3. The ideals $J_{<\lambda}$

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Now we continue with our main topic, started at the end of section 1: properties of pcf.

Theorem 3.1. *Suppose that A is a progressive set, and λ is a regular cardinal such that $\sup(A) < \lambda$. Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $(\prod A, <_I)$ is λ-directed. Then there exist a set A' of regular cardinals and a proper ideal J over A' such that the following conditions hold:*

(i) $A' \subseteq [\min(A), \sup(A))$ and A' is cofinal in $\sup(A)$.

(ii) $|A'| \le |A|$.

(iii) J contains all bounded subsets of A'.

(iv) $\lambda = \mathrm{tcf}(\prod A', <_J)$.

Proof. First we note:

$(*)$ A does not have a largest element.

For, suppose that a is the largest element of A. Note that then $I = \mathscr{P}(A\backslash\{a\})$. For each $\xi < a$ define $f_\xi \in \prod A$ by setting

$$f_\xi(b) = \begin{cases} 0 & \text{if } b \neq a, \\ \xi & \text{if } b = a. \end{cases}$$

Since $a < \lambda$, choose $g \in \prod A$ such that $f_\xi <_I g$ for all $\xi \in a$. Thus $\{b \in A : f_\xi(b) \ge g(b)\} \in I$, so $f_\xi(a) < g(a)$ for all $\xi < a$. This is clearly impossible. So $(*)$ holds.

Now by 2.13 there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ in $\prod A$ which satisfies $(*)_\kappa$ for every $\kappa \in A$. Hence by 2.9 and 2.10, f has an exact upper bound $h \in {}^A\mathrm{Ord}$ such that

$$\{a \in A : h(a) \text{ is non-limit or } \mathrm{cf}(h(a)) < \kappa\} \in I \tag{1}$$

for every $\kappa \in A$. Now the identity function k on A is clearly an upper bound for f, so $h \le_I k$; and by (1), $\{a \in A : h(a)$ is non-limit or $\mathrm{cf}(h(a)) < \min(A)\} \in I$. Hence by changing h on a set in the ideal we may assume that

$$\min(A) \le \mathrm{cf}(h(a)) \le a \quad \text{for all } a \in A. \tag{2}$$

Now f shows that $(\prod h, <_I)$ has true cofinality λ. Let $A' = \{\mathrm{cf}(h(a)) : a \in A\}$. By 1.26–1.28, there is a proper ideal J on A' such that $(\prod A', <_J)$ has true cofinality λ; namely,

$$X \in J \quad \text{iff} \quad X \subseteq A' \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in I.$$

Clearly (ii) and (iv) hold. By (2) we have $A' \subseteq [\min(A), \sup(A))$. Now to show that A' is cofinal in $\sup(A)$, suppose that $\kappa \in A$; we find $\mu \in A'$ such that $\kappa \le \mu$. In fact, $\{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I$ by (1). Let $X = \{b \in A' : b < \kappa\}$. Then

$$h^{-1}[\mathrm{cf}^{-1}[X]] = \{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I,$$

and so $X \in J$. Taking any $\mu \in A' \backslash X$ we get $\kappa \leq \mu$. Thus (i) holds. Finally, for (iii), suppose that $\mu \in A'$; we want to show that $Y \stackrel{\text{def}}{=} \{b \in A' : b < \mu\} \in J$. By (i), choose $\kappa \in A$ such that $\mu \leq \kappa$. Then $Y \subseteq \{b \in A' : b < \kappa\}$, and by the argument just given, the latter set is in J. So (iii) holds. $\square$

Corollary 3.2. *Suppose that A is progressive, is an interval of regular cardinals, and λ is a regular cardinal $> \sup(A)$. Assume that I is a proper ideal over A such that $(\prod A, <_I)$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that I contains all proper initial segments of A. For, suppose that this is not true, and let a be the smallest element of A such that $A \cap a \notin I$. We claim that $A \cap a$ is infinite. For, suppose that it is finite, and let b be its greatest member. Thus $A \cap b \in I$ and $A \cap a = (A \cap b) \cup \{b\}$, so $\{b\} \notin I$. For each $\xi < b$ define $f_\xi \in \prod A$ by

$$f_\xi(c) = \begin{cases} 0 & \text{if } c \neq b, \\ \xi & \text{if } c = b. \end{cases}$$

Now $b < \sup(A) < \lambda$, so by λ-directedness choose $g \in \prod A$ such that $f_\xi <_I g$ for all $\xi < b$. Thus $\{c \in A : f_\xi(c) \geq g(c)\} \in I$, so $\xi = f_\xi(b) < g(b)$ for all $\xi < b$, contradiction. Thus $A \cap a$ is infinite. Clearly then the hypotheses of the corollary hold for $A \cap a$ in place of A. So, we may make the indicated assumption about I.

The desired conclusion now follows by 3.1. $\square$

The ideal $J_{<\lambda}$

Let A be a set of regular cardinals. We define

$$J_{<\lambda}[A] = \{X \subseteq A : \mathrm{pcf}(X) \subseteq \lambda\}.$$

In words, $X \in J_{<\lambda}[A]$ iff X is a subset of A such that for any ultrafilter D over A, if $X \in D$, then $\mathrm{cf}(\prod A, <_D) < \lambda$. Thus X "forces" the cofinalities of ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal of A. If $\lambda < \min(A)$, then $J_{<\lambda}[A] = \{0\}$ by 1.30(vii). If $\lambda < \mu$, then $J_{<\lambda}[A] \subseteq J_{<\mu}[A]$. If $\lambda \notin \mathrm{pcf}(A)$, then $J_{<\lambda}[A] = J_{<\lambda^+}[A]$. If λ is greater than each member of $\mathrm{pcf}(A)$, then $J_{<\lambda}[A$ is the improper ideal $\mathscr{P}(A)$. If $\lambda \in \mathrm{pcf}(A)$, then $A \notin J_{<\lambda}[A]$.

If A is clear from the context, we simply write $J_{<\lambda}$.

Lemma 3.3. *If A is an infinite set of regular cardinals and B is a finite subset of A, then for any cardinal λ we have*

$$J_{<\lambda}[A] = J_{<\lambda}[A \backslash B] + (B \cap \lambda).$$

Proof. Let $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. By 1.30(vi) we have $\mathrm{pcf}(X) = \mathrm{pcf}(X \backslash B) \cup (X \cap B)$, so $X \backslash B \in J_{<\lambda}[A \backslash B]$ and $X \cap B \subseteq B \cap \lambda$, and it follows that $X \in J_{<\lambda}[A \backslash B] + (B \cap \lambda)$.

