$\boxed{2.1}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{SHK} \subseteq \mathbf{HSK}$.

Suppose that $\overline{M} \in \mathbf{SHK}$. Then there are $\overline{N} \in \mathbf{K}$ and $\overline{P}$ such that there is a homomorphism f from $\overline{N}$ onto $\overline{P}$ and $\overline{M}$ is a subalgebra of $\overline{P}$. Let $Q = f^{-1}[M]$. Clearly Q is a subuniverse of $\overline{N}$. Let $\overline{Q}$ be the associated subalgebra of $\overline{N}$. Then $f \restriction Q$ is a homomorphism from $\overline{Q}$ onto $\overline{M}$. Hence $\overline{M} \in \mathbf{HSK}$.

$\boxed{2.2}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{SHK} \neq \mathbf{HSK}$.

Let $\mathbf{K} = \{\mathbb{Q}\}$ in the signature of rings. Then $\mathbb{Z}$ is a subalgebra of $\mathbb{Q}$, and $\mathbb{Z}_4$ is a homomorphic image of $\mathbb{Z}$. Thus $\mathbb{Z}_4 \in \mathbf{HSK}$. But $\mathbf{HK}$ has only two elements, $\mathbb{Q}$ and the one-element ring, and $\mathbf{SHK}$ does not have any finite member with more than one element.

$\boxed{2.3}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{PHK} \subseteq \mathbf{HPK}$.

Let $\overline{M} \in \mathbf{PHK}$. Then there is a system $\langle \overline{N}_i : i \in I \rangle$ of members of $\mathbf{K}$, another system $\langle \overline{P}_i : i \in I \rangle$ of algebra, a system $\langle f_i : i \in I \rangle$ of functions, and a function g, such that $\forall i \in I[f_i$ is a homomorphism from $\overline{N}_i$ onto $\overline{P}_i$, and g is an isomorphism from $\prod_{i\in I} \overline{P}_i$ onto $\overline{M}$. Define $h : \prod_{i\in I} N_i \to \overline{M}$ by setting

$$h(x) = g(\langle f_i(x_i) : i \in I \rangle)$$

for any $x \in \prod_{i\in I} N_i$. Then h is a homomorphism from $\prod_{i\in I} \overline{N}_i$ onto $\overline{M}$, hence $\overline{M} \in \mathbf{HPK}$.

$\boxed{2.4}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{PHK} \neq \mathbf{HPK}$. *Hint: let* $\mathbf{K}$ *consist of all fields* $\mathbb{Z}_p$ *for* p *a prime, and take an ultraproduct of them with a nonprincipal ultrafilter. Show that the result is an infinite field.*

Let $I = \{p : p \text{ is a prime}\}$, and let D be a nonprincipal ultrafilter on I. We use the signature $+, \cdot, -, 0, 1$ for the fields $\mathbb{Z}_p$. Now $A \stackrel{\text{def}}{=} \prod_{i\in I} \mathbb{Z}_p/D$ is a commutative ring with identity, since equations are used to define such. Now take a nonzero element $[x]$ of A. Then $M \stackrel{\text{def}}{=} \{p \in I : x(p) \neq 0\}$ is in D. There is a y such that $y(p) = 1/x(p)$ for all $p \in M$. It follows that $\{p \in I : x(p) \cdot y(p) = 1\} \in D$; hence $[x] \cdot [y] = [1]$. So A is a field. Now suppose that A is finite; say $A = \{[x_i] : i < n\}$. Let $N = \{p \in I : n < p\}$. If $p \in N$, then there is a $y(p) \in \mathbb{Z}_p$ such that $y(p) \notin \{x_i(p) : i < n\}$. Define $y(p) = 0$ for $p \in I \backslash N$. Then for any $i < n$,

$$N \subseteq \{p \in I : y(p) \neq x_i(p)\},$$

hence the set on the right is in D, and so $[y] \neq [x_i]$. This contradiction shows that A is infinite after all.

Note that $A \in \mathbf{HPK}$. Now a product of two or more fields in not a field, as there are zero divisors in such a product. Hence the only fields in $\mathbf{PHK}$ are isomorphic to members of $\mathbf{K}$, and so are finite. So $A \notin \mathbf{PHK}$.

