Model theory

(Math 6000)

November 19, 2012

These notes form an introduction to model theory. The topics are: first-order structures; terms and varieties; first-order languages, satisfaction, and truth; elimination of quantifiers; Löwenheim-Skolem theorems; ultraproducts; the compactness theorem; diagrams; Ehrenfeucht-Fraissé games; interpretations; saturated structures; omitting types; Morley's categoricity theorem; Morley rank; interpolation; countable models; the number of types and models.

1. Structures

The basic notions of model theory are *structures* and *first-order logic*; model theory is essentially the study of the relationships between these two notions. In this chapter we define the notion of structure, and give several examples which will be discussed later.

Structures consist of a nonempty set A together with certain elements of A, operations on A, and relations on A, with various numbers of arguments. See the precise definition below. Some important examples are: groups, rings, fields, lattices, Boolean algebras, posets, and linearly ordered sets. We will illustrate the general notions in this chapter with some of these special cases.

Some simple set-theoretic background is needed first. We use ω to denote the set of natural numbers $0, 1, 2, \ldots$. Each natural number is considered to actually be the set of all preceding natural numbers. Thus 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}$, etc. Moreover, $\omega\backslash\{0\}$ is the set of all positive integers. The domain of a function f is denoted by $\mathrm{dmn}(f)$. A *finite sequence* is a function whose domain is some positive integer. For any set A and any postive integer m, we denote by ${}^mA$ the set of all finite sequences of members of A, of length m. That is, ${}^mA$ is the set of all functions mapping m into A. An m-ary *relation* on A is a subset of ${}^mA$. An m-ary *operation* on A is a function mapping ${}^mA$ into A. These are general notions which are used in what follows.

A *signature* or *similarity type* or (first-order) *language* is an ordered quadruple $\sigma = (\mathrm{Fcn}, \mathrm{Rel}, \mathrm{Cn}, \mathrm{ar})$ such that Fcn, Rel, Cn are pairwise disjoint sets, and ar is a function mapping $\mathrm{Fcn} \cup \mathrm{Rel}$ into $\omega\backslash 1$. Members of Fcn, Rel, Cn are called function (or operation) symbols, relation symbols, and individual constants respectively. The positive integer $\mathrm{ar}(S)$ is called the *arity* of the function or relation symbol S, and S is said to be $\mathrm{ar}(S)$-ary. We also say that S has *rank* $\mathrm{ar}(S)$. We sometimes say *unary* rather than 1-ary, and *binary* rather than 2-ary. The members of Cn are called *individual constants.* One might think of the function and relation symbols as constants of a different sort.

A *(first-order) structure* with signature σ is a quadruple $\overline{A} = (A, a, b, c)$ such that:

• A is a nonempty set.

• a is a function with $\mathrm{dmn}(a) = \mathrm{Rel}$, and a_R is an $ar(R)$-ary relation on A for every $R \in \mathrm{dmn}(a)$.

• b is a function with $\mathrm{dmn}(b) = \mathrm{Fcn}$, and b_F is an $ar(F)$-ary operation on A for every $F \in \mathrm{dmn}(b)$.

• c is a function mapping Cn into A.

We say then that (A,a,b,c) has signature σ. The set A is called the *universe* of (A,a,b,c). Frequently we just write A instead of (A,a,b,c) if the appropriate a,b,c are clear. Frequently we write $R^{\overline{A}}$, $F^{\overline{A}}$, or $k^{\overline{A}}$ in place of $a(R), b(F), c(k)$. The functions, relations, constants $R^{\overline{A}}$, $F^{\overline{A}}$, $k^{\overline{A}}$ are called *fundamental functions, relations, constants* of A, to distinguish them from arbitrary functions, relations, constants on A. In practice, structures are of three sorts as far as motivation is concerned. If there are no relation symbols, a structure is called an *algebra*. This is in a general sense, not the same as the notion of algebra considered in ring and field theory. If there are no function symbols, or individual constants the structure is called a *relational structure*. The general notion is needed in many special cases, though, for example in talking about ordered fields. The associated signatures of these three sorts are called *algebraic*, *relational*, or *mixed*.

EXAMPLES

For the examples it is convenient to use a looser notation. In particular, if one of Fcn, Rel, Cn is empty, we simply omit it.

• Partial orderings. These are structures $(A,<)$ such that $<$ is irreflexive (for no $x \in A$ is $x < x$) and transitive (for any $x,y,z \in A$, if $x < y$ and $y < z$ then $x < z$). Officially we are dealing with a signature (Fcn,Rel,Cn,ar), where Fcn $= \emptyset$, Rel is a one element set $\{R\}$, Cn $= \emptyset$, and ar is the function with domain the one-element set Rel with $\mathrm{ar}(R) = 2$. The particular set R which is the only member of Rel is rather irrelevant.

