A structure $\overline{M}$ is *atomic* iff $\mathrm{tp}^{\overline{M}}(\overline{a})$ is isolated, for all positive integers m and all $\overline{a} \in {}^nM$.

Theorem 10.1. *Let $\mathscr{L}$ be a countable language, and T a complete theory in $\mathscr{L}$ with infinite models. Let $\overline{M}$ be a model of T. Then $\overline{M}$ is prime iff it is countable and atomic.*

Proof. $\Rightarrow$: Assume that $\overline{M}$ is prime. If t is a type of T which is not isolated, then by the omitting types theorem 6.26, T has a model $\overline{N}$ which omits t. Since $\overline{M}$ can be elementarily embedded in $\overline{N}$, it follows that $\overline{M}$ also omits t. Thus for any type t of T, t isolated $\Rightarrow$ M omits t. Hence any type which $\overline{M}$ realizes is isolated; this means that $\overline{M}$ is atomic. since T has countable models by the downward Löwenheim-Skolem theorem, and $\overline{M}$ can be elementarily embedded in any model of T, $\overline{M}$ is countable.

$\Leftarrow$: Suppose that $\overline{M}$ is countable and atomic, and $\overline{N}$ is any model of T; we want to construct an elementary embedding of $\overline{M}$ into $\overline{N}$. Let $\langle a_i : i \in \omega\rangle$ enumerate the elements of M, and for each $i \in \omega$ let $\theta_i(\overline{v})$ isolate $\mathrm{tp}^{\overline{M}}(a_0, \ldots, a_i)$. We will now construct elementary maps $f_0 \subseteq f_1 \subseteq \ldots$ from subsets of M into N, where the domain of f_i is $\{a_0, \ldots, a_{i-1}\}$. Let $f_0 = \emptyset$. It is elementary since $\overline{M} \equiv \overline{N}$ (because T is complete). Suppose that f_i has been constructed, an elementary map. Then $\overline{M} \models \theta_i(a_0, \ldots, a_i)$, hence $\overline{M} \models \exists v\theta_i(a_0, \ldots, a_{i-1}, v)$. Since f_i is an elementary map, it follows that $\overline{N} \models \exists v\theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), v)$. Choose $b \in N$ such that $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), b)$, and let $f_{i+1} = f_i \cup \{(i, b)\}$. To see that f_{i+1} is elementary, suppose that $\overline{M} \models \psi(a_0, \ldots, a_i)$, Thus $\psi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(a_0, \ldots, a_i)$, so $T \models \theta_i \to \psi$. Since $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), b)$, it follows that $\overline{N} \models \psi(f_i(a_0), \ldots, f_i(a_{i-1}), b)$. Thus f_{i+1} is elementary.

Now $\bigcup_{i\in\omega} f_i$ is an elementary, as desired. □

Corollary 10.2. *If $\mathscr{L}$ is a countable language and T is a complete theory with infinite models, then T has a prime model iff T has an atomic model.*

Proof. $\Rightarrow$: by Theorem 10.1.

$\Leftarrow$: Suppose that $\overline{M}$ is an atomic model of T. Let $\overline{N}$ be a countable elementary substructure of $\overline{M}$. Then for any $n \in \omega$ and any $\overline{a} \in {}^nN$ we have $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{a})$, and hence $\mathrm{tp}^{\overline{N}}(\overline{a})$ is isolated. So $\overline{N}$ is prime by Theorem 10.1. □

Proposition 10.3. *If $\overline{M}$ is an atomic model of T, then it is ω-homogeneous.*

Proof. See just before Lemma 7.28 for the definition of ω-homogeneous. Suppose that A is a finite subset of M and $f : A \to M$ is a partial elementary map. Let $\overline{a}$ enumerate A. Let $b \in M$. Let $\varphi(\overline{v}, w)$ isolate the type $\mathrm{tp}^{\overline{M}}(\overline{a}, b)$. Then $\overline{M} \models \exists w\varphi(\overline{a}, w)$. So, since f is partial elementary, we get $\overline{M} \models \exists w\varphi(f \circ \overline{a}, w)$ Choose $d \in M$ such that $\overline{M} \models \varphi(f \circ \overline{a}, d)$. Let $g = f \cup \{(b, d)\}$. To see that g is partial elementary, suppose that $\overline{M} \models \psi(\overline{a}, b)$. Then $T \models \varphi \to \psi$ and $\overline{M} \models \varphi(f \circ \overline{a}, d)$, so $\overline{M} \models \psi(f \circ \overline{a}, d)$, as desired. □

