4. The main pcf theorems

The sets H_Ψ

We will shortly give several proofs involving the important general idea of making elementary chains inside the sets H_Ψ. Recall that H_Ψ, for an infinite cardinal Ψ, is the collection of all sets hereditarily of size less than Ψ, i.e., with transitive closure of size less than Ψ. We consider H_Ψ as a structure with $\in$ together with a well-ordering $<^*$ of it, possibly with other relations or functions, and consider elementary substructures of such structures.

Recall that A is an *elementary substructure* of B iff A is a subset of B, and for every formula $\varphi(x_0,\ldots,x_{m-1})$ and all $a_0,\ldots,a_{m-1}\in A$, $A\models\varphi(a_0,\ldots,a_{m-1})$ iff $B\models\varphi(a_0,\ldots,a_{m-1})$.

The basic downward Löwenheim-Skolem theorem will be used a lot. This theorem depends on the following lemma.

Lemma 4.1. (Tarski) *Suppose that A and B are first-order structures in the same language, with A a substructure of B. Then the following conditions are equivalent:*

(i) A is an elementary substructure of B.

(ii) For every formula of the form $\exists y\varphi(x_0,\ldots,x_{m-1},y)$ and all $a_0,\ldots,a_{m-1}\in A$, if $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$ then there is a $b\in A$ such that $B\models\varphi(a_0,\ldots,a_{m-1},b)$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Then by (i) we see that $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, so we can choose $b\in A$ such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $B\models\varphi(a_0,\ldots,a_{m-1},b)$, as desired.

(ii)⇒(i): Assume (ii). We show that for any formula $\varphi(x_0,\ldots,x_{m-1})$ and any elements $a_0,\ldots,a_{m-1}\in A$, $A\models\varphi(a_0,\ldots,a_{m-1})$ iff $B\models\varphi(a_0,\ldots,a_{m-1})$, by induction on φ. It is true for φ atomic by our assumption that A is a substructure of B. The induction steps involving $\neg$ and $\vee$ are clear. Now suppose that $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, with $a_0,\ldots,a_{m-1}\in A$. Choose $b\in A$ such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$. By the inductive assumption, $B\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, as desired.

Conversely, suppose that $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$. By (ii), choose $b\in A$ such that $B\models\varphi(a_0,\ldots,a_{m-1},b)$. By the inductive assumption, $A\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, as desired. □

Theorem 4.2. *Suppose that A is an L-structure, X is a subset of A, κ is an infinite cardinal, and κ is $\geq$ both $|X|$ and the number of formulas of $\mathscr{L}$, while $\kappa\leq|A|$. Then A has an elementary substructure B such that $X\subseteq B$ and $|B|=\kappa$.*

Proof. Let a well-order $\prec$ of A be given. We define $\langle C_n:n\in\omega\rangle$ by recursion. Let C_0 be a subset of A of size κ with $X\subseteq C_0$. Now suppose that C_n has been defined. Let M_n be the collection of all pairs of the form $(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)$ such that a is a sequence of elements of C_n of length m. For each such pair we define $f(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)$ to be the $\prec$-least element b of A such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$, if there is such an element, and otherwise let it be the least element of C_n. Then we define

$$C_{n+1}=C_n\cup\{f(\exists y\varphi(x_0,\ldots,x_{m-1},y),a):(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)\in M_n\}.$$

Finally, let $B=\bigcup_{n\in\omega}C_n$.

By induction it is clear that $|C_n| = \kappa$ for all $n \in \omega$, and so also $|B| = \kappa$.

Now to show that B is an elementary substructure of A we apply Lemma 4.1. To show that B is a substructure of A, let $\mathbf{F}$ be a fundamental operation of A, say of arity n, and suppose that $a \in {}^nB$. Choose m so that $a \in {}^nC_m$. Then

$$\mathbf{F}^A(a) = f(\exists y[\mathbf{F}(x_0, \ldots, x_n) = y], a) \in C_{m+1} \subseteq B,$$

as desired.

Now suppose that we are given a formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and elements $a_0, \ldots, a_{m-1}$ of B, and $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. Clearly there is an $n \in \omega$ such that $a_0, \ldots, a_{m-1} \in C_n$. Then $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n$, and the element $b \stackrel{\text{def}}{=} f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ is in $C_{n+1} \subseteq B$ and is such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. This is as desired in Lemma 4.1. $\square$

Given an elementary substructure A of a set H_Ψ, we will frequently use an argument of the following kind. A set theoretic formula holds in the real world, and involves only sets in A. By absoluteness, it holds in H_Ψ, and hence it holds in A. Thus we can transfer a statement to A even though A may not be transitive; and the procedure can be reversed.

This kind of argument depends on properties of transitive closures. For any set A, define a sequence $\langle B_{An} : n \in \omega\rangle$ as follows: $B_{A0} = A$, and $B_{A,n+1} = \bigcup B_{An}$. Then we define $\operatorname{tr\,cl}(A) = \bigcup_{n\in\omega} B_{An}$, the *transitive closure* of A. Thus $\operatorname{tr\,cl}(A)$ is a transitive set which contains A and is included in any transitive set whtih contains A. We summarize some properties of transitive closures:

Lemma 4.3. *(i) If $X \subseteq A$, then $\operatorname{tr\,cl}(X) \subseteq \operatorname{tr\,cl}(A)$.*
(ii) $\operatorname{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \operatorname{tr\,cl}(A)$.
(iii) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\mathscr{P}(A))| \leq 2^{|\operatorname{tr\,cl}(A)|}$.
(iv) $\operatorname{tr\,cl}(A \cup B) = \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.
(v) $\operatorname{tr\,cl}(A\times B) = (A\times B)\cup\{\{a\} : a \in A\}\cup\{\{a, b\} : a \in A, b \in B\}\cup\operatorname{tr\,cl}(A)\cup\operatorname{tr\,cl}(B)$.
(vi) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}(A \times B)| \leq \max(\operatorname{tr\,cl}(A), \operatorname{tr\,cl}(B)$.
(vii) $\operatorname{tr\,cl}({}^AB) \subseteq ({}^AB) \cup \operatorname{tr\,cl}(A \times B)$.
(viii) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^AB)| \leq 2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(A)|)}$.
(ix) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\prod A)| \leq 2^{|\operatorname{tr\,cl}(A)|}$.
(x) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^A(\prod B))| \leq 2^{2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(B)|)}}$.
(xi) If A is a set of infinite cardinals, then $\operatorname{tr\,cl}(A) = \bigcup A$.
(xii) If A is an infinite set of regular cardinals, then $|\operatorname{tr\,cl}(\operatorname{pcf}(A))| \leq 2^{|\operatorname{tr\,cl}(A)|}$.

Proof. (i)–(viii) are clear. For (ix), note that $\prod A \subseteq {}^A\bigcup A$, so (ix) follows from (viii). For (x),

$$\begin{aligned}\left|\operatorname{tr\,cl}\left({}^A\left(\prod B\right)\right)\right| &\leq 2^{\max(|\operatorname{tr\,cl}(A),|\operatorname{tr\,cl}(\prod B))} \quad \text{by (viii)}\\ &\leq 2^{\max(|\operatorname{tr\,cl}(A),2^{|\operatorname{tr\,cl}(B)})}\\ &\leq 2^{2^{\max(|\operatorname{tr\,cl}(A)|,|\operatorname{tr\,cl}(B)|)}}.\end{aligned}$$

(xi) is clear.

Finally, for (xii), by (xi) it suffices to show that $\bigcup \mathrm{pcf}(A) \leq 2^{\mathrm{tr\,cl}(A)}$. So, let $\alpha \in \bigcup \mathrm{pcf}(A)$. Choose $\lambda \in \mathrm{pcf}(A)$ such that $\alpha < \lambda$. Now there is a one-one function mapping λ into $\prod A/F$ for some ultrafilter F on A; hence $\lambda \leq |\prod A|$. So, using (ix),

$$\alpha < \lambda \leq \left|\prod A\right| \leq \left|\mathrm{tr\,cl}\left(\prod A\right)\right| \leq 2^{\mathrm{tr\,cl}(A)}. \qquad \square$$

We also need the fact that some rather complicated formulas and functions are absolute for sets H_Ψ. Note that H_Ψ is transitive. Many of the indicated formulas are not absolute for H_Ψ in general, but only under the assumptions given that Ψ is much larger than the sets in question.

Lemma 4.4. *Suppose that Ψ is an uncountable regular cardinal. Then the following formulas (as detailed in the proof) are absolute for H_Ψ.*

(i) $B = \mathscr{P}(A)$.

(ii) "D is an ultrafilter on A".

(iii) κ is a cardinal.

(iv) κ is a regular cardinal.

(v) "κ and λ are cardinals, and $\lambda = \kappa^+$".

(vi) $\kappa = |A|$.

(vii) $B = \prod A$.

(viii) $A = {}^B C$.

(ix) "A is infinite", if Ψ is uncountable.

(x) "A is an infinite set of regular cardinals and D is an ultrafilter on A and λ is a regular cardinal and $f \in {}^\lambda \prod A$ and f is strictly increasing and cofinal modulo D", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xi) "A is an infinite set of regular cardinals, and $B = \mathrm{pcf}(A)$", if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xii) "A is an infinite set of regular cardinals and $f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xiii) "A is an infinite set of regular cardinals and $B = \langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ and $\forall \lambda \in \mathrm{pcf}(A)(B_\lambda$ is a λ-generator)", if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

Proof. Absoluteness follows by easy arguments upon producing suitable formulas, as follows.

(i): Suppose that $A, B \in H_\Psi$. We may take the formula $B = \mathscr{P}(A)$ to be

$$\forall x \in B[\forall y \in x(y \in A)] \wedge \forall x[\forall y \in x(y \in A) \rightarrow x \in B].$$

The first part is obviously absolute for H_Ψ. If the second part holds in V it clearly holds in H_Ψ. Now suppose that the second part holds in H_Ψ. Suppose that $x \subseteq A$. Hence $x \in H_\Psi$ and it follows that $x \in B$.

(ii): Assume that $A, D \in H_\Psi$. We can take the statement "D is an ultrafilter on A" to be the following statement:

$$\begin{aligned}&\forall X \in D(X \subseteq A) \wedge A \in D \wedge \forall X, Y \in D(X \cap Y \in D) \wedge \emptyset \notin D \\ &\wedge \forall Y \forall X \in D[X \subseteq Y \wedge Y \subseteq A \rightarrow Y \in D] \wedge \forall Y[Y \subseteq A \rightarrow Y \in D \vee (A \backslash Y) \in D].\end{aligned}$$

Again this is absolute because $Y \subseteq A$ implies that $Y \in H_\Psi$.