Now suppose that $X \in J_{<\lambda}[A\backslash B] + (B \cap \lambda)$. Then there is a $Y \in J_{<\lambda}[A\backslash B]$ such that $X \subseteq Y \cup (B \cap \lambda)$. Hence by 1.30(vi) again, $\mathrm{pcf}(X) \subseteq \mathrm{pcf}(Y) \cup (B \cap \lambda) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. $\square$

Theorem 3.4. *Assume that A is progressive. Then for every cardinal λ, the partial order $(\prod A, <_{J_{<\lambda}[A]})$ is λ-directed.*

Proof. We may assume that there are infinitely many members of A less than λ. For, suppose not. Let $F \subseteq \prod A$ with $|F| < \lambda$. We define $g \in \prod A$ by setting, for any $a \in A$,

$$g(a) = \begin{cases} \sup\{f(a) : f \in F\} & \text{if } |F| < a, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f \leq g \bmod J_{<\lambda}[A]$ for all $f \in F$. For, if $f(a) > g(a)$, then $\lambda > |F| \geq a$; thus $\{a : f(a) > g(a)\} \subseteq \lambda \cap A$. Now $\mathrm{pcf}(\lambda \cap A) = \lambda \cap A \subseteq \lambda$, so $\{a : f(a) > g(a)\} \in J_{<\lambda}[A]$.

So, we make the indicated assumption. It follows that $X \overset{\text{def}}{=} \{|A|^+, |A|^{++}, |A|^{+++}\} \in J_{<\lambda}[A]$. Note that $\prod A/J_{<\lambda} \cong \prod(A\backslash X)/(J_{<\lambda}[A] \cap \mathscr{P}(A\backslash X))$. Now

$$\begin{aligned} Y \in J_{<\lambda}[A] \cap \mathscr{P}(A\backslash X) \quad &\text{iff} \quad \mathrm{pcf}(Y) \subseteq \lambda \text{ and } Y \subseteq A\backslash X \\ &\text{iff} \quad Y \in J_{<\lambda}[A\backslash X]. \end{aligned}$$

Hence we may assume that $|A|^{+3} < \min(A)$.

Now we prove by induction on the cardinal λ_0 that if $\lambda_0 < \lambda$ and $F = \{f_i : i < \lambda_0\} \subseteq \prod A$ is a family of functions of size λ_0, then F has an upper bound in $(\prod A, <_{J_{<\lambda}})$. So, we assume that this is true for all cardinals less than λ_0. If $\lambda_0 < \min(A)$, then $\sup(F)$ is as desired. So, assume that $\min(A) \leq \lambda_0$.

First suppose that λ_0 is singular. Let $\langle \alpha_i : i < \mathrm{cf}(\lambda_0)\rangle$ be increasing and cofinal in λ_0, each α_i a cardinal. By the inductive hypothesis, let g_i be a bound for $\{f_\xi : \xi < \alpha_i\}$ for each $i < \mathrm{cf}(\lambda_0)$, and then let h be a bound for $\{g_i : i < \mathrm{cf}(\lambda_0)\}$. Clearly h is a bound for F.

So assume that λ_0 is regular. We are now going to define a $<_{J_{<\lambda}}$-increasing sequence $\langle f'_\xi : \xi < \lambda_0\rangle$ which satisfies $(*)_\kappa$, with $\kappa = |A|^+$, and such that $f_i \leq f'_i$ for all $i < \lambda_0$. To do this choose for every $\delta \in S^\lambda_{\kappa^{++}}$ a club $E_\delta \subseteq \delta$ of order type κ^{++}. Now for such a δ we define

$$f'_\delta = \sup(\{f'_j : j \in E_\delta\} \cup \{f_\delta\}).$$

For ordinals $\delta < \lambda_0$ of cofinality $\neq \kappa^{++}$ we apply the inductive hypothesis to get f'_δ such that $f'_\xi <_{J_{<\lambda}} f'_\delta$ for every $\xi < \delta$ and also $f_\delta <_{J_{<\lambda}} f'_\delta$.

This finishes the construction. By 2.12, $(*)_{|A|^+}$ holds for f, and hence by 2.10, f has an exact upper bound $g \in {}^A\mathrm{Ord}$ with respect to $<_{J_{<\lambda}}$. The identity function on A is an upper bound for f, so we may assume that $g(a) \leq a$ for all $a \in A$. Now we shall prove that $B \overset{\text{def}}{=} \{a \in A : g(a) = a\} \in J_{<\lambda}[A]$, so a further modification of g yields the desired upper bound for f.

To get a contradiction, suppose that $B \notin J_{<\lambda}[A]$. Hence $\mathrm{pcf}(B) \not\subseteq \lambda$, and so there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Clearly $D \cap J_{<\lambda}[A] = \emptyset$, as otherwise $\mathrm{cf}(\prod A/D) < \lambda$. Now f has length $\lambda_0 < \lambda$, and so it is bounded in $\prod A/D$; say that $f_i <_D h \in \prod A$ for all $i < \lambda_0$. Thus $h(a) < a = g(a)$ for all $a \in B$. Now we define $h' \in \prod A$ by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h' <_{J_{<\lambda}} g$, since

$$\{a \in A : h'(a) \geq g(a)\} = \{a \in A : g(a) = 0\} \subseteq \{a \in A : f_0(a) \geq g(a)\} \in J_{<\lambda}.$$

Hence by the exactness of g it follows that $h' <_{J_{<\lambda}} f_i$ for some $i < \lambda_0$. But $B \in D$ and hence $h =_D h'$. So $h <_D f_i$, contradiction. □

Corollary 3.5. *Suppose that A is progressive, D is an ultrafilter over A, and λ is a cardinal. Then:*
(i) $\mathrm{cf}(\prod A/D) < \lambda$ *iff* $J_{<\lambda}[A] \cap D \neq \emptyset$.
(ii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \neq \emptyset = J_{<\lambda} \cap D$.
(iii) λ^+ *is the first cardinal* μ *such that* $J_{<\mu} \cap D \neq \emptyset$.

Proof. (i): $\Rightarrow$: Assuming that $J_{<\lambda}[A] \cap D = \emptyset$, the fact from 3.2 that $<_{J_{<\lambda}}$ is λ-directed implies that also $\prod A/D$ is λ-directed, and hence $\mathrm{cf}(\prod A/D) \geq \lambda$.
$\Leftarrow$: Assume that $J_{<\lambda}[A] \cap D \neq \emptyset$. Choose $X \in J_{<\lambda} \cap D$. Then by definition, $\mathrm{pcf}(A) \subseteq \lambda$, and hence $\mathrm{cf}(\prod A/D) < \lambda$.
(ii): Immediate from (i).
(iii): Immediate from (ii). □

The following theorem is very important for what follows.

Theorem 3.6. *If A is progressive, then* $|\mathrm{pcf}(A)| \leq 2^{|A|}$.

Proof. By 3.5, for each $\lambda \in \mathrm{pcf}(A)$ we can select an element $f(\lambda) \in J_{<\lambda^+} \backslash J_{<\lambda}$. Clearly f is a one-one function from $\mathrm{pcf}(A)$ into $\mathscr{P}(A)$. □

Notation. We write $J_{\leq\lambda}$ in place of $J_{<\lambda^+}$.

The following theorem is also very important in what follows.