• Groups. We define a group to be a structure $(G, \cdot, {}^{-1}, e)$ such that $\cdot$ is a binary operation on G, ${}^{-1}$ is a one-place operation on G, and $e \in G$, satisfying the familiar laws. Officially our signature is (Fcn,Rel,Cn,ar), where Fcn $= \{F, G\}$, Rel $= \emptyset$, Cn $= \{e\}$, and ar is the function with domain $\{F, G\}$ with $\mathrm{ar}(F) = 2$ and $\mathrm{ar}(G) = 1$.

• Rings. A ring is a structure $(R, +, \cdot, -, 0)$ satisfying the usual properties, where $-$ is the operation of forming the additive inverse of an element. The official definition should be clear.

• Ordered fields. We add to the fundamental operations for rings a binary relation, and thus consider structures of the form $(F, +, \cdot, -, 0, <)$.

• Vector spaces over a field. Since there are two kinds of objects here, it is not immediately clear how to consider vector spaces as structures in our sense. But a standard for this has arisen. We work over a field F which is the same for all structures. The structures have the form $(V, +, -, 0, f_\alpha)_{\alpha\in F}$, where $(V, +, -, 0)$ is an abelian group and for each $\alpha \in F$, f_α is scalar multiplication by α. Officially the signature is $(\{p, m\} \cup F, \emptyset, \{z\}, ar)$ with $ar(p) = 2$, $ar(m) = 1$, and $ar(\alpha) = 1$ for every $\alpha \in F$. Some of the axioms are as follows: $f_\alpha(f_\beta(a)) = f_{\alpha\cdot\beta}(a)$, $f\alpha(a + b) = f_\alpha(a) + f_\alpha(b)$.

SUBSTRUCTURES

We now consider generalizations of several notions of abstract algebra. These involve more than one structure, but all structures will have the same signature. We say that $\overline{A}$ and $\overline{B}$ are *similar* if they have the same signature.

We say that $\overline{A}$ is a *substructure* of $\overline{B}$ iff the following conditions hold:

$A \subseteq B$

For every relation symbol R, $R^{\overline{A}} = R^{\overline{B}} \cap {}^{\mathrm{ar}(R)}A$.

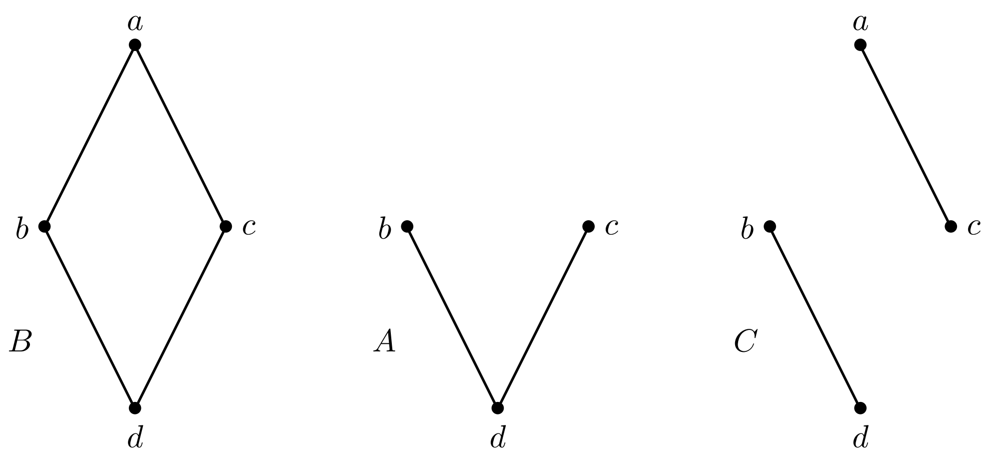
For every function symbol F and all $x \in {}^{\mathrm{ar}(F)}A[F^{\overline{A}}(x) = F^{\overline{B}}(x)]$.

For every individual constant k we have $k^{\overline{A}} = k^{\overline{B}}$.

We write $\overline{A} \leq \overline{B}$ to mean that $\overline{A}$ is a substructure of $\overline{B}$. If $\overline{A} \leq \overline{B}$, then we also call $\overline{B}$ a *superstructure* of $\overline{A}$.