(iii): Suppose that $\kappa \in H_\Psi$. Then

$$\kappa \text{ is a cardinal} \quad \text{iff} \quad \begin{aligned}[t] &\kappa \text{ is an ordinal and } \forall f[f \text{ is a function and} \\ &\text{dmn}(f) = \kappa \text{ and } \text{rng}(f) \in \kappa \to f \text{ is not one-to-one}]. \end{aligned}$$

Note here that if f is a function with $\text{dmn}(f) = \kappa$ and $\text{rng}(f) \subseteq \kappa$, then $f \subseteq \kappa \times \kappa$, and hence $f \in H_\Psi$.

(iv): Assume that $\kappa \in H_\Psi$. Then

$$\kappa \text{ is a regular cardinal} \quad \text{iff} \quad \begin{aligned}[t] &\kappa \text{ is a cardinal, } 1 < \kappa, \text{ and } \forall f[f \text{ is a function} \\ &\text{and } \text{dmn}(f) \in \kappa \text{ and } \text{rng}(f) \subseteq \kappa \text{ and} \\ &\forall \alpha, \beta \in \text{dmn}(f)(\alpha < \beta \to f(\alpha) < f(\beta)) \\ &\to \exists \gamma < \kappa \forall \alpha \in \text{dmn}(f)(f(\alpha) \in \gamma)]. \end{aligned}$$

(v): Assume that $\kappa, \lambda \in H_\Psi$. Then ($\kappa$ and λ are cardinals and $\lambda = \kappa^+$) iff

$$\begin{aligned} &\kappa \text{ is a cardinal and } \lambda \text{ is a cardinal and } \kappa < \lambda \\ &\text{and } \forall \alpha < \lambda[\kappa < \alpha \to \exists f[f \text{ is a function and } \text{dmn}(f) = \kappa \\ &\text{and } \text{rng}(f) = \alpha \text{ and } f \text{ is one-one and } \text{rng}(f) = \alpha]]. \end{aligned}$$

(vi): Suppose that $\kappa, A \in H_\Psi$. Then

$$\kappa = |A| \quad \text{iff} \quad \begin{aligned}[t] &\kappa \text{ is a cardinal and } \exists f[f \text{ is a function} \\ &\text{and } \text{dmn}(f) = \kappa \text{ and } \text{rng}(f) = A \text{ and } f \text{ is one-to-one}] \end{aligned}$$

(vii): Assume that $A, B \in H_\Psi$. Then

$$B = \prod A \quad \text{iff} \quad \begin{aligned}[t] &\forall f \in B[f \text{ is a function and } \text{dmn}(f) = A \text{ and} \\ &\forall x \in A[f(x) \in x]] \text{ and } \forall f[f \text{ is a function and} \\ &\text{dmn}(f) = A \text{ and } \forall x \in A[f(x) \in x] \to f \in B]. \end{aligned}$$

Note that if f is a function with domain A and $f(x) \in x$ for all $x \in A$, then $f \subseteq A \times \bigcup A$, and hence $f \in H_\Psi$.

(viii): Suppose that $A, B, C \in H_\Psi$. Then

$$A = {}^B C \quad \text{iff} \quad \begin{aligned}[t] &\forall f \in A[f \text{ is a function and } \text{dmn}(f) = B \\ &\text{and } \text{rng}(f) \subseteq C] \text{ and } \forall f[f \text{ is a function} \\ &\text{and } \text{dmn}(f) = B \text{ and } \text{rng}(f) \subseteq C \to f \in A]. \end{aligned}$$

(ix): "A is infinite" iff $\exists f(f$ is a one-one function, $\text{dmn}(f) = \omega$, and $\text{rng}(f) \subseteq A)$.

(x): Suppose that $A, D, \lambda, f \in H_\Psi$, and $2^{|\mathrm{tr\,cl}(A)|}) < \Psi$. Then $\prod A \in H_\Psi$ by Lemma 4.3(ix). Now

$$\begin{aligned}&A \text{ is an infinite set of regular cardinals and } D \text{ is an ultrafilter on } A\\&\text{and } \lambda \text{ is a regular cardinal and } f \in {}^{\lambda}\prod A \text{ and } f \text{ is strictly}\\&\text{increasing and cofinal modulo } D\end{aligned}$$

iff

$$\begin{aligned}&A \text{ is infinite and } \forall x \in A[x \text{ is a regular cardinal}] \text{ and } D \text{ is an ultrafilter on } A \text{ and}\\&\lambda \text{ is a regular cardinal and } \exists B\Bigg[B = \prod A \text{ and } f \text{ is a function}\\&\qquad \text{and } \mathrm{dmn}(f) = \lambda \text{ and } \mathrm{rng}(f) \subseteq B \text{ and}\\&\qquad \forall \xi, \eta < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow f_\xi(a) < f_\eta(a)] \to X \in D]\\&\qquad \text{and } \forall g \in B \exists \xi < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow g(a) < f_\xi(a)] \to X \in D]\Bigg].\end{aligned}$$

(xi): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$, and $A, B \in H_\Psi$. Let $\varphi(A, D, \lambda, f)$ be the statement of (xi). Note:

(1) If $\varphi(A, D, \lambda, f)$, then $D, \lambda, f \in H_\Psi$, and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \leq 2^{|\mathrm{tr\,cl}(A)|}$.

In fact, $D \subseteq \mathscr{P}(A)$, so $\mathrm{tr\,cl}(D) \subseteq \mathrm{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \mathrm{tr\,cl}(A)$, and so $|\mathrm{tr\,cl}(D)| < \Psi$ by Lemma 4.3(iii); so $D \in H_\Psi$. Now f is a one-one function from λ into $\prod A$, so $\lambda \leq |\prod A| < \Psi$, and hence $\lambda \in H_\Psi$ and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \leq 2^{|\mathrm{tr\,cl}(A)|}$. Finally, $f \subseteq \lambda \times \prod A$, so it follows that $f \in H_\Psi$.

Thus (1) holds. Hence the following equivalence shows the absoluteness of the statement in (xi):

$$A \text{ is an infinite set of regular cardinals and } B = \mathrm{pcf}(A)$$

iff

$$\begin{aligned}&A \text{ is infinite, and } \forall \mu \in A(\mu \text{ is a regular cardinal}) \wedge \forall \lambda \in B \exists D \exists f \varphi(A, D, \lambda, f)\\&\wedge \forall D \forall \lambda \forall f[\varphi(A, D, \lambda, f) \to \lambda \in B].\end{aligned}$$

(xii): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$. By Lemma 4.3(xii) we have $\mathrm{pcf}(A) \in H_\Psi$. Hence

$$A \text{ is an infinite set of regular cardinals } \wedge f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$$

iff

A is infinite and $\forall\kappa\in A(\kappa$ is a regular cardinal and

f is a function and $\exists B[B=\mathrm{pcf}(A)\wedge B=\mathrm{dmn}(f)]$

$\forall\lambda\in\mathrm{dmn}(f)\forall X\subseteq A[A\in f(\lambda)$ iff $\exists C[C=\mathrm{pcf}(X)\wedge C\subseteq\lambda]]$

(xiii): Assume that $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$, and $A,B\in H_\Psi$. Note as above that $\mathrm{pcf}(A)\in H_\Psi$. Note that for any cardinal λ we have $J_{<\lambda}[A]\subseteq\mathscr{P}(A)$ and, with f as in (xi), $f\subseteq\mathrm{pcf}(A)\times\mathscr{P}(\mathscr{P}(A))$; so $f\in H_\Psi$. Let $\varphi(f,A)$ be the formula of (xii). Thus

A is a set of regular cardinals and $B=\langle B_\lambda:\lambda\in\mathrm{pcf}(A)\rangle$

and $\forall\lambda\in\mathrm{pcf}(A)(B_\lambda$ is a λ-generator)

iff

B is a function and $\exists C[C=\mathrm{pcf}(A)\wedge C=\mathrm{dmn}(B)]\wedge\exists f[\varphi(f,A)\wedge$

$\forall\lambda\in\mathrm{dmn}(B)\forall\mu\in\mathrm{dmn}(B)[\lambda$ is a cardinal and μ is a cardinal and

$\mu=\lambda^+\rightarrow B_\lambda\in f(\mu)\wedge\forall X\subseteq A[X\in f(\mu)$ iff $X\backslash B_\lambda\in f(\lambda)]]]$ □

Now we turn to the consideration of elementary substructures of H_Ψ. The following lemma gives basic facts used below.

Lemma 4.5. *Suppose that Ψ is an uncountable cardinal, and N is an elementary substructure of H_Ψ (under $\in$ and a well-order of H_Ψ).*

(i) For every ordinal α, $\alpha\in N$ iff $\alpha+1\in N$.

(ii) $\omega\subseteq N$.

(iii) If $a\in N$, then $\{a\}\in N$.

(iv) If $a,b\in N$, then $\{a,b\},(a,b)\in N$.

(v) If $A,B\in N$, then $A\times B\in N$.

(vi) If $A\in N$ then $\bigcup A\in N$.

(vii) If $f\in N$ is a function, then $\mathrm{dmn}(f),\mathrm{rng}(f)\in N$.

(viii) If $f\in N$ is a function and $a\in N\cap\mathrm{dmn}(f)$, then $f(a)\in N$.

(ix) If $X,Y\in N$, $X\subseteq N$, and $|Y|\le|X|$, then $Y\subseteq N$.

(x) If $X\in N$ and $X\neq\emptyset$, then $X\cap N\neq\emptyset$.

(xi) $\mathscr{P}(A)\in N$ if $A\in N$ and $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xii) If ρ is an infinite ordinal, $|\rho|^+<\Psi$, and $\rho\in N$, then $|\rho|\in N$ and $|\rho|^+\in N$.

(xiii) If $A\in N$, then $\prod A\in N$ if $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xiv) If $A\in N$, A is a set of regular cardinals, and $A\subseteq H_\Psi$, then $\mathrm{pcf}(A)\in N$ if $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xv) If $A\in N$, A is a set of regular cardinals, then $\langle J_{<\lambda}[A]:\lambda\in\mathrm{pcf}(A)\rangle\in N$ if $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$.

(xvi) If $A\in N$ and A is a set of regular cardinals, then there is a function $\langle B_\lambda:\lambda\in\mathrm{pcf}(A)\rangle\in N$, where for each $\lambda\in\mathrm{pcf}(A)$, the set B_λ is a λ-generator, if $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$.

Proof. (i): Let α be an ordinal, and suppose that $\alpha \in N$. Then $\alpha \in H_\Psi$, and hence $\alpha \cup \{\alpha\} \in H_\Psi$. By absoluteness, $H_\Psi \models \exists x(x = \alpha \cup \{\alpha\})$, so $N \models \exists x(x = \alpha \cup \{\alpha\})$. Choose $b \in N$ such that $N \models b = \alpha \cup \{\alpha\}$. Then $H_\Psi \models b = \alpha \cup \{\alpha\}$, so by absoluteness, $b = \alpha \cup \{\alpha\}$. This proves that $\alpha \cup \{\alpha\} \in N$.

