2. Terms and varieties

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We now make one important step towards full model theory: terms and equations. Let σ be any signature. We assume given now a simple infinite sequence $v_0, v_1, \ldots$ of distinct objects called *variables*, different from the relation symbols, function symbols, and individual constants of σ. We define the notion of a *term* over σ; these are certain finite sequences of members of the set

$$\mathrm{Fcn} \cup \mathrm{Rel} \cup \mathrm{Cn} \cup \{v_i : i < \omega\}.$$

The collection of terms is the intersection of all sets A of nonzero sequences of members of this set such that:

(1) For any variable v_i, the sequence $\langle v_i \rangle$ is in A.

(2) For any constant k, the sequence $\langle k \rangle$ is in A.

(3) For any m-ary function symbol F and any members $\tau_0, \ldots, \tau_{m-1}$ of A, the sequence

$$\langle F \rangle^\frown \tau_0^\frown \cdots ^\frown \tau_{m-1}$$

is also in A.

Note that we distinguish between an object a and the sequence $\langle a \rangle$ consisting of that object alone. The operation $^\frown$ of *concatenation* is defined by

$$\langle a_0, \ldots, a_{m-1} \rangle^\frown \langle b_0, \ldots, b_{n-1} \rangle = \langle a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1} \rangle.$$

We will usually write $F\sigma_0 \ldots \sigma_{m-1}$ rather than $\langle F \rangle^\frown \tau_0^\frown \cdots ^\frown \tau_{m-1}$.

In our examples, terms look like this. For partial orderings, the only terms are the variables (or rather the sequences $\langle v_i \rangle$). For groups some examples of terms are Fv_0e, $Fv_0Fv_1v_2$, $FFv_0v_1v_2$, GFv_0v_1. In more familiar terminology these are $v_0 \cdot e$, $v_0 \cdot (v_1 \cdot v_2)$, $(v_0 \cdot v_1) \cdot v_2$, and $(v_0 \cdot v_1)^{-1}$. For rings, the terms are the polynomials. Examples, with F, G the binary function symbols, H the unary one, and Z the constant, are Z, HZ, Hv_3, $FFGv_0v_0Fv_0v_0v_1$, which one would normally write as 0, -0, $-v_3$, $(v_0^2 + 2v_0) + v_1$. The terms for ordered fields are the same as those for rings.

Proposition 2.1. *(i) No proper initial segment of a term is a term.*

(ii) If τ is a term, then exactly one of the following conditions holds:

(a) τ is a constant.

(b) τ is a variable.

(c) There exist a function symbol F, say of arity m, and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $F\sigma_0 \ldots \sigma_{m-1}$.

(iii) If F and G are function symbols, say of arities m and n respectively, and if $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1}$ are terms, and if $F\sigma_0 \ldots \sigma_{m-1}$ is equal to $G\tau_0, \ldots \tau_{n-1}$, then $F = G$, $m = n$, and $\sigma_i = \tau_i$ for all $i < m$.

Proof.

(i) holds by "induction on terms": the set of terms for which it is true clearly contains the variables and constants and an easy argument shows that if it holds for $\sigma_0, \ldots, \sigma_{m-1}$, then it also holds for $F\sigma_0 \ldots \sigma_{m-1}$.

(ii) is clear, and (iii) follows from (i). $\square$

The terms form a rudimentary kind of logic. We now make the first connection between logic and structures. Let σ be a term and $\overline{A}$ a structure. For each $a \in {}^{\omega}A$ we define the *value of σ under the assignment a*, denoted by $\sigma^{\overline{A}}(a)$, recursively as follows:

$$\begin{aligned} k^{\overline{A}}(a) &= k^{\overline{A}} \quad \text{for each constant } k \\ v_i^{\overline{A}}(a) &= a_i; \\ (F\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= F^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \\ &\qquad \text{with } F \text{ an } m\text{-ary function symbol and } \sigma_0, \ldots, \sigma_{m-1} \text{ terms.} \end{aligned}$$

Note that Proposition 2.1(iii) is needed to see that this definition is unambigous.

In concrete cases we can just use the intuition that $\sigma^{\overline{A}}(a)$ is the result of replacing each variable v_i in σ by a_i, each individual constant $\mathbf{k}$ by $\mathbf{k}^{\overline{A}}$, each function symbol $\mathbf{F}$ by $\mathbf{F}^{\overline{A}}$, and evaluating the result. The precise definition above is needed when we want to prove something about the evaluation of terms in general.

We give some examples in the case of the group Z of integers under addition, using a looser notation than the official one.

$$\begin{aligned} (v_0 + -v_1)(0, 1, 2, \ldots) &= 0 - 1 = -1; \\ (v_0 + -v_1)(0, -1, -2, \ldots) &= 0 + 1 = 1; \\ ((v_5 + 0) + -v_3)(0, 1, 2, \ldots) &= 5 - 3 = 2; \\ ((v_5 + 0) + -v_3)(0, -1, -2, \ldots) &= -5 + 3 = -2. \end{aligned}$$

Proposition 2.2. *If a and b are assignments, τ is a term, the variables occurring in τ are among $v_0, \ldots, v_{m-1}$, and $a_i = b_i$ for all $i < m$, then $\tau^A(a) = \tau^A(b)$.*

Proof. Induction on terms. $\square$

Because of this proposition, we can write $\tau^A(a)$ for a finite sequence a which covers all the variables of τ.