For our main examples we have the following:

Partial orderings. If $(A, <)$ is a substructure of $(B, \prec)$ and $(B, \prec)$ is a partial ordering, then so is $(A, <)$, and $<$ is the intersection of $\prec$ with $A \times A$. This means that $A \subseteq B$ and for any $x, y \in A$, $x < y$ iff $x \prec y$. This is illustrated in the following diagrams, where $(A, <)$ is a substructure of $(B, \prec)$, but $(C, <')$ is not a substructure of $(B, \prec)$.



Groups. If $(H, \circ, f, e')$ is a substructure of a group $(G, \cdot, {}^{-1}, e)$, then $(H, \circ, f, e')$ is also a group, H is closed under $\cdot$ and ${}^{-1}$, $e = e'$, $\circ$ is the restriction of $\cdot$ to $H \times H$, and f is the restriction of ${}^{-1}$ to H. Note that if we had treated a group as a structure $(G, \cdot)$, then there would be substructures which are not groups. For example, $(\omega, +)$ is a substructure of $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$ is a group but $(\omega, +)$ is not.

Rings. If $(R', +', \cdot', -', 0')$ is a substructure of a ring $(R, +, \cdot, -, 0)$ then $(R', +', \cdot', -', 0')$ is a ring. Note that R might have an identity while R' does not. An example is given by $R = \mathbb{Z}$ and $R' = \{2m : m \in \mathbb{Z}\}$.

Ordered fields. If $(F, +, \cdot, -, 0, <)$ is an ordered field and $(F', +', \cdot', -', 0', <')$ is a substructure of it, then F' is not necessarily an ordered field. An example is given by $F = \mathbb{Q}$ and $F' = \mathbb{Z}$.

Vector spaces. If V is a vector space over a field F, a substructure of V is just a subspace in the usual sense.

If $\overline{A}$ is a structure and $B \subseteq A$ is closed under the fundamental operations of A, then we call B a *subuniverse* of $\overline{A}$. (Saying that B is closed under the fundamental operations of A is supposed to include the assumption that the fundamental constants of A are in B.) Any subuniverse B can be made into a substructure of A in the natural way:

$R^{\overline{B}} = {}^{\mathrm{ar}(R)}B \cap R^{\overline{A}}$;

$F^{\overline{B}}(x) = F^{\overline{A}}(x)$ for all $x \in {}^{\mathrm{ar}(F)}B$.

$k^{\overline{B}} = k^{\overline{A}}$.

Proposition 1.1. *If $\overline{A}$ is a structure and K is a nonempty collection of subuniverses of $\overline{A}$ having at least one common element, then $\bigcap K$ is a subuniverse of $\overline{A}$.* $\square$

This proposition justifies the following definition: if $\emptyset \neq X \subseteq A$, then $\langle X\rangle_A$ is the intersection of all subuniverses of A which contain X. This is called the subuniverse *generated* by X; the associated structure is the substructure *generated* by X. In case the signature has at least one individual constant, we can define $\langle\emptyset\rangle_A$ in the same way.

Note that if there are no function symbols or individual constants, then any nonempty subset of A is a subuniverse.

The following characterization of $\langle X\rangle_A$ is frequently useful.

Proposition 1.2. *Suppose that $\overline{A}$ is a structure and X is a nonempty subset of A. Define $\langle Y_i : i \in \omega\rangle$ by recursion, as follows:*

$$\begin{aligned} Y_0 &= X \cup \{k^{\overline{A}} : k \textit{ an individual constant}\};\\ Y_{i+1} &= Y_i \cup \{F^{\overline{A}}(x) : F \textit{ a function symbol}, x \in {}^{\mathrm{ar}(F)}Y_i\}. \end{aligned}$$

Then $\langle X\rangle_A = \bigcup_{i\in\omega} Y_i$.

Proof. An easy induction shows that $Y_i \subseteq \langle X\rangle_A$ for all $i \in \omega$; so $\supseteq$ holds. Hence it suffices to show that $\bigcup_{i\in\omega} Y_i$ is a subuniverse which contains X. Obviously it contains X and all fundamental constants of $\overline{A}$. Now suppose that F is a function symbol, and $x \in {}^{\mathrm{ar}(F)}\bigcup_{i\in\omega} Y_i$. For each $k < \mathrm{ar}(F)$ let $s(k) < \omega$ be such that $x_k \in Y_{s(k)}$. Let l be the maximum of all $s(k)$ for $k < n_j$. Then $x \in {}^{\mathrm{ar}(F)}Y_l$, and it follows that $F^{\overline{A}}(x) \in Y_{l+1}$, as desired. $\square$

Note that if V is a vector space over F and $X \subseteq V$, then $\langle X\rangle$ is the subspace of V spanned by X.