The method used in proving (i) can be used in the other parts; so it suffices in most other cases just to indicate a formula which can be used.

(ii): An easy induction, using the formulas $\exists x \forall y \in x(y \neq y)$ and $\exists x[a \subseteq x \wedge a \in x \wedge \forall y \in x[y \in a \vee y = a]]$.

(iii): Use the formula $\exists x[\forall y \in x(y = a) \wedge a \in x]$.

(iv): Similar to (iii).

(v): Use the formula

$$\exists C[\forall a \in A \forall b \in B[(a,b) \in C] \wedge \forall x \in C \exists a \in A \exists b \in B[x = (a,b)]].$$

(vi): Use the formula $\exists B[\forall x \in A[x \subseteq B] \wedge \forall y \in B \exists x \in A(y \in x)]$.

(vii): Use the formula $\exists A[\forall x \forall y[(x,y) \in f \to x \in A] \wedge \forall x \in A \exists y[(x,y) \in f]]$. Note that this formula is absolute for H_Ψ for example $(x,y) \in f \in H_\Psi$ implies that $x, y \in H_\Psi$.

(viii): Use the formula $\exists x[(a,x) \in f]$.

(ix): Let f be a function mapping X onto Y (assuming, as we may, that $Y \neq \emptyset$). Then $f \in H_\Psi$, so by the above method, we get another function $g \in N$ which maps X onto Y. Now (viii) gives the conclusion of (ix).

(x): Use the formula $\exists x \in X[x = x]$.

(xi): $\mathscr{P}(A) \in H_\Psi$ by Lemma 4.3(iii). Hence we can use the formula

$$\exists B[\forall x \in B(x \subseteq A) \wedge \forall x[x \subseteq A \to x \in B]].$$

(xii): Assume that ρ is an infinite ordinal and $\rho \in N$. Then

$$H_\Psi \models \exists \alpha \leq \rho[(\exists f : \rho \to \alpha, \text{ a bijection}) \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta, \text{ a bijection}) \to \alpha \leq \beta]].$$

Hence by the standard argument, there are $\alpha, f \in N$ such that

$$H_\Psi \models f : \rho \to \alpha \text{ is a bijection } \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta, \text{ a bijection}) \to \alpha \leq \beta].$$

Clearly then $\alpha = |\rho|$.

For $|\rho|^+$, use the formula

$$\begin{aligned}\exists \beta \exists \Gamma \Big[&\forall \gamma \in \Gamma \exists f[f \text{ is a bijection from } \rho \text{ onto } \gamma] \\ &\wedge \forall \gamma \forall f[f \text{ is a bijection from } \rho \text{ onto } \gamma \to \gamma \in \Gamma] \\ &\wedge \beta = \bigcup \Gamma \Big].\end{aligned}$$

(xiii): Note that $\prod A \in H_\Psi$ by Lemma 4.3(ix). Then use the formula

$$\begin{aligned}\exists B \Big[&\forall f \in B(f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a)) \\ &\wedge \forall f[f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a) \to f \in B] \Big].\end{aligned}$$

(xiv): $\mathrm{pcf}(A) \in H_\Psi$ by Lemma 4.3(xi), so by Lemma 4.4(xi) we can use the formula $\exists B[B = \mathrm{pcf}(A)]$.

(xv): We have $\mathrm{pcf}(A) \in H_\Psi$, and hence easily $\langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle \in H_\Psi$. Hence by Lemma 4.4(xii) we can use the formula $\exists f[f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle]$.

(xvi): By Lemma 4.3(iii),(xi) and Lemma 4.4(xiii) we can use the formula

$$\exists B[B : \mathrm{pcf}(A) \to \mathscr{P}(A) \wedge \forall \lambda \in \mathrm{pcf}(A)[B_\lambda \text{ is a } \lambda \text{ generator for } A]]. \qquad \square$$

Definition. Let κ be a regular cardinal. An elementary substructure N of H_Ψ is κ-*presentable* iff there is an increasing and continuous chain $\langle N_\alpha : \alpha < \kappa\rangle$ of elementary substructures of H_Ψ such that:

(1) $|N| = \kappa$ and $\kappa + 1 \subseteq N$.

(2) $N = \bigcup_{\alpha<\kappa} N_\alpha$.

(3) For every $\alpha < \kappa$, the function $\langle N_\beta : \beta \le \alpha\rangle$ is a member of $N_{\alpha+1}$.

It is obvious how to construct a κ-presentable substructure of H_Ψ.

Lemma 4.6. *If N is a κ-presentable substructure of H_Ψ, with notation as above, and if $\alpha < \kappa$, then:*

(i) $\alpha + 1 \in N_{\alpha+1}$;

(ii) $\alpha + \omega \subseteq N_{\alpha+1}$;

(iii) $N_\alpha \in N_{\alpha+1}$.

Proof. Since $\langle N_\beta : \beta \le \alpha\rangle \in N_{\alpha+1}$, (i) follows from Lemma 4.5 (vii). Hence $\alpha + 1 \subseteq N_{\alpha+1}$ by induction, and (ii) follows by Lemma 4.5(i). (iii) holds by Lemma 4.5(viii). $\square$

For any set M, we let $\overline{M}$ be the set of all ordinals α such that $\alpha \in M$ or $M \cap \alpha$ is unbounded in α.

Lemma 4.7. *If N is a κ-presentable substructure of H_Ψ, with notation as above, then*

(i) If $\alpha < \kappa$, then $\overline{N_\alpha} \subseteq N$.

(ii) If $\kappa < \alpha \in \overline{N}\backslash N$, then α is a limit ordinal and $\mathrm{cf}(\alpha) = \kappa$, and in fact there is a closed unbounded subset E of α such that $E \subseteq N$ and E has order type κ.

Proof. First we consider (i). Suppose that $\gamma \in \overline{N_\alpha}$. We may assume that $\gamma \notin N_\alpha$.

Case 1. $\gamma = \sup(N_\alpha \cap \mathrm{Ord})$. Then

$$H_\Psi \models \exists\gamma'[\forall\delta(\delta \in N_\alpha \to \delta \le \gamma') \wedge \forall\varepsilon[\forall\delta(\delta \in N_\alpha \to \delta \le \varepsilon) \to \gamma' \le \varepsilon]];$$

in fact, our given γ is the unique γ' for which this holds. Hence this statement holds in N, as desired.

Case 2. $\exists\theta \in N_\alpha(\gamma < \theta)$. We may assume that θ is minimum with this property. Now for any $\beta \in N_\alpha$ we can let $\rho(\beta)$ be the supremum of all ordinals in N_α which are less than β. So $\rho(\theta) = \gamma$. By absoluteness we get

$$\begin{aligned} H_\Psi \models &\forall\beta \in N_\alpha \exists\rho[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \rho) \\ &\wedge \forall\chi[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \chi) \to \rho \le \chi]]; \end{aligned}$$

Hence N models this formula too; applying it to θ in place of β, we get $\rho \in N$ such that

$$
\begin{aligned}
N \models & \forall \varepsilon \in N_\alpha(\varepsilon < \theta \rightarrow \varepsilon < \rho) \\
& \wedge \forall \chi[\forall \varepsilon \in N_\alpha(\varepsilon < \theta \rightarrow \varepsilon < \chi) \rightarrow \rho \leq \chi].
\end{aligned}
$$

Thus $\gamma = \rho \in N$, as desired. This proves (i).

For (ii), suppose that $\kappa < \alpha \in \overline{N} \backslash N$. Let $E = \{\sup(\alpha \cap N_\xi) : \xi < \kappa\}$. Note that if $\xi < \kappa$, then by (i), $\sup(\alpha \cap N_\xi) \in N$. So $E \subseteq N$. It is clearly closed in α. It is unbounded, since for any $\beta \in \alpha \cap N$ there is a $\xi < \kappa$ such that $\beta \in N_\xi$, and so $\beta \leq \sup(\alpha \cap N_\xi) \in N$. $\square$

For any set N we define the *characteristic function* of N; it is defined for each regular cardinal μ as follows:

$$\mathrm{Ch}_N(\mu) = \sup(N \cap \mu).$$

Proposition 4.8. *Let κ be a regular cardinal, let N be a κ-presentable substructure of H_Ψ, and let μ be a regular cardinal.*

(i) If $\mu \leq \kappa$, then $\mathrm{Ch}_N(\mu) = \mu \in N$.

(ii) If $\kappa < \mu$, then $\mathrm{Ch}_N(\mu) \notin N$, $\mathrm{Ch}_N(\mu) < \mu$, and $\mathrm{Ch}_N(\mu)$ has cofinality κ.

(iii) For every $\alpha \in \overline{N} \cap \mu$ we have $\alpha \leq \mathrm{Ch}_N(\mu)$.

Proof. (i): True since $\kappa + 1 \subseteq N$.

(ii): Since $|N| = \kappa < \mu$ and μ is regular, we must have $\mathrm{Ch}_N(\mu) \notin N$ and $\mathrm{Ch}_N(\mu) < \mu$. Then $\mathrm{Ch}_N(\mu)$ has cofinality κ by Lemma 4.7.

(iii): clear. $\square$

Theorem 4.9. *Suppose that M and N are elementary substructures of H_Ψ and $\kappa < \mu$ are cardinals, with $\mu < \Psi$.*

(i) If $M \cap \kappa \subseteq N \cap \kappa$ and $\sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu \subseteq N \cap \mu$.

(ii) If M and N are both κ-presentable and if $\sup(M \cap \nu^+) = \sup(N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu = N \cap \mu$.

Proof. (i): Assume the hypothesis. We prove by induction on cardinals δ in the interval $[\kappa, \mu]$ that $M \cap \delta \subseteq N \cap \delta$. This is given for $\delta = \kappa$. If, inductively, δ is a limit cardinal, then the desired conclusion is clear. So assume now that δ is a cardinal, $\kappa \leq \delta < \mu$, and $M \cap \delta \subseteq N \cap \delta$. If $\delta^+ \notin M$, then by Lemma 4.5(xii), $[\delta, \delta^+] \cap M = \emptyset$, so the desired conclusion is immediate from the inductive hypothesis. So, assume that $\delta^+ \in M$. Then the hypothesis of (i) implies that there are ordinals in $[\delta, \delta^+]$ which are in $M \cap N$, and hence by Lemma 4.5(xii) again, $\delta^+ \in N$. Now to show that $M \cap [\delta, \delta^+] \subseteq N \cap [\delta, \delta^+]$, take any ordinal $\gamma \in M \cap [\delta, \delta^+]$. We may assume that $\gamma < \delta^+$. Since $\sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$ by assumption, we can choose $\beta \in M \cap N \cap \delta^+$ such that $\gamma < \beta$. Let f be the $<^*$-smallest bijection from β to δ. So $f \in M \cap N$. Since $\gamma \in M$, we also have $f(\gamma) \in M$ by Lemma 4.5(viii). Now $f(\gamma) < \delta$, so by the inductive assumption that $M \cap \delta \subseteq N \cap \delta$, we have $f(\gamma) \in N$. Since $f \in N$, so is f^{-1}, and $f^{-1}(f(\gamma)) = \gamma \in N$, as desired. This finishes the proof of (i).