Proposition 2.3. *If $\overline{A}$ is generated by X, then every element of A has the form $\tau^{\overline{A}}(a)$ for some sequence a of elements of X and some term τ. If $\overline{x}$ is a sequence of elements of A whose range generates $\overline{A}$, then every element of A has the form $\tau^{\overline{A}}(\overline{x})$ for some term τ.* $\square$

Equations and varieties

We introduce now one more part of logic: equality. For a given signature (Fcn,Rel,Cn,ar), we suppose that $\mathbf{e}$ is an object not in the set

$$\mathrm{Fcn} \cup \mathrm{Rel} \cup \mathrm{Cn} \cup \{v_i : i < \omega\}.$$

An *equation* is a sequence of the form $\langle\mathbf{e}\rangle^\frown\sigma^\frown\tau$ with σ and τ terms. This sequence is denoted by $\sigma=\tau$. If $\overline{A}$ is a structure, then we say that $\sigma=\tau$ *holds in* $\overline{A}$ iff $\sigma^{\overline{A}}(a)=\tau^{\overline{A}}(a)$ for every $a\in{}^\omega A$.

For the rest of this chapter, suppose that our signature is algebraic. For any set E of equations, we let $\mathbf{Mod}(E)$ be the class of all structures in which every member of E holds. Note that $\mathbf{Mod}(E)$ is a proper class in the set-theoretic sense; the class of all groups is an example. It is not necessary to completely understand the difference between ordinary sets and proper classes in order to follow the development here. For $\mathbf{K}$ a class of structures, we define

$$\begin{aligned}
\mathbf{SK}&=\{\overline{A}:\overline{A}\text{ is a subalgebra of some member of }\mathbf{K}\};\\
\mathbf{HK}&=\{\overline{A}:\overline{A}\text{ is a homomorphic image of some member of }\mathbf{K}\};\\
\mathbf{PK}&=\{\overline{A}:\overline{A}\text{ is isomorphic to a product of members of }\mathbf{K}\}.
\end{aligned}$$

A class $\mathbf{K}$ of structures is a *variety* iff there is a set E of equations such that $\mathbf{K}=\mathbf{Mod}(E)$.

We are now going to work towards Birkhoff's theorem that $\mathbf{K}$ is a variety iff it is closed under the operations $\mathbf{H},\mathbf{S},\mathbf{P}$. This is a typical model-theoretic result, showing the equivalence between a logical notion (a variety) and a notion not involving logic (closure under $\mathbf{H},\mathbf{S},\mathbf{P}$).

Proposition 2.4. *Let $\overline{A}$ be a substructure of $\overline{B}$.*

(i) If σ is a term and $a\in{}^\omega A$ then $\sigma^{\overline{A}}(a)=\sigma^{\overline{B}}(a)$.

(ii) If $\sigma=\tau$ is an equation holding in $\overline{B}$, then it also holds in $\overline{A}$. □

Proposition 2.5. *Suppose that h is a homomorphism from $\overline{A}$ into $\overline{B}$.*

(i) If τ is a term and $a\in{}^\omega A$ is an assignment, then $h(\tau^{\overline{A}}(a))=\tau^{\overline{B}}(h\circ a)$.

(ii) If $\sigma=\tau$ holds in $\overline{A}$ and h is a surjection, then $\sigma=\tau$ holds in $\overline{B}$.

Proof. (i): by induction on τ:

$$\begin{aligned}
h(v_i^{\overline{A}}(a)&=h(a_i)=(h\circ a)_i=v_i^{\overline{B}}(h\circ a);\\
h(k^{\overline{A}}(a))&=h(k^{\overline{A}})=k^{\overline{B}}=k^{\overline{B}}(a)\quad\text{for }k\text{ an individual constant}\\
h((F\sigma_0\ldots\sigma_{m-1})^{\overline{A}}(a))&=h(F^{\overline{A}}(\sigma_0^{\overline{A}}(a),\ldots,\sigma_{m-1}^{\overline{A}}(a)))\\
&=F^{\overline{B}}(h(\sigma_0^{\overline{A}}(a)),\ldots,h(\sigma_{m-1}^{\overline{A}}(a)))\\
&=F^{\overline{B}}(\sigma_0^{\overline{B}}(h\circ a),\ldots,\sigma_{m-1}^{\overline{B}}(h\circ a))\\
&=(F\sigma_0\ldots\sigma_{m-1})^{\overline{B}}(h\circ a)\\
&\qquad\text{for }F\text{ an }m\text{-ary function symbol}\\
&\qquad\text{and }\sigma_0,\ldots,\sigma_{m-1}\text{ terms.}
\end{aligned}$$

(ii): Assume that $\sigma=\tau$ holds in $\overline{A}$ and h is a surjection. Take any $b\in{}^\omega B$. Choose $a\in{}^\omega A$ such that $h\circ a=b$. Then, using (i),

$$\sigma^{\overline{B}}(b)=\sigma^{\overline{B}}(h\circ a)=h(\sigma^{\overline{A}}(a))=h(\tau^{\overline{A}}(a))=\tau^{\overline{B}}(h\circ a)=\tau^{\overline{B}}(b).$$ □

Proposition 2.6. *Let $\langle \overline{A}_i : i \in I \rangle$ be a system of similar structures. Let $\overline{B} = \prod_{i \in I} \overline{A}_i$.*

(i) If σ is a term, $a \in {}^{\omega}B$. and $i \in I$, then $(\sigma^{\overline{B}}(a))_i = \sigma^{\overline{A}_i}(\mathrm{pr}_i \circ a)$.

