HW 5

Due Monday, February 26

Exercises: (5.2.1), **5.2.2**, **5.3.2**, (5.3.3), **5.3.4**, (5.5.1), **5.5.3**, (5.6.1), **5.6.2**, **5.6.6**, (5.7.2), **5.7.3**, **5.7.4**

(To pass this assignment, submit well-written, complete, correct solutions to at least **four** exercises from among the non-parenthesized numbers above.)

Section 5.2

(5.2.1) Prove that the class of finite cyclic groups is not axiomatizable.

5.2.2. Prove that the class of cyclic groups is not axiomatizable.

Section 5.3

5.3.2. Find a finite system of axioms for the class of rings without zero-divisors. Do the same for the class of commutative rings without zero-divisors (such rings are known **integral domains**).

(5.3.3) Define the characteristic of an arbitrary ring (as before) and show that, for rings without zero-divisors, this number is 0 or a prime.

The next exercise concerns *nilpotent* elements of a ring.

Definition. An non-zero element r in a ring is called **nilpotent** if there exists $n \in \mathbb{N}$ such that $r^n = 0$.

5.3.4. Show that the class of all rings without nilpotent elements is finitely axiomatizable.

Section 5.5

(5.5.1) Show that every dense linear ordering is infinite. What would happen if the nontriviality axiom, (5), were dropped?

5.5.3. Prove that homomorphic images of linear orderings need not be partial orderings (in our sense). Show that, nevertheless, every homomorphic image of a given linear ordering X which is itself a partial ordering is isomorphic to X.

Section 5.6

(5.6.1) Prove—without using the Axiom of Choice (or any equivalent, like Zorn's Lemma)—that in a well-ordered boolean algebra, every filter can be extended to an ultrafilter. (Note that this applies to any countable boolean algebra, in particular to the Lindenbaum-Tarski algebras of countable languages (or theories).)

5.6.2. Prove the following

Theorem 5.6.1. (The Stone Representation Theorem)

1. If $\mathcal{B} \models \mathrm{BA}$ then $S(\mathcal{B})$ is a Stone space, the so-called Stone space of the boolean algebra $\mathcal{B}$.
2. If S is a Stone space, then the clopen subsets of S form a boolean set algebra $\mathrm{B}(S)$.
3. Every boolean algebra $\mathcal{B}$ is isomorphic to the boolean algebra $\mathrm{B}(\mathrm{S}(\mathcal{B}))$ via the map $a \mapsto \langle a \rangle$. Hence $\mathcal{B}$ is isomorphic to a subalgebra of the boolean algebra of all subsets of $\mathrm{S}(\mathcal{B})$.
4. Every Stone space S is homeomorphic to the Stone space $\mathrm{S}(\mathrm{B}(S))$ via the map $x \mapsto \{a \in \mathrm{B}(S) : x \in a\}$.

5.6.6. Prove the aforementioned fact that the L-theories are (up to logical equivalence) the filters of the algebra $\mathcal{B}_L$, while its ultrafilters are the complete L-theories (so the Stone space of $\mathcal{B}_L$ is S_L).

Section 5.7

(5.7.2) Derive the finiteness theorem from the compactness theorem.

5.7.3. Show that a complete L-theory is finitely axiomatizable if and only if it is isolated as a point in S_L.

5.7.4. Derive from the previous exercise that $T_{=}^{\infty}$ is not finitely axiomatizable (a result we already know from Exercise 4.3.4).