April 2024 Jane Street Puzzle Solution

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1 Question

It's been a while and change, but the Robot Games are back once again. This time it's Capture the Flag!

Two robots, Aaron and Erin, have made it to this year's final! Initially they are situated at the center of a unit circle. A flag is placed somewhere inside the circle, at a location chosen uniformly at random. Once the flag is placed, Aaron is able to deduce its distance to the flag, and Erin is only able to deduce its direction to the flag. (Equivalently: if (r, θ) are the polar coordinates of the flag's location, Aaron is told r and Erin is told θ.)

Both robots are allowed to make a single move after the flag is placed, if they wish. Any move they make is without knowledge of what the other robot is doing. (And they may not move outside the circle.)

Whichever robot is closer to the flag after these moves captures the flag and is declared the winner!

During the preliminaries it was discovered that Erin is programmed to play a fixed distance along the detected angle θ. Assuming otherwise optimal play by both robots, can you determine the probability that Aaron will win? (Please express your answer to 10 decimal places.)

2 Aaron's Optimal Move

Aaron only knows r: the distance from his starting point, the origin, to the flag. The flag could be at any angle α from his starting point.

Let the distance Aaron moves from the origin be s and the direction he moves from the origin β. We show that Aaron's optimal move is to always stay at the origin.

We call Aaron's 2-dimensional distance from the flag after his move vector $\overrightarrow{AF}$, which evaluates to

$$\overrightarrow{AF} = \begin{bmatrix} s\cos\beta - r\cos\alpha \\ s\sin\beta - r\sin\alpha \end{bmatrix}$$

Let the magnitude of Aaron's new distance from his move be called d_A and $A = [d_A]^2$

$$|\overrightarrow{AF}| = \sqrt{(s\cos\beta - r\cos\alpha)^2 + (s\sin\beta - r\sin\alpha)^2}$$

$$\begin{aligned}
[|\overrightarrow{AF}|]^2 = A &= (s\cos\beta - r\cos\alpha)^2 + (s\sin\beta - r\sin\alpha)^2 \\
&= s^2\cos^2\beta + r^2\cos^2\alpha + s^2\sin^2\beta + r^2\sin^2\alpha - 2rs\cos\alpha\cos\beta - 2rs\sin\alpha\sin\beta \\
&= r^2(\sin^2\alpha + \cos^2\alpha) + s^2(\sin^2\beta + \cos^2\beta) - 2rs(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
A &= r^2 + s^2 - 2rs\cos(\alpha - \beta)
\end{aligned}$$

Our domain D for r, α, and β in the unit circle is

$$D \begin{cases} r \in [0,1] \\ \alpha \in [0, 2\pi] \\ \beta \in [0, 2\pi] \end{cases}$$

Since Aaron does not know the flag's direction, I will evaluate A_{avg}, the average distance of Aaron to the flag, with respect to the distance Aaron moves from the origin, s, across the domain D. To simplify calculations, we square $|\overrightarrow{AF}|$, Aaron's new distance to the flag, preserving all of $|\overrightarrow{AF}|$'s extreme values in the domain D to optimize it for s.