(ii) If $\sigma = \tau$ holds in each $\overline{A}_i$, then it also holds in $\overline{B}$.

Proof. (i) holds by Propositions 1.9 and 2.5. For (ii), suppose that $\sigma = \tau$ holds in each $\overline{A}_i$. Take any $a \in {}^{\omega}B$ and $i \in I$. Then, using (i),

$$(\sigma^{\overline{B}}(a))_i = \sigma^{\overline{A}_i}(\mathrm{pr}_i \circ a) = \tau^{\overline{A}_i}(\mathrm{pr}_i \circ a) = (\tau^{\overline{B}}(a))_i;$$

since i is arbitrary, it follows that $\sigma^{\overline{B}}(a) = t^{\overline{B}}(a)$. $\square$

Corollary 2.7. *If* **K** *is a variety, then it is closed under* **H**, **S**, **P**. $\square$

Theorem 2.8. (Birkhoff) *suppose that* **K** *is a class of similar algebraic structures. Then* **K** *is a variety iff it is closed under* **H**, **S**, *and* **P**.

Proof. $\Rightarrow$ holds by Corollary 2.7. Now suppose that **K** is closed under **H**, **S**, and **P**. Let Γ be the set of all equations holding in every member of **K**. We claim that $\mathbf{K} = \mathbf{Mod}(\Gamma)$. $\subseteq$ is clear. Now suppose that $\overline{A}' \in \mathrm{Mod}(\Gamma)$; we want to show that $\overline{A}' \in \mathbf{K}$. By Corollary 1.17 we may assume that $\overline{A}'$ is finitely generated.

An auxiliary role will now be played by the following algebra $\mathfrak{F}$, called the absolutely free algebra. Its elements are the terms in our language, and if F is an m-ary function symbol and $\sigma_0, \ldots, \sigma_{m-1}$ are terms, then $F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}) = F\sigma_0 \ldots \sigma_{m-1}$. For any individual constant k, we define $k^{\mathfrak{F}} = k$. Let $L = \{(\sigma, \tau) : \sigma, \tau$ are terms and $\sigma = \tau$ holds in every member of $\mathbf{K}\}$.

(1) $L = \bigcap\{R : R$ is a congruence on $\mathfrak{F}$ and $\mathfrak{F}/R$ can be isomorphically embedded in some member of $\mathbf{K}\}$.

In fact, for $\supseteq$ suppose that (σ, τ) is a member of the right side; we want to show that $(\sigma, \tau) \in L$, i.e., that $\sigma = \tau$ holds in every member of **K**. So suppose that $\overline{A} \in K$ and $a \in {}^{\omega}A$. Define $f(\sigma) = \sigma^{\overline{A}}(a)$ for every term σ. Then f is a homomorphism from $\mathfrak{F}$ into $\overline{A}$:

$$\begin{aligned} f(F^{\mathfrak{F}}(\rho_0, \ldots, \rho_{m-1})) &= f(F\rho_0 \ldots \rho_{m-1}) \\ &= (F\rho_0 \ldots \rho_{m-1})^{\overline{A}}(a) \\ &= F^{\overline{A}}(\rho_0^{\overline{A}}(a), \ldots, \rho_{m-1}^{\overline{A}}(a)) \\ &= F^{\overline{A}}(f(\rho_0), \ldots, f(\rho_{m-1})). \end{aligned}$$

For k an individual constant, $f(k^{\mathfrak{F}}) = f(k) = k^{\overline{A}}(a) = k^{\overline{A}}$. Say that $\overline{B}$ is the range of f. Then f is a homomorphism from $\mathfrak{F}$ onto $\overline{B}$, and so by Theorem 1.8, $\mathfrak{F}/\ker(f)$ can be isomorphically embedded in $\overline{A}$. It follows that $(\sigma, \tau) \in \ker(f)$. This means that $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, as desired.

Conversely, suppose that $(\sigma,\tau) \in L$, R is a congruence relation on $\mathfrak{F}$, and f is an isomorphism from $\mathfrak{F}/R$ into $\overline{A} \in K$. We want to show that $(\sigma,\tau) \in R$. Define $a_i = f([v_i]_R)$ for every $i < \omega$.

(2) $\rho^{\overline{A}}(a) = f([\rho]_R)$ for every term ρ.

We prove (2) by induction on ρ. $v_i^{\overline{A}}(a) = a_i = f([v_i]_R)$, as desired. If k is an individual constant, then $k^{\overline{A}}(a) = k^{\overline{A}} = f([k^{\mathfrak{F}}]_R$. Now assume that $\sigma_i^{\overline{A}}(a) = f([\sigma_i]_R)$ for every $i < m$. Then

$$\begin{aligned}
(F\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= F^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \\
&= F^{\overline{A}}(f([\sigma_0]_R), \ldots, f([\sigma_{m-1}]_R)) \\
&= f(F^{\overline{A}/R}([\sigma_0]_R, \ldots, [\sigma_{m-1}]_R)) \\
&= f([F\sigma_0 \ldots \sigma_{m-1}]_R),
\end{aligned}$$

finishing the inductive proof of (2).