Theorem 3.7. *If A is progressive, then* $\mathrm{pcf}(A)$ *has a largest element.*

Proof. Let

$$I = \bigcup_{\lambda \in \mathrm{pcf}(A)} J_{<\lambda}[A].$$

Now clearly each ideal $J_{<\lambda}$ is proper (since for example $\{\lambda\} \notin J_{<\lambda}$), so I is also proper. Extend the dual of I to an ultrafilter D, and let $\mu = \mathrm{cf}(\prod A/D)$. Then for each $\lambda \in \mathrm{pcf}(A)$ we have $J_{<\lambda} \cap D = \emptyset$ since $I \cap D = \emptyset$, and by 3.5 this means that $\mu \geq \lambda$. □

Corollary 3.8. *Suppose that A is progressive. If λ is a limit cardinal, then*

$$J_{<\lambda}[A] = \bigcup_{\theta < \lambda} J_{<\theta}[A].$$

Proof. The inclusion $\supseteq$ is clear. Now suppose that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Let μ be the largest element of $\mathrm{pcf}(X)$. Then $\mu \in \lambda$, and $\mathrm{pcf}(X) \subseteq \mu^+$, so $X \in J_{<\mu^+}$, and the latter is a subset of the right side. $\square$

Theorem 3.9. *If A is a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is an interval of regular cardinals.*

Proof. By Proposition 3.2, the cardinal $\mu = \sup(A)$ is either a successor cardinal which is a member of A, or is singular. Let $\lambda_0 = \max(\mathrm{pcf}(A))$. Thus we want to show that every regular cardinal λ in (μ, λ_0) is in $\mathrm{pcf}(A)$. By 3.2, the partial order $(\prod A, <_{J_{<\lambda}})$ is λ-directed. Hence $\lambda \in \mathrm{pcf}(A)$ by 3.4. $\square$

Definition. If κ is a cardinal $\leq |A|$, then we define

$$\mathrm{pcf}_\kappa(A) = \bigcup\{\mathrm{pcf}(X) : X \subseteq A \text{ and } |X| = \kappa\}.$$

Theorem 3.10. *If A is a progressive interval of regular cardinals and $\kappa \leq |A|$, then* $\mathrm{pcf}_\kappa(A)$ *is an interval of regular cardinals.*

Proof. Let $\lambda_0 = \sup(\mathrm{pcf}_\kappa(A))$, and suppose that λ is a regular cardinal such that $\sup A < \lambda < \lambda_0$. Again, 3.2 implies that we just need to show that $\lambda \in \mathrm{pcf}_\kappa(A)$. Now there is an $X \subseteq A$ with $|X| = \kappa$ such that $\sup(X) < \lambda \leq \max(\mathrm{pcf}(X))$. Hence $J_{<\lambda}[X]$ is a proper ideal, so by 3.2 and 3.4 we get $\lambda \in \mathrm{pcf}(X)$. $\square$

Theorem 3.11. *Suppose that A is progressive, $B \subseteq \mathrm{pcf}(A)$, and B is progressive. Then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A)$.

Proof. Suppose that $\mu \in \mathrm{pcf}(B)$, and let E be an ultrafilter on B such that $\mu = \mathrm{cf}(\prod B/E)$. For every $b \in B$ fix an ultrafilter D_b on A such that $b = \mathrm{cf}(\prod A/D_b)$. Define F by

$$X \in F \quad \text{iff} \quad X \subseteq A \text{ and } \{b \in B : X \in D_b\} \in E.$$

It is straightforward to check that F is an ultrafilter on A. The rest of the proof consists in showing that $\mu = \mathrm{cf}(\prod A/F)$.

Now by 1.29 we have

$$\mu = \mathrm{cf}\left(\prod_{b \in B}\left(\prod A/D_b\right)/E\right).$$

Hence it remains to show that this iterated ultraproduct is isomorphic to the simple ultraproduct $\prod A/F$. To do this, we consider the Cartesian product $B \times A$ and define

$$H \in P \quad \text{iff} \quad H \subseteq B \times A \text{ and } \{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} \in E.$$

Again it is straightforward to check that P is an ultrafilter over $B \times A$. Let $r(b,a) = a$ for any $(b,a) \in B \times A$. Then

$$(*) \qquad \prod_{(b,a) \in B \times A} a/P \cong \prod_{b \in B}\left(\prod A/D_b\right)/E.$$

To prove $(*)$, for any $f \in \prod_{\langle b,a\rangle \in B\times A} a$ we define $f' \in \prod_{b\in B}(\prod A/D_b)$ by setting

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b.$$

Then for any $f, g \in \prod_{\langle b,a\rangle\in B\times A} a$ we have

$$\begin{aligned} f =_P g \quad &\text{iff} \quad \{\langle b,a\rangle : f(b,a) = g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) = g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) = g'(b)\} \in E \\ &\text{iff} \quad f' =_E g'. \end{aligned}$$

Hence we can define $k(f/P) = f'/E$, and we get a one-one function. To show that it is a surjection, suppose that $h \in \prod_{b\in B}(\prod A/D_b)$. For each $b \in B$ write $h(b) = h'_b/D_b$ with $h'_b \in \prod A$. Then define $f(b,a) = h'_b(a)$. Then

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b = \langle h'_b(a) : a \in A\rangle/D_b = h'_b/D_b = h(b),$$

as desired. Finally, k preserves order, since

$$\begin{aligned} f/P < g/P \quad &\text{iff} \quad \{(b,a) : f(b,a) < g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) < g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) < g'(b)\} \in E \\ &\text{iff} \quad k(f/P) < k(g/P). \end{aligned}$$

So $(*)$ holds.

Now we apply 1.28, with $r, B\times A, A, P$ in place of c, A, B, F respectively. Then F is the Rudin-Keisler projection on A, since for any $X \subseteq A$,

$$\begin{aligned} X \in F \quad &\text{iff} \quad \{b \in B : X \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : r(b,a) \in X\} \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : (b,a) \in r^{-1}[X]\} \in D_b\} \in E \\ &\text{iff} \quad r^{-1}[X] \in P. \end{aligned}$$

Thus by 1.28 we get an isomorphism h of $\prod A/F$ into $\prod_{(b,a)\in B\times A} a/P$ such that $h(e/F) = \langle e(r(b,a)) : (b,a) \in B\times A\rangle/P$ for any $e \in \prod A$. By 1.16 it suffices now to show that the range of h is cofinal in $\prod_{(b,a)\in B\times A} a/P$. Let $g \in \prod_{(b,a)\in B\times A} a$. For every $b \in B$ define $g_b \in \prod A$ by $g_b(a) = g(b,a)$. Let $\lambda = \min(B)$. Since B is progressive, we have $|B| < \lambda$. Hence by the λ-directness of $\prod A/J_{<\lambda}[A]$ (3.4), there is a function $k \in \prod A$ such that $g_b <_{J_{<\lambda}} k$ for each $b \in B$. Now $\lambda \le b$ for all $b \in B$, so $J_{<\lambda} \cap D_b = \emptyset$, and so $g_b <_{D_b} k$. It follows that $g/P <_P h(k/D)$. In fact, let $H = \{(b,a) : g(b,a) < k(r(b,a))\}$. Then

$$\{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} = \{b \in B : \{a \in A : g_b(a) < k(a)\} \in D_b\} = B \in E,$$

as desired. $\square$

Generators for $J_{<\lambda}$

Let A be a set of regular cardinals, and λ a cardinal. A subset $B \subseteq A$ is a *λ-generator over A* iff $J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$. We omit the qualifier "over A" if A is understood from the context.