(ii): Assume the hypothesis. Now we want to check the hypothesis of (i). By the definition of κ-presentable we have $\kappa = M \cap \kappa = N \cap \kappa$. Now suppose that ν is a cardinal and $\nu^+ \le \mu$ with $\nu^+ \in M$. We may assume that $\kappa < \nu^+$. Let $\gamma = \mathrm{Ch}_M(\nu^+)$; this is the same as $\mathrm{Ch}_N(\nu^+)$ by the hypothesis of (ii). By Lemma 4.8 we have $\gamma \notin M \cup N$; hence by Lemma 4.7 there are clubs P, Q in γ such that $P \subseteq M$ and $Q \subseteq N$. Moreover, $\mathrm{cf}(\gamma) = \kappa$, so that $M \cap N$ is also club in γ. Hence $\sup(M \cap \nu^+) = \sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$. This verifies the hypothesis of (i) for the pair M, N and also for the pair N, M. So our conclusion follows. $\square$

Minimally obedient sequences

Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B is a λ-generator for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is called *persistently cofinal* for λ, B provided that $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. Recall that this means that for all $h \in \prod B$ there is a $\xi_0 < \lambda$ such that for all ξ, if $\xi_0 \le \xi < \lambda$, then $h <_{J_{<\lambda}[B]} (f_\xi \restriction B)$.

Lemma 4.10. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B and C are λ-generators for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is persistently cofinal for λ, B iff it is persistently cofinal for λ, C.*

Proof. Suppose that $\langle f_\xi : \xi < \lambda \rangle$ is persistently cofinal for λ, B, and suppose that $h \in \prod C$. Let $k \in \prod B$ be any function such that $h \restriction (B \cap C) = k \restriction (B \cap C)$. Choose $\xi_0 < \lambda$ such that for all $\xi \in [\xi_0, \lambda)$ we have $k <_{J_{<\lambda}[B]} (f_\xi \restriction B)$. Then for any $\xi \in [\xi_0, \lambda)$ we have

$$\begin{aligned}\{a \in C : h(a) \ge f_\xi(a)\} &= \{a \in B \cap C : h(a) \ge f_\xi(a)\} \cup \{a \in C \backslash B : h(a) \ge f_\xi(a)\} \\ &\subseteq \{a \in B : k(a) \ge f_\xi(a)\} \cup (C \backslash B);\end{aligned}$$

Now $(C \backslash B) \in J_{<\lambda}[A]$ by Lemma 3.17(xi), so $h <_{J_{<\lambda}[C]} (f_\xi \restriction C)$. By symmetry the lemma follows. $\square$

Because of this lemma we say that f is persistently cofinal for a $\lambda \in \mathrm{pcf}(A)$ iff it is persistently cofinal for λ, B for some λ-generator B.

Lemma 4.11. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $A \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$.*

Then for every $\xi < \lambda$ there is an $\alpha < \kappa$ such that for any $a \in A$,

$$f_\xi(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a).$$

Proof.

$$\begin{aligned}\mathrm{Ch}_N(a) &= \sup(N \cap a) \\ &= \bigcup (N \cap a) \\ &= \bigcup \left(a \cap \bigcup_{\alpha < \kappa} N_\alpha \right) \\ &= \bigcup_{\alpha < \kappa} \bigcup (N_\alpha \cap a) \\ &= \bigcup_{\alpha < \kappa} \mathrm{Ch}_{N_\alpha}(a).\end{aligned}$$

Hence for every $a \in A$ for which $f_\xi(a) < \mathrm{Ch}_N(a)$, there is an $\alpha_a < \kappa$ such that $f_\xi(a) < Ch_{N_{\alpha_a}}(a)$. Hence the existence of α as indicated follows. $\square$

Lemma 4.12. *Suppose that A is progressive, κ is regular, $\lambda \in \mathrm{pcf}(A)$, and $A, \lambda \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and Ψ is big. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle \in N$ is a sequence of functions in $\prod A$ which is persistently cofinal in λ. Then for every $\xi \geq \mathrm{Ch}_N(\lambda)$ the set*

$$\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\}$$

is a λ-generator for A.

Proof. Assume the hypothesis, including $\xi \geq \mathrm{Ch}_N(\lambda)$. Let α be as in Lemma 4.11. We are going to apply Lemma 3.17(ix). Since $A, f, \lambda \in N$, we may assume that $A, f, \lambda \in N_0$, by renumbering the elementary chain if necessary. Now $\kappa \subseteq N$, and $|A| < \kappa$, so we easily see that there is a bijection $f \in N$ mapping an ordinal $< \kappa$ onto A; hence $A \subseteq N$ by Lemma 4.5(viii), and so $A \subseteq N_\beta$ for some $\beta < \kappa$. We may assume that $A \subseteq N_0$. By Lemma 4.5(xvi),(viii), there is a λ-generator B which is in N_0.

Now the sequence f is persistently cofinal in $\prod B/J_{<\lambda}$, and hence

$$H_\Psi \models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B]; \quad \text{hence}$$
$$N \models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B];$$

Hence for every $h \in N$, if $h \in \prod B$ then there is an $\eta < \lambda$ with $\eta \in N$ such that $N \models \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\varphi \restriction B]$; going up, we see that really for every $h \in N \cap \prod A$ there is an $\eta_h \in N \cap \lambda$ such that for all ρ with $\rho \geq \eta_h$ we have $h \restriction B <_{J_{<\lambda}} f_\rho \restriction B$. Since ξ, as given in the statement of the Lemma, is $\geq$ each member of $N \cap \lambda$, hence $\geq \eta_h$ for each $h \in N \cap \prod A$, we see that

$$(1) \qquad h \restriction B <_{J_{<\lambda}} f_\xi \restriction B \text{ for every } h \in N \cap \prod A.$$

Now we can apply (1) to $h = \mathrm{Ch}_{N_\alpha}$, since this function is clearly in N. So $\mathrm{Ch}_{N\alpha} \restriction B <_{J_{<\lambda}[B]} f_\xi \restriction B$. Hence by the choice of α (see Lemma 4.11)

$$(2) \qquad \mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}[B]} f_\xi \restriction B.$$

Note that (2) says that $B \backslash \{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[B]$.

Now $\lambda \notin \mathrm{pcf}(A \backslash B)$ by Lemma 3.17(ii), and hence $J_{<\lambda}[A \backslash B] = J_{\leq\lambda}[A \backslash B]$. So by Theorem 3.4 we see that $\prod(A \backslash B)/J_{<\lambda}[A \backslash B]$ is λ^+-directed, so $\langle f_\eta \restriction (A \backslash B) : \eta < \lambda \rangle$ has an upper bound $h \in \prod(A \backslash B)$. We may assume that $h \in N$, by the usual argument. Hence

$$f_\xi \restriction (A \backslash B) <_{J_{<\lambda}[A \backslash B]} h < \mathrm{Ch}_N \restriction (A \backslash B);$$

hence $\{a \in A \backslash B : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[A]$, and together with (2) and using Lemma 3.17(ix) this finishes the proof. $\square$

Now suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \le \mathrm{cf}(\delta) < \min(A)$, and E is a club of δ of order type $\mathrm{cf}(\delta)$. Then we define

$$h_E = \sup\{f_\xi : \xi \in E\}.$$

We call h_E the *supremum along E of f*. Thus $h_E \in \prod A$, since $\mathrm{cf}(\delta) < \min(A)$. Note that if $E_1 \subseteq E_2$ then $h_{E_1} \le h_{E_2}$.

Lemma 4.14. *Let A, δ, f be as above. Then there is a unique function g in $\prod A$ such that the following two conditions hold.*

(i) There is a club C of δ of order type $\mathrm{cf}(\delta)$ such that $g = h_C$.

(ii) If E is any club of δ of order type $\mathrm{cf}(\delta)$, then $g \le h_E$.

Proof. Clearly such a function g is unique if it exists.

Now suppose that there is no such function g. Then for every club C of δ of order type $\mathrm{cf}(\delta)$ there is a club D of order type $\mathrm{cf}(\delta)$ such that $h_C \not\le h_D$. Thus there is an $a \in A$ such that $h_C(a) > h_D(a)$, i.e., $\sup\{f_\xi(a) : \xi \in C\} > \sup\{f_\xi(a) : \xi \in D$, hence also $\sup\{f_\xi(a) : \xi \in C\} > \sup\{f_\xi(a) : \xi \in C \cap D$. So $h_C \not\le h_{C\cap D}$. Hence there is a decreasing sequence $\langle E_\alpha : \alpha < |A|^+\rangle$ of clubs of δ such that for every $\alpha < |A|^+$ we have $h_{E_\alpha} \not\le h_{E_{\alpha+1}}$. Now note that

$$|A|^+ = \bigcup_{a\in A}\{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}.$$

Hence there is an $a \in A$ such that $M \stackrel{\text{def}}{=} \{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}$ has size $|A|^+$. Now $h_{E_\alpha}(a) \ge h_{E_\beta}(a)$ whenever $\alpha < \beta < |A|^+$, so this gives an infinite decreasing sequence of ordinals, contradiction. □

The function g of this lemma is called the *minimal club-obedient bound* of f.

Corollary 4.15. *Suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \le \mathrm{cf}(\delta) < \min(A)$, J is an ideal on A, and f is $<_J$-increasing. Let g be the minimal club-obedient bound of f. Then g is a $\le_J$-bound for f.* □

Now suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. We say that $f = \langle f_\alpha : \alpha < \lambda\rangle$ is *κ-minimally obedient for λ* iff f is a universal sequence for λ and for every $\delta < \lambda$ of cofinality κ, f_δ is the minimal club-obedient bound of f.

A sequence f is *minimally obedient for* λ iff $|A|^+ < \min(A)$ and f is minimally obedient for every regular κ such that $|A| < \kappa < \min(A)$.