By (2), since $\sigma = \tau$ holds in $\overline{A}$,

$$f([\sigma]_R) = \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) = f([\tau]_R).$$

Since f is one-one, it follows that $[\sigma]_R = [\tau]_R$, so that $(\sigma,\tau) \in R$, finishing the proof of (1).

(3) $\mathfrak{F}/L \in \mathbf{K}$.

Let $M = \{R : R$ is a congruence relation on $\mathfrak{F}$ such that $\mathfrak{F}/R$ can be isomorphically embedded in a member of $\mathbf{K}\}$. For each $R \in M$, let f_R be an isomorphism from $\mathfrak{F}/R$ into $\overline{A}_R \in \mathbf{K}$. Now if $R \in M$ and $(\sigma,\tau) \in L$, then also $(\sigma,\tau) \in R$, by (1). Hence there is a function g from $\mathfrak{F}/L$ into $\prod_{R\in M} \overline{A}_R$ such that $(g([\sigma]_L))_R = f_R([\sigma]_R)$ for every $\sigma \in \mathfrak{F}$ and $R \in M$. Now g is a homomorphism. To see this, let $\overline{B} = \prod_{R\in M} \overline{A}_R$. Then for an individual constant k we have $(g([k^{\mathfrak{F}}]_L))_R = f_R([k^{\mathfrak{F}}]_R) = k^{\overline{A}_R}$. For an m-ary operation symbol F,

$$\begin{aligned}
g(F^{\mathfrak{F}/L}([\sigma_0]_L, \ldots, [\sigma_{m-1}]_L) &= g([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1})]_L) \\
&= \langle (g([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))]_L))_R : R \in M\rangle \\
&= \langle f_R([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1})]_R) : R \in M\rangle \\
&= \langle f_R(F^{\mathfrak{F}/R}([\sigma_0]_R, \ldots, [\sigma_{m-1}]_R)) : R \in M\rangle \\
&= \langle F^{\overline{A}_R}(f_R([\sigma_0]_R), \ldots, f_R([\sigma_{m-1}]_R)) : R \in M\rangle \\
&= \langle F^{\overline{A}_R}((g([\sigma_0]_L))_R, \ldots, (g([\sigma_{m-1}]_L)_R) : R \in M\rangle \\
&= F^{\overline{B}}(g([\sigma_0]_L), \ldots, g([\sigma_{m-1}]_L)).
\end{aligned}$$

Moreover, g is one-one, for if $g([\sigma]_L) = g([\tau]_L)$, then for each $R \in M$ we have $f_R([\sigma]_R) = (g([\sigma]_L))_R = (g([\tau]_L))_R = f_R([\tau]_R)$, hence $[\sigma]_R = [\tau]_R$, hence $(\sigma,\tau) \in R$. This being true for all $R \in M$, it follows that $(\sigma,\tau) \in L$ by (1). So $[\sigma]_L = [\tau]_L$. Now (3) follows.

Now we are ready for the final argument. Choose $x \in {}^{\omega}A$ such that $\mathrm{rng}(x)$ generates $\overline{A}$. Now if $(\sigma, \tau) \in L$, then $\sigma = \tau$ holds in $\overline{A}$, and hence $\sigma^{\overline{A}}(x) = \tau^{\overline{A}}(x)$. It follows that there is a function $f : \mathfrak{F}/L \to \overline{A}$ such that $f([\sigma]_L) = \sigma^{\overline{A}}(x)$ for every $\sigma \in \mathfrak{F}$. Now f is a homomorphism. In fact, if k is an individual constant, then $f(k^{\mathfrak{F}}]_L) = k^{\overline{A}}(x) = k^{\overline{A}}$. If F is an m-ary operation symbol, then

$$\begin{aligned} f(F^{\mathfrak{F}/L}([\sigma_0]_L, \ldots, [\sigma_{m-1}]_L)) &= f([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))]_L) \\ &= (F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))^{\overline{A}}(x) \\ &= F^{\overline{A}}(\sigma_0^{\overline{A}}(x), \ldots, \sigma_{m-1}^{\overline{A}}(x)) \\ &= F^{\overline{A}}(f([\sigma_0]_L), \ldots, f([\sigma_{m-1}]_L)). \end{aligned}$$

Furthermore, f maps onto A. For, $f([v_i]_L) = v_i^{\overline{A}}(x) = x_i$, so $\mathrm{rng}(x) \subseteq \mathrm{rng}(f)$. Since $\mathrm{rng}(f)$ is a subalgebra of $\overline{A}$ it follows that $\mathrm{rng}(f) = A$.

So f is a homomorphism from the member $\mathfrak{F}/L$ of **K** (by (3)) onto $\overline{A}$, hence $\overline{A} \in$ **K**. $\square$

We now give an application of Theorem 2.8 of a fairly concrete nature. This is to the notion of relation algebra, which has been intensively studied.