Suppose that $\lambda \in \mathrm{pcf}(A)$. A *universal sequence for* λ is a sequence $f = \langle f_\xi : \xi < \lambda\rangle$ which is $<_{J_{<\lambda}[A]}$-increasing, and for every ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$, the sequence f is cofinal in $\prod A/D$.

Theorem 3.12. (Universal sequences) *Suppose that A is progressive. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. First,

(1) We may assume that $|A|^+ < \min(A) < \lambda$.

In fact, suppose that we have proved the theorem under the assumption (1), and now take the general situation.

If $\lambda = \min(A)$, define $f_\xi \in \prod A$, for $\xi < \lambda$, by $f_\xi(a) = \xi$ for all $a \in A$. So $\langle f_\xi : \xi < \lambda\rangle$ is obviously $<_{J_{<\lambda}[A]}$-increasing. It is also cofinal. For, note that $\{\lambda\} \in D$, as otherwise $\mathrm{cf}(\prod A/D) > \lambda$ by 1.30(vii). If $g \in \prod A$, then $g(\lambda) < f_{g(\lambda)+1}(\lambda)$, and hence $[g] < [f_{g(\lambda)+1}]$, proving that $\langle f_\xi : \xi < \lambda\rangle$ is cofinal.

Now suppose that $\min(A) < \lambda$. Let $a_0 = \min A$. Let $A' = A\backslash\{a_0\}$. If D is an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$, then $A' \in D$ since $a_0 < \lambda$, hence $\{a_0\} \notin D$. It follows that $\lambda \in \mathrm{pcf}(A')$. Clearly $|A'|^+ < \min A' \leq \lambda$. Hence by assumption we get a system $\langle f_\xi : \xi < \lambda\rangle$ of members of $\prod A'$ which is increasing in $<_{J_{<\lambda}}$ such that for every ultrafilter D over A' such that $\lambda = \mathrm{cf}(\prod A'/D)$, f is cofinal in $\prod A'/D$. Extend each f_ξ to $g_\xi \in \prod A$ by setting $g_\xi(a_0) = 0$. If $\xi < \eta < \lambda$, then

$$\{a \in A : g_\xi(a) \geq g_\eta(a)\} \subseteq \{a \in A' : f_\xi(a) \geq f_\eta(a)\} \cup \{a_0\},$$

and $\{a_0\} \in J_{<\lambda}$ since $a_0 < \lambda$, so $g_\xi <_{J_{<\lambda}} g_\eta$. Now let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\prod A/D)$. As above, $A' \in D$; let $D' = D \cap \mathscr{P}(A')$. Then $\lambda = \mathrm{cf}(\prod A'/D')$. To show that g is cofinal in $\prod A/D$, take any $h \in \prod A$. Choose $\xi < \lambda$ such that $(h \restriction A')/D' < f_\xi/D'$. Then

$$\{a \in A : h(a) \geq g_\xi(a)\} \supseteq \{a \in A' : h(a) \geq f_\xi(a)\},$$

so $h/D < g_\xi/D$, as desired.

Thus we can make the assumption as in (1). Suppose that there is no universal sequence for λ. Thus

(2) For every $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ but f is bounded in $\prod A/D$.

We are now going to construct a $<_{J_{<\lambda}}$-increasing sequence $f^\alpha = \langle f^\alpha_\xi : \xi < \lambda\rangle$ for each $\alpha < |A|^+$. We use the fact that $\prod A/J_{<\lambda}$ is λ-directed (Theorem 3.4).

Using this directedness, we start with any $<_{J_{<\lambda}}$-increasing sequence $f^0 = \langle f^0_\xi : \xi < \lambda\rangle$.

For δ limit $< |A|^+$ we define f^δ_ξ by induction on ξ so that the following conditions hold:

(3) $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$,

(4) $\sup\{f^\alpha_\xi : \alpha < \delta\} \le f^\delta_\xi$.

Suppose that f^δ_i has been defined for all $i < \xi$. By λ-directedness, choose g such that $f^\delta_i <_{J_{<\lambda}} g$ for all $i < \xi$. Now for any $a \in A$ we have $\sup\{f^\alpha_\xi(a) : \alpha < \delta\} < a$, since $\delta < |A|^+ < \min A \le a$. Hence we can define

$$f^\delta_\xi(a) = \max\{g(a), \sup\{f^\alpha_\xi(a) : \alpha < \delta\}\}.$$

Clearly the conditions (3), (4) hold.

Now suppose that f^α has been defined and is $<_{J_{<\lambda}}$-increasing; we define $f^{\alpha+1}$. By (2), choose an ultrafilter D_α over A such that

(5) $\mathrm{cf}(\prod A/D_\alpha) = \lambda$;

(6) The sequence f^α is bounded in $<_{D_\alpha}$.

By (6), choose $f^{\alpha+1}_0$ which bounds f^α in $<_{D_\alpha}$; in addition, $f^{\alpha+1}_0 \ge f^\alpha_0$. Let $\langle h_\xi/D_\alpha : \xi < \lambda\rangle$ be strictly increasing and cofinal in $\prod A/D_\alpha$. Now we define $f^{\alpha+1}_\xi$ by induction on ξ when $\xi > 0$. First, by $<_{J_{<\lambda}}$-directness, choose k such that $f^{\alpha+1}_i <_{J_{<\lambda}} k$ for all $i < \xi$. Then for any $a \in A$ let

$$f^{\alpha+1}_\xi(a) = \max(k(a), h_\xi(a), f^\alpha_\xi(a)).$$

Then the following conditions hold:

(7) $f^{\alpha+1}$ is increasing and cofinal in $\prod A/D_\alpha$;

(8) $f^{\alpha+1}_i \ge f^\alpha_i$ for every $i < \lambda$.

This finishes the construction. Clearly we then have

(9) If $i < \lambda$ and $\alpha_1 < \alpha_2 < |A|^+$, then $f^{\alpha_1}_i \le f^{\alpha_2}_i$.

(10) f^α is bounded in $\prod A/D_\alpha$ by $f^{\alpha+1}_0$.

(11) $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

Now let $h = \sup_{\alpha<|A|^+} f^\alpha_0$. Then $h \in \prod A$, since $|A|^+ < \min(A)$. By (11), for each $\alpha < |A|^+$ choose $i_\alpha < \lambda$ such that $h <_{D_\alpha} f^{\alpha+1}_{i_\alpha}$. Since $\lambda > |A|^+$ is regular, we can choose $i < \lambda$ such that $i_\alpha < i$ for all $\alpha < |A|^+$. Now define

$$A^\alpha = \le (h, f^\alpha_i).$$

By (9) we have $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$. We are going to get a contradiction by showing that $A^\alpha \subset A^{\alpha+1}$ for every $\alpha < |A|^+$.