$$\begin{aligned}
A_{avg} &= \frac{1}{(1)(2\pi)(2\pi)} \int_0^1 \int_0^{2\pi} \int_0^{2\pi} [r^2 + s^2 - 2rs\cos(\alpha - \beta)]rs \,\mathrm{d}\alpha \,\mathrm{d}\beta \,\mathrm{d}r \\
&= \frac{1}{4\pi^2} \int_0^1 \int_0^{2\pi} \int_0^{2\pi} r^3 s + rs^3 - 2r^2 s^2 \cos(\alpha - \beta) \,\mathrm{d}\alpha \,\mathrm{d}\beta \,\mathrm{d}r \\
&= \frac{1}{4\pi^2} \int_0^1 \int_0^{2\pi} \left[\alpha r^3 s + \alpha r s^3 - 2r^2 s^2 \sin(\alpha - \beta)\right]_0^{2\pi} \,\mathrm{d}\beta \,\mathrm{d}r \\
&= \frac{1}{4\pi^2} \int_0^1 \int_0^{2\pi} 2\pi r^3 s + 2\pi r s^3 - 2r^2 s^2 [\sin(2\pi - \beta) - \sin(0 - \beta)] \,\mathrm{d}\beta \,\mathrm{d}r \\
&= \frac{1}{4\pi^2} \int_0^1 \int_0^{2\pi} 2\pi r^3 s + 2\pi r s^3 \,\mathrm{d}\beta \,\mathrm{d}r \\
&= \frac{1}{4\pi^2} \int_0^1 4\pi^2 r^3 s + 4\pi^2 r s^3 \,\mathrm{d}r \\
&= \int_0^1 r^3 s + r s^3 \,\mathrm{d}r \\
&= \left[\frac{1}{4} r^4 s + \frac{1}{2} r^2 s^3\right]_0^1 \\
A_{avg}(s) &= \frac{1}{2} s^3 + \frac{1}{4} s
\end{aligned}$$

I will show that Aaron's optimal move is staying put at $s = 0$.

$$\frac{\mathrm{d}A_{avg}}{\mathrm{d}s} = \frac{3}{2} s^2 + \frac{1}{4}$$

$$\underline{\frac{\mathrm{d}A_{avg}}{\mathrm{d}s} = 0 \implies}$$

$$\begin{aligned}
\frac{3}{2} s^2 + \frac{1}{4} &= 0 \\
s &= \sqrt{-\frac{1}{6}}
\end{aligned}$$

Since $\frac{\mathrm{d}A_{avg}}{\mathrm{d}s}$ has no real roots and $\frac{\mathrm{d}A_{avg}}{\mathrm{d}s} > 0$ when s is in range $s \in [0, 1]$, A_{avg} is minimized at $s = 0$.

3 Erin's Optimal Move

Erin can only align his direction with the direction of the flag α. His distance moved is predetermined. His distance moved is uniformly distributed across 0 and 1. Let Erin's distance moved be t.

4 Aaron's Winning Probability

Given both robots play optimally without knowledge of the other robot's plays, both robots' moves will lie along the same radial line of the flag on the unit circle.

Aaron's distance from the flag is the same as the flag's distance from the origin r. Erin's distance from the flag is $|r - s|$. Let the function $a(r, s)$ be $r - |r - s|$. a is the sample space of winning and losing outcomes, and when $a < 0$ Aaron wins.

I will evaluate a in the average case across all possible positions of the flag, from 0 to 1, uniformly probable

To quantify the event that Aaron wins, we must find the average d_E distance of Erin to the flag (which is not uniformly distributed along its radial line) across all of Erin's uniformly likely positions across the flag's radial line.

$$
\begin{aligned}
d_{E,avg} &= \frac{1}{1-0}\int_0^1 r - |r-s|\,\mathrm{d}s \\
&= r - \int_0^1 |r-s|\,\mathrm{d}s \\
&= r - \left[\int_0^r r - s\,\mathrm{d}s + \int_r^1 s - r\mathrm{d}s\right] \\
&= r - \left[\left[sr - \frac{1}{2}s^2\right]_0^r + \left[\frac{1}{2}s^2 - sr\right]_r^1\right] \\
&= r - \left[r^2 - \frac{1}{2}r^2 + \left(\frac{1}{2} - r\right) - \left(\frac{1}{2}r^2 - r^2\right)\right] \\
&= r - \left[r^2\left(1 - \frac{1}{2} - \frac{1}{2} + 1\right) + \frac{1}{2} - r\right] \\
&= r - \left[r^2 - r + \frac{1}{2}\right] \\
d_{E,avg(r)} &= -r^2 - +2r - \frac{1}{2}
\end{aligned}
$$

The roots of $d_{E,avg}$ demarcate the domain of Aaron winning when $d_{E,avg} > 0$ no matter the probability density function of r. $d_{E,avg}$ has 1 root in the domain $[0, 1]$, $\boxed{1 - \frac{1}{\sqrt{2}}}$. To 10 decimal places, this is $\boxed{0.2928932188}$.