If R is a binary relation, then $R^{-1} = \{(a, b) : (b, a) \in R\}$. If R and S are binary relations, then $R|S = \{(a, c) : \exists b[(a, b) \in R$ and $(b, c) \in S$. A *proper relation algebra* is a structure $\overline{M} = (M, \cup, \cap, {}_R-, \emptyset, R, |, {}^{-1}, id)$ such that R is a binary relation, M is a collection of binary relations contained in R, M is closed under $\cup$ and $\cap$, $\emptyset, R, id \in M$, $\forall S \in M[{}_R-S = R \backslash S \in M]$, M is closed under $|$ and ${}^{-1}$, and id is the set of all ordered pairs $(a, a) \in R$. We define $U(\overline{M}) = \{a : (a, b) \in S$ for some $S \in M\}$. Thus $S \subseteq U(\overline{M}) \times U(\overline{M})$ for all $S \in M$. Note that R is an equivalence relation on $U(\overline{M})$. In fact, $R^{-1} \subseteq R$ and $R|R \subseteq R$, so R is symmetric and transitive. If $a \in U(\overline{M})$, say with $(a, b) \in S \in M$, then $(b, a) \in S^{-1}$, $(a, a) \in S|S^{-1}$ and hence $(a, a) \in R$; so R is reflexive on $U(\overline{M})$.

Let **RRA** be the class of all algebras isomorphic to a proper relation algebra; **RRA** abreviates "representable relation algebra".

Clearly **RRA** is closed under **S**.

Lemma 2.9. *Suppose that $\overline{M}$ is a proper relation algebra, as above, and f is a bijection from $U(\overline{M})$ onto a set V. Then there exist a proper relation algebra $\overline{N}$ and a function g such that $U(\overline{N}) = V$ and g is an isomorphism from $\overline{M}$ onto $\overline{N}$.*

Proof. For any $S \in M$ let

$$g(S) = \{(f(u), f(v)) : (u, v) \in S\},$$

and $\overline{N} = (g[M], \cup, \cap, {}_{g(R)}-, \emptyset, g(R), |, {}^{-1})$. We check the required conditions.

- N is closed under $\cup$, and $g(S \cup T) = g(S) \cup g(T)$ for all $S, T \in M$. In fact,

$$\begin{aligned} (f(u), f(v)) \in g(S \cup T) \quad &\text{iff} \quad (u, v) \in S \cup T \\ &\text{iff} \quad (u, v) \in S \text{ or } (u, v) \in T \\ &\text{iff} \quad (f(u), f(v)) \in g(S) \text{ or } (f(u), f(v)) \in g(T) \\ &\text{iff} \quad (f(u), f(v)) \in g(S) \cup g(T). \end{aligned}$$

- N is closed under $\cap$, and $g(S \cap T) = g(S) \cap g(T)$ for all $S, T \in M$. The proof is similar to that for $\cup$.
- N is closed under ${}_{g(R)}-$, and $g({}_R - S) = {}_{g(R)} - g(S)$. For,

$$\begin{array}{rll} (f(u), f(v)) \in g({}_R - S) & \text{iff} & (f(u), f(v)) \in g(R) \backslash g(S) \\ & \text{iff} & (f(u), f(v)) \notin g(S) \\ & \text{iff} & (u, v) \notin S \\ & \text{iff} & (f(u), f(v)) \in {}_{g(R)} - g(S). \end{array}$$

- g preserves $\emptyset$ and R. Obvious.
- N is closed under $|$, and $g(S|T) = g(S)|g(T)$. For,

$$\begin{array}{rll} (f(u), f(v)) \in g(S|T) & \text{iff} & (u, v) \in S|T \\ & \text{iff} & \exists w[(u, w) \in S \text{ and } (w, v) \in T] \\ & \text{iff} & \exists f(w)[(f(u), f(w)) \in g(S) \text{ and } (f(w), f(v)) \in g(T) \\ & \text{iff} & ((f(u), f(v)) \in g(S)|g(T) \end{array}$$

- N is closed under ${}^{-1}$, $g(S^{-1}) = (g(S))^{-1}$, and $g(id) = id$. This is clear.

Finally, g is clearly one-one. □

Lemma 2.10. RRA *is closed under* **P**.

Proof. Let $\langle \overline{M}_i : i \in I \rangle$ be a system of proper relation algebras. By Lemma 2.9 we may assume that $U(\overline{M}_i) \cap U(\overline{M}_j) = \emptyset$ for $i \neq j$. Write $\overline{M}_i = \langle M_i, \cup, \cap, {}_{R_i}, \emptyset, R_i, |, {}^{-1}, id \rangle$. For brevity let $\overline{B} = \prod_{i \in I} \overline{M}_i$. The operations of $\overline{B}$ are denoted by $+, \cdot, -, 0, 1, ;, {}^{\cup}, I$. Now for any $x \in B$ we define

$$f(x) = \bigcup_{i \in I} x_i.$$

We claim that f is an isomorphism from B onto a proper relation algebra of the form $(f[B], \cup, \cap, {}_T-, \emptyset, T, |, {}^{-1}, id)$, where $T = \bigcup_{i \in I} R_i$.

Clearly $f(x)$ is a binary relation. f preserves $+$:

$$f(x + y) = \bigcup_{i \in I} (x_i \cup y_i) = \bigcup_{i \in I} x_i \cup \bigcup_{i \in I} y_y = f(x) \cup f(y).$$

f preserves $\cdot$:

$$f(x \cdot y) = \bigcup_{i \in I} (x_i \cap y_i) = \bigcup_{i \in I} x_i \cap \bigcup_{i \in I} y_y = f(x) \cap f(y).$$

f preserves $-$:

$$f(-x) = \bigcup_{i \in I} ({}_{R_i} - x_i) = T \backslash \bigcup_{i \in I} x_i = {}_T - f(x).$$

Obviously $f(0)=\emptyset$ and $f(1)=T$.
f preserves ;:

$$\begin{aligned} f(x;y) &= \bigcup_{i\in I}(x_i|y_i) \\ &= f(x)|f(y). \end{aligned}$$

