6. Evaluation system and Students' Responsibilities

Evaluation System

In addition to the formal exam(s), the internal evaluation of a student may consist of quizzes, assignments, lab reports, projects, class participation, etc. The tabular presentation of the internal evaluation is as follows.

Internal Evaluation	Marks	External Evaluation	Weight	Marks
Attendance & Class Participation	10%	Semester End Board Examination	50%	50
Assignments	20%			
Presentations/Quizzes	10%			
Term exam	60%			
Total Internal	50			
Full Marks: 50 + 50 = 100				

Student Responsibilities:

Each student must secure at least 45% marks in internal evaluation with 80% attendance in the class in order to appear in the Semester End Examination. Failing to get such score will be given NOT QUALIFIED (NQ) and the student will not be eligible to appear the Semester-End Examinations. Students are advised to attend all the classes, formal exam, test, etc. and complete all the assignments within the specified time period. Students are required to complete all the requirements defined for the completion of the course.

7. Prescribed Books and References

Text Books

1. Kreyszig, E. *Advance Engineering Mathematics*, New Delhi: John Wiley and Sons Inc.
2. Stewart, J. *Calculus, Early Transcendental*. India; Cengage Learning.

References

1. Dass, H. K. & Verma R. *Higher Engineering Mathematics*. New Delhi: S Chand Publishing.
2. Mishra, P., Mishra, R., Mishra, V. P., & Mishra, M. *Advance Engineering Mathematics*. New Delhi: V. P. Mishra Publication.
3. Thomas, G. & Finney, R. *Calculus and Analytical Geometry*. New Delhi: Narosa Publishing House.

CONTENTS

Unit 1 Multiple Integrals ✓

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Unit 2 Series Solution of Differential Equations and Special Functions

Unit 3 Laplace Transform and its Application

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Unit 4 Advanced Vector Calculus

Unit 5 Fourier Series

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Unit 6 Partial Differential Equations

Unit 1

Multiple Integrals

Pre-requisite knowledge	Expected learning outcomes
Before starting this unit, students are expected to have fundamental concepts and evaluation skills on • defining integration as a limit of a sum • definite integral as area • evaluating simple integrals in Cartesian coordinates. • evaluating the integration of the single variables. • solving simple applied problems in integration.	After completion of this unit, student will develop sufficient knowledge and evaluation skills on • expressing double integral as an area and volume. • evaluation of the double integrals involving Cartesian and polar coordinates. • evaluation of the area and volume through double integral. • defining Fubini's theorem and integrate by changing the order of integration. • evaluation of triple integrals.

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1.1 Introduction

Integration is the act of bringing smaller components into a single system. Newton was the first person to develop this idea about the concept of integration. Leibniz also worked on integration independently. Newton neglected to publish his work until after Leibniz. Leibniz work was well known as compared to Newton's work.

After Newton and Leibniz brought the concept of integration through the fundamental theorem of calculus, it was seen as anti-differentiation, rather than its own concept. The notion of the integral as the limit of a summation process was certainly something that had been explored but it was treated more as a convenient way to treat difficult integrals, rather than as a fundamental support for integration theory.

Newton and Leibniz

In Calculus-I, we have studied integration of function of a single variable and its applications. In this chapter, we will extend the concept of integration of functions of a single variable to the functions of two and three variables known as double and triple integrals respectively.

1.2 Review on integration of function of a single variable

Let $y = f(x)$ be a continuous function in $[a, b]$. We partition the interval $[a, b]$ into n-subintervals of lengths Δx_i by the points $a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$ such that $\Delta x_i = x_i - x_{i-1}$, the largest of such Δx_i is taken as Δx and $\Delta x \to 0$ as $n \to \infty$.

Length of ith subinterval $= x_i - x_{i-1} = \Delta x_i$

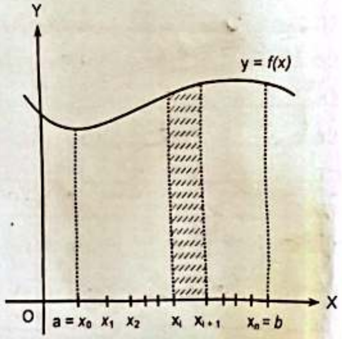
Average height of i^{th} rectangle $= f(x_i^*)$, where $x_i^* \in [x_{i-1}, x_i]$

$\therefore$ Area of i^{th} rectangle $= f(x_i^*)\Delta x_i$

$\therefore$ Total area bounded by $y = f(x)$ above x-axis from $x = a$ to $x = b$

$$= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots + f(x_n^*)\Delta x_n$$

$$= \sum_{n=1}^{\infty} f(x_i^*)\,\Delta x_i$$



The above sum gives approximate area bounded by $y = f(x)$ above x-axis, from $x = a$ to $x = b$. For the exact area we take the limit $n \to \infty$ for larger the number of partition points, the area gets closer to exact area.

i.e., The exact area (A) bounded by $y = f(x)$ above x-axis, from $x = a$ to $x = b$

$$A = \lim_{n\to\infty} \sum_{n=1}^{\infty} f(x_i^*)\Delta x_i.$$

The limit in the R.H.S is known as definite integral of $f(x)$ from $x = a$ to $x = b$ and written as $\int_a^b f(x)dx$.

i.e., $A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{n=1}^{\infty} f(x_i^*)\Delta x_i.$

1.3 Functions of two and three variables

1.3.1 Function of two variable

A function of two variables is a function of the form $z = f(x, y)$, where x and y are independent variables and z is a dependent variable, (depends on x and y) known as function of x and y. The domain of function of two variables $f(x, y)$ is a region (area) R in xy-plane and the graph of $z = f(x, y)$ is a surface in three dimension.

1.3.2 Function of three variables

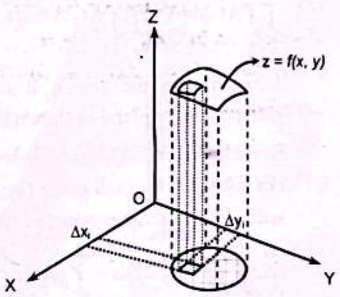
A function of three variables is of the form $u = f(x, y, z)$. The domain of function of three variables is a three dimensional region (volume) and its graph is a surface in 4-dimension known as hypersurface.

1.4 Definition of double integral as limit of a sum

Let $z = f(x, y)$ be a single valued and continuous function of two variables x, y defined over the plane region R. Let the region R be subdivided in any manner into n elementary sub regions $R_1, R_2, R_3, \ldots, R_n$ of areas $\Delta A_1, \Delta A_2 \ldots, \Delta A_n$ respectively. Let R_i encloses the point (x_i, y_i) and ΔA_i is the area of R_i.

Considering, the sum $\sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$.

As the number of sub division $n \to \infty$, then each $\Delta A_i \to 0$.



$\Delta A_i = \Delta x_i\,\Delta y_i, (x_i, y_i) \in \Delta A_i$

And if the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$, exists, it is called the double integral of $f(x, y)$ over the region R and it is denoted by $\iint_R f(x, y)\,dA$.

$$\text{Thus, } \iint_R f(x, y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \qquad \ldots (1)$$

In order to simplify the evaluation of double integral, we take $dA = dx.dy$ as the approximate value of $\Delta A = \Delta x.\,\Delta y$ in general. Therefore, it follows from (1) that

$$\iint_R f(x, y)\,dx.dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i\,\Delta y_i \qquad \ldots (2)$$

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1.5 Geometrical interpretation of double integrals

We have $z = f(x, y)$ which is a surface over the region (area) R in xy-plane. The double integral

$$\iint_R f(x, y)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$$

represents the volume of solid bounded above by the surface $z = f(x, y)$, below by the region R i xy-plane as shown in above figure.

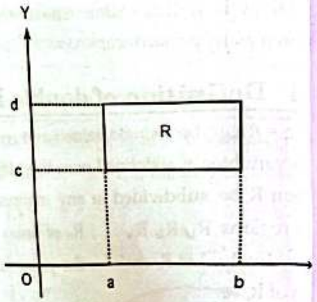
1.6 Evaluation of double integral

The evaluation of double integral is the iterative integral of function of single variabl considering the other variable fixed. This idea of iterative integral was suggested by Fubini theorem depending on the boundary of the region R in xy-plane. In order to evaluate we nee to consider the limits of x and y, that bounds R in xy-plane so for limits of x and y we need t draw the sketch of the region R if limits of integration are not given.

1.6.1 Fubini's Theorem

1. (For rectangular region R)

   If $f(x, y)$ is continuous in a rectangular region R, R: $a \le x \le b$, $c \le y \le d$ in xy-plane. Then



$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

   In such case the region R of integration is a rectangle in xy-plane as shown in figure.

   Since limits of both x and y are constant and are not related with each other (by a function), so we can separate the integrand for x and with their limits as product of two integrals.

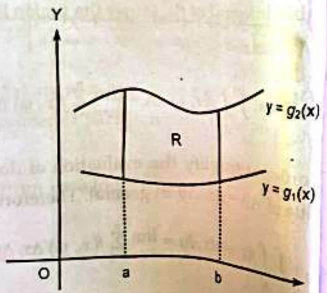
   i.e., $\displaystyle\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b g(x)dx \int_c^d h(y)dy$

   where $g(x) = x$-terms in $f(x, y)$

   $g(y) = y$-terms in $f(x, y)$

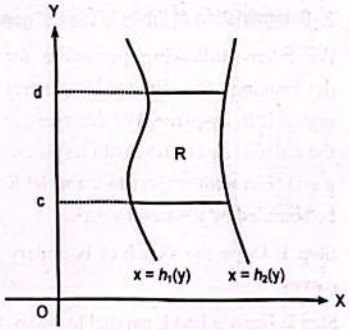
2. For general region R:

a. If $f(x, y)$ is continuous in a general region R which is not a rectangular region and R: $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ then



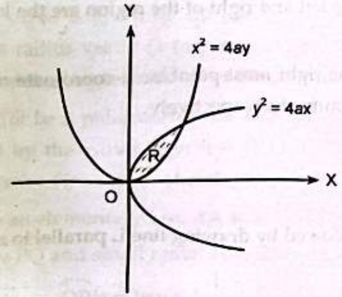
$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

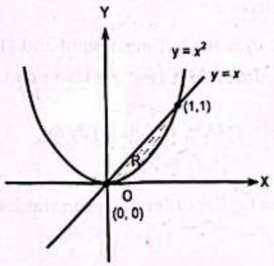
b. If $f(x, y)$ is continuous in a general region R which is not a rectangular region and R: $h_1(y) \le x \le h_2(y)$, $c \le y \le d$, then



$$\iint_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

Note: *The region R may be a closed region bounded by two intersecting curves.*





1.6.2 Properties of double integral:

If $f(x, y)$ and $g(x, y)$ are continuous function on the region R, then

i. Constant multiple: $\displaystyle\iint_R c\,f(x, y)dA = c\iint_R f(x, y)dA$ for any constant c.

ii. Sum and difference: $\displaystyle\iint_R [f(x, y) \pm g(x, y)]dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)dA$

iii. Positivity: If $f(x, y) \ge 0$ on R, then $\displaystyle\iint_R f(x, y)dA \ge 0$.

iv. Additivity: If $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \phi$, then

$$\iint_R f(x, y)dA = \iint_{R_1} f(x, y)dA + \iint_{R_2} f(x, y)dA$$

In such case we can use any of theorems (a) and (b).

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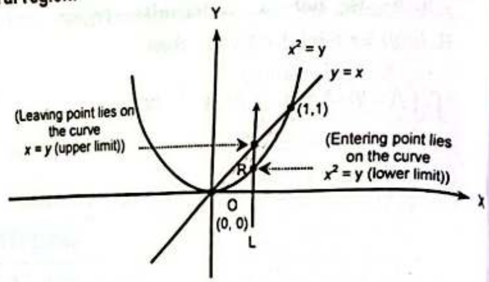
2. Determination of limits in case of general region:

We follow following procedure for determining the limits of general region R in xy-plane. We describe for the order $dy\,dx$ i.e., first with respect to y and then with respect to x and let R be bounded by $y = x$ and $y = x^2$.



Step 1: Draw the sketch of boundary curves.

Step 2: Draw a line L parallel to y-axis, the boundary through which L enters the region and the boundary from which L leaves the region are taken as lower and upper limits for y. The minimum and maximum values that x-takes in the left and right of the region are the lower and upper limits of x.

Since O(0, 0) is the left most point and (1, 1) is the right most point, so x-coordinate of O i.e., 0 and x-coordinate of A i.e. 1 are lower and upper limits of x respectively.

$$\therefore \quad \iint_R f(x, y)\,dA = \int_0^1 \int_{x^2}^{x} f(x, y)\,dy\,dx$$

The process for the order $dx\,dy$ can similarly be followed by drawing line L parallel to x-axis.

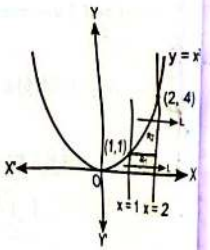
Remarks:

1. In order to apply above process for determining limits of integration of $f(x, y)$ over R the leaving and entering curve boundaries must be same for the line L through the region in R. We draw line L parallel to y-axis for $dy\,dx$ order and parallel to x-axis for $dx\,dy$ order.

   If either entering or leaving curves are different then we divide the region R into sub-regions R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \phi$.

   $$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$$

   For example if R is bounded by $y = 0$, $y = x^2$ and $x = 1$, $x = 2$, then

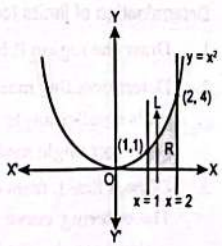
   $$\iint_R f(x, y)\,dx\,dy = \iint_{R_1} f(x, y)\,dx\,dy + \iint_{R_2} f(x, y)\,dx\,dy$$

   as the line L parallel to x-axis has different entering curves $x = 1$ and $x^2 = y$ *for* R_1 and R_2.

   $$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dx\,dy + \iint_{R_2} f(x, y)\,dx\,dy$$

   $$= \int_0^1 \int_1^2 f(x, y)\,dx\,dy + \int_1^4 \int_{\sqrt{y}}^{2} f(x, y)\,dx\,dy$$

2. But we may overcome the process of dividing the region R into sub-regions by changing the order of integration. If we consider the integral $\iint_R f(x, y)\,dA$ in $dy\,dx$ order the entering and the leaving cures for line L parallel to y-axis will be the same and the integral is given by

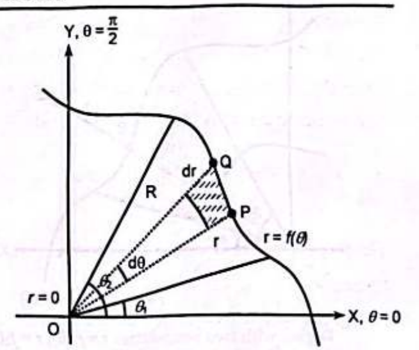
   

   $$\iint_R f(x, y)\,dy\,dx = \int_1^2 \int_0^{x^2} f(x, y)\,dy\,dx.$$

   We see that $dy\,dx$ order is better choice than $dx\,dy$.

1.7 Double integral for polar functions

A polar curve is given by the equation $r = f(\theta)$ with r as radius vector (a line from origin to the curve) and θ as angle made by r with polar axis.

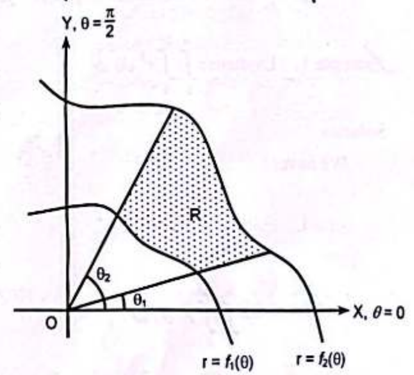


Let $r = f(\theta)$ be a polar curve and R is the region bounded by the curve from $\theta = \theta_1$ to $\theta = \theta_2$ in anticlockwise direction as shown in figure.

Consider an elementary area ΔA with small curve boundary $\overparen{PQ}$ and small radial boundary dr.

$$\text{Also, } d\theta = \frac{\overparen{QP}\text{(arc length)}}{r}$$

$$\therefore \quad \overparen{PQ} = r\,d\theta$$



For very large number of partition or number of division of R we can consider dA as rectangle with length $\overparen{PQ}$ and breadth dr.

and $dA = \overparen{PQ}\,dr = r\,d\theta\,dr = r\,dr\,d\theta$.

Thus, the double integral of any function $F(r, \theta)$ in the region R bounded by the curve $r = f(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$ is given by

$$\iint_R F(r, \theta)\,dA = \iint_R F(r, \theta)\,r\,dr\,d\theta = \int_{\theta_1}^{\theta_2} \int_0^{f(\theta)} F(r, \theta)\,r\,dr\,d\theta$$

Remark: If R is a region bounded by two polar curves $r = f_1(\theta)$ and $r = f_2(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$

$$\text{then} \iint_R F(r, \theta)\,dA = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} F(r, \theta)\,r\,dr\,d\theta.$$

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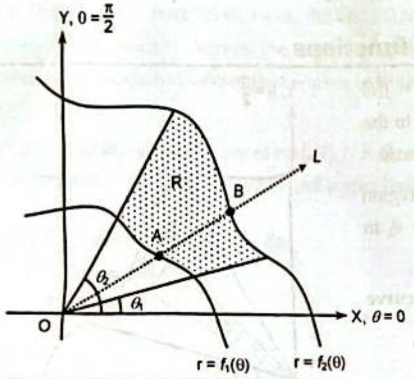
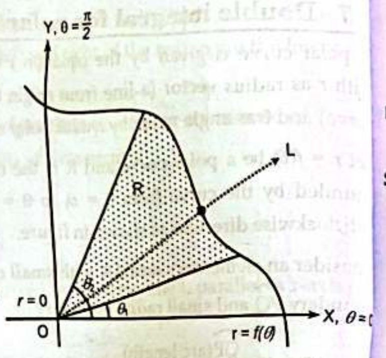
Determination of limits for polar double integral

1. Draw the region R bounded by given curve (curves.)
2. Determine the maximum and minimum values of θ for upper and lower limits of θ (θ_1 is smaller angle made by line L at nearer point of R with polar axis and similarly θ_2 is the larger angle made by line L at farthest point of R with polar axis.)
3. Draw a line L from origin through the region pointing outward directions from the origin. The entering curve and leaving curve for L are lower and upper limits of r. If R is the region bounded by a single curve $r = f(\theta)$ then the lower limit is zero and the upper limit is $r = f(\theta)$.



Region with two boundaries $r = f_1(\theta)$, $r = f_2(\theta)$



Region with a boundary $r = f_1(\theta)$

Example 1. Evaluate: $\int_0^{\pi}\int_0^{a\theta} r^3\, d\theta\, dr$

Solution

We have,

$$I = \int_0^{\pi}\int_0^{a\theta} r^3\, d\theta\, dr$$

$$= \int_0^{\pi}\int_0^{a\theta} r^3\, d\theta\, dr$$

$$= \int_0^{\pi}\left[\int_0^{a\theta} r^3\, dr\right] d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} a^4\theta^4\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi} \theta^4\, d\theta$$

$$= \frac{a^4}{4}\left[\frac{\theta^5}{5}\right]_0^{\pi}$$

$$= \frac{a^4\pi^5}{20}$$

Hence, $\int_0^{\pi}\int_0^{a\theta} r^3\, d\theta\, dr = \frac{a^4\pi^5}{20}$.

Example 2. Evaluate: $\int_1^2\int_0^4 2xy\, dy\, dx$

Solution

We have,

$$\int_1^2\int_0^4 2xy\, dy\, dx = 2\int_1^2 x\, dx \int_0^4 y\, dy$$

$$= 2\left[\frac{x^2}{2}\right]_1^2\left[\frac{y^2}{2}\right]_0^4$$

$$= \frac{1}{2}[4-1][16-0]$$

$$= \frac{1}{2}(48)$$

$$\therefore \int_1^2\int_0^4 2xy\, dy\, dx = 24$$

Example 3. $\int_1^4\int_1^e \frac{\log x}{xy}\, dx\, dy$

Solution

Since both limits for x and y are constants, we can write

$$\int_1^4\int_1^e \frac{\log x}{xy}\, dx\, dy = \int_1^4 \frac{\log x}{x}\, dx \int_1^e \frac{1}{y}\, dy$$

$$= \int_1^4 \log x\, d(\log x) \int_1^e \frac{1}{y}\, dy$$

$$= \left[\frac{(\log x)^2}{2}\right]_1^4 [\log y]_1^e$$

$$= \frac{1}{2}[(\log 4)^2 - (\log 1)^2]\,[\log e - \log 1]$$

$$= \frac{1}{2}[(\log 2^2)^2 - 0]\,[1-0]$$

$$= \frac{1}{2}[(2\log 2)^2]$$

$$\therefore \int_1^4\int_1^e \frac{\log x}{xy}\,dx\,dy = 2(\log 2)^2.$$

Example 4. Evaluate: $\displaystyle\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}}$

Solution

Since both limits for x and y are constants, we can write

$$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}}\int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= [\sin^{-1}x]_0^1\,[\sin^{-1}y]_0^1$$

$$= \left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi^2}{4}$$

Hence, $\displaystyle\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}} = \frac{\pi^2}{4}.$

Example 5. Evaluate:

a. $\displaystyle\int_1^2\int_0^x \frac{1}{x^2+y^2}\,dy\,dx$ b. $\displaystyle\int_0^\pi\int_0^x \sin y\,dy\,dx.$

Solution

a. Here, I $\displaystyle = \int_1^2\int_0^x \left[\frac{1}{x^2+y^2}\right] dy\,dx$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}1 - \frac{1}{x}\tan^{-1}0\right] dx$$

$$= \int_1^2 \frac{1}{x}\cdot\frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\,[\log x]_1^2$$

$$= \frac{\pi}{4}[\log 2 - \log 1]$$

$$= \frac{\pi}{4}\log 2$$

Hence, $\displaystyle\int_1^2\int_0^x \frac{1}{x^2+y^2}\,dy\,dx = \frac{\pi}{4}\log 2.$

b. Here, I $\displaystyle = \int_0^\pi\int_0^x \sin y\,dy\,dx$

$$= \int_0^\pi\left[\int_0^x \sin y\,dy\right] dx$$

$$= \int_0^\pi [-\cos y]_0^x\,dx$$

$$= -\int_0^\pi (\cos x - \cos 0)\,dx$$

$$= -\int_0^\pi \cos x\,dx + \int_0^\pi dx$$

$$= -[\sin x]_0^\pi + [x]_0^\pi$$

$$= -(0-0) + \pi$$

$$= \pi$$

Hence, $\displaystyle\int_0^\pi\int_0^x \sin y\,dy\,dx = \pi.$

Example 6. Evaluate $\displaystyle\iint_R r^3\,dr\,d\theta$, where R is the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution

The boundaries of the region R are $r = 2\sin\theta$ and $r = 4\sin\theta$.

For the circle $r = 2\sin\theta$ (changing into Cartesian form)

or, $r^2 = 2r\sin\theta$

or, $x^2 + y^2 = 2y$

or, $(x-0)^2 + y^2 - 2y + 1 = 0$

$\therefore$ $(x-0)^2 + (y-1)^2 = 1$

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$\therefore$ center at (0, 1) and radius $r = 1$.

Similarly, for $r = 4\sin\theta$

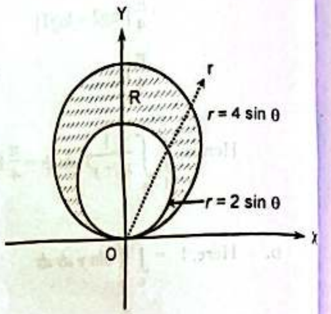
Center at (0, 2) and radius $r = 2$

The region bounded by $r = 2\sin\theta$ and $r = 4\sin\theta$ is shown in figure.

The region R is symmetrical in first and second quadrants. Considering integral in first quadrant.

Limits of r are: $r = 2\sin\theta$ to $r = 4\sin\theta$

and limits of θ are: $\theta = 0$ to $\theta = \frac{\pi}{2}$ (for first quadrant)



$$\therefore \iint_R r^3\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^4\theta\,(256-16)\,d\theta$$

$$= \frac{240}{2}\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta$$

$$= 120\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{6}{2}\right)}$$

$$= \frac{120\,\frac{3}{2}\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\cdot 2}$$

$$= \frac{120\times 3}{16}\sqrt{\pi}\sqrt{\pi}$$

$$\therefore \iint_R r^3\,dr\,d\theta = \frac{45}{2}\pi$$

Example 7. Evaluate: $\displaystyle\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy.$

Solution

We have,

$$I = \int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$$

$$= \int_1^{\log 8}\int_0^{\log y} e^x e^y\,dx\,dy$$

$$= \int_1^{\log 8}\left[\int_0^{\log y} e^x\,dx\right] e^y\,dy$$

$$= \int_1^{\log 8} [e^x]_0^{\log y}\, e^y\,dy$$

$$= \int_1^{\log 8} e^y\,(e^{\log y} - e^0)\,dy$$

$$= \int_1^{\log 8} e^y\,(y-1)\,dy$$

$$= [(y-1)\,e^y - 1\,e^y]_1^{\log 8}$$

$$= [e^y.y - 2e^y]_1^{\log 8}$$

$$= \{e^{\log 8}\log 8 - 2e^{\log 8}\} - \{e - 2e\}$$

$$= e^{\log 8}(\log 8 - 2) + e$$

$$= 8\,(\log 8 - 2) + e$$

Hence, $\displaystyle\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy = 8\,(\log 8 - 2) + e.$

Example 8. Evaluate: $\displaystyle\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy.$

Solution

We have,

$$I = \int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$$

$$= \int_0^{\sqrt{2}}\left[\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dx\right] y\,dy$$

$$= \int_0^{\sqrt{2}} [x]_{-(4-2y^2)^{1/2}}^{(4-2y^2)^{1/2}}\, y\,dy$$

$$= \int_0^{\sqrt{2}} 2\sqrt{4-2y^2}\,y\,dy$$

Put $4 - 2y^2 = t$, $y\,dy = -\frac{1}{4}\,dt$.

When $y = 0$, $t = 4$ and when $y = \sqrt{2}$, $t = 0$.

$$= -\frac{1}{2}\int_4^0 \sqrt{t}\,dt$$

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$$= \frac{1}{2}\int_4^0 t^{1/2}\,dt$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[t^{3/2}\right]_4^0$$

$$= -\frac{1}{3}[0-8]$$

$$= \frac{8}{3}$$

Hence, $\displaystyle\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy = \frac{8}{3}.$

Example 9. Evaluate: $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\,d\theta dr.$

Solution

Let, $\displaystyle I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\,d\theta dr$

$$= \int_0^{\pi}\sin\theta\cos\theta\left[\int_0^{a(1+\cos\theta)} r^3\,dr\right]d\theta$$

$$= \int_0^{\pi}\sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi}(1+\cos\theta)^4\sin\theta\cos\theta\,d\theta.$$

Put $1+\cos\theta = t$ therefore, $-\sin\theta\,d\theta = dt$

Also, if $\theta = 0$, $t = 2$ and if $\theta = \pi$, $t = 0$.

$$\therefore\quad I = \frac{a^4}{4}\int_2^0 t^4(t-1)(-dt)$$

$$= \frac{a^4}{4}\int_0^2 (t^5 - t^4)\,dt$$

$$= \frac{a^4}{4}\left[\frac{t^6}{6}-\frac{t^5}{5}\right]_0^2$$

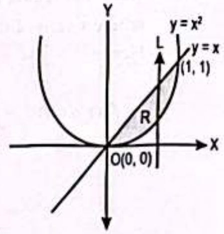
$$= \frac{a^4}{4}\left[\frac{64}{6}-\frac{32}{5}\right]$$

$$= \frac{16}{15}a^4$$

Hence, $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\,d\theta dr = \frac{16}{15}a^4.$

Example 10. Evaluate $\displaystyle\iint_R xy(x+y)\,dx\,dy$ over the area between the parabola $y = x^2$ and the straight line $y = x$.

Solution



The region R is bounded by the curves

$y = f_1(x) = x^2$, $\quad y = f_2(x) = x$.

At the point of intersection, $f_1(x) = f_2(x)$ i.e. $x^2 = x$.

i.e. $x(x-1) = 0$

or, $x = 0$ and $x = 1$.

The region R is shown in figure.

Using figure for the limits

$$\therefore\quad \iint_R xy(x+y)\,dx\,dy = \int_0^1\left[\int_{x^2}^{x} xy(x+y)\,dy\right]dx$$

$$= \int_0^1\left[\int_{x^2}^{x}(x^2y+xy^2)\,dy\right]dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1\left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \int_0^1\left[\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx$$

$$= \left[\frac{x^5}{6}-\frac{x^7}{14}-\frac{x^8}{24}\right]_0^1$$

$$= \left[\frac{1}{6}-\frac{1}{14}-\frac{1}{24}\right]$$

$$= \frac{3}{56}.$$

Hence, $\displaystyle\iint_R xy(x+y)\,dx\,dy = \frac{3}{56}$ sq. units.

Example 11. a. Evaluate: $\displaystyle\iint_R xy\,dy\,dx$ over the circle $x^2+y^2 = a^2$, in the first quadrant.

b. Evaluate $\displaystyle\iint_R \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ where the region of integration of the circle $x^2+y^2 = 1$, in first quadrant.

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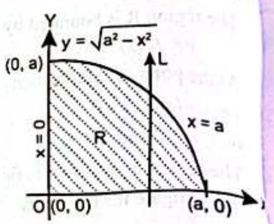
Solution

a. Here, the region of integration R is first quadrant of the circle $x^2 + y^2 = a^2$ i.e. $y = \sqrt{a^2 - x^2}$ where x varies from 0 to a and y varies from 0 to $\sqrt{a^2 - x^2}$.

Hence,



$$\iint_R xy\, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} y\, dy \right] x\, dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\, dx$$

$$= \frac{1}{2}\int_0^a x(a^2 - x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

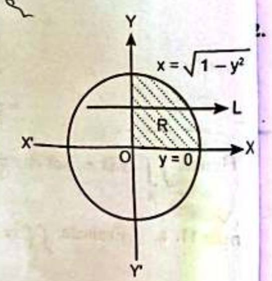
$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]_0^a$$

$$= \frac{1}{8}a^4.$$

Hence, $\iint_R xy\, dy\, dx = \frac{1}{8}a^4$.

b. Here, the region of integration R is the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Using figure for the limits of integration

We have,



$$\iint_R \frac{xy}{\sqrt{1-y^2}}\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{y}{\sqrt{1-y^2}}\, x\, dx\, dy$$

$$= \int_0^1 \frac{y}{\sqrt{1-y^2}} \left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{y}{\sqrt{1-y^2}} \left[\left(\frac{1-y^2}{2}\right) - 0\right] dy$$

$$= \frac{1}{2}\int_0^1 y\sqrt{1-y^2}\, dy$$

Put $1 - y^2 = t^2$, so that $-2y\, dy = 2t\, dt$

$\therefore$ $y\, dy = -t\, dt$

When $y = 0, t = 1$, when $y = 1, t = 0$

$$\therefore \quad \frac{1}{2}\int_0^1 y\sqrt{1-y^2}\, dy = -\frac{1}{2}\int_1^0 t\, t\, dt$$

$$= \frac{1}{2}\int_0^1 t^2\, dt$$

$$= \frac{1}{2}\left[\frac{t^3}{3}\right]_0^1 = \frac{1}{6}[1 - 0]$$

$$\therefore \quad \iint_R \frac{xy}{\sqrt{1-y^2}}\, dx\, dy = \frac{1}{6}$$

Remark: It is better to observe the functional limits and consider order instead of the order of x and y from $dy\, dx$ or $dx\, dy$. For example if we are asked to evaluate the integral $\int_1^2 \int_0^{y^2} f(x, y)\, dy\, dx$, it seems in y and x order by the quantity $dy\, dx$ but the limit does not match because we are getting limit of y in terms y which does not terminate the result. So we first arrange the order of $dy\, dx$ to $dx\, dy$ so that the limit are proper.

i.e., $\int_1^2 \int_0^{y^2} f(x, y)\, dy\, dx = \int_1^2 \int_0^{y^2} f(x, y)\, dx\, dy.$

We changed the order of dy and dx not limit.

Exercise 1.1

1. Evaluate the following double integrals:

a. $\int_0^3 \int_{-1}^0 (x^2y - 2xy)\, dy\, dx$ b. $\int_1^2 \int_{-1}^2 (12xy - 8x^2)\, dy\, dx$

c. $\int_0^1 \int_0^1 \frac{y}{1+xy}\, dx\, dy$ d. $\int_0^{\log 2} \int_0^{\log 5} e^{2x+y}\, dy\, dx$

2. a. $\int_1^2 \int_0^{3y} y\, dx\, dy$ b. $\int_0^1 \int_0^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$

c. $\int_1^2 \int_0^x \frac{1}{x^2+y^2}\, dy\, dx$ d. $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

e. $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\, dy\, dx$ f. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2}\, dy\, dx$

g. $\int_0^{\pi} \int_0^x x \sin y\, dy\, dx$ h. $\int_0^2 \int_{-1}^{2y} (4y - y)^2\, dx\, dy$

i. $\int_0^1 \int_y^{y^2+1} x^2 y\, dx\, dy$ j. $\int_0^{\pi/2} \int_0^{\pi/2} \cos(x + y)\, dy\, dx$

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3. a. Evaluate: $\iint_R 5x^3 \cos(y^3)\, dA$

where R is region bounded by the curves $y = 2, y = \frac{1}{4}x^2$ and y-axis.

b. Evaluate $\iint_R \frac{dA}{y^{1/3}(x^3+1)}$

where R is region bounded by the curves $x = -y^{1/3}, x = 3$ and x-axis.

c. Evaluate $\iint_R e^{y^4}\, dA$

where R is region bounded by the curves $x = y^3$ and line $y = 1$.

4. a. Calculate: $\iint_R x^2y^2\, dx\, dy$ over the region $x^2 + y^2 \le 1$.

b. Evaluate: $\iint_R xy\, dx\, dy$ over the region in the positive quadrant for which $x + y \le 1$.

c. Evaluate $\iint_R xy\, dx\, dy$ where R is region of quarter circle $x^2 + y^2 = a^2$ with $x \ge 0, y \ge 0$.

d. Evaluate $\iint_R \sqrt{xy - y^2}\, dA$, where R is the region of triangle with vertices O(0, 0), A(1, 1), B(10, 1).

5. Evaluate the following double integrals:

a. $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$ b. $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta$

c. $\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta$

6. a. Evaluate $\iint_R r^3\, dr\, d\theta$ over the area bounded by the circle $r = \cos\theta$ and $r = 2\cos\theta$.

b. Evaluate $\iint_R \frac{r\, dr\, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of $r^2 = a^2\cos 2\theta$.

Answer

1.	a. 0	b. 36	c. $2\log 2 - 1$	d. $\frac{3}{2}(5 - e)$	
2.	a. 7	b. $\frac{44}{105}$	c. $\frac{\pi}{4}\log 2$	d. $\frac{1}{2}$	e. $\frac{2}{3}$
	f. $\log(1 + \sqrt{2})$	g. $\left(\frac{\pi^2}{2} + 2\right)$	h. $\frac{36}{5}$	i. $\frac{67}{120}$	j. -2
3.	a. $\frac{20}{3}(\sin 8)$	b. $-\frac{1}{2}\log(28)$	c. $\frac{1}{4}(e - 1)$		
4.	a. $\frac{\pi}{24}$	b. $\frac{1}{24}$	c. $\frac{a^4}{4}$	d. 6	
5.	a. $\frac{\pi a^2}{4}$	b. $\frac{a^2}{6}$	c. 1		
6.	a. $\frac{45}{32}\pi$	b. $\frac{a}{2}(4 - \pi)$			

1.8 Change of order of integration

In the evaluation of double integral for general region R bounded by curves we can change the order of integration so that the evaluation process is easier than the given order.

We follow following steps to change the order of integration.

i. Draw the sketches of the boundaries of the region R.

ii. Determine the limits in different order using increasing directional line L.

iii. If $\int_c^d\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$ is given order than $\int_a^b\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$ will be the integral with reverse order.

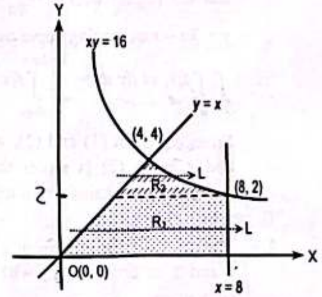
The region R bounded by the boundaries $x = h_1(y)$ and $x = h_2(y)$ from $y = c$ to $y = d$ will be the same region R, bounded by $y = g_1(x)$ and $y = g_2(x)$ from $x = a$ to $x = b$.

Following example will make it clear.

Example 12. If R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and lines $y = x$, $y = 0$ and $x = 8$. Write the limits for both $\iint_R x^2\, dx\, dy$ and $\iint_R x^2\, dy\, dx$.

Solution

The region R in the first quadrant bounded by the hyperbola $xy = 16$ and lines $y = x$, $y = 0$ and $x = 8$ is shown in figure.

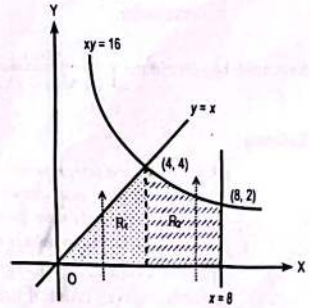


For the integral $\iint_R x^2\, dx\, dy$

The region R has two leaving boundaries for the line L parallel to x in increasing order. So we need to divide the region R into R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \phi$.

$$\iint_R x^2\, dx\, dy = \iint_{R_1} x^2\, dx\, dy + \iint_{R_2} x^2\, dx\, dy$$

$$\therefore \iint_R x^2\, dx\, dy = \int_0^2\int_y^8 x^2\, dx\, dy + \int_2^4\int_y^{16/y} x^2\, dx\, dy \quad \ldots .(1)$$



For the integral $\iint_R x^2\, dy\, dx$

$$= \iint_{R_1} x^2\, dy\, dx + \iint_{R_2} x^2\, dy\, dx$$

$$= \int_0^4\int_0^x x^2\, dy\, dx + \int_4^8\int_0^{16/x} x^2\, dy\, dx$$

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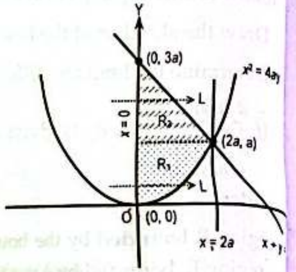
Example 13. If R is a region bounded by $x^2 = 4ay$, $x + y = 3a$, $x = 0$ and $x = 2a$ in first quadrant. D… the region R and write the limits for the integrals $\iint_R f(x, y)\, dx\, dy$ and $\iint_R f(x, y)\, d$… which order x and y or y and x will be easier to evaluate and why ?

Solution

The region R bounded by the curves $x^2 = 4ay$, $x + y = 3a$, $x = 0$ and $x = 2a$ in first quadrant is shown in figure.

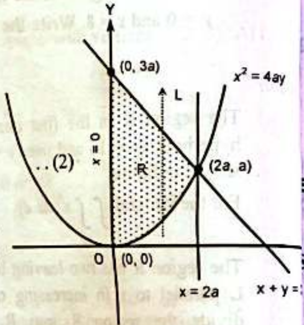


For the order in x and y, the line parallel to x-axis has same entering curve for whole R but has different leaving curve for the regions R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \phi$. So the integral is sum of two integrals

$$\iint_R f(x, y)\, dx\, dy = \iint_{R_1} f(x, y)\, dx\, dy + \iint_{R_2} f(x, y)\, dx\, dy$$

$$\therefore \quad \iint_R f(x, y)\, dx\, dy = \int_0^a \int_0^{2\sqrt{ay}} f(x, y)\, dx\, dy + \int_a^{3a} \int_0^{3a-y} f(x, y)\, dx\, dy \quad \ldots (1)$$

Again, for the order y and x, the line parallel to y-axis has same entering and leaving curves through out the region with $y = \frac{x^2}{4a}$ as entering curve and $y = 3a - x$ as leaving curve as shown in figure.



$$\therefore \quad \iint_R f(x, y)\, dy\, dx = \int_a^{3a} \int_{x^2/4a}^{3a-x} f(x, y)\, dy\, dx \quad \ldots (2)$$

From equation (1) and (2), we see that the order y and x as in (2) is easier than x and y as in (1) because (1) contains two integrals.

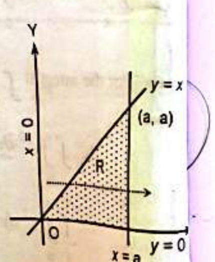
Remarks:

1. The result of integration in x and y order in (1) and y and x as order in (2) will have same result, we can use any one of the form as per convenience.
2. In some cases it may not be possible to integrate in one order but it can be easily integrate… reverse order.

Example 14. Evaluate $\int_0^a \int_0^x \frac{\cos y\, dy\, dx}{\sqrt{(a-x)(a-y)}}$.

Solution

[Note: If we first integrate with respect to y then we have function of y in the numerator and denominator, so it will not be possible to evaluate as we do not have quotient rule in integration. But we will see it is easily evaluated in reverse order.]



We will evaluate the integral by changing order as it is not possible in given order of y and x.

To change the order of integration, we sketch the region R of integration bounded by y = … $y = x$ and $x = 0$, $x = a$ as shown in figure.

Using figure for the limits of integration.

$$\int_0^a \int_0^x \frac{\cos y\, dy\, dx}{\sqrt{(a-x)(a-y)}} = \int_0^a \int_y^a \frac{\cos y}{\sqrt{a-y}} (a-x)^{-1/2}\, dx\, dy$$

$$= \int_0^a \frac{\cos y}{\sqrt{a-y}} \left[\frac{(a-x)^{1/2}}{\frac{1}{2}(-1)}\right]_y^a dy$$

$$= -2\int_0^a \frac{\cos y}{\sqrt{a-y}} [0 - (a-y)^{1/2}]\, dy$$

$$= 2\int_0^a \frac{\cos y}{\sqrt{a-y}} \sqrt{a-y}\, dy$$

$$= 2\int_0^a \cos y\, dy$$

$$= 2[\sin y]_0^a$$

$$= 2\,[\sin a - \sin 0]$$

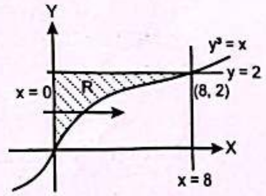
$$= 2\sin a$$

$$\therefore \int_0^a \int_0^x \frac{\cos y\, dy\, dx}{\sqrt{(a-x)(a-y)}} = \int_0^a \int_y^a \frac{\cos y\, dx\, dy}{\sqrt{(a-x)(a-y)}} = 2\sin a$$

Example 15. Evaluate $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4 + 1}$

Solution

The region R of integration is bounded by $y = \sqrt[3]{x}$ i.e., $y^3 = x$, $y = 2$ and $x = 0$, $x = 8$, which is shown in figure.



Using figure for limit of integration

$$\therefore \quad I = \int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4 + 1} = \int_0^2 \int_0^{y^3} \frac{dx\, dy}{y^4 + 1}$$

$$= \int_0^2 \frac{1}{1 + y^4} \left(\int_0^{y^3} dx\right) dy$$

$$= \int_0^2 \frac{1}{1 + y^4} y^3\, dy$$

Let $1 + y^4 = t$

then $4y^3\, dy = dt$

or, $y^3\, dy = \frac{dt}{4}$

If $y = 0$ then $t = 1$ If $y = 2$ then $t = 17$

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$$\therefore \quad I = \int_1^{17} \frac{1}{t}\cdot\frac{1}{4}\,dt$$

$$= \frac{1}{4}\left[\log t\right]_1^{17}$$

$$= \frac{1}{4}(\log 17 - \log 1) = \frac{1}{4}\log 17$$

Hence, $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1} = \frac{1}{4}\log 17.$

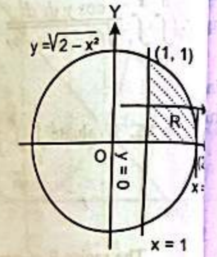
Example 16. Evaluate $\displaystyle\int_1^2 \int_0^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}.$

Solution

The given integral is, $\displaystyle I = \int_1^2 \int_0^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}.$

The region R of integration is bounded by $y = 0$, $y = \sqrt{2-x^2}$, i.e., $x^2 + y^2 = 2$, $x = 1$, x which is shown in figure.

Using figure for the limits of integration in reverse order.



$$I = \int_1^2 \int_0^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1 \int_1^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{2x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\int_0^1 \int_1^{\sqrt{2-y^2}} (x^2+y^2)^{-1/2}\,d(x^2+y^2)\,dy,$$

since $d(x^2+y^2) = 2x\,dx$

$$= \frac{1}{2}\int_0^1 \left[\frac{(x^2+y^2)^{1/2}}{1/2}\right]_1^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left[\sqrt{2-y^2+y^2} - \sqrt{1+y^2}\right] dy$$

$$= \int_0^1 \left[\sqrt{2} - \sqrt{1+y^2}\right] dy$$

$$= \sqrt{2}\int_0^1 dy - \int_0^1 \sqrt{1+y^2}\,dy$$

Using $\displaystyle\int \sqrt{a^2+y^2}\,dy = \frac{y}{2}\sqrt{a^2+y^2} + \frac{a^2}{2}\log\left(y + \sqrt{a^2+y^2}\right)$

$$I = \sqrt{2}\,[y]_0^1 - \left[\frac{y}{2}\sqrt{1+y^2} + \frac{1^2}{2}\log\left(y+\sqrt{1+y^2}\right)\right]_0^1$$

$$= \sqrt{2}(1-0) - \left[\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(1+\sqrt{2})\right\} - \left\{0 + \frac{1}{2}\log 1\right\}\right]$$

$$= \sqrt{2} - \frac{\sqrt{2}}{2} - \frac{1}{2}\log(1+\sqrt{2})$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2}\log(1+\sqrt{2})$$

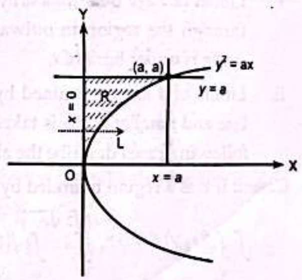
$$\therefore \int_1^2 \int_0^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1 \int_1^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} = \frac{1}{2}\left[\sqrt{2} - \log(1+\sqrt{2})\right]$$

Example 17. Evaluate: $\displaystyle\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$

Solution

Here the integrand is in quotient form, so we cannot integrate in given order. But by changing the order we can easily integrate it.

The region R of integration is bounded by $y = \sqrt{ax}$, i.e., $y^2 = ax$, $y = a$, $x = 0$ and $x = a$, which is shown in figure. Using figure for limits.



$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}} = \int_0^a \int_0^{y^2/a} \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$$

$$= \int_0^a \left[\frac{\sin^{-1}\left(\frac{ax}{y^2}\right)}{a}\right]_0^{y^2/a} y^2\,dy$$

$$= \frac{1}{a}\int_0^a y^2\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$= \frac{1}{a}\int_0^a y^2\left[\frac{\pi}{2} - 0\right] dy$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{6a}(a^3 - 0)$$

$$\therefore \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}} = \frac{\pi a^2}{6}.$$

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1.9 Change of variables in double integral

As we have seen that the change of order in double integral makes easy to evaluate it. In s cases the change of variable also make the evaluation more easier. A common practic changing variable is to change from Cartesian variable i.e., x and y to polar variable i.e., r a We can also change from polar to Cartesian.

We describe the process as follows.

Consider the integral: $\iint_R f(x, y)\, dA$

where, R is the region of integration.

In order to change into polar we suppose $x = r\cos\theta$ and $y = r\sin\theta$ as usual relation betw polar and Cartesian coordinates.

Also, we have the area dA as $dx\, dy$ or $dy\, dx$ in Cartesian form and $r\, dr\, d\theta$ in polar f (see previous section 'double integral for polar functions' 1.7).

i.e., $dA = dx\, dy = dy\, dx = r\, dr\, d\theta$.

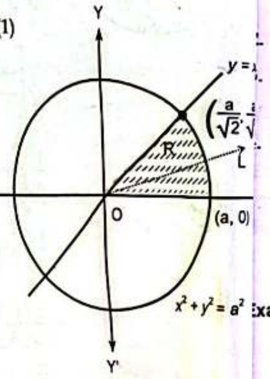
For the limits in polar form, we sketch R in xy-plane with given Cartesian boundaries.

i. Limits of r are determined by entering and leaving curves of the radial line L from or through the region in outward direction. The entering curve is lower limit and leav curve is upper limit of r.

ii. Limits of θ are determined by the angle made by the radius vector (or line L) with in line and smaller angle is taken as lower limit and larger angle is taken as upper limit. following cases describe the above process.

Case I: If R is a region bounded by $x^2 + y^2 = a^2$, $y = x$, $x = 0$, $x = a$ in first quadrant, then

$$\iint_R f(x, y)\, dA = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy \quad \ldots(1)$$



The above integral can be changed into polar form by taking $x = r\cos\theta$, $y = \sin\theta$ so that

$x^2 + y^2 = r^2$

$\therefore$ $r^2 = a^2$, i.e., $r = a$

and $dA = dx\, dy = r\, dr\, d\theta$.

a. The limits of r are $r = 0$ and $r = a$

b. The limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{4}$ (slope of $y = x$)

Thus, The polar form of above integral is

$$\iint_R f(x, y)\, dA = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{a} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Note that variable limit in Cartesian form reduced into constant limit in polar form.

Case II: If R is bounded by the lines $y = x$, $x = a$, $y = 0$, $y = a$ in first quadrant, then

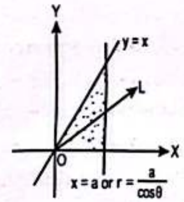
$$\iint_R f(x, y)\, dA = \int_0^{a} \int_y^{a} f(x, y)\, dx\, dy$$

The above integral can be changed into polar form by taking $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dA = dx\, dy = r\, dr\, d\theta$.

The line $x = a$ reduces in polar form as $r\cos\theta = a$, i.e., $r = \frac{a}{\cos\theta}$.

a. The limits of r are $r = 0$ to $r = \frac{a}{\cos\theta}$

b. The limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{4}$ (slope of $y = x$)



∴ The polar form of above integrals is

$$\iint_R f(x, y)\, dA = \int_0^{a} \int_y^{a} f(x, y)\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{a/\cos\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

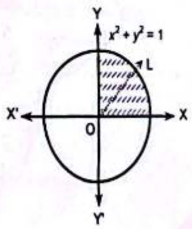
Remarks:

1. If the region R is symmetrical in different quadrants then we can take the portion of R in one quadrant and multiply by number of symmetrical parts.
2. We can also change from polar to Cartesian with $r\cos\theta = x$, $r\sin\theta = y$, $r\, dr\, d\theta = dx\, dy = dy\, dx$.
3. For the problems that contain $(x^2 + y^2)$ expression in the integrand or in the limit, the evaluation in polar form is easy and more useful.
4. We often use the formula

$$\int_0^{\pi/2} \cos^n\theta \sin^m\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{n+m+2}{2}\right)} \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Example 18. By transferring into polar coordinates, evaluate $\iint_R (x^2 + y^2)^{3/2}\, dx\, dy$ over the region bounded by the circle $x^2 + y^2 = 1$.

Solution



The region R of integration is bounded by circle $x^2 + y^2 = 1$ which is symmetrical about all four quadrants. So we can consider the portion in first quadrant for region of integration.

In order to change into polar, put $x = r\cos\theta$, $y = r\sin\theta$, so that $x^2 + y^2 = r^2$.

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Also, $dx\,dy = r\,dr\,d\theta$.

The limits of r are $r = 0$ to $r = 1$ and the limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Thus, the polar equivalent of given integral is

$$\therefore \iint_R (x^2+y^2)^{5/2}\,dx\,dy = 4\int_0^{\pi/2}\int_0^1 (r^2)^{5/2}\,r\,dr\,d\theta, \text{ [} \because \text{ factor 4 by symmetry]}$$

$$= 4\int_0^{\pi/2}\int_0^1 r^6\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{r^7}{7}\right)_0^1 d\theta$$

$$= \frac{4}{7}\int_0^{\pi/2}(1-0)\,d\theta$$

$$= \frac{4}{7}[\theta]_0^{\pi/2}$$

$$= \frac{4}{7}\left(\frac{\pi}{2}-0\right)$$

$$\therefore \iint_R (x^2+y^2)^{5/2}\,dx\,dy = \frac{2\pi}{7}$$

Example 19. Evaluate $\iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy$ over the semicircle $x^2+y^2 = ax$ in first quadrant.

Solution

We have, $\iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy$, where R is the semicircle

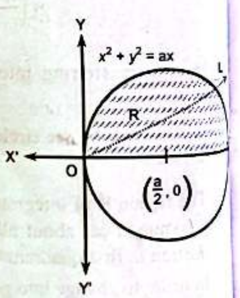
$x^2+y^2 = ax$

or, $x^2 - 2\cdot\frac{a}{2}x + \left(\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

or, $\left(x-\frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$

$\therefore$ Centre of circle is $\left(\frac{a}{2}, 0\right)$ and radius of circle is $\frac{a}{2}$.

The region R is shown in figure.



In order to change into polar put $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$.

For limits of r, we have equation of circle, $x^2+y^2 = ax$, changing into polar form

or, $r^2 = ar\cos\theta$

$\therefore$ $r = 0$ and $r = a\cos\theta$

The limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{2}$. (as the circle touches y-axis)

Using figure for limits, the polar equivalent of given Cartesian integral is,

$$\iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,(-2r\,dr)\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\int_0^{a\cos\theta} (a^2-r^2)^{1/2}\,d(a^2-r^2)\,d\theta \qquad [\because d(a^2-r^2) = -2r\,dr]$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\left(\frac{(a^2-r^2)^{3/2}}{3/2}\right)\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}[a^2(1-\cos^2\theta))^{3/2} - (a^2)^{3/2}]\,d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi/2}[(\sin^2\theta)^{3/2} - 1]\,d\theta$$

$$= -\frac{a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\,d\theta - \int_0^{\pi/2}d\theta\right]$$

$$= -\frac{a^3}{3}\left[\frac{\Gamma\left(\frac{4}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{2}\right)} - [\theta]_0^{\pi/2}\right]$$

$$= -\frac{a^3}{3}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{2\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \left(\frac{\pi}{2}-0\right)\right]$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]$$

$$\iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy = \frac{a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right]$$

ample 20. Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$ by changing into polar coordinates.

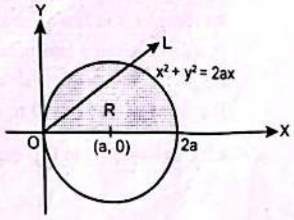
ution

The region R of integration is bounded by $y = 0$, $y = \sqrt{2ax - x^2}$

i.e., $x^2 - 2ax + a^2 + y^2 = a^2$

i.e., $(x-a)^2 + (y-0)^2 = a^2$ (circle), $x = 0$ and $x = 2a$.

The region is shown in figure.



In order to change into polar put $x = r\cos\theta$, $y = r\sin\theta$ so that

$x^2+y^2 = r^2$ and $dy\,dx = r\,dr\,d\theta$

For limits

From $x^2+y^2 = 2ax$, $r^2 = 2ar\cos\theta$

$\therefore$ $r = 0$, $r = 2a\cos\theta$ and limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Hence the given integral becomes

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\,(r\,dr\,d\theta)$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 16a^4\cos^4\theta\,d\theta$$

$$= 4a^4\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= \frac{4a^4\,\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{6}{2}\right)}$$

$$= \frac{4a^4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\cdot 2}$$

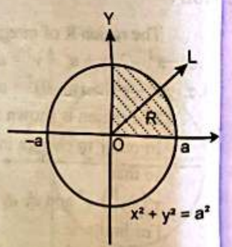
$$= \frac{3}{4}a^4\sqrt{\pi}\sqrt{\pi}$$

Hence, $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx = \frac{3}{4}\pi a^4.$

Example 21. Evaluate $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx$

Solution

The region R of integration is bounded by $y = 0$, $y = \sqrt{a^2-x^2}$ i.e., $x^2+y^2 = a^2$, $x = -a$ and $x = a$, which is shown in figure



The region is symmetric about first and second quadrants, so we consider R in first quadrant.

In order to change into polar put $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$ and $dy\,dx = r\,dr\,d\theta$.

The limits of r are $r = 0$ to $r = a$ and limits of θ are $\theta = 0$ to $\theta = \frac{\pi}{2}$ (considering in first quadrant).

$$\therefore \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx = 2\int_0^{\pi/2}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} e^{-r^2}\,d(r^2)\,d\theta \qquad [\because d(r^2) = 2r\,dr]$$

$$= \int_0^{\pi/2} [-e^{-r^2}]_0^a\,d\theta$$

$$= -\int_0^{\pi/2} [-e^{-a^2} - 1]\,d\theta$$

$$= (1-e^{-a^2})\,[\theta]_0^{\pi/2}$$

$$= (1-e^{-a^2})\left(\frac{\pi}{2}-0\right)$$

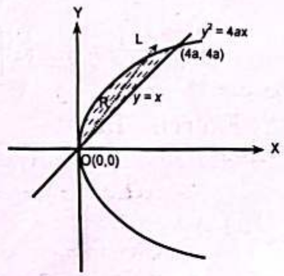
Hence, $\displaystyle\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx = \frac{\pi}{2}(1-e^{-a^2}).$

Example 22. Change the Cartesian Integral $\displaystyle\int_0^{4a}\int_y^{y^2/4a} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$ to polar form and integrate the polar integral.

Solution

We have the integral $\displaystyle\int_0^{4a}\int_y^{y^2/4a} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

The region R of integration is bounded by the curves $x = y$, $x = \frac{y^2}{4a}$, i.e., $y^2 = 4ax$, $y = 0$, $y = 4a$, which is shown in figure.



In order to change into polar, put $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$.

For the limits of r: we have

$y^2 = 4ax$

or, $(r\sin\theta)^2 = 4\,ar\cos\theta$

or, $r^2\sin^2\theta = 4\,ar\cos\theta$

$\therefore$ $r = 0$ and $r = \dfrac{4a\cos\theta}{\sin^2\theta} = 4a\cot\theta\,\mathrm{cosec}\theta$

and the limits of θ are $\theta = \frac{\pi}{4}$ (slope of $y = x$) and $\theta = \frac{\pi}{2}$

(as the parabola touches y-axis).