In fact, this follows from the following two statements.

(12) $A^\alpha \notin D_\alpha$.

This holds because $f_i^\alpha <_{D_\alpha} f_i^{\alpha+1} \leq h$.

(13) $A^{\alpha+1} \in D_\alpha$.

This holds because $h <_{D_\alpha} f_i^{\alpha+1}$ by the choice of i and (7). $\square$

Theorem 3.13. *If A is progressive, then* $\mathrm{cf}(\prod A, <) = \max(\mathrm{pcf}(A))$. *In particular,* $\mathrm{cf}(\prod A, <)$ *is regular.*

Proof. First we prove $\geq$. Let $\lambda = \max(\mathrm{pcf}(A))$, and let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Now for any $f, g \in \prod A$, if $f < g$ then $f <_D g$. Hence any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and so $\lambda = \mathrm{cf}(\prod A, <_D) \leq \mathrm{cf}(\prod A, <)$.

To prove $\leq$, we exhibit a cofinal subset of $(\prod A, <)$ of size λ. For every $\mu \in \mathrm{pcf}(A)$ fix a universal sequence $f^\mu = \langle f_i^\mu : i < \mu\rangle$ for μ, by 3.12. Let F be the set of all functions of the form

$$\sup\{f_{i_1}^{\mu_1}, f_{i_2}^{\mu_2}, \ldots, f_{i_n}^{\mu_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that F is cofinal in $(\prod A, <)$; this will complete the proof.

To prove this claim, let $g \in \prod A$. Let

$$I = \{>(f, g) : f \in F\}.$$

(Recall that $>(f, g) = \{a \in A : f(a) > g(a)\}$.) Now I is closed under unions, since

$$>(f_1, g) \cup >(f_2, g) = >(\sup(f_1, f_2), g).$$

If $A \in I$, then $A = >(f, g)$ for some $f \in F$, as desired. So, suppose that $A \notin I$. The dual of I has fip since I is closed under unions, and hence that dual can be extended to an ultrafilter D over A. Let $\mu = \mathrm{cf}(\prod A/D)$. Then f^μ is cofinal in $(\prod A, <_D)$ since it is universal for μ. But $f_i^\mu \leq_I g$ for all $i < \mu$, since $f_i^\mu \in F$ and so $>(f_i^\mu, g) \in I$. This is a contradiction. $\square$

Lemma 3.14. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $f' = \langle f'_\xi : \xi < \lambda\rangle$ is a universal sequence for λ. Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is $<_{J_{<\lambda}}$-increasing, and for every $\xi' < \lambda$ there is a $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$. Then f is universal for λ.*

Proof. This is clear, since for any ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ we have $D \cap J_{<\lambda} = \emptyset$, and hence $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$ implies that $f'_{\xi'} \leq_D f_\xi$. $\square$

Lemma 3.15. *If A is progressive and $\lambda \in \mathrm{pcf}(A)$, then there is a universal sequence for λ that satisfies $(*)_{|A|^+}$.*

If we assume in addition that every bounded subset of A is in $J_{<\lambda}$, then there is a universal sequence for λ that satisfies $(*)_\kappa$ for every regular cardinal κ such that $|A| < \kappa < \sup(A)$.

Proof. Since $\lambda \in \mathrm{pcf}(A)$, we have $\mathrm{pcf}(A) \not\subseteq \lambda$, and so $A \notin J_{<\lambda}$. Thus there is an ordinal μ such that $A \cap \mu \notin J_{<\lambda}$, and we let μ be the least such.

Suppose first that $A \cap \mu$ has a largest element ρ. Then

(1) $\rho = \lambda$.

In fact, since $A \cap \mu \notin J_{<\lambda}$, it follows that $\mathrm{pcf}(A \cap \mu) \not\subseteq \lambda$, and hence there is an ultrafilter D on A such that $A \cap \mu \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Now $A \cap \rho \in J_{<\lambda}$ by the minimality of μ, so $A \cap \rho \notin D$. Note that $A \cap \mu = (A \cap \rho) \cup \{\rho\}$. It follows that $\{\rho\} = (A \cap \mu) \backslash (A \cap \rho) \in D$, and hence $\rho = \mathrm{cf}(\prod A/D) \geq \lambda$. On the other hand, $\mathrm{cf}(\prod A/E) = \lambda$ for some ultrafilter E. Now $A \cap \rho \in J_{<\lambda}$ by the minimality of μ, so $\mathrm{pcf}(A \cap \rho) \subseteq \lambda$, and hence $A \cap \rho \notin E$. Hence $A \backslash \rho \in E$. Now $\lambda \in \mathrm{pcf}(A \backslash \rho)$, and hence by 1.30(vii) we get $\rho \leq \lambda$. So (1) holds.

Now for each $\xi < \lambda$ define for each $a \in A$,

$$f_\xi(a) = \begin{cases} \xi & \text{if } \lambda \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $\xi < \eta < \lambda$ we have

$$\{a : f_\xi(a) \geq f_\eta(a)\} \subseteq A \cap \rho \in J_{<\lambda},$$

so $\langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}}$-increasing. If D is any ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$, then $A \cap \lambda \notin D$ because $A \cap \lambda = A \cap \rho \in J_{<\lambda}$, and so $A \backslash \lambda \in D$ If $\{\lambda\} \notin D$, then $\lambda = \mathrm{cf}(\prod A/D) \in \mathrm{pcf}(A \backslash (\lambda \cup \{\lambda\}))$, hence $\lambda \geq \min \mathrm{pcf}(A \backslash (\lambda \cup \{\lambda\}) = \min(A \backslash (\lambda \cup \{\lambda\})) > \lambda$, contradiction. So $\{\lambda\} \in D$ and so clearly $\langle f_\xi : \xi < \lambda \rangle$ is cofinal in $(\prod A, <_D)$. So it is universal. Since the sequence is strictly increasing outside the member $A \cap \lambda$ of $J_{<\lambda}$, all the assertions about $(*)_\kappa$ hold.

Second, suppose that $A \cap \mu$ does not have a largest element. We are going to apply 2.13 to A and $J_{<\lambda}$. Clearly $|A| < \lambda$. By 3.4, $\prod A/J_{<\lambda}$ is λ-directed. By 3.12, let $\langle f'_\xi : \xi < \lambda \rangle$ be a universal sequence for λ. So by 2.13 we get a $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ in $\prod A/J_{<\lambda}$ such that $f'_\xi < f_{\xi+1}$ for every $\xi < \lambda$ and $(*)_\kappa$ holds for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in J_{<\lambda}$. Now we claim

(1) $\mu \leq \lambda + 1$.