Here the second equality is seen as follows. If $(a,b)\in\bigcup_{i\in I}(x_i|y_i)$, choose $i\in I$ such that $(a,b)\in(x_i|y_i)$. Then there is a $c\in U(\overline{M}_i)$ such that $(a,c)\in x_i$ and $(c,b)\in y_i$. So $(a,c)\in\bigcup_{i\in I}x_i$ and $(c,b)\in\bigcup_{i\in I}y_i$, so $(a,b)\in f(x)|f(y)$. On the other hand, suppose that $(a,b)\in f(x)|f(y)$. Say $(a,c)\in f(x)$ and $(c,b)\in f(y)$. Choose $i,j\in I$ such that $(a,c)\in x_i$ and $(c,b)\in y_j$. Then $c\in U(\overline{M}_i)\cap U(\overline{M}_j)$, and it follows that $i=j$. Hence $(a,b)\in x_i|y_i$, so that $(a,b)\in f(x;y)$.

f preserves $^\cup$:

$$f(x^\cup)=\bigcup_{i\in I}x_i^{-1}=\left(\bigcup_{i\in I}x_i\right)^{-1}=(f(x))^{-1}.$$

Clearly $f(id)=id$. Finally, it is clear that f is one-one. □

To deal with **H** we first have to consider ultraproducts.

Lemma 2.11. *An ultraproduct of members of* **RRA** *is again a member of* **RRA**.

Proof. Let $\langle\overline{M}_i : i\in I\rangle$ be a system of proper relation algebras, and let D be an ultrafilter on I. For brevity let $\overline{B}=\prod_{i\in I}\overline{M}_i$, and $\overline{C}=\overline{B}/D$, the ultraproduct. Members of C will be denoted by $[x]$ with $x\in B$. Also, let $V=\prod_{i\in I}U(\overline{M}_i)$, and $W=V/D$, the ultraproduct of these sets. Members of W will be denoted by $[v]'$, with $v\in V$. Now for any $x\in B$ let

$$F([x])=\{([u]',[v]') : u,v\in V \text{ and } \{i\in I : (u_i,v_i)\in x_i\}\in D\}.$$

First we need to see that F is well-defined. So, suppose that $[x]=[y]$; we want to show that the expressions on the right for $F([x])$ and $F([y])$ are the same. Suppose that $u,v\in V$ and $M\overset{\text{def}}{=}\{i\in I : (u_i,v_i)\in x_i\}\in D$. Let $N=\{i\in I : x_i=y_i\}$. Then $N\in D$. For any $i\in M\cap N$ we have $(u_i,v_i)\in y_i$, and hence $([u]',[v]')$ is in the right side for $F([y])$. By symmetry, then, F is well-defined.

Now we claim

$(*)$ If $x\in B$, $u,v\in V$, and $([u]',[v]')\in F([x])$, then $\{i\in I : (u_i,v_i)\in x_i\}\in D$.

In fact, assume the hypothesis of $(*)$. Then there exist $u',v'\in V$ such that $[u]'=[u']'$, $[v]'=[v']'$, and $\{i\in I : (u'_i,v'_i)\in x_i\}\in D$. Now take any $i\in I$ such that $u_i=u'_i$, $v_i=v'_i$, and $(u'_i,v'_i)\in x_i$. Then $(u_i,v_i)\in x_i$. So the conclusion of $(*)$ follows.

F preserves $+$: Suppose that $x, y \in B$ and $u, v \in V$. Then

$$\begin{array}{rll}
([u]', [v]') \in F([x]+[y]) & \text{iff} & ([u]', [v]') \in F([x+y]) \\
& \text{iff} & \{i \in I : (u_i, v_i) \in x_i \cup y_i\} \in D \\
& \text{iff} & \{i \in I : (u_i, v_i) \in x_i\} \in D \text{ or } \{i \in I : (u_i, v_i) \in y_i\} \in D \\
& \text{iff} & ([u]', [v]') \in F([x]) \text{ or } ([u]', [v]') \in F([y]) \\
& \text{iff} & ([u]', [v]') \in F([x]) \cup F([y]).
\end{array}$$

The proofs for $\cdot$ and $-$ are similar. Clearly $F(0) = \emptyset$ and $f(1) = 1$.

To show that F preserves ;, suppose that $x, y \in B$, $u, v \in V$. First suppose that $([u]', [v]') \in F(x; y)$. Thus by $(*)$, $M \stackrel{\text{def}}{=} \{i \in I : (u_i, v_i) \in x_i | y_i\} \in D$. Take any $i \in M$. Choose $z_i \in U(\overline{M}_i)$ such that $(u_i, z_i) \in x_i$ and $(z_i, v_i) \in y_i$. Let $z_i \in U(\overline{M}_i)$ be arbitrary for $i \notin M$. Then

$$M \subseteq \{i \in I : (u_i, z_i) \in x_i\} \cap \{i \in I : (z_i, v_i) \in y_i\},$$

so it follows that $\{i \in I : (u_i, z_i) \in x_i\} \in D$ and $\{i \in I : (z_i, v_i) \in y_i\} \in D$. Hence $([u]', [z]') \in F([x])$ and $([z]', [v]') \in F([y])$, so that $([u]', [v]') \in F([x])|F([y])$.