Thus the polar equivalent of given Cartesian integral is,

$$\int_0^{4a}\int_y^{y^2/4a} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{4a\cot\theta\,\mathrm{cosec}\theta} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}\,r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta-\sin^2\theta)\left[\frac{r^2}{2}\right]_0^{4a\cot\theta\,\mathrm{cosec}\theta} d\theta$$

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$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\, 16a^2(\cot^2\theta\, \mathrm{cosec}^2\theta - 0)$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta\, (\cot^2\theta - 1)\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta\, (\mathrm{cosec}^2\theta - 1 - 1)\, d\theta$$

$$= 8a^2 \left[\int_{\pi/4}^{\pi/2} \cot^2\theta\, \mathrm{cosec}^2\theta\, d\theta - 2\int_{\pi/4}^{\pi/2} \cot^2\theta\, d\theta\right]$$

$$= 8a^2 \left[-\int_{\pi/4}^{\pi/2} \cot^2\theta\, d(\cot\theta) d - 2\int_{\pi/4}^{\pi/2} (\mathrm{cosec}^2\theta - 1)\, d\theta\right]$$

$$= -8a^2 \left[\left\{\frac{\cot^3\theta}{3}\right\}_{\pi/4}^{\pi/2} + 2\left\{-\cot\theta - \theta\right\}_{\pi/4}^{\pi/2}\right]$$

$$= -8a^2 \left[\frac{1}{3}(0-1) + 2\left\{\left(0 + \frac{\pi}{2}\right) - \left(1 + \frac{\pi}{4}\right)\right\}\right]$$

$$= -8a^2 \left[-\frac{1}{3} + 2\left\{\frac{\pi}{2} - 1 - \frac{\pi}{4}\right\}\right]$$

$$= 8a^2 \left[\frac{1}{3} - 2\left(\frac{\pi}{4} - 1\right)\right]$$

$$= 8a^2 \left[\frac{1}{3} - \frac{\pi}{2} + 2\right]$$

$$= 8a^2 \left[\frac{7}{3} - \frac{\pi}{2}\right]$$

$$\therefore \int_0^{4a}\int_y^{y^2/4a} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy = 8a^2\left[\frac{7}{3} - \frac{\pi}{2}\right].$$

Exercise 1.2

A. Evaluate the given integrals by changing the order of integration if necessary.

i. $\int_0^2\int_1^{e^x} dy\, dx$ ii. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$ iii. $\int_0^2\int_x^2 y^2 \sin xy\, dy\, dx$

iv. $\int_0^1\int_{2y}^2 \cos x^2\, dx\, dy$ v. $\int_0^2\int_{y^2}^4 y\cos x^2\, dx\, dy$ vi. $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$

vii. $\int_0^1\int_y^1 x^2 e^{xy}\, dx\, dy$ viii. $\int_0^4\int_y^4 \frac{x\,dx\, dy}{x^2 + y^2}$ ix. $\int_0^b\int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\, dx\, dy$

x. $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$ xi. $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} x\, dx\, dy$ xii. $\int_0^{\infty}\int_0^{x} e^{-xy}\, y\, dy\, dx$

xiii. $\int_0^{\infty}\int_0^{x} x\, e^{\frac{-x^2}{y}}\, dy\, dx$ xiv. $\int_0^1\int_x^1 \sin(y^2)\, dy\, dx$

B. **Evaluate the following integrals by changing into polar form.**

1. $\int_0^2\int_0^x y\, dy\, dx$
2. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$
3. $\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} x\, dx\, dy$
4. $\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2(x^2+y^2)^{1/2}\, dx\, dy$
5. $\int_0^3\int_0^{\sqrt{3x}} \frac{dy\, dx}{\sqrt{x^2+y^2}}$
6. $\int_0^2\int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}\, dy\, dx$
7. $\int_0^a\int_y^a \frac{x\, dx\, dy}{x^2+y^2}$
8. $\int_0^a\int_y^a \frac{x^2\, dx\, dy}{\sqrt{x^2+y^2}}$
9. $\int_0^2\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)\, dx\, dy$
10. $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$
11. $\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\, dy\ (a > 0)$
12. $\int_0^a\int_y^a \frac{1}{x^2+y^2}\, dx\, dy$
13. $\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$
14. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx$
15. $\int_0^1\int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Answer

A. i. $e^2 - 3$ ii. 2   iii. $\frac{4 - \sin 4}{2}$ iv. $\frac{\sin 4}{4}$ v. $\frac{1}{4}\sin 16$

vi. $\frac{\pi}{16}$ vii. $\frac{e-2}{2}$ viii. π ix. $\frac{a^2b^2}{8}$ x. 1

xi. $\frac{a^3\sqrt{2}}{6}$ xii. $\frac{\sqrt{\pi}}{2}$ xiii. $\frac{1}{2}$ xiv. $\frac{1}{2}$

B. 1. $\frac{4}{3}$ 2. πa^2 3. $\frac{a^3\sqrt{2}}{6}$ 4. $\frac{\pi a^5}{20}$ 5. $\frac{3}{2}\log 3$

6. $\frac{4}{3}$ 7. $\frac{\pi a}{4}$ 8. $\frac{a^3}{3}\log(\sqrt{2}+1)$ 9. $\frac{\pi}{4}\sin 4$ 10. $\frac{\pi}{4}$

11. $\frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right)$ 12. $\frac{\pi a}{4}$ 13. $\frac{\pi a^5}{20}$

14. $\frac{3}{4}\pi a^4$ 15. $\left(\frac{3\pi}{8} - 1\right)$

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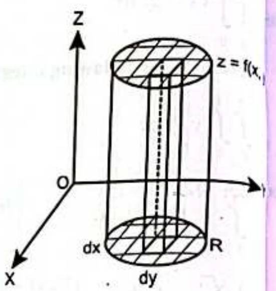
1.10 Application of double integral to determine area of plane region and volume of a solid

1. The double integral $\iint_R f(x, y)\, dA$ is evaluated on the region R in xy-plane.

   If $z = f(x, y)$, then $f(x, y)$ gives a surface at height z from the base R of the region of integration.

   Thus, $f(x, y)\, dA$ [height $z = f(x, y)$ times area dA] an elementary volume of the solid above the element dA of R bounded above by the surface $z = f(x, y)$, so $\iint_R f(x, y)\, dA$ is the volume of solid bounded above by surface $z = f(x, y)$ over the region R.

   i.e., Volume of solid (V) = $\iint_R f(x, y)\, dA$.



2. If $z = f(x, y) = 1$ (constant), then

   $\iint_R f(x, y)\, dA = \iint_R dA = \iint_R dx\, dy = \iint_R dy\, dx$ gives the area of region R in xy plane

   $z = f(x, y) = 1$.

3. The polar double integral $\iint_R f(r, \theta)\, dA = \iint_R f(r, \theta)\, rd\, r\, d\theta$ gives volume of solid above th

   region R with upper boundary $f(r, \theta)$.

4. If $f(r, \theta) = 1$, then the polar double integral $\iint_R f(r, \theta)\, dA = \iint_R dA = \iint_R rd\, r\, d\theta$ represent

   area of R with boundaries as polar curves.

Example 23. Show that the area of the region bounded by the line $x = \frac{1}{4}$ and the parabola $y^2 = 4x$ is $\frac{1}{3}$ by using double integration.

Solution

The region R of integration bounded by the lines $x = \frac{1}{4}$, $y^2 = 4x$ is shown in figure.

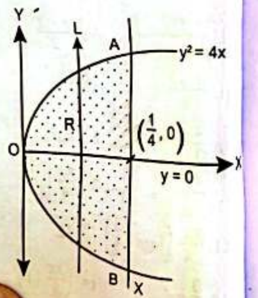
The area of the region R is

$$A = \iint_R dx\, dy$$

Since R is symmetric in two quadrants

$$\therefore \quad A = 2\int_0^{1/4}\int_0^{2\sqrt{x}} dy\, dx$$

$$= 2\int_0^{1/4} [y]_0^{2\sqrt{x}}\, dx$$



$$= 2\int_0^{1/4} \{2\sqrt{x} - 0\}\, dx$$

$$= 4\left[\frac{x^{3/2}}{3/2}\right]_0^{1/4}$$

$$= \frac{8}{3}\left[\left(\frac{1}{4}\right)^{3/2} - 0\right]$$

$$= \frac{8}{3}\left[\left(\frac{1}{8}\right) - 0\right]$$

$$= \frac{1}{3}$$

$$\therefore \quad A = \frac{1}{3}$$

Hence, required area = $\frac{1}{3}$ sq. units.

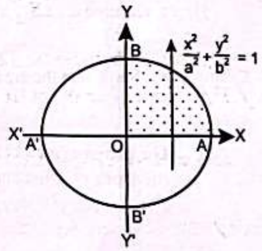
Example 24. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, by double integration.

Solution

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is symmetrical about all four quadrants is shown in figure.

The required area of the ellipse = 4 (area of the first quadrant OABO of the ellipse.)

$$\therefore \quad \text{Total area} = 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dy\, dx$$



$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2 - x^2}}\, dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_0^a \qquad [\because \text{formula of } \int\sqrt{a^2 - x^2}\, dx]$$

$$= \frac{2b}{a}\,[a^2 \sin^{-1}(1)]$$

$$= \frac{2b}{a} \cdot a^2 \cdot \frac{\pi}{2}$$

$$= \pi ab.$$

Hence, required area is πab sq. units.

Remark: We can also integrate by substituting $x = a \sin\theta$.

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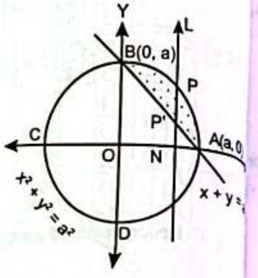
Example 25. Find by double integration, the area of the region enclosed by the curves $x^2 + y^2 = a^2, x + y$ in the first quadrant.

Solution

The area enclosed by the curves $x^2 + y^2 = a^2$, $x + y = a$ in first quadrant is shown in fi
Using figure for limits we can write the area A as,

$$\therefore \quad A = \iint_R dy\, dx$$



$$= \int_{x=0}^{a} \int_{y=a-x}^{\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a [y]_{a-x}^{\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2} - (a-x)\right] dx$$

$$= \left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{1}{2}x^2\right]_0^a$$

$$= \left[\frac{1}{2}a^2\left(\frac{\pi}{2}\right) - a^2 + \frac{1}{2}a^2\right] = \frac{1}{2}a^2\left(\frac{1}{2}\pi - 1\right) = \frac{1}{4}a^2(\pi - 2).$$

Hence, required area is $\frac{1}{4}a^2(\pi - 2)$ sq. units.

Example 26. Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution

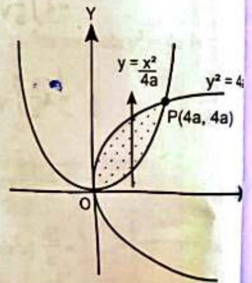
The common area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is shown in figure. Solving equations $y^2 = 4ax$ and $x^2 = 4ay$ for the points of intersection, we have

$x^2 = 4ay$ and $y^2 = 4ax$,

i.e., $x = \frac{y^2}{4a}$

$\therefore \quad \left(\frac{y^2}{4a}\right)^2 = 4ay$

Also, $x = \frac{y^2}{4a}$, $x = 0$, $x = 4a$



$\therefore$ The points of intersection are O(0, 0) and P(4a, 4a).
Therefore, the required area A is given by

$$A = \int_0^{4a} \int_{x^2/4a}^{\sqrt{4ax}} dy\, dx$$

$$= \int_0^{4a} [y]_{x^2/4a}^{\sqrt{4ax}}\, dx$$

$$= \int_0^{4a} [y]_{x^2/4a}^{\sqrt{4ax}}\, dx$$

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$= 2\sqrt{a}\int_0^{4a} x^{1/2}\, dx - \frac{1}{4a}\int_0^{4a} x^2\, dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}(a^{1/2})(8a^{3/2}) - \frac{1}{12a}[(4a)^3 - 0]$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Thus the area between parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Example 27. Find the area bounded by the curves $x = y^2$, $x = 2y - y^2$.

Solution

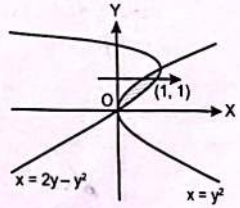
The given equations are $x = y^2$ and $x = 2y - y^2$.

or, $y^2 - 2y + 1 = -x + 1$

$\therefore \quad (y - 1)^2 = -(x + 1)$

Which are parabolas as shown in figure,
Solving the equations for point of intersection, we get (0, 0) and (1, 1) as point of intersection.
The required area is given by,

$$\therefore \quad \text{Area} = \int_0^1 \int_{y^2}^{2y-y^2} dx\, dy$$



$$= \int_0^1 [x]_{y^2}^{2y-y^2}\, dy$$

$$= \int_0^1 (2y - y^2 - y^2)\, dy$$

$y(2-y)$

$$= \left[\frac{2y^2}{2} - \frac{2y^3}{3}\right]_0^1$$

$$= \left[\frac{2}{2} - \frac{2}{3}\right]$$

$$= \frac{1}{3}$$

Hence, the required area $= \frac{1}{3}$ sq. units.

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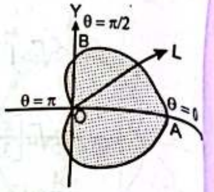
Example 28. Using double integration, find the area of the curve $r = a(1 + \cos\theta)$.

Solution

The curve is symmetrical with respect to initial line OX. Hence we need to consider only portion of the area above the line OX. The region of integration is $0 \le \theta \le \pi$ and $0 \le r \le a(1 + \cos\theta)$.



$\therefore$ Required area $= 2 \times$ area of OABO.

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$\therefore$ Total area

$$= 2\int_{\theta=0}^{\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} [a(1+\cos\theta)]^2\,d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta = 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

Let $\frac{\theta}{2} = \phi$

$\Rightarrow$ $d\theta = 2\,d\phi$. If $\theta = 0$, $\phi = 0$ and if $\theta = \pi$, $\phi = \frac{\pi}{2}$.

$\therefore$ Required area

$$= 8a^2\int_0^{\pi/2}\cos^4\phi\,d\phi$$

$$= 8a^2\,\frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{6}{2}}$$

$$= \frac{8a^2\cdot\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2(2)}$$

$$= \frac{3}{2}a^2\sqrt{\pi}\sqrt{\pi}$$

$$= \frac{3}{2}a^2\pi.$$

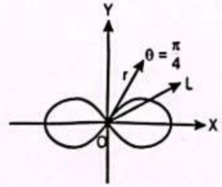
Hence, required area is $\frac{3}{2}a^2\pi$ sq. units.

Example 29. Find the area of the lemniscate of Bernoulli's $r^2 = a^2\cos 2\theta$ by double integration.

Solution

The curve is symmetrical about the initial line OX. Here, we consider only the portion of the area above the line OX in the first quadrant. The limits of r are $r = 0$ and $r = a\sqrt{\cos 2\theta}$ and θ are $\theta = 0$ and $\theta = \frac{\pi}{4}$.

$$0 \le \theta \le \frac{\pi}{4} \text{ and } 0 \le r \le a\sqrt{\cos 2\theta}$$



Hence, required area

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} (r\,dr)\,d\theta$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2\left[\sin\frac{\pi}{2} - 0\right]$$

$$= a^2$$

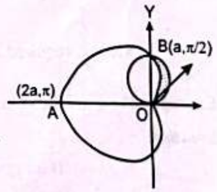
Hence, required area is a^2 sq. units.

Example 30. Show by double integration that the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$ is $\frac{a^2}{4}(4 - \pi)$.

Solution

The shaded region in the figure is the required area.

The required area $= \iint_R r\,dr\,d\theta$, the limits of integration are



Limits of r are $r = a(1 - \cos\theta)$ to $r = a\sin\theta$ and limits of θ are 0 to $\frac{\pi}{2}$.

Hence, the required area

$$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}[a^2\sin^2\theta - a^2(1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} [(1-\cos^2\theta)-(1-\cos\theta)^2]\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1-\cos\theta)\,(1+\cos\theta-1+\cos\theta)\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 2\cos\theta\,(1-\cos\theta)$$

$$= a^2\int_0^{\pi/2} (\cos\theta-\cos^2\theta)\, d\theta$$

$$= a^2\int_0^{\pi/2}\left[\cos\theta-\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= a^2\int_0^{\pi/2}\left[\cos\theta-\frac{\cos 2\theta}{2}-\frac{1}{2}\right] d\theta$$

$$= a^2\left[\sin\theta-\frac{1}{2}\sin 2\theta-\frac{\theta}{2}\right]_0^{\pi/2}$$

$$= a^2\left[\sin\frac{\pi}{2}-\frac{1}{2}\sin\left(2\times\frac{\pi}{2}\right)-\frac{\pi}{4}\right]$$

$$= a^2\left\{(1-0)-\frac{\pi}{4}\right\}$$

$$= a^2\left(1-\frac{\pi}{4}\right)$$

$$= \frac{a^2}{4}(4-\pi)$$

17.06

Hence, required area is $\frac{a^2}{4}(4-\pi)$ sq. units.

Example 31. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

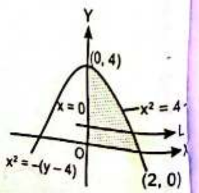
Solution

We have the upper surface of the solid $z = 4 - x^2 - y$.

For the base of solid (region of integration) projecting the surface $z = 4 - x^2 - y$ to xy-pl[ane] where $z = 0$.

$\therefore$ $x^2 = -(9-4)$ is the (projection) base of solid in xy-plane, which is a parabola as show[n in] figure.

The required volume V of solid in first octant is



$$V = \iint_R f(x, y)\, dA$$

$$= \int_0^4\int_0^{\sqrt{4-y}} (4-x^2-y)\, dx\, dy$$

$$= \int_0^4\int_0^{\sqrt{4-y}} [(4-y)-x^2]\, dx\, dy$$

256

$$= \int_0^4\left[(4-y)x-\frac{x^3}{3}\right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4\left[(4-y)^{3/2}-\frac{(4-y)^{3/2}}{3}\right] dy$$

$$= \frac{2}{3}\int_0^4 (4-y)^{3/2}\, dy$$

$$= \frac{2}{3}\left[\frac{(4-y)^{5/2}}{\left(-\frac{5}{2}\right)}\right]_0^4$$

$$= -\frac{4}{15}[(4-y)^{5/2}]_0^4$$

$$= -\frac{4}{15}[0-4^{5/2}]$$

$$= \frac{128}{15}$$

Hence, the required volume is $\frac{128}{15}$ cubic units.

Example 32. Find the volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

Solution

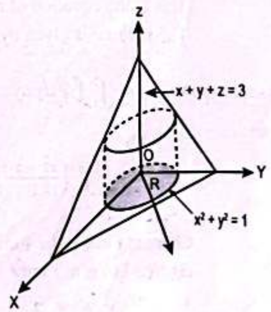
The equation of cylinder is $x^2+y^2=1$ and the equation of plane is

$$x+y+z=3 \Rightarrow z=3-x-y$$

The solid (cylinder) has circular base $x^2+y^2=1$ in xy-plane and bounded above by the plane $x+y+z=3$

i.e., $z=3-x-y$

$\therefore$ The required volume V,



$$V = \iint_R z\, dx\, dy$$

$$= \iint_R (3-x-y)\, dx\, dy$$

We evaluate the value of $\iint_R (3-x-y)\, dx\, dy$ by changing into polar.

For this put $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2+y^2=r^2$ and $dx\, dy = r\, dr\, d\theta$. Since the base is circular, so the limits of r are $r = 0$ and $r = 1$(from $x^2+y^2=1$) and limits of θ are $\theta = 0$ to $\theta = 2\pi$.

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$$\therefore\ V = \int_0^{2\pi}\int_0^1 [3 - r(\cos\theta + \sin\theta)]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r\, dr\, d\theta - \int_0^{2\pi}\int_0^1 (\cos\theta + \sin\theta)\, r^2\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta - \int_0^{2\pi}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \frac{3}{2}\int_0^{2\pi}(1-0)\, d\theta - \frac{1}{3}\int_0^{2\pi}(\cos\theta+\sin\theta)(1-0)\, d\theta$$

$$= \frac{3}{2}[\theta]_0^{2\pi} - \frac{1}{3}[\sin\theta - \cos\theta]_0^{2\pi}$$

$$= \frac{3}{2}2\pi - \frac{1}{3}[(0-\cos 2\pi) - (0-\cos 0)]$$

$$= 3\pi - \frac{1}{3}[-1+1]$$

$$= 3\pi$$

Hence, the required volume is 3π cubic units.

Example 33. Find the volume bounded by the paraboloid $x^2 + y^2 = az$ as the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

Solution

The equation of the paraboloid is $x^2 + y^2 = az$

i.e., $z = \dfrac{x^2+y^2}{a}$ and the cylinder is $x^2 + y^2 = 2ay$

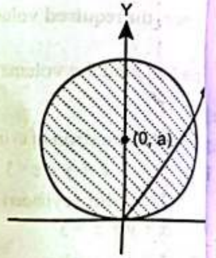
In the xy-plane, the circular base of cylinder is

$x^2 + y^2 - 2ay = 0$

$x^2 + y^2 + 2ay = 0$

or, $(x-0)^2 + (y-a)^2 = a^2$

This is the equation of a circle with centre at $(0, a)$ and radius a.



The volume is given by

$$V = \iint_R z\, dx\, dy$$

1 imp

$$= \iint_R \left(\frac{x^2+y^2}{a}\right) dx\, dy$$

Changing into polar form, by substituting $x = r\cos\theta$, $y = r\sin\theta$. So that $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$. The limits of r are $r = 0$ to $r = 2a\sin\theta$ and limits of θ are $\theta = 0$ to $\theta = \pi$.

$$V = \int_0^{\pi}\int_0^{2a\sin\theta} \frac{r^2}{a}\, r\, dr\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta \qquad \left[\begin{array}{l} x^2+y^2 = 2ay \\ r^2 = 2ar\sin\theta \\ \therefore r = 2a\sin\theta \end{array}\right]$$

$$= \frac{1}{a}\int_0^{\pi} 4a^4 \sin^4\theta\, d\theta$$

$$= 4a^3 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta, \text{ by symmetry}$$

$$= 8a^3 \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}$$

$$= 8a^3 \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot 2!}$$

$$= 8a^3 \frac{3}{16}\sqrt{\pi}\sqrt{\pi} = \frac{3}{2}\pi a^3$$

Hence, the required volume is $\frac{3}{2}\pi a^3$ cubic units.

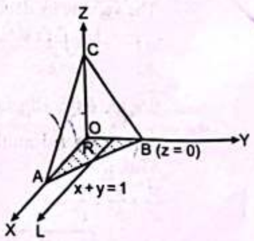
Example 34. Find the volume enclosed by the coordinate planes and that portion of the plane $x + y + z = 1$, which lies in the first octant.

Solution

The required volume is given by $V = \iint_R z\, dx\, dy$

where $z = 1 - x - y$ and R is the region bounded by $x = 0$, $y = 0$, and $x + y = 1$ (see figure alongside).

Hence the required volume is given by



$$V = \int_0^1\int_0^{1-y} (1 - x - y)\, dx\, dy$$

$$= \int_0^1\int_0^{1-y} [(1-y) - x]\, dx\, dy$$

$$= \int_0^1 \left[(1-y)x - \frac{x^2}{2}\right]_{x=0}^{x=1-y} dy$$

$$= \int_0^1 \left[(1-y)^2 - \frac{(1-y)^2}{2}\right] dy$$

$$= \frac{1}{2}\int_0^1 (1-y)^2\, dy$$

$$= \frac{1}{2}\left[\frac{(1-y)^3}{3(-1)}\right]_0^1$$

$$= -\frac{1}{6}[0 - 1] = \frac{1}{6}$$

Hence, the required volume is $\frac{1}{6}$ cubic units.

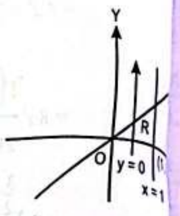
Example 35. Find the volume of the solid whose base is in the xy-plane and is the triangle bounded by the x-axis, the line $y = x$ and the line $x = 1$ while the top of the solid is in the plane $z = x + y,$

Solution

Volume of the elementary strip of altitude z is

$dV = z\, dy\, dx = (x + y + 1)\, dy\, dx.$

For any x between 0 and 1, y varies from $y = 0$, $y = x$. Therefore,



$$V = \int_0^1\int_0^x (x + y + 1)\, dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} + y\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left[\frac{3x^2}{2} + x\right] dx$$

$$= \left[\frac{3x^3}{6} + \frac{x^2}{2}\right]_0^1$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) - (0 + 0)$$

$$= 1$$

Hence, the required volume is 1 cubic units.

Example 36. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by double integral.

Solution

The volume by double integral,

$$I = \iint_R z\, dA \qquad \ldots(1)$$

We have the ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is symmetric about all eight octant, so we evaluate first octant and multiply by 8.

Also, from

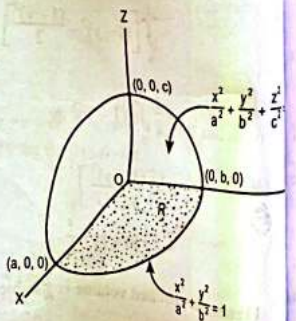
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\therefore \quad z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

The region of integration R is the projection of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ to xy-plane, i.e., $z = 0$

and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Which is an ellipse.



$$\therefore \quad I = \iint_R z\, dA$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(\sqrt{1 - \frac{x^2}{a^2}}\right)^2 - \left(\frac{y}{b}\right)^2}\, dy\, dx$$

$$= 8c\int_0^a \frac{1}{1/b}\left[\frac{y}{2b}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} + \frac{\left(1 - \frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{\frac{y}{b}}{\sqrt{1 - \frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 8bc\int_0^a \left[\left\{0 + \left(\frac{a^2 - x^2}{2a^2}\right)\sin^{-1}(1)\right\} - \{0\}\right] dx$$

$$= 8\,\mathrm{bc}\int_0^a \left(\frac{a^2 - x^2}{2a^2}\right)\sin^{-1}(1)\, dx \qquad \left[\because \int\sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{8\,\mathrm{bc}\pi}{4a^2}\int_0^a (a^2 - x^2)\, dx$$

$$= \frac{2\pi \mathrm{bc}}{a^2}\left[a^2\,[x]_0^a - \left[\frac{x^3}{3}\right]_0^a\right]$$

$$= \frac{2\pi \mathrm{bc}}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

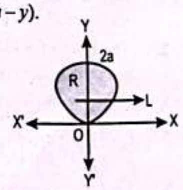
$$= \frac{2\pi \mathrm{bc}}{a^2}\frac{[3a^3 - a^3]}{3}$$

$$= \frac{4}{3}\pi abc$$

$$\therefore \quad \iint_R z\, dx\, dy\, dz = \frac{4}{3}\pi abc.$$

Example 37. Find, by double integration the whole area of the curve $a^2x^2 = y^3(2a - y)$.

Solution



The equation of the curve is $a^2x^2 = y^3(2a - y)$

$$\text{or,} \quad x^2 = \frac{y^3(2a - y)}{a^2}$$

$$\therefore \quad x = \frac{y^{3/2}\sqrt{(2a - y)}}{a}$$

The curve is symmetrical about y-axis and the curve is bounded between $y = 0$ and $y = 2a$

Required area $= 2 \times$ Area in first quadrant

$$= 2\int_0^{2a}\int_0^{\frac{y^{3/2}\sqrt{2a-y}}{a}} dx\, dy$$

$$= 2\int_0^{2a} [x]_0^{\frac{y^{3/2}\sqrt{2a-y}}{a}} dy$$

$$= \frac{2}{a}\int_0^{2a} y^{3/2}\sqrt{2a - y}\, dy$$

Put $y = 2a\sin^2\theta$ so that $dy = 4a\sin\theta\cos\theta\, d\theta$

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When $y = 0$, then $\theta = 0$ and when $y = 2a$ then $\theta = \frac{\pi}{2}$.

$\therefore$ Required area $= \frac{2}{a}\int_0^{\pi/2} (2a\sin^2\theta)^{3/2}\sqrt{2a - 2a\sin^2\theta}\,.\,4a\sin\theta\cos\theta\, d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta\sqrt{2a}\,\cos\theta\,4a\sin\theta\cos\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta.\cos^2\theta\, d\theta$$

$$= 32a^2\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{8}{2}\right)}$$

$$= 32a^2.\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3.2.1},\text{ using Gamma function}$$

$$= \pi a^2$$

Hence, the area of the curve $a^2x^2 = y^3(2a - y)$ is πa^2.

Exercise 1.3

1. Sketch the region bounded by the equation and find its area

   i. $y^2 = -x, x - y = 4, y = -1, y = 2$ ii. $y = \frac{1}{x^2}, y = -x^2, x = 1, x = 2$
   iii. $x = y - y^2, x + y = 0$ iv. $y = x, y = x^2$ in first quadrant
   v. $y = e^x, x = 0$ and $x = 1$ vi. $2y = 16 - x^2,\ x + 2y = 4$

2. The base of a solid is the region in the xy-Plane that is bonded by the circle $x^2 + y^2 = a^2$, whil top of the solid is bounded by the paraboloid $az = x^2 + y^2$, find the volume.
3. Find the volume bounded by the xy-Plane, the parabolic $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$
4. Find the volume of the solid whose base in the region in the xy-Plane. that is bonded by the par $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bonded by the Plane $z = x + y$.
5. Find the volume of the cylinder $x^2 + y^2 = 4$ intercepted by the plane $z + y = 3$ in first octant.
6. The prism is such that its triangular base is in xy-Plane bonded by $y = x$ and $x = 1$. If its top li the Plane $z = 3 - x - y$, find its volume.
7. a. Find by double integration, the area of the circle $x^2 + y^2 = a^2$.
   b. Show that by double integration, the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.
8. a. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.
   b. Find the area of the region bounded by quadrant of the circle $x^2 + y^2 = a^2$ and the straight l $x + y = a$.
   c. Find the area of the region bounded by the parabola $y^2 = x$ and the straight line $y = x$.
   d. Find the area of the region bounded by the parabola $y = 4x - x^2$ and line $y = x$.
   e. Use double integration, to find the area of the region bounded by the parabolas $y^2 = 4x$ $x^2 = 4y$.

9. Find the volume bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
10. Find volume of sphere $x^2 + y^2 + z^2 = a^2$ by double integration.
11. Find the common area between the cardioids $r = a(1 - \cos\theta)$ and $r = a(1 + \cos\theta)$.

Answer

1. i. $\frac{33}{2}$	ii. $\frac{17}{6}$	iii. $\frac{4}{3}$	iv. $\frac{1}{6}$	v. $e - 1$	vi. $\frac{343}{12}$
2. $\frac{2\pi a^3}{4}$	3. 4π	4. $\frac{625}{12}$	5. $3\pi - \frac{8}{3}$	6. 1	7. a. πa^3
8. a. $\frac{8a^3}{3}$	b. $(\pi - 2)\frac{a^2}{4}$	c. $\frac{1}{10}$	d. $\frac{9}{2}$	e. $\frac{16}{3}$	9. a. $\frac{abc}{6}$
10. $\frac{4}{3}\pi a^3$	11. $a^2\left(\frac{3\pi}{2} - 4\right)$				

1.11 Triple Integration

Function of three variable: A function of three variables is of the form $u = f(x, y, z)$ and its domain is a three dimensional region i.e., volume and range is set of value of function f at the points (x, y, z) in the domain.

Geometrically, a function of three variable is possible to represent in 4-dimension, which we can realize but can not draw physically. The graph (geometrical representation) will be a surface in 4-dimension known as hyper surface and the 4-dimension (and higher spaces) are known as hyper spaces.

1.11.1 Integration of function of three variables:

Let $u = f(x, y, z)$ be a function of three variable with the domain V (for volume) in space. Then consider the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$, where $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$

Where (x_i, y_i, z_i) is any point in the elementary volume ΔV_i obtaining by portioning (dividing) V into n-sub-volumes by the planes parallel to the coordinate planes, $z = 0$ (xy-plane), $y = 0$ (xz-plane) and $x = 0$ (yz-plane).

If the limit, $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$ exists (i.e., finite), it is called triple integral of $f(x, y, z)$ over V

and written as $\iiint_v f(x, y, z)\, dV = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$, where $dV = dx\, dydz$

1.11.2 Evaluation of triple integral

The triple integral of $f(x, y, z)$ over a volume V is evaluated by the iteration (repetition) of integrals considering one of the variables at a time keeping other two unchanged and again the resulting double integral is evaluated by iteration considering one of the variable at a time keeping other variable unchanged and finally the last integration is done for the remaining variable.

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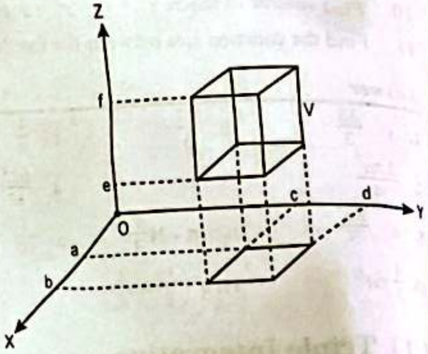
We need to observe base surface of V for lower limit of z, top surface of V for upper limit of z and project the base surface of V to xy-plane and apply the idea of double integral in the projected region in xy-plane for limits of x and y.

A. Integration in Cartesian form:

1. If all the boundaries are planes i.e., V is a cuboid with V: $a < x < b, c < y < d, e < z < f$, then

$$\iiint_v f(x, y, z)\, dv = \int_a^b \int_c^d \int_e^f f(x, y, z)\, dzdy\, dx$$



2. If some of the boundaries are curved then we may have following case,

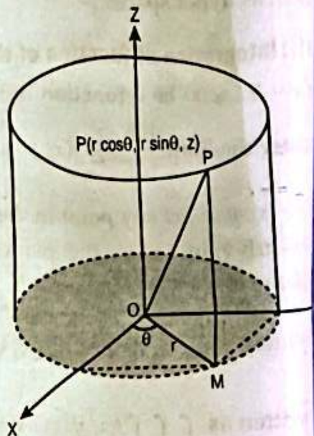
i. $$\iiint_v f(x, y, z)\, dv = \int_a^b \int_c^d \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dzdy\, dx$$

ii. $$\iiint_v f(x, y, z)\, dv = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dzdy\, dx$$

B. Integration by changing variables

1. **Use of cylindrical polar coordinates:**

If we rotate vertical line PM in xy-plane, the locus of M will be a circle $x^2 + y^2 = r^2$ in xy-plane and hence PM will generate a circular cylinder with circular base $x^2 + y^2 = r^2$ and height z.



We can evaluate triple integral by changing the Cartesian variables x, y, z in cylindrical coordinates by taking

$x = r\cos\theta, y = r\sin\theta, r = \sqrt{x^2 + y^2}, 0 \le \theta \le 2\pi.$

Keeping height z as z i.e., $(x, y, z) = (r\cos\theta, r\sin\theta, z)$

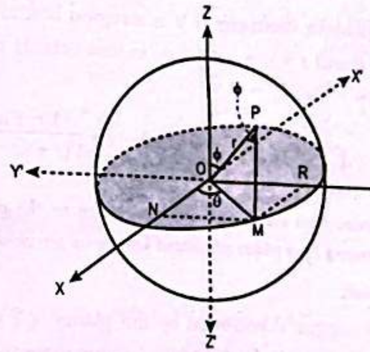
And $dV = dx\, dydz = r dr d\theta dz$ (in double integral we had $dx\, dy = r dr d\theta$)

Thus, $$\iiint_V f(x, y, z)\, dV = \int_e^f \int_{\theta_1}^{\theta_2} \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta, z)\, r dr d\theta dz$$

The set of equations $x = r\cos\theta, y = r\sin\theta, z = z, x^2 + y^2 = r^2, 0 \le \theta \le 2\pi$ is known as param[etric] equation of cylinder.

2. **Use of spherical polar coordinates:**

In this case we project OP on xy-plane represented by OM and rotate OM in xy-plane so that locus of M is a circle in xy-plane and OP about z-axis so that locus of P in vertical plane is a semi-circle then the combined locus of P and M generate a sphere in space.



Let $P(x, y, z)$ be the point such that $OP = r$.

Then in right angled triangle OMP,

$OM = OP\cos\phi = r\cos\phi$ and

$PM = z = OP$

$\sin\phi = r\sin\phi$

Now, In ΔOMN,

$ON = x = OM\cos\theta$

$\therefore x = r\cos\phi\cos\theta$

And $OR = MN = y = OM\sin\theta$

$\therefore y = r\cos\phi\sin\theta$

Thus, the coordinate (variables) of a point $P(x, y, z)$ in spherical polar coordinate become $(x, y, z) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$, where

$r^2 = x^2 + y^2 + z^2,\ 0 < \phi < \pi$ or $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ and $0 \le \theta \le 2\pi.$

The set of equations

$x = r\cos\phi\cos\theta,\ y = r\cos\phi\sin\theta,\ z = r\sin\phi$

$0 \le \theta \le 2\pi,\ -\frac{\pi}{2} < \phi < \frac{\pi}{2}$ or $0 < \phi < \pi$

Completely describe the sphere and θ is known as the parametric equation of sphere.

Also, $dV = dx\, dydz = r^2\sin\phi\, dr d\theta d\phi$

Thus, $$\iiint_V f(x, y, z)\, dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1 = h_1(\theta, \phi)}^{r_2 = h_2(\theta, \phi)} f(r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)\, r^2\sin\phi\, dr d\theta d\phi$$

Remark:

In a triple integral if

i. $(x^2 + y^2)$ is present in limit or function we use cylindrical polar coordinates.

ii. $(x^2 + y^2 + z^2)$ is present in limit or function we use spherical coordinates.

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Example 39. Evaluate: $\iiint_V (x-y-z)\,dx\,dydz$, where V: $1 \le x \le 2,\ 2 \le y \le 3,\ 1 \le z \le 3.$

Solution

We have,

$$\iiint_V (x-y-z)\,dx\,dydz = \int_1^3\int_2^3\int_1^2 (x-y-z)\,dx\,dydz$$

$$= \int_1^3\int_2^3 \left(\frac{3}{2}-y-z\right) dydz$$

$$= \int_1^3 \left[\frac{3}{2}[y]_2^3 - \left[\frac{y^2}{2}\right]_2^3 - z\,[y]_2^3\right] dz$$

$$= \int_1^3 \left[\frac{3}{2}(3-2) - \frac{1}{2}(9-4) - z(3-2)\right] dz$$

$$= \int_1^3 \left(\frac{3}{2}-\frac{5}{2}-z\right) dz$$

$$= \int_1^3 (-1-z)\,dz$$

$$= -\left[[z]_1^3 + \left[\frac{z^2}{2}\right]_1^3\right]$$

$$= -\left[(3-1)+\frac{1}{2}(9-1)\right]$$

$$= -[2+4]$$

$$= -6$$

$$\therefore \iiint_V (x-y-z)\,dx\,dydz = -6.$$

Example 40. $\displaystyle\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

Solution

We have,

$$I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^x e^y e^z\,dz\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y\,[e^z]_0^{x+\log y}\,dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y\,[y\,e^x - 1]\,dy\,dx$$

$$= \int_0^{\log 2} e^x\left[\int_0^x (y\,e^x - 1)\,e^y\,dy\right] dx$$

$$= \int_0^{\log 2} e^x\left[(y\,e^x - 1)\,e^y - e^x e^y\right]_0^x dx$$

$$= \int_0^{\log 2} e^x\left[\{(x\,e^x - 1)\,e^x - e^x e^x\} - \{(0-1)e^0 - e^x e^0\}\right] dx$$

$$= \int_0^{\log 2} e^x\left[x\,e^{2x} - e^x - e^{2x} + 1 + e^x\right] dx$$

$$= \int_0^{\log 2} e^x\left[(x-1)\,e^{2x} + 1\right] dx$$

$$= \int_0^{\log 2} (x\,e^{3x} - e^{3x} + e^x)\,dx$$

$$= \left[x\frac{e^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2}$$

$$= \left\{\log 2\left(\frac{8}{3}\right) - \frac{8}{9} - \frac{8}{3} + 2\right\} - \left\{-\frac{1}{9} - \frac{1}{3} + 1\right\}$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3}\log 2 - \frac{7}{9} - \frac{7}{3} + 1$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

$$\therefore \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx = \frac{8}{3}\log 2 - \frac{19}{9}.$$

Example 41. Compute $\iiint_V x^2yz\ dx\ dy\ dz$ over the volume of tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$

Solution

We have the triple integral

$$I = \iiint_V x^2yz\,dx\,dy\,dz \qquad \ldots(1)$$

where, V is the tetrahedron in first octant bounded by planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1.$

We will use Dirichlet's integral to reduce $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ to the form $x + y + z = 1$.

Put $x = au$, $y = bv$ and $z = cw$ so that $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$ and $u + v + w = 1$.

When $x = 0$, $u = 0$, $y = 0$, $v = 0$, $z = 0$, $w = 0$

So applying Dirichlet's integral

Thus, we have

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$$\iiint_V x^2yz\,dx\,dy\,dz = \iiint_V (au)^2(bv)(cw)\,adu\,bdv\,cdw$$

$$= a^3b^2c^2 \iiint_V u^2 v\,w\,du\,dv\,dw$$

$$= a^3b^2c^2 \iiint_V u^{3-1}v^{2-1}w^{2-1}du\,dv\,dw$$

$$= a^3b^2c^2\frac{\Gamma3\,\Gamma2\,\Gamma2}{\Gamma(3+2+2+1)}$$

$$= a^3b^2c^2\frac{\Gamma(2+1)\,\Gamma(1+1)\,\Gamma(1+1)}{\Gamma(7+1)}$$

$$= \frac{a^3b^2c^2\,2\cdot1\cdot1}{7!}$$

$$= \frac{2a^3b^2c^2}{1\times2\times3\times4\times5\times6\times7}$$

$$= \frac{a^3b^2c^2}{2520}$$

$$\therefore \iiint_V x^2yz\,dx\,dy\,dz = \frac{a^3b^2c^2}{2520}.$$

Example 42. Evaluate $\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume of the sphere $x^2+y^2+z^2 = 1$

Solution

The sphere $x^2+y^2+z^2 = 1$ is symmetrical about 8 octants and the integrand $(x^2+y^2+z^2)$ symmetric in x, y and z. So, the given triple integral is 8 times the integral in first octant. Changing into spherical polar coordinates with $x = r\cos\phi\cos\theta$, $y = \cos\phi\sin\theta$ and $z = r\sin\phi$ so that $x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$ with $r = 0$ to $r = 1$ and $0 \le \theta = \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}$

$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^2 r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2}\sin\phi\,d\phi\int_0^1 r^4\,dr$$

$$= 8\,[\theta]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^1$$

$$= 8\left(\frac{\pi}{2}-0\right)\left\{-\left(\cos\frac{\pi}{2}-\cos0\right)\right\}\frac{(1-0)}{5}$$

$$= 4\pi\cdot1\cdot\frac{1}{5}$$

$$= \frac{4\pi}{5}$$

$$\iint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{4\pi}{5}$$

Example 43. Evaluate $\iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ over the sphere $x^2+y^2+z^2 = a^2$.

Solution

The $x^2+y^2+z^2 = a^2$ is symmetric in all 8 octants and the integrand is symmetric in x, y and z. Using spherical polar coordinates consistency first octant with $x = r\cos\phi\cos\theta$, $y = \cos\phi\sin\theta$, $z = r\sin\phi$ so that $x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi$ and $r = 0$ to $r = a$, $0 \le \theta\,\theta\frac{\pi}{2}$,

$0 \le \phi\,\phi\frac{\pi}{2}$ for first octant.

$$\therefore \iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{r^2}$$

$$= 8\int_0^{\pi/2} d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^a dr$$

$$= 8[\theta]_0^{\pi/2}\,[-\cos\phi]_0^{\pi/2}\,[r]_0^a$$

$$= 8\left(\frac{\pi}{2}-0\right)\left\{-\cos\frac{\pi}{2}-\cos0\right\}(a-0)$$

$$= 4\pi a$$

$$\therefore \iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2} = 4\pi a.$$

Example 44. Evaluate $\iiint_V xyz\,dx\,dy\,dz$ over the sphere $x^2+y^2+z^2 = a^2$ in first octant.

Solution

We have,

$$I = \iiint_V xyz\,dx\,dy\,dz \qquad \ldots(1)$$

and the volume is part of sphere $x^2+y^2+z^2 = a^2$ in first octant.

We apply Dirichlet's integral for this we need to change the sphere $x^2+y^2+z^2 = a^2$ into plane.

Put $x^2 = a^2u$, $y^2 = a^2v$, $z^2 = a^2w$

$\therefore$ $2xdx = a^2du$, $2ydy = a^2dv$, $2zdz = a^2dw$

$\therefore$ $xdx = \frac{a^2}{2}du$, $ydy = \frac{a^2}{2}dv$, $zdz = \frac{a^2}{2}dw$

So that $x^2+y^2+z^2 = a^2$ becomes

$a^2(u+v+w) = a^2$

$\therefore$ $u+v+w = 1$

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$$\therefore \quad I = \iiint_V x dx \, y dy \, z dz$$

$$= \iiint_V \frac{a^2}{2} du \frac{a^2}{2} dv \frac{a^2}{2} dw$$

$$= \frac{a^6}{8} \iiint u^{1-1} v^{1-1} w^{1-1} du \, dv \, dw$$

$$= \frac{a^6}{8} \frac{\Gamma 1 \Gamma 1 \Gamma 1}{\Gamma(1+1+1+1)}$$

$$= \frac{a^6}{8\Gamma(3+1)}$$

$$= \frac{a^6}{8 \times 3!}$$

$$\iiint_V dyz \, dx \, dy \, dz = \frac{a^6}{48}.$$

Example 45. Evaluate the integral $\iiint_v x \, dx \, dy \, dz$ over the region v in first octant bounde[d by] $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

Solution

Let $I = \iiint_V x \, dx \, dy \, dz$ ....(1)

We apply Dirichlet's integral, for this we need to change the surface $x^{2/3} + y^{2/3} + z^{2/3} = 1$ in octant into plane. For this put,

$x = u^{3/2}, y = v^{3/2}, z = w^{3/2}$

$dx = \frac{3}{2} u^{1/2} du, \; dy = \frac{3}{2} v^{1/2} dv, \; dz = \frac{3}{2} w^{1/2} dw$

and $x^{2/3} + y^{2/3} + z^{2/3} = 1$ becomes $u + v + w = 1$.

Thus, the equivalent value of the region in first octant bounded by $x^{2/3} + y^{2/3} + z^{2/3} = 1$ is region in first octant bounded by $u + v + w = 1$. So applying Dirichlet's integral, we have

$$U = \iiint_V x \, dx \, dy \, dz$$

$$= \iiint_V u^{3/2} \cdot \frac{3}{2} u^{1/2} du \cdot \frac{3}{2} v^{1/2} dv \cdot \frac{3}{2} w^{1/2} dw$$

$$= \frac{27}{8} \iiint_V u^2 v^{1/2} w^{1/2} du \, dv \, dw$$

$$= \frac{27}{8} \iiint_V u^{3-1} v^{3/2-1} w^{3/2-1} du \, dv \, dw$$

$$= \frac{27}{8} \cdot \frac{\Gamma 3 \, \Gamma\frac{3}{2} \Gamma\frac{3}{2}}{\Gamma\left(3 + \frac{3}{2} + \frac{3}{2} + 1\right)}$$

$$= \frac{27}{8} \frac{\Gamma(2+1) \frac{1}{2} \Gamma\frac{1}{2} \frac{1}{2} \Gamma\frac{1}{2}}{\Gamma(6+1)}$$

$$= \frac{27}{8} \frac{2 \frac{1}{2} \sqrt{\pi} \frac{1}{2} \sqrt{\pi}}{6!}$$

$$= \frac{9\pi}{16(1 \times 2 \times 3 \times 4 \times 5 \times 6)}$$

$$= \frac{\pi}{1280}.$$

$$\therefore \iiint_V x \, dx \, dy \, dz = \frac{\pi}{1280}.$$

Example 46. Evaluate $\iiint_B 12xy^2z^3 \, dv$ over a rectangular box given by $B = \{(x, y, z) \in R^3; -1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2\}$.

Solution

We have,

$$\iiint_B 12xy^2z^3 \, dv = \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3 \, dz dy \, dx$$

$$= \int_{-1}^{2}\int_0^3 12xy^2 \left[\frac{z^4}{4}\right]_0^2 dy \, dx$$

$$= 12 \int_{-1}^{2}\int_0^3 xy^2 \left(\frac{16-0}{4}\right) dy \, dx$$

$$= 48 \int_{-1}^{2} x \left[\frac{y^3}{3}\right]_0^3 dx$$

$$= 48 \int_{-1}^{2} x \left(\frac{27-0}{3}\right) dx$$

$$= 48 \times 9 \int_{-1}^{2} x \, dx$$

$$= 48 \times 9 \left[\frac{x^2}{2}\right]_{-1}^{2}$$

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$$= 48 \times 9 \left(\frac{4-(-1)^2}{2}\right)$$

$$= 48 \times 9 \times \frac{3}{2}$$

$$= 24 \times 9\sqrt{3}$$

$$= 216 \times 3 = 648$$

Example 47. Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and the planes $z = 1$ and $x + z = 5$.

Solution

We have

$$x^2 + y^2 = 9$$

$$\therefore \quad y = \pm\sqrt{9-x}$$

$$\Rightarrow \quad \sqrt{9-x^2} \le y \le \sqrt{9-x^2}$$

Also $z = 5 - x$

$$\therefore \quad 1 \le z \le 5 - x$$

Also $x^2 + y^2 = 9$ is a circle of radius 3.

$$\therefore \quad -3 \le y \le 3$$

$$\therefore \quad \text{Volume} \iiint_V dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-x} dz\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [z]_1^{5-x}\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [5-x-1]\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx$$

$$= \int_{-3}^{3}(4-x)\,[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx$$

$$= \int_{-3}^{3}(4-x)\,2\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx$$

$$= 8 \times 2\int_{0}^{3}\sqrt{9-x^2}\,dx - 0 \qquad [\text{as } 2x\sqrt{9-x^2} \text{ is an odd function}]$$

$$= 16\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= 16\left[\left\{0 + \frac{9}{2}\sin^{-1}(1)\right\} - 0\right]$$

$$= 16\left(\frac{9}{2}\,\frac{\pi}{2}\right)$$

$$= 36\pi$$

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - 0$$

$$= 36\pi$$

Exercise 1.4

Evaluate

i. $\int_2^3\int_1^2\int_0^1 (x + y + z)\,dx\,dy\,dz$

ii. $\int_1^2\int_0^1\int_{-1}^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$

iii. $\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$

iv. $\int_0^2\int_1^3\int_1^2 xy^2z\,dx\,dy\,dz$

Evaluate

i. $\int_{-1}^1\int_0^2\int_{x-z}^{x+z} (x - y + z)\,dy\,dx\,dz$

ii. $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

iii. $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,dz\,dx\,dy$

iv. $\int_0^2\int_0^y\int_{x-y}^{x+y} (x + y + z)\,dx\,dy\,dz$

v. $\int_0^a\int_0^{a-x}\int_0^{a-x-y} x^2\,dx\,dy\,dz$

Find volume of hemisphere $x^2 + y^2 + z^2 = a^2$, by triple integral

Compute $\iiint_V \frac{dx\,dy\,dz}{(x + y + z + 1)^3}$; $V : 0 \le x \le 1, 0 \le y \le (1 - x), 0 \le z \le (1 - x - y)$

Evaluate $\iiint_V x^2\,dx\,dy\,dz$ over the region v bounded by $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Evaluate $\iiint_V x^2\,dx\,yz\,dz$ over the region v bounded by the places $x = 0, y = 0, z = 0$ and $x + y + z = a$.

Evaluate $\iiint_V y\,dx\,dy\,dz$ where v is the region in first octant bounded by $x + y + z = 1$.

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by triple integral.

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Answer

1. i. $\frac{9}{2}$ ii. 6 iii. 1 iv. 8
2. i. 0 ii. $\frac{5}{8}$ iii. $\frac{1}{2}(e^2 - 8e + 13)$ iv. 16 v. $\frac{a^5}{60}$
3. $\frac{2}{3}\pi a^3$
4. $\frac{1}{2}\log 2 - \frac{5}{16}$
5. $\frac{a^3bc}{60}$
6. $\frac{a^5}{6}$
7. $\frac{1}{2a}$
8. $\frac{4}{3}\pi abc$

Unit 2

Series Solution of Differential Equations and Special Functions

Pre-requisite knowledge	Expected learning outcomes
Before starting this unit, students are expected to have fundamental concepts and evaluation skills on • concept of infinite series, power series and their convergence • recurrence formula • different ways of solving first and second order differential equations	After completion of this unit, student will develop sufficient knowledge and evaluation skills on • solving the differential equations of first and second order through series solution. • stating Bessel's function and Legendre's equations and Rodrigue's formula. • proving the identities related to Bessel's function. • proving the identities related to Legendre's polynomial.

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2.1 Introduction

In Calculus-I we have studied several algebraic methods that involve elementary functio solve differential equations with constant coefficients and some equations (Euler-Cauchy) variable coefficients reducible to constant coefficients by substitution. Differential equa with variable coefficients, in general, are not solvable by algebraic methods. We need methods to solve such equations. We will develop the idea of series solution for solving equations.

The solutions of differential equations by series method provide solutions to the differen equations of engineering importance. The Legendre's and Bessel's equations are two impo equation that we solve by series method and give rise new kind of functions that are know special functions. We will discuss two standard series method for solving ordinary differe equations.

The first method is known as power series method because it gives solutions in the form power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

The second method is the extension of power series in the form $x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$, whi a power series multiplied by ℓnx or fractional power x^r. This method is applicable to Bes type equations. The solutions of Legendre's and Bessel equations are new functions which not elementary known as special functions.

2.2 Power series

We recall from the infinite series in the course 'Algebra and Geometry' that a power series i infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

where, $a_0, a_1, a_2, \ldots$ are constants known as coefficients (to be determined) and x_0 is called centre about which the series converges i.e., centre of convergence. The distance of any p other than x_0 such that the series is convergent is known as radius of convergence. If such se converges inside an interval we can compute values, graph the solutions and explore proper of the solutions of the differential equations.

The power series solutions of differential equations may match with the Taylor's (Maclaur for $x_0 = 0$) series of some elementary functions for simple differential equations.

Some familiar Maclaurin's series (power series) are:

i. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots ; |x| < 1$

ii. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$; for all x

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, for all x

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$, for all x

.1 Method of power series described

p 1 : Take the power series $y = \sum_{n=0}^{\infty} a_n x^n$ as the solution of given differential equation.

p 2 : Differentiate the series in up to the order of differential equation, i.e., find $y', y'', \ldots$, etc.

p 3 : Substitute y and its derivatives $y', y'', \ldots$, in given differential equation.

p 4 : Simplify the series in each terms making power of x same x^n in all series. For this purpose you may need to replace index n in some series by suitable $n+1$, $n+2$, etc. to make each series with x^n. In some case, you may need to separate some of the terms to make initial index n same in all series.

p 5 : Find the recurrence (recursive, algorithm) relation of the coefficients by equating the coefficients of combined series to zero. Determine $a_0, a_1, a_2, a_3, \ldots$ from the recurrence relation taking $n = 0, 1, 2, \ldots$ successively. Note that the power series solution will contain an arbitrary constant a_0 in case of first order equation and a_0, a_1 in case of second order equation, i.e., containing the same number of arbitrary constants as order of differential equation.

p 6 : Substitute all coefficients in the power series $y = \sum_{n=0}^{\infty} a_n x^n$, which will be the required general power series solution of given equation. In simpler case the power series may be same as Maclaurin's series of elementary functions e^x, $\sin x$, $\cos x$, $\frac{1}{1-x}$, etc.

mple 1. Solve the equation $y' = y$ by power series method.

ution

Given,

$y' = y$

$\therefore \quad y' - y = 0 \qquad \ldots(1)$

Let $y = \sum_{n=0}^{\infty} a_n x^n \qquad \ldots(2)$

be the power series solution of (1), then

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

- The derivative of first constant term a_0 is zero so the series starts from $n = 1$.

Substituting y and y' in (1)

$$\sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

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Let's make powers of x in both series same i.e., x^n by replacing n by $n+1$ in first seri

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or, $$\sum_{n=0}^{\infty} [a_{n+1}(n+1) - a_n]x^n = 0$$

which shows that the series in L.H.S. is *identically* equal to zero, *which is* true coefficients of x^n are zero.

i.e., $a_{n+1}(n+1) - a_n = 0$

$\therefore \quad a_{n+1} = \dfrac{a_n}{n+1}$

which is known as recursive formula (recurrence relation) from which we get all coeffic

We have,

$$a_{n+1} = \frac{a_n}{n+1}$$

Taking $n = 0, 1, 2, 3, \ldots$, successively

$$a_1 = \frac{a_0}{0+1} = a_0$$

$$a_2 = \frac{a_1}{1+1} = \frac{a_0}{2} = \frac{a_0}{2!}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}$$

$$a_4 = \frac{a_3}{4} = \frac{a_0}{3!\,4} = \frac{a_0}{4!}$$

and so on and $a_n = \dfrac{a_0}{n!}$

Thus, the power series solution (2) becomes

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{a_0 x^n}{n!}$$

$$\therefore \quad y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \ldots\right)$$

$$\therefore \quad y = a_0 e^x$$

which is the general solution of (1), as it contains one arbitrary constant and the equatio order one.

Example 2. Use the power series to solve the equation $y'' + y = 0$.

Solution

Given equation is

$$y'' + y = 0 \qquad \ldots(1)$$

which is a second order *differential* equation, so its general solution will contain two ar constants.

$$\text{Let } y = \sum_{n=0}^{\infty} a_n x^n \qquad \ldots(2)$$

be the power series solution of (1). Then differentiating two times, we get

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}.$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

Substituting y, y'' in (1), we get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

To simplify the series we make same exponent (power) of x i.e., x^n in both series. Replacing n by $n+2$ in first series then n will start from 0.

$$\therefore \quad \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore \quad \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + a_n]x^n = 0$$

The series in L.H.S. is identically zero, so for this the coefficients must be identically zero.

$$\therefore \quad a_{n+2}(n+2)(n+1) + a_n = 0$$

$$\therefore \quad a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad \ldots(3)$$

which is a recursive (recurrence, algorithm) relation for determining the coefficients in the power series solution.

Since the equation is of second order, so its general solution contains two arbitrary constants, namely a_0 and a_1 which are arbitrary.

Now, from (3)

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

Taking $n = 0, 1, 2, 3, \ldots$, successively, we get

$$a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!} \qquad a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!}$$

$$a_4 = -\frac{a_2}{12} = \frac{a_0}{2!\,3\cdot 4} = \frac{a_0}{4!} \qquad a_5 = -\frac{a_3}{20} = \frac{a_1}{3!\,20} = \frac{a_1}{3!\,4\cdot 5} = \frac{a_1}{5!}$$

$$a_6 = -\frac{a_4}{30} = -\frac{a_0}{4!\,5\cdot 6} = -\frac{a_0}{6!} \qquad a_7 = -\frac{a_5}{42} = -\frac{a_1}{5!\,6\cdot 7} = -\frac{a_1}{7!}$$

and all pattern follows same.

Thus, substituting all a_n's in (2) we get the series solution of (1) as,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_1}{4!}x^4 + \frac{a_1}{5!}x^5 + \ldots$$

$$\therefore \quad y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

i.e., $y = a_0 \cos x + a_1 \sin x$

Thus, the power series solution of (1) reduces to the elementary functions $\sin x$ and $\cos x$.

Note: It is not always the case that power series solution of a differential equation reduces to elementary functions. In such case the solution will be in the form of infinite power series.

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Example 3. Solve $y'' = 2y$ by power series method.

Solution

Given equation is

$$y'' = 2y$$

or, $y'' - 2y = 0$ ....(1)

Let $y = \sum_{n=0}^{\infty} a_n x^n$ ....(2)

be solution of (1). Differentiating (2) successively,

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substituting y and y'' in (1), we get

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Replacing n by $n+2$ in first series so that n starts from 0. We have,

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore \quad \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 2a_n] x^n = 0$$

The series in L.H.S. is identically zero, so we must have the coefficients in L.H.S. equal to zero.

$$\therefore \quad a_{n+2}(n+2)(n+1) - 2a_n = 0$$

$$\therefore \quad a_{n+2} = \frac{2a_n}{(n+2)(n+1)} \quad \text{....(3)}$$

Taking $n = 0, 1, 2, 3, \ldots$ successively in (3), we get

$$a_2 = \frac{2a_0}{2} = a_0$$

$$a_3 = \frac{2a_1}{6} = \frac{a_1}{3}$$

$$a_4 = \frac{2a_2}{12} = \frac{a_0}{6}$$

$$a_5 = \frac{2a_3}{20} = \frac{1}{10}\left(\frac{a_1}{3}\right) = \frac{a_1}{30}$$

Substituting values of all a_n's, the series solution of (1) is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + a_0 x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{6}x^4 + \frac{1}{30}a_1 x^5 + \ldots$$

$$\therefore \quad y = a_0\left(1 + x^2 + \frac{x^4}{6} + \ldots\right) + a_1\left(x + \frac{x^3}{3} + \frac{1}{30}x^5 + \ldots\right)$$

$$y = a_0 y_1(x) + a_1 y_2(x)$$

where, $y_1 = 1 + x^2 + \frac{x^4}{6} + \ldots$ and $y_2(x) = x + \frac{x^3}{3} + \frac{a_1}{30}x^5 + \ldots$

which is the required general power series solution of (1).

Example 4. Solve $y'' - 2xy' + y = 0$

Solution

Given equation is $y'' - 2xy' + y = 0$ ....(1)

Let $y = \sum_{n=0}^{\infty} a_n x^n$ ....(2)

be the power series solution of equation.