HOMOMORPHISMS

Again, let $\overline{A}$ and $\overline{B}$ be similar structures. A *homomorphism* from $\overline{A}$ to $\overline{B}$ is a function f mapping the set A into the set B such that the following conditions hold:

- For R an m-ary relation symbol and any sequence $\overline{a}$ of length m of elements of A, $\overline{a} \in R^{\overline{A}}$ iff $f \circ \overline{a} \in R^{\overline{B}}$.

- For F an m-ary operation symbol, if $\overline{a} \in {}^mA$, then $f(F^{\overline{A}}(\overline{a})) = F^{\overline{B}}(f \circ \overline{a})$.

• For any individual constant k we have $f(k^{\overline{A}}) = k^{\overline{B}}$.

A *weak homomorphism* from $\overline{A}$ to $\overline{B}$ is a mapping f satisfying the above conditions for individual constants and function symbols, but with the following weaker condition for relation symbols: For R an m-ary relation symbol and any sequence $\overline{a}$ of length m of elements of A, if $\overline{a} \in R^{\overline{A}}$ then $f \circ \overline{a} \in R^{\overline{B}}$.

In our examples, homomorphisms mean the following.

A homomorphism from a partial ordering $(P, <)$ to a partial ordering $(Q, \prec)$ is a function f mapping P into Q such that for all $a, b \in P$, $a < b$ iff $f(a) \prec f(b)$. Thus in the examples concerning substructures, the identity on A is a homomorphism from A into B, but the identity on C is not a homomorphism from C into B. The identity on C is a weak homomorphism from C into B.

For groups, rings, and ordered fields, homomorphisms have the same meaning as in elementary algebra. For vector spaces, homomorphisms are the same as linear transformations.

An *embedding* or *isomorphism into* or *monomorphism* from $\overline{A}$ to $\overline{B}$ is a one-one homomorphism from $\overline{A}$ into $\overline{B}$. An *isomorphism* is an embedding which is onto. An isomorphism of $\overline{A}$ onto $\overline{A}$ is called an *automorphism* of $\overline{A}$.

We write $\overline{A} \cong \overline{B}$ to indicate that there is an isomorphism from $\overline{A}$ onto $\overline{B}$.

Proposition 1.3. *Suppose that $\overline{A}$ and $\overline{B}$ are similar structures and $f : A \to B$. Then the following conditions are equivalent:*

(i) f is a embedding from $\overline{A}$ into $\overline{B}$.

(ii) f is an isomorphism from $\overline{A}$ onto a substructure of $\overline{B}$.

Proof. (i)⇒(ii): Assume (i). Let $C = \mathrm{rng}(f)$. We check that C is closed under the operations of $\overline{B}$. If F is an m-ary function symbol and $\overline{a} \in {}^mA$, then

$$F^{\overline{B}}(f \circ \overline{a}) = f(F^{\overline{A}}(\overline{a})) \in C.$$

For any individual constant k, $k^{\overline{B}} = f(k^{\overline{A}}) \in C$. Thus C is a subuniverse of $\overline{B}$, and we have a structure $\overline{C}$ which is a substructure of $\overline{B}$. f is an isomorphism from $\overline{A}$ onto $\overline{C}$ since

$$\begin{aligned} f(F^{\overline{A}}(\overline{a})) &= F^{\overline{B}}(f \circ \overline{a}) = F^{\overline{C}}(f \circ \overline{a}); \\ f(k^{\overline{A}}) &= k^{\overline{B}} = k^{\overline{C}}; \\ \overline{a} \in R^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in R^{\overline{B}} \\ &\text{iff} \quad f \circ \overline{a} \in R^{\overline{C}}. \end{aligned}$$

(ii)⇒(i): Assume (ii); say that f is an isomorphism from $\overline{A}$ onto a structure $\overline{C}$ which is a substructure of $\overline{B}$. Then f is an embedding from $\overline{A}$ into $\overline{B}$:

$$\begin{aligned} f(F^{\overline{A}}(\overline{a})) &= F^{\overline{C}}(f \circ \overline{a}) = F^{\overline{B}}(f \circ \overline{a}); \\ f(k^{\overline{A}}) &= k^{\overline{C}} = k^{\overline{B}}; \\ \overline{a} \in R^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in R^{\overline{C}} \\ &\text{iff} \quad f \circ \overline{a} \in R^{\overline{B}}. \end{aligned}$$

□

Proposition 1.4. *Suppose that $\overline{A}$ is a structure, and f is a bijection from A onto a set B. Then there is a structure $\overline{B}$ with universe B such that f is an isomorphism from $\overline{A}$ onto $\overline{B}$.*