Conversely, suppose that $([u]', [v]') \in F([x])|F([y])$. Choose $z \in V$ with $([u]', [z]') \in F([x])$ and $([z]', [v]') \in F([y])$. Thus

$$\{i \in I : (u_i, z_i) \in x_i\} \in D \quad \text{and} \quad \{i \in I : (z_i, v_i) \in y_i\} \in D,$$

so the intersection N of these two sets is in D. Now

$$N \subseteq \{i \in I : (u_i, v_i) \in x_i | y_i\},$$

so the set on the right is in D, and hence $([u]', [v]') \in F(x; y)$. This finishes the proof that F preserves ;.

Clearly F preserves $^\cup$ and id.

If $[x] \neq [y]$, then $P \stackrel{\text{def}}{=} \{i \in I : x_i \neq y_i\} \in D$. Take any $i \in P$, and choose $(u_i, v_i) \in x_i \triangle y_i$. Let u_i and v_i be any members of $U(\overline{M}_i)$ for $i \notin P$. Then $([u]', [v]') \in F([x \triangle y]) = F([x]) \triangle F([y])$. So $F([x]) \neq F([y])$. □

Lemma 2.12. *If $\overline{M}$ is a proper relation algebra with unit element R, then for any $X \in M$ we have $R|(R\backslash(R|X|R))|R = (R\backslash(R|X|R))$.*

Proof. Suppose that $(a, b) \in R|(R\backslash(R|X|R))|R$ but also $(a, b) \in (R|X|R)$. Say $aRc(R\backslash(R|X|R))dRb$ and $aReXfRb$. Then $cRaReXfRbRd$, hence $cReXfRd$, so that $c(R|X|R)d$, contradiction. Thus $\subseteq$ holds.

If $(a, b) \in (R\backslash(R|X|R))$, then $aRa(R\backslash(R|X|R))bRb$, and so (a, b) is a member of $(R|(R\backslash(R|X|R))|R)$, proving $\supseteq$. □

Lemma 2.13. *A homomorphic image of a member of* **RRA** *is again a member of* **RRA**.

Proof. Let $\overline{M}$ be a proper relation algebra with unit element R, and let f be a homomorphism from $\overline{M}$ onto $\overline{N}$. Then $\overline{M}/\ker(f) \cong \overline{N}$ by Theorem 1.8, so it suffices to show that $\overline{M}/\ker(f)$ is isomorphic to a proper relation algebra.

(1) For any $Z \in M$, the set $N_Z \stackrel{\text{def}}{=} \{y \cap (R|Z|R) : y \in M\}$ is a proper relation algebra, and $\langle y \cap (R|Z|R) : y \in M\rangle$ is a homomorphism from $\overline{M}$ onto the associated structure.

In fact, clearly N_Z is closed under $\cup$, $\cap$, complementation relative to $R|Z|R$, and $\emptyset \in N_Z$. To show that it is closed under $|$, suppose that $x, y \in M$; we want to show that $(x|y) \cap (R|Z|R) = (x \cap (R|Z|R))|(y \cap (R|Z|R))$. First suppose that $(u, v) \in (x|y) \cap (R|Z|R)$. Choose w such that $(u, w) \in x$ and $(w, v) \in y$. Also choose s, t so that $(u, s) \in R$, $(s, t) \in Z$, and $(t, v) \in R$. Then $(u, s) \in R$ and $(t, w) \in R|y^{-1} \subseteq R$, so $(u, w) \in (R|Z|R)$. So $(u, w) \in (x \cap (R|Z|R))$. Similarly, $wRuRs$. so $wRsZtRv$. hence $(w, v) \in (y \cap (R|Z|R))$, so $(u, v) \in (x \cap (R|Z|R))|(y \cap (R|Z|R))$.

Conversely, suppose that $(u, v) \in (x \cap (R|Z|R))|(y \cap (R|Z|R))$. Choose w so that $(u, w) \in (x \cap (R|Z|R))$ and $(w, v) \in (y \cap (R|Z|R))$. Then $(u, v) \in (x|y)$. Also choose p, q, r, s so that $(u, p) \in R$, $(p, q) \in Z$, $(q, w) \in R$, $(w, r) \in R$, $(r, s) \in Z$, and $(Z, v) \in R$. Then $(u, r) \in R|Z|R|R \subseteq R$, so $(u, v) \in (R|Z|R)$. Thus $(u, v) \in ((x|y) \cap (R|Z|R))$, as desired.

To show that N_Z is closed under $^{-1}$, suppose that $x \in M$; we want to show that $(x \cap R|Z|R)^{-1} = (x^{-1} \cap R|Z|R)$. First suppose that $(u, v) \in (x \cap R|Z|R)^{-1}$. Then $(v, u) \in (x \cap R|Z|R)$. Choose s, t so that $(v, s) \in R$, $(s, t) \in Z$, and $(t, u) \in R$. Then $(u, s) \in R^{-1}|Z^{-1} \subseteq R$ and $(t, v) \in Z^{-1}|R^{-1} \subseteq R$, so $(u, v) \in (R|Z|R)$. So $(u, v) \in (x^{-1} \cap R|Z|R)$. This shows that $(x \cap R|Z|R)^{-1} \subseteq (x^{-1} \cap R|Z|R)$. Conversely, suppose that $(u, v) \in (x^{-1} \cap R|Z|R)$. Then $(v, u) \in x$. Also, $(u, v) \in R|Z|R$, so choose s, t so that $uRsZtRv$. Then $(v, s) \in x|R \subseteq R$ and $(t, u) \in R|x \subseteq R$, so $(v, u) \in R|Z|R$. Hence $(v, u) \in x \cap (R|Z|R)$, and it follws that $(u, v) \in (x \cap (R|Z|R))^{-1}$.