Corollary 10.4. *If T is a complete theory in a countable language, then any two prime models of T are isomorphic.*

Proof. Let $\overline{M}$ and $\overline{N}$ be prime models of T. Then by Theorem 10.1 they are both countable and atomic. By Proposition 10.3 they are also both ω-homogeneous. Now every type realized in $\overline{M}$ is isolated. Also, if t is an isolated type, say isolated by $\varphi(\overline{v})$, then $T \models \exists\overline{v}\varphi(\overline{v})$, hence $\overline{M}$ realizes t. So a type is isolated iff it is realized in $\overline{M}$. The same is true of $\overline{N}$, so $\overline{M}$ and $\overline{N}$ realize the same types. Hence they are isomorphic by Theorem 7.31. $\square$

Theorem 10.5. *If $\overline{M}$ is κ-saturated, then $\overline{M}$ is κ-homogeneous.*

Proof. Suppose that $A \in [M]^{<\kappa}$, $f : A \to M$ is partial elementary, and $b \in M \backslash A$. Let

$$\Gamma = \{\varphi(v, f \circ \overline{a}) : \exists m \in \omega[\overline{a} \in {}^m A \text{ and } \overline{M} \models \varphi(b, \overline{a})]\}.$$

Let Δ be a finite subset of Γ. For each member χ of Δ, choose $m_\chi \in \omega$, φ_χ, and $\overline{a}_\chi \in {}^m A$ such that $\overline{M} \models \varphi_\chi(b, \overline{a}_\chi)$ and $\chi = \varphi_\chi(v, f \circ \overline{a}_\chi)$. Then there is an $n \in \omega$, $\overline{a}' \in {}^n A$, and a formula ψ such that the following conditions hold:

(1) $\overline{M} \models \psi(b, \overline{a}')$.

(2) $\models \bigwedge \Delta \leftrightarrow \psi(v, f \circ \overline{a}')$.

Now from (1) we get $\overline{M} \models \exists v\psi(v, \overline{a}')$. Hence, since f is elementary, $\overline{M} \models \exists v\psi(v, f \circ \overline{a}')$. Hence by (2), $\overline{M} \models \exists v \bigwedge \Delta$. Thus Γ is finitely satisfiable. So since $\overline{M}$ is ω-saturated we get $c \in M$ such that $\overline{M} \models \varphi(c, f \circ \overline{a})$ for each $\varphi(v, f \circ \overline{a}) \in \Gamma$. So $f \cup \{(b, c)\}$ is elementary. $\square$

Theorem 10.6. *Suppose that $\overline{M}$ is a model of T. Then $\overline{M}$ is ω-saturated iff $\overline{M}$ is ω-homogeneous and for every $m \in \omega$, $\overline{M}$ realizes all types in $S_m(T)$.*

Proof. $\Rightarrow$: by Theorem 10.5.

$\Leftarrow$: Let $m, n \in \omega$, $\overline{a} \in {}^m M$, and $p \in S_n^{\overline{M}}(\overline{a})$. Let $q \in S_{n+m}(T)$ be the type $\{\varphi(\overline{v}, \overline{w}) : \varphi(\overline{v}, \overline{a}) \in p\}$. Since $\overline{M}$ realizes all types in $S_{m+n}(T)$, choose $(\overline{b}, \overline{c})$ realizing q. Now

(∗) $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{c})$.

In fact, let $\psi(\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a})$. Let ψ' be $\overline{v} = \overline{v} \wedge \psi$. Then $\psi'(\overline{v}, \overline{a}) \in p$, since otherwise $\neg\psi'(\overline{v}, \overline{a}) \in p$ and hence, since every finite subset of p is satisfiable in $\overline{M}$, we get $\overline{e}$ such that $\overline{M} \models \neg\psi'(\overline{e}, \overline{a})$, i.e., $\overline{M} \models \neg\psi(\overline{a})$, contradiction. So $\psi'(\overline{v}, \overline{a}) \in p$, hence $\psi'(\overline{v}, \overline{w}) \in q$, hence $\overline{M} \models \psi'(\overline{b}, \overline{c})$, so $\overline{M} \models \psi(\overline{c})$. This proves (∗).