Differentiating (2) successively, we get

$$y' = \sum_{n=1}^{\infty} a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substituting y, y' and y'' in (1)

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

or, $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

Replacing n by $n+2$ in first series so that we have x^n in the series.

i.e., $\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$$\therefore \quad \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 2na_n + a_n] x^n = 0$$

Which is an identity, so coefficient of x^n must be zero.

$$\therefore \quad a_{n+2}(n+2)(n+1) - (2n-1)a_n = 0$$

$$\therefore \quad a_{n+2} = \frac{(2n-1)a_n}{(n+2)(n+1)} \quad \text{....(3)}$$

Which is recurrence relation of coefficients.

Taking $n = 0, 1, 2, 3, \ldots$ successively in (3), we get

$$a_2 = -\frac{a_0}{2!}$$

$$a_3 = \frac{a_1}{6} = \frac{a_1}{3!}$$

$$a_4 = \frac{3a_2}{12} = \frac{3}{12}\left(-\frac{a_0}{2}\right) = -\frac{3a_0}{4!}$$

$$a_5 = \frac{5a_3}{20} = \frac{5}{20}\left(\frac{a!}{3!}\right) = \frac{5}{5!}a_1$$

Substituting a_n's in (2), the series solution of (1) is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

$$= a_0 + a_1 x + \left(-\frac{a_0}{2!}\right)x^2 + \left(\frac{a_1}{3!}\right)x^3 + \left(\frac{3a_0}{-4!}\right)x^4 + \frac{5}{5!}a_1 x^5 + \ldots$$

$$\therefore \quad y = a_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 + \ldots\right) + a_1\left(x - \frac{1}{3}x^3 + \frac{5}{5!}x^5 + \ldots\right)$$

$$\therefore \quad y = a_0 y_1(x) + a_2 y_2(x)$$

is the general solution of (1)

Where, $y_1(x) = 1 - \frac{x^2}{2!} - \frac{3}{4!}x^4 - \ldots$

$$y_2(x) = x - \frac{x^3}{3!} - \frac{5}{5!}x^5 + \ldots$$

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Example 5. Solve $y'' - 4xy' + (4x^2 - 2)\,y = 0$ by power series method solution.

Solution

The given differential equation is

$$y'' - 4xy' + (4x^2 - 2)\,y = 0 \qquad \ldots(1)$$

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots(2)$

be the solution of equation (1).

Then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots \qquad \ldots(3)$$

Also

$$y'' = 2a_2 + 2.3a_3x + 4.3a_4x^2 + \ldots$$
$$= 2a_2 + 6a_3x + 12a_4x^2 + \ldots \qquad \ldots(4)$$

Now, replacing the values of (2), (3) and (4) in (1)

$(2a_2 + 6a_3x + 12a_4x^2 + \ldots) - 4x\,(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots) + (4x^2 - 2)\,(a_0 + a_1x +$

$a_3x^3 + \ldots) = 0$

or, $(2a_2 - 2a_0) + (6a_3 - 4a_1 - 2a_1)\,x + (12a_4 - 8a_2 + 4a_0 - 2a_2)\,x^2 + (20a_5 - 12a_3 - 4a_1 - 2a_3)x^3$

Now comparing the coefficients of like terms:

$2a_2 - 2a_0 = 0$ [coefficient of x^0]

$\Rightarrow$ $a_2 = a_0$

and $6a_3 - 4a_1 - 2a_1 = 0$ [coefficient of x^1]

or, $6a_3 - 6a_1 = 0$

$\Rightarrow$ $a_3 = a_1$

Again,

$12a_4 - 8a_2 + 4a_0 - 2a_2 = 0$ [coefficient of x^2]

or, $12a_4 = 6a_0 \Rightarrow a_4 = \frac{1}{2}a_0$

Then,

$20a_5 - 12a_3 - 4a_1 - 2a_3 = 0$

or, $20a_5 = 14a_1 - 4a_1 = 10a_1$

$\Rightarrow$ $a_5 = \frac{1}{2}a_1$

Now replacing the value of $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ we get

$$y = a_0 + a_1x + a_0x^2 + a_1x^3 + \frac{1}{2}a_0x^4 + \frac{1}{2}a_1x^5 + \ldots$$

or, $y = a_0\left(1 + x^2 + \frac{1}{2}x^4 + \ldots\right) + a_1\left(1 + x^3 + \frac{1}{2}x^5 + \ldots\right)$

is the required solution.

2.3 Theory of power series method

2.3.1 Analytical function

A function $f(x)$ is said to be analytic at $x = x_0$ if it ca[...] a power series i[...] powers of $(x - x_0)$ as the Taylor's series (Maclaurin's [...] interval containi[...] i.e., a function $f(x)$ is analytic at $x = x_0$, if a power [...] onverges to $f(x)$ interval of convergence centered at x_0 and positive

Theorem: (Existence of power series solutions)

The differential equation

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

with $p(x)$, $q(x)$, $r(x)$ are analytical at $x = x_0$, then every solution of the differential equation (1) is analytical at $x = x_0$ and hence can be represented by a power series about $x = x_0$ with an interval of convergence containing x_0 and a positive radius.

Note: If $r(x) = 0$, which is obviously analytical at every point $x = x_0$, then the equation (1) is homogeneous and has a power series solution.

2.3.2 Regular (ordinary) and singular points of differential equations

Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad \ldots(1)$$

or, $y'' + \frac{b(x)}{a(x)}y' + \frac{c(x)}{a(x)}y = 0$

i.e., $y'' + p(x)y' + q(x)y = 0 \qquad \ldots(2)$

where $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$

If $p(x)$ and $q(x)$ are analytical at $x = x_0$ i.e., $a(x_0) \neq 0$, then the point $x = x_0$ is called **regular (ordinary) point** of the equation (2).

If at least one of $p(x)$ and $q(x)$ is not analytical at $x = x_0$ i.e., $a(x_0) = 0$, then the point $x = x_0$ is said to be **singular point.**

The singular point $x = x_0$ is said to be regular singular point if the functions $(x - x_0)\,p(x)$ and $(x - x_0)^2\,q(x)$ are analytical at x_0 (i.e., have Taylor's series about $x = x_0$.)

If at least one of $(x - x_0)\,p(x)$ and $(x - x_0)^2\,q(x)$ is not analytical, then the singular point is called **irregular singular point.**

Remarks:

In general we find the power series solution is in the form $\sum_{n=0}^{\infty} a_n x^n$ assuming $x = 0$ as an ordinary point. We will use power series method to solve Legendre differential equation at ordinary point $x = 0$.

In the case $x = x_0 \neq 0$ is an ordinary point and we are required to find series solution in the power of $(x - x_0)$, we use change of variable by substituting $u = x - x_0$ and find series solution in the form $\sum_{n=0}^{\infty} a_n u^n$ and substitute back $u = x - x_0$ in the series of u to get series in $(x - x_0)$.

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3. The power series solution can be extended to obtain series solution at the regular singular point $x = 0$ in the Frobenius form $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$, where r is positive or negative or fraction or complex number. We will use this extended power series method to solve Bessel differential equation at a regular singular point $x = 0$.
4. If $x = 0$ is irregular singular point, then the differential equation has series solution in the Frobenius form multiplied by an exponential function and has the form $\sum_{n=0}^{\infty} a_n x^{n+r}$, the power r is due to logarithmic or exponential factor.

Example 6. Identify ordinary and singular points and classify the singular points.

i. $(1 + x^2)y'' + xy' - y = 0$
ii. $x^2y'' + xy' + (x^2 - n^2)y = 0$
iii. $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$
iv. $(x - 1)^3y'' + 3(x - 1)^2 y' + y = 0$

Solution

i. We have,

$$(1 + x^2)y'' + xy' - y = 0$$

i.e., $y'' + \left(\frac{x}{1 + x^2}\right) y' - \frac{1}{1 + x^2} y = 0$

Comparing with $y'' + p(x)y' + q(x)y = 0$

both $p(x) = \frac{x}{1 + x^2}$ and $q(x) = -\frac{1}{1 + x^2}$ are defined and hence analytical everywhere, so value of x is an ordinary point.

ii. We have,

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

i.e., $y'' + \frac{1}{x}y' + \left(\frac{x^2 - n^2}{x^2}\right) y = 0$

Comparing with $y'' + p(x)y' + q(x)y = 0$

both $p(x) = \frac{1}{x}$ and $q(x) = \frac{(x^2 - n^2)}{x^2}$ are not analytic at $x = 0$, so $x = 0$ is a singular point.

Moreover, $x\, p(x) = x \frac{1}{x} = 1$ and $x^2q(x) = x^2 \frac{(x^2 - n^2)}{x^2} = x^2 - n^2$ are analytic at $x = 0$, so x regular singular point.

iii. We have

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

i.e., $y'' - \frac{2x}{1 - x^2} y' + \frac{n(n + 1)}{1 - x^2} y = 0$

Comparing with $y'' + p(x)y' + q(x)y = 0$

both $p(x) = -\frac{2x}{1 - x^2}$ and $q(x) = \frac{n(n + 1)}{1 - x^2}$

which are not defined and not analytic at $x = \pm 1$, so $x = \pm 1$ are singular points.

Moreover at $x = 1$

$$(x - 1)\, p(x) = (x - 1)\left(-\frac{2x}{1 - x^2}\right)$$

$$= \frac{2x}{x + 1}$$

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and $(x-1)^2 q(x) = (x-1)^2 \dfrac{n(n+1)}{1-x^2}$

$$= \frac{(x-1)\,n(n+1)}{1+x}$$

are analytic at $x = 1$, so $x = 1$ is a regular singular point.

Also, at $x = -1$ both

$$(x+1)\,p(x) = (x+1)\left(-\frac{2x}{1-x^2}\right)$$

$$= -\frac{2x}{1-x}$$

and $(x+1)^2 q(x) = (x+1)^2 \dfrac{n(n+1)}{1-x^2}$

$$= \frac{(x+1)\,n(n+1)}{1-x}$$

are analytic at $x = -1$, so $x = -1$ is a regular singular point.

iv. Given equation is,

$$(x-1)^3 y'' + 3(x-1)^2 y' + y = 0$$

i.e., $y'' + \dfrac{3(x-1)^2}{(x-1)^3}\, y' + y = 0$

Comparing with $y'' + p(x)y' + q(x)y = 0$

both $p(x) = \dfrac{3}{x-1}$ and $q(x) = \dfrac{1}{(x-1)^3}$

are not analytic at $x = 1$, so $x = 1$ is a singular point.

Moreover,

$$(x-1)\,p(x) = (x-1)\frac{3}{(x-1)} = 3, \text{ analytic at } x = 1$$

and $(x-1)^2 q(x) = (x-1)^2 \dfrac{1}{(x-1)^3} = \dfrac{1}{x-1}$, not analytic at $x = 1$.

So $x = 1$ is an irregular singular point.

xample 7. Solve $y'' - y = 0$ in the power of $(x-1)$.

olution

Given equation is, $y'' - y = 0$

i.e., $\dfrac{d^2y}{dx^2} - y = 0$ ....(1)

To find series solution of (1) in the power of $(x-1)$.

Let $u = x - 1$

$\therefore\ \dfrac{du}{dx} = 1.$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{dy}{du}$

and, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{du}\right) = \dfrac{d}{du}\left(\dfrac{dy}{du}\right)\dfrac{du}{dx} = \dfrac{d^2y}{du^2}.$

Hence equation (1) becomes

$\dfrac{d^2y}{du^2} - y = 0$ .....(2)

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Let $y = \sum_{n=0}^{\infty} a_n u^n = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 + a_6 u^6 + \dots$ ....(3)

be power series solution of (2).

$\therefore \quad \frac{dy}{du} = a_1 + 2a_2 u + 3a_3 u^2 + 4a_4 u^3 + 5a_5 u^4 + 6a_6 u^5 + \dots$

and $\frac{d^2y}{du^2} = 2a_2 + 6a_3 u + 12a_4 u^2 + 20a_5 u^3 + 30a_6 u^4 + \dots$

Substituting y and $\frac{d^2y}{du^2}$ in (2), we get

$(2a_2 + 6a_3 u + 12a_4 u^2 + 20a_5 u^3 + 30a_6 u^4 + \dots) - (a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a4 u^4 + \dots) = 0$

$\therefore \quad (2a_2 - a_0) + (6a_3 - a_1)u + (12u_4 - a_2)u^2 + (20a_5 - a_3)u^3 + (30a_6 - a_4)u^4 + \dots = 0$

Which is an identity, so equating coefficients of each term to zero.

u^0: $2a_2 - a_0 = 0; \quad \therefore a_2 = \frac{a_0}{2}$

u: $6a_3 - a_1 = 0; \quad \therefore a_3 = \frac{a_1}{3}$

u^2: $12a_4 - a_2 = 0; \quad \therefore a_4 = \frac{a_2}{12} = \frac{a_0}{24}$

u^3: $20a_5 - a_3 = 0; \quad \therefore a_5 = \frac{a_3}{20} = \frac{1}{20}\left(\frac{a_1}{3}\right) = \frac{a_1}{60}$

Substituting the coefficients $a_2, a_3, a_4 \dots$ in (3)

$$y = a_0 + a_1 u + \frac{a_0}{2} u^2 + \frac{a^1}{3} u^3 + \frac{a_0}{24} + \frac{a_1}{60} u^5 + \dots$$

$$\therefore \quad y = a_0\left(1 + \frac{1}{2}u^2 + \frac{1}{24}u^4 + \dots\right) + a_1\left(u + \frac{1}{3}u^3 + \frac{1}{60}u^5 + \dots\right).$$

The series solution of (1) in the power $(x - 1)$ is obtained by substituting $u = (x - 1)$ in the se

$$y = a_0\left\{1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{24} + \dots\right\} + a_1\left\{(x-1) + \frac{(x-3)^3}{3} + \frac{(x-1)^5}{60} + \dots\right\}.$$

2.4 Frobenius method

Frobenius method is an extension of power series in the case of regular singular point.

Recall that, for the equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

or, $\quad y'' + \frac{b(x)}{a(x)} y' + \frac{c(x)}{a(x)} y = 0$

i.e., $\quad y'' + p(x)y' + q(x)y = 0$ ....(1)

Where, $p(x) = \frac{b(x)}{a(x)}, q(x) = \frac{c(x)}{a(x)}$

A point $x = x_0$ is a regular singular point if at least one of $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$ are not analyti

(due to $a(x_0) = 0$ at $x = x_0$) but $(x - x_0)\,p(x)$ and $(x - x_0)^2 q(x)$ are analytical $x = x_0$.

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The coefficient corresponding to the lowest power x^r of x is

$$[r(r-1) + b_0r + c_0]a_0 = 0.$$

For (2) to be solution of (1), we must choose $a_0 \neq 0$.

$$\therefore \quad r(r\ 1) + b_0(r) + c_0 = 0 \qquad \ldots(3)$$

This is quadratic equation in r known as indicial equation of equation (1).

By Frobenius theorem, at least one solution of equation (1) must be of the form (2). The sol of equation (1) depends on the nature of roots of (3) with following cases.

Case 1: Distinct two roots not differing by an integer 1, 2, 3, … .

Case 2: Two roots are equal.

Case 3: Roots differing by an integer 1, 2, 3, … .

The solutions corresponding to the above cases will be as follows:

1. If roots r_1 and r_2 do not differ by an integer then the two solutions of (1) are

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$$

$$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$$

2. If roots r_1 and r_2 are same, i.e., $r = r_1 = r_2$, then two solutions are

$$y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots) \qquad \left[\because r = \tfrac{1}{2}(1 - b_0) \text{ by Euler Cauchy method, Calculus-I}\right]$$

and, $y_2(x) = y_1(x)\ \ell n x + x^r(A_1x + A_2x^2 + \ldots), x > 0.$

3. Roots r_1 and r_2 differ by an integer and $r_1 - r_2 > 0$, then the solutions are

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots) \text{ and}$$

$$y_2(x) = k\, y_1(x)\ \ell n(x) + x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$$

in some cases the arbitrary constant k may be zero.

Remark: Frobenius will be used to solve Bessel differential equation $x^2y'' + xy' + (x^2 - v^2)y = 0$.

Example 8. Find the series solution of the differential equation $3xy'' + 2y' + y = 0$.

Solution

Given equation is

$$3xy'' + 2y' + y = 0 \qquad \ldots(1)$$

i.e., $y'' + \dfrac{2}{3x}y' + \dfrac{1}{3x}y = 0$

Comparing with $y'' + p(x)y' + q(x)y = 0$

$$p(x) = \frac{2}{3x} \text{ and } q(x) = \frac{1}{3x}$$

Here $x\,p(x) = \frac{2}{3}$ and $x^2q(x) = x$, which are analytic at $x = 0$, so $x = 0$ is a regular singular point of (1). Hence it has series solution in Frobenius form.

$$\text{Let } y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \ldots(2)$$

be the solution of (1).

$$\therefore \quad y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting y, y', y'' in (1), we get

$$3\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\therefore \quad \sum_{n=0}^{\infty} [3(n+r)(n+r-1) + 2(n+r)]\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \ldots(3)$$

When $n = 0$, equating the coefficient of lowest degrees term x^{r-1} in (3) to zero, the indicial equation is,

$$a_0[3r(r-1) + 2r] = 0$$

$$a_0(3r^2 - r) = 0$$

Since $a_0 \neq 0$, so

$$3r^2 - r = 0$$

$$r = 0, r = \frac{1}{3}$$

The coefficient of next lowest degree term x^r obtained from (3) by taking $n = 1$ in first term and $n = 0$ in second term.

$$a_1[3r(r+1) + 2(r+1)] + a_0 = 0$$

$$\therefore \quad a_1 = -\frac{a_0}{(3r+2)(r+1)}.$$

Equating coefficient of general term x^{n+r}, [for this taking $n = n+1$ in first term and $n = n$ in second term.]

$$a_{n+1}[3(r+n+1)(r+n) + 2(r+n+1)]a_n = 0$$

$$a_{n+1}(r+n+1)[3r+3n+2] = -a_n$$

$$\therefore \quad a_{n+1} = -\frac{a_n}{(r+n+1)(3r+3n+2)}$$

Now taking $n = 0, 1, 2, 3, \ldots$

$$\text{For } n = 0,\ a_1 = -\frac{-a_0}{3(r+1)(3r+2)}$$

$$\text{For } n = 1,\ a_2 = -\frac{a_1}{(r+2)(3r+5)} = \frac{a_0}{(r+1)(r+2)(3r+2)(3r+5)}$$

$$\text{For } n = 2,\ a_3 = -\frac{a_2}{(r+3)(3r+8)} = \frac{-a_0}{(r+1)(r+2)(r+3)(3r+2)(3r+5)(3r+8)}$$

Again, for $r = 0$, $a_1 = -\dfrac{a_0}{2}$, $a_2 = \dfrac{1}{20}a_0$, $a_3 = -\dfrac{1}{480}a_0$.

Thus, for $r = 0$, $y_1(x) = a_0\left(1 - \dfrac{1}{2}x + \dfrac{1}{20}x^2 - \dfrac{1}{480}x^3 + \ldots\right)$

Also for $r = \dfrac{1}{3}$, $a_1 = -\dfrac{1}{4}a_0$, $a_2 = \dfrac{1}{56}a_0$, $a_3 = -\dfrac{1}{1680}a_0, \ldots$

Thus, for $r = \dfrac{1}{3}$, $y_2(x) = a_0\left(x^{1/3} + \dfrac{1}{4}x^{7/3} - \dfrac{1}{1680}x^{10/3} + \ldots\right)$

Thus, the general solution of (1) is, $y = Ay_1(x) + By_2(x)$ as $y_1(x)$ and $y_2(x)$ already contain arbitrary constants a_0 and a_2.

The general solution is given by

$$\therefore \quad y(x) = Aa_0\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \ldots\right) + Ba_0x^{1/3}\left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \ldots\right).$$

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Exercise 2.1

A. Applying power series method solve the following differential equations.

1. $y' = 3y$
2. $y' = ky$
3. $y' = 2xy$
4. $y' = 3x^2y$
5. $y'' + 4y = 0$
6. $y'' + 9y = 0$
7. $y'' + y = 0$
8. $(1-x^2)y'' - 2xy' + 2y = 0$
9. $y'' + (1-x^2)y = 0$
10. $(1-x^2)y'' + 2y = 0$

B. Applying Frobenius method to solve the following equations:

1. $xy'' + 2y' + xy = 0$
2. $y'' + \left(x - \frac{1}{2}\right)y = 0$

Answer

A. 1. $y = a_0e^{3x}$ 2. $y = a_0e^{kx}$ 3. $y = a_0e^{x^2}$ 4. $y = a_0e^{x^3}$ 5. $y = a_0\cos 2x + \frac{1}{2}a_1\sin 2x$

6. $y = a_0\cos 3x + a_1\sin 3x$ 7. $y = a_0(\cos x) + a_1(\sin x)$ 8. $y = a_0\left(1 - x - \frac{x^4}{3}\right) + a_1$

9. $y = a_0\left(1 - \frac{x^2}{2} - \frac{x^4}{24} - \ldots\right) + a_1\left(x - \frac{x^3}{6} - \ldots\right)$

10. $y = (1-x^2)\,a_0 + \left(x - \frac{x^3}{3} - \frac{x^5}{15} - \frac{x^7}{35} \ldots\right)$

B. 1. $y_1 = \left(1 + \frac{1}{6}x^2 + \frac{1}{120}x^4 + \ldots\right) = \frac{\sinh x}{x}$, $y_2 = \frac{1}{x}\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right) = \frac{\cosh x}{x}$

2. $y_1 = \left(1 + \frac{1}{4}x^2 - \frac{1}{6}x^3 + \frac{1}{96}x^4 \ldots\right)$, $y_2 = \left(x + \frac{1}{12}x^3 - \frac{1}{12}x^4 + \frac{1}{480}x^5 - \ldots\right)$

2.5 Legendre's Equation

The second order differential equation of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where n is a parameter and can be taken to be an integer or any real number.

In most of case of application in physics and engineering the parameter n are taken to be non-negative integers.

Comparing with $y'' + p(x)y' + q(x)y = 0$

The coefficient function $p(x) = -\frac{2x}{1-x^2}$ and $q(x) = \frac{n(n+1)}{1-x^2}$ both are analytic at $x = 0$, so $x = 0$ is an ordinary point and hence Legendre's equation has power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

2.6 Solution of Legendre's differential equation

The Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

or, $(1-x^2)y'' - 2xy' + ky = 0$ ....(1), $k = n(n+1)$

Here, n is involved in equation, So we take m as index of the coefficients in the ser

Let $y = \sum_{m=0}^{\infty} a_m x^m$ ....(2)

be the power series solution of (1). Differentiating (2) successively, we get

$$y' = \sum_{m=1}^{\infty} a_m m x^{m-1} = \sum_{m=0}^{\infty} a_m m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} a_m m(m-1)x^{m-2} = \sum_{m=0}^{\infty} a_m m(m-1)x^{m-2}$$

Substituting y, y', y'' in (1), we get

$$(1-x^2)\sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} - 2x\sum_{m=0}^{\infty} a_m m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

sabar coeff x^m banauna

or, $\sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} - \sum_{m=0}^{\infty} m(m-1)x^m - 2\sum_{m=0}^{\infty} a_m m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$

To make x^m in all series, replace m by $(m+2)$ in first series

or, $\sum_{m=-2}^{\infty} a_{m+2}(m+2)(m+1)x^m - \sum_{m=0}^{\infty} m(m-1)x^m - 2\sum_{m=0}^{\infty} a_m m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$

The first two terms of first series are 0, when $m = -2$ and -1. So we can write it as

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m$$

$\therefore$ $\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m - \sum_{m=0}^{\infty} a_m m(m-1)x^m - 2\sum_{m=0}^{\infty} a_m m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$

$\therefore$ $\sum_{m=0}^{\infty} [a_{m+2}(m+2)(m+1) - \{m^2 - m + 2m - k\}a_m]x^m = 0$

The series in L.H.S. is identically zero, which is possible only when the coefficients of x^m are zero.

$\therefore$ $a_{m+2}(m+2)(m+1) - \{m^2 + m - n(n+1)\}a_m = 0$ $[\because k = n(n+1)]$

or, $a_{m+2}(m+2)(m+1) - \{m^2 + m - n^2 - n\}a_m = 0$

or, $a_{m+2}(m+2)(m+1) - \{(m-n)(m+n) + (m-n)\}a_m = 0$

or, $a_{m+2}(m+2)(m+1) - (m-n)(m+n+1)a_m = 0$

$$a_{m+2} = -\frac{(n-m)(m+n+1)}{(m+2)(m+1)}a_m, \text{ for } m \geq 2.$$

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which is the recursive formula for determining the coefficients a_m for $m \geq 2$ in terms of a_0 ar

Taking $m = 0, 1, 2, 3, 4, \ldots$, successively we have

$$a_2 = \frac{-n(n+1)a_0}{2}$$

$$\therefore \quad a_2 = \frac{-n(n+1)a_0}{2!}$$

$$a_3 = -\frac{(n-1)(n+2)a_1}{6}$$

$$\therefore \quad a_3 = \frac{-(n-1)(n+2)a_1}{3!}$$

$$a_4 = -\frac{(n-2)(n+3)}{12}a_1$$

$$\therefore \quad a_4 = \frac{(n-2)n(n+1)(n+3)a_0}{4!}$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3$$

$$\therefore \quad a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

and so on. Substituting these coefficients in solution (2) we get the general solution Legendre's differential equation.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$$

$$= a_0 + a_1x - \frac{n(n+1)}{2!}a_0x^2 - \frac{(n-1)(n+2)}{3!}a_1x^3 + \frac{n(n+1)(n+3)(n-2)}{4!}a_0x^4$$

$$+ \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1x^5 + \ldots$$

$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 - \ldots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \right]$$

Which can be written as

$$y = a_0\,y_1(x) + a_1\,y_2(x)$$

where, $y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots$

and $\quad y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5$ -

emarks:

The series $y_1(x)$ and $y_2(x)$ both converge for $|x| < 1$. App ratio test for alternating infinite series we can prove $y_1(x)$ and $y_2(x)$ are $|x| < 1$ i.e., in $-1 < x < 1$ (interval of convergence).

2. Also none of $y_1(x)$ and $y_2(x)$ is a scalar multiple of other, so $y_1(x)$ and $y_2(x)$ are independent solutions and hence $y = a_0y_1(x) + a_1y_2(x)$ is the general solution of Legendre's equation.
3. Legendre equation models boundary value problems for spherical symmetry.
4. Legendre's equation is used in solving Laplace equation in spherical coordinate system for heat conduction.

2.7 Legendre's function and Legendre's polynomials

2.7.1 Legendre functions

The independent solutions $y_1(x)$ and $y_2(x)$ of Legendre equation are given by

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots$$

and $y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots$

These two infinite series, which are the two independent solution of Legendre equation and are known as Legendre functions.

2.7.2 Legendre polynomial

In most of application of Legendre equation to solve physical or engineering problems that involves spherical symmetry (boundaries) the parameter n is an non-negative integer. The solution of Laplace equation in spherical coordinates reduces into Legendre equation and hence its solution are the solutions of Legendre equation and the parameter n to be non-negative integer.

We have the recurrence relation for the coefficients in the power series solution of Legendre equation,

$$a_{m+2} = \frac{-(n-m)(m+n+1)}{(m+2)(m+1)}\,a_m,\ m \geq 2$$

When the parameter n is non-negative inter then the R.H.S. of recurrence relation will be zero for $m = n$, so that the coefficients $a_{m+2}, a_{m+4}, a_{m+6}, \ldots$ are all zero. i.e., $a_{m+2} = a_{m+4} = a_{m+6} = \ldots = 0$.

Thus, if $n = m$ is even, the series with even powers i.e., $y_1(x)$ of Legendre solution will reduce into a polynomial of degree n in even powers of x.

Also, if $n = m$ is odd, the series with odd powers i.e., $y_2(x)$ of Legendre solution will reduce into a polynomial of degree n in odd powers of x.

2.8 Solution of Legendre equation in polynomial form

We have the solution of Legendre equation as

$$y(x) = a_0y_1(x) + a_1y_2(x)$$

where, $y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots$

and $\quad y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots$

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Thus, the solution $y(x)$ can be written as

$$y(x) = \begin{cases} a_0\left[1 - \frac{n(n+2)}{2!}x^2 + \frac{(n-2)\,n(n+1)\,(n+3)}{4!}x^4 - \dots\right] & \text{for even } n \\ a_1\left[x - \frac{(n-1)\,(n+2)}{3!}x^3 + \frac{(n-3)\,(n-1)\,(n+2)\,(n+4)}{5!}x^5 - \dots\right] & \text{for odd } n \end{cases}$$

i. When $n = 0$, $y(x) = a_0$

ii. When $n = 1$, $y(x) = a_1 x$

iii. When $n = 2$, $y(x) = a_0(1 - 3x^2)$

iv. When $n = 3$, $y(x) = a_1\left(x - \frac{5}{3}x^3\right)$

v. When $n = 4$, $y(x) = a_0\left(1 - 12x^2 + \frac{35}{3}x^4\right)$

Polynomial solution in higher degree can similarly be obtained.

2.8.1 Legendre polynomials

We have seen that for non-negative integer vale of n. The solution of Legendre differentia equation reduces into polynomials of degree n.

For application purpose and to write in standard form in computation, we generally use some suitable scaling for the above polynomial solutions and the scaled polynomial is denoted by $P_n(x)$ and are known as (standard) Legendre polynomials.

The standard practice to choose the scaling factor is such that $a_0 = P_0(x) = 1$. For this condition we choose the coefficient of highest power of x in $P_n(x)$.

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot3\cdot5\dots(2n-1)}{n!}, \text{ for all positive integer } n.$$

Such that $a_0 = P_0(x) = 1$.

For other coefficients, other than $a_0 = P_0(x) = 1$ and a_n already chosen as $a_n = \frac{(2n)!}{2^n (n!)^2}$.

We have from recursive formula,

$$a_{m+2} = -\frac{(n-m)\,(m+n+1)}{(m+2)\,(m+1)}\, a_m$$

or determining lower coefficients, we rewrite above recursive formula as,

$$a_m = -\frac{(m+2)\,(m+1)}{(n-m)\,(m+n+1)}\, a_{m+2}$$

king $m = n - 2$, for the coefficient of second highest degree term in $P_n(x)$.

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}\, a_n$$

$$= -\frac{n(n-1)}{2(2n-1)}\, \frac{(2n)!}{2^n (n!)^2}$$

Note that $(2n)! = 2n(2n-1)(2n-2)!$, $n! = n(n-1)!$ and $n! = n(n-1)\,(n-2)!$, we have

$$a_{n-2} = -\frac{n(n-1)\,2n\,(2n-1)\,(2n-2)!}{2(2n-1)\,2^n\, n(n-1)!\; n(n-1)\,(n-2)!}$$

$$\therefore \quad a_{n-2} = -\frac{(2n-2)!}{2^n\,(n-1)!\,(n-2)!}$$

Similarly, $a_{n-4} = -\frac{(n-2)\,(n-3)}{4(2n-3)}\, a_{n-2}$

$$= \frac{(2n-4)!}{2^n\, 2!\,(n-2)!\,(n-4)!}$$

and so on, and in general, when $n - 2m \geq 0$

$$a_{n-2m} = \frac{(-1)^m\,(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}$$

The solution of Legendre's equation with above coefficients for non-negative integer value of n always reduce into polynomial, known as Legendre polynomials of degree n. These polynomials are denoted by $P_n(x)$ with $P_0(x) = 1$. The polynomial $P_n(x)$ of degree n are the solutions of Legendre differential equation.

With above coefficients a_{n-2m}, the solution $y(x) = P_n(x)$ is given by the polynomial,

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}\, x^{n-2m}$$

$$= \frac{(2n)!}{2^n (n!)^2}\, x^n - \frac{(2n-2)!}{2^n\, 1!\,(n-1)!\,(n-2)!}\, x^{n-2} + \dots$$

Where, $M = \frac{n}{2}$ or $\left(\frac{n-1}{2}\right)$, whichever is an integer.

The first few of these $P_n(x)$ for $n = 0, 1, 2, 3, 4, 5$ are

$P_0(x) = 1$, $\quad P_1(x) = x$

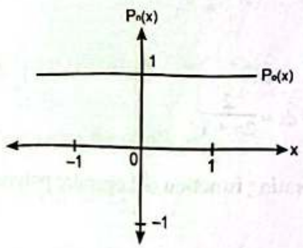
$P_2(x) = \frac{1}{2}(3x^2 - 1)$, $\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $\quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

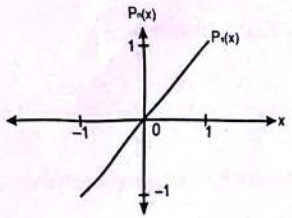
and so on.

Graphs of $P_n(x)$

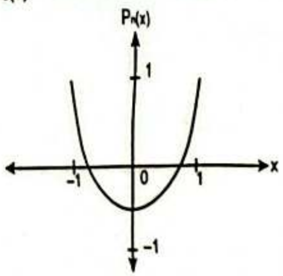
i. $P_0(x)$



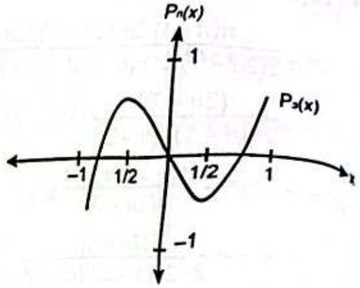
ii. $P_1(x)$



iii. $P_2(x)$



iv. $P_3(x)$



2.9 Orthogonal property of Legendre polynomial

1. If $P_n(x)$ and $P_m(x)$ are Legendre polynomials then $P_n(x)$ and $P_m(x)$ satisfy orthogonal property with respect to integration in the interval of convergence (-1, 1) of solution of Legendre equation.

i.e., $\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0$, for $m \neq n$.

Proof: Since $P_n(x)$ and $P_m(x)$ are the polynomial solution of Legendre differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

For integer values of n for $P_n(x)$, m for $P_m(x)$.

Substituting $P_n(x)$ and $P_m(x)$ in (1)

$$(1-x^2)\, P''_n(x) - 2x\, P'_n(x) + n(n+1)\, P_n(x) = 0 \qquad \ldots(2)$$

$$(1-x^2)\, P''_m(x) - 2x\, P'_m(x) + m(m+1)\, P_m(x) = 0 \qquad \ldots(3)$$

Multiplying (2) by $P_m(x)$ and (3) by $P_n(x)$, by subtraction we get

$$(1-x^2)\,[P_m(x)\, P''_n(x) - P_n(x)\, P''_m(x)] - 2x[P_m(x)\, P'_n(x) + P_n(x)\, P'_m(x)] = (n^2 + n - m^2 - m)\, P_n(x)\, P_m(x)$$

or, $$\frac{d}{dx}[(1-x^2)\{P_m(x)\, P'_n(x) - P_n(x)\, P'_m(x)\}] = (n-m)(m+n+1)\, P_n(x)\, P_m(x)$$

Integrating both sides from $x = -1$ to $x = 1$,

$$[(1-x^2)\{P_m(x)\, P'_n(x) - P_n(x)\, P'_m(x)\}]_{-1}^{1} = (n-m)(n+m+1)\int_{-1}^{1} P_n(x)\, P_m(x)\, dx$$

L.H.S. is zero at both $x = -1$ and $x = 1$.

$\therefore$ $$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0.$$

If $P_n(x)$ is the Legendre polynomial then, $\int_{-1}^{1} \{P_n(x)\}^2\, dx = \frac{2}{2n+1}$.

Proof: If $P_n(x)$ is Legendre polynomial then by generating function of Legendre polynomial,

$$(1 - 2ux + u^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, u^n.$$

$\therefore$ $$\sum_{n=0}^{\infty} P_n(x)\, u^n = (1 - 2ux + u^2)^{-1/2} \qquad \ldots(1)$$

Also, we can write

$$\sum_{m=0}^{\infty} P_m(x)\, u^m = (1 - 2ux + u^2)^{-1/2} \qquad \ldots(2)$$

Multiplying both sides of (1) and (2)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)\, P_m(x)\, u^{m+n} = \frac{1}{(1 - 2ux + u^2)}$$

Integrating both sides with respect to x from $x = -1$ to $x = 1$.

We have,

$$\int_{-1}^{1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\, P_n(x)\, u^{m+n} = \int_{-1}^{1} \frac{1}{(1 - 2ux + u^2)}$$

or, $$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x)\, P_n(x)\, u^{m+n}\, dx = \left(-\frac{1}{2u}\right)[\log(1 - 2ux + u^2]_{-1}^{1}$$

If $m \neq n$ the integral in L.H.S. is zero. So taking $m = n$, we have

$$\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_n(x)\, P_n(x)\, u^{2n}\, dx = -\frac{1}{2u}[\log(1 - 2u + u^2) - \log(1 + 2u + u^2)]$$

or, $$\sum_{n=0}^{\infty}\int_{-1}^{1} \{P_n(x)\}^2\, u^{2n}\, dx = -\frac{1}{2u}[\log(1-u)^2 - \log(1+u)^2]$$

$$= -\frac{1}{2u}[2\log(1-u) - 2\log(1+u)]$$

$$= \frac{1}{u}[\log(1+u) - \log(1-u)]$$

$$= \frac{1}{u}\left[\left\{u - \frac{u^2}{2} + \frac{u^3}{3} - \ldots\right\} - \left\{-u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots\right\}\right]$$

$$= \frac{1}{u}\left[2u + \frac{2u^3}{3} + \frac{2u^5}{5} + \ldots\right]$$

$$= \frac{2u}{u}\left[1 + \frac{u^2}{3} + \frac{u^4}{5} + \ldots\right]$$

$\therefore$ $$\sum_{n=0}^{\infty}\int_{-1}^{1} \{P_n(x)\}^2\, u^{2n}\, dx = 2\left[1 + \frac{u^2}{3} + \frac{u^4}{5} + \ldots + \frac{u^{2n}}{2n+1} + \ldots\right]$$

Comparing the coefficients of u^{2n} on both sides, we get

$$\int_{-1}^{1} \{P_n(x)\}^2\, dx = \frac{2}{2n+1}.$$

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2.10 Rodrigue's formula for Legendre polynomials

Mathematician Rodrigue suggested that the Legendre polynomials can be expressed in n^{th} order derivative of (x^2-1) given by

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}[(x^2-1)^n].$$

We show that the polynomials $P_n(x)$ given by above formula do satisfy Legendre diff. equation and hence $P_n(x)$ given by Rodrigue's formula are Legendre polynomials.

Let $y = (x^2-1)^n$

$\therefore \quad y_1 = n(x^2-1)^{n-1}\, 2x$

or, $(x^2-1)y_1 = 2nxy$

or, $(x^2-1)y_1 - 2nxy = 0$

$\therefore \quad (1-x^2)\, y_1 + 2nxy = 0$

Differentiating $(n+1)$ times by using Leibniz theorem, we get

$$(1-x^2)y_{n+2} + (n+1)(-2x)y_{n+1} + \frac{(n+1)\,n}{2}(-2)\, y_n + 2n\,[xy_{n+1} + (n+1)y_n] = 0$$

$$\therefore \quad (1-x^2)\frac{d^2(y_n)}{dx^2} - 2x\frac{d(y_n)}{dx} + n(n+1)y_n = 0$$

which Legendre differential equation in y_n and has its polynomial solution $P_n(x)$ given by

$$\begin{aligned} P_n(x) &= cy_n \\ &= c\frac{d^n}{dx^n}[(x^2-1)^n], \text{ where } c \text{ is an arbitrary constant} \\ &= c\frac{d^n}{dx^n}[(x-1)^n(x+1)^n] \\ &= c\left[(x-1)^n\frac{nd^n}{dx^n}(x+1)^n + C(n,1)\, n(x-1)^{(n-1)}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \ldots + (x+1)^n\frac{d^n}{dx^n}(x-1)^n\right] \\ &= c\left[(x-1)^n\, n! + C(n,1)\, n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \ldots + (x+1)^n\, n!\right] \end{aligned}$$

To determine constant c, put $x = 1$ (since it is an identity we can take any value for x),

$P_n(1) = c[0 + \ldots + 2^n\, n!]$

or, $1 = c\, 2^n\, n!$

$\therefore \quad c = \frac{1}{2^n\, n!}$

Substituting value of c in $P_n(x) = cy_n$

We have,

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(y)$$

$$\therefore \quad P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}[(x^2-1)^n].$$

Which is Rodrigues formula for Legendre polynomial $P_n(x)$.

2.11 Generating function for Legendre polynomials

Generating functions are functions in infinite series form that describe the special functions (solutions of Legendre, Bessel, Gauss hyper-geometric equations). These functions generate same functions as in the solution of above differential equations.

The general form of generating function is $G(u, x) = \sum_{n=0}^{\infty} f_n(n)\, u^n$.

The coefficients of $f_n(n)$ of u^n in the generating functions are turned to be the solutions of the above differential equations.

2.11.1 Generating function of Legendre equation

The generating function of solution (Legendre polynomial) of Legendre differential equation is given by

$$G(u, x) = \frac{1}{(1-2ux+u^2)^{1/2}}$$

i.e., $(1-2xu+u^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, u^n$.

Here, the coefficients of u^n in the expansion of $(1-2ux+u^2)^{-1/2}$ using binomial theorem give $P_n(x)$, i.e., Legendre polynomial of degree n.

2.12 Recurrence relation for Legendre polynomial, $P_n(x)$

The Legendre polynomials satisfy several recurrence relation which are useful to express a function (solutions of complex differential equations) in terms of Legendre polynomial which are incorporated in several computing software and hence the functions can easily be handled during computation.

Following are few important recurrence relations on Legendre polynomials.

1. **Bonnet's recurrence formula:** $(n+1)\, P_{n+1}(x) = (2n+1)x\, P_n(x) - n\, P_{n-1}(x)$

   Proof: We have the generating function for Legendre polynomial $P_n(x)$ is given by,

   $$\sum_{n=0}^{\infty} P_n(x)\, u^n = (1-2xu+u^2)^{-1/2}$$

   Differentiating both sides partially with respect to u, we get

   $$\Sigma n\, P_n(x)u^{n-1} = -\frac{1}{2}(1-2xu+u^2)^{-3/2}(2u-2x)$$

   Multiplying both sides by $(1-2xu+u^2)$, we get

   $$(1-2xu+u^2)\, \Sigma n\, P_n(x)u^{n-1} = -\frac{2}{2}(1-2xu+u^2)^{-1/2}(u-x)$$

   or, $(1-2xu+u^2)\, \Sigma n\, P_n(x)u^{n-1} = (x-u)(1-2xu+u^2)^{-1/2}$

   or, $(1-2xu+u^2)\, \Sigma n\, P_n(x)u^{n-1} = (x-u)\, \Sigma P_n(x)u^n$

   or, $\Sigma n\, P_n(x)u^{n-1} - \Sigma 2xn\, P_n(x)u^n + \Sigma n\, P_n(x)u^{n+1} = x\, \Sigma P_n(x)u^n - \Sigma P_n(x)u^{n+1}$

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Making all terms in powers of u^n by suitable form of n in each terms.

or, $\Sigma(n+1)\, P_{n+1}(x)u^n + \Sigma(-2xn)\, P_n(x)u^n + \Sigma(n-1)\, P_n(x)u^n = \Sigma x\, P_n(x)u^n + \Sigma(-P_{n-1}(x))$

Equating coefficients of u^n on both sides we get

$$(n+1)\, P_{n+1}(x) - 2xn\, P_n(x) + (n-1)\, P_{n-1}(x) = x\, P_n(x) - P_{n-1}(x)$$

or, $(n+1)\, P_{n+1}(x) = x\, P_n(x) - P_{n-1}(x) + 2nx\, P_n(x) - (n-1)P_{n-1}(x)$

$$= (2n+1)\, xP_n(x) + P_{n-1}(x)\,(-n+1-1)$$

$\therefore$ $(n+1)\, P_{n+1}(x) = (2n+1)\, xP_n(x) - nP_{n-1}(x)$

2. $nP_n(x) = xP_n'(x) - P'_{n-1}(x)$

Proof: We have,

$$(1-2xu+u^2)^{-1/2} = \Sigma P_n(x)u^n \qquad \ldots(1)$$

Differentiating both sides partially with respect to x

$$-\frac{1}{2}(1-2xu+u^2)^{-3/2}(-2u) = \Sigma P_n'(x)u^n$$

or, $u(1-2xu+u^2)^{-3/2} = \Sigma P_n'(x)u^n \qquad \ldots(2)$

Again differentiating (1) partially with respect to u,

$$\left(-\frac{1}{2}\right)(1-2xu+u^2)^{-3/2}(-2x+2u) = \Sigma\, P_n(x)n\, u^{n-1}$$

or, $(x-u)\,(1-2xu+u^2)^{-3/2} = \Sigma n\, P_n(x)n\, u^{n-1} \qquad \ldots(3)$

Dividing equation (3) by equation (2), we get

$$\frac{x-u}{u} = \frac{\Sigma n\, P_n(x)u^{n-1}}{\Sigma P_n'(x)u^n}$$

$\therefore$ $\Sigma n\, P_n(x)u^n = (x-u)\, \Sigma P_n'(x)u^n$

or, $\Sigma n\, P_n(x)u^n = \Sigma x\, P'_n(x)u^n - \Sigma\, P'_n(x)u^{n+1}$

Replacing n by $(n-1)$ in last series to make the term with u^n.

$\therefore$ $\Sigma n\, P_n(x)u^n = \Sigma x\, P'_n(x)u^n - \Sigma\, P'_{n-1}(x)u^n$

Equating the coefficient of u^n on both sides, we get

$n\, P_n(x) = x\, P'_n(x) - P'_{n-1}(x)$.

3. $(2n+1)\, P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Proof: We have from (1),

$$(n+1)\, P_{n+1}(x) = (2n+1)x\, P_n(x) - n\, P_{n-1}(x) \qquad \ldots(1)$$

Differentiating both sides with respect to x,

$$(n+1)\, P'_{n+1}(x) = (2n+1)\, P_n(x) + (2n+1)x\, P'_n(x) - n\, P'_{n-1}(x) \qquad \ldots(2)$$

Also, we have from recurrence relation (2), we have

$$xP'_n(x) = n\, P_n(x) + P'_{n-1}(x) \qquad \ldots(3)$$

Using (3) in (2), we have

$$(n+1)\, P'_{n+1}(x) = (2n+1)\, P_n(x) + (2n+1)\, [nP_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x)$$

or, $(2n+1)\, P_n(x) = (n+1)\, P'_{n+1}(x) - n(2n+1)P_n(x) - (2n+1)\, P'_{n-1}(x) + n\, P'_{n-1}(x)$

or, $(2n+1)\, P_n(x) + n(2n+1)\, P_n(x) = (n+1)P'_{n+1}(x) - P'_{n-1}(x)\,(2n+1-n)$

or, $(2n+1)\, P_n(x)\,(n+1) = (n+1)\, [P'_{n+1}(x) - P'_{n-1}(x)]$

$\therefore$ $(2n+1)\, P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

4. $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$

Proof: We have from recurrence relation (1),

$$(n+1)P_{n+1}(x) = (2n+1)\, xP_n(x) - n\, P_{n-1}(x)$$

Differentiating both sides with respect to x, we get

$$(n+1)\, P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)x\, P'_n(x) - n\, P'_{n-1}(x)$$

or, $(n+1)\, P'_{n+1}(x) = (2n+1)P_n(x) + (n+1+n)x\, P'_n(x) - n\, P'_{n-1}(x)$

$$= (2n+1)P_n(x) + (n+1)x\, P'_n(x) + nx\, P'_n(x) - n\, P'_{n-1}(x)$$

$$= (2n+1)P_n(x) + (n+1)x\, P'_n(x) + n[x\, P'_n(x) - P'_{n-1}(x)]$$

$$= (2n+1)P_n(x) + (n+1)x\, P'_n(x) + n(nP_n(x)) \qquad [\because x\, P'_n(x) - P'_{n-1}(x) = n\, P_n(x)]$$

$$= (2n+1)P_n(x) + (n+1)x\, P'_n(x) + n^2P_n(n)$$

$$= (n^2+2n+1)\, P_n(x) + (n+1)x\, P'_n(x)$$

$$= (n+1)^2\, P_n(x) + (n+1)x\, P'_n(x)$$

$\therefore$ $(n+1)\, P'_{n+1}(x) = (n+1)\, [(n+1)P_n(x) + x\, P'_n(x)]$

i.e., $P'_{n+1}(x) = (n+1)\, P_n(x) + x\, P'_n(x)$

On replacing n by $(n-1)$ we get

$$P'_n(x) = nP_{n-1}(x) + x\, P'_{n-1}(x).$$

Example 9. Express $2x^2 - 4x + 2$ as Legendre polynomial.

Solution

We have, $P_0(x) = 1$, $P_1(x) = x$, $\left(P_2(x) = \frac{3x^2-1}{2}\right)$ $x^2 = \frac{2P_2(x)+1}{3}$

So, $x^2 = \frac{1}{3}[2P_2(x) + 1] = \frac{1}{3}[2\, P_2(x) + P_0(x)]$

$\therefore$ $2x^2 - 4x + 2 = [2\, P_2(x) + P_0(x)] - 4\, P_1(x) + 2\, P_0(x)$

$$= \frac{4}{3}\, P_2(x) + \frac{2}{3}\, P_0(x) - 4\, P_1(x) + 2\, P_0(x)$$

$$= \frac{4}{3}\, P_2(x) - 4\, P_1(x) + \frac{8}{3}\, P_0(x)$$

$$= \frac{4}{3}[\, P_2(x) - P_1(x) + 2\, P_0(x)]$$

$\therefore$ $2x^2 - 4x + 2 = \frac{4}{3}[\, P_2(x) - P_1(x) + 2\, P_0(x)]$.

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Example 10. Express $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of Legendre polynomials.

Solution

We know,

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3x^2-1}{2},\ P_3(x) = \frac{5x^3-3x}{2}$$

Let $f(x) = A\,P_3(x) + B\,P_2(x) + C\,P_1(x) + D\,P_0(x)$ ....(1)

$$\therefore\ 4x^3 - 2x^2 - 3x + 8 = A\left(\frac{5x^3-3x}{2}\right) + B\left(\frac{3x^2-1}{2}\right) + Cx + D$$

Comparing coefficients like powers of x in both sides

Coefficient of x^3: $4 = \frac{5A}{2}$; $\therefore A = \frac{8}{5}$

Coefficient of x^2: $-2 = \frac{3B}{2}$; $\therefore B = -\frac{4}{3}$

Coefficient of x: $-3 = -\frac{3}{2}A + C$; $\therefore C = -3 + \frac{3}{2}A = -3 + \frac{12}{5} = -\frac{3}{5}$

Constant: $8 = -\frac{B}{2} + D$; $\therefore D = 8 + \frac{B}{2} = 8 - \frac{2}{3} = \frac{22}{3}$

Substituting values of A, B, C, D in (1), we get

$$4x^3 - 2x^2 - 3x + 8 = \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{3}P_0(x)$$

2.13 Bessel differential equation

The second order differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 1 \quad ....(1)$$

Where, ν is a non-negative parameter.

This equation has diverse application range in vibrating system, electric fields and h conduction, etc. It is mainly applied the problems that shows cylindrical symmetry cylindrical boundaries.

2.14 Solution of Bessel Equation

We have the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad ....(1)$$

$$\therefore\ y'' + \frac{1}{x}y' + \frac{(x^2-\nu^2)}{x^2}y = 0$$

Comparing with $y'' + p(x)y + q(x)y = 0$

We have $xp(x) = 1$ and $x^2q(x) = x^2\frac{(x^2-\nu^2)}{x^2} = x^2 - \nu^2$, which shows that both $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$, so $x = 0$ is a regular singular point. So Bessel equation has series solution in Frobenius form.

Let $y = x^r \sum_{m=0}^{\infty} x^m = \sum_{m=0}^{\infty} x^{m+r}$ ....(2)

be the series solution of (1).

Differentiating (2) successively, we get

$$y' = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}$$

Substituting y, y', y'' in (1), we get

$$x^2\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + x\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + (x^2-\nu^2)\sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

or, $$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2} - \nu^2\sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

$\therefore$ $$\sum_{m=0}^{\infty}[(m+r)(m+r-1) + (m+r) - \nu^2]a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2} = 0$$

or, $$\sum_{m=0}^{\infty}[(m+r)\{(m+r-1)+1\} - \nu^2]a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2} = 0$$

or, $$\sum_{m=0}^{\infty}[(m+r)^2 - \nu^2]a_m x^{m+r} + \sum_{m=0}^{\infty}a_m x^{m+r+2}$$

$\therefore$ $$\sum_{m=0}^{\infty}[(m+r)^2 - \nu^2]a_m x^{m+r} + \sum_{m=2}^{\infty}a_{m-2}x^{m+r} = 0 \quad ....(3)$$

We have replaced m by $(m-2)$ in second series to make x^{m+r}.

The lowest power of x in the above series equation is x^r, when $m = 0$. Second term starts only after x^{r+2}, so it will not contribute coefficieents for x^r and x^{r+1}.

Equating coefficients of x^r to zero, we have $(r^2 - \nu^2)a_0 = 0$.

Since, $a_0 \neq 0$

$\therefore$ $r = \nu$ and $-\nu$.

Thus the roots of indicial equation (1) are

$r_1 = \nu$ and $r_2 = -\nu$ so that $r_1 - r_2 = \nu + \nu = 2\nu > 0$.

Recursive formula for coefficient when $r = r_1 = \nu$.

When $m = 1$, the coefficient of x^{r+1} from equation (3), we have

$$[(1+r)^2 - \nu^2]a_1 = 0$$

or, $[1 + 2\nu + \nu^2 - \nu^2]a_1 = 0$ $[\because r = \nu]$

$$(2\nu + 1)a_1 = 0$$

We must have, $a_1 = 0$, otherwise $2\nu + 1 = 0$ gives $\nu = -\frac{1}{2}$ but ν is non-negative parameter in Bessel differential equation.

Again, taking $r = \nu$ in (3) and equating the coefficient of general term x^{m+r} for $m \geq 2$, we have

$$[(m+r)^2 - \nu^2]a_m = -a_{m-2}$$

i.e., $[(m+\nu)^2 - \nu^2]a_m = -a_{m-2}$

or, $[m^2 + 2m\nu + \nu^2 - \nu^2]a_m = -a_{m-2}$

$$\therefore\ a_m = -\frac{a_{m-2}}{m(m+2\nu)} \quad ....(4)$$

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Since $a_1 = 0$ and $\nu \geq 0$, equation (4) shows that $a_3 = a_5 = \ldots = 0$. Hence there will be only even terms in the series $\sum_{m=0}^{\infty} a_m x^m$ part of $x^r \sum_{m=0}^{\infty} a_m x^m$ and corresponding coefficients in the series solution. Hence there will be corresponding non-zero coefficients x^{2m} for the terms x^{2m+r} in the series. We now consider even coefficients only with suffices $2m$, i.e., a_{2m}.

Replacing m by $2m$ in (4), we get

$$\therefore \quad a_{2m} = -\frac{a_{2m-2}}{2m(2m+2\nu)}$$

$$\therefore \quad a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+\nu)} \qquad \ldots(5)$$

Taking $m = 1, 2, 3, \ldots$, we have

$$a_2 = -\frac{a_0}{2^2(\nu+1)},$$

$$a_4 = -\frac{a_2}{2^2 2(\nu+2)}$$

$$= \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$$

and $a_{2m} = \dfrac{(-1)^m a_0}{2^{2m} m! (\nu+1)(\nu+2)\ldots(\nu+m)}$, $m = 1, 2, \ldots$

In summary, $a_1 = a_3 = a_5 = \ldots = 0$ for all odd coefficients and even coefficients are given by,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (\nu+1)(\nu+2)\ldots(\nu+m)}, \quad m = 1, 2, 3, \ldots$$

Thus, the solution of Bessel equation for $r = \nu$ is,

$$y_1 = a_0 x^{\nu}\left[1 - \frac{x^2}{2^2 1!(\nu+1)} + \frac{x^4}{2^4 2!(\nu+1)(\nu+2)} - \frac{x^6}{2^6 3!(m+1)(m+2)(m+3)} + \ldots\right] \ldots(6)$$

For $r = -\nu$, the second solution of Bessel equation is a descending series in x.

2.14.1 Bessel Functions $J_\nu(x)$ for integer value of ν

If $\nu = n$ is an integer in Bessel equation, we have from recurrence relation,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! ((m+1)(m+2)\ldots(m+n))}, \quad m = 1, 2, 3, \ldots$$

Here a_0 is arbitrary, so the series

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} = \sum_{m=0}^{\infty} a_m x^{m+\nu}$$

will contain the arbitrary constant a_0.

For Standard Computational result we choose, a_0 as

$$a_0 = \frac{1}{2^n n!}$$

so that $m!(m+1)(m+2)\ldots(m+n) = (m+n)!$

and, $a_{2m} = \dfrac{(-1)^m}{2^{2m+n} m!(n+m)!}$, $m = 1, 2, 3, \ldots$

with these coefficients and $r_1 = \nu = n$ we get a particular solution of Bessel equation (1), which is denoted by $J_n(x)$ and is given by

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

$J_n(x)$ is called Bessel function of the first kind of order n.

The Bessel function of order 0 and 1 (J_0, J_1)

a. for $n = 0$, i.e. $J_0(x)$ Bessel function of first kind of order 0.

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)(m!)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \ldots$$

Which resembles (looks like) to the series of cosine function and is shown in figure.

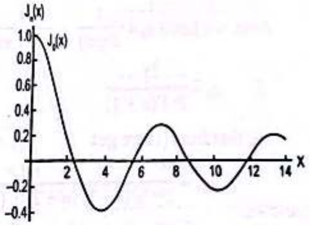

Figure: Bessel functions of order 0 (J_0)

b. For $m = 1$, i.e. $J_1(x)$ Bessel function of first kind of order 1.

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!}$$

$$= \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + \ldots$$

which resembles to the series of sine function and is shown in figure,

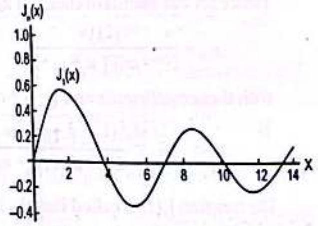

Figure: Bessel functions of order 1(J_1)

(c) The Bessel function $J_n(x)$ is asymptotically equal with $\sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right)$

i.e. $J_n(x) \sim \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right)$

By asymptotically equal we mean, for a fixed n as $x \to \infty$ both $J_n(x)$ and $\sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{n\pi}{2} - \dfrac{\pi}{4}\right)$ are 1.

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(d) Using gamma function

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1} dt,\ n > 0$$

$$\Gamma(n+1) = n!,\ \sqrt{\frac{1}{2}} = \sqrt{\pi},\ \Gamma(n+1) = n\Gamma n, \text{ for fraction } n.$$

The Bessel function of first kind can be written as

$$J_n = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}}\ \Gamma(m+n+1)$$

Proof: We have the Bessel function of first kind as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!(m+n)!} \quad \ldots(1)$$

Also, we have $a_0 = \dfrac{1}{2^n(n!)} = \dfrac{1}{2^n\,\Gamma(n+1)}$

$\therefore \quad a_0 = \dfrac{1}{2^n\,\Gamma(n+1)}$

So that from (1) we get

$$a_{2m} = \frac{(-1)^m}{2^{2m}\, m!(n+1)(n+2)\ldots(n+m)\, 2^n\,\Gamma(n+1)}$$

Using $(n+1)\Gamma(n+1) = \Gamma(n+2)$, $(n+2)\Gamma(n+2) = \Gamma(n+3)$ and so on, we have

$(n+1)(n+2)\ldots(n+m)\Gamma(n+m+1) = \Gamma(n+m+1)$.

Hence for our standard choice of a_0 the coefficients a_{2m} are given by

$$a_{2m} = \frac{(-1)^m}{2^{2m+n}\, m!\,\Gamma(n+m+1)}$$

with these coefficients and $r = r_1 = n$, we get particular solution of Bessel's function as,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!\,\Gamma(n+m+1)}$$

The function $J_n(x)$ is called Bessel's function of first kind.

