Solutions of exercises in Chapter 5

$\boxed{\text{E5.1}}$ *Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Let F be a nonprincipal ultrafilter on a set I. For each $a \in A$ let $f(a) = [\langle a : i \in I\rangle]_F$. Show that f is an embedding of $\overline{A}$ into ${}^I\overline{A}/F$, and $\overline{A}$ is elementarily equivalent to ${}^I\overline{A}/F$.*

For brevity let $\overline{B} = {}^I\overline{A}$ and $\overline{C} = \overline{B}/F$. See the definition of $\overline{C}$ following Theorem 1.15.

Suppose that $f(a) = f(b)$. Then $[\langle a : i \in I\rangle]_F = [\langle b : i \in I\rangle]_F$, hence $\{i \in I : a = b\} \in F$. Since the empty set is not in F, it follows that $a = b$. So f is one-one.

If k is an individual constant, obviously $f(k^{\overline{A}}) = k^{\overline{B}}$.

Suppose that G is an m-ary function symbol. Then

$$\begin{aligned} f(G^{\overline{A}}(a^0, \ldots, a^{m-1})) &= [\langle G^{\overline{A}}(a^0, \ldots, a_{m-1})]_F \\ &= [G^{\overline{B}}(\langle a^0 : i \in I\rangle, \ldots, \langle a^{m-1} : i \in I\rangle)]_F \\ &= G^{\overline{C}}(f(a^0), \ldots, f(a^{m-1})). \end{aligned}$$

If R is an m-ary relation symbol, then

$$\begin{aligned} \langle f(a^0), \ldots, f(a^{m-1})\rangle \in R^{\overline{C}} \quad &\text{iff} \quad \{i \in I : \langle a^0, \ldots, a^{m-1}\rangle \in R^{\overline{A}}\} \in F \\ &\text{iff} \quad \langle a^0, \ldots, a^{m-1}\rangle \in R^{\overline{A}}. \end{aligned}$$

Hence f is an isomorphism of $\overline{A}$ into $\overline{C}$. The last statement of the exercise is true by Corollary 5.2.

$\boxed{5.2}$ *We work in the language for ordered fields; see Chapter 1. In general, an element $a \in M$ is definable iff there is a formula $\varphi(x)$ with one free variable x such that $\{b \in M : \overline{M} \models \varphi[b]\} = \{a\}$.*

(i) Show that 1 is definable in $\mathbb{R}$.

(ii) Show that every positive integer is definable in $\mathbb{R}$.

(iii) Show that every positive rational is definable in $\mathbb{R}$.

(iv) If $\overline{M}$ is an extension of $\mathbb{R}$, an element ε of M is infinitesimal iff $0 < \varepsilon < r$ for every positive rational r. Let $\overline{M}$ be an ultrapower of $\mathbb{R}$ using a nonprincipal ultrafilter on ω. Thus $\overline{M}$ is isomorphic to an extension of $\mathbb{R}$ by exercise 5.1. Show that $\overline{M}$ has an infinitesimal.

(v) Use the compactness theorem to show the existence of an ordered field $\overline{M}$ which has an infinitesimal, and is elementarily equivalent to $\mathbb{R}$.

(i): Let $\varphi(x)$ be the formula $\forall y[x \cdot y = y]$.

(ii): Let φ be as in (i). By induction we define a formula ψ_m which defines m, for each positive integer m. Let ψ_1 be φ. Having defined ψ_m, let ψ_{m+1} be the formula $\exists y\exists z[\psi_m(y) \wedge \varphi(z) \wedge x = y + z]$.

(iii) Let r be a positive rational. Say $r = m/n$ with m and n positive integers. Let χ_r be the formula $\exists y\exists z[\psi_m(y) \wedge \psi_n(z) \wedge y = x \cdot z]$.

(iv) Let F be a nonprincipal ultrafilter on ω. Define $e \in {}^\omega\mathbb{R}$ by setting $e(n) = 1/(n+1)$ for every $n \in \omega$. We claim that $[e]$ is an infinitesimal. To prove this, take any positive rational r. Choose $p \in \omega$ with $\frac{1}{p} < r$. Let $x(m) = r$ for all $m \in \omega$. Thus $[x]$ is the image of r under the isomorphism of exercise 5.1, so it suffices to show that $[0] < [e] < [r]$. We have

$$\{m \in \omega : 0 < e(m)\} = \omega \in F$$

and

$$\{m \in \omega : e(m) < r\} \supseteq \{m \in \omega : m \geq p\} \in F;$$

hence $[0] < [e] < [r]$.

(v) Adjoin a new individual constant $\mathbf{c}$ to our language, and consider the following set of sentences:

$$\begin{aligned}&\{\varphi : \varphi \text{ is a sentence and } \mathbb{R} \models \varphi\}\\ &\cup \{0 < \mathbf{c}\} \cup \{\forall x[\mathbf{c} < \chi_r(x)] : r \text{ a positive rational}\}.\end{aligned}$$

Clearly every finite subset of this set has a model; the compactness theorem gives a model of the whole set, and this give the desired conclusion. (The denotation of the constant $\mathbf{c}$ is ignored in order to make the final model an ordered field with no extra fundamental constant.)