For, suppose that $\lambda + 1 < \mu$. Write $\lambda = \mathrm{cf}(\prod A/D)$ for some ultrafilter D over A. Clearly $A \cap (\lambda + 1) \in D$, so we have $\lambda \in \mathrm{pcf}(A \cap (\lambda + 1))$. But by the choice of μ we have $A \cap (\lambda + 1) \in J_{<\lambda}$, contradiction.

By (1) we have $|A|^{+++} < \lambda$ and $\{a \in A : a \leq |A|^{+++}\} \in J_{<\lambda}$, so $(*)_{|A|^+}$ holds, as desired.

Now assume additionally that every bounded subset of A is in $J_{<\lambda}$, still under the assumption that A is unbounded in μ. Then $\mu = \sup A$, and the additional conclusion is true. $\square$

Theorem 3.16. *If A is progressive, then every member of* $\mathrm{pcf}(A)$ *has a generator.*

Proof. First suppose that we have shown the theorem if $|A|^+ < \min(A)$. We show how it follows when $|A|^+ = \min(A)$. The least member of $\mathrm{pcf}(A)$ is $|A|^+$ by 1.30(vii). We have $J_{<|A|^+} = \{\emptyset\}$ and $J_{\leq|A|^+} = \{\emptyset, \{|A|^+\}\} = J_{<|A|^+} + \{|A|^+\}$, so $|A|^+$ has a generator, namely $\{|A|^+\}$. Now suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda > |A|^+$. Let $A' = A \backslash \{|A|^+\}$. By

1.30(vi) we also have $\lambda \in \mathrm{pcf}(A')$. By the supposed result there is a $b \subseteq A'$ such that $J_{\leq\lambda}[A'] = J_{<\lambda}[A'] + b$. Using 3.3,

$$\begin{aligned} J_{\leq\lambda}[A] &= J_{\leq\lambda}[A'] + \{|A|^+\} \\ &= J_{<\lambda}[A'] + b + \{|A|^+\} \\ &= J_{<\lambda}[A] + b, \end{aligned}$$

as desired.

Thus we assume henceforth that $|A|^+ < \min(A)$. Suppose that $\lambda \in \mathrm{pcf}(A)$. First we take the case $\lambda = |A|^{++}$. Hence by 1.30(vii) we have $\lambda \in A$. Clearly

$$J_{\leq\lambda}[A] = \{\emptyset, \{\lambda\}\} = \{\emptyset\} + \{\lambda\} = J_{<\lambda}[A] + \{\lambda\},$$

so λ has a generator in this case. So henceforth we assume that $|A|^{++} < \lambda$.

By 3.15, there is a universal sequence $f = \langle f_\xi : \xi < \lambda \rangle$ such that $(*)_{|A|^+}$ holds. Hence by 2.10, f has an exact upper bound h with respect to $<_{J_{<\lambda}}$. Since h is a least upper bound for f and the identity function on A is an upper bound for f, we may assume that $h(a) \leq a$ for all $a \in A$. We now define

$$B = \{a \in A : h(a) = a\}.$$

Thus we can finish the proof by showing that

$$J_{\leq\lambda}[A] = J_{<\lambda}[A] + B \tag{$\star$}$$

First we show that $B \in J_{\leq\lambda}[A]$, i.e., that $\mathrm{pcf}(B) \subseteq \lambda^+$. Let D be any ultrafilter over A having B as an element; we want to show that $\mathrm{cf}(\prod A/D) \leq \lambda$. If $D \cap J_{<\lambda} \neq \emptyset$, then $\mathrm{cf}(\prod A/D) < \lambda$ by the definition of $J_{<\lambda}$. Suppose that $D \cap J_{<\lambda} = \emptyset$. Now since f is $<_{J_{<\lambda}}$-increasing and $D \cap J_{<\lambda} = \emptyset$, the sequence f is also $<_D$-increasing. It is also cofinal; for let $g \in \prod A$. Define

$$g'(a) = \begin{cases} g(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\} \subseteq \{a \in A : f_0(a) \geq h(a)\} \in J_{<\lambda}$. So $g' <_{J_{<\lambda}} h$. Since h is an exact upper bound for f, there is hence a $\xi < \lambda$ such that $g' <_{J_{<\lambda}} f_\xi$. Hence $g' <_D f_\xi$, and clearly $g =_D g'$, so $g <_D f_\xi$. This proves that $\mathrm{cf}(\prod A/D) = \lambda$. So we have proved $\supseteq$ in ($\star$).

For $\subseteq$, we argue by contradiction and suppose that there is an $X \in J_{\leq\lambda}$ such that $X \notin J_{<\lambda}[A] + B$. Hence (by 1.2), $X \backslash B \notin J_{<\lambda}$. Hence $J^{fi}_{<\lambda} \cup \{X \backslash B\}$ has fip, so we extend it to an ultrafilter D. Since $D \cap J_{<\lambda} = \emptyset$, we have $\mathrm{cf}(\prod A/D) \geq \lambda$. But also $X \in D$ since $X \backslash B \in D$, and $X \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$. By the universality of f it follows that f is cofinal in $\mathrm{cf}(\prod A/D)$. But $A \backslash B \in D$, so $\{a \in A : h(a) < a\} \in D$, and so there is a $\xi < \lambda$ such that $h <_D f_\xi$. This contradicts the fact that h is an upper bound of f under $<_{J_{<\lambda}}$. □

Now we state some important properties of generators.

Lemma 3.17. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $B \subseteq A$.*

(i) If B is a λ-generator, D is an ultrafilter on A, and $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

(ii) If B is a λ-generator, then $\lambda \notin \mathrm{pcf}(A \backslash B)$.

(iii) If $B \in J_{\leq\lambda}$ and $\lambda \notin \mathrm{pcf}(A\backslash B)$, then B is a λ-generator.

(iv) If $\lambda = \max(\mathrm{pcf}(A))$, then A is a λ-generator on A.

(v) If B is a λ-generator, then the restrictions to B of any universal sequence for λ are cofinal in $(\prod B, <_{J_{<\lambda}[B]})$.

(vi) If B is a λ-generator, then $\mathrm{tcf}(\prod B, <_{J_{<\lambda}[B]}) = \lambda$.

(vii) If B is a λ-generator on A, then $\lambda = \max(\mathrm{pcf}(B))$.

(viii) If B is a λ-generator on A and D is an ultrafilter on A, then $\mathrm{cf}(\prod A/D) = \lambda$ iff $B \in D$ and $D \cap J_{<\lambda} = \emptyset$.

(ix) If B is a λ-generator on A and $B =_{J_{<\lambda}} C$, then C is a λ-generator on A. [Here $X =_I Y$ means that the symmetric difference of X and Y is in I, for any ideal I.]

(x) If B is a λ-generator, then so is $B \cap (\lambda + 1)$.

(xi) If $\lambda = \max(\mathrm{pcf}(A))$ and B is a λ-generator, then $A\backslash B \in J_{<\lambda}$.