2.14.2 General solution of Bessel equation:

The particular solution or Bessel function of first kind is,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!\,\Gamma(n+m+1)} \quad \ldots(1)$$

For the root $r = -\nu$ of indicial equation of Bessel differential equation, when ν is not an intege[r] we replace n in (1) by $-\nu$, so that

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu}\, m!\,\Gamma(m-\nu+1)}$$

$\therefore$ The general solution of Bessel equation is

$y(x) = C_1 J_\nu(x) = C_2 J_{-\nu}(x)$

[R]emark: The Bessel function,

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu}\, m!\,\Gamma(\nu+m+1)}$$

[C]an also be written as,

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{2m+\nu} \frac{1}{m!\,\Gamma(\nu+m+1)}.$$

2.15 Properties and recurrence relations of Bessel functions

The Bessel function (i.e., solution of Bessel equation) is given by,

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{\nu+2m} \frac{1}{m!\,\Gamma(\nu+m+1)}.$$

Where, ν is a non-negative parameter.

The Bessel functions satisfy following recurrence relations.

1. $x J'_\nu(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$.

Proof: We have, $J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{\nu+2m} \frac{1}{m!\,\Gamma(\nu+m+1)}$.

Where, ν is a positive integer. Differentiating with respect to x, we have

$$J'_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(\nu+2m)}{m!\,\Gamma(\nu+m+1)} \frac{1}{2}\left(\frac{x}{2}\right)^{\nu+2m-1}$$

Multiplying both sides by x, we get

$$\Rightarrow\quad x J'_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(\nu+2m)}{m!\,\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{\nu+2m}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\nu}{m!\,\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{\nu+2m} + \sum_{m=0}^{\infty} (-1)^m \frac{2m}{m!\,\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{\nu+2m-1}$$

$$= \nu J_\nu(x) + x \sum_{m=1}^{\infty} (-1)^m \frac{1}{(m-1)!\,\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{\nu+2m-1}$$

$$= \nu J_\nu(x) - x \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{(m-1)!\,\Gamma(\nu+m+1)} \left(\frac{x}{2}\right)^{\nu+2m-1}$$

Since $(m-1)!$ will not be defined for $m < 0$.

Taking $(m-1) = n$, we have

$$x J'_\nu(x) = \nu J_\nu(x) - x \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,\Gamma((\nu+1)+n+1)} \left(\frac{x}{2}\right)^{(\nu+1)+2n}$$

$\therefore\quad x J'_\nu(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$

Exercise 2.2

1. Show that $(n+1)\,P_n(x) = P_{n+1}'(x) = xP_n'(x)$.
2. Show that $(1-x^2)\,P_n' = n\,[P_{n-1}(x) - x\,P_n'(x)]$
3. Show that $(1-x^2)\,P_n'(x) = (n+1)\,[x\,P_n(x) - P_{n+1}(x)]$
4. Prove that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.
5. Express $x^3 - 5x^2 + x + 2$ in terms of Legendre polynomials.
6. Express $x^3 - 5x^2 + x + 2$ in terms of Legendre polynomials.
7. Express $(x^3 + x + 1)$ in terms of Legendre polynomials.
8. For Bessel function $J_\nu(x)$, show that $\frac{d}{dx}(x^2 J_\nu) = x^\nu J_{\nu-1}(x)$
9. For Bessel function $J_\nu(x)$, show that $2x\,J'_{\nu-1}(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$.
10. For Bessel function $J_\nu(x)$, show that $2\nu\,J_\nu(x) = x[J_{\nu-1}(x) + J_{\nu+1}(x)]$.
11. For Bessel function $J_\nu(x)$, prove that $4J''_\nu(x) = J_{\nu-2}(x) - 2J_\nu(x) + J_\nu + 2(x)$.

    [Hint: use the relation $2\,J_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$ differentiate it and replace ν by $\nu - 1$ ad $\nu +$ simplify to get the result.
12. Show that $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$.

Answers

5. $\frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$ 6. $P_3(x) + \frac{3}{5}P_1(x)$ 7. $P_3(x) + \frac{8}{5}P_1(x) + P_0(x)$

Unit 3

Laplace Transform and its Application

Pre-requisite knowledge	Expected learning outcomes
Before starting this unit, students are expected to have fundamental concepts and evaluation skills on • solve simple differential equations of first and second order • understand transform as the change of variables in different mathematical domains. • shifting theorems and their application.	After completion of this unit, student will develop sufficient knowledge and evaluation skills on • evaluating Laplace transform of some algebraic and trigonometric function. • evaluating inverse Laplace transform of function F(s). • defining shifting theorems, initial value theorems and final value theorem. • solving IVP on differential equations through Laplace transform by using and without using Laplace transform. • defining convolution and use it to evaluate Laplace transform of some algebraic, trigonometric and exponential functions.

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2. $\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}$

Proof: From $x\,J'_\nu(x) = \nu\,J_\nu(x) - x\,J_{\nu+1}(x)$

Multiplying by $x^{-\nu}$, we get

$$x^{-\nu+1}J'_\nu(x) = \nu\,x^{-\nu}J_\nu(x) - x\,x^{-\nu}J_{(\nu+1)}(x)$$

$$x^{-\nu+1}J'_\nu(x) - \nu\,x^{-\nu}J_\nu(x) = -x\,x^{-\nu}J_{\nu+1}(x)$$

or, $x^{-\nu}\,x\,J'_\nu(x) - \nu\,x^{-\nu-1}\,x\,J_\nu(x) = -x\,x^{-\nu}J_{\nu+1}(x)$

or, $x[x^{-\nu}J'_\nu(x) - \nu\,x^{-\nu-1}J_\nu(x)] = -x\,x^{-\nu}J_{\nu+1}(x)$

$\therefore$ $\frac{d}{dx}[x^{-\nu}J\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$

3. $x\,J'_\nu(x) = -\nu\,J_\nu(x) + x\,J_{\nu-1}(x)$

Proof: We have

$$J_\nu(x) = \sum_{m=0}^{\infty}(-1)^m\left(\frac{x}{2}\right)^{\nu+2m}\frac{1}{m!\,\Gamma(\nu+m+1)}$$

Differentiating with respect to x, we have

$$J'_\nu(x) = \sum_{m=0}^{\infty}(-1)^m\frac{(\nu+2m)}{m!\,\Gamma(\nu+m+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{\nu+2m-1}$$

Multiplying by x on both sides

$$x\,J'_\nu(x) = \sum_{m=0}^{\infty}(-1)^m\frac{(\nu+2m)}{m!\,\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{\nu+2m}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{(2\nu+2m-\nu)}{m!\,\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{\nu+2m}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{2(\nu+m)}{m!\,\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{\nu+2m} - \sum_{m=0}^{\infty}(-1)^m\frac{\nu}{m!\,\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{\nu+2m}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{2(\nu+m)}{m!\,\Gamma(\nu+m+1)}\left(\frac{x}{2}\right)^{\nu+2m-1}\left(\frac{x}{2}\right) - \nu\,J_\nu(x)$$

$$= x\sum_{m=0}^{\infty}(-1)^m\frac{(\nu+m)}{m!\,(\nu+m)\,\Gamma(\nu+m)}\left(\frac{x}{2}\right)^{\nu+2m-1} - \nu\,J_\nu(x)$$

$$= x\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\,\Gamma((\nu-1)+m+1)}\left(\frac{x}{2}\right)^{(\nu-1)+2m} - \nu\,J_\nu(x)$$

$\therefore$ $x\,J'_\nu(x) = x\,J_{\nu-1} - \nu\,J_\nu(x)$

4. Show that $\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$

Proof: We have

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -\nu\,x^{-\nu-1}J_\nu(x) + x^{-\nu}J'_\nu(x)$$

$$= x^{-\nu-1}[\nu\,J_\nu(x) + xJ'_\nu(x)]$$

Using $x\,J'_\nu(x) = \nu\,J_\nu(x) - x\,J_{\nu+1}(x)$, we have

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = x^{-\nu+1}[-\nu\,J_\nu(x) + \nu\,J_\nu(x) - x\,J_{\nu+1}(x)]$$

$$= x^{-\nu+1}[-x\,J_{\nu+1}(x)]$$

$$= -x^{-\nu}J_{\nu+1}(x)$$

$\therefore$ $\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x).$

5. $\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu}J_{\nu-1}(x)$

Proof: We have

$$\frac{d}{dx}[x^{\nu}J_\nu(x)] = \nu\,x^{\nu-1}J_\nu(x) + x^{\nu}J'\nu(x)$$

$$= x^{\nu-1}[\nu\,J_\nu(x) + x\,J_\nu(x)]$$

Using $x\,J'_\nu(x) = -\nu\,J_\nu(x) + x\,J'_{\nu-1}(x)$, we have

$$\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu-1}[\nu\,J_\nu(x) - \nu\,J_\nu(x) + x\,J_{\nu-1}(x)]$$

$$= x^{\nu-1}[x\,J_{\nu-1}]$$

$\therefore$ $\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^{\nu}J_{\nu-1}.$

6. $2\nu\,J_\nu(x) = x\,[J_{\nu-1}(x) + J_{\nu+1}(x)]$

Proof: We have

$x\,J'_\nu(x) = x\,J_\nu(x) - x\,J_{\nu+1}(x)$

and $x\,J'_\nu(x) = -\nu\,J_\nu(x) + x\,J_{\nu-1}(x)$

By subtraction, we have

$0 = 2\nu\,J'_\nu(x) - x\,[J_{\nu-1}(x) + J_{\nu+1}(x)]$

$\therefore$ $2\nu\,J_\nu(x) = x\,[J_{\nu+1}(x) + J_{\nu-1}(x)]$

Remark: We can write above formula as

$$J_{n+1}(x) = \frac{2\nu}{x}J_\nu(x) - J'_{\nu-1}(x)$$

Which shows that, the Bessel function of higher order can be expressed in terms of two consecutive lower order Bessel function.

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3.0 Introduction

Pierre-Simon, marquis de La

The Laplace transform is named after mathematician and astronomer Pierre-Simon, marquis de Laplace, who used a similar transform in his work on probability theory. Laplace wrote extensively about the use of generating functions in *Essaiphilosophique sur les probabilités* (1814), and the integral form of the Laplace transform.

Laplace's use of generating functions was similar to the z-transform, he gave little attention to the continuous variable case which was discussed by Niels Henrik Abel. The theory was further developed in the 19th and early 20th centuries by Mathias Lerch, Oliver Heaviside and Thomas Bromwich.

The current widespread use of the transform (mainly in engineering) came after World War I replacing the earlier Heaviside operational calculus. The advantages of the Laplace transform emphasized by Gustav Doetsch, to whom the name Laplace transform is apparently due. Lapl transform is a strong tool to solve differential equations.

3.1 Laplace transform as generalization of power series

In Calculus-I we have used the operational method i.e., integration as a tool for solving differential equation. In the chapter 'The Power Series Solution of Differential Equation a Special Functions', we learned the power series method to solve a differential equation. In operational method the variable remains same after operation. Differentiation and integrati are examples of operational method.

Transformation is a mathematical method in which we get the result in terms of a new variab from a given function. Laplace transformation is mainly a tool to solve differential equations. can also be used to evaluate complex integrals.

Laplace transform is obtained by the generalization of power series.

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots = A(x)$$

which can be written as

$$\sum_{n=0}^{\infty} a(n) x^n = A(x); \text{ where } a(n) = a_n$$

i. If $a(n) = 1$ for all n then

$$\sum_{n=0}^{\infty} a(n) x^n = 1 + x + x^2 + \ldots$$

$$\therefore \sum_{n=0}^{\infty} a(n) x^n = \frac{1}{1-x} = A(x), \ |x| < 1$$

ii. If $a(n) = \frac{1}{n!}$, then

$$\sum_{n=0}^{\infty} a(n) x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= e^x$$

$$\therefore \sum_{n=0}^{\infty} a(n) x^n = A(x) = e^x$$

In the power series, we have integer values for n, i.e., $n = 0, 1, 2, \ldots$

If we let n to be any real number (i.e., continuous) such that $n > 0$, then we represent n by 't', where t represents time and $a(n)$ by $f(t)$ then the above power series can be written as

$$\sum_{n=0}^{\infty} a(n) x^n = \lim_{n \to \infty} \sum_{t=0}^{n} f(t)\, x^t$$

which can be written as Riemann integral of $f(t)$ from $t = 0$ to $t = \infty$.

i.e., $$\lim_{n \to \infty} \sum_{t=0}^{n} f(t)\, x^t = \int_0^{\infty} f(t)\, x^t\, dt$$

$$= A(x) \quad \ldots(1)$$

In order to simplify x^t, where variable is in exponent we proceed as follows

We know

$$x = e^{\ell nx}$$

$$\therefore x^t = (e^{\ell nx})^t$$

$$= e^{t\ell nx}, \ 0 < x < 1$$

Also, if $0 < x < 1$, then $\ell nx < 0$

Let $-s = \ell nx < 0$ for $s > 0$, so that $x^t = e^{t\ell nx} = e^{-st}$

Thus, (1) becomes

$$\int_0^{\infty} f(t)\, e^{-st}\, dt = F(s) = A(e^{-s})$$

Thus, we see that we can have an equivalent integral representation for continuous variable t obtained by generalizing the integer variable n (discrete) of the power series variable.

The integral of $\int_0^{\infty} f(t)\, e^{-st}\, dt$ is known as Laplace transform F(s) of $f(t)$.

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Definition: Let $t > 0$ and $f(t)$ be continuous function of t, then the integral

$\int_0^\infty f(t)\, e^{-st}\, dt$ is defined as the Laplace transform of $f(t)$. It is denoted by F(s).

i.e., $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \qquad \ldots(1)$$

The function $f(t)$ is also known as inverse Laplace transform of F(s) and written as

$f(t) = \mathcal{L}^{-1}$ [F(s)]

i.e., If $\mathcal{L}\{f(t)\} = F(s)$

Then $f(t) = \mathcal{L}^{-1}$ [F(s)] $\qquad \ldots(2)$

3.2 Existence of Laplace transform

In the definition of Laplace transform we have $f(t)$ a continuous function for $t \geq 0$. The Lapla[ce] transform can be extended for piecewise continuous functions too.

3.2.1 Piecewise continuous function

Let $f(t)$ be a function defined on a finite interval $a \leq t \leq b$. If the function is continuous except [at] some finite number of points inside $a \leq t \leq b$, then the function is said to be piecew[ise] continuous function

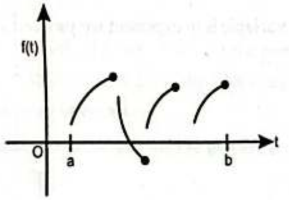

Figure: *Graph of piecewise continuous function $f(t)$. The dots show the functional values at jump discontinuities.*

3.2.2 Existence theorem for Laplace transform

Let $f(t)$ is defined and piecewise continuous function on every finite interval on the semi-axis $t \geq 0$, i.e., $0 \leq t < \infty$ and $|f(t)| \leq Me^{kt}$ for some constants M and k for all $t \geq 0$, then Laplace transform $\mathcal{L}\{f(t)\}$ exists for all $s > k$.

i.e., $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, for all $s > k$, exists.

Proof: Let $f(t)$ is piecewise continuous on every finite interval for $t \geq 0$, and s[ince e^{-st} is] continuous function for all real numbers t. So, $e^{-st} f(t)$ is integrable over every finite [interval] $t \geq 0$.

Also since,

$|f(t)| \leq Me^{kt} \qquad \ldots.(1)$

for $t \geq 0$ and for some constants M and k.

Now, $$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-st} f(t)\, dt\right|$$

$$\leq \int_0^\infty |f(t)|\, e^{-st}\, dt$$

$$\leq M \int_0^\infty e^{kt} e^{-st}\, dt \qquad [\because \text{using (1)}]$$

$$= M \int_0^\infty e^{-(s-k)t}\, dt \qquad \text{which exists only for } s > k.$$

$$= M\left[\frac{e^{-(s-k)t}}{-(s-k)}\right]_0^\infty$$

$$= \frac{M}{s-k} \qquad \text{which is a finite quantity.}$$

Thus, $\mathcal{L}\{f(t)\}$ converges absolutely for $s > k$. So, the Laplace transform of $f(t)$ exists for $s > k$.

Remark: The requirement (Sufficient condition) for a function $f(t)$ for its Laplace transform to exists is $|f(t)| \leq Me^{kt}$ for some constants M and k. This condition is mathematically known as exponential growth restriction.

Illustrative example

1. The functions $f(t) = e^{at}$ automatically satisfies condition $|f(t)| \leq Me^{kt}$ for M = 1 and $k = a$.
2. The functions $f(t) = \sin at, \cos at, \sinh at, \cosh at \leq e^{at}$
3. The function $f(t) = t^n < n!\, e^t$, $M = n!$, $k = 1$.

Remarks

1. The function $f(t) = e^{t^2}$ does not satisfy the exponential growth restriction for any M and k, so it has no Laplace transform.
2. Uniqueness: if the Laplace transform of given function exists, it is unique.

3.3 Transform of elementary functions

By direct use of definition, we find the Laplace transform of some of the simple functions.

1. $\mathcal{L}\{1\} = \dfrac{1}{s}$ $\quad (s > 0)$
2. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ $\quad (s > a)$
3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ $\quad$ where $n = 0, 1, 2, \ldots.$
4. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ $\quad (s > 0)$
5. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$ $\quad (s > 0)$
6. $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$ $\quad (s > |a|)$
7. $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$ $\quad (s > |a|)$

Proof:

1. $\mathcal{L}\{1\} = \displaystyle\int_0^\infty 1.e^{-st}\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left(\frac{e^{-\infty} - e^0}{-s}\right)$$

$$= \left(\frac{0-1}{-s}\right)$$

$$= \frac{1}{s}$$

Hence, $\mathcal{L}\{1\} = \dfrac{1}{s}$

2. $\mathcal{L}\{e^{at}\} = \displaystyle\int_0^\infty e^{-st} e^{at}\,dt$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a}$$

Hence, $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$.

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}.\,t^n\,dt$$

$$= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty n t^{n-1}\frac{e^{-st}}{-s}\,dt$$

$$= \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

$$= \frac{n}{s}.\frac{n-1}{s}\mathcal{L}\{t^{n-2}\}$$

$$= \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}\cdots\frac{1}{s}\mathcal{L}\{t^0\}$$

$$= \frac{n!}{s^n}\mathcal{L}\{1\}$$

$$= \frac{n!}{s^n}.\frac{1}{s}$$

Hence, $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$.

lternatively

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}.\,t^n\,dt$$

et $p = st$

hen, $\dfrac{dp}{dt} = s$

$\therefore\ dt = \dfrac{1}{s}dp$

hen, $\mathcal{L}\{t^n\} = \displaystyle\int_0^\infty e^{-p}\left(\frac{p}{s}\right)^n \frac{dp}{s}$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p} p^n\,dp$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p} p^{(n+1)-1}\,dp$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \qquad \text{if } n > -1 \text{ and } s > 0$$

$$\therefore \quad \mathcal{L}\{t^n\} = \begin{cases} \dfrac{n!}{s^{n+1}} & \text{if } n = 0, 1, 2, 3, \ldots \\ \dfrac{\Gamma(n+1)}{s^{n+1}} & \text{if } n = \text{posiitve rational number} \end{cases}$$

Remark:

i. The Laplace transform is an improper integral which converges for $s > 0$ and we gene[rally] use limit process to evaluate improper integral.

ii. We have used the gamma function to find Laplace transform of t^n. The gamma functio[n is] defined by

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

iii. We can also compute Laplace transform of t^n by using integration by parts formula.

4. $\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2}[-s \sin at - a \cos at]\right]_0^\infty$$

$$= 0 - \left\{\frac{1}{s^2 + a^2}[0 - a]\right\}$$

$$= \frac{a}{s^2 + a^2}$$

Hence, $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$.

5. $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2}[-s \cos at - a \sin at]\right]_0^\infty$$

$$= 0 - \left\{\frac{1}{s^2 + a^2}[-s - 0]\right\}$$

$$= \frac{s}{s^2 + a^2}$$

Hence, $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$.

Alternately

Since $e^{iat} = \cos at + i \sin at$, both of 4 and 5 can be obtained at the same time ... t $\mathcal{L}\{e^{iat}\}$ and by separating the real and imaginary parts.

Now, $\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat}\, dt$

$$= \int_0^\infty e^{-(s - ia)t}\, dt$$

$$= \left[\frac{e^{(s-ia)t}}{-(s - ia)}\right]_0^\infty$$

$$= \frac{-1}{(s - ia)}(e^{-\infty} - e^0)$$

$$= \frac{-1}{s - ia}(0 - 1)$$

$$= \frac{1}{s - ia} \times \frac{s + ia}{s + ia}$$

$$\therefore \quad \mathcal{L}\{e^{iat}\} = \frac{s + ia}{s^2 - i^2 a^2}$$

$$\mathcal{L}\{\cos at + i \sin at\} = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2} \qquad [\because i^2 = -1]$$

Therefore, $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$.

6. $\mathcal{L}\{\sinh at\} = \int_0^\infty e^{-st}.\sinh at\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty (e^{-(s-a)t} - e^{-(s+a)t})\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s - a)} - \frac{e^{-(s+a)t}}{-(s + a)}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{-(s - a)} - \frac{e^{-\infty}}{-(s + a)} + \frac{e^0}{-(s - a)} - \frac{e^0}{(s + a)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s - a)} - \frac{1}{s + a}\right] = \frac{1}{2}\left(\frac{s + a - s + a}{s^2 - a^2}\right)$$

Hence, $\mathcal{L}\ (\sinh at) = \frac{a}{s^2 - a^2}$, $s > |a|$

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7. $\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} + e^{-(s+a)t}\right) dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-a)} + \frac{e^{-\infty}}{-(s+a)} + \frac{e^{0}}{-(s-a)} + \frac{e^{0}}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$\therefore \quad \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}, s > |a|$

3.4 Properties of Laplace transforms

3.4.1 Linearity of the Laplace transform

Theorem: The Laplace transform is a linear operator i.e., if $f(t)$ and $g(t)$ be two functions whose Laplace transforms exist, then for any constants a and b, the transform of $af(t) + bg(t)$ exists and $\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$.

Proof:

By the definition of Laplace transform

$$\mathcal{L}\{a f(t) + b g(t)\} = \int_0^\infty e^{-st} [af(t) + bg(t)] \, dt$$

$$= a\int_0^\infty e^{-st} f(t) \, dt + b\int_0^\infty e^{-st} g(t) \, dt$$

$\therefore \quad \mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$

3.4.2 First shifting theorem or s-shifting [multiplication of a function by e^{at}]

the Laplace transform of $f(t)$ is F(s), (where $s > k$ for some k) then Laplace transform of $[e^{at} f(t)]$ $F(s-a)$, where $s - a > k$, i.e., if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Proof: By the definition of Laplace transform

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} [e^{at} f(t)] \, dt$$

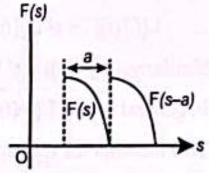
$$= \int_0^\infty e^{-(s-a)t} f(t) \, dt$$

$$= F(s-a)$$

Hence $\mathcal{L}\{e^{at} f(t)\} = F(s-a), s > a$

Remarks:

1. The result can be stated as, if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(t)\} = [F(s-a)] = [F(s)]_{s \to s-a}$ that is first find $\mathcal{L}\{f(t)\}$, and replace s by $(s-a)$



2. We have $\mathcal{L}[e^{at}f(t)] = F(s-a)$, so $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$.

Application

1. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ as $\mathcal{L}\{1\} = \frac{1}{s}$
2. $\mathcal{L}\{e^{at}\, t^n\} = \frac{n!}{(s-a)^{n+1}}$ as $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
3. $\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$ as $\mathcal{L}\{\sin bt\} = \frac{a}{s^2 + b^2}$
4. $\mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$ as $\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}$
5. $\mathcal{L}\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$ as $\mathcal{L}\{\sinh bt\} = \frac{b}{s^2 - b^2}$
6. $\mathcal{L}\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2 - b^2}$ as $\mathcal{L}\{\cosh bt\} = \frac{s}{s^2 - b^2}$

where in each case $s > a$

3.5 Laplace transform of the derivative of a function

Theorem (Laplace transform of derivative): Let $f(t)$ be continuous for $t \geq 0$ and let $f'(t)$ be piecewise continuous on every finite intervals contained in $t \geq 0$. If $\mathcal{L}\{f(t)\} = F(s)$ then

$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$

Proof: Let $f'(t)$ be continuous for all $t > 0$, then definition of Laplace transform

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) \, dt$$

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Integrating by parts,

$$L\{f'(t)\} = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} e^{-st}(-s) f(t)\, dt = [0 - f(0)] + s\int_0^{\infty} e^{-st} f(t)\, dt$$

Thus, $L\{f'(t)\} = s\,L\{f(t)\} - f(0)$ .... (1)

Remark: The above theorem may be extended higher order derivative of $f(t)$. Applying the second derivative, we get

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s\,[s\,L\{f(t)\} - f(0)] - f'(0)$$

$$\therefore \quad L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

Similarly, $L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(t)$

In general form, $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$

This formula for transform of the derivative is highly useful in solving differential equation.

3.6 Theorems on Laplace transform

3.6.1 Laplace transform of the integral of a function

If the function $f(t)$ is piecewise continuous and is of exponential order (i.e., $f(t)$ has Laplace transform) and $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left[\int_0^t f(u)\, du\right] = \frac{1}{s} F(s)$$

Proof:

Let $g(t) = \int_0^t f(u)\, du$, then,

$$\mathcal{L}\{g(t)\} = \int_0^{\infty} e^{-st} g(t)\, dt$$

$$= \left[g(t)\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{d}{dt} g(t) \frac{e^{-st}}{-s}\, dt$$

Since $g(t) = \int_0^t f(u)\, du$, therefore $\frac{d}{dt} g(t) = f(t)$.

Also, $f(t)$ is of exponential order, the first term is zero at both uppe[...] mits.

Therefore, $\mathcal{L}\{G(t)\} = \frac{1}{s}\int_0^{\infty} e^{-st} f(t)\, dt$

$$= \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$= \frac{1}{s} F(s)$$

Hence, $\mathcal{L}\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s} F(s)$

Remark: From above theorem $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\, du.$

3.6.2 Change of scale

If $\mathcal{L}\{f(t)\} = F(s)$, and $k > 0$, then for $t > 0$, $\mathcal{L}\{kt\} = \frac{1}{k} F\left(\frac{s}{k}\right)$.

Proof: By the definition of Laplace transform

$$\mathcal{L}\{f(kt)\} = \int_0^{\infty} e^{-st} f(kt)\, dt$$

Let $u = kt$, then $t = \frac{u}{k}$.

Now, differentiating with respect to t, we get

$$dt = \frac{du}{k},$$

When $t = 0$, then $u = 0$ and when $t \to \infty$ then $u \to \infty$.

$$\therefore \quad \mathcal{L}\{f(kt)\} = \int_0^{\infty} e^{-s\left(\frac{u}{k}\right)} f(u) \frac{du}{k}$$

or $\mathcal{L}\{f(kt)\} = \frac{1}{k}\int_0^{\infty} e^{-\left(\frac{s}{k}\right)u} f(u)\, du$

Hence, $\mathcal{L}\{f(kt)\} = \frac{1}{k} F\left(\frac{s}{k}\right)$. $\left[\because F(s) = \int_0^{\infty} e^{-su} f(u)\, du\right]$

$L\{f(kt)\}$

$= \frac{1}{8k} F\left\{\frac{s}{k}\right\}$

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3.6.3 Change of scale with shifting

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(kt)\} = \frac{1}{k} F\left(\frac{s-a}{k}\right)$.

Proof: We have,

$$\mathcal{L}\{e^{at} f(kt)\} = \int_0^\infty e^{-st} e^{at} f(kt)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(kt)\, dt$$

Let $u = kt$, then $t = \frac{u}{k}$.

Now, differentiating with respect to t, we get

$$dt = \frac{du}{k},$$

When $t = 0$, then $u = 0$ and when $t \to \infty$ then $u \to \infty$.

$$\text{Then, } \mathcal{L}\{e^{at} f(kt)\} = \int_0^\infty e^{-(s-a)\left(\frac{u}{k}\right)} f(u) \frac{du}{k}$$

$$= \int_0^\infty e^{-\left(\frac{s-a}{k}\right)u} f(u)\, du$$

$$\therefore \mathcal{L}\{e^{at} f(kt)\} = \frac{1}{k} F\left(\frac{s-a}{k}\right)$$

3.6.4 Multiplication of a function by t^n

If $\mathcal{L}\{f(t)\} = F(s)$, and n is positive integer, then for $t > 0$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof: By definition of Laplace of $f(t)$, we have

$$\int_0^\infty e^{-st} f(t)\, dt = F(s) \qquad \ldots (1)$$

Differentiating equation (1) with respect to s, we get

$$\frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds} F(s)$$

By Leibniz rule of differentiation under integral sign, (2) reduces to

$$\int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t)\, dt = \frac{d}{ds} F(s)$$

$$\text{or, } \int_0^\infty -te^{-st} f(t)\, dt = \frac{d}{ds} F(s)$$

$$\therefore \int_0^\infty e^{-st} (t.f(t))\, dt = -\frac{d}{ds} F(s) \qquad \ldots (3)$$

This proves the theorem for $n = 1$ (i.e., first result)

Now, assume the theorem is true for $n = m$ (say) so that equation (3) gives

$$\int_0^\infty e^{-st} (t^m f(t))\, dt = (-1)^m \frac{d^m}{ds^m} F(s) \qquad \ldots (4)$$

$$\text{Then, } \frac{d}{ds} \int_0^\infty e^{-st}\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s) \qquad \ldots (5)$$

Again, by Leibniz rule of differentiation under integral sign, (5) becomes

$$\int_0^\infty -t.e^{-st}\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$$

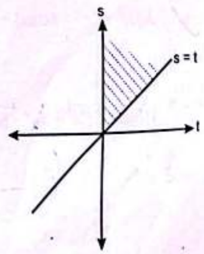
$$\therefore \int_0^\infty e^{-st}\, t^{m+1} f(t)\, dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s) \qquad \ldots (6)$$

which shows that if the theorem is true for $n = m$, then it is true for $n = m + 1$, i.e., for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, so on. Hence the theorem is true for all positive integer value n.

$$\therefore \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

3.6.5 Division of a function by t

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$.



Proof: By definition of Laplace of $f(t)$,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

Changing the order of integration, using figure for limits of integration.

$$\int_s^\infty\int_0^\infty e^{-st} f(t)\,dt\,ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t)\left[-\frac{1}{t}(0-e^{-st})\right] dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt$$

or, $\displaystyle\int_s^\infty F(s)\,ds = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$

Hence, $\displaystyle\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

e can use different variable symbol for F(s) i.e., F(k) inside integral, but it is common practice use s in both limit and variable so that the result will be in terms of s after integration.

ample 1. Find the Laplace transform of the following.

a. $3t^2 + 2e^{-t} - 5\cos t$ b. $\sin^3 2t$

c. $e^{-t}\sin^2 t$ d. $\sin 2t.\cos 3t$

e. $\sin at.\sin bt$ f. $\cosh at.\sin bt$

ion

a. $\mathcal{L}\{3t^2 + 2e^{-t} - 5\cos t\} = 3\mathcal{L}\{t^2\} + 2\mathcal{L}\{e^{-t}\} - 5\mathcal{L}\{\cos t\}$

$$= 3\left(\frac{2}{s^3}\right) + 2\left(\frac{1}{s+1}\right) - 5\left(\frac{s}{s^2+1}\right)$$

Hence, $\mathcal{L}\{3t^2 + 2e^{-t} - 5\cos t\} = \dfrac{6}{s^3} + \dfrac{2}{s+1} - \dfrac{5s}{s^2+1}$.

b. $\mathcal{L}\{\sin^3 2t\} = \frac{1}{4}\{3\sin 2t - \sin 6t\}$ $\left[\because \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}\right]$

$$= \frac{3}{4}\mathcal{L}\{\sin 2t\} - \frac{1}{4}\{\sin 6t\}$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{3}{2}\times\frac{32}{(s^2+4)(s^2+36)}$$

Hence, $\mathcal{L}\{\sin^3 2t\} = \dfrac{48}{(s^2+4)(s^2+36)}$.

c. We know,

$$\mathcal{L}(\sin^2 t) = \mathcal{L}\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(1) - \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}$$

$$= F(s)$$

So, by first shifting theorem,

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\therefore$ $\mathcal{L}\{e^{-t}\sin^2 t\} = F(s+1)$

$$= \frac{1}{2(s+1)} - \frac{1}{2}\left(\frac{s+1}{(s+1)^2+4}\right)$$

$$= \frac{1}{2}\left(\frac{s^2+2s+1+4-s^2-2s-1}{(s+1)((s+1)^2+4)}\right)$$

$\therefore$ $\mathcal{L}\{e^{-t}\sin^2 t\} = \dfrac{2}{(s+1)(s^2+2s+5)}$

Hence, $\mathcal{L}\{e^{-t}\sin^2 t\} = \dfrac{2}{(s+1)(s^2+2s+5)}$.

d. $\mathcal{L}\{\sin 2t.\cos 3t\} = \frac{1}{2}\mathcal{L}\{\sin 5t - \sin t\}$

$$= \frac{1}{2}\mathcal{L}\{\sin 5t\} - \frac{1}{2}\mathcal{L}\{\sin t\}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\frac{4s^2-20}{(s^2+25)(s^2+1)}$$

Hence, $\mathcal{L}\{\sin 2t.\cos 3t\} = \dfrac{2(s^2-5)}{(s^2+25)(s^2+1)}$.

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e. $\mathcal{L}\{\sin at.\sin bt\} = \frac{1}{2}\mathcal{L}\{\cos(a-b)t - \cos(a+b)t\}$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+(a+b)^2 - \{s^2+(a-b)^2\}}{[s^2+(a-b)^2][s^2+(a+b)^2]}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+a^2+2ab+b^2-s^2-a^2+2ab-b^2}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right]$$

$$= \frac{s}{2}\left[\frac{4ab}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right]$$

Hence, $\mathcal{L}\{\sin at\ \sin bt\} = \frac{2abs}{[s^2+(a+b)^2][s^2+(a-b)^2]}$.

f. $\mathcal{L}\{\cosh at \sin at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\sin at\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\sin at\} + \frac{1}{2}\mathcal{L}\{e^{-at}\sin at\}$$

$$= \frac{1}{2}[\mathcal{L}\{\sin at\}]_{s\to s-a} + \frac{1}{2}[\mathcal{L}\{\sin at\}]_{s\to s+a}$$

$$= \frac{1}{2}\left[\frac{a}{s^2+a^2}\right]_{s\to s-a} + \frac{1}{2}\left[\frac{a}{s^2+a^2}\right]_{s\to s+a}$$

$$= \frac{1}{2}\left(\frac{a}{(s-a)^2+a^2}\right) + \frac{1}{2}\left(\frac{a}{(s+a)^2+a^2}\right)$$

$$= \frac{a}{2}\left[\frac{1}{(s^2+2a^2)-2as} + \frac{1}{(s^2+2a^2)+2as}\right]$$

$$= \frac{a}{2}\left[\frac{s^2+2a^2+2as+s^2+2a^2-2as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{s^4+4a^4}\right]$$

Hence, $\mathcal{L}\{\cosh at \sin at\} = \frac{a(s^2+2a^2)}{s^4+4a^4}$.

Alternately: We can also find above result by taking the imaginary part of $\mathcal{L}\{e^{iat}$ cosh a

when $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > |a|$.

Example 2. Find the Laplace transform of following functions:

a. $\int_0^t \cos at\, dt$ b. $\int_0^t t^3 e^{-2t}\, dt$ c. $\int_0^t te^t \sin t\, dt$

Solution

a. $\mathcal{L}\left\{\int_0^t \cos at\, dt\right\} = \frac{1}{s}\mathcal{L}\{\cos at\}$

$$= \frac{1}{s}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{1}{s^2+a^2}$$

Hence, $\mathcal{L}\left\{\int_0^t \cos at\, dt\right\} = \frac{1}{s^2+a^2}$.

b. $\mathcal{L}\left\{\int_0^t t^3 e^{-2t}\, dt\right\} = \frac{1}{s}\mathcal{L}\{t^3 e^{-2t}\}$

$$= \frac{1}{s}[\mathcal{L}\{t^3\}]_{s\to s+2}$$

$$= \frac{1}{s}\left[\frac{3!}{s^4}\right]_{s\to s+2}$$

$$= \frac{1}{s}\left[\frac{6}{(s+2)^4}\right]$$

Hence, $\mathcal{L}\left\{\int_0^t t^3 e^{-2t}\, dt\right\} = \frac{6}{s(s+2)^4}$.

c. $\mathcal{L}\left\{\int_0^t te^t \sin t\, dt\right\} = \frac{1}{s}\mathcal{L}\{te^t \sin t\}$

$$= \frac{1}{s}[\mathcal{L}\{t \sin t\}]_{s\to s-1}$$

$$= \frac{1}{s}\left[\frac{-d}{ds}\mathcal{L}\{\sin t\}\right]_{s\to s-1}$$

$$= \frac{1}{s}\left[\frac{-d}{ds}\left(\frac{1}{s^2+1}\right)\right]_{s\to s-1}, \text{ since } -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$\therefore$ $\mathcal{L}\left\{\int_0^t te^t \sin t\, dt\right\} = \frac{1}{s}\left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1}$

$$= \frac{1}{s}\left[\frac{2(s-1)}{[(s-1)^2+1]^2}\right]$$

Hence, $\mathcal{L}\left\{\int_0^t te^t \sin t\, dt\right\} = \left[\frac{2(s-1)}{s(s^2-2s+2)^2}\right]$

Example 3. Find the Laplace transform of the following functions:

a. $e^{at}\cos^2 t$ b. $e^{-3t}\underbrace{(2\cos 5t - 3\sin 5t)}_{s\to s+3}$ c. $t^2 e^{-3t}$

Solution

a. $\mathcal{L}\{e^{at}\cos^2 t\} = [\mathcal{L}\{\cos^2 t\}]_{s\to s-a}$

$$= \left[\mathcal{L}\left\{\frac{1+\cos 2t}{2}\right\}\right]_{s\to s-a}$$

$$= \frac{1}{2}[\mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}]_{s\to s-a}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]_{s\to s-a}$$

Hence, $\mathcal{L}\{e^{at}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-a} + \frac{s-a}{(s-a)^2+4}\right]$.

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b. $\mathcal{L}\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = [\mathcal{L}\{2\cos 5t - 3\sin 5t\}]_{s\to s+3}$

$$= [2\mathrm{L}\{\cos 5t\} - 3\mathrm{L}\{\sin 5t\}]_{s\to s+3}$$

$$= \left[2\left(\frac{s}{s^2+25}\right) - 3\left(\frac{5}{s^2+25}\right)\right]_{s\to s+3}$$

$$= \left[\frac{2s-15}{s^2+25}\right]_{s\to s+3}$$

$$= \frac{2(s+3)-15}{(s+3)^2+25}$$

$$= \frac{2s+6-15}{s^2+6s+9+25}$$

Hence, $\mathcal{L}\{e^{-3t}(2\cos 5t - 3\sin 5t) = \dfrac{2s-9}{s^2+6s+34}$.

c. By using shifting rule

$$\mathcal{L}\{t^2 e^{-3t}\} = [\mathcal{L}\{t^2\}]_{s\to s+3}$$

$$= \left[\frac{2!}{s^3}\right]_{s\to s+3}$$

Hence, $\mathcal{L}\{t^2 e^{-3t}\} = \dfrac{2}{(s+3)^3}$.

Example 4. Find the Laplace transform of the following functions

a. $t.\sin 3t$ b. $t\sin^2 t$ c. $t^2\cos at$

d. $\sinh 3t\cos^2 t$ e. $te^{-4t}\sin 3t$ f. $t^2 - e^{-t}\sin t$

g. $\dfrac{1-\cos t}{t}$

Solution

Using $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d}{ds}$ F(s), where $\mathcal{L}\{(t)\} = F(s)$

a. $\mathcal{L}\{t\sin 3t\} = -\dfrac{d}{ds}\mathcal{L}\{\sin 3t\}$

$$= \frac{-d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{3}{(s^2+9)^2}.2s$$

$$= \frac{6s}{(s^2+9)^2}$$

$\therefore$ $\mathcal{L}[t\sin 3t] = \dfrac{6s}{(s^2+9)^2}$

b. $\mathcal{L}\{t\sin^2 t\} = \dfrac{-d}{ds}\mathcal{L}\{\sin^2 t\}$

$$= \frac{-d}{ds}\mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{-d}{ds}\frac{1}{2}[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}]$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \frac{(s^2+4).1 - s.2s}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^4+8s^2+16+4s^2-s^4}{s^2(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{12s^2+16}{s^2(s^2+4)^2}\right]$$

Hence, $\mathcal{L}\{t\sin^2 t\} = \dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$.

c. $\mathcal{L}\{t^2\cos at\} = (-1)^2\dfrac{d^2}{ds^2}\mathcal{L}\{\cos at\}$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{(s^4+2a^2s^2+a^4)(-2s) - 4s(a^4-s^4)}{(s^2+a^2)^4}$$

$$= \frac{-2s^5-4a^2s^3-2a^4s-4a^4s+4s^5}{(s^2+a^2)^4}$$

$$= \frac{2s^5-4a^2s^3-6a^4s}{(s^2+a^2)^4}$$

$$= \frac{2s^5-6a^2s^3+2a^2s^3-6a^4s}{(s^2+a^2)^4}$$

$$= \frac{2s^3(s^2-3a^2)+2a^2s(s^2-3a^2)}{(s^2+a^2)^4}$$

$$= \frac{2s(s^2-3a^2)(s^2+a^2)}{(s^2+a^2)^4}$$

Hence, $\mathcal{L}\{t^2\cos at\} = \dfrac{2s(s^2-3a^2)}{(s^2+a^2)^3}$.

d. $\mathcal{L}\{\sinh 3t\cos^2 t\} = \mathcal{L}\left\{\left(\dfrac{e^{3t}-e^{-3t}}{2}\right)\left(\dfrac{1+\cos 2t}{2}\right)\right\}$

$$= \frac{1}{4}\mathcal{L}\{(e^{3t}-e^{-3t}) + (e^{3t}-e^{-3t})\cos 2t\}$$

$$= \frac{1}{4}[\mathcal{L}\{e^{3t}\} - \mathcal{L}\{e^{-3t}\} + \mathcal{L}\{e^{3t}\cos 2t\} - \mathcal{L}\{e^{-3t}\cos 2t\}]$$

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$$=\frac{1}{4}\left[\left(\frac{1}{s-3}-\frac{1}{s+3}\right)+\left(\frac{s-3}{(s-3)^3+4}-\frac{s+3}{(s+3)^2+4}\right)\right]$$

$$=\frac{1}{4}\left[\frac{6}{s^2-9}+\frac{(s-3)\{(s+3)^2+4\}-(s+3)\{(s-3)^2+4\}}{[(s-3)^2+4]\,[(s+3)^2+4]}\right]$$

On simplifying, we get, $\mathcal{L}\{\sinh 3t.\cos^2 t\}=\frac{3}{2}\left[\frac{1}{s^2-9}+\frac{s^2-13}{s^4-10s^2+169}\right].$

e. $\mathcal{L}\{te^{-4t}\sin 3t\}=[\mathcal{L}\{t\sin 3t\}]_{s\to s+4}$

$$=\left[-\frac{d}{ds}\mathcal{L}\{\sin 3t\}\right]_{s\to s+4}$$

$$=\left[-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right]_{s\to s+4}$$

$$=\left[\frac{3}{(s^2+9)^2}.2s\right]_{s\to s+4}$$

$$=\left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$$

$$=\frac{6(s+4)}{[(s+4)^2+9]^2}$$

Hence, $\mathcal{L}\{te^{-4t}\sin 3t\}=\frac{6(s+4)}{(s^2+8s+25)}$

f. $\mathcal{L}\{t^2e^{-t}\sin t\}=[\mathcal{L}\{t^2\sin t\}]_{s\to s+1}$

$$=\left[(-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\sin t\}\right]_{s\to s+1}$$

$$=\left[\frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)\right]_{s\to s+1}$$

$$=\frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right]_{s\to s+1}$$

$$=-2\left[\frac{d}{ds}\frac{s}{(s^2+1)^2}\right]_{s\to s+1}$$

$$=-2\left[\frac{[(s^2+1)^2.1-s.2(s^2+1).2s]}{(s^2+1)^4}\right]_{s\to s+1}$$

$$=-2\left[\frac{1(s^2+1)\,[s^2+1-4s^2]}{(s^2+1)^4}\right]_{s\to s+1}$$

$$=2\left[\frac{(s^2+1)\,(3s^2-1)}{(s^2+1)^4}\right]_{s\to s+1}$$

$$=\left[\frac{2(3(s+1)^2-1)}{((s+1)^2+1)^3}\right]$$

$$=\frac{2[(3s^2+6s+3)-1]}{(s^2+2s+1+1)^3}$$

Hence, $\mathcal{L}\{t^2e^{-t}\sin t\}=\frac{2(3s^2+6s+2)}{(s^2+2s+2)^3}$

ample 5. Find the Laplace transform of following functions:

a. $\frac{e^{-t}\sin at}{t}$ b. $\frac{e^{-at}-e^{-bt}}{t}$ c. $\frac{\sin 3t\cos t}{t}$

d. $\frac{\sin^2 t}{t}$ e. $\frac{1-\cos t}{t}$

lution

a. $\mathcal{L}\left\{\frac{e^{-t}\sin at}{t}\right\}=\left[\mathcal{L}\left\{\frac{\sin at}{t}\right\}\right]_{s\to s+1}$

$$=\left[\int_s^\infty \mathcal{L}\{\sin at\}\,ds\right]_{s\to s+1}$$

$$=\left[\int_s^\infty \frac{a}{s^2+a^2}\,ds\right]_{s\to s+1}$$

$$=\left[\left[\tan^{-1}\frac{s}{a}\right]_s^\infty\right]_{s\to s+1}$$

$$=\left[\tan^{-1}\infty-\tan^{-1}\frac{s}{a}\right]_{s\to s+1}$$

$$=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{a}\right]_{s\to s+1}$$

Hence, $\mathcal{L}\left\{\frac{e^{-t}\sin at}{t}\right\}=\frac{\pi}{2}-\tan^{-1}\left(\frac{s+1}{a}\right)$

b. $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty \mathcal{L}\{e^{-at}-e^{-bt}\}\,ds$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=[\log(s+a)-\log(s+b)]_s^\infty$$

$$=\left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$=\left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^\infty$$

$$=\left[0-\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]$$

$$=-\log\left(\frac{s+a}{s+b}\right)$$

Hence, $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\log\left(\frac{s+b}{s+a}\right).$

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c. $\mathcal{L}\left\{\frac{\sin 3t . \cos t}{t}\right\} = \int_s^\infty \mathcal{L}[\sin 3t \cos t\} \, ds$

$$= \frac{1}{2}\int_s^\infty (\sin 4t + \sin 2t)\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{4}{s^2+4^2} + \frac{2}{s^2+2^2}\right) ds$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\pi}{2}\right) - \left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)\right]$$

Hence, $\mathcal{L}\left\{\frac{\sin 3t . \cos t}{t}\right\} = \frac{\pi}{2} - \frac{1}{2}\left(\tan^{-1}\frac{s}{4} + \tan^{-1}\frac{s}{2}\right)$

d. $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \int_0^\infty \mathcal{L}\{\sin^2 t\}\, ds$

$$= \int_s^\infty \mathcal{L}\left(\frac{1-\cos 2t}{2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty [\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}]\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+2^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}[2\log s - \log(s^2+4)]_s^\infty$$

$$= \frac{1}{4}[\log s^2 - \log(s^2+4)]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+\frac{4}{s^2}}\right)\right]_s^\infty$$

$$= \left[\log(1) - \log\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= \frac{1}{4}\left[0 - \log\left(\frac{s^2}{s^2+4}\right)\right]$$

Hence, $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$.

e. Taking $f(t) = \frac{1-\cos t}{t}$

Using $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$

$$\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \mathcal{L}\{(1-\cos t)\, ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1^2}\right) ds \qquad \left[\because \mathcal{L}\{1\} = \frac{1}{s} \text{ and } \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}\right]$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\sqrt{s^2} - \log(s^2+1)^{1/2}\right]_s^\infty$$

$$= \left[\log\left(\sqrt{\frac{s^2}{s^2+1}}\right)\right]_s^\infty$$

$$= \left[\log\sqrt{\frac{1}{1+\frac{1}{s^2}}}\right]_s^\infty$$

or, $\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \log 1 - \log\sqrt{\frac{s^2}{s^2+1}}$

$$= \log\sqrt{\frac{(s^2+1)}{s^2}} \qquad \left[\because \frac{1}{1+\frac{1}{s^2}} \to \frac{1}{1+\frac{1}{\infty}} = 1 \text{ when } s \to \infty\right]$$

$\therefore \mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \frac{1}{2}\log\frac{(s^2+1)}{s^2}$.

Exercise 3.1

1. Find the Laplace transform of following functions $f(t)$.
   a. $t^{5/2}$ b. $\sinh^3 2t$
   c. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$ d. $t - \sinh 2t$
   e. $\sin 2t \sin 3t$ f. $\sin at \cos at$
2. Find the Laplace transform of following functions
   a. $\cosh at . \cos at$ b. $\cosh at . \sin at$
   c. $\sin 2t . \cos 3t$ d. $\sin at \cosh at - \cos at \sinh at$
   e. $\cos^3 2t$ f. $\cosh^2 3t$
   g. $\sin(wt + \theta)$ h. $\cos\left(3t + \frac{\pi}{4}\right)$
3. Find the Laplace transform of following functions
   a. $t^2 e^{-3t}$ b. $t . \sinh at$
   c. $t . \cos at$ d. $e^{2t} \sin t . \sin 2t . \sin 3t$
   e. $e^{3t}\sin^2 t$ f. $t\sin^3 at$
   g. $t^2 \cos wt$ h. $t . e^{-t}\cosh t$
   i. $t . e^{-3t}\cos 2t$

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4. Find the Laplace transform of following functions

a. $\frac{\sin at}{t}$

b. $\frac{1-\cos 2t}{t}$

c. $\frac{1-e^{t}}{t^2}$

d. $\frac{\cos at-\cos bt}{t}$

e. $\frac{e^{t}\sin t}{t}$

5. Find the Laplace transform of the following functions

a. $\int_0^t e^{-t}\cos t\,dt$

b. $\int_0^t \frac{\cos at-\cos bt}{t}\,dt$

c. $\int_0^t \frac{\sin t}{t}\,dt$

d. $\int_0^t \frac{1-\cos t}{t}\,dt$

e. $\int_0^t \frac{1-e^{t}}{t}\,dt$

f. $\int_0^t t^2e^{-t}\,dt$

Answer

1. a. $\frac{15\sqrt{\pi}}{8s^{7/2}}$ b. $\frac{48}{(s^2-4)(s^2-36)}$ c. $\frac{1}{s-2}+\frac{24}{s^5}+\frac{3(s-2)}{s^2+9}$

d. $\frac{4+s^2}{s^2(4-s^2)}$ e. $\frac{12s}{(s^2+1)(s^2+25)}$ f. $\frac{a(s^2-2a^2)}{s^4+4a^4}$

2. a. $\frac{s^2}{s^4+4a^4}$ b. $\frac{a(s^2+2a^2)}{s^4+4a^4}$ c. $\frac{2(s^2-5)}{(s^2+1)(s^2+25)}$

d. $\frac{4a^3}{s^4+4a^4}$ e. $\frac{s(s^2+28)}{(s^2+4)(s^2+36)}$ f. $\frac{1}{4}\left(\frac{1}{s-6}+\frac{1}{s+6}+\frac{2}{s}\right)$

g. $\frac{w\cos\theta+s\sin\theta}{s^2+w^2}$ h. $\frac{s-3}{\sqrt{2}(s^2+9)}$

3. a. $\frac{2}{(s+3)^3}$ b. $\frac{s^2-a^2}{(s^2+a^2)^2}$ c. $\frac{2as}{(s^2-a^2)^2}$

d. $\frac{1}{2}\left[\frac{1}{s^2-4s+8}-\frac{3}{s^2-4s+40}+\frac{2}{s^2-4s+2}\right]$

e. $\frac{2}{(s-3)(s^2-6s+13)}$ f. $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$ g. $\frac{s^2+2s+2}{(s^2+2s)^2}$

h. $\frac{s^2+6s+5}{(s^2+6s+13)^2}$ i. $\frac{s^2-4}{(s^2+4)^2}$

4. a. $\cos^{-1}\left(\frac{s}{a}\right)$ b. $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$ c. doesn't exist

d. $\log\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right)$ e. $\cot^{-1}(s+1)$

5. a. $\frac{1}{s}\left(\frac{s+1}{s^2+2s+2}\right)$ b. $\frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$ c. $\frac{1}{s}\cot^{-1}(s)$

d. $\frac{1}{s}\log\left(\frac{s+1}{s}\right)$ d. $\frac{1}{2s}\log\left(\frac{s^2+1}{s^2}\right)$ f. $\frac{2}{s(s+1)^3}$

3.7 Determination of inverse Laplace transforms

To find the inverse Laplace transform $f(t)$ of $F(s)$, we try to recognize the given function either in the given form comparable to some standard expression, whose inverse function is a known standard function or split the given function of s into number of expressions in s (*with the help of partial fractions*) comparable again to some standard functions of s of which inverse functions are known.

Note that, if $\mathcal{L}[f(t)] = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is defined by $f(t) = \mathcal{L}^{-1}[F(s)]$.

The table given below lists the inverse Laplace transform of some standard elementary functions and associated functions.

1	$\mathcal{L}^{-1}\left(\frac{1}{s}\right)=1$	$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=e^{at}$
2	$\mathcal{L}^{-1}\left(\frac{1}{s^n}\right)=\frac{t^{n-1}}{(n-1)!}$	$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right)=\frac{e^{at}t^{n-1}}{(n-1)!}$
3	$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$	$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$
4	$\mathcal{L}^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$	$\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right)=\cosh at$
5	$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2+b^2}\right)=\frac{1}{b}e^{at}\sin bt$	$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right)=e^{at}\cos bt$
6	$\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)=\frac{1}{2a}t\sin at$	$\mathcal{L}^{-1}\left(\frac{s}{(s^2-a^2)^2}\right)=\frac{1}{2a}t\sinh at$
7	$\mathcal{L}^{-1}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)=t\cos at$	$\mathcal{L}^{-1}\left(\frac{s^2+a^2}{(s^2-a^2)^2}\right)=t\cosh at$
8	$\mathcal{L}^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)=\frac{1}{2a^3}(\sin at-at\cos at)$	$\mathcal{L}^{-1}\left(\frac{1}{(s^2-a^2)^2}\right)=-\frac{1}{2a^3}(\sinh at-at\cosh at)$
9	$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right)=\frac{1}{2a}(\sin at+at\cos at)$	$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2-a^2)^2}\right)=\frac{1}{2a}(\sinh at+at\cosh at)$
10	$\mathcal{L}^{-1}\left(\frac{s^3}{(s^2+a^2)^2}\right)=\left(\cos at-\frac{1}{2}at\sin at\right)$	$\mathcal{L}^{-1}\left(\frac{s^3}{(s^2-a^2)^2}\right)=\left(\cosh at+\frac{1}{2}-at\sinh at\right)$
11	$\mathcal{L}^{-1}\left(\frac{s}{s^4+4a^4}\right)=\frac{1}{2a^2}\sinh at\,.\,\sin at$	$\mathcal{L}^{-1}\left(\frac{s^3}{s^4+4a^4}\right)=\cosh at\,.\,\cos at$
12	$\mathcal{L}^{-1}\left(\frac{s^2+2a^2}{s^4+4a^4}\right)=\frac{1}{a}\cosh at\,.\,\sin at$	$\mathcal{L}^{-1}\left(\frac{s^2-2a^2}{s^4+4a^4}\right)=\frac{1}{a}\sinh at\,.\,\cos at$
13	$\mathcal{L}^{-1}\left(\frac{1}{s^4+4a^4}\right)=\frac{1}{4a^3}(\cosh at\sin at-\sinh at\cos at)$	
14	$\mathcal{L}^{-1}\left(\frac{s^2}{s^4+4a^4}\right)=\frac{1}{2a}(\cosh at\sin at+\sinh at\cos at)$	

nverse transforms tabulated above will be frequently used as ready reference, particularly for olving differential equations using Laplace transforms. Without going into details of derivation f these, we may discuss some of them as below.

$$\frac{s^3}{(s^2+a^2)^2} = \frac{s(s^2+a^2-a^2)}{(s^2+a^2)^2}$$

$$= \frac{s}{(s^2+a^2)} - a^2\frac{s}{(s^2+a^2)^2}$$

Then, taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\right] - a^2\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= \cos at - a^2\frac{t\sin at}{2a}$$

$$= \cos at - \frac{1}{2}at\sin at$$

Similarly, $$\mathcal{L}^{-1}\left[\frac{s^3}{(s^2-a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s^2-a^2)}\right] + a^2\mathcal{L}^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$$

$$= \cosh at + a^2\frac{t\sinh at}{2a}$$

$$= \cosh at - \frac{1}{2}at\sinh at$$

As already stated, in deducting these results, we need to make use of **partial fractions**. Some the properties of inverse Laplace transforms and Convolution Theorem are taken up subsequent discussions.

Tips for Partial Fractions: In any general expression containing:

$$\frac{A_1}{s-a}, \frac{A_2}{(s-a)^r}, \frac{A_3}{s^2+as+b}, \frac{A_4}{(s^2+as+b)^r}$$

i. Corresponding to a non-repeated linear factor $(s-a)$, write $\frac{A}{(s-a)}$

ii. Corresponding to a repeated linear factor $(s-a)^r$, write

$$\frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_r}{(s-a)^r}$$

iii. Corresponding to a non-repeated quadratic factor (s^2+as+b), write $\frac{As+B}{(s^2+as+b)}$

iv. Corresponding to repeated quadratic factors (s^2+as+b), write

$$\frac{A_1s+B_1}{(s^2+as+b)} + \frac{A_2s+B_2}{(s^2+as+b)^2} + \ldots + \frac{A_rs+B_r}{(s^2+as+b)^r}$$

Then, determine unknown constant $A_1, A_2, A_3, \ldots, A_r, B_1, B_2, B_3, \ldots, B_r$ by equating the coefficients of equal powers of s on both sides.

Example 6. Find the inverse Laplace transform of the following functions:

a. $\frac{6}{s^2+4}$ b. $\frac{5s}{s^2-36}$ c. $\frac{s+2}{(s+2)^2-25}$

d. $\frac{s+9}{s^2+6s+13}$ e. $\frac{60+6s^2+s^4}{s^7}$ f. $\frac{1}{(s+a)^n}$

Solution

a. We have $F(s) = \frac{6}{s^2+4}$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3\times2}{s^2+4}\right\}$$

$$= 3L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

Hence, $f(t) = 3\sin 2t$

b. We have $F(s) = \frac{5s}{s^2-36}$

Taking inverse Laplace transform a both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = 5L^{-1}\left\{\frac{s}{s^2-6^2}\right\}$$

Hence, $f(t) = 5\cosh 6t$

c. We have $F(s) = \frac{s+2}{(s+2)^2-25}$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2-25}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2-5^2}\right\}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\}$$

Hence, $f(t) = e^{-2t}\cosh 5t$

d. We have, $F(s) = \frac{s+9}{s^2+6s+13}$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+9}{s^2+6s+13}\right\}.$$

Then, $$f(t) = \mathcal{L}^{-1}\left\{\frac{s+9}{s^2+2.s.3+3^2-3^2+13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+3+6}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3\times2}{(s+3)^2+2^2}\right\}$$

$$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3e^{-3t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

Hence, $f(t) = e^{-3t}(\cos 2t + 3\sin 2t)$.