Proof. With obvious assumptions, we define

$$\begin{aligned} \mathbf{F}^{\overline{B}}(\overline{b}) &= f(F^{\overline{A}}(f^{-1} \circ \overline{b})); \\ \overline{b} \in \mathbf{R}^{\overline{B}} \quad \text{iff} &\quad f^{-1} \circ \overline{b} \in \mathbf{R}^{\overline{A}}; \\ \mathbf{k}^{\overline{B}} &= f(\mathbf{k}^{\overline{A}}). \end{aligned}$$

The conclusion is easy to check. $\square$

Proposition 1.5. *Suppose that f is an isomorphism from a structure $\overline{A}$ into a structure $\overline{B}$. Then there exist a structure $\overline{C}$ and a function g such that $\overline{A}$ is a substructure of $\overline{C}$, g is an isomorphism from $\overline{C}$ onto $\overline{B}$, and $f \subseteq g$.*

Proof. Let X be a set disjoint from A with the same number of elements as $B\backslash\mathrm{rng}(f)$, and let h be a bijection from X onto $B\backslash\mathrm{rng}(f)$. Let $g = f \cup h$. So g is a bijection from $A \cup X$ onto B. We now apply Proposition 1.4 to $\overline{B}$ and g^{-1} to get a structure $\overline{C}$ with universe $A \cup X$ such that g^{-1} is an isomorphism from $\overline{C}$ onto $\overline{B}$. Hence g is an isomorphism from $\overline{B}$ onto $\overline{C}$. So it remains only to check that $\overline{A}$ is a substructure of $\overline{C}$. With obvious assumptions, we have:

$$\begin{aligned} \mathbf{F}^{\overline{A}}(\overline{a}) &= f^{-1}(f(\mathbf{F}^{\overline{A}}(\overline{a}) \\ &= f^{-1}(\mathbf{F}^{\overline{B}}(f \circ \overline{a})) \\ &= g^{-1}(\mathbf{F}^{\overline{B}}(f \circ \overline{a})) \\ &= \mathbf{F}^{\overline{C}}(g^{-1} \circ f \circ \overline{a}) \\ &= \mathbf{F}^{\overline{C}}(\overline{a}); \\ \overline{a} \in \mathbf{R}^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad g \circ \overline{a} \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad \overline{a} \in \mathbf{R}^{\overline{C}}; \\ \mathbf{k}^{\overline{A}} &= f^{-1}(f(\mathbf{k}^{\overline{A}})) \\ &= f^{-1}(k^{\overline{B}}) \\ &= g^{-1}(k^{\overline{B}}) \\ &= k^{\overline{C}}. \end{aligned}$$

$\square$

Proposition 1.5 is justification for the common procedure in algebra when one has an isomorphism from a structure $\overline{A}$ into a structure $\overline{B}$, and assumes "without loss of generality" that $\overline{A}$ is actually a substructure of $\overline{B}$.

Let $\overline{A}$ be a structure. A *congruence relation* on $\overline{A}$ is an equivalence relation $\equiv$ on A such that if F is an m-ary function symbol, $\overline{a}$ and $\overline{b}$ are sequences of elements of A of length m, and $a_j \equiv b_j$ for each $j < m_i$, then $F^{\overline{A}}(\overline{a}) \equiv F^{\overline{A}}(\overline{b})$; and if R is an n-ary relation symbol, $\overline{a}$ and $\overline{b}$ are sequences of elements of A of length m, and $a_j \equiv b_j$ for each $j < m_i$, then $\overline{a} \in R^A$ iff $\overline{b} \in R^A$.

The equivalence class of an element a under an equivalence relation R is denoted by $[a]_R$, or simply $[a]$ if R is clear.

Proposition 1.6. *The intersection of a nonempty family of congruence relations on a structure is again a congruence relation on the structure.* □

Proposition 1.7. *If A is a structure and $\equiv$ is a congruence relation on A, then the set $B \stackrel{\text{def}}{=} A/{\equiv}$ of all equivalence classes under $\equiv$ can be given a structure $\overline{B}$ similar to that of A, such that $k^{\overline{B}} = [k^A]_{\equiv}$ for any individual constant k, $F^{\overline{B}}([a_0]_{\equiv}, \ldots, [a_{m-1}]_{\equiv}) = [F^{\overline{A}}(a_0, \ldots, a_{m-1})]_{\equiv}$ for any m-ary operation symbol F, and for any n-ary relation symbol R and any $a \in {}^mA$, $\langle [a_0]_{\equiv}, \ldots, [a_{n-1}]_{\equiv}\rangle \in R^{\overline{B}}$ iff $\langle a_0, \ldots, a_{m-1}\rangle \in R^A$. Moreover, the function assigning to each $a \in A$ its equivalence class $[a]_{\equiv}$ is a homomorphism from A onto $\overline{B}$.*