Lemma 4.16. *Suppose that $|A|^+ < \min(A)$ and $\lambda \in \mathrm{pcf}(A)$. Then there is a minimally obedient sequence for λ.*

Proof. By Theorem 4.16 let $\langle f^0_\xi : \xi < \lambda\rangle$ be a universal sequence for λ. Now by induction we define functions f_ξ for $\xi < \lambda$. Let $f_0 = f^0_0$, and choose $f_{\xi+1}$ so that $\max(f_\xi, f^0_\xi) < f_{\xi+1}$.

For limit $\delta < \lambda$ such that $|A| < \mathrm{cf}(\delta) < \min(A)$, let f_δ be the minimally club-obedient bound of $\langle f_\xi : \xi < \delta\rangle$.

For other limit $\delta < \lambda$, use the λ-directedness (Theorem 9.8) to get f_δ as a $<_{J_{<\lambda}}$-bound of $\langle f_\xi : \xi < \delta\rangle$.

Thus we have assured the minimally obedient property, and it is clear that $\langle f_\xi : \xi < \lambda\rangle$ is universal. $\square$

Lemma 4.17. *Suppose that A is progressive, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. Also assume the following:*

(i) $\lambda \in \mathrm{pcf}(A)$.

(ii) $f = \langle f_\xi : \xi < \lambda\rangle$ is a κ-minimally obedient sequence for λ.

(iii) N is a κ-presentable elementary substructure of H_Ψ, with Ψ large, such that $\lambda, f, A \in N$.

Then the following conditions hold:

(iv) For every $\gamma \in \overline{N} \cap \lambda \backslash N$ we have:

(a) $\mathrm{cf}(\gamma) = \kappa$.

(b) There is a club C of γ of order type κ such that $f_\gamma = \sup\{f_\xi : \xi \in C\}$ and $C \subseteq N$.

(c) $f_\gamma(a) \in \overline{N} \cap a$ for every $a \in A$.

(v) If $\gamma = \mathrm{Ch}_N(\lambda)$, then:

(a) $\gamma \in \overline{N} \cap \lambda \backslash N$; hence we let C be as in (iv)(b), with $f_\gamma = \sup\{f_\xi : \xi \in C\}$.

(b) $f_\xi \in N$ for each $\xi \in C$.

(c) $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$.

(vi) $\gamma = \mathrm{Ch}_N(\lambda)$ and C is as in (iv)(b), with $f_\gamma = \sup\{f_\xi : \xi \in C\}$, and B is a λ generator, then for every $h \in N \cap \prod A$ there is a $\xi \in C$ such that $(h \restriction B) <_{J_{<\lambda}} (f_\xi \restriction B)$.

Proof. Assume (i)–(iii). Note that $A \subseteq N$, by Lemma 4.5(ix), applied to κ, A in place of X, Y.

For (iv), suppose also that $\gamma \in \overline{N} \cap \lambda \backslash N$. Then by Lemma 4.7 we have $\mathrm{cf}(\gamma) = \kappa$, and there is a club E in γ of order type κ such that $E \subseteq N$. By (ii), we have $f_\gamma = f_C$ for some club C of γ of order type κ. By the minimally obedient property we have $f_C = f_{C\cap E}$, and thus we may assume that $C \subseteq E$. For any $\xi \in C$ and $a \in A$ we have $f_\xi(a) \in N$ by Lemma 4.5(viii). So (iv) holds.

For (v), suppose that $\gamma = \mathrm{Ch}_N(\lambda)$. Then $\gamma \in \overline{N} \cap \lambda \backslash N$ because $|N| = \kappa < \min(A) \leq \lambda$. For each $\xi \in C$ we have $f_\xi \in N$ by Lemma 4.5(viii). For (c), if $a \in A$, then $f_\gamma(a) = \sup_{\xi \in C} f_\xi(a) \leq \mathrm{Ch}_N(a)$, since $f_\xi(a) \in N \cap a$ for all $\xi \in C$.

Next, assume the hypotheses of (vi). By Lemma 4.11, f is persistently cofinal in λ, so by Lemma 4.13, $B' \stackrel{\text{def}}{=} \{a \in A : \mathrm{Ch}_N(a) \leq f_\gamma(a)\}$ is a λ-generator. By Lemma 4.23(v) there is a $\xi \in C$ such that $h \restriction B' <_{J_{<\lambda}} f_\xi \restriction B'$. Now $B =_{J_{<\lambda}[A]} B'$ by Lemma 4.23(xi), so

$$\{a \in B : h(a) \geq f_\xi(b)\} \subseteq (B \backslash B') \cup \{a \in B' : h(a) \geq f_\xi(b)\} \in J_{<\lambda}[A]. \qquad \square$$

We now define some abbreviations.

$H_1(A, \kappa, N, \Psi)$ abbreviates
A is a progressive set of regular cardinals, κ is a regular cardinal such that $\lvert A\rvert < \kappa < \min(A)$, and N is a κ-presentable elementary substructure of H_Ψ, with Ψ big and $A \in N$.

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_1(A, \kappa, N, \Psi)$, $\lambda \in \mathrm{pcf}(A)$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of members of $\prod A$, $f \in N$, and $\gamma = \mathrm{Ch}_N(\lambda)$.

$H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $\{a \in A : \mathrm{Ch}_N(a) \leq f_\gamma(a)\}$ is a λ-generator.

$H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and the following hold:

(i) $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$.

(ii) For every $h \in N \cap \prod A$ there is a $d \in N \cap \prod A$ such that for any λ-generator B,

$$(h \restriction B) <_{J_{<\lambda}} (d \restriction B) \quad \text{and} \quad d \leq f_\gamma.$$

Thus $H_1(A, \kappa, N, \Psi)$ is part of the hypothesis of Lemma 4.17, and $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ is a part of the hypotheses of Lemma 4.17(v).

Lemma 4.18. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is persistently cofinal for λ, then $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds.*

Proof. This follows immediately from Lemma 4.13. $\square$

Lemma 4.19. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is κ-minimally obedient for λ, then both $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold.*

Proof. Since f is κ-minimally obedient for λ, it is a universal sequence for λ, by definition. Hence by Lemma 4.11 f is persistently cofinal for λ, and so property H_3 follows from Lemma 4.18.

For H_4, note that $\lambda, A \in N$ since $f \in N$, by Lemma 4.5(vii),(ix). Hence the hypotheses of Lemma 4.17(v) hold. So (i) in H_4 holds by Lemma 4.17(v)(c). For condition (ii), suppose that $h \in N \cap \prod A$. Take B and C as in Lemma 4.17(vi), and choose $\xi \in C$ such that $h \restriction B <_{J_{<\lambda}} f_\xi \restriction B$. Let $d = f_\xi$. Clearly this proves condition (ii). $\square$

The following obvious extension of Lemma 4.19 will be useful below.

Lemma 4.20. *Assume $H_1(A, \kappa, N, \Psi)$, and also assume that $\gamma = \mathrm{Ch}_N(\lambda)$ and*

(i) $f \overset{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A) \rangle$ is a sequence of sequences $\langle f^\lambda_\xi : \xi < \lambda \rangle$ each of which is κ-minimally obedient for λ.

Then for each $\lambda \in N \cap \mathrm{pcf}(A)$, $H_3(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma)$ and $H_4(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma)$ hold. $\square$

Lemma 4.21. *Suppose that $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold. Then*

(i) $\{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}$ is a λ-generator.

(ii) If $\lambda = \max(\mathrm{pcf}(A))$, then $\{a \in A : f_\gamma(a) < \mathrm{Ch}_N(a)\} \in J_{<\lambda}[A]$.

Proof. By (i) of $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \upharpoonright A)$, so (i) holds by $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$. (ii) follows from $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and Lemma 4.23(xii). $\square$

Lemma 4.22. *Assume that $H_3(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold. Let*

$$b = \{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}.$$

Then

(i) b is a λ-generator.

(ii) There is a set $b' \subseteq b$ such that:

(a) $b' \in N$;

(b) $b \backslash b' \in J_{<\lambda}[A]$;

(c) b' is a λ-generator.

Proof. (i) holds by Lemma 4.21(i). For (ii), by Lemma 4.12 choose $\alpha < \kappa$ such that, for every $a \in A$,

$$f_\gamma(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\gamma(a) < \mathrm{Ch}_{N_\alpha}(a). \tag{1}$$

Now by (i) of $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \upharpoonright A)$. Hence by (1) we see that for every $a \in A$,

$$a \in b \quad \text{iff} \quad \mathrm{Ch}_{N_\alpha}(a) \leq f_\gamma(a). \tag{2}$$

Now by (ii) of $H_4(A, \kappa, N, \Psi, \lambda, f, \gamma)$ applied to $h = \mathrm{Ch}_{N_\alpha} \upharpoonright A$, there is a $d \in N \cap \prod A$ such that the following conditions hold:

(3) $(\mathrm{Ch}_{N_\alpha} \upharpoonright b) <_{J_{<\lambda}} (d \upharpoonright b)$.

(4) $d \leq f_\gamma$.

Now we define

$$b' = \{a \in A : \mathrm{Ch}_{N_\alpha}(a) \leq d(a)\}.$$

Clearly $b' \in N$. Also, by (3),

$$b \backslash b' = \{a \in b : d(a) < \mathrm{Ch}_{N_\alpha}(a)\} \in J_{<\lambda},$$

and so (ii)(b) holds. If $a \in b'$, then $\mathrm{Ch}_{N_\alpha}(a) \leq d(a) \leq f_\gamma(a)$ by (4), so $a \in b$ by (2). Thus $b' \subseteq b$. Now (ii)(c) holds by Lemma 4.23(ix). $\square$

Lemma 4.23. *Assume $H_1(A, \kappa, N, \Psi)$ and $A \in N$. Suppose that $\langle f^\lambda : \lambda \in \mathrm{pcf}(A) \rangle \in N$ is an array of sequences $\langle f^\lambda_\xi : \xi < \lambda \rangle$ with each $f^\lambda_\xi \in \prod A$. Also assume that for every $\lambda \in N \cap \mathrm{pcf}(A)$, both $H_3(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ and $H_4(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ hold.*

Then there exist cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $\mathrm{pcf}(A) \cap N$ such that

$$(\mathrm{Ch}_N \upharpoonright A) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}, \ldots, f^{\lambda_n}_{\gamma(\lambda_n)}\}.$$

Proof. We will define by induction a descending sequence of cardinals $\lambda_i \in \mathrm{pcf}(A) \cap N$ and sets $A_i \in \mathscr{P}(A) \cap N$ (strictly decreasing under inclusion as i grows) such that if $A_i \neq \emptyset$ then $\lambda_i = \max(\mathrm{pcf}(A_i))$ and

$$(1) \qquad (\mathrm{Ch}_N \restriction (A \backslash A_{i+1})) = \sup\{(f^{\lambda_0}_{\gamma(\lambda_0)} \restriction (A \backslash A_{i+1})), \ldots, (f^{\lambda_i}_{\gamma(\lambda_i)} \restriction (A \backslash A_{i+1}))\}.$$

Since the cardinals are decreasing, there is a first i such that $A_{i+1} = \emptyset$, and then the lemma is proved. To start, $A_0 = A$ and $\lambda_0 = \max(\mathrm{pcf}(A))$. Clearly $\lambda_0 \in N$. Now suppose that λ_i and A_i are defined, with $A_i \neq 0$. By Lemma 4.22(i) and Lemma 4.23(x), the set

$$\{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$$

is a λ_i-generator. Hence by Lemma 4.22(ii) we get another λ_i-generator b'_{λ_i} such that

(2) $b'_{\lambda_i} \in N$.