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e. We have, $F(s)=\dfrac{60+6s^2+s^4}{s^7}$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{60}{s^7}+\frac{6s^2}{s^7}+\frac{s^4}{s^7}\right\}$$

$$=\frac{1}{12}L^{-1}\left\{\frac{6!}{s^7}\right\}+\frac{1}{4}L^{-1}\left\{\frac{4!}{s^5}\right\}+\frac{1}{2}L^{-1}\left\{\frac{2!}{s^3}\right\}$$

Hence, $f(t)=\dfrac{1}{12}t^6+\dfrac{1}{4}t^4+\dfrac{1}{2}t^2$

f. We have, $F(s)=\dfrac{1}{(s+a)^n}$

Taking inverse Laplace on both sides, we get

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+a)^n}\right\}$$

$$=e^{-at}L^{-1}\left\{\frac{1}{s^n}\right\} \quad \text{.... (1)}$$

Also, we have

$$L\{t^n\}=\frac{n!}{s^{n+1}}$$

So, $L\{t^{n-1}\}=\dfrac{(n-1)!}{s^n}$

i.e., $L^{-1}\left\{\dfrac{1}{s^n}\right\}=\dfrac{t^{n-1}}{(n-1)!}$ .... (2)

From equation (1) and (2)

$$L^{-1}\left\{\frac{1}{(s+a)^n}\right\}=e^{-at}\frac{t^{n-1}}{(n-1)!}.$$

Example 7. Find the inverse Laplace transform of following functions:

a. $\dfrac{1}{s^2-5s+6}$ b. $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ c. $\dfrac{1}{4s+s^3}$

d. $\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$ e. $\dfrac{s}{(s+1)^2(s^2+1)}$

Solution

a. We have, $F(s)=\dfrac{1}{s^2-5s+6}$

$$=\frac{1}{s^2-3s-2s+6}$$

$$=\frac{1}{s(s-3)-2(s-3)}$$

$$=\frac{1}{(s-3)(s-2)}$$

Now, let

$$F(s)=\frac{1}{(s-3)(s-2)}$$

$$=\frac{A}{s-3}+\frac{B}{s-2} \quad \text{....}$$

Then, $A(s-2)+B(s-3)=1$

A(s-2)+B(s-3)

putting $s=3 \Rightarrow A=1$ putting $s=2 \Rightarrow B=-1$

Then, equation (1) becomes

$$F(s)=\frac{1}{s-3}-\frac{1}{s-2}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s-3}\right\}-L^{-1}\left\{\frac{1}{s-2}\right\}.$$

Hence, $f(t)=e^{3t}-e^{2t}$

b. We have, $F(s)=\dfrac{s^2+s-2}{s(s+3)(s-2)}$

Let, $F(s)=\dfrac{s^2+s-2}{s(s+3)(s-2)}$

$$=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2} \quad \text{.... (1)}$$

Then, $s^2+s-2=A(s+3)(s-2)+B(s-2)s+C(s+3)s$

When, $s=0 \Rightarrow -2=-6A+0+0 \Rightarrow A=\dfrac{1}{3}$

Also, put $s=2 \Rightarrow 4=0+0+10C \Rightarrow C=\dfrac{2}{5}$

Again, put $s=-3 \Rightarrow 4=0+15B+0 \Rightarrow B=\dfrac{4}{15}$

Then, equation (1) becomes

$$F(s)=\frac{\frac{1}{3}}{s}+\frac{\frac{4}{15}}{s+3}+\frac{\frac{2}{5}}{s-2}$$

$$\therefore \quad F(s)=\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}$$

Taking inverse Laplace transform as both sides, we get

$$L^{-1}\{F(s)\}=\frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\}+\frac{4}{15}L^{-1}\left\{\frac{1}{s+3}\right\}+\frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$$

Hence $f(t)=\dfrac{1}{3}+\dfrac{4}{15}e^{-3t}+\dfrac{2}{5}e^{2t}$

c. We have $F(s)=\dfrac{1}{4s+s^3}$

$$=\frac{1}{s(4+s^2)}$$

Let, $F(s)=\dfrac{1}{s(4+s^2)}$

$$=\frac{A}{s}+\frac{Bs+C}{4+s^2} \quad \text{.... (1)}$$

Then, $1=A(4+s^2)+(Bs+C)s$

or, $1=A(4+s^2)+(Bs+C)s$

or, $1=4A+As^2+Bs^2+Cs$

or, $s^2(A+B)+Cs+4A=1$

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Equating coefficient of s and ...

A + B = 0, C = 0 and 4A = 1

Solving those, $A = \frac{1}{4}$, $B = -\frac{1}{4}$ and $C = 0$

Now, equation (1) becomes

$$F(s) = \frac{1}{4s} - \frac{s}{4(s^2+4)}$$

Taking inverse Laplace transformation on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$= \frac{1}{4}.1 - \frac{1}{4}\cos 2t$$

$$= \frac{1-\cos 2t}{4}$$

$$= \frac{1-(1-2\sin^2 t)}{4}$$

Hence, $f(t) = \frac{1}{2}\sin^2 t$.

d. We have $F(s) = \frac{s+1}{s^2(s^2+1)}$

Let, $F(s) = \frac{s+1}{s^2(s^2+1)}$

$$= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

Then,

$s + 1 = (As + B)(s^2 + 1) + (Cs + D)s^2$

or, $s + 1 + As^3 + Bs^2 + As + B + Cs^3 + Ds^2$

or, $s + 1 = (A + C)s^3 + (B + D)s^2 + As + B$

Equating coefficient of the like term on both sides, we get

A + C = 0, B + D = 0, A = 1 & B = 1

Solving, A = 1, B = 1, C = –1, D = –1

Equation (1) becomes

$$F(s) = \left(\frac{s+1}{s^2} - \frac{s+1}{s^2+1}\right)$$

or, $F(s) = \left(\frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$f(t) = 1 + t - \cos t - \sin t$

e. We have $F(s) = \frac{s}{(s+1)^2(s^2+1)}$

Let, $F(s) = \frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$ .... (1)

Then,

$s = A(s + 1)(s^2 + 1) + B(s^2 + 1) + (Cs + d.(s + 1)^2$

or, $s = As^3 + As^2 + As + A + Bs^2 + B + Cs^3 + Ds^2 + 2Cs^2 + 2Ds + Cs + D$

or, $(A + c)s^3 + (A + B + D + 2C)s^2 + (A + 2D + C)s + (A + B + D) = s$

Equating coefficient of s and constant,

A + C = 0, A + B + D + 2C = 0

A + 2D + C = 1 and A + B + D = 0

Solving these equation, we get

$A = 0, B = -\frac{1}{2}, C = 0$ and $D = \frac{1}{2}$

Therefore, equation (1) becomes

$$F(s) = \frac{-\frac{1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}$$

$$= \frac{-1}{2(s+1)^2} + \frac{1}{2(s^2+1)}$$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

or, $\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\sin t$

or, $f(t) = \frac{-1}{2}e^{-t}t + \frac{1}{2}\sin t$

$\therefore$ $f(t) = \frac{1}{2}(\sin t - te^{-t})$

Example 8. Find the inverse Laplace transform of following functions

a. $\frac{s}{s^4+s^2+1}$ b. $\frac{s^2}{s^4+4a^4}$

Solution

a. We have, $\frac{s}{s^4+s^2+1} = \frac{s}{(s^4+2s^2+1)-s^2}$

$$= \frac{s}{(s^2+1)^2 - s^2}$$

$$= \frac{s}{(s^2+1+s)(s^2+1-s)}$$

Let, $\frac{s}{(s^2+1-s)(s^2+1+s)} = \frac{As+B}{s^2+1-s} + \frac{Cs+D}{s^2+1+s}$

Then,

$s = (As + B)(s^2 + 1 + s) + (Cs + D)(s^2 + 1 - s)$

or, $s = (A + c)s^3 + (A + B - C + D)s^2 + (A + B + C - D)s + (B + D)$

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Equating coefficient of s and constant form on both sides, we get.

$$A + C = 0 \quad \ldots\ldots (1)$$
$$A + B - C + D = 0 \quad \ldots\ldots (2)$$
$$A + B + C - D = 1 \quad \ldots\ldots (3)$$
$$B + D = 0 \quad \ldots\ldots (4)$$

Solving all these equations, we get

$$A = 0,\ B = \frac{1}{2},\ C = 0 \text{ and } D = \frac{-1}{2}$$

The given function becomes

$$\frac{s}{s^4+s^2+1} = \frac{1}{2}\times\frac{1}{(s^2+1-s)} - \frac{1}{2}\times\frac{1}{(s^2+1+s)}$$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1-s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1+s}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2-s+\frac{1}{4}\right)+\frac{3}{4}}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+s+\frac{1}{4}\right)+\frac{3}{4}}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}e^{\frac{1}{2}t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}e^{-\frac{1}{2}t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} \quad [\because \text{ by first shifting property}]$$

$$= \frac{1}{2}e^{\frac{1}{2}t}\left[\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right] - \frac{1}{2}e^{-\frac{1}{2}t}\left[\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\left(\frac{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}}{2}\right)$$

$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sinh\left(\frac{t}{2}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}t}{2}$$

The given function is $\frac{s^2}{s^4+4a^4} = \frac{s^2}{s^4+4a^4+4a^2s^2-4a^2s^2}$

$$= \frac{s^2}{(s^2+2a^2)^2-(2as)^2}$$

$$= \frac{s^2}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

Let, $\frac{s^2}{s^4+4a^4} = \frac{As+B}{s^2+2a^2+2as} + \frac{Cs+D}{s^2+2a^2-2as}$

Then,

$= As^3+2a^2As-2aAs^2+Bs^2+2a^2B-2aBs+Cs^3+2a^2Cs+2aCs^2+Ds^2+$

$= (A+C)s^3+(B-2aA+2aC+D)s^2+(2a^2A-2aB+2a^2C+2aD)s+(2$

$$A + C = 0 \quad \ldots\ldots (1)$$
$$B - 2aA + 2aC + D = 1 \quad \ldots\ldots (2)$$
$$2a^2A - 2aB + 2a^2C + 2aD = 0 \quad \ldots\ldots (3)$$
and $2a^2B + 2a^2D = 0 \quad \ldots\ldots (4)$

Solving all these equations, we get

$$A = -\frac{1}{4a},\ B = 0,\ C = \frac{1}{4a} \text{ and } D = 0$$

So, given function becomes

$$\frac{s^2}{s^4+4a^4} = -\frac{1}{4a}\times\frac{s}{s^2+2a^2+2as} + \frac{1}{4a}\times\frac{s}{s^2+2a^2-2as}$$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \frac{1}{4a}\left[\mathcal{L}^{-1}\left\{\frac{-s}{s^2+2as+2a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2-2as+2a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\mathcal{L}^{-1}\left\{\frac{-s}{(s+a)^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s-a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\mathcal{L}^{-1}\left\{\frac{s-a+a}{(s-a)^2+a^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s+a-a}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{a}{(s+a)^2+a^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{a}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + e^{at}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} - e^{-at}\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + e^{-at}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}[e^{at}\cos at + e^{at}\sin at - e^{-at}\cos at + e^{-at}\sin at]$$

$$= \frac{1}{2a}\left[\cos at\left(\frac{e^{at}-e^{-at}}{2}\right) + \sin at\left(\frac{e^{at}+e^{-at}}{2}\right)\right]$$

$$= \frac{1}{2a}(\cos at \sinh at + \sin at \cosh at)$$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \frac{1}{2a}(\cos at \sinh at + \sin at \cosh at)$$

Inverse Laplace transforms by using theorems

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

i. $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$, from first shifting theorem

ii. $\mathcal{L}^{-1}\{s\,F(s)\} = \frac{d}{dt}f(t)$ provided $f(0) = 0$, from Laplace transform of derivative of a function

iii. $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$, from Laplace transform of integration

iv. $\mathcal{L}^{-1}\left\{-\frac{d}{ds}F(s)\right\} = t\,f(t)$

v. $\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$

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Example 9. Find the inverse Laplace transform of following functions:

a. $\dfrac{2as}{(s^2+a^2)^2}$ b. $\dfrac{s^3}{(s^2+a^2)^2}$ c. $\dfrac{s}{s^2+a^2}$

d. $\log\left(\dfrac{s(s+1)}{s^2+4}\right)$ e. $\dfrac{s+2}{(s^2+4s+5)^2}$ f. $s\log\left(\dfrac{s-1}{s+1}\right)$

g. $\dfrac{1}{s^3(s^2+1)}$

Solution

a. We know that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

$\therefore\ \mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$

or, $\mathcal{L}^{-1}\left\{\dfrac{-d}{ds}\left(\dfrac{a}{s^2+a^2}\right)\right\} = t\sin at$ $[\because \mathcal{L}^{-1}\{F'(s)\} = -t\,\mathcal{L}\{F(s)\} = -t f(t)]$

or, $\mathcal{L}^{-1}\left\{\dfrac{a.2s}{(s^2+a^2)^2}\right\} = t\sin at$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{2as}{(s^2+a^2)^2}\right\} = t\sin at$

b. We have, $F(s) = \dfrac{s^3}{(s^2+a^2)^2}$

$$= \frac{s.s^2}{(s^2+a^2)^2}$$

$$= \frac{s(s^2+a^2-a^2)}{(s^2+a^2)^2}$$

Then, $F(s) = \dfrac{s(s^2+a^2)}{(s^2+a^2)^2} - \dfrac{a^2s}{(s^2+a^2)^2}$

$$= \frac{s}{s^2+a^2} - \frac{a^2s}{(s^2+a^2)^2}$$

Taking inverse Laplace transform on both sides, we get

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} - \mathcal{L}^{-1}\left\{\frac{a^2s}{(s^2+a^2)^2}\right\}$$

$$\mathcal{L}^{-1}\{F(s)\} = \cos at - \frac{a}{2}\,\mathcal{L}^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\}$$

Hence, $f(t) = \cos at - \dfrac{a}{2}t\sin at$ [Already proved in (a)]

c. We know that $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$

Then,

$\mathcal{L}^{-1}\left\{\dfrac{d}{ds}\left(\dfrac{s}{s^2+a^2}\right)\right\} = -t\cos at$ $\left[\because \mathcal{L}^{-1}f'(s) = \dfrac{-d}{ds}F(s) = -tf(t)\right]$

or, $\mathcal{L}^{-1}\left\{\dfrac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right\} = -t\cos at$

or, $\mathcal{L}^{-1}\left\{\dfrac{(a^2-s^2)}{(s^2+a^2)^2}\right\} = -t\cos at$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{(s^2-a^2)}{(s^2+a^2)^2}\right\} = t\cos at.$

d. We have,

$$F(s) = \log\left(\frac{s(s+1)}{s^2+4}\right)$$

$$= \log s + \log(s+1) - \log(s^2+4)$$

Differentiating with respect to s then

$$F'(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4}$$

or, $-\mathcal{L}\{t f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\} - 2\mathcal{L}\{\cos 2t\}$

Now, taking inverse Laplace on both sides, we get

$$-t f(t) = 1 + e^{-t} - 2\cos 2t$$

$$\therefore\ f(t) = \frac{2\cos 2t - e^{-t} - 1}{t}$$

e. We have, $\dfrac{s+2}{(s^2+4s+5)^2} = \dfrac{-1}{2}\dfrac{d}{ds}\left[\dfrac{1}{s^2+4s+5}\right]$

$$= -\frac{1}{2}\frac{d}{ds}F(s)$$

where $F(s) = \dfrac{1}{s^2+4s+5}$

Taking inverse Laplace on both sides, we get

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{2}\frac{d}{ds}F(s)\right\}$$

$$= \frac{-1}{2}\mathcal{L}^{-1}\{F'(s)\}$$

Now, using the property $\mathcal{L}^{-1}\{F'(s)\} = -t\,\mathcal{L}^{-1}\{F(s)\}$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{-1}{2}[-t\,\mathcal{L}^{-1}\{F(s)\}]$$

$$= \frac{-1}{2}t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$$

$$= \frac{-1}{2}t\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}$$

$$= \frac{-1}{2}t\,e^{-2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s^2+4s+5)^2}\right\} = \dfrac{-1}{2}t\,e^{-2t}\sin t.$

f. We have, $F(s) = s\log\left(\dfrac{s-1}{s+1}\right)$

$$\frac{-d}{ds}F(s) = \frac{-d}{ds}\left[s\log\left(\frac{s-1}{s+1}\right)\right]$$

$$= \frac{-d}{ds}[s\log(s+1) - s\log(s-1)]$$

$$= \left[\log(s+1) + \left(\frac{s}{s+1}\right) - \log(s-1) - \left(\frac{s}{s-1}\right)\right]$$

$$= \left[\{\log(s+1) - \log(s-1)\} + \left(\frac{1}{s+1} - \frac{s}{s-1}\right)\right]$$

or, $\dfrac{-d}{ds}F(s) = \phi(s) - \dfrac{2s}{s^2-1}$ where $\phi(s) = \log(s+1) - \log(s-1)$

Taking inverse Laplace on both sides,

$$\mathcal{L}^{-1}\left\{\frac{-d}{ds}f(s)\right\} = \mathcal{L}^{-1}\{\phi(s)\} - \mathcal{L}^{-1}\left\{\frac{2s}{s^2-1}\right\}$$

or, $$t f(t) = \mathcal{L}^{-1}\{\phi(s)\} - 2\cosh t \qquad \ldots (1) \qquad \left[\because t\phi(t) = \mathcal{L}^{-1}\left\{\frac{-d}{ds}\ldots\right.\right.$$

or, $$t\phi(t) = \mathcal{L}^{-1}\left\{\frac{d}{ds}[\log(s-1) - \log(s+1)]\right\}$$

or, $$t\phi(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^2-1}\right\} = 2\sinh t$$

or, $$\phi(t) = \frac{2\sinh t}{t} \qquad \ldots (2)$$

From (1) and (2), we get

$$t f(t) = \frac{2\sinh t}{t} - 2\cosh t$$

Hence, $$f(t) = 2\left\{\frac{\sinh t - t\cosh t}{t^2}\right\}.$$

g. We know that: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\} \qquad \left[\because \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du\right]$$

$$= \int_0^t \sin u\,du = [-\cos u]_0^t$$

$$= [-\cos t + \cos 0] = 1 - \cos t$$

Implying: $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s^2}\right\}$$

$$= \int_0^t (1 - \cos u)\,du$$

$$= [t - \sin u]_0^t$$

$$= t - \sin t$$

and $$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u - \sin u)\,du$$

$$= \left[\frac{u^2}{2} + \cos u\right]_0^t$$

$$= \left[\frac{t^2}{2} + \cos t - 0 - \cos 0^\circ\right]$$

Hence, $$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \left(\frac{t^2}{2} + \cos t - 1\right)$$

3.8 Convolution of Laplace transform

3.8.1 Convolution of two functions

Let $f(t)$ and $g(t)$ be two functions defined for $t \geq 0$, then the convolution of the two functions $f(t)$ and $g(t)$ denoted by $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\,du$$

The convolution $f * g$ is

i. Commutative *i.e.,* $f * g = g * f$

ii. Associative *i.e.,* $(f * g) * h = f * (g * h)$

iii. Distributive with respect to addition *i.e.,* $f * (g + h) = f * g + f * h$

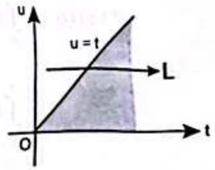
Convolution theorem

Statement: If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then $\mathcal{L}\{(f * g)(t)\} = F(s)\,G(s)$ and

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= (f * g)(t)$$

Proof: From definition of Laplace transform



$$\mathcal{L}\{(f * g)\} = \int_0^\infty e^{-st}(f * g)\,dt$$

and we have from definition of convolution

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,g(t-u)\,du\right]dt$$

or, $$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \qquad \ldots (1)$$

The R.H.S. of (1) is a double integral with the region of integration as shown in figure. By interchanging the order of integration we can write,

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty \left(\int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\right)du$$

$$= \int_0^\infty \left(\int_u^\infty e^{-s(t-u+u)} f(u)\,g(t-u)\,dt\right)du$$

$$= \int_0^\infty e^{-su} f(u)\left[\int_u^\infty e^{-s(t-u)}\,g(t-u)\,dt\right]du$$

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Taking $t-u=p$, so that $dt=dp$ and $p=0$ for $t=u$, $p\to\infty$ for $t\to\infty$.

$$\mathcal{L}\{(f*g)(t)\} = \int_0^{\infty} e^{-su} f(u)\left[\int_0^{\infty} e^{-sp} g(p)\,dp\right] du$$

$$\mathcal{L}\{(f*g)(t)\} = \int_0^{\infty} e^{-su} f(u)\,G(s)\,du) \quad \text{[by definition of Laplace transform]}$$

$$= \left(\int_0^{\infty} e^{-su} f(u)\,du\right) G(s)$$

Hence, $\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$.

Example 10. Find the convolution of following functions.

a. $t*e^t$ b. $\sin at * \cos at$ c. $u(t-3)*e^{-2t}$

Solution

a. Let $f(t)=t$ and $g(t)=e^t$

$\therefore$ The convolution of $f*g$ is

$$(f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t (t-u)\,e^u\,du$$

$$= \int_0^t te^u\,du - \int_0^t ue^u\,du$$

$$= [te^u]_0^t - [ue^u]_0^t + [e^u]_0^t$$

$$= t(e^t-1) - (te^t-0) + e^t - 1$$

$$= te^t - t - te^t + e^t - 1$$

$$\therefore (f*g)(t) = e^t - t - 1$$

b. Let $f(t)=\sin at$ and $g(t)=\cos at$

Then, the convolution of $f*g$ is

$$(f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t \sin a(t-u).\cos at\,du$$

$$= \int_0^t (\sin at.\cos au - \cos at.\sin au)\cos$$

$$= \sin at\int_0^t \cos^2 au\,du - \cos at\int_0^t \sin au\cos au\,du$$

$$= \sin at\int_0^t \left(\frac{1+\cos 2au}{2}\right) du - \cos at\int_0^t \frac{\sin 2au}{2}\,du$$

$$= \frac{\sin at}{2}\left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{\cos at}{2}\left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2}\left(t + \frac{\sin 2at}{2a}\right) + \frac{\cos at}{4a}(\cos 2at - 1)$$

$$= \frac{t}{2}\sin at + \frac{1}{4a}(\sin at\sin 2at + \cos at\cos 2at) - \frac{\cos at}{4a}$$

$$= \frac{t}{2}\sin at + \frac{1}{4a}\cos(2at - at) - \frac{\cos at}{4a}$$

$$= \frac{t}{2}\sin at + \frac{1}{4a}\cos at - \frac{1}{4a}\cos at$$

$$\therefore \sin at * \cos at = \frac{t}{2}\sin at$$

c. Let $f(u)=u(t-3)$ and $g(t)=e^{-2t}$

Then, the convolution of $f*g$ is

$$(f*g)(t) = \int_0^t u(t-3)\,e^{-2(t-u)}.du$$

$$= \int_0^t u(t-3)\,e^{-2t}.e^{2u}.du$$

$$= \int_0^3 u(t-3)\,e^{-2t}.e^{2u}.du + \int_3^t u(t-3)\,e^{-2t}.e^{2u}.du \quad \left[\because u(t-a) = \begin{cases}0 & t<a\\ 1 & t\ge a\end{cases}\right]$$

$$= e^{-2t}\int_3^t e^{2u}\,du \quad [t\ge 3]$$

$$= e^{-2t}\left[\frac{e^{2u}}{2}\right]_3^t$$

$$= \frac{1}{2}e^{-2t}\left[e^{2t} - e^{2\times 3}\right]$$

or, $$(f*g)(t) = \frac{1}{2}\left[1 - e^{-2(t-3)}\right]$$

Hence, $\sin at * \cos at = \frac{1}{2}\left[1 - e^{-2(t-3)}\right]$

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Example 11. Find in inverse Laplace transform of following functions:

a. $\frac{s}{(s^2+a^2)^2}$ b. $\frac{s^2}{s^4-a^4}$ c. $\frac{1}{s(s^2+4)}$

Solution

a. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}$

where, $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{\sin at}{a}$

Using convolution theorem

$$\mathcal{L}^{-1}\{F(s).\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

or, $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,du$

$$= \frac{1}{a}\int_0^t \cos au\,(\sin at\cos au - \cos at\sin au)\,du$$

$$= \frac{1}{a}\sin at\int_0^t \cos^2 au\,du - \frac{1}{a}\cos at\int_0^t \sin au\cos au\,du$$

$$= \frac{1}{2a}\sin at\int_0^t (1+\cos 2au)\,du - \frac{1}{2a}\cos at\int_0^t \sin 2au\,du$$

$$= \frac{1}{2a}\sin at\left[u+\frac{\sin 2au}{2a}\right]_0^t - \frac{1}{2a}\cos at\left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{1}{2a}\sin at\left[t+\frac{\sin 2at}{2a}\right] + \frac{\cos at}{4a}(\cos 2at-1)$$

$$= \frac{t\sin at}{2a} + \frac{\sin at\sin 2at}{4a^2} + \frac{1}{4a^2}\cos at\cos 2at - \frac{\cos at}{4a^2}$$

$$= \frac{t\sin at}{2a} + \frac{1}{4a^2}(\cos 2at\cos at + \sin 2at\sin at) - \frac{\cos at}{4a^2}$$

$$= \frac{t\sin at}{2a} + \frac{\cos(2at-at)}{4a^2} - \frac{\cos at}{4a^2}$$

Hence, $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{t}{2a}\sin at$

b. $\mathcal{L}^{-1}\left\{\frac{s^2}{s^4-a^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\times\frac{s}{s^2+a^2}\right\}$

where $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

using convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\times\frac{s}{s^2+a^2}\right\} = \int_0^t \cosh au.\cos a(t-u)\,du = I$$

$$I = \left[\cosh au\,\frac{\sin a(t-u)}{-a}\right]_0^t - \int_0^t a\sinh at\,\frac{\sin a(t-u)}{-a}\,du$$

$$= -\frac{1}{a}(0-\sin at) + \int_0^t \sinh at\sin at\,(t-u)\,du$$

$$= -\frac{\sin at}{a} + \left[\sinh at\,\frac{-\cos(t-u)}{-a}\right]_0^t - \int_0^t a\cosh au\left\{\frac{-\cos a(t-u)}{-a}\right\}du$$

$$= \frac{\sin at}{a} + \left[\frac{\sinh at}{a} - 0\right] - \int_0^t \cosh au.\cos a(t-u)\,du$$

or, $I = \frac{\sin at}{a} + \frac{\sinh at}{a} - I$

or, $2I = \frac{1}{a}(\sin at + \sinh at)$

or, $I = \frac{1}{2a}(\sin at + \sinh at)$

Hence, $\mathcal{L}^{-1}\left\{\frac{s^2}{s^4-a^4}\right\} = \frac{1}{2a}(\sin at + \sinh at)$

Note: Alternately, may take $\cosh au = \frac{e^{au}+e^{-au}}{2}$ in integral I.

c. We have $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2+4}\right)$

where $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$

Using convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{2}\int_0^t 1.\sin 2(t-u)\,du$$

$$= \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}[\cos(2t-2u)]_0^t$$

$$= \frac{1}{4}[\cos 0 - \cos 2t]$$

Hence, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}(1-\cos 2t)$

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Exercise 3.2

1. Find the inverse Laplace transform of the following functions

   a. $\frac{3s+2}{s^2+2s+2}$ b. $\frac{2s+3}{s^2+4}$ c. $\frac{1}{s^2-9}$

   d. $\frac{s-1}{(s-1)^2+4}$ e. $\frac{1}{s^2+2s+5}$ f. $\frac{s-4}{s^2-4}$

   g. $\frac{2s^3}{s^4-1}$ h. $\frac{s+1}{s^2-6s+25}$

2. Find the inverse Laplace transform of the following functions

   a. $\frac{s+1}{s(s^2+4)}$ b. $\frac{1}{(s-1)(s^2+1)}$

   c. $\frac{1-7s}{(s-1)(s+2)(s-3)}$ d. $\frac{9(s+1)}{s^2(s^2+9)}$

   e. $\frac{s}{(s+1)^2(s^2+1)}$ f. $\frac{s}{(s^2-1)^2}$

   g. $\frac{s}{(s+a)^3}$ h. $\frac{2s+1}{(s+2)^2(s-1)^2}$

3. Find the inverse Laplace transform of the following function

   a. $\frac{1}{(As+B)^n}$ b. $\frac{1}{s^4+4a^4}$ c. $\frac{s}{s^4+4a^4}$

   d. $\frac{s^2}{s^4+4a^4}$ e. $\frac{a(s^2-2a^2)}{s^4+4a^4}$ f. $\frac{s^3}{s^4-a^4}$

   g. $\frac{\pi^5}{s^4(s^2+\pi^2)}$

4. Find inverse Laplace transform of the following function

   a. $\frac{s}{(s^2+a^2)^2}$ b. $\frac{1}{(s^2+a^2)^2}$ c. $\frac{s^2}{(s^2+a^2)^2}$

   d. $\tan^{-1}\left(\frac{2}{s^2}\right)$ e. $\log\left(1-\frac{a^2}{s^2}\right)$ f. $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

   g. $\frac{1}{s^2(s^2+a^2)}$ h. $\log\frac{s(s+1)}{s^2+4}$

5. Use convolution theorem to find the Laplace transform of following function

   a. $t * 1$ b. $e^t * e^{-t}$

   c. $1 * \sin t$ d. $\sin t * \sin t$

. Use convolution theorem to find the inverse Laplace transform of following function

   a. $\frac{1}{(s^2+a^2)^2}$ b. $\frac{1}{s(s^2+4)}$ c. $\frac{1}{s^2(s-1)}$

   d. $\frac{5}{(s^2+\pi^2)^2}$ e. $\frac{s}{(s^2+1)(s^2+4)}$ f. $\frac{1}{s^2(s^2+w^2)}$

   g. $\frac{w}{s^2(s^2+w^2)}$ h. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ i. $\frac{1}{(s+1)(s+9)^2}$

Answer

1. a. $(3\cos t-\sin t)e^{-t}$ b. $2\cos 2t+\frac{3}{2}\sin 2t$ c. $\frac{1}{3}\sin 3t$

   d. $e^t\cos 2t$ e. $e^{-t}\sin 2t$ f. $\cosh 2t-2\sinh 2t$

   g. $\cos t+\cosh t$ h. $e^{3t}(\cos 4t+\sin 4t)$

2. a. $\frac{1}{4}-\frac{1}{4}\cos 2t+\frac{1}{2}\sin 2t$ b. $\frac{e^t}{2}-\cos t-\sin t$ c. $-2e^{3t}+e^t+e^{-2t}$

   d. $1+t-\cos 3t-\frac{1}{3}\sin 3t$ e. $\frac{\sin t-te^{-t}}{2}$ f. $\frac{t\sinh t}{2}$

   g. $\frac{e^{-at}}{2}(a^2t^2-4at+2)$ h. $\frac{e^t-e^{-2t}}{3}$

3. a. $\frac{1}{A^n}e^{\frac{-Bt}{A}}\frac{t^{n-1}}{(n-1)!}$ b. $\frac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$

   c. $\frac{1}{2a^2}\sin at\sinh at$ d. $\cosh at\cos at$ e. $\sinh at\cos at$

   f. $\frac{1}{2}(\cosh at+\cos at)$ g. $\sin \pi t+\frac{\pi^3}{6}t^3-\pi t$

4. a. $\frac{1}{2a}t\sin at$ b. $\frac{1}{2a^3}(\sin at-at\cos at)$ c. $\frac{1}{2a}(\sin at+at\cos at)$

   d. $\frac{2\sinh t\sin t}{t}$ e. $\frac{2(1-\cosh at)}{t}$ f. $\frac{\cos at-\cos bt}{t}$

   g. $\frac{1}{a^3}(at-\sin at)$ h. $\frac{2\cos 2t-1-e^t}{t}$

5. a. $\frac{t^2}{2}$ b. $\sinh t$ c. $\frac{1-\cos at}{a}$ d. $\frac{\sin t-t\cos t}{2}$

6. a. $\frac{1}{2a^3}(\sin at-at\cos at)$ b. $\frac{1}{4}(1-\cos 2t)$ c. $(1-e^t-t)$

   d. $\frac{\sin \pi t-\pi t\cos \pi t}{2\pi}$ e. $\frac{1}{3}(\cos t-\cos 2t)$ f. $\frac{at-\sin at}{a^3}$

   g. $\frac{wt-\sin wt}{w^2}$ h. $\frac{1}{3}(\cos t-\cos 2t)$ i. $\frac{e^{-t}}{64}[1-e^{-8t}(1+8t)]$

3.9 Unit step function (Heaviside function)

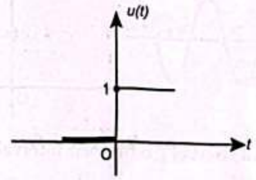
The unit step function or Heaviside function $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave if undefined), and is 1 for $t > a$.

i.e., $u(t-a) = u_a(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$, where $a \geq 0$

Illustration

i. If $a = 0$ then

$$u(t-a) = u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

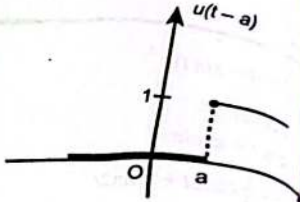


From the graph, it is apparent that function is zero for all values of $t < 0$.

ii. For any $a > 0$

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

which is shown in figure



The unit step or shifted unit step function are typical engineering applications, which o involve mechanical or electrical signal that are either 'off' or 'on' or required to control syste By multiplying functions (signals) $f(t)$ by $u(t)$ or $u(t-a)$m we can produce all sorts of effects practical (application) purpose.

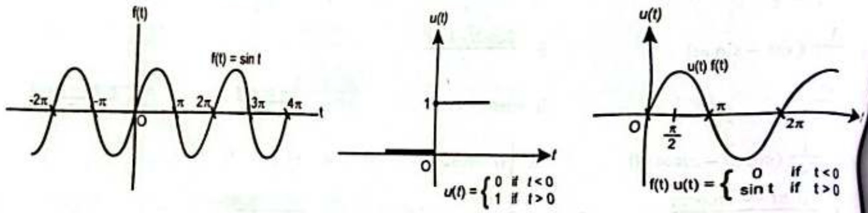
Following cases illustrate the practical purpose of unit step functions.

Case I: If we need effect of $f(t)$ for $t > 0$, we multiply $f(t)$ by $u(t)$

Let $f(t) = \sin t$ and $u(t)$ is its unit step function then

$$u(t)\, f(t) = \begin{cases} 0 & \text{if } t < a \\ \sin t & \text{if } t > a \end{cases}$$

Which is shown in figure.

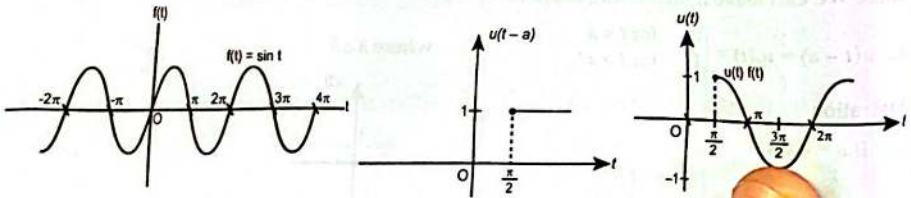


Case II: If we need effect of $f(t)$ for $t > a \neq 0$, we multiply $f(t)$ by $u(t-a)$

Let $f(t) = \sin t$ and $u(t-a)$ be shifted unit step function. Then,

$$f(t)\, u(t-a) = \begin{cases} 0 & \text{if } t < a \\ \sin t & \text{if } t > a \end{cases}$$

Which is shown in figure.

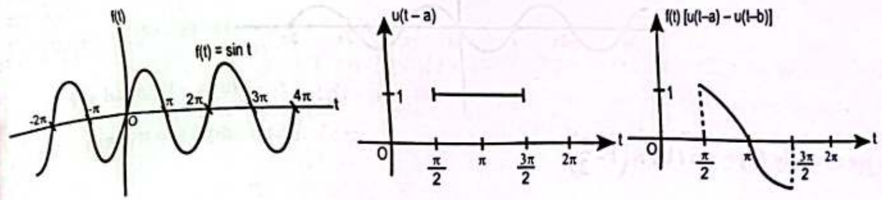


Case III: If we need effect of $f(t)$ for some interval of time from $t = a$ to $t =$ ... ply $f(t)$ by $[u(t-a) - u(t-b)]$

Let $f(t) = \sin t$ and $u(t-a)$ and $u(t-b)$ are shifted step functions, then

$$f(t)\,[u(t-a) - u(t-b)] = \begin{cases} 0 & \text{if } t < a \\ \sin t & \text{if } a < t < b \\ 0 & \text{if } t > b \end{cases}$$

Which is shown in figure for $a = \dfrac{\pi}{2}$ and $b = \dfrac{3\pi}{2}$.



The above three cases are applicable to switch 'off' or 'on' or control the system (signal).

3.9.1 Laplace transform of unit step function

We determine Laplace transform for shifted unit step function.

From the definition of Laplace transform, we have

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^\infty e^{-st}\, .0\, dt + \int_a^\infty e^{-st}\, .1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \left[0 - \frac{e^{-st}}{-s}\right]$$

$$\therefore \quad \mathcal{L}\{u(t-a)\} = \frac{e^{-sa}}{s}$$

Remark: If $a = 0$ then, $\mathcal{L}\{u(t-a)\} = \mathcal{L}\{u(t)\} = \dfrac{1}{s}$.

3.9.2 Shifted function

Let $f(t)$ be a continuous function for all t,

$$g(t) = f(t-a)\, u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

is known as shifted function of $f(t)$ with shift $t = a$.

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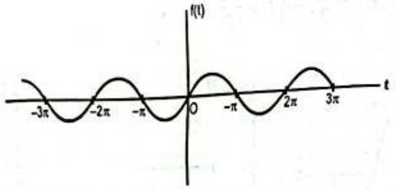
Illustrative example

Consider $f(t) = \sin t$ ....(1)

Then its graph is



If we multiply $f(t) = \sin t$ by $u\left(t-\frac{\pi}{2}\right)$

where,

$$u\left(t-\frac{\pi}{2}\right)=\begin{cases}0 & \text{for } t<\frac{\pi}{2}\\ 1 & \text{for } t>\frac{\pi}{2}\end{cases} \quad \text{....(2)}$$

Then,

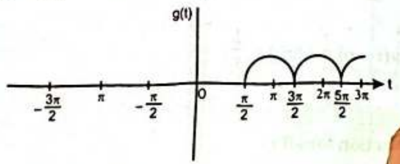
$$g(t) = f\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right) \quad \text{....(3)}$$

$$=\begin{cases}0 & \text{if } t<\frac{\pi}{2}\\ f\left(t-\frac{\pi}{2}\right) & \text{if } t>\frac{\pi}{2}\end{cases}$$

$$=\begin{cases}0 & \text{if } t<\frac{\pi}{2}\\ \sin\left(t-\frac{\pi}{2}\right) & \text{if } t>\frac{\pi}{2}\end{cases}$$

$$=\begin{cases}0 & \text{if } t<\frac{\pi}{2}\\ -\cos t & \text{if } t>\frac{\pi}{2}\end{cases}$$

The graph of function $g(t)$ in (3) is as shown in figure below.



This shows that by multiplying a function $f(t)$ by $u(t-a)$ with su... ...e of a we c... eliminate effect of $f(t)$ in unwanted part (interval).

3.9.3 Second shifting theorem (t-shifting)

Theorem: If F(s) is the Laplace transform of $f(t)$ then the Laplace transform of

$$g(t) = f(t-a)\,u(t-a) = \begin{cases}0 & \text{if } t<a\\ f(t-a) & \text{if } t>a\end{cases}$$

is e^{-as} F(s).

i.e., $\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,F(s)$

Proof: We have, $f(t-a)\,u(t-a) = \begin{cases}0 & \text{if } t<a\\ f(t-a) & \text{if } t>a\end{cases}$

By the definition of Laplace transform

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^\infty e^{-st}\,[f(t-a)\,u(t-a)]\,dt$$

$$= \int_0^a e^{-st}\,[f(t-a)\,u(t-a)]\,dt + \int_a^\infty e^{-st}\,[f(t-a)\,u(t-a)]\,dt$$

$$= \int_0^a e^{-st}.0.\,dt + \int_a^\infty e^{-st} f(t-a).1\,dt$$

$$\therefore \quad \mathcal{L}\{f(t-a)\,u(t-a)\} = \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $t-a=p$ then $dt = dp$.

When $t=a$ then $p=0$ and when $t\to\infty$, then $p\to\infty$.

Then, $\mathcal{L}\{f(t-a).u(t-a)\} = \int_0^\infty e^{-s(a+p)} f(p)\,dp$

$$\therefore \quad \mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\int_0^\infty e^{-sp} f(p)\,dp$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt, \quad [\because p \text{ is dummy variable}]$$

$$= e^{-as}\,F(s), \qquad \text{where } F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

$\therefore \quad \mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\,F(s).$

Which completes the proof.

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Note:

1. If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then
$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$
2. The second shifting theorem is also known as t-shifting, i.e., t is replaced by $(t-a)$ i...

Remark: Application of second shifting theorem for determining inverse transfo...

We have the second shifting theorem.

$\mathcal{L}\{f(t-a)\, u(t-a)\} = e^{-as} F(s),\ F(s) = \mathcal{L}[f(t)]$

i.e., $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$

Here the effect of e^{-as} is handled by $u(t-a)$ but we need to find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(...$ $\mathcal{L}^{-1}[F(s)] = f(t)$ so that we can replace t in $f(t)$ by $(t-a)$ while applying the second s... theorem.

Example 12. Find Laplace inverse transform of $\frac{e^{-2s}}{s-3}$.

Solution

We have, $e^{-2s}\frac{1}{s-3}$, taking $F(s) = \frac{1}{s-3}$

We know,

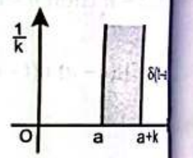
$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t} = f(t)$$

Now, applying second shifting theorem

$$\begin{aligned}
\mathcal{L}^{-1}\{e^{-as} F(s)\} &= f(t-a)\, u(t-a)\\
\therefore\ \mathcal{L}^{-1}\{e^{-2s} F(s)\} &= e^{3(t-2)} u(t-2)\\
&= e^{3t} e^{-6} u(t-2)\\
\therefore\ \mathcal{L}^{-1}\{e^{-2s} F(s)\} &= e^{-6} e^{3t} u(t-2)
\end{aligned}$$

3.10 Dirac's Delta function (unit impulse function)

Impulse is considered as a force of very high magnitude applied for very short time as in case of hitting of runway by aeroplane during landing and function representing the impulse is called *Dirac-Delta function* and is given by $\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$.



Mathematically, it is the limiting form of the function

$$\delta_k(t-a) = \begin{cases} \frac{1}{k} & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$

It is apparent from the above figure, when $k \to 0$, the height of the strip increases indefinit... and the width decreases in such a way that the area of the strip is always unity. The above f... an notionally be stated as

$$\lim_{k\to 0} \delta_k(t-a) = \delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

such that $\int_0^\infty \delta(t-a)\, dt = 1$ for $a \ge 0$.

Theorem (Statement only): (Filtering or shifting property of Dirac Delta function)

Let $f(t)$ be defined and integrable over all intervals contained within $0 \le t < \infty$ and let $f(t)$ is continuous in a neighbourhood of a. Then for $a \ge 0$.

$$\int_0^\infty f(t)\, \delta(t-a)\, dt = f(a)$$

3.10.1 Transform of unit impulse (Dirac Delta) function

$$\begin{aligned}
\mathcal{L}\{\delta(t-a)\} &= \int_0^\infty e^{-st}\, \delta(t-a)\, dt\\
&= \int_0^a e^{-st}\delta_k(t-a)\, dt + \int_a^{a+k} e^{-st}\delta_k(t-a)\, dt + \int_{a+k}^\infty e^{-st}\delta_k(t-a)\, dt\\
&= \int_0^a e^{-st}.0.\, dt + \int_a^{a+k} e^{-st}.\frac{1}{k}.\, dt + \int_{a+k}^\infty e^{-st}.0.\, dt\\
&= \frac{1}{k}\left[\frac{e^{-st}}{-s}\right]_a^{a+k}\\
&= \frac{1}{sk}\left[-e^{-s(a+k)} + e^{-as}\right]\\
&= \frac{e^{-as}}{s}\left[\frac{1-e^{-sk}}{k}\right]
\end{aligned}$$

Taking limit as $k \to 0$

$$\begin{aligned}
\mathcal{L}\{\delta(t-a)\} &= \lim_{k\to 0} \frac{e^{-as}}{s}\left(\frac{1-e^{-sk}}{k}\right)\\
&= e^{-as} \lim_{k\to 0}\left(\frac{1-e^{-sk}}{sk}\right)\\
&= e^{-as} \lim_{k\to 0}\left[\frac{0+s\,e^{-sk}}{s}\right] \quad \text{[L-Hospital rule]}\\
&= e^{-as}\\
\mathcal{L}\{\delta(t-a)\} &= e^{-as}
\end{aligned}$$

Remark: If $a = 0$, then $\mathcal{L}\{\delta(t)\} = 1$.

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3.11 Laplace transform of periodic functions

If $f(t)$ be a periodic function with period $T > 0$, i.e., $f(t + T) = f(t)$, then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{(1 - e^{-sT})}$$

Proof: By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

or, $\mathcal{L}\{f(t)\} = \int_0^t e^{-st} f(t)\, dt + \int_t^{2t} e^{-st} f(t)\, dt + \int_{2t}^{3t} e^{-st} f(t)\, dt + \ldots$

Let $t = u$, $t = u + T$, $t = u + 2T$, ... in the successive integrals. Then,

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-su} f(u)\, du + \int_0^T e^{-s(u+T)} f(u+T)\, du + \int_0^T e^{-s(u+2T)} f(u+2T)\, du$$

Since $f(u) = f(u+T) = f(u+2T) = \ldots$

$$\therefore \quad \mathcal{L}\{f(t)\} = \int_0^T e^{-su} f(u)\, du + e^{-sT}\int_0^T e^{-su} f(u)\, du + e^{-2sT}\int_0^T e^{-su} f(u)\, du + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots)\int_0^T e^{-su} f(u)\, du = \frac{1}{1 - e^{-sT}}\int_0^T e^{-su} f(u)\, du$$

On changing the dummy variable u to t inside the integral sign.

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{(1 - e^{-sT})}$$

Example 13. Find Laplace transform of the following functions:

a. $e^{-3t}\, u_2(t)$ b. $\sin 2t.u(t - \pi)$ c. $t^2 u(t - 1)$

Solution

a. The given function is

$$e^{-3t} u_2(t) = e^{-3(t-2)-6)}.u_2(t)$$
$$= e^{-6} e^{-3(t-2)}.u(t-2)$$

Then, Laplace transform of the function is

$$\mathcal{L}\{e^{-3t} u_2(t)\} = e^{-6}\, \mathcal{L}\{e^{-3(t-2)}\, u(t-2)\}$$

By the second shifting theorem,

$$\mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\, \mathcal{L}\{f(t)\}$$
$$\mathcal{L}\{e^{-3t} u_2(t)\} = e^{-6} e^{-2s}\, \mathcal{L}\{e^{3t}\}$$

or, $\mathcal{L}\{e^{-3t} u_2(t)\} = e^{-2s-6} \frac{1}{s+3}$

$\therefore \quad \mathcal{L}\{e^{-3t} u_2(t)\} = \frac{e^{-2s-6}}{s+3}$

b. The given function is $\sin 2t.u(t - \pi) = \sin(2t + 2\pi - 2\pi).u(t - \pi)$

$$= \sin\{2\pi + (2t - 2\pi)\}.u(t - \pi)$$
$$= \sin(2t - 2\pi)\, u(t - \pi)$$
$$= \sin 2(t - \pi).u(t - \pi)$$

Then, Laplace transform of the function is

$$\mathcal{L}\{\sin 2t\, .\, u(t - \pi)\} = \mathcal{L}\{\sin 2(t - \pi).u(t - \pi)\}$$
$$= e^{-\pi s}\, \mathcal{L}\{f(t)\} \qquad \text{[by the second shifting theorem]}$$
$$= e^{-\pi s}\, \mathcal{L}(\sin 2t)$$
$$= e^{-\pi s} \frac{2}{s^2 + 2^2}$$

Hence, $\mathcal{L}\{\sin 2t\, .\, u(t - \pi)\} = \frac{2e^{-\pi s}}{s^2 + 4}$

c. The given function is $t^2 u(t - 1) = (t - 1 + 1)^2 u(t - 1)$

$t^2 u(t - 1) = \{(t - 1) + 1)\}^2 u(t - 1)\}$

or, $t^2 u(t - 1) = [(t - 1)^2 + 2(t - 1)1 + 1^2]\, u(t - 1)$

or, $t^2 u(t - 1) = (t - 1)^2 u(t - 1) + 2(t - 1)\, u(t - 1) + u(t - 1)$

The Laplace transform of the function is

$$\mathcal{L}\{t^2.u(t - 1)\} = \mathcal{L}\{(t - 1)^2.u(t - 1)\} + 2\mathrm{L}\{(t - 1).u(t - 1)\} + \mathcal{L}\{u(t - 1)\}$$

By the second shifting theorem,

$$\mathcal{L}\{t^2.\, u(t - 1)\} = e^{-s}\, \mathcal{L}\{t^2\} + 2e^{-s}\mathrm{L}\{t\} + \int_1^\infty e^{-st}\, dt \qquad \left[\because u(t-1) = \begin{cases} 0 & \text{for } t < 1 \\ 1 & \text{for } t \geq 1 \end{cases}\right]$$

$$= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s} + \left[\frac{e^{-st}}{-s}\right]_1^\infty$$

$$= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s} + \left[\frac{e^{-\infty} - e^{-s}}{-s}\right]$$

$$= \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s} + \frac{e^{-s}}{s} \qquad [\because e^{-\infty} = 0]$$

Hence, $\mathcal{L}\{t^2.u(t - 1)\} = e^{-s}\left(\frac{2}{s^3} + \frac{3}{s}\right)$

Example 14. Find the inverse Laplace transform using second shifting theorem:

a. $\frac{se^{\frac{-s}{2}} + \pi e^{-s}}{s^2 + \pi^2}$ b. $\frac{e^{-as}}{s^2(s + b)}$

Solution

a. The given function is $\frac{se^{\frac{-s}{2}} + \pi e^{-s}}{s^2 + \pi^2} = \frac{se^{\frac{-s}{2}}}{s^2 + \pi^2} + \frac{\pi e^{-s}}{s^2 + \pi^2}$

$$= e^{\frac{-s}{2}}\left(\frac{s}{s^2 + \pi^2}\right) + e^{-s}\left(\frac{\pi}{s^2 + \pi^2}\right)$$

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For first term

$$f(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos \pi t$$

and for second term

$$f(t) = \mathcal{L}^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t$$

We have second shifting theorem

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$$

So, $$\mathcal{L}^{-1}\left\{\frac{se^{\frac{-s}{2}} + \pi e^{-s}}{s^2+\pi^2}\right\} = \mathcal{L}^{-1}\left\{e^{\frac{-s}{2}}\frac{s}{s^2+\pi^2}\right\} + \mathcal{L}^{-1}\left\{e^{\frac{-s}{2}}\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right).u\left(t-\frac{1}{2}\right) + \sin\pi(t-1).u(t-1)$$

$$= \cos\left(\pi t-\frac{\pi}{2}\right).u\left(t-\frac{1}{2}\right) + \sin(\pi t-\pi).u(t-1)$$

Hence, $$\mathcal{L}^{-1}\left\{\frac{se^{\frac{-s}{2}} + \pi e^{s}}{s^2+\pi^2}\right\} = \sin \pi t\left\{u\left(t-\frac{1}{2}\right) - u(t-1)\right\}$$

b. The given function is $\frac{e^{as}}{s^2(s+b)}$

Let, $$\frac{1}{s^2(s+b)} = \frac{A}{s+b} + \frac{B}{s} + \frac{C}{s^2}$$

Then,

$1 = As^2 + B(s+b)s + C(s+b)$

or, $1 = As^2 + Bs^2 + Bbs + Cs + Cb$

or, $(A+B)s^2 + (Bb+C)s + Cb = 1$

Equating coefficient of s and constant terms on both sides, we get

$A + B = 0$, $Bb + C = 0$ and $Cb = 1$

Solving above equations, we get

$A = \frac{1}{b^2}$, $B = \frac{-1}{b^2}$ and $C = \frac{1}{b}$

Then,

$$\frac{e^{as}}{s^2(s+b)} = e^{-as}\left(\frac{1}{b^2(s+b)} - \frac{1}{b^2}\frac{1}{s} + \frac{1}{b}\frac{1}{s^2}\right)$$

or, $$\frac{e^{as}}{s^2(s+b)} = \frac{e^{-as}}{b^2}\frac{1}{s+b} - \frac{e^{-as}}{b^2}\frac{1}{s} + \frac{e^{-as}}{b}\frac{1}{s^2}$$

For $\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\} = f(t) = e^{-bt}$, $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = f(t) = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = f(t) = t$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{e^{as}}{s^2(s+b)}\right\} = \frac{1}{b^2}\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s+b}\right\} - \frac{1}{b^2}\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s}\right\} + \frac{1}{b}\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s^2}\right\},$$

By second shifting theorem,

$$\mathcal{L}^{-1}\{e^{as} F(s)\} = f(t-a).u(t-a)$$

or, $$\mathcal{L}^{-1}\left\{\frac{e^{as}}{s^2(s+b)}\right\} = \frac{1}{b^2}e^{-b(t-a)}.u(t-a) - \frac{1}{b^2}.1.u(t-a) + \frac{1}{b}.(t-a).u$$

Hence, $$\mathcal{L}^{-1}\left\{\frac{e^{as}}{s^2(s+b)}\right\} = \frac{1}{b^2}\left[e^{-b(t-a)} - 1 + b(t-a)\right] u(t-a)$$

Exercise 3.3

1. Find the Laplace transform of the following

a. $(t-1).u(t-1)$ b. $e^{-2t}.u(t-3)$

c. $tu(t-1)$ d. $u(t-\pi)\cos t$

e. $(t-1)^2.u(t-1)$ f. $t^2.u(t-3)$

2. Find the Laplace transform of following functions

a. $f(t) = \begin{cases} \sin t & \text{for } 0<t<2\pi \\ 0 & \text{for } t>\pi \end{cases}$ b. $f(t) = \begin{cases} e^{2t} & \text{for } 0<t<1 \\ 0 & \text{for } t>1 \end{cases}$

c. $f(t) = \begin{cases} t-1 & \text{for } 1<t<2 \\ 0 & \text{otherwise} \end{cases}$ d. $f(t) = \begin{cases} 1-e^{-t} & 0<t<2 \\ 0 & \text{otherwise} \end{cases}$

e. $f(t) = \begin{cases} 1 & \text{if } 0<t<\pi \\ 0 & \text{if } \pi<t<2\pi \\ \sin t & \text{if } t<2\pi \end{cases}$

3. Find inverse Laplace transform using second shifting theorem

a. $\frac{e^{-2s}}{s^2}$ b. $\frac{e^{-3s}}{(s-1)^3}$ c. $\frac{[e^{-2s} - 2e^{-5s}]}{s}$

d. $\frac{3(1-e^{-\pi s})}{s^2+9}$ e. $\frac{se^{-as}}{s^2-w^2}$ for $a>0$ f. $\frac{se^{-2s}}{s^2+\pi^2}$

Answer

1. a. $\frac{e^{-s}}{s^2}$ b. $\frac{e^{-6-3s}}{(s+2)}$ c. $\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$

d. $\frac{se^{-\pi s}}{s^2+1}$ e. $\frac{2e^{-s}}{s^3}$ f. $e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

2. a. $\frac{e^{-2\pi s} - e^{-4\pi s}}{s^2+1}$ b. $\frac{1-e^{-(s-2)}}{s-2}$ c. $\frac{e^{-s} - e^{-2s}}{s^2} - \frac{e^{-2s}}{s}$

d. $\frac{1}{s}(1-e^{-2s}) + \frac{1}{s+1}[e^{-2(s+1)} - 1]$ e. $\frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$

3. a. $(t-2)\, u(t-2)$ b. $\frac{t^2}{2}e^t u_3(t)e^{-3}$ c. $u_2(t) - 2u_5(t)$

d. $\sin 3t - \sin 3(t-\pi)\, u_\pi(t)$ e. $\cosh w(t-a)\, u_a(t)$ f. $u_2(t)\cos\pi t$

3.12 Application of Laplace transform for evaluating improper integral

Following theorems on Laplace transform can be used to evaluate improper integrals.

1. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$.

2. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$.

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Example 15. $\int_0^\infty t e^{-3t} \cos t \, dt$

Solution

We have the integral

$$I = \int_0^\infty t e^{-3t} \cos t \, dt \qquad \ldots(1)$$

We know,

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\therefore \quad \mathcal{L}\{t \cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1) - s \cdot 2s}{(s^2+1)^2}\right]$$

$$\therefore \quad \therefore \{t \cos t\} = F(s) = \frac{s^2-1}{(s^2+1)^2}.$$

Now, we use the definition of Laplace transform

$$\mathcal{L}\{t \cos t\} = \int_0^\infty e^{-st} t \cos t \, dt = F(s) \qquad \ldots(2)$$

Comparing (1) and (2), we take $s = 3$ in (2)

$$I = \int_0^\infty t e^{-3t} t \cos t \, dt$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=3}$$

$$\therefore \quad \int_0^\infty t e^{-3t} \cos t \, dt = \frac{8}{100}$$

$$= \frac{4}{25}$$

$$\therefore \quad \int_0^\infty t e^{-3t} \cos t \, dt = \frac{4}{25}.$$

Example 16. $\int_0^\infty \frac{e^{-2t} \sin 2t}{t} \, dt$

Solution

$$\text{Let } I = \int_0^\infty \frac{e^{-2t} \sin 2t}{t} \, dt \qquad \ldots(1)$$

$$\text{Now, } \mathcal{L}\{\sin 2t\} = \frac{2}{s+2+4}$$

$$\therefore \quad \mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = 2\int_s^\infty \frac{1}{s^2+4} \, ds$$

$$= 2 \cdot \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$\therefore \quad \mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$\therefore \quad F(s) = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} \qquad \ldots(2)$$

Now by definition of Laplace transform,

$$\mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \int_0^\infty e^{-st}\left(\frac{\sin 2t}{t}\right) dt$$

Comparing (1) and (2), we take $s = 2$

$$\therefore \quad I = \int_0^\infty \frac{e^{-2t} \sin 2t}{t} \, dt$$

$$= [F(s)]_{s=2}$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)\right]_{s=2}$$

$$= \left[\frac{\pi}{2} - \tan^{-1}(1)\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\therefore \quad \int_0^\infty \frac{e^{-2t} \sin 2t}{t} \, dt = \frac{\pi}{2}.$$

3.13 Application of Laplace transform to solve the differential equation

The Laplace transform is a well-established mathematical technique for solving a differential equation. Many mathematical problems are solved using transformation. The idea is to transform the problem into another problem that is easier to solve. On the other side, the inverse transform is helpful to calculate the solution to the given problem.

3.13.1 Working procedure on Laplace transform to solve differential equation (initial value problem)

Laplace transform can be used to solve ordinary and partial differential equation.

Steps to solve the differential equation are

i. First, we take Laplace on both sides of the given differential equation and represent $\mathcal{L}(y) = Y(s)$, so that $y = \mathcal{L}^{-1}(Y)$.

ii. Use the formulae of Laplace transform of derivative with given initial conditions.

   a. $\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y(t)\} - y(0)$

   b. $\mathcal{L}\{y''(t)\} = s^2\,\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)$

   c. $\mathcal{L}\{y^n(t)\} = s^n\,\mathcal{L}\{y(t)\} - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \ldots - y^{n-1}(0)$

iii. Finally, we find the inverse Laplace transform of the result to get the solution y of the equation.