Proof. The definition of congruence relation assures that the definition of $F^{\overline{B}}$ is unambiguous, for any function symbol F. Now for an n-ary relation symbol R, let

$$R^{\overline{B}} = \{\langle [a_0]_{\equiv}, \ldots, [a_{n-1}]_{\equiv}\rangle : \langle a_0, \ldots, a_{m-1}\rangle \in R^{\overline{A}}\}.$$

Thus for any $a \in {}^nA$, obviously $a \in R^{\overline{A}}$ implies that $\langle [a_0]_{\equiv}, \ldots, [a_{n-1}]_{\equiv}\rangle \in R^{\overline{B}}$. Conversely, suppose that $\langle [a_0]_{\equiv}, \ldots, [a_{n-1}]_{\equiv}\rangle \in R^{\overline{B}}$. Then there is a $b \in R^{\overline{A}}$ such that $[a_i]_{\equiv} = [b_i]_{\equiv}$ for all $i < n$. So $a_i \equiv b_i$ for all $i < n$, and hence $\langle a_0, \ldots, a_{n-1}\rangle \in R^{\overline{A}}$.

Now the function described at the end of the statement of the proposition is clearly a homomorphism. □

The structure in Proposition 1.7 is denoted by $\overline{A}/{\equiv}$. This is the *quotient structure* of A under $\equiv$.

If f is a homomorphism from $\overline{A}$ to $\overline{B}$, the *kernel* of f is $\{(a_0, a_1) : a_0, a_1 \in A$ and $f(a_0) = f(a_1)\}$; we denote it by $\ker(f)$.

Proposition 1.8. *If f is a homomorphism from $\overline{A}$ to $\overline{B}$, then $\ker(f)$ is a congruence relation on $\overline{A}$, and $\overline{A}/\ker(f)$ can be isomorphically embedded in $\overline{B}$.*

Proof. Clearly $\ker(f)$ is an equivalence relation on A. Suppose that F is an m-ary function symbol and $\overline{a}$ and $\overline{c}$ are sequences of elements of A of length m, with $(a_i, c_i) \in \ker(f)$ for each $i < m$. Thus $f(a_i) = f(c_i)$ for each $i < m$, and so

$$\begin{aligned} f(F^{\overline{A}}(a_0, \ldots, a_{m-1})) &= F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1})) \\ &= F^{\overline{B}}(f(c_0), \ldots, f(c_{m-1})) \\ &= f(F^{\overline{A}}(c_0, \ldots, c_{m-1})); \end{aligned}$$

thus $(F^{\overline{A}}(a_0, \ldots, a_{m-1}), F^{\overline{A}}(c_0, \ldots, c_{m-1}) \in \ker(f)$. Now suppose that R is an m-ary relation symbol and $(a_i, b_i) \in \ker(f)$ for all $i < m$. Thus $f(a_i) = f(b_i)$ for all $i < m$. Suppose that $\langle a_0, \ldots, a_{m-1}\rangle \in R^{\overline{A}}$. Then $\langle f(a_0), \ldots, f(a_{m-1})\rangle \in R^{\overline{B}}$, hence $\langle f(b_0), \ldots, f(b_{m-1})\rangle \in R^{\overline{B}}$. Hence $\langle b_0, \ldots, b_{m-1}\rangle \in R^{\overline{A}}$. The converse holds by the same argument. So $\ker(f)$ is a congruence relation on $\overline{A}$.

Now let $g = \{([a]_{\ker(f)}, f(a)) : a \in A\}$. We claim that g is an isomorphic embedding from $A/\ker(f)$ into $\overline{B}$. First of all, g is a function and is one-one, since for any $a, c \in A$,

$$[a]_{\ker(f)} = [c]_{\ker(f)} \quad \text{iff} \quad f(a) = f(c).$$

g is a homomorphism, since for an individual constant k we have

$$g(k^{\overline{A}/\ker(f)}) = g([k^{\overline{A}}]_{\ker(f)}) = f(k^{\overline{A}}) = k^{\overline{B}},$$

and for F an m-ary function symbol,

$$\begin{aligned} g(F^{\overline{A}/\ker(f)}([a_0]_{\ker(f)}, \ldots, [a_{m-1}]_{\ker(f)}) &= g([F^{\overline{A}}(a_0, \ldots, a_{m-1})]_{\ker(f)}) \\ &= f(F^{\overline{A}}(a_0, \ldots, a_{m-1})) \\ &= F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1})) \\ &= F^{\overline{B}}(g([a_0]_{\ker(f)}), \ldots, g([a_{m-1}]_{\ker(f)})), \end{aligned}$$