(3) $b'_{\lambda_i} \subseteq \{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$.

Note that $b'_{\lambda_i} \neq \emptyset$. Let $A_{i+1} = A_i \backslash b'_{\lambda_i}$. Thus $A_{i+1} \in N$. Furthermore,

(4) $A \backslash A_{i+1} = (A \backslash A_i) \cup b'_{\lambda_1}$.

Now by Lemma 4.23(ii) and $\lambda_i = \max(\mathrm{pcf}(A_i))$ we have $\lambda_i \notin \mathrm{pcf}(A_{i+1})$. If $A_{i+1} \neq \emptyset$, we let $\lambda_{i+1} = \max(\mathrm{pcf}(A_{i+1}))$. Now by (i) of $H_4(A, \kappa, N, \Psi, \lambda, f^{\lambda_j}, \gamma(\lambda_j))$ we have

(5) $f^{\lambda_j}_{\gamma(\lambda_j)} \leq (\mathrm{Ch}_N \restriction A)$ for all $j \leq i$.

Now suppose that $a \in A \backslash A_{i+1}$. If $a \in A_i$, then by (4), $a \in b'_{\lambda_1}$, and so by (3), $\mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_1)}(a)$, and (1) holds for a. If $a \notin A_i$, then $A \neq A_i$, so $i \neq 0$. Hence by the inductive hypothesis for (1),

$$\mathrm{Ch}_N(a) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}(a), \ldots, f^{\lambda_{i-1}}_{\gamma(\lambda_{i-1})}(a)\},$$

and (1) for a follows by (5). □

Theorem 4.24. *Suppose that μ is singular and $\kappa < \mu$ is an uncountable regular cardinal such that $A \stackrel{\text{def}}{=} (\kappa, \mu)_{\mathrm{reg}}$ has size $\leq \kappa$. Then*

$$\mathrm{cf}([\mu]^\kappa, \subseteq) = \max(\mathrm{pcf}(A)).$$

Proof. Note by the progressiveness of A that every limit cardinal in the interval (κ, μ) is singular, and hence every member of A is a successor cardinal.

First we prove $\geq$. Suppose to the contrary that $\mathrm{cf}([\mu]^\kappa, \subseteq) < \max(\mathrm{pcf}(A))$. For brevity write $\max(\mathrm{pcf}(A)) = \lambda$. let $\{X_i : i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality less than λ. Pick a universal sequence $\langle f_\xi : \xi < \lambda\rangle$ for λ by Theorem 4.16. For every $\xi < \lambda$, $\mathrm{rng}(f_\xi)$ is a subset of μ of size $\leq |A| \leq \kappa$, and hence $\mathrm{rng}(f_\xi)$ is covered by some X_i. Thus $\lambda = \bigcup_{i \in I}\{\xi < \lambda : \mathrm{rng}(f_\xi) \subseteq X_i\}$, so by $|I| < \lambda$ and the regularity of λ we get an $i \in I$ such that $|\{\xi < \lambda : \mathrm{rng}(f_\xi) \subseteq X_i\}| = \lambda$. Now define for any $a \in A$,

$$h(a) = \sup(a \cap X_i).$$

Since $\kappa < a$ for each $a \in A$, we have $h \in \prod A$. Now the sequence $\langle f_\xi : \xi < \lambda \rangle$ is cofinal in $\prod A$ under $<_{J_{<\lambda}}$ by Lemma 4.17. So there is a $\xi < \lambda$ such that $h <_{J_{<\lambda}} f_\xi$. Thus there is an $a \in A$ such that $h(a) < f_\xi(a) \in X_i$, contradicting the definition of h.

Second we prove $\leq$, by exhibiting a cofinal subset of $[\mu]^\kappa$ of size at most $\max(\mathrm{pcf}(A))$. Take N and Ψ so that $H_1(A, \kappa, N, \Psi)$. Let $\mathscr{M}$ be the set of all κ-presented elementary substructures M of H_Ψ such that $A \subseteq M$, and let

$$F = \{M \cap \mu : M \in \mathscr{M}\} \backslash [\mu]^{<\kappa}.$$

Since $|M| = \kappa$, we have $|M \cap \mu| \leq \kappa$, and so $\forall M \in F(|M \cap \mu| = \kappa)$.

(1) F is cofinal in $[\mu]^\kappa$.

In fact, for any $X \in [\mu]^\kappa$ we can find $M \in \mathscr{M}$ such that $X \subseteq M$, and (1) follows.

By (1) it suffices to prove that $|F| \leq \max(\mathrm{pcf}(A))$.

Claim. If $M, N \in \mathscr{M}$ are such that $\mathrm{Ch}_M \restriction A = \mathrm{Ch}_N \restriction A$, then $M \cap \mu = N \cap \mu$.

For, if ν^+ is a successor cardinal $\leq \mu$, then $\sup(M \cap \nu^+) = \mathrm{Ch}_M(\nu^+) = \mathrm{Ch}_N(\nu^+) = \sup(N \cap \nu^+)$. So the claim holds by Theorem 4.9.

Now for each $M \in \mathscr{M}$, let $g(M)$ be the sequence $\langle (\lambda_0, \gamma_0), \ldots, (\lambda_n, \gamma_n) \rangle$ given by Lemma 4.23. Clearly the range of g has size $\leq \max(\mathrm{pcf}(A))$. Now for each $X \in F$, choose $M_X \in \mathscr{M}$ such that $X = M_X \cap \mu$. Then for $X, Y \in F$ and $X \neq Y$ we have $M_X \cap \mu \neq M_Y \cap \mu$, hence by the claim $\mathrm{Ch}_{M_X} \restriction A \neq \mathrm{Ch}_{M_Y} \restriction A$, and hence by Lemma 4.23, $g(M_X) \neq g(M_Y)$. This proves that $|F| \leq \max(\mathrm{pcf}(A))$. □

Corollary 4.25. *Let $A = \{\aleph_m : 1 < m < \omega\}$. Then for any $m \in \omega$ we have $\mathrm{cf}([\aleph_\omega]^{\aleph_m}) = \max(\mathrm{pcf}(A))$.*

Proof. Immediate from Lemma 2.27(vi) and Theorem 4.24. □

Elevations and transitive generators

We start with some simple general notions about cardinals. If B is a set of cardinals, then a *walk* in B is a sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of members of B. Such a walk is necessarily finite. Given cardinals $\lambda_0 > \lambda$ in B, a *walk from λ_0 to λ* is a walk as above with $\lambda_n = \lambda$. We denote by $F_{\lambda_0,\lambda}(B)$ the set of all walks from λ_0 to λ.

Now suppose that A is progressive and $\lambda_0 \in \mathrm{pcf}(A)$. A *special walk from λ_0 to λ_n in* $\mathrm{pcf}(A)$ is a walk $\lambda_0 > \cdots > \lambda_n$ in $\mathrm{pcf}(A)$ such that $\lambda_i \in A$ for all $i > 0$. We denote by $F'_{\lambda_0,\lambda}(A)$ the collection of all special walks from λ_0 to λ in $\mathrm{pcf}(A)$.

Next, suppose in addition that $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A) \rangle$ is a sequence of sequences, where each f^λ is a sequence $\langle f^\lambda_\xi : \xi < \lambda \rangle$ of members of $\prod A$. If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_0 \in \lambda_0$, then we define an associated sequence of ordinals by setting

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1})$$

for all $i < n$. Note that $\gamma_i < \lambda_i$ for all $i = 0, \ldots, n$. Then we define

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

Now we define the *elevation* of the sequence f, denoted by $f^e \stackrel{\text{def}}{=} \langle f^{\lambda,e} : \lambda \in \mathrm{pcf}(A)\rangle$, by setting, for any $\lambda_0 \in \mathrm{pcf}(A)$, any $\gamma_0 \in \lambda_0$, and any $\lambda \in A$,

$$f^{\lambda_0,e}_{\gamma_0}(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \le \lambda, \\ \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0,\ldots,\lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Note here that the superscript ^{e} is only notational, standing for "elevated".

Lemma 4.26. *Assume the above notation. Then $f^{\lambda_0}_{\gamma_0} \le f^{\lambda_0,e}_{\gamma_0}$ for all $\lambda_0 \in \mathrm{pcf}(A)$ and all $\gamma_0 \in \lambda_0$.*

Proof. Take any $\gamma_0 \in \lambda_0$ and any $\lambda \in A$. If $\lambda_0 \le \lambda$, then $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose that $\lambda < \lambda_0$. If the above maximum does not exist, then again $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose the maximum exists. Now $(\lambda_0,\lambda) \in F'_{\lambda_0,\lambda}(A)$, so

$$f^{\lambda_0}_{\gamma_0}(\lambda) = \mathrm{El}_{\lambda_0,\lambda}(\gamma_0) \le \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0,\ldots,\lambda_n) \in F'_{\lambda_0,\lambda}\}) = f^{\lambda_0,e}_{\gamma_0}(\lambda). \qquad \square$$

Lemma 4.27. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume also $H_1(A,\kappa,N,\Psi)$ and $f \in N$.*

Then also $f^e \in N$.

Proof. The proof is a more complicated instance of our standard procedure for going from V to H_Ψ to N and then back. We sketch the details.

Assume the hypotheses. In particular, $A \in N$. Hence also $\mathrm{pcf}(A) \in N$. Also, $|A| < \kappa$, so $A \subseteq N$. Now clearly $F' \in N$. Also, $\mathrm{El} \in N$. (Note that El depends upon A.) Then by absoluteness,

$$H_\Psi \models \exists g \;\; g \text{ is a function, } \mathrm{dmn}(g) = \mathrm{pcf}(A) \wedge \forall \lambda_0 \in \mathrm{pcf}(A) \forall \gamma_0 \in \lambda_0 \forall \lambda \in A$$

$$g(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \le \lambda, \\ \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0,\ldots,\lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Now the usual procedure can be applied. $\square$

Lemma 4.28. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume $H_1(A,\kappa,N,\Psi)$ and $f \in N$.*

Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, and let $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$.