Example 17. Using Laplace transform solve the initial value problem.

a. $y'' + 2y' + 2y = 0$, $y(0)$ and $y'(0) = 1$

b. $y'' + 4y' + 3y = e^{t}$, $y(0) = y'(0) = 1$

c. $x'' + 2x' + 5x = e^{t}\sin t$, $x(0)$, $x'(0) = 1$

0

a. The given differential equation is $y''+2y'+2y=0$

Taking Laplace transform on both sides, we get

$\mathcal{L}\{y''\}+2L\{y'\}+2L\{y\}=0$

or, $s^2L\{y\}-s\,y(0)-y'(0)+2sL\{y\}-2y(0)+2L\{y\}=0$

Using initial value condition

$s^2L\{y\}-s\times 0-1+2sL\{y\}-2\times 0+2L\{y\}=0$

or, $\mathcal{L}\{y\}\,(s^2+2s+2)=1$

or, $\mathcal{L}\{y\}=\dfrac{1}{s^2+2s+2}$ $L\{y\}(s^2$

$=\dfrac{1}{(s+1)^2+1^2}$

$=\left[\dfrac{1}{s^2+1}\right]_{s\to s+1}$

$=[\mathcal{L}\{\sin t\}]_{s\to s+1}$

∴ $\mathcal{L}\{y\}=\mathcal{L}\{e^{-t}\sin t\}$

Now, taking inverse Laplace transform on both sides, we get

$y=e^{-t}\sin t$. Which is the required solution.

b. The given differential equation is

$y''+4y'+3y=e^{-t}$

Taking Laplace transform on both sides, we get

$\mathcal{L}\{y''\}+4L\{y'\}+3L\{y\}=\mathcal{L}\{e^{-t}\}$

or, $s^2L\{y\}-s\,y(0)-y'(0)+4sL\{y\}-4\,y(0)+3L\{y\}=\dfrac{1}{s+1}$

Using initial condition, $y(0)=y'(0)=1$

$s^2L\{y\}-s\times 1-1+4sL\{y\}-4\times 1+3L\{y\}=\dfrac{1}{s+1}$

or, $\mathcal{L}\{y\}\,(s^2+4s+3)=\dfrac{1}{s+1}+s+5$

or, $\mathcal{L}\{y\}\,(s^2+4s+3)=\dfrac{1+s^2+5s+s+5}{s+1}$

or, $\mathcal{L}\{y\}=\dfrac{s^2+6s+6}{(s+1)\,(s^2+4s+3)}$

or, $\mathcal{L}\{y\}=\dfrac{s^2+6s+6}{(s+1)\,(s+1)\,(s+3)}$

Using partial fraction, we get

Let, $\dfrac{s^2+6s+6}{(s+3)\,(s+1)^2}=\dfrac{A}{s+3}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$ .... (1)

Then,

$s^2+6s+6=A\,(s+1)^2+B\,(s+3)\,(s+1)+C(s+3)$

$s^2+6s+6=As^2+2\,As+A+Bs^2+4Bs+3B+Cs+3C$

$s^2+6s+6=(A+B)\,s^2+(2A+4B+C)\,s+(A+3B+3C)$

Equating coefficient of s and the constant term on both sides, we get

$A+B=1,\ 2A+4B+C=6$ and $A+3B+3C=6$

Solving these equations, we get

$A=\dfrac{-3}{4},\ B=\dfrac{7}{4}$ and $C=\dfrac{1}{2}$

Then, equation (1) becomes

$\mathcal{L}\{y\}=\dfrac{-3}{4(s+3)}+\dfrac{7}{4(s+1)}+\dfrac{1}{2(s+1)^2}$

or, $\mathcal{L}\{y\}=-\dfrac{3}{4}\left[\dfrac{1}{s}\right]_{s\to s+3}+\dfrac{7}{4}\left[\dfrac{1}{s}\right]_{s\to s+1}+\dfrac{1}{2}\left[\dfrac{1}{s^2}\right]_{s\to s+1}$

$=-\dfrac{3}{4}[\mathcal{L}\{1\}]_{s\to s+3}+\dfrac{7}{4}[\mathcal{L}\{1\}]_{s\to s+1}+\dfrac{1}{2}[\mathcal{L}\{t\}]_{s\to s+1}$

$=\mathcal{L}\left\{-\dfrac{3}{4}e^{-3t}\right\}+\mathcal{L}\left\{\dfrac{7}{4}e^{-t}\right\}+\mathcal{L}\left\{\dfrac{1}{2}te^{-t}\right\}$

∴ $\mathcal{L}\{y\}=\mathcal{L}\left\{\dfrac{-3e^{-3t}+7e^{-t}+2te^{-t}}{4}\right\}$

Now, taking inverse Laplace transform on both sides, we get

$y=\dfrac{-3e^{-3t}+7e^{-t}+2te^{-t}}{4}$

c. The given differential equation is

$x''+2x'+5x=e^{-t}\sin t$

Taking Laplace transform on both sides, we get

$\mathcal{L}\{x''\}+2L\{x'\}+5L\{x\}=\mathcal{L}\{e^{-t}\sin t\}$

or, $s^2L\{x\}-s\,x(0)-x'(0)+2sL\{x\}-2x(0)+5L\{x\}=[\mathcal{L}\{\sin t\}]_{s\to s+1}$

Using initial condition, $x(0)=0$ and $x'(0)=1$, we get

$s^2L\{x\}-s\times 0-1+2sL\{x\}-2\times 0+5L\{x\}=\left[\dfrac{1}{s^2+1}\right]_{s\to s+1}$

or, $\mathcal{L}\{x\}\,(s^2+2s+5)=\dfrac{1}{(s+1)^2+1}+1$

or, $\mathcal{L}\{x\}\,(s^2+2s+5)=\dfrac{1+s^2+2s+2}{(s^2+2s+1+1)}$

or, $\mathcal{L}\{x\}=\dfrac{s^2+2s+3}{(s^2+2s+2)\,(s^2+2s+5)}$ .... (1)

Using partial fraction

or, $\dfrac{s^2+2s+3}{(s^2+2s+2)\,(s^2+2s+5)}=\dfrac{As+B}{s^2+2s+2}+\dfrac{Cs+D}{s^2+2s+5}$

Then,

$s^2+2s+3=(As+B)\,(s^2+2s+5)+(Cs+D)\,(s^2+2s+2)$

or, $s^2+2s+3=As^3+2As^2+5As+Bs^2+2Bs+5B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D$

or, $s^2+2s+3=(A+C)\,s^3+(2A+B+2C+D)^2+(5A+2B+2C+2D)\,s+(5B+2D)$

Equating coefficient of s and the constant term on both sides, we get

$A+C=0,\ \ 2A+B+2C+D=1,\ \ 5A+2B+2C+2D=2$ and $5B+2D=3$

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Solving these equations, we get

$A = 0,\ B = \frac{1}{3},\ C = 0$ and $D = \frac{2}{3}$

Then, equation becomes

$$\mathcal{L}\{x\} = \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3[(s+1)^2+1]} + \frac{1}{3[(s+1)^2+4]}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3}[\mathcal{L}\{\sin t\}]_{s\to s+1} + \frac{2}{3}[\mathcal{L}\{\sin 2t\}]_{s\to s+1}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3}\mathcal{L}\{(\sin t + \sin 2t)\, e^{t}$$

Now, taking inverse Laplace transform on both sides, we get

$$x = \frac{1}{3}e^{t}(\sin t + \sin 2t)$$

This is the required solution of the given differential equation.

Example 18. Solve the simultaneous differential equations $y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$ given that $x(0) = 1,\ y(0) = 0$.

Solution:

The given simultaneous differential equation are

$$y' + 2x = \sin 2t \text{ and } x' - 2y = \cos 2t$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{x\} = \mathcal{L}\{\sin 2t\} \text{ and } \mathcal{L}\{x'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}$$

or, $$s\mathcal{L}\{y\} + y(0) + 2\mathcal{L}\{x\} = \frac{2}{s^2+4} \text{ and } s\mathcal{L}\{x\} - x(0) - 2\mathcal{L}\{y\} = \frac{s}{s^2+4}$$

Using initial conditions $x(0) = 1, y(0) = 0$, we get,

$$s\mathcal{L}\{y\} + 2\mathcal{L}\{x\} = \frac{s}{s^2+4} \text{ and } s\mathcal{L}\{x\} - 2\mathcal{L}\{y\} = \frac{s}{s^2+4} + 1 \quad \ldots\ (1)$$

or, $$s\mathcal{L}\{x\} - 2\mathcal{L}\{y\} = \frac{s^2+s+4}{s^2+4} \quad \ldots\ (2)$$

Solving equation (1) and (2), we get

$$\mathcal{L}\{x\} = \frac{1+s}{s^2+4} \text{ and } \mathcal{L}\{y\} = \frac{-2}{s^2+4}$$

Taking inverse Laplace on both sides we get

$$x = \mathcal{L}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}\left\{\frac{s}{s^2+4}\right\} \text{ and } y = \mathcal{L}\left\{\frac{-2}{s^2+2^2}\right\}$$

or, $$x = \frac{1}{2}\sin 2t + \cos 2t \text{ and } y = \sin 2t$$

Hence, required solution of given differential equations is

$$x = \frac{1}{2}(\sin 2t + 2\cos 2t) \text{ and } y = \sin 2t.$$

Example 19. Using Laplace transform, solve the initial value problem.

a. $ty'' + 2y' + ty = \sin t,\ y(0) = 1,\ y'(0) = 0$

b. $\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1,\ x(0) = 1,\ x'(0) = 2$

Solution

a. The given differential equation is $ty'' + 2y' + ty = \sin t$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{ty''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{ty\} = \mathcal{L}\{\sin t\}$$

or, $$\frac{-d}{ds}[s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)] + 2s\mathcal{L}\{y\} - 2y(0) + \frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1}$$

or, $$-2s\mathcal{L}\{y\} - s^2\frac{d}{ds}\mathcal{L}\{y\} + y(0) + 2s\mathcal{L}\{y\} - 2y(0) - \frac{d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1}$$

or, $$-(s^2+1)\frac{d}{ds}\mathcal{L}\{y\} - y(0) = \frac{1}{s^2+1}$$

Using initial condition $y(0) = 1, y'(0) = 0$ we get

or, $$-(s^2+1)\frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1} + 1$$

or, $$\frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$$

or, $$\mathcal{L}\{y\} = -\int\frac{1}{(s^2+1)^2}ds - \int\frac{1}{s^2+1}ds$$

or, $$\mathcal{L}\{y\} = -\int\frac{1}{(s^2+1)^2}ds - \tan^{-1}s$$

or, $$\mathcal{L}\{y\} = -\mathrm{I} - \tan^{-1}s \quad \ldots\ (1)$$

where, $\mathrm{I} = \int\frac{1}{(s^2+1)^2}ds$

Let $s = \tan\theta$ so that $ds = \sec^2\theta\, d\theta$

Then, $$\mathrm{I} = \int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}d\theta$$

$$= \int\frac{\sec^2\theta}{(\sec^2\theta)^2}d\theta$$

$$= \int\frac{1}{\sec^2\theta}d\theta$$

$$= \int\cos^2\theta\, d\theta$$

$$= \int\frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{2\tan\theta}{4(1+\tan^2\theta)} \quad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\therefore\ \mathrm{I} = \frac{\tan^{-1}s}{2} + \frac{s}{2(s^2+1)} \quad [\because s = \tan\theta \text{ so } \theta = \tan^{-1}s]$$

Equation (1) becomes

$$\mathcal{L}\{y\} = -\frac{1}{2}\tan^{-1}s - \frac{s}{2(s^2+1)} - \tan^{-1}s$$

or, $$\mathcal{L}\{y\} = -\frac{3}{2}\tan^{-1}s - \frac{s}{2(s^2+1)}$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = -\frac{3}{2}\mathcal{L}^{-1}\{\tan^{-1}s\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \qquad \ldots\ (2)$$

Let $g(t) = \mathcal{L}^{-1}(\tan^{-1}s)$. Then, $\mathcal{L}\{t\,g(t)\} = -\frac{d}{ds}\tan^{-1}s$

or, $\mathcal{L}\{t\,g(t)\} = \frac{1}{s^2+1}$

Taking inverse Laplace on both sides

$$t\,g(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\}$$
$$= -\sin t$$

$\therefore\ g(t) = -\frac{\sin t}{t}$

Putting value of $g(t)$ in equation (2), we get

$$y(t) = -\frac{3}{2}\left(-\frac{\sin t}{t}\right) - \frac{1}{2}\cos t$$

or, $y(t) = \frac{3\sin t}{2t} - \frac{\cos t}{2}$

Hence, $y(t) = \frac{3\sin t - t\cos t}{2t}$ is required solution.

b. The given differential equation is

$$\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1$$

or, $x'' - tx' + x = 1$

Taking Laplace Transform on both side we get

$\mathcal{L}\{x''\} - \mathcal{L}\{tx'\} + \mathcal{L}\{x\} = \mathcal{L}\{1\}$

or, $s^2L\{x\} - sx(0) - x'(0) + \frac{d}{ds}[sL\{x\} - x(0)\}] + \mathcal{L}\{x\} = \frac{1}{s}$

Using initial condition $x(0) = 1$, $x'(0) = 2$, we get

or, $s^2L\{x\} - s \times 1 - 2 + \frac{d}{ds}[sL\{x\} - 1] + \mathcal{L}\{x\} = \frac{1}{s}$

or, $s^2L\{x\} - s - 2 + s\frac{d}{ds}\mathcal{L}\{x\} + \mathcal{L}\{x\}.1 + \mathcal{L}\{x\} = \frac{1}{s}$

or, $s\frac{d}{ds}\mathcal{L}\{x\} + (s^2+2)\,\mathcal{L}\{x\} = \frac{1}{s} + s + 2$

or, $\frac{d}{ds}\mathcal{L}\{x\} + \left(\frac{s^2+2}{s}\right)\mathcal{L}\{x\} = \frac{1}{s^2} + 1 + \frac{2}{s}$

Now, this equation can be treated as Leibniz linear differential equation in $\mathcal{L}\{x\}$.

where $P = \frac{s^2+2}{s}$, $Q = 1 + \frac{2}{s} + \frac{1}{s^2}$

Then, I.F. $= e^{\int p\,ds}$

$$= e^{\int \frac{s^2+2}{s}ds}$$
$$= e^{\int\left(\frac{s^2+2}{s}\right)ds}$$
$$= e^{\frac{s^2}{2} + 2\log s}$$
$$= e^{\frac{s^2}{2}}e^{\log s^2}$$
$$= s^2 e^{\frac{s^2}{2}}$$

Now for solution,

$$\mathcal{L}[x] \times \text{I.F.} = \int \text{I.F.} \times Q\,ds + C$$
$$= \int s^2 e^{\frac{s^2}{2}}\left(1 + \frac{2}{s} + \frac{1}{s^2}\right)ds + C$$
$$= \int s^2 e^{\frac{s^2}{2}}ds + 2\int s e^{\frac{s^2}{2}}ds + \int e^{\frac{s^2}{2}}ds + C$$
$$= \int\left(se^{\frac{s^2}{2}}\right)s\,ds + 2\int\left(se^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + C$$
$$= \int d\left(e^{\frac{s^2}{2}}\right).s\,ds + 2\int d\left(e^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + C$$

or, $\mathcal{L}\{x\} \times \text{I.F.} = \left[se^{\frac{s^2}{2}} - \int 1.e^{\frac{s^2}{2}}ds\right] + 2e^{\frac{s^2}{2}} + \int e^{\frac{s^2}{2}}ds + C$

or, $\mathcal{L}\{x\}s^2e^{\frac{s^2}{2}} = (s+2)e^{\frac{s^2}{2}} + C$

or, $\mathcal{L}\{x\} = \frac{1}{s} + \frac{2}{s^2} + \frac{Ce^{\frac{-s^2}{2}}}{s^2}$

Hence, $\mathcal{L}\{x\} = \mathcal{L}\left\{\frac{1}{s}\right\} + 2L\left\{\frac{2}{s^2}\right\} + \mathcal{L}\left\{\frac{Ce^{\frac{-s^2}{2}}}{s^2}\right\}$

Taking Inverse Laplace transform on both sides, we get $x(t) = 1 + 2t + C\,F(t)$.

where $F(t)$ is some function of t on using boundary conditions,

Where $x(0) = 1$, then

$x(0) = 1 + 2.0 + C.F.(0)$, *i.e.*, $F(0) = 0$

When, $x'(0) = 2$ then

$x'(t) = 2 + CF'(t)$ $\quad x'(0) = 2 + CF'(0)$ $\quad 2 = 2 + C\,F'(0)$

$C.F'(0) = 0 \quad C = 0$

Hence, required solution of the given differential equation is

$x(t) = 1 + 2t$

Exercise 3.4

1. Apply Laplace transform to evaluate following integrals.

a. $\int_0^\infty e^{-3t}\,t\sin t\,dt = \frac{3}{50}$ b. $\int_0^\infty \frac{e^t\sin^2 t}{t}dt = \frac{1}{4}\log 5$ c. $\int_0^\infty e^{-t}t^3\sin t\,dt = 0$

2. Solve the following initial value problem using the Laplace transform.

a. $y'' - 4y = 0,\ y(0) = 1,\ y'(0) = 2$
b. $y'' + 2y' + 17y = 0,\ y(0) = 0,\ y'(0) = 12$
c. $y'' - 2y' - 3y = 0,\ y(0) = 1,\ y'(0) = 7$
d. $y'' - y' - 2y = 0,\ y(0) = 8,\ y'(0) = 7$
e. $y'' - 2y' + 10y = 0,\ y(0) = 3, y'(0) = 3$
f. $y'' + \pi^2 y = 0, y(0) = 2, y'(0) = 0$
g. $y'' + 2y' + 2y = 0, y(0) = 0, y'(0) = 1$
h. $4y'' + 8y' + 5y = 0, y(0) = 0, y'(0) = 1$

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j. $9y'' - 6y' + y = 0, y(0) = 3, y'(0) = 1$

k. $4y'' - 4y' + 37y = 0, y(0) = 3, y'(0) = 1.5$

l. $y'' + ky' - 2k^2y = 0, y(0) = 6, y'(0) = 0$

3. Solve the following initial value problem using the Laplace transform.

a. $y'' - 3y' + 2y = 4t + e^{3t}, y(0) = 1,\ y'(0) = -1$

b. $y' - 4y = 5e^{-4t}, y(0) = 1$

c. $y'' + 4y' + 4y = e^{t}, y(0) = y'(0) = 0$

d. $y'' + 4y' + 3y = e^{t}, y(0) = y'(0) = 1$

e. $x'' + 2x' + 5x = e^{t} \sin t, x(0) = 0, x'(0) = 1$

f. $y'' + 2y' + 2y = 5\sin t, y(0) = y'(0) = 0$

g. $y'' - 2y' + y = e^{t}, y(0) = 2, y'(0) = 1$

h. $y'' + y' - 2y = t, y(0) = 1, y'(0) = 0$

i. $y'' + 2y' + y = e^{t}, y(0) = -1, y(0) = 1$

j. $y'' + y = 2\cos t, y(0) = 3, y'(0) = 4$

k. $y'' + 2y' - 3y = 6e^{-2t}, y(0) = 2, y'(0) = -14$

l. $y'' + 4y' + 4y = \sin t, y(0) = 1, y'(0) = 3$

m. $x'' + x = t\cos 2t, x(0) = x'(0) = 0$

n. $y'' - y' - 2y = 3e^{2t}, y(0) = 0, y'(0) = -2$

o. $y'' - 2y + y = e^{t}, y(0) = 2, y'(0) = -1$

Answer

2. a. $y = e^{2t}$ b. $y = e^{\pi t} + 1$ c. $y = e^{t} + 2e^{3t}$ d. $y = 3e^{t} + 5e^{2t}$ e. $y = 3e^{t} \cos 3t$

f. $y = 2\cos \pi t$ g. $y = e^{-t} \sin t$ h. $y = 2e^{t}\sin \frac{t}{2}$ i. $y = e^{-t}(2\cos 2t - \sin 2t)$

j. $y = 3e^{t/3}$ k. $y = 3e^{t/2}\cos 3t$ l. $y = 2e^{-2kt} + 4e^{kt}$

3. a. $y = 3 + 2t + \frac{1}{2}(e^{3t} - e^{t}) - 2e^{2t}$ b. $y = \frac{5}{4}\sinh 4t + 4e^{4t}$

c. $y = e^{-t} - e^{-2t}(1 + t)$ d. $y = \frac{7e^{-t} + 2te^{-t} + 3e^{-3t}}{4}$

e. $x = e^{-t}\left(\frac{\sin t + \sin 2t}{3}\right)$ f. $y = 2\cos t(e^{-t} - 1) + (e^{-t} + 1)\sin t$

g. $y = e^{t}\left(2 - 3t + \frac{t^2}{2}\right)$ h. $y = \frac{-1 - 2t + 4e^{t} + e^{-2t}}{4}$

k. $y = t\sin t + 3\cos t + 4\sin t$ l. $y = \frac{16e^{-2t} - 3e^{-3t} + 7e^{t}}{3}$

m. $y = \frac{1}{29}(4\cos t - 3\sin t + 25e^{-2t} + 165te^{-2t})$

n. $x = \frac{1}{9}(4\sin 2t - 5\sin t - 3t\cos 2t)$

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Unit 4

Advanced Vector Calculus

Pre-requisite knowledge	Expected learning outcomes
Before starting this unit, students are expected to have fundamental concepts and evaluation skills on • defining vector, unit vector, magnitude and direction of vector • multiplication of a vector by scalar quantity • dot and cross product of two or more vectors • understanding concept of derivatives and slope of tangent and rate measure • rules of differentiation • application of derivatives • defining integration • evaluation of integrals on function of a single variable • evaluation of double and triple integration • define work and evaluate the work done	After completion of this unit, student will develop sufficient knowledge and evaluation skills on • defining gradient, divergence and curl of vector • understanding the application of gradient, divergence and curl in practical application • evaluating gradient, divergence and curl of the given vectors • defining and evaluation of the potential function • able to define line integral • defining and using Green's theorem • express surface integral in physical term • evaluation of Green's theorem, Stoke's theorem and Gauss divergence theorem • geometrical interpretation of Gauss theorem, Stoke's theorem and Gauss divergence theorem

Solution

a. The given differential equation is $y'' + 2y' + 2y = 0$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{y''\} + 2L\{y'\} + 2L\{y\} = 0$$

or, $s^2L\{y\} - s\,y(0) - y'(0) + 2sL\{y\} - 2y(0) + 2L\{y\} = 0$

Using initial value condition

$$s^2L\{y\} - s \times 0 - 1 + 2sL\{y\} - 2 \times 0 + 2L\{y\} = 0$$

or, $\mathcal{L}\{y\}\,(s^2 + 2s + 2) = 1$

or, $\mathcal{L}\{y\} = \dfrac{1}{s^2 + 2s + 2}$ $\quad$ $L\{y\}(\ s^2$

$$= \frac{1}{(s+1)^2 + 1^2}$$

$$= \left[\frac{1}{s^2+1}\right]_{s \to s+1}$$

$$= [\mathcal{L}\{\sin t\}]_{s \to s+1}$$

$\therefore$ $\mathcal{L}\{y\} = \mathcal{L}\{e^{-t} \sin t\}$

Now, taking inverse Laplace transform on both sides, wet get

$y = e^{-t} \sin t$. Which is the required solution.

b. The given differential equation is

$$y'' + 4y' + 3y = e^{-t}$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{y''\} + 4L\{y'\} + 3L\{y\} = \mathcal{L}\{e^{-t}\}$$

or, $s^2L\{y\} - s\,y(0) - y'(0) + 4sL\{y\} - 4\,y(0) + 3L\{y\} = \dfrac{1}{s+1}$

Using initial condition, $y(0) = y'(0) = 1$

$$s^2L\{y\} - s \times 1 - 1 + 4sL\{y\} - 4 \times 1 + 3L\{y\} = \frac{1}{s+1}$$

or, $\mathcal{L}\{y\}\,(s^2 + 4s + 3) = \dfrac{1}{s+1} + s + 5$

or, $\mathcal{L}\{y\}\,(s^2 + 4s + 3) = \dfrac{1 + s^2 + 5s + s + 5}{s+1}$

or, $\mathcal{L}\{y\} = \dfrac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)}$

or, $\mathcal{L}\{y\} = \dfrac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)}$

Using partial fraction, we get

Let, $\dfrac{s^2 + 6s + 6}{(s+3)(s+1)^2} = \dfrac{A}{s+3} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$ .... (1)

Then,

$$s^2 + 6s + 6 = A(s+1)^2 + B(s+3)(s+1) + C(s+3)$$

or, $s^2 + 6s + 6 = As^2 + 2As + A + Bs^2 + 4Bs + 3B + Cs + 3C$

or, $s^2 + 6s + 6 = (A + B)s^2 + (2A + 4B + C)s + (A + 3B + 3C)$

Equating coefficient of s and the constant term on both sides, we get

$$A + B = 1,\ 2A + 4B + C = 6 \text{ and } A + 3B + 3C = 6$$

Solving these equations, we get

$$A = \frac{-3}{4},\ B = \frac{7}{4} \text{ and } C = \frac{1}{2}$$

Then, equation (1) becomes

$$\mathcal{L}\{y\} = \frac{-3}{4(s+3)} + \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2}$$

or, $\mathcal{L}\{y\} = -\dfrac{3}{4}\left[\dfrac{1}{s}\right]_{s \to s+3} + \dfrac{7}{4}\left[\dfrac{1}{s}\right]_{s \to s+1} + \dfrac{1}{2}\left[\dfrac{1}{s^2}\right]_{s \to s+1}$

$$= -\frac{3}{4}[\mathcal{L}\{1\}]_{s \to s+3} + \frac{7}{4}[\mathcal{L}\{1\}]_{s \to s+1} + \frac{1}{2}[\mathcal{L}\{t\}]_{s \to s+1}$$

$$= \mathcal{L}\left\{-\frac{3}{4}e^{-3t}\right\} + \mathcal{L}\left\{\frac{7}{4}e^{-t}\right\} + \mathcal{L}\left\{\frac{1}{2}te^{-t}\right\}$$

$\therefore$ $\mathcal{L}\{y\} = \mathcal{L}\left\{\dfrac{-3e^{-3t} + 7e^{-t} + 2te^{-t}}{4}\right\}$

Now, taking inverse Laplace transform on both sides, we get

$$y = \frac{-3e^{-3t} + 7e^{-t} + 2te^{-t}}{4}$$

c. The given differential equation is

$$x'' + 2x' + 5x = e^{-t} \sin t$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{x''\} + 2L\{x'\} + 5L\{x\} = \mathcal{L}\{e^{-t} \sin t\}$$

or, $s^2L\{x\} - s\,x(0) - x'(0) + 2sL\{x\} - 2x(0) + 5L\{x\} = [\mathcal{L}\{\sin t\}]_{s \to s+1}$

Using initial condition, $x(0) = 0$ and $x'(0) = 1$, we get

$$s^2L\{x\} - s \times 0 - 1 + 2sL\{x\} - 2 \times 0 + 5L\{x\} = \left[\frac{1}{s^2+1}\right]_{s \to s+1}$$

or, $\mathcal{L}\{x\}\,(s^2 + 2s + 5) = \dfrac{1}{(s+1)^2 + 1} + 1$

or, $\mathcal{L}\{x\}\,(s^2 + 2s + 5) = \dfrac{1 + s^2 + 2s + 2}{(s^2 + 2s + 1 + 1)}$

or, $\mathcal{L}\{x\} = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$ .... (1)

Using partial fraction

or, $\dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \dfrac{As + B}{s^2 + 2s + 2} + \dfrac{Cs + D}{s^2 + 2s + 5}$

Then,

$$s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$$

or, $s^2 + 2s + 3 = As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$

or, $s^2 + 2s + 3 = (A + C)s^3 + (2A + B + 2C + D)^2 + (5A + 2B + 2C + 2D)s + (5B + 2D)$

Equating coefficient of s and the constant term on both sides, we get

$$A + C = 0,\quad 2A + B + 2C + D = 1,\quad 5A + 2B + 2C + 2D = 2 \text{ and } 5B + 2D = 3$$

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Solving these equations, we get

$$A = 0,\ B = \frac{1}{3},\ C = 0 \text{ and } D = \frac{2}{3}$$

Then, equation becomes

$$\mathcal{L}\{x\} = \frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3[(s+1)^2+1} + \frac{1}{3[(s+1)^2+4}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3}[\mathcal{L}\{\sin t\}]_{s\to s+1} + \frac{2}{3}[\mathcal{L}\{\sin 2t\}]_{s\to s+1}$$

or, $$\mathcal{L}\{x\} = \frac{1}{3}\mathcal{L}\{(\sin t + \sin 2t)\, e^{t}$$

Now, taking inverse Laplace transform on both sides, we get

$$x = \frac{1}{3}\, e^{t}(\sin t + \sin 2t)$$

This is the required solution of the given differential equation.

Example 18. Solve the simultaneous differential equations $y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$ given that $x(0) = 1,\ y(0) = 0$.

Solution:

The given simultaneous differential equation are

$y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$

Taking Laplace transform on both sides, we get

$\mathcal{L}\{y'\} + 2L\{x\} = \mathcal{L}\{\sin 2t\}$ and $\mathcal{L}\{x'\} - 2L\{y\} = \mathcal{L}\{\cos 2t\}$

or, $sL\{y\} + y(0) + 2L\{x\} = \frac{2}{s^2+4}$ and $sL\{x\} - x(0) - 2L\{y\} = \frac{s}{s^2+4}$

Using initial conditions $x(0) = 1, y(0) = 0$, we get,

$$sL\{y\} + 2L\{x\} = \frac{s}{s^2+4} \text{ and } sL\{x\} - 2L\{y\} = \frac{s}{s^2+4} + 1 \quad \ldots.\ (1)$$

or, $$sL\{x\} - 2L\{y\} = \frac{s^2+s+4}{s^2+4} \quad \ldots.\ (2)$$

Solving equation (1) and (2), we get

$$\mathcal{L}\{x\} = \frac{1+s}{s^2+4} \text{ and } \mathcal{L}\{y\} = \frac{-2}{s^2+4}$$

Taking inverse Laplace on both sides we get

$$x = \mathcal{L}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}\left\{\frac{s}{s^2+4}\right\} \text{ and } y = \mathcal{L}\left\{\frac{-2}{s^2+2^2}\right\}$$

or, $$x = \frac{1}{2}\sin 2t + \cos 2t \text{ and } y = \sin 2t$$

Hence, required solution of given differential equation

$$x = \frac{1}{2}(\sin 2t + 2\cos 2t) \text{ and } y = \sin 2t.$$

Example 19. Using Laplace transform, solve the *initial* value problem.

a. $ty'' + 2y' + ty = \sin t,\ y(0) = 1,\ y'(0) = 0$

b. $\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1,\ x(0) = 1,\ x'(0) = 2$

Solution

a. The given differential equation is $ty'' + 2y' + ty = \sin t$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{ty''\} + 2L\{y'\} + \mathcal{L}\{ty\} = \mathcal{L}\{\sin t\}$$

or, $$\frac{-d}{ds}[s^2L\{y\} - s\,y(0) - y'(0)] + 2sL\{y\} - 2y(0) + \frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1}$$

or, $$-2sL\{y\} - s^2\frac{d}{ds}\mathcal{L}\{y\} + y(0) + 2sL\{y\} - 2y(0) - \frac{d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1}$$

or, $$-(s^2+1)\frac{d}{ds}\mathcal{L}\{y\} - y(0) = \frac{1}{s^2+1}$$

Using initial condition $y(0) = 1, y'(0) = 0$ we get

or, $$-(s^2+1)\frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{s^2+1} + 1$$

or, $$\frac{-d}{ds}\mathcal{L}\{y\} = \frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$$

or, $$\mathcal{L}\{y\} = -\int\frac{1}{(s^2+1)^2}\,ds - \int\frac{1}{s^2+1}\,ds$$

or, $$\mathcal{L}\{y\} = -\int\frac{1}{(s^2+1)^2}\,ds - \tan^{-1}s$$

or, $$\mathcal{L}\{y\} = -I - \tan^{-1}s \quad \ldots.\ (1)$$

where, $I = \int\frac{1}{(s^2+1)^2}\,ds$

Let $s = \tan\theta$ *so that* $ds = \sec^2\theta\, d\theta$

Then, $$I = \int\frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$$

$$= \int\frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= \int\frac{1}{\sec^2\theta}\,d\theta$$

$$= \int\cos^2\theta\, d\theta$$

$$= \int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{2\tan\theta}{4(1+\tan^2\theta)} \quad \left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\therefore\ I = \frac{\tan^{-1}s}{2} + \frac{s}{2(s^2+1)} \quad [\because s = \tan\theta \text{ so } \theta = \tan^{-1}s]$$

Equation (1) becomes

$$\mathcal{L}\{y\} = -\frac{1}{2}\tan^{-1}s - \frac{s}{2(s^2+1)} - \tan^{-1}s$$

or, $$\mathcal{L}\{y\} = -\frac{3}{2}\tan^{-1}s - \frac{s}{2(s^2+1)}$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = -\frac{3}{2}\mathcal{L}^{-1}\{\tan^{-1}s\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \quad \ldots\ldots (2)$$

Let $g(t) = \mathcal{L}^{-1}(\tan^{-1}s)$. Then, $\mathcal{L}\{t\,g(t)\} = -\frac{d}{ds}\tan^{-1}s$

or, $\mathcal{L}\{t\,g(t)\} = \frac{1}{s^2+1}$

Taking inverse Laplace on both sides

$$t\,g(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\}$$
$$= -\sin t$$

$\therefore$ $g(t) = -\frac{\sin t}{t}$

Putting value of $g(t)$ in equation (2), we get

$$y(t) = -\frac{3}{2}\left(-\frac{\sin t}{t}\right) - \frac{1}{2}\cos t$$

or, $y(t) = \frac{3\sin t}{2t} - \frac{\cos t}{2}$

Hence, $y(t) = \frac{3\sin t - t\cos t}{2t}$ is required solution.

b. The given differential equation is

$$\frac{d^2x}{dt^2} - t\frac{dx}{dt} + x = 1$$

or, $x'' - tx' + x = 1$

Taking Laplace Transform on both side we get

$$\mathcal{L}\{x''\} - \mathcal{L}\{tx'\} + \mathcal{L}\{x\} = \mathcal{L}\{1\}$$

or, $s^2L\{x\} - sx(0) - x'(0) + \frac{d}{ds}[sL\{x\} - x(0)\}] + \mathcal{L}\{x\} = \frac{1}{s}$

Using initial condition $x(0) = 1$, $x'(0) = 2$, we get

or, $s^2L\{x\} - s \times 1 - 2 + \frac{d}{ds}[sL\{x\} - 1] + \mathcal{L}\{x\} = \frac{1}{s}$

or, $s^2L\{x\} - s - 2 + s\frac{d}{ds}\mathcal{L}\{x\} + \mathcal{L}\{x\}.1 + \mathcal{L}\{x\} = \frac{1}{s}$

or, $s\frac{d}{ds}\mathcal{L}\{x\} + (s^2+2)\,\mathcal{L}\{x\} = \frac{1}{s} + s + 2$

or, $\frac{d}{ds}\mathcal{L}\{x\} + \left(\frac{s^2+2}{s}\right)\mathcal{L}\{x\} = \frac{1}{s^2} + 1 + \frac{2}{s}$

Now, this equation can be treated as Leibniz linear differential equation in $\mathcal{L}\{x\}$.

where $P = \frac{s^2+2}{s}$, $Q = 1 + \frac{2}{s} + \frac{1}{s^2}$

Then, I.F. $= e^{\int p\,ds}$

$$= e^{\int \frac{s^2+2}{s}ds}$$
$$= e^{\int\left(\frac{s^2+2}{s}\right)ds}$$
$$= e^{\frac{s^2}{2} + 2\log s}$$
$$= e^{\frac{s^2}{2}}e^{\log s^2}$$
$$= s^2 e^{\frac{s^2}{2}}$$

Now for solution,

$$\mathcal{L}[x] \times \text{I.F.} = \int \text{I.F.} \times Q\,ds + C$$
$$= \int s^2 e^{\frac{s^2}{2}}\left(1 + \frac{2}{s} + \frac{1}{s^2}\right)ds + C$$
$$= \int s^2 e^{\frac{s^2}{2}}ds + 2\int s e^{\frac{s^2}{2}}ds + \int e^{\frac{s^2}{2}}ds + C$$
$$= \int\left(se^{\frac{s^2}{2}}\right)s\,ds + 2\int\left(se^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + C$$
$$= \int d\left(e^{\frac{s^2}{2}}\right).s\,ds + 2\int d\left(e^{\frac{s^2}{2}}\right)ds + \int e^{\frac{s^2}{2}}ds + C$$

or, $\mathcal{L}\{x\} \times \text{I.F.} = \left[se^{\frac{s^2}{2}} - \int 1.e^{\frac{s^2}{2}}ds\right] + 2e^{\frac{s^2}{2}} + \int e^{\frac{s^2}{2}}ds + C$

or, $\mathcal{L}\{x\}s^2e^{\frac{s^2}{2}} = (s+2)e^{\frac{s^2}{2}} + C$

or, $\mathcal{L}\{x\} = \frac{1}{s} + \frac{2}{s^2} + \frac{Ce^{\frac{-s^2}{2}}}{s^2}$

Hence, $\mathcal{L}\{x\} = \mathcal{L}\left\{\frac{1}{s}\right\} + 2L\left\{\frac{2}{s^2}\right\} + \mathcal{L}\left\{\frac{Ce^{\frac{-s^2}{2}}}{s^2}\right\}$

Taking Inverse Laplace transform on both sides, we get $x(t) = 1 + 2t + C\,F(t)$.

where $F(t)$ is some function of t on using boundary conditions,

Where $x(0) = 1$, then

$x(0) = 1 + 2.0 + C.F.(0)$, i.e., $F(0) = 0$

When, $x'(0) = 2$ then

$x'(t) = 2 + CF'(t)$ $\quad x'(0) = 2 + CF'(0)$ $\quad 2 = 2 + C\,F'(0)$

$C.F'(0) = 0$ $\quad C = 0$

Hence, required solution of the given differential equation is

$x(t) = 1 + 2t$

Exercise 3.4

1. Apply Laplace transform to evaluate following integrals.

a. $\int_0^\infty e^{-3t}\,t\sin t\,dt = \frac{3}{50}$ $\quad$ b. $\int_0^\infty \frac{e^{t}\sin^2 t}{t}dt = \frac{1}{4}\log 5$ $\quad$ c. $\int_0^\infty e^{-t}t^3\sin t\,dt = 0$

2. Solve the following initial value problem using the Laplace transform.

a. $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$

b. $y'' + 2y' + 17y = 0$, $y(0) = 0$, $y'(0) = 12$

c. $y'' - 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 7$

d. $y'' - y' - 2y = 0$, $y(0) = 8$, $y'(0) = 7$

e. $y'' - 2y' + 10y = 0$, $y(0) = 3$, $y'(0) = 3$

f. $y'' + \pi^2 y = 0$, $y(0) = 2$, $y'(0) = 0$

g. $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$

h. $4y'' + 8y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$

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i. $y''+2y'+5y=0, y(0)=2, y'(0)=-4$

j. $9y''-6y'+y=0, y(0)=3, y'(0)=1$

k. $4y''-4y'+37y=0, y(0)=3, y'(0)=1.5$

l. $y''+ky'-2k^2y=0, y(0)=6, y'(0)=0$

3. Solve the following initial value problem using the Laplace transform.

a. $y''-3y'+2y=4t+e^{3t}, y(0)=1, y'(0)=-1$

b. $y'-4y=5e^{-4t}, y(0)=1$

c. $y''+4y'+4y=e^{t}, y(0)=y'(0)=0$

d. $y''+4y'+3y=e^{t}, y(0)=y'(0)=1$

e. $x''+2x'+5x=e^{t}\sin t, x(0)=0, x'(0)=1$

f. $y''+2y'+2y=5\sin t, y(0)=y'(0)=0$

g. $y''-2y'+y=e^{t}, y(0)=2, y'(0)=1$

h. $y''+y'-2y=t, y(0)=1, y'(0)=0$

i. $y''+2y'+y=e^{t}, y(0)=-1, y(0)=1$

j. $y''+y=2\cos t, y(0)=3, y'(0)=4$

k. $y''+2y'-3y=6e^{-2t}, y(0)=2, y'(0)=-14$

l. $y''+4y'+4y=\sin t, y(0)=1, y'(0)=3$

m. $x''+x=t\cos 2t, x(0)=x'(0)=0$

n. $y''-y'-2y=3e^{2t}, y(0)=0, y'(0)=-2$

o. $y''-2y+y=e^{t}, y(0)=2, y'(0)=-1$

Answer

2. a. $y=e^{2t}$ b. $y=e^{\pi t}+1$ c. $y=e^{t}+2e^{3t}$ d. $y=3e^{t}+5e^{2t}$ e. $y=3e^{t}\cos 3t$

f. $y=2\cos\pi t$ g. $y=e^{-t}\sin t$ h. $y=2e^{t}\sin\frac{t}{2}$ i. $y=e^{-t}(2\cos 2t-\sin 2t)$

j. $y=3e^{t/3}$ k. $y=3e^{t/2}\cos 3t$ l. $y=2e^{-2kt}+4e^{kt}$

3. a. $y=3+2t+\frac{1}{2}(e^{3t}-e^{t})-2e^{2t}$ b. $y=\frac{5}{4}\sinh 4t+4e^{4t}$

c. $y=e^{-t}-e^{-2t}(1+t)$ d. $y=\frac{7e^{-t}+2te^{-t}+3e^{-3t}}{4}$

e. $x=e^{-t}\left(\frac{\sin t+\sin 2t}{3}\right)$ f. $y=2\cos t(e^{-t}-1)+(e^{-t}+1)\sin t$

g. $y=e^{t}\left(2-3t+\frac{t^2}{2}\right)$ h. $y=\frac{-1-2t+4e^{t}+e^{-2t}}{4}$

k. $y=t\sin t+3\cos t+4\sin t$ l. $y=\frac{16e^{-2t}-3e^{-3t}+7e^{t}}{3}$

m. $y=\frac{1}{29}(4\cos t-3\sin t+25e^{-2t}+165te^{-2t})$

n. $x=\frac{1}{9}(4\sin 2t-5\sin t-3t\cos 2t)$

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Unit 4

Advanced Vector Calculus

Pre-requisite knowledge

Before starting this unit, students are expected to have fundamental concepts and evaluation skills on

- defining vector, unit vector, magnitude and direction of vector
- multiplication of a vector by scalar quantity
- dot and cross product of two or more vectors
- understanding concept of derivatives and slope of tangent and rate measure
- rules of differentiation
- application of derivatives
- defining integration
- evaluation of integrals on function of a single variable
- evaluation of double and triple integration
- define work and evaluate the work done

Expected learning outcomes

After completion of this unit, student will develop sufficient knowledge and evaluation skills on

- defining gradient, divergence and curl of vector
- understanding the application of gradient, divergence and curl in practical application
- evaluating gradient, divergence and curl of the given vectors
- defining and evaluation of the potential function
- able to define line integral
- defining and using Green's theorem
- express surface integral in physical term
- evaluation of Green's theorem, Stoke's theorem and Gauss divergence theorem
- geometrical interpretation of Gauss theorem, Stoke's theorem and Gauss divergence theorem

4.1 Introduction

In the previous semester, we have introduced the vector quantity. Our study was centered on defining the vector, sum, difference, and product of two or three vectors. Further we have studied the meaning, rules and application of derivatives. Now we study the vector-valued functions and their derivatives.

J. Willard Gibbs
1839-1903

Vector calculus deals with the application of calculus in the context of vector function. Vector differentiation is the process of finding the rate of change of a vector function with respect to its independent variable. In geometry, it helps us understand how a vector field changes in different directions. A two-dimensional vector field is a function that maps each point (x, y) in R^2 to a two dimensional vector (u, v) and similarly a three-dimensional vector field maps (x, y, z) to (u, v, w). Vector fields have many applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid). Vector differentiation deals with differentiation of the vector function, in two or three-dimensional Euclidean space. The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus. Vector calculus was invented by two mathematicians J. Willard Gibbs and Oliver Heaviside at the end of the 19th century

Oliver Heaviside
1850-1925

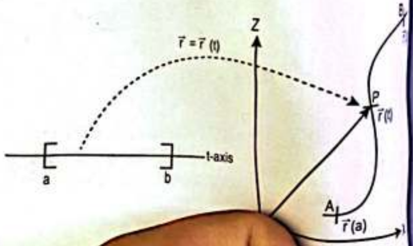
Vector calculus allows scientists and mathematicians to calculate things such as speed and direction from a graph. It also allows for a great visualization of what is happening with these numbers.

4.2 Vector function of scalar variable

A vector quantity that depends on some scalar variable t is known as vector function of scalar variable. In general the scalar variable is time variable.

A vector function of scalar variable t is written as,

$$\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}.$$



Where the set of values of scalar variable t i.e., domain of $\vec{r}$ is taken on some interval $[a, b]$ and is known as domain of $\vec{r} = \vec{r}(t)$.

Geometrically, a vector function $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$ represents a curve in three dimensional space as shown in figure.

Remarks:

1. A vector function will be written as $\vec{r}(t)$ or simply $\vec{r}$ with the variable to be understood.
2. If $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$ then the x-, y-, z- coordinates $r_1(t)$, $r_2(t)$, $r_3(t)$ are real valued function of real variable t.
3. Concepts of limit, continuity and derivative of Calculus of function of real variable can be applied for each components $r_1(t)$, $r_2(t)$ and $r_3(t)$ so that limit, continuity and derivative of $\vec{r}(t)$ is obtained from $r_1(t)$, $r_2(t)$, $r_3(t)$.

4.3 Limit of vector function of scalar variable

Let $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$

and $t \in [a, b]$.

Let $t = t_0$ be a point in $[a, b]$, then a vector $\vec{L}$ is said to be limit of $\vec{r}(t)$ if we can bring $\vec{r}(t)$ very close to $\vec{L}$ by brining t sufficiently close to t_0.

i.e., If for some $\in > 0$, we can find $\delta > 0$ such that $|\vec{r}(t) - \vec{L}| < \in$ whenevr $|t - t_0| < \delta$.

Mathematically, we write as, $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$.

Remark: If $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$ then $\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} r_1(t)\hat{i} + \lim_{t \to t_0} r_2(t)\hat{j}, \lim_{t \to t_0} r_3(t)\hat{k}$,

i.e., limit of $\vec{r} = \vec{r}(t)$ can always be determined in terms of limit of coordinate functions $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}_3(t)$ of $\vec{r}(t)$.

Example 1. If $\vec{r}(t) = 2t^2\hat{i} + \sin t\hat{j} + e^t$, find $\lim_{t \to \pi/2} \vec{r}(t)$.

Solution

We have, $\vec{r} = \vec{r}(t) = 2t^2\hat{i} + \sin t\hat{j} + e^t\hat{k}$

$$\therefore \lim_{t \to \pi/2} \vec{r}(t) = \lim_{t \to \pi/2} (2t^2)\hat{i} + \lim_{t \to \pi/2} \sin t\hat{j} + \lim_{t \to \pi/2} e^t\hat{k}$$

$$= 2\left(\frac{\pi}{2}\right)^2 \hat{i} + \sin\frac{\pi}{2}\hat{j} + e^{\frac{\pi}{2}}\hat{k}$$

$$\therefore \lim_{t \to \pi/2} \vec{r}(t) = \frac{\pi^2}{2}\hat{i} + \hat{j} + e^{\frac{\pi}{2}}\hat{k}.$$

4.4 Continuity of vector function

A vector function $\vec{r} = \vec{r}(t) = r_1(t)\hat{i}, r_2(t)\hat{j}, r_3(t)\hat{k}$ is said to be continuous at $t = t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

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i.e., If limiting value of $\vec{r}(t)$ as $t \to t_0$ is same as thee functional value $\vec{r}(t_0)$ at $t = t_0$

function $\vec{r} = \vec{r}(t)$ is continuous at $t = t_0$ and so it has continuous graph at $t = t_0$.

Remark: If $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$ then $\vec{r}(t)$ is continuous if each of coordinates $r_1(t), r_2(t), r_3(t)$ are continuous, i.e., $\lim_{t \to t_0} r_1(t) = r_1(t_0)$, $\lim_{t \to t_0} r_2(t) = r_2(t_0)$ and $\lim_{t \to t_0} r_3(t) = r_3(t_0)$.

4.5 Derivative of vector function

Let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable t. The limit, $\lim_{\Delta t \to 0} \dfrac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$,

if exists (finite) is known as derivative of $\vec{r}(t)$ and is denoted by $\dfrac{d\vec{r}(t)}{dt}$ or $\vec{r}'(t)$.

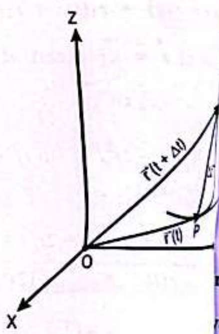
i.e., $\vec{r}'(t) = \dfrac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \dfrac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$.

We note that the derivative is rate of change of $\vec{r}(t)$ with respect to the independent variable.

Note: If $\vec{r} = \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}$ then $\dfrac{d\vec{r}(t)}{dt} = \dfrac{dr_1(t)}{dt}\hat{i} + \dfrac{dr_2(t)}{dt}\hat{j} + \dfrac{dr_3(t)}{dt}\hat{k}$.

4.5.1 Geometrical (physical) interpretation of derivative $\dfrac{d\vec{r}}{dt}$

Let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable t, whose graph is as shown in figure.



Let $\vec{OP} = \vec{r}(t)$, $\vec{OQ} = \vec{r}(t + \Delta t)$, then

$\vec{PQ} = \vec{OQ} - \vec{OP}$

or, $\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$

$$\therefore \quad \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \qquad \ldots(1)$$

Equation (1) represents average rate of change of $\vec{r}$ with respect to t.

If $\Delta t \to 0$, then $\vec{PQ}$ becomes a tangential vector $\vec{T}$ to the curve $\vec{r} = \vec{r}(t)$ at point P.

i.e., $\lim_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \vec{T}$.

Thus, the derivative of a vector function $\vec{r} = \vec{r}(t)$ gives the tangential vector $\vec{T}$ to the curve $\vec{r} = \vec{r}(t)$.

If $\vec{r} = \vec{r}(t)$ is position vector and t is time then $\vec{T} = \dfrac{d\vec{r}}{dt}$ gives tangential velocity vector $\vec{v}$ of a particle moving along the curve.

i.e., velocity $\vec{v} = \dfrac{d\vec{r}}{dt}$.

4.5.2 Higher order derivative

If $\vec{r} = \vec{r}(t)$ and $\vec{r}'(t) = \dfrac{d\vec{r}}{dt}$ then the higher order derivatives of $\vec{r}$ are defined by

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right), \quad \frac{d^3\vec{r}}{dt^3} = \frac{d}{dt}\left(\frac{d^2\vec{r}}{dt^2}\right) \text{ and so on.}$$

Mathematically, higher order derivative of $\vec{r} = \vec{r}(t)$ give higher order rate of change. Physically second order derivative gives acceleration of a moving particle.

i.e., Acceleration $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right) = \dfrac{d}{dt}(\vec{v})$

The third order deriavtive give jerk $\vec{J}$ i.e., jerk $\vec{J} = \dfrac{d^3\vec{r}}{dt^3} = \dfrac{d}{dt}\left(\dfrac{d^2\vec{r}}{dt^2}\right) = \dfrac{d}{dt}(\vec{a})$.

4.5.3 Partial derivative

Let $\vec{r} = \vec{r}(t_1, t_2, \ldots t_n)$ $\qquad \ldots(1)$

be a vector function of several scalar variables, $t_1, t_2, \ldots, t_n$, then the rate of change of $\vec{r}$ with respect to any variable t_m is defined to be partial derivative of $\vec{r}$ with respect to t_m and is given by

$$\frac{\partial\vec{r}}{\partial t_m} = \frac{\partial}{\partial t_m}\vec{r}(t_1, t_2, \ldots, t_n)$$

In order to evaluate $\dfrac{\partial\vec{r}}{\partial t_m}$ we use the same formula for ordinary derivative $\dfrac{d\vec{r}}{dt_m}$ keeping all other variables $t_1, t_2, \ldots t_n$ except t_m unchanged (treated as constant).

4.5.4 Derivative of different functions (rules of differentiation)

1. $\dfrac{d\vec{C}}{dt} = 0$, where $\vec{C}$ is constant vector.

2. $\dfrac{d}{dt}(\vec{r}_1 \pm \vec{r}_2) = \dfrac{d\vec{r}_1}{dt} \pm \dfrac{d\vec{r}_2}{dt}$, $\vec{r}_1 = \vec{r}_1(t)$, $\vec{r}_2 = \vec{r}_2(t)$

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3. $\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\cdot\vec{r_2}$

If $\vec{r_1} = \vec{r_2} = \vec{r}$, then

$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\cdot\vec{r_2}$ becomes

$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \vec{r}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{r}$

or, $\frac{dr^2}{dt} = \vec{r}\cdot\frac{d\vec{r}}{dt} + \vec{r}\cdot\frac{d\vec{r}}{dt}$, $r = |\vec{r}|$ and $\vec{r}\cdot\vec{r} = r^2$

or, $2r\frac{dr}{dt} = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$

$\therefore\ \vec{r}\cdot\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$

Note that in R.H.S. we have magnitude r and its derivative.

Remark: If $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then $r = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$

which is a scalar function of scalar vartiable and may not be constant.

e.g. Let $\vec{r} = t\hat{i} + \sin t\hat{j} + \hat{k}$

Then, $r = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$

$\therefore\quad r = \sqrt{t^2 + \sin^2 t + 1}$

4. $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$

If $\vec{r_2} = \frac{d\vec{r_1}}{dt}$, then $\frac{d\vec{r_2}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r_1}}{dt}\right) = \frac{d^2\vec{r_1}}{dt^2}$

and $\frac{d}{dt}\left(\vec{r_1}\times\frac{d\vec{r_1}}{dt}\right) = \vec{r}\times\frac{d^2\vec{r_1}}{dt^2} + \frac{d\vec{r_1}}{dt}\times\frac{d\vec{r_1}}{dt}$

$\therefore\ \frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \vec{r}\times\frac{d^2\vec{r}}{dt^2}$ $\qquad\left[\because \frac{d\vec{r_1}}{dt}\times\frac{d\vec{r_1}}{dt} = 0\right]$

Without loss of generality, we have removed suffix 1 because both side involve only r.

5. $\frac{d}{dt}[\vec{r_1}\ \vec{r_2}\ \vec{r_3}] = \left[\frac{d\vec{r_1}}{dt}\ \vec{r_2}\ \vec{r_3}\right] + \left[\vec{r_1}\ \frac{d\vec{r_2}}{dt}\ \vec{r_3}\right] + \left[\vec{r_1}\ \vec{r_2}\ \frac{d\vec{r_3}}{dt}\right]$

6. $\frac{d}{dt}[\vec{r_1}\times(\vec{r_2}\times\vec{r_3})] = \frac{d\vec{r_1}}{dt}\times(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\times\left(\frac{d\vec{r_2}}{dt}\times\vec{r_3}\right) + \vec{r_1}\times\left(\vec{r_2}\times\frac{d\vec{r_3}}{dt}\right)$.

Theorem (constant magnitude theorem)

The necessary and sufficient condition for a vector function $\vec{r} = \vec{r}(t)$ of scalar variable to have constant magnitude is $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$.

Proof: Necessary condition,

Let $\vec{r} = \vec{r}(t)$ has constant magnitude, then to show $\vec{r}(t)$ satisfies $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$

We have, $\vec{r}\cdot\vec{r} = r^2 =$ constant, $r = |\vec{r}|$

or, $\vec{r}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{r} = \frac{d(r^2)}{dt}$

or, $\vec{r}\cdot\frac{d\vec{r}}{dt} + \vec{r}\cdot\frac{d\vec{r}}{dt} = \frac{d}{dt}$(constant)

or, $2\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$

$\therefore\ \vec{r}\cdot\frac{d\vec{r}}{dt} = 0$

Sufficient condition, let $\vec{r} = \vec{r}(t)$ satisfies $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$, then to prove $\vec{r}$ has constant magnitude.

We know for any vector $\vec{r} = \vec{r}(t)$,

$\vec{r}\cdot\vec{r} = r^2$

Differentiating both sides with respect to t, we get

$\vec{r}\cdot\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$

But by supposition,

$\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$

$\therefore\ r\frac{dr}{dt} = 0$

Since $r \neq 0$, i.e., $\vec{r} = \vec{r}(t)$ is a non-zero vector.

$\therefore\ \frac{dr}{dt} = 0$.

Which shows that derivative of magnitude r of $\vec{r} = \vec{r}(t)$ is zero. Hence the magnitude of r of $\vec{r} = \vec{r}(t)$ must be constant.

Note: If $\vec{r}$ is any vector then the unit vector $\hat{r}$ along the direction of $\vec{r}$ is given by $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and both $\vec{r}$ and $\hat{r}$ will have same direction.

Theorem: (Constant direction theorem): The necessary and sufficient condition for a function $\vec{r} = \vec{r}(t)$ to have constant direction is $\vec{r} \times \frac{d\vec{r}}{dt} = 0$.

Proof: Necessary condition

Let $\vec{r} = \vec{r}(t)$ be a vector function of scalar variable and has constant direction, then to prove

$$\vec{r} \times \frac{d\vec{r}}{dt} = 0.$$

For any vector $\vec{r}$, we have

$$\vec{r} = r\hat{r} \qquad \ldots(1)$$

where, $r = |\vec{r}|$ and $\hat{r}$ is unit vector along $\vec{r}$.

Differentiating (1) with respect to t

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}.$$

Taking cross-product with $\vec{r} = r\hat{r}$ on both sides.

$$\vec{r} \times \frac{d\vec{r}}{dt} = (r\hat{r}) \times \left(r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right) \qquad [\because \hat{r} \text{ has constant direction, so it is constant}]$$

$$\therefore \quad \vec{r} \times \frac{d\vec{r}}{dt} = (r\hat{r}) \times \left[0 + \frac{dr}{dt}\hat{r}\right]$$

$$= r\frac{dr}{dt}(\hat{r} \times \hat{r})$$

$$\therefore \quad \vec{r} \times \frac{d\vec{r}}{dt} = 0 \qquad [\because \hat{r} \times \hat{r} = 0]$$

Sufficient condition,

Suppose $\vec{r} = \vec{r}(t)$ satisfies $\vec{r} \times \frac{d\vec{r}}{dt} = 0$, then to prove $\vec{r}$ has constant direction.

We have,

$$\vec{r} \times \frac{d\vec{r}}{dt} = 0, \text{ using } \vec{r} = r\hat{r}$$

$$\text{or,} \quad r\hat{r} \times \frac{d(r\hat{r})}{dt} = 0$$

$$\text{or,} \quad r\hat{r} \times \left(r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right) = 0$$

$$\text{or,} \quad r^2\hat{r} \times \frac{d\hat{r}}{dt} + r\frac{dr}{dt}(\hat{r} \times \hat{r}) = 0$$

$$\text{or,} \quad r^2\hat{r} \times \frac{d\hat{r}}{dt} = 0 \qquad [\because \hat{r} \times \hat{r} = 0]$$

Since $r^2 \neq 0$.

$$\therefore \quad \hat{r} \times \frac{d\hat{r}}{dt} = 0 \qquad \ldots(2)$$

Since $\hat{r}$ has constant magnitude 1, so by constant magnitude theorem

$$\hat{r} \cdot \frac{d\hat{r}}{dt} = 0 \qquad \ldots(3)$$

Thus, the same vectors $\hat{r}$ and $\frac{d\hat{r}}{dt}$ satisfy (2) and (3), in (2)) we have sine of angle between $\hat{r}$ and $\frac{d\hat{r}}{dt}$ whereas in (3) the consine of angle between them. But we have no common angle for both sine and cosine to be zero. Hence we must have,

$$\frac{d\hat{r}}{dt} = 0.$$

i.e., derivative of $\hat{r}$ is zero, so it should be a constant vector and hence has both magnitude and direction constant. But $\vec{r}$ and $\hat{r}$ has same direction, so $\vec{r}$ must have constant direction.