$\boxed{2.5}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{PSK} \subseteq \mathbf{SPK}$.

Let $\overline{M} \in \mathbf{PSK}$. Then there exist a system $\langle \overline{N}_i : i \in I \rangle$ of members of $\mathbf{K}$, a system $\langle \overline{P}_i : i \in I \rangle$ of algebras such that $\overline{P}_i$ is a subalgebra of $\overline{N}_i$ for each $i \in I$, and an isomorphism f from $\overline{M}$ onto $\prod_{i \in I} \overline{P}_i$. Then $\prod_{i \in I} \overline{P}_i$ is a subalgebra of $\prod_{i \in I} \overline{N}_i$. By Proposition 1.5 there exist an algebra $\overline{C}$ and an isomorphism g from $\overline{C}$ onto $\prod_{i \in I} \overline{N}_i$ such that $\overline{M}$ is a subalgebra of $\overline{C}$ and $f \subseteq g$. Thus $\overline{C} \in \mathbf{PK}$, so $\overline{M} \in \mathbf{SPK}$.

$\boxed{2.6}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{PSK} \neq \mathbf{SPK}$.

Let $\mathbf{K} = \{\mathbb{Z}_2\}$, the 2-element field. Then every member of $\mathbf{PSK}$ is finite or nondenumerable, but $\mathbf{SPK}$ has a countably infinite member.

$\boxed{2.7}$ *Prove that* $\mathbf{HSPK}$ *is closed under* $\mathbf{H}$, $\mathbf{S}$, *and* $\mathbf{P}$. *Infer that for any class* $\mathbf{K}$ *of structures,* $\mathbf{HSPK}$ *is the smallest variety containing* $\mathbf{K}$.

$\mathbf{HHSPK} = \mathbf{HSPK}$. Next, using exercise 2.1, $\mathbf{HSPK} \subseteq \mathbf{SHSPK} \subseteq \mathbf{HSSPK} = \mathbf{HSPK}$. Finally, using exercises 2.3 and 2.5,

$$\mathbf{HSPK} \subseteq \mathbf{PHSPK} \subseteq \mathbf{HPSPK} \subseteq \mathbf{HSPPK} = \mathbf{HSPK}.$$

For the second part of the exercise, suppose that $\mathbf{K}$ is a class of algebras, $\mathbf{L}$ is a variety, and $\mathbf{K} \subseteq \mathbf{L}$. Then $\mathbf{HSPK} \subseteq \mathbf{HSPL} = \mathbf{L}$.

$\boxed{2.8}$ *Prove that the following hold in any proper relation algebra with unit* R*:*
(i) $S^{-1}|[-(S|T)] \subseteq -T$.
(ii) $((S|R) \cap id)|R = S|R$.
(iii) $S \subseteq S|S^{-1}|S$.

(i): suppose that $(a,b) \in S^{-1}|[-(S|T)]$ and also $(a,b) \in T$; we want to get a contradiction. Choose c such that $(a,c) \in S^{-1}$ and $(c,b) \in -(S|T)$. Then $(c,a) \in S$ and $(a,b) \in T$, so $(c,b) \in (S|T)$, contradiction.

(ii): We have

$$\begin{aligned} (a,b) \in ((S|R) \cap id)|R \quad & \text{iff} \quad \exists c[(a,c) \in ((S|R) \cap id) \text{ and } (c,b) \in R \\ & \text{iff} \quad (a,a) \in (S|R) \text{ and } (a,b) \in R \\ & \text{iff} \quad (a,a) \in (S|R) \\ & \text{iff} \quad \exists d[(a,d) \in S \text{ and } (d,a) \in R] \\ & \text{iff} \quad \exists d[(a,d) \in S \text{ and } (d,b) \in R] \\ & \text{iff} \quad (a,b) \in (S|R). \end{aligned}$$

(iii): suppose that $(a,b) \in S$. Then $(b,a) \in S^{-1}$, hence $(a,a) \in (S|S^{-1})$, hence $(a,b) \in [S|S^{-1}|S]$.