$\boxed{5.3}$ *Consider the structure $\overline{N} = (\omega, +, \cdot, 0, 1, <)$. We look at models of $\Gamma = \{\varphi : \varphi$ is a sentence and $\overline{N} \models \varphi\}$.*

(i) For every $m \in \omega$ there is a formula φ_m with one free variable x such that $\overline{N} \models \varphi_m[m]$ and $\overline{N} \models \exists! x \varphi_m(x)$.

(ii) $\overline{N}$ can be embedded in any model of Γ.

(iii) Show that Γ has a model with an infinite element in it, i.e., an element greater than each $m \in \omega$.

(i): We define φ_m by recursion; clearly the ones defined work: φ_0 is $x = 0$. Having defined φ_m, φ_{m+1} is the formula $\exists y[\varphi_m(y) \wedge x = y + 1]$.

(ii) For each $m \in \omega$, let $f(m)$ be the unique $a \in M$ such that $\overline{M} \models \varphi_m[a]$. If $m \neq n$, then $\overline{N} \models \neg(\varphi_m(x) \wedge \varphi_n(x))$, so also $\overline{m} \models \neg(\varphi_m(x) \wedge \varphi_n(x))$, hence $f(m) \neq f(n)$.

Next, $\overline{N} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow \varphi_{m+n}(x+y)$, so also $\overline{M} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow \varphi_{m+n}(x+y)$. Now $\overline{M} \models \varphi_m[f(m)]$ and $\overline{M} \models \varphi_m[f(n)]$, so $\overline{M} \models \varphi_{m+n}[f(m) + f(n)]$. Hence $f(m+n) = f(m) + f(n)$.

Similarly for $\cdot$.

If $m < n$, then $\overline{N} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow x < y$, so $\overline{M} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow x < y$, so $f(m) < f(n)$. It folows that $m < n$ iff $f(m) < f(n)$.

This finishes the proof of (ii).

For (iii), adjoin a new individual constant $\mathbf{c}$ and consider the set

$$\begin{aligned}\Gamma = &\{\varphi : \varphi \text{ is a sentence and } \overline{N} \models \varphi\}\\ &\cup \{\forall x[\varphi_m(x) \rightarrow x < \mathbf{c}] : m \in \omega\}.\end{aligned}$$

By the compactness theorem, Γ has a model, which gives the desired conclusion.

$\boxed{5.4}$ *(Continuing exercise 5.3.) An element p of a model $\overline{M}$ of Γ is a prime iff $p > 1$ and for all $a, b \in M$, if $p = a \cdot b$ then $a = 1$ or $a = p$.*

(i) Prove that if $\overline{M}$ is a model of Γ with an infinite element, then it has an infinite prime element.

(ii) Show that the following conditions are equivalent:

(a) There are infinitely many (ordinary) primes p such that $p + 2$ is also prime. (The famous twin prime conjecture, unresolved at present.)

(b) There is a model $\overline{M}$ of Γ having at least one infinite prime p such that $p+2$ is also a prime.

(c) For every model $\overline{M}$ of Γ having an infinite element, there is an infinite prime p such that $p + 2$ is also a prime.

(i): The sentence $\forall m \exists p[m < p \wedge$ "p is a prime"$]$ holds in $\overline{N}$ and hence in $\overline{M}$. Applying this with m an infinite element yields an infinite prime.

(ii): (a)$\Rightarrow$(c): Assume (a), and let $\overline{M}$ be any model of Γ having an infinite element e. Now $\forall m \exists p[p$ "is a prime, and also $p + 2$ is a prime$]$ holds in $\overline{N}$, and hence also in $\overline{M}$. Applying this with m an infinite element gives the desired conclusion.

(c)$\Rightarrow$(b): By exercise 5.3, there is a model $\overline{M}$ of Γ having an infinite element. Hence (c) gives the conclusion of (b).

(b)$\Rightarrow$(a): Suppose that (a) is false. Choose $m \in \omega$ such that if p and $p+2$ are primes, then $p < m$. Then the sentence

$$\forall x[\varphi_m(x) \rightarrow \forall p[p \text{ "is a prime, and also } p + 2 \text{ is a prime"} \rightarrow p < x]]$$

holds in $\overline{N}$, and hence also in $\overline{M}$. But then there cannot exist an infinite prime p of $\overline{M}$ such that $p + 2$ is also a prime.

$\boxed{5.5}$ *Let G be a group which has elements of arbitrarily large finite order. Show that there is a group H elementarily equivalent to G which has an element of infinite order.*

Add an individual constant $\mathbf{c}$ to our language. Then the set

$$\{\varphi : \varphi \text{ is a sentence and } G \models \varphi\} \cup \{\neg(\mathbf{c} = e\} \cup \{\neg(\mathbf{c}^m = e) : m \in \omega \backslash 1\}$$

has a model by the compactness theorem, giving the desired result.

$\boxed{5.6}$ *Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

We prove the contrapositive: Suppose that for every finite subset Δ of Γ, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg\varphi\}$ has a model, so $\Gamma \cup \{\neg\varphi\}$ has a model, proving that $\Gamma \not\models \varphi$.