(i) If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$*, and $\gamma_1, \ldots, \gamma_n$ are formed as above, then $\gamma_i \in \overline{N}$ for all $i = 0, \ldots, n$.*

(ii) For every $\lambda \in A \cap \lambda_0$ we have $f^{\lambda_0,e}_{\gamma_0}(\lambda) \in \overline{N}$.

Proof. (i): By induction, using Lemma 4.17(iv)(c).

(ii): immediate from (i). □

Lemma 4.29. *Assume the hypotheses of Lemma 4.28. Then*

(i) For any special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$, we have

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \leq \mathrm{Ch}_N(\lambda).$$

(ii) $f^{\lambda_0,e}_{\gamma_0} \leq \mathrm{Ch}_N \restriction A$ for every $\gamma_0 < \lambda_0$.

(iii) If there is a special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$ such that

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \mathrm{Ch}_N(\lambda),$$

then

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda).$$

(iv) Suppose that $\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda) = \gamma$. If there is an $a \in A \cap \lambda$ such that $f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a)$, then also $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$.

Proof. (i) is immediate from Lemma 4.28(i) and Lemma 4.8(iii). (ii) and (iii) follow from (i). For (iv), by Lemma 4.28(i) and (i) there are special walks $\lambda_0 > \cdots > \lambda_n = \lambda$ and $\lambda = \lambda'_0 > \cdots > \lambda'_m = a$ such that

$$\begin{aligned} f^{\lambda_0,e}_{\gamma_0}(\lambda) &= \mathrm{Ch}_N(\lambda) = \mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \quad \text{and} \\ f^{\lambda,e}_{\gamma}(a) &= \mathrm{Ch}_N(a) = \mathrm{El}_{\lambda'_0,\ldots,\lambda'_m}(a). \end{aligned}$$

It follows that

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n,\lambda'_1,\ldots,a}(\gamma_0) = \mathrm{Ch}_N(a),$$

and (iii) then gives $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$. □

Definition. Suppose that A is progressive and $A \subseteq P \subseteq \mathrm{pcf}(A)$. A system $\langle b_\lambda : \lambda \in P\rangle$ of subsets of A is *transitive* iff for all $\lambda \in P$ and all $\mu \in b_\lambda$ we have $b_\mu \subseteq b_\lambda$.

Theorem 4.30. *Suppose that $H_1(A, \kappa, N, \Psi)$, $f = \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a system of functions, and each f^λ is κ-minimally obedient for λ. Let f^e be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A : \mathrm{Ch}_N(a) = f^{\lambda_0,e}_{\gamma_0}(a)\}.$$

Then the following hold for each $\lambda_0 \in \mathrm{pcf}(A) \cap N$:

(i) b_{λ_0} is a λ_0-generator.

(ii) There is a $b'_{\lambda_0} \subseteq b_{\lambda_0}$ such that

(a) $b_{\lambda_0} \backslash b'_{\lambda_0} \in J_{<\lambda_0}[A]$.

(b) $b'_{\lambda_0} \in N$ *(each one individually, not the sequence).*

(c) b'_{λ_0} *is a* λ_0*-generator.*

(iii) The system $\langle b_\lambda : \lambda \in \mathrm{pcf}(A) \cap N\rangle$ *is transitive.*

Proof. Note that $H_2(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemma 4.27. By definition, minimally obedient implies universal, so f^{λ_0} is persistently cofinal by Lemma 4.11. Hence by Lemma 4.26, $f^{\lambda_0,e}$ is persistently cofinal, and so $H_3(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemma 4.18. Also, by Lemma 4.19 $H_4(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0}, \gamma_0)$ holds, so the condition $H_4(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemmas 4.26 and 4.29(ii). Now (i) and (ii) hold by Lemma 4.22.

Now suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\lambda \in b_{\lambda_0}$. Thus

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda),$$

where $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$. Write $\gamma = \mathrm{Ch}_N(\lambda)$. We want to show that $b_\lambda \subseteq b_{\lambda_0}$. Take any $a \in b_\lambda$. So $\mathrm{Ch}_N(a) = f^{\lambda,e}_\gamma(a)$. By Lemma 4.29(iv) we get $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$, so $a \in b_{\lambda_0}$, as desired. ∎

Localization

Theorem 4.31. *Suppose that A is a progressive set. Then there is no subset $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$ and, for every $b \in B$, $b > \max(\mathrm{pcf}(B \cap b))$.*

Proof. Assume the contrary. We may assume that $|A|^+ < \min(A)$. In fact, if we know the result under this assumption, and now $|A|^+ = \min(A)$, suppose that $B \subseteq \mathrm{pcf}(A)$ with $|B| = |A|^+$ and $\forall b \in B[b > \max(\mathrm{pcf}(B \cap b))]$. Let $A' = A\backslash\{|A|^+\}$. Then let $B' = B\backslash\{|A|^+\}$. So by Proposition 9.1(vi) we have $B' \subseteq \mathrm{pcf}(A')$. Clearly $|B'| = |A'|^+$ and $\forall b \in B'[b > \max(\mathrm{pcf}(B' \cap b))]$, contradiction.

Also, clearly we may assume that B has order type $|A|^+$.

Let $E = A \cup B$. Then $|E| < \min(E)$. Let $\kappa = |E|$. By Lemma 4.16, we get an array $\langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$, with each f^λ κ-minimally obedient for λ. Choose N and Ψ so that $H_1(A, \kappa, N, \Psi)$, with N containing $A, B, E, \langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$. Now let $\langle b_\lambda : \lambda \in \mathrm{pcf}(E) \cap N\rangle$ be the set of transitive generators as guaranteed by Theorem 4.30. Let $b'_\lambda \in N$ be such that $b'_\lambda \subseteq b_\lambda$ and $b_\lambda\backslash b'_\lambda \in J_{<\lambda}$.

Now let F be the function with domain $\{a \in A : \exists\beta \in B(a \in b_\beta)\}$ such that for each such a, $F(a)$ is the least $\beta \in B$ such that $a \in b_\beta$. Define $B_0 = \{\gamma \in B : \exists a \in \mathrm{dmn}(F)(\gamma \le F(a)\}$. Thus B_0 is an initial segment of B of size at most $|A|$. Clearly $B_0 \in N$. We let $\beta_0 = \min(B\backslash B_0)$; so $B_0 = B \cap \beta_0$.

Now we claim

(1) There exists a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

We prove more: we find a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b'_{\lambda_0} \cup \ldots \cup b'_{\lambda_n}$. Let $\lambda_0 = \max(\mathrm{pcf}(B_0))$. Since $B_0 \in N$, we clearly have $\lambda_0 \in N$ and hence $b'_{\lambda_0} \in N$. So $B_1 \stackrel{\mathrm{def}}{=} B_0\backslash b'_{\lambda_0} \in N$. Now suppose that $B_k \subseteq B_0$ has been defined so that $B_k \in N$. If $B_k = \emptyset$, the construction stops. Suppose that

$B_k \neq \emptyset$. Let $\lambda_k = \max(\mathrm{pcf}(B_k))$. Clearly $\lambda_k \in N$, so $b'_{\lambda_k} \in N$ and $B_{\kappa+1} \stackrel{\mathrm{def}}{=} B_k \backslash b'_{\lambda_k} \in N$. Since $B_{\kappa+1} = B_k \backslash b'_{\lambda_k}$ and b'_{λ_k} is a λ_k-generator, from Lemma 4.23(xii) it follows that $\lambda_0 > \lambda_1 > \cdots$. So the construction eventually stops; say that $B_{n+1} = \emptyset$. So $B_n \subseteq b'_{\lambda_n}$. So

$$\begin{aligned} B_0 &\subseteq b'_{\lambda_0} \cup (B_0 \backslash b'_{\lambda_0}) \\ &= b'_{\lambda_0} \cup B_1 \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup B_2 \\ &\ldots\ldots\ldots \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup B_n \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup b'_{\lambda_n}. \end{aligned}$$

This proves (1).

Note that $\beta_0 > \max(\mathrm{pcf}(B \cap \beta_0) = \max(\mathrm{pcf}(B_0)) = \lambda_0$ by the initial assumption of the proof. Next, we claim

(2) $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

Consider any cardinal $a \in b_{\beta_0} \cap A$. Since $\beta_0 \in B$, we have $a \in \mathrm{dmn}(F)$, and since $\beta_0 \notin B_0$ we have $F(a) < \beta_0$. Let $\beta = F(a)$. So $a \in b_\beta$, and $\beta < \beta_0$, so by the minimality of β_0, $\beta \in B_0$. Since $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$, it follows that $\beta \in b_{\lambda_i}$ for some $i = 0, \ldots, n$. But transitivity implies that $b_\beta \subseteq b_{\lambda_i}$, and hence $a \in b_{\lambda_i}$, as desired. So (2) holds.

Using Lemma 2.27(ii),(iii), by (2) we have

$$\mathrm{pcf}(b_{\beta_0}) \subseteq \mathrm{pcf}(b_{\lambda_0}) \cup \ldots \cup \mathrm{pcf}(b_{\lambda_n}). \tag{3}$$

Now $\beta_0 \notin \mathrm{pcf}(A \backslash b_{\beta_0})$ by Lemma 4.23(ii), but $\beta_0 \in B \subseteq \mathrm{pcf}(A)$, so $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A$. Hence $\beta_0 \in \mathrm{pcf}(b_{\lambda_i}$ for some $i \leq n$, so $\beta_0 \leq \lambda_i$ by Lemma 4.23(vii). This contradicts what was stated before (2) above. □

Theorem 4.32. (Localization) *Suppose that A is a progressive set of regular cardinals. Suppose that $B \subseteq \mathrm{pcf}(A)$ is also progressive. Then for every $\lambda \in \mathrm{pcf}(B)$ there is a $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.*

Proof. We prove by induction on λ that if A and B satisfy the hypotheses of the theorem, then the conclusion holds. Let C be a λ-generator over B. Thus $C \subseteq B$ and $\lambda = \max(\mathrm{pcf}(C))$ by Lemma 4.23(vii). Now $C \subseteq \mathrm{pcf}(A)$ and C is progressive. It suffices to find $B_0 \subseteq C$ with $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.

Let $C_0 = C$ and $\lambda_0 = \lambda$. Suppose that $C_0 \supseteq \cdots \supseteq C_i$ and $\lambda_0 > \cdots > \lambda_i$ have been constructed so that $\lambda = \max(\mathrm{pcf}(C_i))$ and C_i is a λ-generator over B. If there is no maximal element of $\lambda \cap \mathrm{pcf}(C_i)$ we stop the construction. Otherwise, let λ_{i+1} be that maximum element, let D_{i+1} be a λ_{i+1}-generator over B, and let $C_{i+1} = C_i \backslash D_{i+1}$. Now $D_{i+1} \in J_{\leq \lambda_{i+1}}[B] \subseteq J_{<\lambda}[B]$, so C_{i+1} is still a λ-generator of B by Lemma 4.23(ix), and $\lambda = \max(\mathrm{pcf}(C_{i+1}))$ by Lemma 4.23(vii). Note that $\lambda_{i+1} \notin \mathrm{pcf}(C_{i+1})$, by Lemma 4.23(ii).