Proof. (i): By 3.5(ii), choose $C \in J_{\leq\lambda} \cap D$. Hence by 1.2, $C \subseteq X \cup B$ for some $X \in J_{<\lambda}$. By 3.5(ii) again, $J_{<\lambda} \cap D = \emptyset$, so $X \notin D$. Thus $C\backslash X \subseteq B$ and $C\backslash X \in D$, so $B \in D$.

(ii): Clear by (i).

(iii): Assume the hypothesis. We need to show that every member C of $J_{\leq\lambda}$ is a member of $J_{<\lambda} + B$. Now $\mathrm{pcf}(C) \subseteq \lambda^+$. Hence $\mathrm{pcf}(C\backslash B) \subseteq \lambda$, so $C\backslash B \in J_{<\lambda}$, and the desired conclusion follows from 1.2.

(iv): By (iii).

(v): Suppose not. Let $f = \langle f_\xi : \xi < \lambda\rangle$ be a universal sequence for λ such that there is an $h \in \prod B$ such that h is not bounded by any $f_\xi \restriction B$. Thus $\leq (f_\xi \restriction B, h) \notin J_{<\lambda}[B]$ for all $\xi < \lambda$. Now suppose that $\xi < \eta < \lambda$. Then

$$\begin{aligned}\leq (f_\eta \restriction B, h)\backslash(\leq (f_\xi \restriction B, h)) &= \{a \in B : f_\eta(a) \leq h(a) < f_\xi(a)\}\\ &\subseteq \{a \in A : f_\eta(a) < f_\xi(a)\} \in J_{<\lambda}.\end{aligned}$$

It follows that

$$(\leq (f_\eta \restriction B, h))/J_{<\lambda}[B] \leq (\leq (f_\xi \restriction B, h)/J_{<\lambda}[B].$$

hence if N is a finite subset of λ with largest element η we get

$$(*) \qquad (\leq (f_\eta \restriction B, h)) =_{J_{<\lambda}[B]} \bigcap_{\xi\in N} (\leq (f_\xi \restriction B, h)).$$

We claim now that

$$M \stackrel{\text{def}}{=} \{\leq (f_\xi \restriction B, h) : \xi < \lambda\} \cup (J_{<\lambda}[B])^{fi}$$

has fip. Otherwise, there is a finite subset N of λ and a set $C \in J_{<\lambda}[B]$ such that

$$\left(\bigcap_{\xi\in N} \leq (f_\xi \restriction B, h)\right) \cap (B\backslash C) = \emptyset;$$

hence if ξ is the largest member of N we get $\leq (f_\xi \restriction B, h) \in J_{<\lambda}[B]$ by $(*)$, contradiction. So we extend the set M to an ultrafilter D on B, then to an ultrafilter on A. Note that $B \in D$ and $D \cap J_{<\lambda} = \emptyset$. So $\mathrm{cf}(\prod A/D) = \lambda$, and h bounds all f_ξ in this ultraproduct, contradicting the universality of f.

(vi): By (v).

(vii): By (i) we have $\lambda \in \mathrm{pcf}(B)$. Now $B \in J_{\leq\lambda}[A]$, so $\mathrm{pcf}(B) \subseteq \lambda^+$. The desired conclusion follows.

(viii): For $\Rightarrow$, suppose that $\mathrm{cf}(\prod A/D) = \lambda$. Then $B \in D$ by (i), and obviously $D \cap J_{<\lambda} = \emptyset$. For $\Leftarrow$, assume that $B \in D$ and $D \cap J_{<\lambda} = \emptyset$. Now $B \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$ by 3.5(ii).

(ix): We have $B \in J_{\leq\lambda}$ and $C = (C\backslash B) \cup (C \cap B)$, so $C \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash C)$. Let D be an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$ and $A\backslash C \in D$. Now $B \in D$ by (i), so $B\backslash C \in D$. This contradicts $B\backslash C \in J_{<\lambda}$. So $\lambda \notin \mathrm{pcf}(A\backslash C)$. Hence C is a λ-generator, by (iii).

(x): Let $B' = B \cap (\lambda + 1)$. Clearly $B' \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash B')$. Say $\lambda = \mathrm{cf}(\prod A/D)$ with $A\backslash B' \in D$. Also $A \cap (\lambda + 1) \in D$. Since clearly

$$(A\backslash B') \cap (A \cap (\lambda + 1)) \subseteq A\backslash B,$$

this yields $A\backslash B \in D$, contradicting (ii). Therefore, $\lambda \notin \mathrm{pcf}(A\backslash B')$. So B' is a λ-generator, by (iii).

(xi): $A \in J_{\leq\lambda}$, so this follows from 3.1. □

Lemma 3.18. *Suppose that A is a progressive set, F is a proper filter over A, and λ is a cardinal. Then the following are equivalent.*

(i) $\mathrm{tcf}(\prod A/F) = \lambda$.

(ii) $\lambda \in \mathrm{pcf}(A)$, *$F$ has a λ-generator on A as an element, and* $J^{fi}_{<\lambda} \subseteq F$.

(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *for every ultrafilter D extending F.*

Proof. (i)$\Rightarrow$(iii): obvious.

(iii)$\Rightarrow$(ii): Obviously $\lambda \in \mathrm{pcf}(A)$. Let B be a λ-generator on A. Suppose that $B \notin F$. Then there is an ultrafilter D on A such that $A\backslash B \in D$ and D extends F. Then $\mathrm{cf}(\prod A/D) = \lambda$ by (iii), and this contradicts 3.17(ii).

(ii)$\Rightarrow$(i): Let $B \in F$ be a λ-generator. By 3.17(vi) we have $\mathrm{tcf}(\prod B/J_{<\lambda}) = \lambda$, and hence $\mathrm{tcf}(\prod A/F) = \lambda$ since $B \in F$ and $J^{fi}_{<\lambda} \subseteq F$. □

Lemma 3.19. *Suppose that A is progressive, $A_0 \subseteq A$, and $\lambda \in \mathrm{pcf}(A_0)$. Suppose that B is a λ-generator on A. Then $B \cap A_0$ is a λ-generator on A_0.*

Proof. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$ and hence $\mathrm{pcf}(B \cap A_0) \subseteq \lambda^+$ and so $B \cap A_0 \in J_{\leq\lambda}[A_0]$. If $\lambda \in \mathrm{pcf}(A_0\backslash B)$, then also $\lambda \in \mathrm{pcf}(A\backslash B)$, and this contradicts 3.17(ii). Hence $\lambda \notin \mathrm{pcf}(A_0\backslash B)$, and hence $B\cap A_0$ is a λ-generator for A_0 by 3.17(iii). □

Definition. If A is progressive, a *generating sequence for A* is a sequence $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that B_λ is a λ-generator on A for each $\lambda \in \mathrm{pcf}(A)$.