Clearly $id \in N_Z$. The homomorphism property in (1) is clear.

We may assume that $f(R) \neq f(\emptyset)$, as otherwise clearly $|\text{rng}(f)| = 1$. Let $I = \{Z \in M : f(Z) = 0\}$. Thus $R \notin I$, and if $Z_1, Z_2 \in I$ then also $Z_1 \cup Z_2 \in I$.

Now let D be an ultrafilter on I containing each set $\{V \in I : Z \leq V\}$ for each $Z \in I$. For each $Z \in I$ let $T(Z) = (R\backslash(R|Z|R))$.

Define $h : M \to \prod_{Z \in I} N_{T(Z)}$ by setting $(h(Y))_Z = Y \cap (R|T(Z)|R)$ for each $Y \in M$ and $Z \in I$. Then define $h'(Y) = h(Y)/D$ for any $Y \in M$.

From (1) it follows that h' is a homomorphism. We claim that $\ker(f) \subseteq \ker(h')$. For, suppose that $(Y, Y') \in \ker(f)$. So $f(Y) = f(Y')$, so $f(Y \triangle Y') = 0$. Hence $(Y \triangle Y', 0) \in \ker(f)$, so $Y \triangle Y' \in I$. For any $W \in I$ such that $(Y \triangle Y') \subseteq W$ we have $T(W) \subseteq T(Y \triangle Y')$. Now $Y \triangle Y' \subseteq R|(Y \triangle Y')|R$, so $(Y \triangle Y') \cap (R\backslash(R|(Y \triangle Y')|R) = \emptyset$, which means that $(Y \triangle Y') \cap T(Y \triangle Y') = \emptyset$, so $(Y \triangle Y') \cap T(W) = \emptyset$. It follows that $Y \cap T(W) = Y' \cap T(W)$. This being true for all $W \in I$ such that $(Y \triangle Y') \subseteq W$, it follows that $h'(Y) = h'(Y')$. Thus we have shown that $\ker(f) \subseteq \ker(h')$.

Now suppose that $h'(Y) = h'(Y')$. Then $h(Y)/D = h(Y')/D$, and so $\{Z \in I : h(Y)_Z = h(Y')_Z\} \in D$. Choose any Z such that $h(Y)_Z = h(Y')_Z$. Thus $Y \cap (R|T(Z)|R) = Y' \cap (R|T(Z)|R)$, so $(Y \triangle Y') \cap (R|T(Z)|R) = 0$. By Lemma 2.12 we have $(R|T(Z)|R) = T(Z)$. Hence $(Y \triangle Y') \cap T(Z) = 0$, and it follows that $(Y \triangle Y') \subseteq (R|Z|R)$. Clearly

$f(R|Z|R) = 0$, so also $f(Y \triangle Y') = 0$. So $f(Y) = f(Y')$. This shows that $\ker(f) = \ker(h')$.

Now define $k(Y/\ker(f)) = h'(Y)$, for any $Y \in M$. Then k is well-defined and one-one by what was just shown. Now $\overline{M} \in \mathbf{RRA}$ by Lemma 2.10. □

Theorem 2.14. (Tarski) **RRA** *is a variety.* □

No very nice system of equations characterizing **RRA** is known. It was proved in an article of Monk in 1964 that no finite system of equations works; the proof involves a construction of a system of non-representable relation algebras whose ultraproduct is representable.

Exc. 2.1. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{SHK} \subseteq \mathbf{HSK}$.

Exc. 2.2. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{SHK} \neq \mathbf{HSK}$.

Exc. 2.3. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{PHK} \subseteq \mathbf{HPK}$.

Exc. 2.4. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{PHK} \neq \mathbf{HPK}$. Hint: let $\mathbf{K}$ consist of all fields $\mathbb{Z}_p$ for p a prime, and take an ultraproduct of them with a nonprincipal ultrafilter. Show that the result is an infinite field.

Exc. 2.5. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{PSK} \subseteq \mathbf{SPK}$.

Exc. 2.6. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{PSK} \neq \mathbf{SPK}$.

Exc. 2.7. Prove that $\mathbf{HSPK}$ is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$. Infer that for any class $\mathbf{K}$ of structures, $\mathbf{HSPK}$ is the smallest variety containing $\mathbf{K}$.

[In 1972, D. Pigozzi determined all possible distinct sequences of $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$; it turns out that there are exactly 18 of them. It would be natural to adjoin an operation $\mathbf{Up}$ such that $\mathbf{UpK}$ is the class of all algebras isomorphic to an ultraproduct of members of $\mathbf{K}$. Then there is an open problem, raised by Henkin, Monk, and Tarski in 1971: does the sequence $\mathbf{P}$, $\mathbf{PUp}$, $\mathbf{PUpP}$, $\mathbf{PUpPUp}$, ... have any repetitions?]

Exc. 2.8. Prove that the following hold in any proper relation algebra with unit R:

(i) $S^{-1}|[-(S|T)] \subseteq -T$.

(ii) $((S|R) \cap id)|R = S|R$.

(iii) $S \subseteq S|S^{-1}|S$.