Now by ω-homogeneity we get $\overline{d}$ such that $\mathrm{tp}^{\overline{M}}(\overline{b}, \overline{c}) = \mathrm{tp}^{\overline{M}}(\overline{d}, \overline{a})$. Thus for any $\varphi(\overline{v}, \overline{a}) \in p$ we have $\varphi(\overline{v}, \overline{w}) \in q$, hence $\overline{M} \models \psi(\overline{b}, \overline{c})$, so $\varphi(\overline{v}, \overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{c})$, hence $\varphi(\overline{v}, \overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{d}, \overline{a})$, so $\overline{M} \models \varphi(\overline{d}, \overline{a})$. This shows that $\overline{d}$ realizes p. $\square$

Corollary 10.7. *If $\overline{M}$ and $\overline{N}$ are countable saturated models of T, then $\overline{M} \cong \overline{N}$.*

Proof. By Theorem 10.6, both $\overline{M}$ and $\overline{N}$ are ω-homogeneous and realize all types in any $S_m(T)$. By Theorem 7.31 they are isomorphic. $\square$

Now we need some variants of 7.38–7.39.

Lemma 10.8. *Suppose that $\overline{M}$ is a model of T and $\overline{a}, \overline{b}, c \in M$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b})$. Then there exist an elementary extension $\overline{N}$ of $\overline{M}$ such that $|M| = |N|$ and an element $d \in N$ such that $\mathrm{tp}^{\overline{N}}(\overline{a}, c) = \mathrm{tp}^{\overline{N}}(\overline{b}, d)$.*

Proof. Apply the compactness theorem to the set

$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{b}, u) : \overline{M} \models \varphi(\overline{a}, c)\}$ (u a new constant) $\square$

Theorem 10.9. *Suppose that $\overline{M}$ is a model of T. Then there exists an elementary extension $\overline{N}$ of $\overline{M}$ such that $|M| = |N|$ and for all $\overline{a}, \overline{b}, c \in M$ there is a $d \in N$ such that $\mathrm{tp}^{\overline{N}}(\overline{a}, c) = \mathrm{tp}^{\overline{N}}(\overline{b}, d)$.*

Proof. Iterate Lemma 10.8. $\square$

Theorem 10.10. *Suppose that $\overline{M}$ is a model of T. Then there is an elementary extension $\overline{N}$ of $\overline{M}$ such that $\overline{N}$ is ω-homogeneous and $|M| = |N|$.*

Proof. Iterate Lemma 10.9 ω times. $\square$

Theorem 10.11. *T has a countable saturated model iff $|S_n(T)| \leq \aleph_0$ for all n.*

Proof. $\Rightarrow$: If $\overline{M}$ is a countable saturated model, then it realizes only countably many types; but it realizes all types, so $|S_n(T)| \leq \aleph_0$ for all n.

$\Leftarrow$: Let $t_0, t_1, \ldots$ list all members of $\bigcup_{n\in\omega} S_n(T)$. By the compactness theorem, for any countable model $\overline{N}$ of T and any $i \in \omega$ there is a countable elementary extension $\overline{P}$ of $\overline{N}$ with an element which realizes t_i. So if we start with $\overline{M}$ and iterate this process ω times we obtain an elementary chain $\overline{M} = \overline{N}_0 \preceq \overline{N}_1 \preceq \cdots$ such that each $\overline{N}_i$ is countable and $\overline{N}_{i+1}$ has an element realizing t_i. Let $\overline{P} = \bigcup_{i\in\omega} \overline{N}_i$. Then $\overline{P}$ is an elementary extension of $\overline{M}$ which realizes every type over T. By Theorem 10.10 let $\overline{Q}$ be a countable elementary extension of $\overline{P}$ which is ω-homogeneous. By Theorem 10.6, $\overline{Q}$ is ω-saturated. $\square$