Theorem 3.20. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Then there is a finite subset N of $\mathrm{pcf}(X)$ such that $X \subseteq \bigcup_{\mu\in N} B_\mu$.*

Proof. We show that for all $X \subseteq A$, if $\lambda = \max(\mathrm{pcf}(X))$, then there is a finite subset N as indicated, using induction on λ. So, suppose that this is true for every cardinal $\mu < \lambda$, and now suppose that $X \subseteq A$ and $\max(\mathrm{pcf}(X)) = \lambda$. Then $\lambda \notin \mathrm{pcf}(X \backslash B_\lambda)$ by 3.17(ii), and so $\mathrm{pcf}(X \backslash B_\lambda) \subseteq \lambda$. Hence $\max(\mathrm{pcf}(X \backslash B_\lambda)) < \lambda$. Hence by the inductive hypothesis there is a finite subset N of $\mathrm{pcf}(X \backslash B_\lambda)$ such that $X \backslash B_\lambda \subseteq \bigcup_{\mu \in N} B_\mu$. Hence

$$X \subseteq B_\lambda \cup \bigcup_{\mu \in N} B_\mu,$$

and $\{\lambda\} \cup N \subseteq \mathrm{pcf}(X)$. $\square$

Lemma 3.21. *Suppose that A is progressive and $\langle B_\lambda : \lambda \in \mathrm{pcf}(A) \rangle$ is a generating sequence for A. Suppose that D is an ultrafilter on A. Then there is a $\lambda \in \mathrm{pcf}(A)$ such that $B_\lambda \in D$, and if λ is minimum with this property, then $\lambda = \mathrm{cf}(\prod A/D)$.*

Proof. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}(A)$ and $B_\mu \in D$ by 3.17(i). Suppose that $B_\lambda \in D$ with $\lambda < \mu$. Now $B_\lambda \in J_{\leq\lambda} \subseteq J_{<\mu}$, contradicting 3.17(viii), applied to μ. $\square$

Lemma 3.22. *If A is progressive and also $\mathrm{pcf}(A)$ is progressive, and if $\lambda \in \mathrm{pcf}(A)$ and B is a λ-generator for A, then $\mathrm{pcf}(B)$ is a λ-generator for $\mathrm{pcf}(A)$.*

Proof. Note by 3.11 that $\mathrm{pcf}(\mathrm{pcf}(B)) = \mathrm{pcf}(B)$ and $\mathrm{pcf}(\mathrm{pcf}(A \backslash B)) = \mathrm{pcf}(A \backslash B)$. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$, and hence $\mathrm{pcf}(\mathrm{pcf}(B)) \subseteq \lambda^+$ and so $\mathrm{pcf}(B) \in J_{\leq\lambda}[\mathrm{pcf}(A)]$. Now suppose that $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A) \backslash \mathrm{pcf}(B))$. Then by 1.30(iv) we have $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A \backslash B)) = \mathrm{pcf}(A \backslash B)$, contradicting 3.17(ii). So $\lambda \notin \mathrm{pcf}(\mathrm{pcf}(A) \backslash \mathrm{pcf}(B))$. It now follows by 3.17(iii) that $\mathrm{pcf}(B)$ is a λ-generator for $\mathrm{pcf}(A)$. $\square$

The following result is relevant to 2.14. By that result we have $J_{<\mu}[C^{(+)}] = J_{\leq\mu}[C^{(+)}] \subseteq J^{\mathrm{bd}}$. In fact, suppose that $B \in J_{<\mu}[C^{(+)}] \backslash J^{\mathrm{bd}}$. Then $(J^{\mathrm{bd}})^{\mathrm{fi}} \cup \{B\}$ has fip, and so is contained in an ultrafilter D on $C^{(+)}$. By 2.14 this yields a strictly increasing cofinal sequence in $\prod C^{(+)}/D$ of order type μ^+, contradicting $B \in J_{<\mu}[C^{(+)}] \cap D$. Hence the following is apparently a generalization of 2.14.

Lemma 3.23. *If μ is a singular cardinal of uncountable cofinality, then there is a club $C \subseteq \mu$ such that $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$.*

Proof. Let C_0 be a club in μ such that such that $\mu^+ = \mathrm{tcf}(\prod C_0^{(+)}/J^{\mathrm{bd}})$, by 2.14. Thus $\mu^+ \in \mathrm{pcf}(C_0^{(+)})$. Let B be a μ^+-generator for $C_0^{(+)}$. Define $C = \{\delta \in C_0 : \delta^+ \in B\}$. Now $C_0 \backslash C$ is bounded. Otherwise, let $X = C_0^{(+)} \backslash B = (C_0 \backslash C)^{(+)}$. So X is unbounded, and hence $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$ by 1.16. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts 3.17(ii).

So, choose $\varepsilon < \mu$ such that $C_0 \backslash C \subseteq \varepsilon$. Hence $C_0 \backslash \varepsilon \subseteq C \backslash \varepsilon \subseteq C_0 \backslash \varepsilon$, so $C_0 \backslash \varepsilon = C \backslash \varepsilon$. It follows that $\mu^+ = \mathrm{tcf}(\prod (C_0 \backslash \varepsilon)^{(+)}/J^{\mathrm{bd}})$ by 3.3, so $\mu^+ \in \mathrm{pcf}((C_0 \backslash \varepsilon)^{(+)})$. We claim that $\mathrm{tcf}(\prod (C_0 \backslash \varepsilon)^{(+)}/J_{<\mu}[(C_0 \backslash \varepsilon)^{(+)}]) = \mu^+$ (as desired). To show this, we apply 3.18(iii). Suppose that D is any ultrafilter on $(C_0 \backslash \varepsilon)^{(+)}$ such that $J_{<\mu}[(C_0 \backslash \varepsilon)^{(+)}] \cap D = \emptyset$. Now by 3.19, $B \cap (C_0 \backslash \varepsilon)^{(+)}$ is a μ^+-generator for $(C_0 \backslash \varepsilon)^{(+)}$. But $B \cap (C_0 \backslash \varepsilon)^{(+)} = B \cap (C \backslash \varepsilon)^{(+)} = (C \backslash \varepsilon)^{(+)}$. It follows by 3.7(viii) that $\mathrm{cf}(\prod (C_0 \backslash \varepsilon)^{(+)}/D) = \mu^+$. $\square$

Lemma 3.24. *Assume that μ is a singular cardinal, that C is a club in μ, and that $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$. Then $\max(\mathrm{pcf}(C^{(+)})) = \mu^+$.*

Proof. Suppose that $\lambda \in \mathrm{pcf}(C^{(+)})$ and $\lambda \geq \mu^+$. Say $\lambda = \mathrm{cf}(\prod C^{(+)}/D)$ with D an ultrafilter on $C^{(+)}$. By 3.5(i) we have $J_{<\lambda} \cap D = \emptyset$ so, since $\mu^+ \leq \lambda$, also $J_{<\mu} \cap D = \emptyset$. Hence by 3.18, $\lambda = \mu^+$. $\square$

Corollary 3.25. *If μ is a singular cardinal of uncountable cofinality, then there is a club $C \subseteq \mu$ such that* $\max(\mathrm{pcf}(\prod C^{(+)})) = \mu^+$. $\square$