A Mathematical Intuition of Common Demand Functions

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Abstract

This document provides a mathematical intuition behind common demand functions derived from different types of utility functions.

1 Cobb-Douglas Utility Function

The Cobb-Douglas utility function is given by:

$$U(x_1, x_2) = x_1^a x_2^b \tag{1}$$

where $a > 0$ and $b > 0$.

The Lagrangian for the utility maximization problem is:

$$\mathcal{L} = x_1^a x_2^b + \lambda(m - p_1 x_1 - p_2 x_2) \tag{2}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = a x_1^{a-1} x_2^b - \lambda p_1 = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = b x_1^a x_2^{b-1} - \lambda p_2 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \tag{5}$$

From the first two equations, we can write:

$$\lambda = \frac{a x_1^{a-1} x_2^b}{p_1} \tag{6}$$

$$\lambda = \frac{b x_1^a x_2^{b-1}}{p_2} \tag{7}$$

Setting these two expressions for λ equal to each other gives:

$$\frac{a x_1^{a-1} x_2^b}{p_1} = \frac{b x_1^a x_2^{b-1}}{p_2} \tag{8}$$

$$\frac{a x_2}{p_1} = \frac{b x_1}{p_2} \implies p_2 x_2 = \frac{b}{a} p_1 x_1 \tag{9}$$

Substitute this into the budget constraint:

$$p_1 x_1 + \frac{b}{a} p_1 x_1 = m \tag{10}$$

$$x_1(p_1 + \frac{b}{a} p_1) = m \tag{11}$$

$$x_1(\frac{a p_1 + b p_1}{a}) = m \tag{12}$$

$$x_1^*(p_1, p_2, m) = \frac{a}{a+b} \frac{m}{p_1} \tag{13}$$

To find the demand for x_2, substitute x_1^* back into the relationship between x_1 and x_2:

$$p_2 x_2 = \frac{b}{a} p_1 \left(\frac{a}{a+b} \frac{m}{p_1} \right) \tag{14}$$

$$p_2 x_2 = \frac{b}{a+b} m \tag{15}$$

$$x_2^*(p_1, p_2, m) = \frac{b}{a+b} \frac{m}{p_2} \tag{16}$$

2 Perfect Substitutes Utility Function

The perfect substitutes utility function is given by:

$$U(x_1, x_2) = ax_1 + bx_2 \tag{17}$$

where $a > 0$ and $b > 0$. The consumer will spend all of their income on the good that provides the most utility per dollar.

The marginal utilities are:

$$MU_1 = a \tag{18}$$

$$MU_2 = b \tag{19}$$

The marginal rate of substitution (MRS) is:

$$MRS = \frac{MU_1}{MU_2} = \frac{a}{b} \tag{20}$$

We compare the MRS to the price ratio $\frac{p_1}{p_2}$:

- If $\frac{a}{b} > \frac{p_1}{p_2}$, then $\frac{MU_1}{p_1} > \frac{MU_2}{p_2}$. The consumer will only purchase good 1.

$$x_1^* = \frac{m}{p_1}, \quad x_2^* = 0 \tag{21}$$

- If $\frac{a}{b} < \frac{p_1}{p_2}$, then $\frac{MU_1}{p_1} < \frac{MU_2}{p_2}$. The consumer will only purchase good 2.

$$x_1^* = 0, \quad x_2^* = \frac{m}{p_2} \tag{22}$$

- If $\frac{a}{b} = \frac{p_1}{p_2}$, the consumer is indifferent between any combination of x_1 and x_2 along the budget line. Any bundle satisfying $p_1 x_1 + p_2 x_2 = m$ is optimal.

3 Perfect Complements Utility Function

The perfect complements utility function is given by:

$$U(x_1, x_2) = \min(ax_1, bx_2) \tag{23}$$

where $a > 0$ and $b > 0$. The consumer will always consume the goods in a fixed ratio.

To maximize utility, the consumer will choose x_1 and x_2 such that:

$$ax_1 = bx_2 \implies x_2 = \frac{a}{b} x_1 \tag{24}$$

Substitute this into the budget constraint:

$$p_1 x_1 + p_2 \left(\frac{a}{b} x_1\right) = m \tag{25}$$
$$x_1 \left(p_1 + \frac{a}{b} p_2\right) = m \tag{26}$$
$$x_1 \left(\frac{bp_1 + ap_2}{b}\right) = m \tag{27}$$
$$x_1^*(p_1, p_2, m) = \frac{bm}{bp_1 + ap_2} \tag{28}$$

To find the demand for x_2, substitute x_1^* back into the relationship between x_1 and x_2:

$$x_2^* = \frac{a}{b} x_1^* = \frac{a}{b} \left(\frac{bm}{bp_1 + ap_2}\right) \tag{29}$$
$$x_2^*(p_1, p_2, m) = \frac{am}{bp_1 + ap_2} \tag{30}$$

4 Quasi-linear Utility Function

A quasi-linear utility function is linear in one good, say x_2:

$$U(x_1, x_2) = v(x_1) + x_2 \tag{31}$$

where $v(x_1)$ is a strictly concave function.