4.5.5 Angle betweeen two curves

Let $\vec{r}_1 = \vec{r}_1(t)$ and $\vec{r}_2 = \vec{r}_2(t)$ be any two curves in space, then the angle between the curves at the point of intersection is defined as the angle between the tangential vectors at the common point.

The tangential vector to $\vec{r}_1 = \vec{r}_1(t)$ and $\vec{r}_2 = \vec{r}_2(t)$ are given by

$$\vec{T}_1 = \frac{d\vec{r}_1}{dt}, \vec{T}_2 = \frac{d\vec{r}_2}{dt}$$

If θ is angle between curves (i.e., tangents), then

$$\cos\theta = \frac{\vec{T}_1 \cdot \vec{T}_2}{|\vec{T}_1|\ |\vec{T}_2|}$$

Remark: If $\vec{T}_1 \cdot \vec{T}_2 = 0$, then the two curves are said to be orthogonal curves.

Example 2. A particle moves along the curve $\vec{r} = [x(t), y(t), z(t)] = [2\cos t, 3\sin t, 5t]$, find the velocity and acceleration at $t = 0$ and $t = \frac{\pi}{2}$.

Solution:

Given,

$$\vec{r} = [x(t), y(t), z(t)]$$

$$\vec{r} = 2\cos t\hat{i} + 3\sin t\hat{j} + 5t\hat{k} \qquad \ldots(1)$$

Differenting with respect to t on both sides

$$\text{velocity } \vec{v} = \frac{d\vec{r}}{dt} = -2\sin t\hat{i} + 3\cos t\hat{j} + 5\hat{k} \qquad \ldots(2)$$

$$\text{and acceleration } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = -2\cos t\hat{i} - 3\sin t\hat{j} + 0\hat{k} \qquad \ldots(3)$$

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i. When $t = 0$

From (2) velocity at $t = 0$

$$\hat{v} = \left[\frac{d\vec{r}}{dt}\right]_{t=0} = -2\sin 0\hat{i} + 3\cos 0\hat{j} + 5\hat{k}$$

$$= 3\hat{j} + 5\hat{k}$$

$\therefore$ Magnitude of velocity,

$$|\hat{v}| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25}$$

$$\therefore |\hat{v}| = \sqrt{34}$$

From (3), the acceleration at $t = 0$

$$\vec{a} = -2\cos 0\,\hat{i} - 3\sin 0\,\hat{j} + 0\hat{k}$$

$$\therefore \vec{a} = -2\hat{i}$$

Magnitude of acceleration,

$$\therefore |\vec{a}| = \sqrt{(-2)^2} = 2$$

ii. When $t = \frac{\pi}{2}$

From (2) velocity at $t = \frac{\pi}{2}$

$$\hat{v} = \left[\frac{d\vec{r}}{dt}\right]_{t=\frac{\pi}{2}} = -2\sin\frac{\pi}{2}\hat{i} + 3\cos\frac{\pi}{2}\hat{j} + 5\hat{k}$$

$$= -2\hat{i} + 5\hat{k}$$

$\therefore$ Magnitude of velocity,

$$|\hat{v}| = \sqrt{(-2)^2 + 5^2}$$

$$\therefore |\hat{v}| = \sqrt{29}$$

From (3), the acceleration at $t = \frac{\pi}{2}$

$$\vec{a} = -2\cos\frac{\pi}{2}\hat{i} - 3\sin\frac{\pi}{2}\,\hat{j} + 0\hat{k}$$

$$\therefore \vec{a} = -3\hat{j}$$

Magnitude of acceleration,

$$\therefore |\vec{a}| = \sqrt{(-3)^2} = 3$$

... is a unit vector, prove that $\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$.

... s a unit vector i.e., has constant magnitude. So constant magn...

or, r^2 ... 0

or, $|\vec{r}|\left|\frac{d\vec{r}}{dt}\right|\cos\theta = 0.$

$\therefore \cos\theta = 0 = \cos\frac{\pi}{2}$

Angle between $\vec{r}$ and $\frac{d\vec{r}}{dt}$ is $\theta = \frac{\pi}{2}$.

Now,

$$\vec{r} \times \frac{d\vec{r}}{dt} = |\vec{r}|\left|\frac{d\vec{r}}{dt}\right|\sin\theta\,\hat{n}$$

$$\therefore \left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|\left|\sin\frac{\pi}{2}\right||\hat{n}|$$

$$= \left|\frac{d\vec{r}}{dt}\right| 1 \cdot 1$$

$$\therefore \left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

Example 4. If $\vec{r} = \vec{a}\cos wt + \vec{b}\sin wt$, show that

i. $\vec{r} \times \frac{d\vec{r}}{dt} = w\,\vec{a} \times \vec{b}$ ii. $\frac{d^2\vec{r}}{dt^2} = -w^2\vec{r}$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution

We have

$$\vec{r} = \vec{a}\cos wt + \vec{b}\sin wt \quad \text{... (1)}$$

Differentiating with respect to t

$$\frac{d\vec{r}}{dt} = -\vec{a}\,w\sin wt + \vec{b}\,w\cos wt$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a}\cos wt + \vec{b}\sin wt) \times (-\vec{a}\,w\sin wt + \vec{b}\,w\cos wt)$$

$$= (\vec{a} \times \vec{b})w\cos^2 wt - (\vec{b} \times \vec{a})w\sin^2 wt$$

$$= (\vec{a} \times \vec{b})w\cos^2 wt + (\vec{a} \times \vec{b})w\sin^2 wt$$

$$= (\vec{a} \times \vec{b})w[\cos^2 wt + \sin^2 wt]$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a} \times \vec{b})w$$

Again differentiating (2) with respect to t

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\,w^2\cos wt - \vec{b}\,w^2\sin wt$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = -w^2\vec{r}$$

i. When $t=0$

From (2) velocity at $t=0$

$$\hat{v}=\left[\frac{d\vec{r}}{dt}\right]_{t=0}=-2\sin 0\hat{i}+3\cos 0\hat{j}+5\hat{k}$$

$$=3\hat{j}+5\hat{k}$$

∴ Magnitude of velocity,

$$|\hat{v}| =\sqrt{3^2+5^2}$$
$$=\sqrt{9+25}$$

∴ $|\hat{v}| =\sqrt{34}$

From (3), the acceleration at $t=0$

$$\vec{a} =-2\cos 0\hat{i}-3\sin 0\hat{j}+0\hat{k}$$

∴ $\vec{a} =-2\hat{i}$

Magnitude of acceleration,

∴ $|\vec{a}| =\sqrt{(-2)^2}=2$

ii. When $t=\frac{\pi}{2}$

From (2) velocity at $t=\frac{\pi}{2}$

$$\hat{v}=\left[\frac{d\vec{r}}{dt}\right]_{t=\frac{\pi}{2}}=-2\sin\frac{\pi}{2}\hat{i}+3\cos\frac{\pi}{2}\hat{j}+5\hat{k}$$

$$=-2\hat{i}+5\hat{k}$$

∴ Magnitude of velocity,

$$|\hat{v}| =\sqrt{(-2)^2+5^2}$$

∴ $|\hat{v}| =\sqrt{29}$

From (3), the acceleration at $t=\frac{\pi}{2}$

$$\vec{a} =-2\cos\frac{\pi}{2}\hat{i}-3\sin\frac{\pi}{2}\hat{j}+0\hat{k}$$

∴ $\vec{a} =-3\hat{j}$

Magnitude of acceleration,

∴ $|\vec{a}| =\sqrt{(-3)^2}=3$

Example 3. If $\vec{r}$ is a unit vector, prove that $\left|\vec{r}\times\frac{d\vec{r}}{dt}\right|=\left|\frac{d\vec{r}}{dt}\right|$

Solution

Since $\vec{r}$ is a unit vector i.e., has constant magnitude. So constant magnitude them,

$$\vec{r}\cdot\frac{d\vec{r}}{dt}=0$$

or, $|\vec{r}|\left|\frac{d\vec{r}}{dt}\right|\cos\theta=0.$

∴ $\cos\theta=0=\cos\frac{\pi}{2}$

Angle between $\vec{r}$ and $\frac{d\vec{r}}{dt}$ is $\theta=\frac{\pi}{2}$.

Now,

$$\vec{r}\times\frac{d\vec{r}}{dt} =|\vec{r}|\left|\frac{d\vec{r}}{dt}\right|\sin\theta\,\hat{n}$$

∴ $\left|\vec{r}\times\frac{d\vec{r}}{dt}\right| =\left|\frac{d\vec{r}}{dt}\right|\left|\sin\frac{\pi}{2}\right||\hat{n}|$

$$=\left|\frac{d\vec{r}}{dt}\right|1\cdot 1$$

∴ $\left|\vec{r}\times\frac{d\vec{r}}{dt}\right| =\left|\frac{d\vec{r}}{dt}\right|$

Example 4. If $\vec{r}=\vec{a}\cos wt+\vec{b}\sin wt$, show that

i. $\vec{r}\times\frac{d\vec{r}}{dt}=w\vec{a}\times\vec{b}$ ii. $\frac{d^2\vec{r}}{dt^2}=-w^2\vec{r}$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution

We have

$$\vec{r}=\vec{a}\cos wt+\vec{b}\sin wt \quad \text{... (1)}$$

Differentiating with respect to t

$$\frac{d\vec{r}}{dt} =-\vec{a}w\sin wt+\vec{b}w\cos wt$$

∴ $\vec{r}\times\frac{d\vec{r}}{dt} =(\vec{a}\cos wt+\vec{b}\sin wt)\times(-\vec{a}w\sin wt+\vec{b}w\cos wt)$

$$=(\vec{a}\times\vec{b})w\cos^2 wt-(\vec{b}\times\vec{a})w\sin^2 wt$$
$$=(\vec{a}\times\vec{b})w\cos^2 wt+(\vec{a}\times\vec{b})w\sin^2 wt$$
$$=(\vec{a}\times\vec{b})w[\cos^2 wt+\sin^2 wt]$$

∴ $\vec{r}\times\frac{d\vec{r}}{dt} =(\vec{a}\times\vec{b})w$

Again differentiating (2) with respect to t

$$\frac{d^2\vec{r}}{dt^2} =-\vec{a}w^2\cos wt-\vec{b}w^2\sin wt$$

∴ $\frac{d^2\vec{r}}{dt^2} =-w^2\vec{r}$

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Example 5. If $\vec{r} = \vec{r}(t) = (3t, 3t^2, 2t^3)$, prove that $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = 216$.

Solution

We have

$$\vec{r}(t) = (3t, 3t^2, 2t^3)$$

$$\therefore\quad \vec{r} = \vec{r}(t) = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k} \qquad \ldots (1)$$

Differentiating (i) with respect to t successively,

$$\dot{\vec{r}} = 3\hat{i} + 6t\hat{j} + 6t^2\hat{k}$$

$$\ddot{\vec{r}} = 0\hat{i} + 6\hat{j} + 12t\hat{k}$$

$$\dddot{\vec{r}} = 0\hat{i} + 0\hat{j} + 12\hat{k}$$

Now,

$$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \begin{vmatrix} 3 & 6t & 6t^2 \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix}$$

$$= 3 \times 6 \times 12$$

$$= 216$$

Example 6. Find $\vec{f}'(t)$ if $\vec{f}(t) = (\sin^3 t)\hat{i} + (\ell n\, t)\hat{j} + \tan^{-1}(3t)\,\hat{k}$.

Solution

We have, $\vec{f}(t) = \sin^3 t\hat{i} + \ell n\, t\hat{j} + \tan^{-1}(2t)\hat{k}$

Then, $\vec{f}'(t) = 3\sin^2 t\cos t\,\hat{i} + \frac{1}{t}\hat{j} + \frac{2}{1+4t^2}\hat{k}$.

Here, the function $f(t)$ is defined and its derivative exists at each point on t except at $t = 0$. The the derivative is also defined for all t except at $t = 0$.

Example 7. Let $\vec{v} = (y^3,\ z^2,\ x^3)$ be a vector function. Evaluate all first order partial derivatives of $\vec{v}$.

Solution

We have,

$$\vec{v} = (y^3,\ z^2,\ x^3)$$

$$= y^3\hat{i} + z^2\hat{j} + x^3\hat{k}$$

Then,

$$\frac{\partial \vec{v}}{\partial x} = 3x^2\hat{k},\quad \frac{\partial \vec{v}}{\partial y} = 3y^2\hat{i} \text{ and } \frac{\partial \vec{v}}{\partial z} = 2z\hat{j}$$

Example 8. Find the unit tangent vector any point on the curve $\vec{r}(x, y, z) = (2\cos t, 2\sin t, 2t)$.

Solution

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$. Then,

$$\vec{r} = (2\cos t, 2\sin t, 2t)$$

$$\therefore\quad \frac{d\vec{r}}{dt} = -2\sin t\hat{i} + 2\cos t\hat{j} + 2\hat{k})$$

This represents the tangential vector $\vec{T}$ at any point in the given curve. Now for the unit tangent vector $\hat{T}$ is,

$$\hat{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + 2\hat{k}}{\sqrt{(-2\sin t)^2 + (2\cos t)^2 + 4}}$$

$$= \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + 2\hat{k}}{4(\sin^2 t + \cos^2 t) + 4}$$

$$= \frac{-2\sin t\hat{i} + 2\cos t\hat{j} + 2\hat{k}}{8}$$

$$= \frac{1}{8}(-2\sin t\hat{i} + 2\cos t\hat{j} + 2\hat{k})$$

is the required unit tangent vector.

Exercise 4.1

1. a. If $\vec{r} = e^{at}\vec{u} + e^{-at}\vec{v}$, where $\vec{u}$ and $\vec{v}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - a^2\vec{r} = 0$.

   b. If $\vec{r} = e^{at}\vec{u} + e^{bt}\vec{v}$, where $\vec{u}$ and $\vec{v}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - (a+b)\frac{d\vec{r}}{dt} + ab\vec{r} = 0$.

2. Find the unit tangent vector at any point on the curve

   a. $x = 3\cos t$, $\quad y = 3\sin t$, $\quad z = 4t$

   b. $x = (t^2 - 1)$, $\quad y = 4t - 3$, $\quad z = 2t^2 - 6t$

   c. $x = t^2 + 2$, $\quad y = 4t - 8$, $\quad z = 2t^2 - 6t$ at $t = 2$.

3. Find the angle between the tangents to the curve at the point $t = \pm 1$.

   a. $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ $\qquad$ b. $\vec{r} = t\hat{i} + t^2\hat{j} - t^3\hat{k}$

4. a. The particle moves along the curve $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, where t is the time. Find the velocity and magnitude of the tangential component of its acceleration at time $t = 2$ sec.

   b. A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{\pi}{2}$.

   c. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$ and $z = 3t - 5$. Find the component of its velocity and acceleration at time $t = 1$ sec in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

   d. A particle moves along the curve $\vec{r}(x, y, z) = (e^{-t}, 2\cos t\, 3t, 2\sin t)$ where t is the time. Determine its velocity and acceleration vectors and the magnitude of velocity and acceleration at $t = 0$ sec.

   e. A particle moves along the curve $(t^3 + 1, t^2, 2t + 5)$. Find the component of velocity and acceleration at $t = 1$ along $\hat{i} + \hat{j} + 3\hat{k}$.

5. Find all partial derivatives of the following vector functions.

   a. $\vec{v} = \cos xyz\,(\hat{i} + \hat{j})$ $\qquad$ b. $\vec{v} = (e^x\cos y\hat{i} + e^x\sin y\hat{j})$

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Answer

2. a. $\frac{1}{5}(-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k})$ b. $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$ c. $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$

3. a. $\cos^{-1}\left(\frac{9}{17}\right)$ b. $\cos^{-1}\left(\frac{3}{7}\right)$

4. a. $\vec{v} = 8\hat{i} + 8\hat{j} - 4\hat{k}$, $\vec{a}$ along tangential component = 16 b. $4\hat{j} + 6\hat{k}, -4\hat{i}, 4\hat{i} + 6\hat{k}, -4\hat{j}$

c. $\frac{8\sqrt{14}}{7}, -\frac{\sqrt{14}}{7}$ d. $\sqrt{37}, 5\sqrt{13}$ e. $\sqrt{11}, \frac{8}{\sqrt{11}}$

5. a. $\vec{v}_r = -yz\sin xyz\,i - yz\sin xyz\,j$, $\vec{v}_y = -xz\sin xy\hat{i} - xz\sin xyz\hat{j}$, $\vec{v}_z = -xz\sin xyz\hat{i} - xz\sin xyz\hat{j}$

b. $\vec{v}_r = e^x\cos y\hat{i} + e^x\sin\hat{j}$, $\vec{v}_y = e^x\cos y\hat{i} - e^x\sin y\hat{j}$

4.6 Point Function

A physical quantity that depends on the position (point) or coordinate in three dimension space is known as point function.

a. Scalar point function

A point function which is scalar in nature is known as scalar point function. For example,

i. Temperature at a point in a conducting body.

ii. Density at a point of a body.

A scalar point function is denoted by $\phi = \phi(x, y, z)$ to mean ϕ is a scalar point function of the variables x, y and z.

Moreover, the graph of a function $\phi = \phi(x, y, z)$ is a surface in three dimensional space described by $z = f(x, y)$, for some function f, or equivalently by $\phi(x, y, z) = 0$.

A scalar point function depends on coordinate of a point in space.

Level surface

Let $\phi = \phi(x, y, z)$ be a scalar point function that represents a surface in three dimensional space. Let $P(x_1, y_1, z_1)$ be any point in space such that

$$\phi(x_1, y_1, z_1) = c_1$$

then the locus of $P(x_1, y_1, z_1)$ given by $\phi(x, y, z) = c_1$ is known as level surface of $P(x_1, y_1, z_1)$.

Note: The equation of the locus of the point $P(x_1, y_1, z_1)$ is obtained by generalizing the coordinate $P(x_1, y_1, z_1)$ by (x, y, z) of the general point on the locus.

i. The level surface of $\phi(x, y, z) = x^2 + y^2 + z^2$.

for different points will be a family of concentric spheres. We can think of different layers of onion as different level surfaces given by $\phi(x, y, z) = x^2 + y^2 + z^2$ at different levels determined by the radius of sphere.

ii. The level of surface $\phi(x, y, z) = 2x + 3y + z + 2$

will be a family of parallel planes. We can think of a pile of books on a table as different level plane (surface).

iii. The contour lines in a map are two dimensional level cu

b. Vector Point Function

A point function which is vector in nature is known as vector point function. For example,

i. The velocity, acceleration of a moving particle.

ii. The gravitational force around a celestial body like earth.

A vector point function is denoted by

$$\vec{V} = \vec{V}(x, y, z) \text{ to mean } \vec{V} \text{ is a vector point function.}$$

Moreover, we note that the graph of a vector point function is surface or curve in space that depends on position vector or coordinate.

4.7 Calculus of Point Function

In calculus of point function we study the effect of differential operator known as vector differential operator on point functions.

4.7.1 Vector Differential Operator

A vector differential operator is defined by,

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

∇ *is pronounced as nabla or del.*

4.8 Gradient of a Scalar Point Function

Let $\phi = \phi(x, y, z)$ be a scalar point function then its gradient is defiend by

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

which can also be written as

$$\nabla\phi = \Sigma\hat{i}\frac{\partial\phi}{\partial x}, \text{ with } \Sigma\hat{i}\frac{\partial\phi}{\partial x} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

From the definition, the gradient of scalar point function is a vector point function.

4.8.1 Geometrical Interpretation of Gradient

Let $\phi = \phi(x, y, z)$ ... (1)

be a scalar point function. Let P and Q be any two points, very close to each other such that the level surfaces of P and Q, determined by $\phi = \phi(x, y, z)$ are $\phi(P)$ and $\phi(Q)$ as shown in figure.

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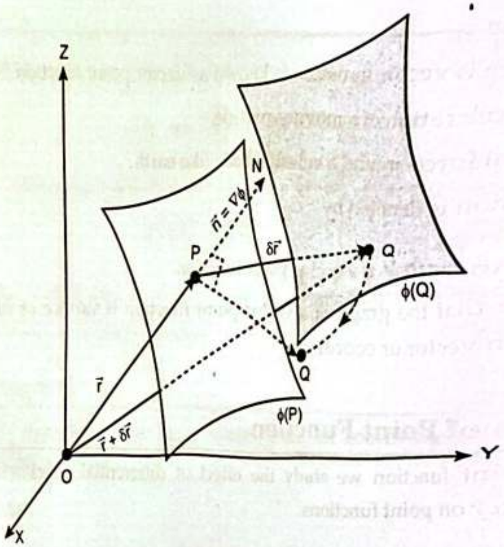


Let the position vectors of P and Q with reference O are $\vec{r}$ and $\vec{r} + \delta\vec{r}$

i.e. $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$

So that, $\overrightarrow{PQ} = \delta\vec{r}$

Also, let $\overrightarrow{OP} = P(x, y, z)$, $\overrightarrow{OQ} = Q(x + \delta x, y + \delta y, z + \delta z)$

Such that

$$\overrightarrow{PQ} = \delta\vec{r} = (\delta x, \delta y, \delta z) = \hat{i}\delta x + \hat{j}\delta y + \hat{k}\delta z$$

If $\nabla\phi$ is gradient of ϕ, then,

$$\nabla\phi \cdot \delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \cdot (\hat{i}\delta x + \hat{j}\delta y + \hat{k}\delta z)$$

$$= \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

$$= d\phi$$

If $Q \to P$, the differential $d\phi$ of ϕ (change in ϕ) will approach to zero and $\overrightarrow{PQ}$ becomes a tangential vector at P to the level surface of P.

i.e. If $Q \to P$, $d\phi \to 0$ and $\delta\vec{r}$ becomes tangential vector at P to its level surface.

$\because \quad \nabla\phi \cdot \delta\vec{r} = 0$

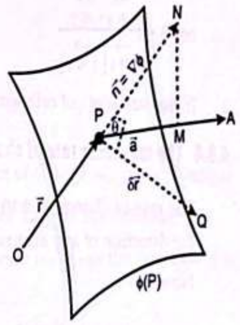
Thus, the gradient $\nabla\phi$ of ϕ gives the rate of change (derivative) of ϕ in normal direction i.e. $\nabla\phi$ at a point P on the level surface of $\phi = \phi(x, y, z)$ gives the rate of change of ϕ in normal direction vector.

Thus, the vector $\vec{n} = \nabla\phi$ is a surface normal vector to the level surface $\phi = \phi(x, y, z)$ at the point P.

4.8.2 Directional Derivative

Let $\phi = \phi(x, y, z)$ be a scalar point function. As we have seen that $\nabla\phi$ gives the rate of change (derivative) of ϕ in normal direction to thee surface $\phi = \phi(x, y, z)$

If we need to measure the rate of change of ϕ in the direction of given vector $\vec{a}$, then we can apply dot product of two vectors $\hat{a}$ and $\nabla\phi$, as dot product gives projection of one vector onto other vector. Also $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ represents unit vector along the direction of $\vec{a}$ so that



$$\nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = |\nabla\phi| \left|\frac{\vec{a}}{|\vec{a}|}\right| \cos\theta = |\nabla\phi| \cos\theta$$

which gives the projection of $\nabla\phi$ (normal rate of change) along the direction of $\vec{a}$. The component of ϕ in the direction of $\vec{a}$ given by $\nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$ is known as directional derivative of ϕ in the direction of vector $\vec{a}$ and is denoted by $D_{\vec{a}}(\phi)$.

i.e. $\quad D_{\vec{a}}(\phi) = \nabla\phi \cdot \hat{a} = \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

Remarks:

1. The gradient $\nabla\phi$ of ϕ at a point gives rate of change (derivative) of ϕ in the normal direction to the *level* surface of the point and $\vec{n} = \nabla\phi$, is known as surface normal vector.

4.8.3 Angle between surfaces

The angle between two surfaces is defined as the angle between the normals at the point of intersection. If $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$ are any two surface and

$\vec{n}_1 = \nabla\phi_1$, $\vec{n}_2 = \nabla\phi_2$

then the angle θ between the surfaces is given by

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Remarks: If $\phi(x, y, z) = 0$ is a surface and $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are any two points on the surface $\phi = \phi(x, y, z)$ and $\vec{n}_1 = [\nabla\phi]_{(x_1, y_1, z_1)}$.

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$\vec{n}_2 = [\nabla\phi]_{(x_2, y_2, z_2)}$

then angles θ between two normal at different points of same surface $\phi(x, y, z) = 0$ is given

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

2. If two surfaces cut orthogonally then $\theta = 90°$ and $\cos\theta = 0$. Therefore, $\nabla\phi_1 \cdot \nabla\phi_2 = 0$.

4.8.4 The maximum rate of change

The rate of change of ϕ in normal direction is $\nabla\phi$. Also, $\nabla\phi \cdot \hat{a}$ gives rate of change of ϕ in the direction of any arbitrary vector.

Now,

$$\nabla\phi \cdot \hat{a} = |\nabla\phi|\ |\hat{a}| \cos\theta$$

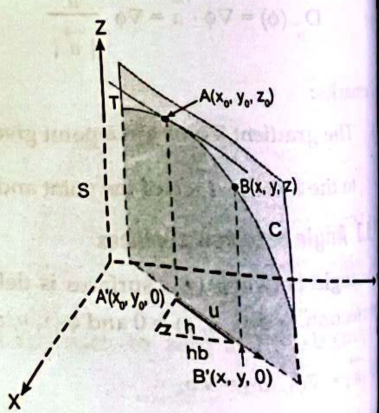
which shows that the rate of change of ϕ in arbitrary direction depends on $\nabla\phi$, $\cos\theta$ and $\hat{a}$.

Thus, $\nabla\phi \cdot \hat{a} = |\nabla\phi| \cos\theta$.

But maximum value of $\cos\theta$ is 1 when $\theta = 0$, so the rate of change of ϕ (directional derivative) will be along the direction of $\nabla\phi$. Hence the maximum rate of change of ϕ is in the normal direction and is given by $\nabla\phi$ and its value is $|\nabla\phi|$.

4.8.5 Directional derivative and the gradient vector

We recall that rate of change of a function is obtained by the derivatives. In this section, we wil discuss a special type of derivative which will enable us to find the rate of change of a function of two or more variables in any direction. This special type of derivative is called the directional derivative. We now define directional derivative.



Let $z = f(x, y)$ be a given function of two variables x and y and $\vec{u} = a\hat{i} + b\hat{j}$ be a unit vector. We want to find the rate of change of z in the direction of $\vec{u}$. Let S denote the surface of $z = f(x, y)$ and take a fixed point $z_0 = f(x_0, y_0)$. Thus (x_0, y_0, z_0) is a point on S. Denote it by $A(x_0, y_0, z_0)$. Then the intersection of S and a vertical plane through the point $A(x_0, y_0, z_0)$ is a curve C as shown in figure.

Then the slope of tangent at $A(x_0, y_0, z_0)$ to the curve C gives the rate of change of z in the directi $\hat{n}$. We now find this precisely. Let $B(x, y, z)$ be another point on C and A' and B' are the projecti A and B on xy-plane as shown in figure. Clearly, vector $\vec{A'B'}$ is parallel to $\hat{n}$ and hence,

$$\vec{A'B'} = h\vec{u} = h(a\hat{i} + b\hat{j}) = ha\hat{i} + hb\hat{j}, \text{ for some scalar } h.$$

Now,

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x, y) - f(x_0, y_0)}{h}$$

Here, $x - x_0 = ha$ and $y - y_0 = hb$

$\therefore\ x = x_0 + ha,\ y = y_0 + hb$

So, $\dfrac{\Delta z}{h} = \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

If we take the limit of average change as $h \to 0$, this gives the rate of change of z in the direction of $\vec{u}$ and is the directional derivative of f in the direction of $\vec{u}$. Now we give the definition.

Definition: The directional derivative of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\vec{u} = a\hat{i} + b\hat{j}$, denoted by $D_u f(x_0, y_0)$, is defined as

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$ provided the limit exists.

Using chain rule, we can write the directional derivative of $z = f(x, y)$ in the direction of unit vector $\vec{u} = (a, b)$ as

$$D_u f(x, y) = \frac{\partial f(x, y)}{\partial x} a + \frac{\partial f(x, y)}{\partial y} b.$$

$\therefore\ D_u f(x, y) = f_x a + f_y b$

So, we can write directional derivative as

$$D_u f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$
$$= (f_x\hat{i} + f_y\hat{j}) \cdot (a\hat{i} + b\hat{j})$$
$$= (f_x, f_y) \cdot \vec{u}.$$

The first term in the dot product occurs in various other contexts and is popularly known as gradient of f. We next define gradient.

We next introduce a notation for vector differential operator.

Definition: The vector differential operator ∇ (read as 'del' or 'nabla') is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The vector operator ∇ can be considered to behave as a ordinary vector. It possesses properties like that of the ordinary vector. The symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its component along $\hat{i}, \hat{j}$ and $\hat{k}$.

Definition: If f is a function of two variables x and y, then the gradient of f, denoted by ∇f or grad f, is a vector function defined by

$$\nabla f(x, y) = \frac{\partial f(x, y)}{\partial x}\hat{i} + \frac{\partial f(x, y)}{\partial y}\hat{j}$$

Thus directional derivative of $z = f(x, y)$ in the direction of unit vector $\vec{u}$ in terms of grad
vector is given by

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

Also, $D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$

$$= |\nabla f(x, y)|\ |\vec{u}| \cos\theta$$

$$= |\nabla f(x, y)| \cos\theta \quad \ldots(1)$$

Thus the directional derivative of $f(x, y)$ in the direction of unit vector $\vec{u}$ is the scalar projec
of $\nabla f(x, y)$ onto the direction of unit vector $\vec{u}$.

For function $f(x, y, z)$ of three variables, we have gradient of f is

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

and directional derivative in the direction of unit vector $\vec{u}$ is

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$$

Now, by equation (1), we have

geometrical meaning?

$$D_u f = \nabla f \cdot \vec{u}$$

$$= |\nabla f|\ |\vec{u}| \cos\theta$$

$$= |\nabla f| \cos\theta.$$

Here, θ is the angle between the gradient of f and unit vector $\vec{u}$. We know that maximum va
of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Then the maximum value of $D_u f$ is $|\nabla f|$ and it occ
when the gradient of f and $\vec{u}$ have the same direction. Thus, the gradient vector ∇f gives
direction of fastest increase of f.

Let $f(x, y, z) = k$ denote a level surface of f of variables x, y, z. Then we show that the gradi
vector at a point (x_0, y_0, z_0) given by $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}'(t$
any curve C on the given surface S that passes through (x_0, y_0, z_0) as shown in figure. T
$\nabla f(x_0, y_0, z_0)$ is always orthogonal to the level surface S of f through (x_0, y_0, z_0).

Example 9. Find the directional derivative of $\phi = 4x^2 + 3y - 4z$ at (1, 2, 1) in the direction $2\hat{i} + 2\hat{j} + \hat{}$

Solution

The given surface is $\phi = 4x^2 + 3y - 4z$ and the direction is $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$.

$$\therefore \quad \hat{a} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4 + 4 + 1}} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{9}} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{3}$$

$$\text{Now, } \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2 + 3y - 4z) = 8x\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\text{At } (1, 2, 1),\ \nabla\phi|_{(1,2,1)} = (8x\hat{i} + 3\hat{j} + 4\hat{k})_{(1,2,1)} = 8\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\therefore \quad \text{Directional derivative } D_{\vec{a}}(\phi) = \nabla\phi \cdot \hat{a} = (8\hat{i} + 3\hat{j} - 4\hat{k}) \cdot \left(\frac{2\hat{i} + 2\hat{j} + \hat{k}}{3}\right) = \frac{16 + 6 - 4}{3} = \frac{18}{3}$$

$$\therefore \quad \text{Directional derivative} = D_{\vec{a}}(\phi) = \frac{18}{3}.$$

Example 10. Find the greatest rate of increase of $\phi = x^2y^3z^2$ at the point (1 −1 0).

Solution

We have $\phi = x^2y^3z^2$

The maximum rate of change of ϕ is given by $\nabla\phi$.

$$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^3z^2) = 2xy^3z^2\hat{i} + 3x^2y^2z^2\hat{j} + 2x^2y^3z\hat{k}$$

Then, $\nabla\phi|_{(1,-1,0)} = 0 + 0 + 0 = 0.$

Example 11. Find the unit vector normal to the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

We have $x^2 + y^2 + z^2 = 1$

$\therefore \quad \phi = x^2 + y^2 + z^2 - 1$

The normal vector $\vec{n}$ to the surface is $\nabla\phi$,

$$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 1) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{1} = 2$$

$$\text{Therefore unit surface normal } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2}$$

$$\therefore \quad \hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$$

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Example 12. Find the rate of change of $f(x, y, z) = xyz$ in the direction normal to the surface $xy + yz + z$... at the point $(1, -1, 1)$.

Solution

We have,

$$f = xyz$$

$$\therefore \quad \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz)$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\text{So, } \nabla f_{(1,-1,1)} = (-1)\cdot 1\hat{i} + 1\cdot 1\hat{j} + 1(-1)\hat{k}$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

Let $\psi(x, y, z)$ denotes the surface, then

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx - 3)$$

$$= (y + z)\hat{i} + (x + z)\hat{j} + (y + x)\hat{k}$$

$$\therefore \quad \nabla\psi|_{(1,-1,1)} = (-1 + 1)\hat{i} + (1 + 1)\hat{j} + (-1 + 1)\hat{k}$$

$$= 2\hat{j}$$

$$= \vec{a} \text{ (say)}$$

So, the directional derivative of $f(x, y, z)$ in the direction normal to the surface $\psi(x, y, z)$ at $(1, -1, 1)$ is

$$\nabla f \cdot \frac{\vec{a}}{|\vec{a}|} = (-\hat{i} + \hat{j} - \hat{k}) \cdot \frac{2\hat{j}}{(\sqrt{2})^2}$$

$$= \frac{2}{2} = 1.$$

Example 13. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution

Let $\phi(x, y, z) = x^2 + y^2 + z^2 - 9$

The normal to the surface ϕ is,

$$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla\phi|_{(2,-1,2)} = 2\cdot 2\hat{i} + 2(-1)\hat{j} + 2(2)\hat{k}$$

$$= 4\hat{i} - 2\hat{j} + 4\hat{k} \qquad \ldots(1)$$

The normal to the surface ϕ is,

Again, let $\psi(x, y, z) = x^2 + y^2 - z - 3$

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\therefore \quad \nabla\psi|_{(2,-1,2)} = 2\times 2\hat{i} + 2(-1)\hat{j} - \hat{k}$$

$$= 4\hat{i} - 2\hat{j} - \hat{k} \qquad \ldots(2)$$

Let θ be the angle between normals (1) and (2), then

$$\cos\theta = \frac{\nabla\phi}{|\nabla\phi|} \cdot \frac{\nabla\psi}{|\nabla\psi|}$$

$$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k})}{\sqrt{(4^2) + (-2)^2 + (4)^2}} \cdot \frac{4\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{(4^2) + (-2)^2 + (1)^2}}$$

$$= \frac{16 + 4 - 4}{6\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

$\therefore$ Required angle is $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

4.9 Application of ∇ to vector fields

In this section, we discuss two operations that can be performed using vector differential operator ∇ on the vector fields. These operations enable us to apply vector calculus in the fluid flow, in the field of magnetism and many more. These two operations are curl and divergence. Each of these operations is similar to differentiation but one gives a vector field whereas the other produces a scalar field.

4.10 Divergence of a vector field

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field on R^3 such that the partial derivatives $\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}$ and $\frac{\partial F_3}{\partial z}$ exist. Then the divergence of $\vec{F}$ denoted by div($\vec{F}$), is a scalar function given by

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$\text{i.e., } \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Thus divergence of $\vec{F}$ *is the scalar product of* ∇ *and* $\vec{F}$.

4.10.1 Second order properties of divergence

1. If ϕ is a scalar point function having second order partial derivatives, then

$$\text{div (grad }\phi) = \nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

$$\text{We have, } \nabla\cdot(\nabla\phi) = \nabla\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

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$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$\therefore$ $\nabla \cdot (\nabla\phi) = \nabla^2\phi.$

Where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

The second order differential operator ∇^2 is known as Laplace operator.

2. If $\vec{F}$ is a vector field then curl $\vec{F}$ is also vector field and hence we can find divergence of curl of $\vec{F}$.

i.e., div (curl $\vec{F}$) $= \nabla . (\nabla \times \vec{F})$

$$\therefore \quad \nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

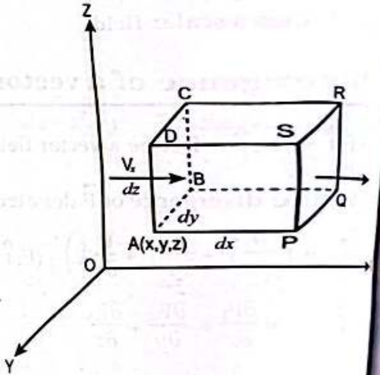
$$= 0$$

This shows that div(curl $\vec{F}$) = 0, which will be used in numerous problems.

4.10.2 Physical interpretation of divergence

If $\vec{V}$ is the velocity of the vector field of the fluid and $\vec{r}$ represent the position vector of the point A in the flow

$$\frac{d\vec{r}}{dt} = \vec{V}.$$



Let us consider the case of a fluid flowing in a small rectangular parallelopiped of dimensions dx, dy, dz parallel to x, y and z axes respectively.

Let $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be the velocity of the fluid at $P(x, y, z)$.

For the flow of fluid in x-direction,

$\therefore$ Mass of fluid flowing in through the face ABCD in unit time

= Velocity × Area of face ABCD

$= V_x\, dy\, dz.$

Similarly mass of fluid flowing out across the face PQRS per unit time

$= V_x (x + dx)\, dy\, dz$, where is the velocity particle at distance $(x + dx)$ and $V_x(x + dx)$ as a function

Expanding V_x using Taylor series up to second term

$$= \left[V_x + \frac{\partial V_x}{\partial x} dx\right] dy\, dz.$$

Net change in mass of fluid in the parallelopiped due to flow in x-direction per unit time

$$= \left[V_x + \frac{\partial V_x}{\partial x} dx\right] dy\, dz - V_x dy dz$$

$$= \frac{\partial V_x}{\partial x} dx\, dy\, dz.$$

Similarly, the change in mass of fluid along y-direction

$$= \frac{\partial V_y}{\partial y} dx\, dy\, dz$$

and along z-direction $= \frac{\partial V_z}{\partial z} dx\, dy\, dz$.

$\therefore$ Total change in the mass of fluid per unit time

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\, dy\, dz$$

Thus, the rate of change in mass per unit volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (V_x\hat{i} + V_y\hat{j} + V_z\hat{k})$$

$$= \nabla \cdot \vec{V} = \text{div}\,(\vec{V})$$

$\therefore$ div $\vec{V}$ gives the rate of change of mass per unit volume.

Thus, if $\vec{V}(x, y, z)$ is the velocity of a fluid, then divergence of $\vec{V}$ gives the net rate of change with respect to time of the mass of fluid flowing from the point (x, y, z) per unit volume. Thus divergence measures the tendency of the fluid to diverge from the point. Because of this diverging tendency, it is called the divergence.

Note:

1. Divergence of a vector point function is a scalar quantity.
2. Physically the divergence of a vector field $\vec{F}$ gives its flux, i.e., rate of flow per unit time per unit volume (for motion in space) or per unit area (for motion in plane).
3. A vector point function $\vec{F}$ is said to be solenoidal if $\nabla \cdot \vec{F} = 0$.
4. If $\vec{V}$ is a constant vector then div $\vec{V} = 0$.

4.11 Curl of a vector field

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field in R^3 such that the partial derivatives of F_1, F_2 and F_3 all exist. Then the curl of $\vec{F}$ is the vector field in R^3 is defined by,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

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i.e., $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \quad \nabla \times \vec{F} = \text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

Remark: If f is a scalar field then its gradeint is a vector field, so we can compute its curl.

i.e., curl $\nabla f = \nabla \times \nabla f$,

$$\therefore \quad \nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0$$

This shows that curl of a gradient of a scalar function is always zero. This can be used redefine a conservative field. We recall that a vector field $\vec{F}$ is called conservative if the work done by a $\vec{F}$ in moving from one point to another point does not depend on path joining the points but on the points only. In such case $\vec{F}$ can be written as ∇f for some scalar point function.

$\vec{F} = \nabla f$ for some scalar function f.

So, curl of $\vec{F} = \nabla \times \vec{F} = \nabla \times \nabla f = 0$.

Thus, we see that for field $\vec{F}$ to be conservative $\vec{F}$ satisfies $\nabla \times \vec{F} = \nabla \times \nabla f = 0$.

4.11.1 Physical interpretation of curl of a vector

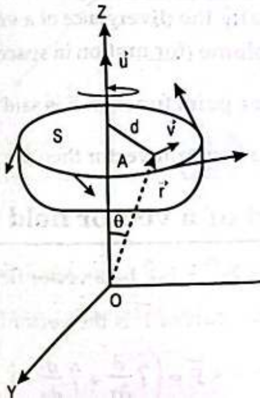
Here we show a connection of curl vector with rotation.

Let us consider a rigid body S rotating about z-axis. This motion of S can be described by

$\vec{u} = \omega\hat{k}$, where ω is the angular speed of S.

We recall that angular speed ω is the ratio of tangential speed and distance d from the z-axis.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of point A as shown in figure and θ be the angle between $\vec{r}$ and z-axis. Let $\vec{v}$ denote the velocity of S.



We have,

$$\omega = \frac{|\vec{v}|}{d}.$$

Also,

$$\sin\theta = \frac{d}{|\vec{r}|} \Rightarrow d = |\vec{r}| \sin\theta$$

So, $|\vec{v}| = \omega d = \omega |\vec{r}| \sin\theta$

$$= |\vec{u}|\,|\vec{r}| \sin\theta$$

$$= |\vec{u} \times \vec{r}|$$

Here, $\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{r}$, so that $\vec{v} = \vec{u} \times \vec{r}$

So, $\vec{v} = \vec{u} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix}$

$$= \hat{i}(-\omega y) - \hat{j}(-x\omega) + \hat{k}(0)$$

$$= -\omega y\hat{i} + x\omega\hat{j}.$$

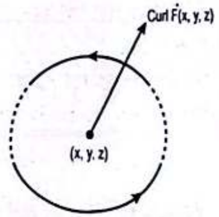
Now, we find curl $\vec{v} = \nabla \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}.$$

$$= \hat{i}(0) - \hat{j}(0 - 0) + \hat{k}(\omega + \omega)$$

$$= 2\omega\hat{k}$$

$$= 2\vec{u}.$$



This shows that curl of tangential velocity (linear velocity) vector is equal to two times the magnitude of rotational velocity vector. Thus curl vector is associated with rotations. This can be explained with another example as follows. Let $\vec{F}(x, y, z)$ denote the velocity field in a fluid flow.

Fluid particles near the point (x, y, z) tend to rotate about axis pointing in the direction of curl $\vec{F}(x, y, z)$ and the length of this curl vector gives the measurement of how fast the particle move around this axis. Thus, if curl $\vec{F}(x, y, z) = 0$ at point P, then the particles are free from rotations at P. i.e., there is no whirlpool at P.

Irrotationa: A vector field $\vec{F}$ is said to be irrotational if curl $\vec{F} = \nabla \times \vec{F} = 0$.

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Example 14. Find the divergence of the vector $\vec{v} = (x^2 + yz)\vec{i} + (y^2 + zx)\vec{j} + (z^2 + xy)\vec{k}$.

Solution

We have,

$$\vec{v} = (x^2 + yz)\vec{i} + (y^2 + zx)\vec{j} + (z^2 + xy)\vec{k}$$

Then, div $\vec{v} = \nabla \cdot \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x^2 + yz)\vec{i} + (y^2 + zx)\vec{j} + (z^2 + xy)\vec{k}]$$

$$= 2x + 2y + 2z$$

$\therefore$ div $\vec{v} = 2(x + y + z)$.

gradient F(all) (i, j, k)
divergence (x) (y) (z)
(value) sep
$\frac{\partial f}{\partial}$

Example 15. If $\vec{v} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$, find div $\vec{v}$.

Solution

We have,

$$\vec{v} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

unit vector $\dfrac{\nabla\phi}{|\nabla\phi|}$

Then, div $\vec{v}$

$= \nabla \cdot \vec{v}$

dire. de
$D_{\vec{a}}(\phi) = \text{grad} f \cdot \dfrac{\vec{a}}{|\vec{a}|}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$= \frac{\partial}{\partial x}\left\{\frac{x}{(x^2 + y^2 + z^2)^{1/2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{(x^2 + y^2 + z^2)^{1/2}}\right\} + \frac{\partial}{\partial z}\left\{\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right\}$$

$$= \frac{(x^2 + y^2 + z^2)^{1/2} - x \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x}{(x^2 + y^2 + z^2)} + \frac{(x^2 + y^2 + z^2)^{1/2} - y \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y}{(x^2 + y^2 + z^2)} +$$

$$\frac{(x^2 + y^2 + z^2)^{1/2} - z \cdot \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2z}{(x^2 + y^2 + z^2)}$$

Scalor pot
$\vec{D} = a(\nabla)\phi$

$$= \frac{(x^2 + y^2 + z^2) - x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - y^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(x^2 + y^2 + z^2) - z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$d\phi = \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy \ldots$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$

integel

$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

Example 16. Find the curl $\vec{v}$ if $\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$ at (2, –1, 1).

Solution

We have, $\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$

Then, curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times [xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & (xz^2 - y^2z) \end{vmatrix}$$

$$= -2yz\vec{i} - (z^2 - xy)\vec{j} + (6xy - xz)\vec{k}$$

At (2, –1, 2)

$$(\nabla \times v)\big|_{(2,-1,1)} = -2(-1)1\vec{i} + \{2(-1) - 1\}\vec{j} + \{6(2)(-1) - 2(1)\}\vec{k}$$

$$= 2\vec{i} - 3\vec{j} - 14\vec{k}$$

Exercise 4.2

1. Find grad ϕ at the given points.
   a. $\phi = \ln(x^2 + y^2 + z^2)$ at P(1, 1, 1)
   b. $\phi = x^2 + y - z - 1$ at P(1, 0, 0)
   c. $\phi = 3x^2y - y^3z^2$ at P(1, –2, –1)
2. Find the unit normal vector to the surface.
   a. $xy^3z^2 = 4$ at (–1, –1, 2)
   b. $z = x^2 + y^2$ at (–1, –2, 5)
   c. $x^2 + y^2 - 2z + 3 = 0$ at (1, 2, –1)
3. a. Find the angle between the normal planes to the surface $x\log z - y^2 + 1 = 0$ and $2 - z - x^2y = 0$ at (1, 1, 1).
   b. Find the angle between the surfaces $xy^2z - 3x - z^2 = 0$ and $3x^2 + 2z = 1 + y^2$ at (1, –2, 1).
4. Find the directional derivatives of the given functions at point P in the given direction.
   a. $\vec{f} = xyz$ at P(–1, 1, 3) along $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$.
   b. $\vec{f} = 2xy + z^2$ at P(1, –1, 3) along $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$.
   c. $\vec{f} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ at P(3, 0, 4) along $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.
   d. $\vec{f} = xyz^2$ at P(1, 0, 3) along $\hat{i} - \hat{j} + \hat{k}$.
   e. $\vec{f} = x^2yz + 4xz^2$ at P(1, –2, 1) in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.
   f. $\vec{f} = x^2 + 3y^2 + 4z^2$ at P(1, 0, 1) in the direction $-\hat{i} - \hat{j} - \hat{k}$.
   g. $\vec{f} = xy + yz + zx$ at P(1, 2, 0) in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
   h. $\vec{f} = x^2y^2z^2$ at (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.
   i. $\vec{f} = \sqrt{x^2 + y^2 + z^2}$ at P(3, 1, 2) in the direction of $yz\hat{i} + zx\hat{j} + y\hat{k}$.
   j. $\vec{f} = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at (–1, 2, 1).
5. Find the divergence of the following vectors.
   a. $\vec{v} = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$
   b. $\vec{v} = e^x\cos y\hat{i} + e^x\sin y\hat{j}$
   c. $\vec{v} = 3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k}$ at (1, 2, 3)
   d. $\vec{v} = xyz\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at (2, –1, 1)
6. Find the curl of the following vectors.
   a. $\vec{v} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$
   b. $\vec{v} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$
   c. $\vec{v} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$
   d. $\vec{v} = (x - y)^2\hat{i} + (y - z)^2\hat{j} + (z - x)^2\hat{k}$

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7. Prove that $\vec F = (y^2 - z^2 + 3yz - 2x)\hat i + (3xz + 2xy)\hat j + (3xy - 2xz + 2z)\hat k$ is both solenoidal and irrotational.

8. a. If $\vec v = x^2yz\hat i + xy^2z\hat j + xyz^2\hat k$ find

   i. curl (grad ϕ)   ii. curl (curl $\vec v$)

   iii. div (curl $\vec v$)

   b. If $\phi = x^3 + y^3 + z^3 - 3xyz$, find:

   i. div (grad ϕ)   ii. curl (grad ϕ)

Answer

1. a. $\frac{2}{3}(\hat i + \hat j + \hat k)$ b. $2\hat i + \hat j - \hat k$ c. $-12\hat i - 9\hat j - 16\hat k$

2. a. $-\frac{1}{\sqrt{11}}(\hat i + 3\hat j - \hat k)$ b. $-\frac{1}{\sqrt{21}}(2\hat i + 4\hat j + \hat k)$ c. $\frac{2\vec i + 4\vec j - 2\vec k}{\sqrt{24}}$

3. a. $\cos^{-1}\frac{1}{\sqrt{30}}$ b. $\cos^{-1}\frac{1}{7\sqrt6}$

4. a. $\frac{7}{3}$ b. $\frac{14}{3}$ c. $-\frac{7}{125\sqrt3}$ d. $-3\sqrt3$ e. $-\frac{13}{3}$

   f. $2\sqrt3$ g. $\frac{37}{3}$ h. $\frac{6}{\sqrt5}$ i. $\frac{18}{7\sqrt{14}}$ j. $\frac{15}{\sqrt{17}}$

5. a. $6xyz$ b. $2e^x \cos y$ c. 80 d. 14

6. a. 0 b. $-(x+y)\hat i + (y+z)\hat k$ c. 0

   d. $2(y-z)\hat i + (z-x)\hat j + (x-y)\hat k$

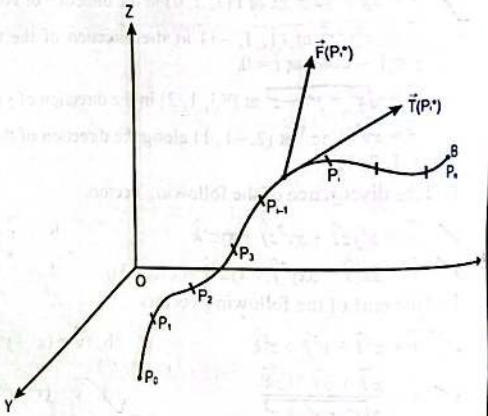
8. a. i. $x(z^2 - y^2)\hat i + y(x^2 - z^2)\hat j + z(y^2 - x^2)\hat k$ ii. $4[yz\hat i + 2xj + xy\hat k]$ iii. 0

   b. i. $6(x+y+z)$ ii. 0

4.12 Vector Integral Calculus

4.12.1 Line integral

Any integral which is evaluated over a curve C is called line integral. Line integral is a simple and natural generalization of definite integral.



Let C be a curve and $\vec r(t)$ denote parametrization of C for $a \le t \le b$ such that $\vec r(a)$ is the initial point and $\vec r(b)$ is the final or terminal point of C. The direction of C from A to B, in which t increases, is called the positive direction on C. Let $\vec F$ be the given vector field which we want to integrate over C. We now divide the interval $[a, b]$ into n sub-intervals $[t_{i-1}, t_i]$, $0 \le i \le n$ of equal width. We denote the end points $\vec r(t_0), \vec r(t_1), \ldots, \vec r(t_n)$ by $P_0, P_1, P_2, \ldots$ and P_n. Let Δs_i denote the arc length from P_{i-1} to P_i. Now for each i, choose a value t_i^* in the $[t_{i-1}, t_i]$. Let $\vec T(P_i^*)$ denote the unit tangent vector and $\vec F(P_i^*)$ is the direction of vector field at point P_i^* lying on the arc from P_{i-1} to P_i.

Take the dot product $\vec F(P_i^*) \cdot \vec T(P_i^*)$ which is a scalar quantity and multiply it by Δs_i and form a sum given by $\sum_{i=1}^{n} [\vec F(P_i^*) \cdot \vec T(P_i^*)]\, \Delta s_i$.

We then take the lim as $n \to \infty$. If this limit exists, then it is called line integral of $\vec F$ over C and is denoted by $\int_C \vec F \cdot \vec T\, ds$.

Hence, $\int_C \vec F \cdot \vec T\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} [\vec F(P_i^*) \cdot \vec T(P_i^*)]\, \Delta s_i$.

Here unit tangent vector $\vec T$ is also given by

$$\vec T = \frac{\vec r'(t)}{|\vec r'(t)|}.$$

So, $(\vec F \cdot \vec T)ds = \vec F \cdot \frac{\vec r'(t)}{|\vec r'(t)|}\, ds.$

Also, $ds = |\vec r'(t)|\, dt$.

We have,

$$(\vec F \cdot \vec T)ds = \vec F \cdot \vec r'(t)\, dt$$

Hence line integral is $\int_C \vec F \cdot \vec T\, ds = \int_C \vec F(\vec r(t)) \cdot \vec r'(t)dt$

Since $\frac{d\vec r(t)}{dt} = \vec r'(t)$, we can write $d\vec r(t) = \vec r'(t)\, dt$.

So the line integral is also expressed as

$$\int_C \vec F \cdot \vec T\, ds = \int_C \vec F(\vec r(t)) \cdot \vec r'(t)dt$$

$$= \int_C \vec F(\vec r(t)) \cdot d\vec r$$

$$= \int_C \vec F \cdot d\vec r$$

The notation $\int_C \vec F \cdot d\vec r$ is commonly used for the line integral.

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4.12.2 Cartesian form of line integral $\int_C \vec{F}\cdot d\vec{r}$

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

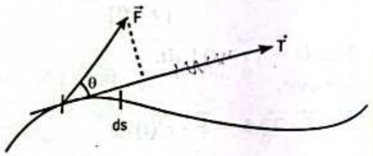
Thus, $\int_C \vec{F}\cdot d\vec{r} = \int_C (F_1dx + F_2dy + F_3dz)$.

This form of line integral is called Cartesian form of line integral.

4.12.3 Physical meaning of line integral $\int_C \vec{F}\cdot d\vec{r}$

Let $\vec{F}$ denote a force field acting on a particle which moves on a path C starting from a point A to another point B on C. Let $\vec{r}(t)$ denote the position vector of a point on path C. Then the vector $\vec{T} = \frac{d\vec{r}}{ds}$ is a unit vector along the tangential direction as shown in figure.

Now effect of force $\vec{F}$ along the tangent $\vec{T}$ is given by



$|\vec{F}| \cos\theta$

$= |\vec{F}|\ |\vec{T}| \cos\theta \qquad (\text{since } |\vec{T}| = 1)$

$= \vec{F}\cdot\vec{T}$

$= \vec{F}\cdot\frac{d\vec{r}}{ds}$

We take ds to be very small so that ds is almost like a straight line along the direction of $\vec{T}$. Then $\left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds$ gives the work done by $\vec{F}$ on the length ds. Then the total work done on the path C is given by

$$\int_C \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F}\cdot d\vec{r}$$

which is line integral.

This shows that if $\vec{F}$ is a force field, then the line integral $\int_C \vec{F}\cdot d\vec{r}$ gives the to[...]one by $\vec{F}$ on path C.

4.12.4 Line integral as path independent

Let A and B be two points in space. There are various paths from A to B. A line integral $\int_C \vec{F}\cdot d\vec{r}$ is said to be path independent if the value of the integral is same for all possible paths starting from point A to B. Not all of line integral of vector field $\vec{F}$ possess this kind of property. The following theorem gives the criteria for a line integral to be path independent.

Theorem 1: A line integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent if and only if $\vec{F} = \nabla\phi$ for some scalar function ϕ.

Proof: We first suppose that $\vec{F} = \nabla\phi$ for some scalar function ϕ. We then show that $\int_C \vec{F}\cdot d\vec{r}$ is path independent. Let A and B be any two points in space and C be a randomly taken path from A to B. Let C be given by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \le t \le b$.

So, $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r}$

$= \int_C \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

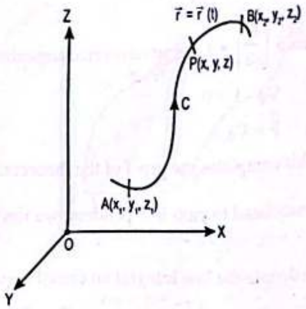
$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$

$= \int_A^B \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right) dt$

$= \int_A^B \frac{d\phi}{dt} dt$

$= [\phi(x(t), y(t), z(t))]_A^B$

$= \phi(B) - \phi(A).$



This shows that the value of $\int_C \vec{F}\cdot d\vec{r}$ is simply the difference of value of ϕ at the end points A and B of path and hence $\int_C \vec{F}\cdot d\vec{r}$ is path independent.

Conversely, suppose that $\int_C \vec{F}\cdot d\vec{r}$ is path independent. We note that $\vec{F}$ is the gradient of some scalar function ϕ. Let $A(x_1, y_1, z_1)$ be the initial point. Then for some point $P(x, y, z)$, $\int_{A(x_1, y[...]}^{P(x,y,z)} \vec{F}\cdot d\vec{r}$ defines a function of x, y, z, say, $\phi(x, y, z)$. So,

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$$\phi(x, y, z) = \int_{A(x_1, y_1, z_1)}^{P(x, y, z)} \vec{F} \cdot d\vec{r}$$

$$\phi(x, y, z) = \int_{A(x_1, y_1, z_1)}^{P(x, y, z)} \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds$$

Differentiating both sides with respect to arc length s.

$$\Rightarrow \frac{d\phi}{ds} = \vec{F} \cdot \frac{d\vec{r}}{ds}.$$

$$\text{So, } \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \vec{F} \cdot \frac{d\vec{r}}{ds}$$

$$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right) = \vec{F} \cdot \frac{d\vec{r}}{ds}$$

$$\nabla\phi \cdot \frac{d\vec{r}}{ds} = \vec{F} \cdot \frac{d\vec{r}}{ds}$$

$$\therefore \quad \nabla\phi \cdot \frac{d\vec{r}}{ds} - \vec{F} \cdot \frac{d\vec{r}}{ds} = 0$$

$$(\nabla\phi - \vec{F}) \cdot \frac{d\vec{r}}{ds} = 0$$

Since $\left|\frac{d\vec{r}}{ds}\right| = 1$, being non-zero tangential vector, we have

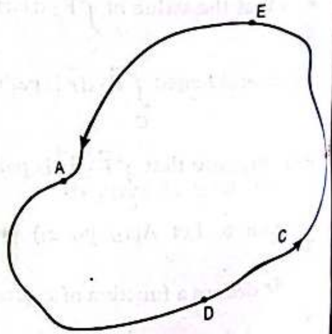
$$\nabla\phi - \vec{F} = 0$$

$$\therefore \quad \vec{F} = \nabla\phi.$$

This completes the proof of the theorem.

Associated to path independent, we next discuss another theorem. We use the notation $\oint_C \vec{F} \cdot$ to denote the line integral on closed region C.

Theorem 2: A line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path joining any two points A and B in space if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ on every simple closed curve C.



Proof: Suppose that $\int \vec{F} \cdot d\vec{r}$ is independent of path joining two points A and B in space. We then show $\oint_C \vec{F} \cdot d\vec{r} = 0$ on every simple closed curve C such that A and B lie on the closed curve C. Let C be such simple closed curve as shown in figure.

We denote C by ADBEA.

