$\boxed{3.1}$ *Define $\varphi|\psi = \neg\varphi \wedge \neg\psi$. (The Sheffer stroke.). Show that $\neg$ and $\wedge$ can be defined in terms of $|$.*

$\neg\varphi$ is equivalent to $\varphi|\varphi$. Then $\varphi \wedge \psi$ is equivalent to $(\neg\varphi)|(\neg\psi)$.

$\boxed{3.2}$ *A formula φ involving only $S_0, \ldots, S_m$ determines a function $t_\varphi : {}^{m+1}2 \to 2$ defined by $t_\varphi(x) = \varphi[x]$ for any $x \in {}^{m+1}2$. Show that any member of $\bigcup_{0<m<\omega} {}^{({}^m2)}2$ can be obtained in this way.*

Let $0 < m < \omega$ and let $f \in {}^{{}^m2}2$. If f takes on only the value 0, then $f = t_\varphi$ with φ the formula $S_0 \wedge \neg S_0$. Suppose that f has at least one value 1. Let $M = \{x \in {}^m2 : f(x) = 1\}$. Consider the following formula φ:

$$\bigvee_{x\in M} \bigwedge_{i<m} S_i^{x(i)}.$$

Note that for any $x, y \in {}^m2$ we have $\left(\bigwedge_{i<m} S_i^{x(i)}\right)[y] = 1$ iff $x = y$. It follows that $\varphi[y] = 1$ iff $y \in M$. Hence $t_\varphi = f$.

$\boxed{3.3}$ *Show that the following formula is a tautology:*

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)]$$

(This formula can be used as a single axiom in an axiomatic development of sentential logic.)

A truth table for this formula would involve 32 rows; we want to avoid that. We argue by contradiction. Suppose that f is an assignment which gives our formula the value 0; we want to get a contradiction. It follows that

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau)[f] = 1 \tag{1}$$

and

$$[(\tau \to \varphi) \to (\theta \to \varphi)][f] = 0;$$

from this last condition we get

$$(\tau \to \varphi)[f] = 1 \tag{2}$$

and

$$(\theta \to \varphi)[f] = 0,$$

and the last condition here yields

$$\theta[f] = 1 \quad \text{and} \quad \varphi[f] = 0. \tag{3}$$

Hence from (2) we get

$$\tau[f] = 0.$$

Then (1) yields

$$\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\}[f] = 0,$$

from which we obtain

$$[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)][f] = 1 \tag{4}$$

and

$$\chi[f] = 0,$$

which yields

$$(\neg\chi)[f] = 1. \tag{5}$$

But from (3) we get $(\neg\theta)[f] = 0$, and hence by (5), $(\neg\chi \to \neg\theta)[f] = 0$. So by (4) we have $(\varphi \to \psi)[f] = 0$, so that $\varphi[f] = 1$ and $\psi[f] = 0$. This contradicts (3).