The Lagrangian is:

$$\mathcal{L} = v(x_1) + x_2 + \lambda(m - p_1 x_1 - p_2 x_2) \tag{32}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = v'(x_1) - \lambda p_1 = 0 \tag{33}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_2 = 0 \tag{34}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \tag{35}$$

From the second equation, we get $\lambda = \frac{1}{p_2}$. Substituting this into the first equation:

$$v'(x_1) - \frac{p_1}{p_2} = 0 \implies v'(x_1) = \frac{p_1}{p_2} \tag{36}$$

This gives us the demand for x_1, which only depends on the prices:

$$x_1^*(p_1, p_2) = (v')^{-1} \left(\frac{p_1}{p_2}\right) \tag{37}$$

The demand for x_2 is then found from the budget constraint:

$$p_1 x_1^* + p_2 x_2 = m \implies x_2^*(p_1, p_2, m) = \frac{m - p_1 x_1^*}{p_2} \tag{38}$$

This is valid for an interior solution, which requires $m > p_1 x_1^*$.

5 Constant Elasticity of Substitution (CES) Utility Function

The CES utility function is defined as:

$$U(x_1, x_2) = (a x_1^\rho + b x_2^\rho)^{\frac{1}{\rho}} \tag{39}$$

where $a, b > 0$ and $\rho \leq 1, \rho \neq 0$. The elasticity of substitution is $\sigma = \frac{1}{1-\rho}$.

The Lagrangian is:

$$\mathcal{L} = (ax_1^\rho + bx_2^\rho)^{\frac{1}{\rho}} + \lambda(m - p_1x_1 - p_2x_2) \tag{40}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{\rho}(ax_1^\rho + bx_2^\rho)^{\frac{1}{\rho}-1}(a\rho x_1^{\rho-1}) - \lambda p_1 = 0 \tag{41}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{\rho}(ax_1^\rho + bx_2^\rho)^{\frac{1}{\rho}-1}(b\rho x_2^{\rho-1}) - \lambda p_2 = 0 \tag{42}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0 \tag{43}$$

From the first two equations, we can find the MRS:

$$\frac{ax_1^{\rho-1}}{bx_2^{\rho-1}} = \frac{p_1}{p_2} \implies \left(\frac{x_1}{x_2}\right)^{\rho-1} = \frac{bp_1}{ap_2} \tag{44}$$

$$\frac{x_1}{x_2} = \left(\frac{bp_1}{ap_2}\right)^{\frac{1}{\rho-1}} = \left(\frac{ap_2}{bp_1}\right)^{\frac{1}{1-\rho}} \tag{45}$$

Let $\sigma = \frac{1}{1-\rho}$, then:

$$x_1 = x_2 \left(\frac{ap_2}{bp_1}\right)^\sigma \tag{46}$$

Substitute this into the budget constraint:

$$p_1 \left(x_2 \left(\frac{ap_2}{bp_1}\right)^\sigma\right) + p_2x_2 = m \tag{47}$$

$$x_2 \left(p_1 \left(\frac{ap_2}{bp_1}\right)^\sigma + p_2\right) = m \tag{48}$$

$$x_2^* = \frac{m}{p_1\left(\frac{ap_2}{bp_1}\right)^\sigma + p_2} = \frac{mp_1^\sigma b^\sigma}{p_1a^\sigma p_2^\sigma + p_2p_1^\sigma b^\sigma} = \frac{mp_1^\sigma b^\sigma}{p_1^\sigma p_2(a^\sigma p_2^{\sigma-1} + b^\sigma p_1^{\sigma-1})} \tag{49}$$

After further simplification:

$$x_1^*(p_1, p_2, m) = \frac{ma^\sigma}{a^\sigma p_1^{1-\sigma} + b^\sigma p_2^{1-\sigma}} \cdot \frac{1}{p_1^\sigma} = \frac{ma^\sigma p_1^{-\sigma}}{a^\sigma p_1^{1-\sigma} + b^\sigma p_2^{1-\sigma}} \tag{50}$$

A more common form is:

$$x_1^*(p_1, p_2, m) = \frac{mp_1^{\frac{1}{\rho-1}}a^{\frac{1}{1-\rho}}}{p_1^{\frac{\rho}{\rho-1}}a^{\frac{1}{1-\rho}} + p_2^{\frac{\rho}{\rho-1}}b^{\frac{1}{1-\rho}}} \tag{51}$$

Let $r = \dfrac{\rho}{\rho-1}$. Then:

$$x_1^*(p_1, p_2, m) = \frac{mp_1^{r-1}a^\sigma}{p_1^r a^\sigma + p_2^r b^\sigma} \tag{52}$$

By symmetry:

$$x_2^*(p_1, p_2, m) = \frac{mp_2^{r-1}b^\sigma}{p_1^r a^\sigma + p_2^r b^\sigma} \tag{53}$$