This construction must eventually stop, when $\lambda \cap C_i$ does not have a maximal element; we fix the index i.

(1) There is an $E \subseteq \lambda \cap \mathrm{pcf}(C_i)$ such that $|E| \leq |A|$ and $\lambda \in \mathrm{pcf}(E)$.

In fact, suppose that no such E exists. We now construct a strictly increasing sequence $\langle \gamma_j : j < |A|^+ \rangle$ of elements of $\lambda \cap \mathrm{pcf}(C_i)$ such that $\gamma_k > \max(\mathrm{pcf}(\{\gamma_j : j < k\})$ for all $k < |A|^+$. (This contradicts Theorem 4.31.) Suppose that $\{\gamma_j : j < k\} = E$ has been defined. Now $\lambda \notin \mathrm{pcf}(E)$ by the supposition after (1), and $\lambda < \max(\mathrm{pcf}(E))$ is impossible since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(C_i)$ and $\lambda = \max(\mathrm{pcf}(C_i))$. So $\lambda > \max(\mathrm{pcf}(E))$. Hence, because $\lambda \cap C_i$ does not have a maximal element, we can choose $\gamma_k \in \lambda \cap C_i$ such that $\gamma_k > \max(\mathrm{pcf}(E))$, as desired. Hence (1) holds.

We take E as in (1). Apply the inductive hypothesis to each $\gamma \in E$ and to A, E in place of A, B; we get a set $G_\gamma \subseteq E$ such that $|G_\gamma| \leq |A|$ and $\gamma \in \mathrm{pcf}(G_\gamma)$. Let $H = \bigcup_{\gamma \in E} G_\gamma$. Note that $|H| \leq |A|$. Thus $E \subseteq \mathrm{pcf}(H)$. Since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(H)$ by Theorem 4.13, we have $\lambda \in \mathrm{pcf}(H)$, completing the inductive proof. $\square$

The size of $\mathrm{pcf}(A)$

Theorem 4.33. *If A is a progressive interval of regular cardinals, then $|\mathrm{pcf}(A)| < |A|^{+4}$.*

Proof. Assume that A is a progressive interval of regular cardinals but $|\mathrm{pcf}(A)| \geq |A|^{+4}$. Let $\rho = |A|$. We will define a set B of size ρ^+ consisting of cardinals in $\mathrm{pcf}(A)$ such that each cardinal in B is greater than $\max(\mathrm{pcf}(B \cap b))$. This will contradict Theorem 4.31.

Let $S = S^{\rho^{+3}}_{\rho^+}$; so S is a stationary subset of ρ^{+3}. By Theorem 1.3, let $\langle C_k : k \in S \rangle$ be a club guessing sequence. Thus

(1) C_k is a club in k of order type ρ^+, for each $k \in S$.

(2) If D is a club in ρ^{+3}, then there is a $k \in D \cap S$ such that $C_k \subseteq D$.

Let σ be the ordinal such that $\aleph_\sigma = \min(A)$, and let $\max(\mathrm{pcf}(A)) = \aleph_\gamma$. Choose δ such that $\sigma + \delta = \gamma$. So $|A|^{+4} \leq |\mathrm{pcf}(A)| \leq \delta$. Now $\mathrm{pcf}(A)$ is an interval of regular cardinals by Theorem 4.12. So $\mathrm{pcf}(A)$ contains all regular cardinals in the set $\{\aleph_{\sigma+\alpha} : \alpha < \rho^{+4}\}$.

Now we are going to define a strictly increasing continuous sequence $\langle \alpha_i : i < \rho^{+3} \rangle$ of ordinals less than ρ^{+4}.

1. Let $\alpha_0 = \rho^{+3}$.
2. For i limit let $\alpha_i = \bigcup_{j<i} \alpha_j$.
3. Now suppose that α_j has been defined for all $j \leq i$; we define α_{i+1}. For each $k \in S$ let $e_k = \{\aleph_{\sigma+\alpha_j+1} : j \in C_k \cap (i+1)\}$. Thus e_k is a subset of $\mathrm{pcf}(A)$. If $\max(\mathrm{pcf}(e_k)) < \aleph_{\sigma+\rho^{+4}}$, let β_k be an ordinal such that $\max(\mathrm{pcf}(e_k)) < \aleph_{\sigma+\beta_k}$ and $\beta_k < \rho^{+4}$; otherwise let $\beta_k = 0$. Let α_{i+1} be greater than α_i and all β_k for $k \in S$, with $\alpha_{i+1} < \rho^{+4}$. This is possible because $|S| = \rho^{+3}$. Thus

(3) For every $k \in S$, if $\max(\mathrm{pcf}(e_k)) < \aleph_{\sigma+\rho^{+4}}$, then $\max(\mathrm{pcf}(e_k)) < \aleph_{\sigma+\alpha_{i+1}}$.

This finishes the definition of the sequence $\langle \alpha_i : i < \rho^{+3} \rangle$. Let $D = \{\alpha_i : i < \rho^{+3}\}$, and let $\delta = \sup(D)$. Then D is club in δ. Let $\mu = \aleph_{\sigma+\delta}$. Thus μ has cofinality ρ^{+3}, and it is singular since $\delta > \alpha_0 = \rho^{+3}$. Now we apply Corollary 4.32: there is a club C_0 in μ such that $\mu^+ = \max(\mathrm{pcf}(C_0^{(+)}))$. We may assume that $C_0 \subseteq [\aleph_\sigma, \mu)$. so we can write

$C_0 = \{\aleph_{\sigma+i} : i \in D_0\}$ for some club D_0 in δ. Let $D_1 = D_0 \cap D$. So D_1 is a club of δ. Let $E = \{i \in \rho^{+3} : \alpha_i \in D_1\}$. It is clear that E is a club in ρ^{+3}. So by (2) choose $k \in E \cap S$ such that $C_k \subseteq E$. Let $C'_k = \{\beta \in C_k$: there is a largest $\gamma \in C_k$ such that $\gamma < \beta\}$. Set $B = \{\aleph^+_{\sigma+\alpha_i} : i \in C'_k\}$. We claim that B is as desired. Clearly $|B| = \rho^+$.

Take any $j \in C'_k$. We want to show that

$$(*) \qquad \aleph^+_{\sigma+\alpha_j} > \max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})).$$

Let $i \in C_k$ be largest such that $i < j$. So $i+1 \leq j$. We consider the definition given above of α_{i+1}. We defined $e_k = \{\aleph_{\sigma+\alpha_l+1} : l \in C_k \cap (i+1)\}$. Now

(4) $B \cap \aleph^+_{\sigma+\alpha_j} \subseteq e_k$.

For, if $b \in B \cap \aleph^+_{\sigma+\alpha_j}$, we can write $b = \aleph^+_{\sigma+\alpha_l}$ with $l \in C'_k$ and $l < j$. Hence $l \leq i$ and so $b = \aleph^+_{\sigma+\alpha_l} \in e_k$. So (4) holds.

Now if $l \in C_k \cap (i+1)$, then $l \in E$, and so $\alpha_l \in D_1 \subseteq D_0$. Hence $\aleph_{\sigma+\alpha_l} \in C_0$. This shows that $e_k \subseteq C_0$. So $\max(\mathrm{pcf}(e_k)) \leq \max(\mathrm{pcf}(C_0)) = \mu^+ < \aleph_{\sigma+\rho^{+4}}$. Hence by (3) we get $\max(\mathrm{pcf}(e_k)) < \aleph_{\sigma+\alpha_{i+1}}$. So

$$\begin{aligned}\max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})) &\leq \max(\mathrm{pcf}(e_k)) \quad \text{by (4)}\\ &< \aleph^+_{\sigma+\alpha_{i+1}}\\ &\leq \aleph^+_{\sigma+\alpha_j},\end{aligned}$$

which proves $(*)$. □

Theorem 4.34. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{|\delta|^{+4}}.$$

Proof. Let $\kappa = |\delta|^+$ and $A = (\kappa, \aleph_\delta)_{\mathrm{reg}}$. By Lemma 2.30(iii) and Lemma 4.24,

$$\begin{aligned}\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) &\leq \max(|\delta|^+, \mathrm{cf}([\aleph_\delta]^{|\delta|^+}, \subseteq))\\ &\leq \max(|\delta|^+, \max(\mathrm{pcf}(A))).\end{aligned}$$

Hence it suffices to show that $\max(\mathrm{pcf}(A)) < \aleph_{|\delta|^{+4}}$.

By Theorem 4.33, $|\mathrm{pcf}(A)| < |A|^{+4}$. Write $\max(\mathrm{pcf}(A)) = \aleph_\alpha$ and $\kappa = \aleph_\beta$. We want to show that $\alpha < |\delta|^{+4}$. Now $\mathrm{pcf}(A) = (\kappa, \max(\mathrm{pcf}(A))]_{\mathrm{reg}} = (\aleph_\beta, \aleph_\alpha]_{\mathrm{reg}}$. By Lemma 1.2, $|(\beta, \alpha)| = |\mathrm{pcf}(A)| < |A|^{+4} \leq |\delta|^{+4}$. Also, $\beta \leq \aleph_\beta = \kappa = |\delta|^+ < |\delta|^{+4}$. So $|\alpha| < |\delta|^{+4}$, and hence $\alpha < |\delta|^{+4}$. □

Theorem 4.35. *If δ is a limit ordinal, then*

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \max\left(\left(|\delta|^{\mathrm{cf}(\delta)}\right)^+, \aleph_{|\delta|^{+4}}\right).$$

Proof. If $\delta = \aleph_\delta$, then $|\delta| = \aleph_\delta$ and the conclusion is obvious. So assume that $\delta < \aleph_\delta$. Now

(1) $\aleph_\delta^{\mathrm{cf}(\delta)} \leq |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$.

In fact, let $B \subseteq [\aleph_\delta]^{|\delta|}$ be cofinal and of size $\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$. Now $\mathrm{cf}(\delta) \leq |\delta|$, so

$$[\aleph_\delta]^{\mathrm{cf}(\delta)} = \bigcup_{Y \in B} [Y]^{\mathrm{cf}(\delta)},$$

and (1) follows. Hence the theorem follows by Theorem 4.34. $\square$

Corollary 4.36. $\aleph_\omega^{\aleph_0} < \max\left((2^{\aleph_0})^+, \aleph_{\omega_4}\right)$. $\square$