and for R an m-ary relation symbol,

$$\begin{aligned} \langle [a_0]_{\ker(f)}, \ldots, [a_{m-1}]_{\ker(f)}\rangle \in R^{\overline{A}/\ker(f)} \quad &\text{iff} \quad \langle a_0, \ldots, a_{m-1}\rangle \in R^{\overline{A}} \\ &\text{iff} \quad \langle f(a_0), \ldots, f(a_{m-1})\rangle \in R^{\overline{B}} \\ &\text{iff} \quad \langle g([a_0]_{\ker(f)}), \ldots, g([a_{m-1}]_{\ker(f)})\rangle \in R^{\overline{B}}. \square \end{aligned}$$

PRODUCTS

Suppose that $\langle \overline{A}_i : i \in I\rangle$ is a system of similar structures. We make the set-theoretic product $B \overset{\text{def}}{=} \prod_{i\in I} A_i$ into a structure $\prod_{i\in I} \overline{A}_i$ similar to the $\overline{A}_i$'s as follows. For each $i \in I$ let pr_i be the projection of B into A_i, defined by $\mathrm{pr}_i(f) = f_i$ for all $f \in B$. For R an m-ary relation symbol,

$$R^{\overline{B}} = \{f \in {}^mB : \forall i \in I[\mathrm{pr}_i \circ f \in R^{A_i}]\}.$$

For F an m-ary fundamental operation and for any $a \in {}^mB$,

$$F^{\overline{B}}(a) = \langle F^{A_i}(\mathrm{pr}_i \circ a) : i \in I\rangle.$$

for k an individual constant, $k^{\overline{B}} = \langle k^{A_i} : i \in I\rangle$.

This definition works out as follows for our examples.

If $\langle (A_i, <_i) : i \in I \rangle$ is a system of partial orders, then $\prod_{i \in I} \overline{A}_i$ has the relation $\prec$, where $f \prec g$ iff $f_i <_i g_i$ for all $i \in I$.

If $\langle (A_i, \cdot, ^{-1}, e_i) : i \in I \rangle$ is a system of groups (where we leave off some of the necessary subscripts $_i$), then the operations in $\prod_{i \in I}$ are defined as follows:

$$\begin{aligned} f \cdot g &= \langle f_i \cdot g_i : i \in I \rangle; \\ f^{-1} &= \langle f_i^{-1} : i \in I \rangle; \\ e &= \langle e_i : i \in I \rangle. \end{aligned}$$

If $\langle (A_i, +, \cdot, -, 0_i) : i \in I \rangle$ is a system of rings, then the operations in $\prod_{i \in I} A_i$ are defined as follows:

$$\begin{aligned} f + g &= \langle f_i + g_i : i \in I \rangle; \\ f \cdot g &= \langle f_i \cdot g_i : i \in I \rangle; \\ -f &= \langle -f_i : i \in I \rangle; \\ 0 &= \langle 0_i : i \in I \rangle. \end{aligned}$$

Ordered fields and vector spaces are treated similarly.

Proposition 1.9. *Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures. Let $\overline{B} = \prod_{i \in I} \overline{A}_i$, and let $i \in I$. Then pr_i is a weak homomorphism from $\overline{B}$ onto $\overline{A}_i$.*

Proof. Clearly pr_i maps onto $\overline{A}_i$. If k is an individual constant, then $\mathrm{pr}_i(k^{\overline{B}}) = k^{\overline{A}_i}$. If F is an m-ary function symbol and $f \in {}^mB$. Then

$$\mathrm{pr}_i(F^{\overline{B}}(f)) = (F^{\overline{B}}(f))_i = F^{\overline{A}_i}(\mathrm{pr}_i \circ f).$$

If R is an m-ary relation symbol and $a \in R^{\overline{B}}$, then $\forall j \in I[\mathrm{pr}_j(a) \in R^{\overline{A}_j}]$, and hence $\mathrm{pr}_i(a) \in R^{\overline{A}_i}$. □

An important special case of products is the product of two structures. We define $\overline{A} \times \overline{B}$ to be $\prod_{i \in 2} \overline{C}_i$, where $\overline{C}_0 = \overline{A}$ and $\overline{C}_1 = \overline{B}$.

UNIONS

We can form unions of structures under special circumstances, given by the following theorem. A collection K of subuniverses of a structure A is *directed by* $\subseteq$ iff $\forall A, B \in K \exists C \in K[A \subseteq C$ and $B \subseteq C]$.