Theorem 10.12. *If $\mathscr{L}$ is countable, T is a theory in $\mathscr{L}$, and $|S_n(T)| \leq \omega$ for all $n \in \omega$, then T has a countable atomic model.*

Proof. By Theorem 6.26 (the omitting types theorem), let $\overline{M}$ be a countable model of T which omits all non-isolated types. Thus $\overline{M}$ is atomic. $\square$

Corollary 10.13. *If $\mathscr{L}$ is countable and T is a theory in $\mathscr{L}$ which has an ω-saturated model, then T has a countable atomic model.*

Proof. By Theorems 10.11 and 10.12. $\square$

Theorem 10.14. *For T a theory in a countable language the following are equivalent:*

(i) T is $\aleph_0$-categorical.

(ii) For every $n < \omega$, every type in $S_n(T)$ is isolated.

(iii) $|S_n(T)| < \aleph_0$ for every $n \in \omega$.

(iv) For every $n \in \omega$ there is a finite set Γ of formulas with free variables among $v_0, \ldots, v_{n-1}$ such that for every formula φ with free variables among $v_0, \ldots, v_{n-1}$ there is a $\psi \in \Gamma$ such that $T \models \varphi \leftrightarrow \psi$.

Proof. (i)$\Rightarrow$(ii): Suppose that (ii) fails: there exist $n < \omega$ and a type $p \in S_n(T)$ which is not isolated. By the omitting types theorem 6.26, there is a countable model $\overline{M}$ of T which omits p. But clearly there is also a countable model $\overline{N}$ which admits p. Thus $\overline{M} \not\cong \overline{N}$, so (i) fails.

(ii)⇒(iii): Assume (ii), but suppose that (iii) fails: there is an $n \in \omega$ such that $S_n(T)$ is infinite. For each $p \in S_n(T)$ let φ_p isolate p. Let $\overline{c}$ be a sequence of new constants of length n, and consider the set

$$T' \stackrel{\text{def}}{=} T \cup \{\neg\varphi_p(\overline{c}) : p \in S_n(T)\}.$$

We claim that T' has a model. For, take any finite subset T'' of T. Let P be the set of all types p such that $\neg\varphi_p(\overline{c})$ is in T''. Let $q \in S_n(T)$ be different from each member of P. Now $T \models \varphi_q \to \neg\varphi_p$ for each $p \in P$; otherwise $T \models \varphi_q \to \varphi_p$ for some $p \in P$ and then $T \models \varphi_q \to \psi$ for each $\psi \in p$, so that $p \subseteq q$, hence $p = q$, contradiction. Now take a model $\overline{M}$ of T which realizes q, say $\overline{M} \models \varphi_q(\overline{a})$. Interpreting $\overline{c}$ by $\overline{a}$ in this model gives a model of T'', as desired.

Thus T' has a model, which gives a model $\overline{N}$ of T with a sequence $\overline{b}$ satisfying in $\overline{N}$ each formula $\neg\varphi_p$. Thus $\mathrm{tp}^{\overline{N}}(\overline{b})$ cannot be in $S_n(T)$, contradiction.

(iii)⇒(iv): Assume (iii), and suppose that $n \in \omega$. For distinct $p, q \in S_n(T)$ choose $\varphi_{pq} \in p \backslash q$. For each $p \in S_n(T)$ let $\psi_p = \bigwedge\{\varphi_{pq} : q \neq p\}$. Thus $\psi_p \in p$ while $\psi_p \notin q$ for all $q \neq p$. Now given χ with variables among $v_0, \ldots, v_{n-1}$ we have $T \models \chi \leftrightarrow \bigvee\{\psi_p : \chi \in p\}$.

iv)⇒(i): Assume (iv). Let $\overline{M}$ be a countable model of T; we show that $\overline{M}$ is atomic; so T is $\aleph_0$-categorical by Theorems 10.1 and 10.4. If $\overline{a} \in {}^nM$, then $\mathrm{tp}^{\overline{M}}(\overline{a})$ is isolated by

$$\bigwedge\{\varphi_i(\overline{v}) : \overline{M} \models \varphi_i(\overline{a})\} \wedge \bigwedge\{\neg\varphi_i(\overline{v}) : \overline{M} \models \neg\varphi_i(\overline{a})\} \qquad \square$$