Now,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{ADBEA} \vec{F} \cdot d\vec{r}$$

$$= \int_{ADB} \vec{F} \cdot d\vec{r} + \int_{BEA} \vec{F} \cdot d\vec{r}$$

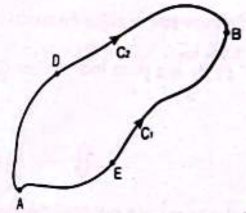
$$= \int_{ADB} \vec{F} \cdot d\vec{r} - \int_{AEB} \vec{F} \cdot d\vec{r}$$

But ADB and AEB are two paths from A to B and $\int \vec{F} \cdot d\vec{r}$ is path independent joining A and B.

$$\text{So, } \oint_C \vec{F} \cdot d\vec{r} = \int_{ADB} \vec{F} \cdot d\vec{r} - \int_{BEA} \vec{F} \cdot d\vec{r} = 0$$

Conversely, we suppose that line integral on every simple closed curve C is zero. i.e., $\oint_C \vec{F} \cdot d\vec{r} = 0.$

We then show $\int \vec{F} \cdot d\vec{r}$ is independent of path joining A and B. Let C_1 and C_2 be any two paths from A to B as shown in figure.



Then by assumption, we have

$$\oint_{AEBDA} \vec{F} \cdot d\vec{r} = 0.$$

$$\int_{AEB} \vec{F} \cdot d\vec{r} + \int_{BDA} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{AEB} \vec{F} \cdot d\vec{r} - \int_{ADB} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{AEB} \vec{F} \cdot d\vec{r} = \int_{ADB} \vec{F} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

This shows that $\int_C \vec{F} \cdot d\vec{r}$ is independent of paths joining A and B. This completes the proof of the theorem.

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4.13 Conservative vector field and potential function

There are certain class of vector fields which have the advantage that they can be obtained from scalar fields. We note that scalar fields are easy to handle in comparison to vector fields. One of the class of vector field of this type is conservative.

A vector field $\vec{F}$ is said to be conservative vector field if it is the gradient of some scalar function. In other words, $\vec{F}$ is conservative if there exists a scalar function f such that $\vec{F} = \nabla f$.

The scalar function so exists is called potential function associated to $\vec{F}$. Gravitational field is an example of conservative vector field. We recall that the kinetic energy is defined as the ability of the body to do work by virtue of its motion. So if a body moves in a conservative field (such as gravitational field) of force, the body will have same kinetic energy after it completes a round-trip.

Let $\vec{F}$ be a conservative vector field. So, we have $\vec{F} = \nabla\phi$ for some potential function ϕ.

Now, curl $\vec{F} = \nabla \times \vec{F} = \nabla \times \nabla\phi = 0$

Hence the curl of $\vec{F}$ is zero. Since the curl characterizes the rotation in a field, we also say that the gradient fields describing a motion are irrotational. Hence conservative field are also known as irrotational.

We now summarize the results of theorem 1 and 2 diagrammatically as

$\int_C \vec{F}.\vec{dr}$ is a path independent $\Leftrightarrow$ $\vec{F} = \nabla\phi$ for some scalar function ϕ.

$$\Updownarrow$$

$\oint_C \vec{F}\cdot\vec{dr} = 0$ on every simple closed curve C.

Thus a vector field $\vec{F}$ is conservative if the line integral is path independent. Alternatively, the field $\vec{F}$ is conservative if the circulation or rotational effect $\oint_C \vec{F}\cdot\vec{dr}$ is zero.

Also by Theorem 1,

$$\int_C \vec{F}\cdot\vec{dr} = \phi(B) - \phi(A)$$

where A and B are initial and terminal point of path C. In conclusion, for a conservative field $\vec{F}$, the value of line integral $\int_C \vec{F}\cdot\vec{dr}$ is given by the difference of potential values at the end points.

Here $\phi(B) - \phi(A)$ is called potential drop.

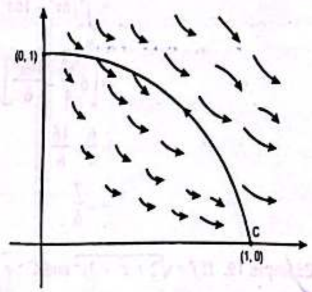
Example 17. Find the work done by $\vec{F} = (x^2, -xy)$ along a path C given by $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

Solution

$\vec{F}(x, y) = x^2\hat{i} - xy\hat{j}$

We write the parameteric form of C

$x = \cos t,\ y = \sin t,\ 0 \le t \le \frac{\pi}{2}$



$$\text{Work done} = \int_C \vec{F}\cdot\vec{dr} = \int_0^{\pi/2} \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

where, $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\therefore\ \vec{r}'(t) = -\sin t\hat{i} + \cos t\hat{j}$

and $\vec{F}(\vec{r}(t)) = \cos^2 t\hat{i} - \cos t \sin t\hat{j}$

$$\text{Work} = \int_C \vec{F}\cdot\vec{dr} = \int_0^{\pi/2} (\cos^2 t\hat{i} - \cos t\sin t\hat{j})\cdot(-\sin t\hat{i} + \cos t\hat{j})\,dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t\,dt$$

$$= 2\left[\frac{\cos^3(t)}{3}\right]_0^{\pi/2}$$

$$= -\frac{2}{3}.$$

Example 18. If $\vec{f} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{f}\cdot\vec{dr}$, where C is the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

Solution

We have,

$\vec{f} = 3xy\vec{i} - y^2\vec{j}$ and the parabola is $y = 2x^2$

$\Rightarrow \frac{dy}{dx} = 4x$

$\therefore\ dy = 4xdx$

Here, x varies from 0 to 1

We have,

$\vec{r} = x\vec{i} + y\vec{j}$

$= x\vec{i} + 2x^2\vec{j}$

$\therefore\ \vec{dr} = dx\vec{i} + 4xdx\vec{j}$

$= (1 + 4xj)\,dx$

Now, $\int_C \vec{f}\cdot\vec{dr} = \int(3xy\vec{i} - y^2\vec{j})\cdot(dx\vec{i} + dy\vec{j})$

$$= \int_0^1 (3x(2x^2)\vec{i} - (2x^2)^2\vec{j})\cdot(dx\vec{i} + 4xdx\vec{j})$$

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$$= \int_0^1 (6x^3\vec{i} - 4x^4\vec{j}) \cdot (\vec{i} + 4x\vec{j})\, dx$$

$$= \int_0^1 (6x^3 - 16x^5) dx$$

$$= \left[6\frac{x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6}$$

$$= -\frac{7}{6}$$

Example 19. If $f = \sqrt{2 + x^2 + 3y^2}$ and C : $\vec{r} = (t, t, t^2)$, $0 \le t \le \pi$, find $\int_C f\, ds$.

Solution

We have,

$$\vec{r} = t\vec{i} + t\vec{j} + t^2\vec{k}$$

$$\frac{d\vec{r}}{dt} = \vec{i} + \vec{j} + 2t\vec{k}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{\left(\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right)}$$

$$= \sqrt{(\vec{i} + \vec{j} + 2t\vec{k}) \cdot (\vec{i} + \vec{j} + 2t\vec{k})}$$

$$= \sqrt{(1)^2 + (1)^2 + 4t^2}$$

$$= \sqrt{2 + 4t^2}$$

Also, $f \; = \sqrt{2 + x^2 + 3y^2}$

$$= \sqrt{2 + t^2 + 3t^2}$$

$$= \sqrt{2 + 4t^2}$$

Now, $\int f ds = \int_0^\pi \sqrt{2 + 4t^2}\sqrt{2 + 4t^2}\, dt$

$$= \int_0^\pi (2 + 4t^2)\, dt$$

$$= \left[2t + \frac{4t^3}{3}\right]_0^\pi$$

$$= 2\pi + \frac{4}{3}\pi^3$$

$$\therefore \quad \int f ds = 2\pi + \frac{4}{3}\pi^3.$$

Example 20. Find the work done in moving a particle once around the circle $x^2 + y^2 = 9$, $z = 0$ by the force $\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$.

Solution

The equation of circle is $x^2 + y^2 = 9$, $z = 0$.

It's parametric equation $x = 3\cos t$, $y = 3\sin t$ $\quad (0 \le t \le 2\pi)$

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j}$$

$$= 3\cos t\,\vec{i} + 3\sin t\,\vec{j}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = -3\sin t\,\vec{i} + 3\cos t\,\vec{j}$$

or, $\quad d\vec{r} = (-3\sin t\,\vec{i} + 3\cos t\,\vec{j})\, dt$

and $\quad \vec{F} = (2x - y + z)\,\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$

$$= (2 \cdot 3\cos t - 3\sin t)\vec{i} + (3\cos t + 3\sin t)\vec{j} + (3 \cdot 3\cos t - 2 \cdot 3\sin t)\vec{k}$$

$$= (6\cos t - 3\sin t)\vec{i} + 3(\cos t + \sin t)\vec{j} + (9\cos t - 6\sin t)\vec{k}$$

Now, $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [(6\cos t - 3\sin t)\vec{i} + 3(\cos t + \sin t)\vec{j} + (9\cos t - 6\sin t)\vec{k}] \cdot [-3\sin t\,\vec{i} + 3\cos t\,\vec{j})\, dt$

$$= \int_0^{2\pi} (-18\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} \left(9 - \frac{9}{2}\sin 2t\right) dt = 18\pi.$$

Example 21. Find the work done $\int_C f ds$ along the circumference of the hypocycloid with $\vec{r} = (\cos^3 t, \sin^3 t)$, $0 \le t \le \pi$ if $f = x^2 + (xy)^{1/3}$.

Solution

We have,

$$C : \vec{r} = (\cos^3 t, \sin^3 t)$$

$$= \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j}$$

$$\frac{d\vec{r}}{dt} = -3\cos^2 t\sin t\,\vec{i} + 3\sin^2 t \cdot \cos t\,\vec{j}$$

Also, $f \; = x^2 + (xy)^{1/3}$

$$= \cos^6 t + \sin t\cos t$$

Now, $\frac{ds}{dt} = \sqrt{\left(\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right)}$

$$= \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}$$

$$= \sqrt{9\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)}$$

$$= 3\cos t\sin t$$

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$$\therefore \int_C f\,ds = \int_C f\frac{ds}{dt}\cdot dt$$

$$= \int_0^{\pi} (\cos^6 t + \sin t\cos t)(3\cos t\sin t)\,dt$$

$$= \int_0^{\pi} (3\cos^7 t\sin t + 3\sin^2 t\cos^2 t)\,dt$$

$$= \int_0^{\pi} \left[3\cos^7 t\sin t + \frac{3}{4}(\sin^2 2t)\right] dt$$

$$= \int_0^{\pi} 3\cos^7 t\sin t + \frac{3}{4}\frac{(1-\cos 4t)}{2}\,dt$$

$$= \int_0^{\pi} \left[3\cos^7 t\sin t + \frac{3}{8}(1-\cos 4t)\right] dt$$

Let $\cos t = p$ Also if $t = 0 \Rightarrow p = 1$

$\therefore$ $-\sin t\,dt = dp$ and if $t = \pi \Rightarrow p = -1$

$$\therefore \int_0^{\pi} 3\cos^7 t\sin t + \frac{3}{8}\int_0^{\pi}(1-\cos 4t)\,dt = -3\int_1^{-1} p^7\,dp + \frac{3}{8}\left[t - \frac{\sin 4t}{4}\right]_0^{\pi}$$

$$= -\frac{3}{8}[p^8]_1^{-1} + \frac{3}{8}[\pi]$$

$$= \frac{3\pi}{8}$$

Example 22. Show that $\int_C \vec{F}\cdot\vec{dr}$ is independent of path $\vec{F} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$.

Solution

We have,

$$\vec{F} = (4xy - 3x^2z^2)\hat{i} + 2x^2\hat{j} - 2x^3z\hat{k}$$

$$\text{Now, curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(-6x^2z + 6x^2z) + \hat{k}(4x - 4x)$$

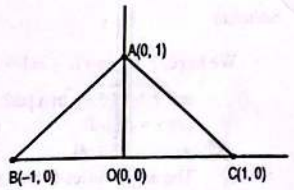
$$= 0$$

Thus, $\nabla\times\vec{F} = 0$, so $\int_C \vec{F}\cdot\vec{dr}$ is independent of path.

curl = 0

Example 23. Compute the integral $\int_C (y^2dx - x^2dy)$, where C is the boundary of the triangle with vertices (1, 0), (0, 1) and (–1, 0).

Solution



The equation of line AC is

$$y - 1 = \frac{0-0}{1-0}(x-0)$$

or, $x + y - 1 = 0$

Similarly equation of AB is $y - x - 1 = 0$.

Here, we need to evaluate $\int_C (y^2dx - x^2dy)$ along AB, BC and CA.

Along BC, $y = 0$, so $dy = 0$

$$\therefore \int_{BC} (y^2dx - x^2dy) = 0$$

Along AC,

$y = -x + 1$

$\therefore$ $dy = -dx$

$$\therefore \int_{AC} (y^2dx - x^2dy) = \int_0^1 (1-x)^2dx - x^2(-dx)$$

$$= \int_0^1 (1 - 2x + 2x^2)\,dx$$

$$= \left[x - \frac{2x^2}{2} + \frac{2x^3}{3}\right]_0^1$$

$$= 1 - 1 + \frac{2}{3}$$

$$= \frac{2}{3}$$

Again, along AB

$y = x + 1$

$= dy$

$= dx$

$$\therefore \int_{AB} (y^2dx - x^2dy) = \int_0^{-1} (x+1)^2dx - x^2dx$$

$$= \int_0^{-1} [(x^2 + 2x + 1) - x^2]dx$$

$$= \left[\frac{2x^2}{2} + x\right]_0^{-1}$$

$$= 1 - 1 - 0$$

$$= 0$$

$$\therefore \int_{AB+BC+CA} y^2dx - x^2dy = 0 + 0 + \frac{2}{3} = \frac{2}{3}.$$

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Example 24. Given that $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ for

$C: x^2 + y^2 = a^2, z = 0.$

Solution

We have, $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$

and $\vec{r} = x\hat{i} + y\hat{j}$ in a path $x^2 + y^2 = a^2$

Let $x = a\cos\theta$ and $y = a\sin\theta$

$\Rightarrow dx = -a\sin\theta\, d\theta \quad \Rightarrow \quad dy = a\cos\theta\, d\theta$

The angle varies from 0 to 2π

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [\sin y\, dx + x(1 + \cos y)\, dy]$$

$$= \int_0^{2\pi} d(x\sin y) + \int_0^{2\pi} (a\cos\theta)\, a\cos\theta\, d\theta$$

$$= [x\sin y]_0^{2\pi} + a^2\int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= [a\cos 2\pi\sin(a\sin 2\pi) - a\cos 0^\circ\sin(a\sin 0^\circ)] + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 0 + \frac{a^2}{2}\left[2\pi + \frac{\sin 4\pi}{2} - 0\right] = \pi a^2$$

Example 25. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x\hat{i} - xy\hat{j}$ from the origin to the point (1, 1) along the parabola $y^2 = x$.

Solution

We have, $\vec{F} = x\hat{i} - xy\hat{j}$

And $\vec{r} = x\hat{i} + y\hat{j}$

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j}$. Also x varies from 0 to 1.

The curve is $y^2 = x$ so we have $2y\,dy = dx$.

$$\text{Now, } \int_C \vec{F}\cdot d\vec{r} = \int_C (x\vec{i} - xy\vec{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_c (x\,dx - xy\,dy)$$

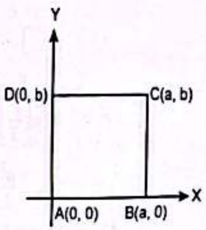
$$= \int_0^1 [y^2.(2y\,dy) - y^2.y\,dy]$$

$$= \int_0^1 (2y^3\,dy - y^3\,dy)$$

$$= \int_0^1 y^3\,dy = \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{4}$$

Example 26. Calculate $\int_C [(x^2 + y^2)\hat{i} - 2xy\hat{j}].\, d\vec{r}$, where C is a rectangle in xy-plane bounded by $y = 0$, $y = b, x = 0, x = a$.

Solution



We have,

$\vec{F} = (x^2 + y^2)\,\hat{i} - 2xy\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j}$

Then

$\vec{F}.d\vec{r} = [(x^2 + y^2)\hat{i} - 2xy\hat{j}].(dx\hat{i} + dy\hat{j})$

$= (x^2 + y^2)\,dx - 2xy\,dy$

The curve is rectangle in xy - plane bounded by

$y = 0, y = b, x = 0, x = a.$

$$\text{Now, } \int_C \vec{F}.d\vec{r} = \int_{AB}\vec{F}.d\vec{r} + \int_{BC}\vec{F}.d\vec{r} + \int_{CD}\vec{F}.d\vec{r} + \int_{DA}\vec{F}.d\vec{r}$$

Along AB, we have

$y = 0$

$\therefore dy = 0$

Also the limit of x is from 0 to a.

$$\therefore \int_{AB}\vec{F}.d\vec{r} = \int_0^a [(x^2 + 0)\,dx - 2x.0]$$

$$= \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along BC, we have

$x = a$

$\therefore dx = 0$

Also, the limit of y is from 0 to b.

$$\therefore \int_{BC}\vec{F}.d\vec{r} = \int_{BC}[(x^2 + y^2)\,dx - 2x\,dy]$$

$$= \int_0^b -2a\,dy$$

$$= -2a\,[y]_0^b$$

$$= -2ab$$

Along CD, we have

$y = b$

$\therefore dy = 0$

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The curve is rectangle in xy - plane bounded by $y = 0, y = b, x = 0, x = a$.

Now,

$$\int_c \vec{F}.d\vec{r} = \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CD} \vec{F}.d\vec{r} + \int_{DA} \vec{F}.d\vec{r}$$

Along AB, we have

$y = 0$

$\therefore$ $dy = 0$

Also the limit of x is from 0 to a.

$$\int_{AB} \vec{F}.d\vec{r} = \int_0^a (x^2)\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}a^3$$

Along BC, we have

$x = a$

$\therefore$ $dx = 0$

Also the limit of y is from 0 to b,

$$\therefore \int_{BC} \vec{F}.d\vec{r} = \int_0^b -2ay\,dy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b$$

$$= -ab^2$$

Along CD, we have

$y = b$

$\therefore$ $dy = 0$

Also the limit of x is from a to 0.

$$\therefore \int_{CD} \vec{F}.d\vec{r} = \int_a^0 (x^2 + b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^0$$

$$= -\frac{1}{3}a^3 - ab^2$$

Along DA, we have

$x = 0$

$\therefore$ $dx = 0$

Also, the limit of y is from b to 0.

$$\therefore \int_{DA} \vec{F}.d\vec{r} = \int_b^0 (-0)\,dy = 0$$

Now, from equation (1), we have

$$\therefore \int_c \vec{F}.d\vec{r} = \frac{1}{3}a^3 - ab^2 - \frac{1}{3}a^3 - ab^2 + 0$$

$$= -2ab^2$$

Example 30. Show, that $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ is irrotational. Find the scalar potential.

Solution

Here $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\hat{k}$

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right]\hat{i} - \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]\hat{j} + \left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]\hat{k}$$

$$= (x - x)\,\hat{i} - (y - y)\,\hat{j} + (z - z)\,\hat{k}$$

$$= 0$$

Hence, $\vec{F}$ is irrotational.

If ϕ is a scalar potential, then

$d\phi = \vec{F}.\,d\vec{r}$

or, $d\phi = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}).\,(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

or, $d\phi = yzdx + zxdy + xydz$

Integrating, we get,

$\phi = xyz + xyz + xyz + C$

$\therefore$ $\phi = 3xyz + C$

Example 31. Prove that $\vec{F} = (2xz^3 + 6y)\,\hat{i} + (6x - 2yz)\,\hat{j} + (3x^2z^2 - y^2)\,\hat{k}$ is a conservative vector field. Also find the scalar potential.

Solution

Here, $\vec{F} = (2xz^3 + 6y)\,\hat{i} + (6x - 2yz)\,\hat{j} + (3x^2z^2 - y^2)\,\hat{k}$

$$\therefore \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xz^3 + 6y) & (6x - 2yz) & (3x^2z^2 - y^2) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(6x - 2yz)\right]\hat{i} - \left[\frac{\partial}{\partial x}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(2xz^3 + 6y)\right]\hat{j} + \left[\frac{\partial}{\partial x}(6x - 2yz) - \frac{\partial}{\partial y}(2xz^3 + 6y)\right]\hat{k}$$

$$= (-2y + 2y)\,\hat{i} - (6xz^2 - 6xz^2)\hat{j} + (6 - 6)\,\hat{k}$$

$$= 0$$

Hence, $\vec{F}$ is a conservative vector field.

If ϕ is a scalar potential then

$d\phi = \vec{F}.\,d\vec{r}$

or, $d\phi = \{(2xz^3 + 6y)\,\hat{i} + (6x - 2yz)\,\hat{j} + (3x^2z^2 - y^2)\,\hat{k}\}.\,(dx\hat{i} + dy\hat{j} + dz\hat{k})$

or, $d\phi = (2xz^3 + 6y)\,dx + (6x - 2yz)dy + (3x^2z^2 - y^2)dz$

Integrating, we get

$$\phi = \frac{2x^2z^3}{2} + 6xy + 6xy - \frac{2y^2z}{2} + \frac{3x^2z^3}{3} - y^2z + c$$

$$\therefore \phi = 12xy + 2x^2z^3 - 2y^2z + c$$

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Example 32. Show that $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ is irrotational also find the scalar potential.

Solution

Here, $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

$$\therefore \text{ Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right]\hat{i} - \left[\frac{\partial}{\partial x}(xy\cos z + y^2) - \frac{\partial}{\partial z}(y\sin z - \sin x)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right]\hat{k}$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - (y\cos z - y\cos z)\hat{j} + (\sin z - \sin z)\hat{k}$$

$$= 0$$

Hence, $\vec{F}$ is irrotational.

If ϕ is a scalar potential, then

$d\phi = \vec{F}.\vec{dr}$

or, $d\phi = \{(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\} \cdot (dx\hat{i} + dy\vec{j} + dz\vec{k})$

or, $d\phi = (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos z + y^2)\,dz$

Integrating, we get

$$\phi = x\,y\sin z + \cos x + xy\sin z + \frac{2y^2z}{2} + xy\sin z + y^2z + c$$

$$\therefore \phi = 3xy\sin z + \cos x + 2y^2z + c$$

Example 33. 14. If $\vec{F} = r^2\vec{r}$, show that $\vec{F}$ is conservative field and scalar potential is $\phi = \frac{r^4}{4}$ + constant.

curl = 0

Solution

We have $\vec{F} = r^2\vec{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Also

$r = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$

Then,

$$\vec{F} = r^2\vec{r} = (x^2 + y^2 + z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= x(x^2 + y^2 + z^2)\hat{i} + y(x^2 + y^2 + z^2)\hat{j} + z(x^2 + y^2 + z^2)\hat{k}$$

Now,

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^3 + xy^2 + xz^2) & (x^2y + y^3 + yz^2) & (x^2z + y^2z + z^3) \end{vmatrix}$$

$$= \hat{i}(2yz - 2yz) - \hat{j}(2xz - 2xz) + \hat{k}(2xy - 2xy)$$

$$= 0$$

Hence, $\vec{F}$ is conservative field.

Again,

If ϕ is a scalar potential, then

$\vec{F} = \nabla\phi$

So, $\vec{F}.\vec{dr} = \nabla\phi.\vec{dr}$

$$= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

or, $\vec{F}.\vec{dr} = d\phi$

$\therefore d\phi = \vec{F}.\vec{dr}$

$$= (x^3 + xy^2 + xz^2)\hat{i} + (x^2y + y^3 + yz^2)\hat{j} + (x^2z + y^2z + z^3)\hat{k}\}.(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= (x^3 + xy^2 + xz^2)\,dx + (x^2y + y^3 + yz^2)\,dy + (x^2z + y^2z + z^3)\,dz$$

$$= x^3dx + y^3dy + z^3dz + (xy^2dx + x^2ydy) + (xz^2dx + x^2zdz) + (yz^2dy + y^2zdz)$$

$$d\phi = x^3dx + y^3dy + z^3dz + \frac{1}{2}d(x^2y^2) + \frac{1}{2}d(x^2z^2) + \frac{1}{2}d(y^2z^2)$$

On integrating, we get

$$\phi = \frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} + \frac{1}{2}x^2y^2 + \frac{1}{2}x^2z^2 + \frac{1}{2}y^2z^2 + c$$

$$= \frac{(x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2)}{4} + c$$

$$= \frac{(x^2 + y^2 + z^2)^2}{4} + c$$

$$= \frac{(r^2)^2}{4} + c$$

$$\therefore \phi = \frac{r^4}{4} + \text{constant}$$

Example 34. If $\vec{f} = 2x\vec{i} + 4y\vec{j} + 8z\vec{k}$, show that $\vec{f}$ is irrotational and find the scalar potential function ϕ so that $\vec{f}$ = grad ϕ.

Solution

We have, $\vec{f} = 2x\vec{i} + 4y\vec{j} + 8z\vec{k}$

Then, curl $\vec{f} = \nabla \times \vec{f}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times [2x\vec{i} + 4y\vec{j} + 8z\vec{k}]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 4y & 8z \end{vmatrix}$$

$$= \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

It implies that $\vec{f}$ is irrotational. If ϕ is its potential function, we have

$\vec{f}$ = grad ϕ

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or, $2x\vec{i} + 4y\vec{j} + 8z\vec{k} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

$\Rightarrow$ $\frac{\partial \phi}{\partial x} = 2x, \frac{\partial \phi}{\partial y} = 4y$ and $\frac{\partial \phi}{\partial z} = 8z.$

Further, we know that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz.$$

or, $d\phi = 2xdx + 4ydy + 8zdz$

or, $d\phi = d(x^2 + 2y^2 + 4z^2)$

$\Rightarrow$ $\phi = x^2 + 2y^2 + 4z^2$

$\therefore$ The required potential function is $\phi = x^2 + 2y^2 + 4z^2$.

Exercise 4.3

1. Evaluate $\int_C \vec{F}\cdot \vec{dr}$ for the following vector fields.
   a. $\vec{F} = (xy, x^2y^2)$, C: AB where coordinates are A(2, 0) and B(0, 2).
   b. $\vec{F} = [(x-y)^2, (y-x)^2]$, C: $xy = 1, 1 \le x \le 4$
   c. $\vec{F} = y^2\hat{i} + 2xy\hat{j}$, C: $y^2 = x$ from (0, 0) to (1, 1)
   d. $\vec{F} = (2z, x, -y)$, C: $\vec{r} = (\cos t, \sin t, 2t)$ from (0, 0, 0) to (1, 0, 4π)
   e. $\vec{F} = (x-y, y-z, z-x)$, C: $(2\cos t, t, 2\sin t)$ from (2, 0, 0) to (2, 2π, 0)
   f. $\vec{F} = (e^x, e^{-y}, e^z)$, c: (t, t^2, t) from (0, 0, 0) to (1, 1, 1)
   g. $\vec{F} = (2xy - z)\hat{i} + yz\hat{j} + x\hat{k}$, C: $(t, 2t, t^2 - 1)$ from $t = 0$ to 1
   h. $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$, C: $x^2 = 4y, 3x^3 = 8z$, from $0 \le x \le 2$
   i. $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$, C: $\vec{r} = (\cos t, \sin t, t), 0 \le t \le 2\pi$
   j. $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, C : $x = t, y = t^2, z = t^3$ from (0, 0, 0) to (1, 1, 1)
2. Calculate $\int_C f ds$ for the following.
   a. $f = x^2y$ C : $\vec{r} = (2\cos t, 2\sin t)$ $(0 \le t \le \pi/2)$
   b. $f = \sqrt{16x^2 + 81y^2}$ C : $\vec{r} = (3\cos t, 2\sin t)$ $(0 \le t \le \pi)$
   c. $f = 1 + y^2 + z^2$ C : $\vec{r} = (t, \cos t, \sin t)$ $(0 \le t \le \pi)$
   d. $f = x^2 + y^2 + z^2$ C : $\vec{r} = (\cos t, \sin t, 2t)$ $(0 \le t \le 4\pi)$
   e. $f = xyz$ C : line joining (0, 0, 0) and (1, 2, 3)
3. Evaluate the line integral along C for the questions given below.
   a. $\int_C [xydx + (x+y)\,dy]$, where C : (0, 0) to (1, 3)
   b. $\int_C [xzdx + (y+z)dy + xdz]$, where C : $(x, y, z) = (e^t, e^t, e^{2t})$ $0 \le t \le 1$
   c. $\int_C [(x-y)dx + xdy]$, where C: $y+2 = x$ from (4, –2) to (4, 2)

4. Prove that the given vectors are conservative (irrotational) and find a scalar function ϕ such that $\vec{F} = \nabla\phi$.
   a. $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$
   b. $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$
   c. $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$
   d. $\vec{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$
   e. $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$
   f. $\vec{F} = 2xyz\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$

Answer

1. a. $-\frac{4}{15}$ b. $\frac{891}{64}$ c. 1   d. 9π e. $2\pi^2 - 8\pi$
   f. $\frac{2e^2 - e - 1}{2}$ g. $\frac{5}{3}$ h. 16   i. 3π j. 5
2. a. $\frac{16}{3}$ b. 39π c. $2\sqrt{2}\pi$ d. $\sqrt{5}\pi\left(\frac{4 + 256\pi^2}{3}\right)$ e. $\frac{3\sqrt{14}}{2}$
3. a. $\frac{15}{2}$ b. $\frac{1}{12}(3e^6 + 8e^5 - 12e^3 - 8e^2 + 6)$ c. $\frac{16}{3}$
4. a. $xyz + c$ b. $\frac{1}{3}(x^3 + y^3 + z^3) - xyz + C)$ c. $\frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + C$
   d. $\cos x + xy\sin z + y^2z + C$ e. $\frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$ f. $x^2yz + y^2$

4.14 Exactness condition of the vector field

Let $\int_C F_1dx + F_2dy + F_3dz$ be the integral in the vector field $\vec{F} = (F_1, F_2, F_3)$ and $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ such that F_1, F_2 and F_3 are continuous and have the continuous partial derivatives of first order *in the* defined domain D. The given integral is exact if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$. In this *case, the integration is independent* of the path in D. The solution is obtained by transforming *the integrand in differential* term.

Example 35. *Show that* the integral $\int_{(0,0,0)}^{(2,4,0)} e^{x-y+z^2}(dx - dy + 2z\,dz)$ is exact and evaluate the exact value.

Solution

We have, $\int_{(0,0,0)}^{(2,4,0)} [(e^{x-y+z^2})dx - (e^{x-y+z^2})dy + 2z(e^{x-y+z^2})dz]$

The integrand value is

$$I = (e^{x-y+z^2})dx - (e^{x-y+z^2})dy + 2z(e^{x-y+z^2})dz \quad \ldots(1)$$

Comparing (1) with $F_1dx + F_2dy + F_3dz$

$F_1 = e^{x-y+z^2}$, and $F_2 = -(e^{x-y+z^2})$ $F_3 = 2z(e^{x-y+z^2})$

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Now, $\dfrac{\partial F_1}{\partial y} = -e^{x-y+z^2}$ and $\dfrac{\partial F_2}{\partial x} = -e^{x-y+z^2}$ $\quad \Rightarrow \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$

$\dfrac{\partial F_3}{\partial x} = 2z\, e^{x-y+z^2}$ and $\dfrac{\partial F_1}{\partial z} = 2ze^{x-y+z^2}$ $\quad \Rightarrow \dfrac{\partial F_3}{\partial x} = \dfrac{\partial F_1}{\partial z}$

$\dfrac{\partial F_2}{\partial z} = -2z\, e^{x-y+z^2}$ and $\dfrac{\partial F_3}{\partial y} = -2z\, e^{x-y+z^2}$ $\quad \Rightarrow \dfrac{\partial F_2}{\partial z} = \dfrac{\partial F_3}{\partial y}$

Thus, the given integral is exact

$$\text{Now, I } = \int_{(0,0,0)}^{(2,4,0)} e^{x-y+z^2}\,(dx - dy + 2zdz)$$

$$= \int_{(0,0,0)}^{(2,4,0)} d(e^{x-y+z^2})$$

$$= [e^{(x-y+z^2)}]_{(0,0,0)}^{(2,4,0)}$$

$$= e^{2-4+0} - 0^{0-0+0}$$

$$= e^{-2} - 1 = \frac{1-e^2}{e^2}$$

Example 36. Evaluate $\displaystyle\int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz + 1)dz]$ by proving given integrant is e

Solution

The integrand is

$I = (6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz + 1)dz$ ....(1)

Comparing (1) with $F_1dx + F_2dy + F_3dz$, we get

$F_1 = 6xy^3 + 2z^2 \qquad F_2 = 9x^2y^2 \qquad F_3 = 4xz + 1$

Now, $\dfrac{\partial F_1}{\partial y} = 18xy^2$, and $\dfrac{\partial F_2}{\partial x} = 18xy^2$ $\quad \Rightarrow \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

$\dfrac{\partial F_1}{\partial z} = 4z$, and $\dfrac{\partial F_3}{\partial x} = 4z$ $\quad \Rightarrow \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$

and $\dfrac{\partial F_2}{\partial z} = 0$, and $\dfrac{\partial F_3}{\partial y} = 0$ $\quad \Rightarrow \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial y}$

Thus, the given integral (1) is exact.

$$\text{Now I } = \int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz + 1)dz)]$$

$$= \int_{(1,0,2)}^{(-2,1,3)} [(6xy^3dx + 2z^2dx + 9x^2y^2dy + 4xzdx + dz]$$

$$= \int_{(1,0,2)}^{(-2,1,3)} [(6xy^3dx + 9x^2y^2dy) + (2z^2dx + 4xzdx) + dz]$$

$$= \int_{(1,0,2)}^{(-2,1,3)} [(d(3x^2y^3) + d(2xz^2) + d(z)]$$

$$= \int_{(1,0,2)}^{(-2,1,3)} d(3x^2y^3 + 2xz^2 + z)$$

$$= [3x^2y^3 + 2xz^2 + z]_{(1,0,2)}^{(-2,1,3)}$$

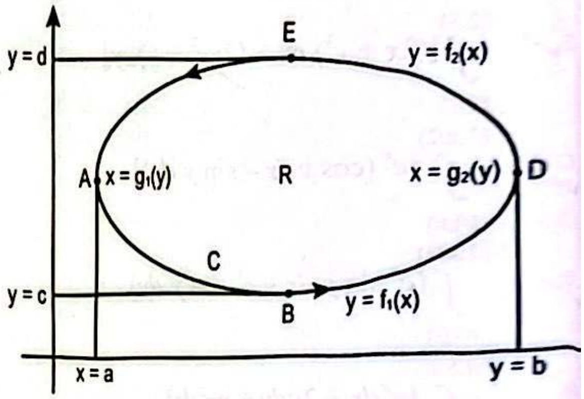
$$= 12 - 36 + 3 - 8 - 2$$

$$= -31$$

-4

3xy2 + x+y+2

Proof: We prove the theorem in a special case in which any line parallel to the coordinate does not intersect the boundary of the closed region R at more than two points as show figure.



Let us represent the curves ABD and AED by the functions $y = f_1(x)$ and $y = f_2(x)$ respectively. Similarly, we represent the curves EAB and EDB by $x = g_1(y)$ and $x = g_2(y)$. Let the region be bounded by lines $x = a$, $x = b$, $y = c$ and $y = d$ as shown in figure. Thus the region R can be described as.

$$\begin{matrix} a \le x \le b \\ f_1(x) \le y \le f_2(x) \end{matrix} \qquad \text{OR} \qquad \begin{matrix} c \le y \le d \\ g_1(y) \le x \le g_2(y) \end{matrix}$$

Now, $$\iint_R \frac{\partial F_2}{\partial x}\,dx\,dy = \int_{y=c}^{y=d}\int_{x=g_1(y)}^{g_2(y)} \frac{\partial F_2(x,y)}{\partial x}\,dx\,dy$$

$$= \int_{y=c}^{y=d} [F_2(x,y)]_{x=g_1(y)}^{x=g_2(y)}\,dy$$

$$= \int_{y=c}^{y=d} [F_2(g_2(y), y) - F_2(g_1(y), y)]dy$$

$$= \int_{y=c}^{y=d} F_2(g_2(y), y),\,dy - \int_{y=c}^{y=d} F_2(g_1(y), y)\,dy$$

$$= \int_{BDE} F_2\,dy - \int_{BAE} F_2\,dy$$

$$= \int_{BDE} F_2\,dy + \int_{EAB} F_2\,dy$$

$$= \int_{BDEAB} F_2\,dy$$

$$= \oint_C F_2\,dy \qquad \ldots.(1)$$

Again,

$$-\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = -\int_{x=a}^{x=b}\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial F_1(x,y)}{\partial y}\,dy\,dx$$

$$= -\int_{x=a}^{x=b} [F_1(x,y)]_{y=f_1(x)}^{y=f_2(x)}\,dx$$

$$= \iint_R 2\,dx\,dy$$

$$= 2\iint_R dx\,dy$$

$$\therefore \quad \iint_R dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$\text{Area of R} = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

This formula expresses the area of a closed region R in terms of line integral and this formula used while measuring the area in the planimeters, a device to measure area. We note that the choice of F_1 and F_2 is not unique.

Again,

$x = r\cos\theta, \qquad y = r\sin\theta$

$\therefore \quad dx = -r\sin\theta\,d\theta, \qquad dy = r\cos\theta\,d\theta$

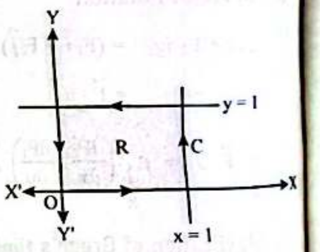
$$xdy - ydx = r\cos\theta \cdot r\cos\theta\,d\theta + r\sin\theta \cdot r\sin\theta\,d\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)\,d\theta$$
$$= r^2 d\theta$$

So, area formula in polar coordinates becomes

$$A = \frac{1}{2}\oint_C r^2\,d\theta$$

Example 37. Evaluate $\oint_C (3ydx + 5xdy)$, where C is the boundary of square $0 \le x \le 1,\ 0 \le y \le$ (counter clockwise)

Solution



The given integral is

$$I = \oint_C (3ydx + 5xdy) \qquad \ldots(1)$$

Comparing (1) with $\int [F_1dx + F_2dy]$, we get

$F_1 = 3y,\ F_2 = 5x$

$$\therefore \quad \frac{\partial F_1}{\partial y} = 3,\ \frac{\partial F_2}{\partial x} = 5$$

Now by Green's theorem,

$$\oint_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$
$$= \int_0^1\int_0^1 (5-3)\,dx\,dy$$
$$= 2 \times 1 \times 1$$
$$= 2.$$

Example 38. Evaluate $\oint_C (x^3 - 3y)dx + (x + \sin y)\,dy$, where C is the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2).

Solution

The given integral is

$$\oint_C [(x^3 - 3y)\,dx + (x + \sin y)\,dy] \qquad \ldots(1)$$

Comparing (1) with $\int [F_1dx + F_2dy]$, we get

$F_1 = x^3 - 3y,\ F_2 = x + \sin y$

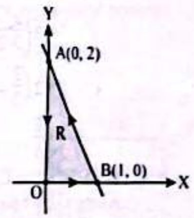
$$\therefore \quad \frac{\partial F_1}{\partial y} = -3,\ \frac{\partial F_2}{\partial x} = 1$$

Now by Green's theorem

$$\oint_C (F_1dx + F_2dy) = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$
$$= \iint (3+1)\,dx\,dy. \qquad \ldots(2)$$

Then, from figure,

The equation of the line joining AB is $y = 2 - 2x$.



The region of integral is

$0 \le x \le 1$ and $0 \le y \le 2 - 2x$. So from (2)

$$\oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$
$$= \int_0^1\int_0^{2-2x} 4\,dx\,dy$$
$$= 4\int_0^1 (2-2x)\,dy$$
$$= 4[2x - x^2]_0^1$$
$$= 4.$$

$$\therefore \quad \oint_C [(x^3 - 3y)\,dx + (x + \sin y)\,dy] = 4$$

Alternatively, $4\iint_R dx\,dy = 4 \times \text{Area of } \Delta AOB$

$$= 4 \times \frac{1}{2} \times 1 \times 2$$
$$= 4.$$

Example 39. Verify Green's theorem for $\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$ over the region R bounded by $y = x^2$ and $y = \sqrt{x}$.

Solution

Here, $\vec{F} = (x-y)\,\vec{i} + (x+y)\,\vec{j}$

So, $F_1 = x - y,\ F_2 = x + y$

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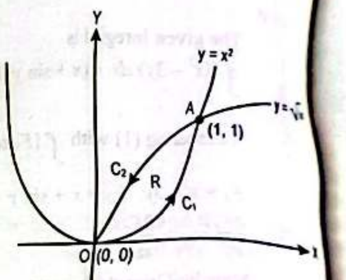
We verify

$$\oint_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \iint_R \left[\frac{\partial (x+y)}{\partial x} - \frac{\partial (x-y)}{\partial y}\right] dx\, dy$$

$$= \iint_R 2\, dx\, dy.$$



Here, $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$

$$\therefore \text{R.H.S.} = 2\int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx$$

$$= 2\int_0^1 [\sqrt{x} - x^2]\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{2}{3} - \frac{1}{3}\right)$$

$$= \frac{2}{3}.$$

For L.H.S., let C_1 and C_2 be the paths which represent OA and AO respectively.
Along C_1, we have
$y = x^2$ so $dy = 2x\, dx$. At O, $x = 0$, $y = 0$ and at A, $x = 1$, $y = 1$.

$$\therefore \int_{C_1} (F_1 dx + F_2 dy) = \int_{C_1} [(x-y)\, dx + (x+y)\, dy]$$

$$= \int_{C_1} x\, dx - \int_{C_1} y\, dx + \int_{C_1} x\, dy + \int_{C_1} y\, dy$$

$$= \int_0^1 x\, dx - \int_0^1 x^2\, dx + \int_0^1 x\, 2x\, dx + \int_0^1 y\, dy$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{2}{3} + \frac{1}{2}$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3} \qquad \ldots (1)$$

Along C_2, we have
$y = \sqrt{x}$ so that $y^2 = x$.

$\therefore$ $2y\, dy = dx$

$$\int_{C_2} (F_1 dx + F_2 dy) = \int_{C_2} (x-y)\, dx + (x+y)\, dy$$

$$= \int_{C_2} x\, dx - \int_{C_2} y\, dx + \int_{C_2} x\, dy + \int_{C_2} y\, dy$$

$$= \int_1^0 x\, dx - \int_1^0 \sqrt{x}\, dx + \int_1^0 y^2\, dy + \int_1^0 y\, dy$$

$$= -\frac{1}{2} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2}$$

$$= -1 + \frac{1}{3}$$

$$= -\frac{2}{3} \qquad \ldots (2)$$

Adding (1) and (2), we get

$$\text{L.H.S.} = \oint_C F_1 dx + F_2 dy$$

$$= \int_{C_1} (F_1 dx + F_2 dy) + \int_{C_2} (F_1 dx + F_2 dy)$$

$$= \frac{4}{3} - \frac{2}{3}$$

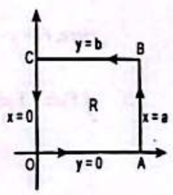
$$= \frac{2}{3}.$$

Example 40. Verify Green's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ over a region R on xy-plane bounded by lines $x = 0$, $y = 0$, $x = a$ and $y = b$.

Solution

We verify:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C (F_1 dx + F_2 dy)$$



Here, $F_1 = x^2 + y^2$, $F_2 = -2xy$

$$\therefore \text{L.H.S.} = \iint_R \left[\frac{\partial (-2xy)}{\partial x} - \frac{\partial (x^2 + y^2)}{\partial y}\right] dx\, dy$$

$$= \iint_R (-2y - 2y)\, dx\, dy$$

$$= \int_0^b \int_0^a -4y\, dx\, dy$$

$$= -4\int_0^b y \cdot a\, dy$$

$$= -4a \cdot \frac{b^2}{2}$$

$$= -2ab^2.$$

$$\text{L.H.S.} = \oint_C (F_1 dx + F_2 dy)$$

$$= \oint_C [(x^2 + y^2)\, dx - 2xy\, dy]$$

Along OA, $y = 0$, $dy = 0$, $0 \le x \le a$
Along AB, $x = a$, $dx = 0$, $0 \le y \le b$

Along BC, $y = b$, $dy = 0$, $a \le x \le 0$

Along CO, $x = 0$, $dx = 0$, $b \le y \le 0$

So, L.H.S. $= \int_{OA} [(x^2+y^2)dx - 2xy\,dy] + \int_{AB} [(x^2+y^2)dx - 2xy\,dy] + \int_{BC} [(x^2+y^2)dx - 2xy\,dy]$

$$+ \int_{CO} [(x^2+y^2)dx - 2xy\,dy]$$

$$= \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2+b^2)\,dx + 0$$

$$= \frac{a^3}{3} - 2a\cdot\frac{b^2}{2} - \frac{a^3}{3} - b^2 a$$

$$= -2ab^2.$$

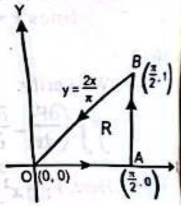
This verifies Green's theorem.

Example 41. Apply Green's theorem to evaluate the integral $\oint_C [(y - \sin x)\,dx + \cos x\,dy]$; where C is the triangle with vertices $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Solution

By Green's theorem, we have

$$\oint_C [F_1 dx + F_2 dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$



Here, $F_1 = y - \sin x$ and $F_2 = \cos x$.

$$\therefore \oint_C F_1 dx + F_2 dy = \iint_R \left[\frac{\partial \cos x}{\partial x} - \frac{\partial (y - \sin x)}{\partial y}\right] dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\iint_R (1 + \sin x)\,dx\,dy$$

On R, $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{2x}{\pi}$.

$$\therefore \oint_C [(y - \sin x)\,dx + \cos x\,dy] = -\int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (1 + \sin x)\,dy\,dx$$

$$= -\int_0^{\pi/2} (1 + \sin x)\left(\frac{2x}{\pi} - 0\right) dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(1 + \sin x)\,dx$$

$$= -\frac{2}{\pi}\left[x(x - \cos x) - 1\left(\frac{x^2}{2} - \sin x\right)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi^2}{8} - 1\right)\right]$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right).$$

Example 42. Apply Green's theorem to evaluate the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution

Area formula using Green's theorem is

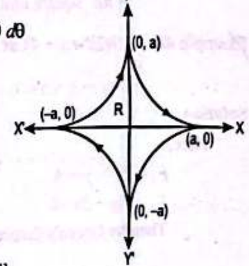
$$A = \frac{1}{2}\oint_C [x\,dy - y\,dx]$$

To find the line integral on the closed path, we use the parametric representation of the path. The parametric form of asteroid is:

$x = a\cos^3\theta$, $y = a\sin^3\theta$; where $0 \le \theta \le 2\pi$

$\therefore$ $dx = -3a\cos^2\theta\sin\theta\,d\theta$

$dy = 3a\sin^2\theta\cos\theta\,d\theta$



So area $A = \frac{1}{2}\int_0^{2\pi} [a\cos^3\theta\,3a\sin^2\theta\cos\theta + a\sin^3\theta\,3a\cos^2\theta\sin\theta\,d\theta$

$$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{3a^2}{2}\cdot 2\int_0^{\pi} \sin^2\theta\cos^2\theta\,d\theta \quad \text{[property of definite integral]}$$

$$= \frac{3a^2}{2}\cdot 4\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 6a^2\cdot\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2+2}{2}\right)} \quad \text{[Gamma Function]}$$

$$= 6a^2\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)}$$

$$= \frac{3a^2\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2}$$

$$= \frac{3a^2\pi}{8} \text{ sq. unit.}$$

Example 43. *Using Green's theorem, find the area of the circle* $x^2 + y^2 = a^2$.

Solution

Here, the circle is $x^2 + y^2 = a^2$

Area of circle $= \frac{1}{2}\int_C (x\,dy - y\,dx)$

Put, $x = a\cos\theta$; $y = a\sin\theta$

$dx = -a\sin\theta$; $dy = a\cos\theta$ and θ varies from 0 to 2π

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$$A = \frac{1}{2}\int_0^{2\pi} a\cos\theta .\ a\cos\theta\, d\theta - a\sin\theta\,(-a\sin\theta\, d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} a^2(\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2}a^2[\theta]_0^{2\pi}$$

$$= \frac{1}{2}a^2 \times 2\pi$$

$= \pi a^2$ square unit.

Example 44. $\int_C [(2x - y + 4)\, dx + (5y + 3x - 6)\, dy]$ around a circle $x^2 + y^2 = 16$.

Solution

Here,

$F_1 = 2x - y + 4$

$F_2 = 5y + 3x - 6$

Then by Green's theorem, we have

$$\int_c (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (3 + 1)\, dx\, dy$$

$$= 4\int_{-4}^{4} [y]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\, dx$$

$$= 4\int_{-4}^{4} (\sqrt{16-x^2} + \sqrt{16 + x^2}\,)\, dx$$

$$= 8\int_{-4}^{4} \sqrt{16-x^2}\, dx$$

$$= 8 \times 2\int_{0}^{4} \sqrt{16-x^2}\, dx$$

$$= 16\left[\frac{x\sqrt{16-x^2}}{2} + 8\sin^{-1}\frac{x}{4}\right]_0^4$$

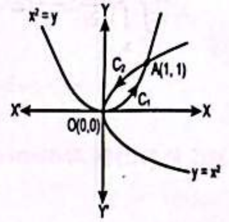
$$= 16\left[0 + 8.\frac{\pi}{2}\right]$$

$$= 64\pi$$

Example 45. Verify Green's theorem in the plane for $\int_c [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is boundary of the region $y = \sqrt{x}, y = x^2$.

Solution

Here, $y = \sqrt{x}$ or $y^2 = x$ and $y = x^2$ are two parabolas intersecting at O(0,0) and A(1, 1).



Now,

$$\therefore \int_C F_1dx + F_2dy = \int_{C_1} (F_1dx + F_2dy) + \int_{C_2} (F_1dx + F_2dy)$$

Along C_1, we have $x^2 = y$

$\therefore$ $2x\, dx = dy$

x varies from 0 to 1.

$$\therefore \text{Line integral along } C_1 = \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)\, 2x\, dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\, dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1$$

$$= 1 + 2 - 4$$

$$= -1$$

Along C_1, we have $y^2 = x$

$\therefore$ $2ydy = dx$

y varies from 1 to 0.

$\therefore$ Line integral along C_2 [y varies from 1 to 0]

$$= -\int_0^1 (3y^4 - 8y^2)\, 2y\, dy + (4y - 6y^3)\, dy$$

$$= -\int_0^1 (6y^5 - 22y^3 + 4y)\, dy$$

$$= -\left[\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right]_0^1$$

$$= -\left(1 - \frac{11}{2} + 2\right)$$

$$= -\left(\frac{2 - 11 + 4}{2}\right)$$

$$= \frac{5}{2}.$$

$$\therefore \int_C (F_1dx + F_2dy) = -1 + \frac{5}{2} = \frac{3}{2} \quad \ldots(1)$$

Now, we have to calculate

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$\therefore$ $F_1 = 3x^2 - 8y^2$, $F_2 = 4x - 6xy$

$$\frac{\partial F_1}{\partial y} = -16y, \frac{\partial F_2}{\partial x} = -6y$$

$$\therefore \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 10y$$

$$\therefore \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx$$

$$= \int_0^1 10\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5} - 0\right] = \frac{3}{2} \qquad \ldots (2)$$

From (1) and (2), we have

$$\int_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

Hence, Green's Theorem is verified.

Exercise 4.5

1. Using Green's theorem, evaluate the following integrals.

   a. $\oint_C [2xy dx + (e^x + x^2) dy]$ C : triangle joining (0, 0), (1, 0) and (1, 1)

   b. $\oint_C [(2xy - x^2)dx + (x + y^2)dy]$

      C : boundary of the region bounded by the curves $y = x^2$ and $x = y$

   c. $\oint_C (x^2 + y^2)dy$ C : square $2 \le x \le 4, 2 \le y \le 4$

   d. $\oint_C [2xy^3\, dx + 3x^2y^2 dy)]$ C : $x^2 + y^2 = 1$

   e. $\oint_C [(x + y)dx + (y + x^2)dy]$ C : area between the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$

   f. $\oint_C [x^2y^2 dx + (x^2 - y^2)dy]$ C : square with vertices (0, 0), (1, 0), (1, 1), (0, 1)

   g. $\int_C \left[\frac{y^2}{1+x^2}dx + 2y\tan^{-1}x\, dy\right]$ C : hypocycloid $x^{2/3} + y^{2/3} = 1$

   h. $\int_C (\sqrt{y}dx + \sqrt{x}dy)$ C: triangle with vertices (1, 1), (3, 1) and (2, 2)

   i. $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ C: region bounded by parabola (i) $y^2 = x$ and (ii) $y = x^2$

   j. $\int_C [(x^2 - \cosh y)\, dx + (y + \sin x)\, dy]$C : triangle with vertices (0, 0), (π, 0), (π, 1), (0, 1)

2. Use Green's theorem to evaluate the integral of $\oint \vec{F} \cdot \vec{dr}$ counter clockwise around the boundary C of the region.

   a. $\vec{F} = (y, -x)$ C : $4x^2 + 4y^2 = 1$

   b. $\vec{F} = (x \cosh 2y, 2x^2 \sinh 2y)$ C : $x^2 \le y \le x$

   c. $\vec{F} = (e^{x+y}, e^{x-y})$ C : boundary of the triangle $x \le y \le 2x, 0 \le x \le 1$

   d. $\vec{F} = \text{grad}\,(\sin x \cos y)$ C : $25x^2 + 9y^2 = 225$

3. Evaluate by using Green's theorem.

   a. $\oint_C (\cos x \sin y - 2xy)dx + \sin x \cos y\, dy$, where C is the circle: $x^2 + y^2 = 1$.

   b. $\oint_C [(3x^2 + y)\, dx + 4y^2\, dy]$, C is the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2) in anti-clockwise direction.

   c. $\oint_C \vec{F} \cdot \vec{dr}$ counter clockwise where $\vec{F}$ = (sin y, cos x) and C is the triangle with vertices (0, 0), (π, 0) and (π, 1).

   d. $\oint_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = (x^2e^y, y^2e^x)$, where C is the rectangle with vertices (0, 0), (2, 0), (2, 3) and (0, 3).

Answer

1. a. -1 b. 0   c. 24   d. 0   e. -3π
   f. $\frac{2}{3}$ g. 0   h. $2 - \frac{3}{\sqrt{2}}$ i. (i) -1; (ii) $\frac{3}{2}$ j. $\pi \cosh 1 - \pi$
2. a. $-\frac{\pi}{2}$ b. $\frac{1}{4}(1 + \sinh 2 - \cosh 2)$ c. $\frac{e^2}{2} - \frac{e^3}{3} + \frac{1}{e} - \frac{1}{6}$ d. 0
3. a. $\frac{8}{3}$ b. -1 c. $\pi \cos 1 - \pi - 1$ d. $[9(e^2 - 1) - \frac{8}{3}(e^3 - 1)$

4.16 Surface integrals

Any integral which is evaluated over a surface is called surface integral. Surface integrals are generally considered as a natural generalization of line integrals in which we integrate over a surface instead of curve. Surface integrals are calculated in various situations in the field of science and engineering such as to calculate moment of inertia and centre of mass, to evaluate the total charge distributed over a surface, to determine pressure, to determine gravitational force, magnetic force, etc.

The surface integral of a continuous function F(x, y, z) over a surface S is denoted by

$$\iint_S F(x, y, z)\, ds.$$

Here, ds is the area of infinitesimal piece of surface S. To define the surface integral precisely, we consider a sum:

$$\sum_{i=1}^{n} F(x_i, y_i, z_i)\, \Delta s_i$$

where Δs_i is the area of small piece after dividing S into n number of small pieces and $F(x_i, y_i, z_i)$ is the value given by F at some point (x_i, y_i, z_i) in Δs_i. Then

$$\iint_S F(x, y, z)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i, y_i, z_i)\, \Delta s_i$$

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$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{2(1-x)^4}{(-1)3 \times 4}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - \left(-\frac{1}{6}\right)$$

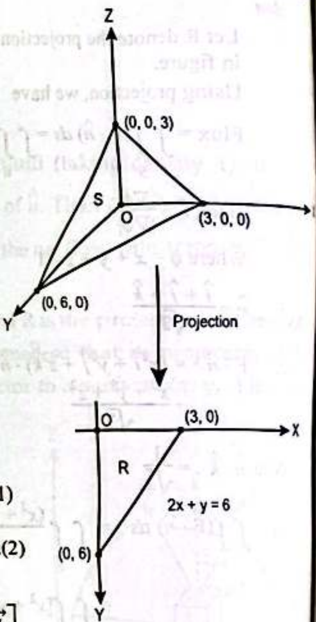
$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{6}$$

$$= \frac{4-3+2}{12}$$

$$= \frac{1}{4}$$

Example 47. Evaluate $\iint_S (\vec{F} \cdot \hat{n})\, ds$ where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Solution



The given equation of the surface is

$2x + y + 2z = 6$

i.e., $\phi = 2x + y + 2z - 6$

Vector normal to S is given by

$$\vec{n} = \nabla\phi$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x + y + 2z - 6)$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \quad \hat{n} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Also, $\hat{k} \cdot \hat{n} = \hat{k} \cdot \left[\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right] = \frac{2}{3}$ ....(1)

Now, by formula $\iint_S (\vec{F} \cdot \hat{n})\, ds = \iint_R \vec{F} \cdot \hat{n} \frac{dx\, dy}{|\hat{k} \cdot \hat{n}|}$ ....(2)

Here, $\vec{F} \cdot \hat{n} = [(x+y^2)\hat{i} - 2x + 2yz\hat{k}] \cdot \left[\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right]$

$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}yz$$

But we have $2x + y + 2z = 6 \Rightarrow z = \frac{6 - 2x - y}{2}$

$$\Rightarrow \quad \vec{F} \cdot \hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right)$$

$$= \frac{2}{3}y(y + 6 - 2x - y)$$

$$= \frac{4}{3}y(3 - x) \quad ....(3)$$

Now, replacing the value of (1) and (3) in (2)

$$\iint_S (\vec{F} \cdot \hat{n})\, ds = \frac{4}{3}\iint y(3-x) \cdot \frac{3}{2}\, dx\, dy$$

$$= 2\int_0^3 \int_0^{6-2x} y(3-x)\, dy\, dx$$

$$= 2\int_0^3 (3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\, dx$$

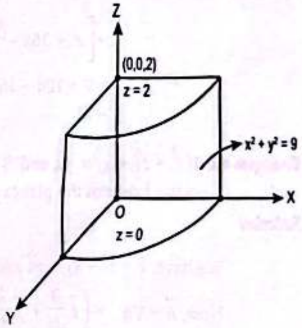
$$= 4\int_0^3 (3-x)^3\, dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= 81.$$

Example 48. Evaluate $\iint_S (\vec{F} \cdot \hat{n})\, ds$ if $\vec{F} = yz\hat{i} + 2y^2\hat{j} + xz^2\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ in first octant between $z = 0$ and $z = 2$.

Solution



We have,

$\vec{F} = yz\hat{i} + 2y^2\hat{j} + xz^2\hat{k}$ and

Let $\phi = x^2 + y^2 - 9$

Then $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - 9)$

$$= 2x\hat{i} + 2y\hat{j}$$

and $|\nabla\phi| = \sqrt{4x^2 + 4y^2}$

$$= 2(3)$$

$$= 6$$

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2(x\hat{i} + y\hat{j})}{6}$$

$$= \frac{x\hat{i} + y\hat{j}}{3}$$

Now, $\vec{F} \cdot \hat{n} = (yz\hat{i} + 2y^2\hat{j} + xz^2\hat{k}) \cdot \left(\frac{x\vec{i} + y\vec{j}}{3}\right)$

$$= \frac{xyz + 2y^3}{3}$$

Also, $|\hat{n} \cdot \hat{j}| = \left|\frac{x\vec{i} + y\vec{j}}{3} \cdot \vec{j}\right|$

$$= \frac{y}{3}$$

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