \underbrace{-x_1 \frac{\Delta x_1^m}{\Delta m}}_{\text{Ordinary Income Effect}} + \underbrace{\omega_1 \frac{\Delta x_1^m}{\Delta m}}_{\text{Endowment Income Effect}}$$

We can combine the two income effects into a single term:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1)\frac{\Delta x_1^m}{\Delta m} \tag{9.1}$$

*Remark* 9.2. The term $(\omega_1 - x_1)$ represents the consumer's net supply of good 1.

- If the consumer is a net seller of good 1 ($\omega_1 > x_1$) and good 1 is a normal good, an increase in $p_1$ has a positive income effect, encouraging more consumption of good 1. This can potentially outweigh the negative substitution effect.

- If the consumer is a net buyer of good 1 ($\omega_1 < x_1$), the term $(\omega_1 - x_1)$ is negative. The total income effect is reinforced, making the demand curve for a normal good unambiguously downward-sloping.

## 9.5 Application: Labor Supply

The model of buying and selling is perfectly suited to analyze a worker's decision on how many hours to work.

### 9.5.1 The Labor-Leisure Choice

Consider a worker who chooses between two goods: a composite consumption good, $C$, and leisure, $R$.

- The price of the consumption good is $p$.

- The "price" of leisure is the wage rate, $w$, as it represents the opportunity cost of not working.

The worker is endowed with some non-labor income, $M$, and a total amount of time, $\bar{R}$ (e.g., 24 hours a day). The endowment bundle is $(\bar{R}, M)$. The worker sells some of their time endowment as labor to fund consumption. Labor supplied, $L$, is the total time endowment minus the amount of leisure consumed: $L = \bar{R} - R$.

The budget constraint can be expressed as:

$$\text{Expenditure} = \text{Income}$$

$$pC = w(\bar{R} - R) + M$$

Rearranging this gives us the standard endowment budget line:

$$pC + wR = w\bar{R} + M$$

The left side is the total expenditure on consumption and leisure. The right side is the total potential income, or the value of the endowment of time and non-labor money. The slope of this budget line is $-w/p$, which is the **real wage**.

The worker chooses the optimal combination $(C^*, R^*)$ that maximizes utility, which in turn determines the amount of labor supplied, $L^* = \bar{R} - R^*$.

### 9.5.2 Backward-Bending Labor Supply Curve

How does the amount of labor supplied change when the wage rate, $w$, increases? We can use the Slutsky equation to analyze the effect of a change in $w$ on the demand for leisure, $R$. Here, leisure is good 1, and its price is $w$. The endowment of leisure is $\bar{R}$.

From Equation (**??**), we have:

$$\frac{\Delta R}{\Delta w} = \underbrace{\frac{\Delta R^s}{\Delta w}}_{\text{(-)}} + \underbrace{(\bar{R} - R)}_{\text{(+)}} \underbrace{\frac{\Delta R^m}{\Delta m}}_{\text{(?)}} \tag{9.2}$$

Let's analyze the signs:

- **Substitution Effect ($\frac{\Delta R^s}{\Delta w}$):** An increase in the wage rate $w$ makes leisure more expensive. The consumer will substitute away from leisure towards consumption. Thus, the substitution effect on the demand for leisure is negative. This effect implies that the worker supplies more labor.
- **Income Effect ($(\bar{R} - R)\frac{\Delta R^m}{\Delta m}$):**
  - $(\bar{R} - R)$ is the amount of labor supplied, which is typically positive.
  - If leisure is a normal good (which is a reasonable assumption), then $\frac{\Delta R^m}{\Delta m}$ is positive. An increase in income leads to a desire for more leisure.
  - Therefore, the entire income effect term is positive. A higher wage makes the worker richer, so they choose to "purchase" more leisure (i.e., work less).

The substitution and income effects work in opposite directions.

- At low wage rates, the substitution effect often dominates. An increase in the wage leads to an increase in labor supply.
- At high wage rates, the income effect may dominate. The worker is already earning a high income, and a further wage increase may lead them to value leisure more, thus reducing their labor supply.

This can lead to a **backward-bending labor supply curve**, where the quantity of labor supplied first increases with the wage and then decreases.

### 9.5.3 Overtime

An overtime wage creates a kink in the budget constraint. Suppose a worker is paid a wage $w$ for the first $L_0$ hours of work, and an overtime wage $w' > w$ for any hours beyond $L_0$. The budget line becomes steeper after the worker has supplied $L_0$ hours of labor.

This non-linear wage structure can induce an employee to work more hours than a simple increase in the flat wage rate. The overtime offer effectively isolates the substitution effect over a certain range, encouraging more work without the large offsetting income effect that a high flat wage for all hours would create.