Theorem 1.10. *Suppose that $\overline{A}$ is a structure, and K is a nonempty collection of subuniverses of $\overline{A}$ directed by $\subseteq$. Then $\bigcup K$ is a subuniverse of A, such that B is a subuniverse of the corresponding structure $\bigcup K$ for all $B \in K$.*

Proof. Take any n-ary function symbol F and any $x \in {}^n\bigcup K$. For each $j < n$ choose $B_j \in K$ such that $x_j \in B_j$. The condition of the theorem then implies that there is a $C \in K$ such that $B_j \subseteq C$ for all $j < n_i$. Hence $F^{\overline{A}}(x) \in C \subseteq \bigcup K$. □

A subuniverse B of a structure $\overline{A}$ is *finitely generated* iff there is a finite nonempty set F such that $B = \langle F \rangle_A$.

Theorem 1.11. *Any structure is the union of its finitely generated substructures.*

Proof. Let K be the set of all finitely generated substructures of A. The condition of Theorem 1.10 clearly holds, and so $\bigcup K$ is a subuniverse of $\overline{A}$. But clearly $A = \bigcup K$, since for any $a \in A$ we have $\langle \{a\} \rangle_A \in K$. □

ULTRAPRODUCTS

Ultraproducts play an important role in model theory. Here we are not dealing with a generalization of a common notion; ultraproducts are a new thing, although related to products and quotients.

First we need to discuss the purely set-theoretic notion of ultrafilters. Let I be a nonempty set. An *ultrafilter* on I is a collection D of subsets of I satisfying the following conditions:

(1) $I \in D$,

(2) $\emptyset \notin D$.

(3) For all $a, b \in D$, also $a \cap b \in D$.

(4) For all $a \in D$ and all $b \subseteq I$ with $a \subseteq b$, also $b \in D$.

(5) For any $a \subseteq I$, either $a \in D$ or $(I \backslash a) \in D$.

Note that in (5) the "or" is exclusive. Indeed, if $a \in D$ and $(I \backslash a) \in D$, then $\emptyset \in D$ by (3), contradicting (2).

If D satisfies (1)–(4), it is called a *filter* on I. A filter is *proper* iff it is different from $\mathscr{P}(I)$. Note that D is proper iff $\emptyset \notin D$, by (4).

Proposition 1.12. *Let D be a proper filter on a nonempty set I. Then D is an ultrafilter iff it is maximal among proper filters.*

Proof. First suppose that D is an ultrafilter, and $D \subset F$ with F a filter. Say $a \in F \backslash D$. Then $(I \backslash a) \in D$ by (5), and so $\emptyset = a \cap (I \backslash a) \in F$ by (3), so that F is not proper.

Second suppose that D is a filter which is maximal among proper filters. We verify condition (5). Suppose that $a \subseteq I$ and $a \notin D$. Let

$$F = \{x \subseteq I : y \backslash a \subseteq x \text{ for some } y \in D\}.$$

Clearly F is a filter. It is proper, since $\emptyset \in F$ would imply that $y \backslash a = \emptyset$ for some $y \in D$, hence $y \subseteq a$, hence $a \in D$ by (4), contradiction. It follows that $D = F$, and hence $(I \backslash a) \in D$. □

A family $\mathscr{A}$ of subset of a set I has the *finite intersection property* (fip) iff every finite subset of $\mathscr{A}$ has nonempty intersection. A connection of this notion with ultrafilters is as follows.

Theorem 1.13. *Let D be a collection of subsets of a set I. Then D is an ultrafilter on I iff D is maximal with respect to the property of having fip.*

Proof. $\Rightarrow$ is clear. Now assume that D is maximal with respect to the property of having fip. $D \cup \{I\}$ clearly still has fip, so it is equal to D, so that $I \in D$. Obviously $\emptyset \notin D$. Suppose that $a, b \in D$. Then $D \cup \{a \cap b\}$ still has fip, and it follows that $a \cap b \in D$. Similarly, $a \in D$ and $a \subseteq b \subseteq I$ imply that $b \in I$. Finally, suppose that $a \subseteq I$ and $a \notin D$. Then $D \cup \{a\}$ no longer has fip, so there is a finite subset F of D such that $a \cap \bigcap F = \emptyset$. Hence $\bigcap F \subseteq (I \backslash a)$, and so $(I \backslash a) \in D$. $\square$

A basic theorem concerning ultrafilters is as follows.

Theorem 1.14. *Suppose that I is a nonempty set, and $\mathscr{A}$ is a family of subsets of I with the fip. Then there is an ultrafilter D on I such that $\mathscr{A} \subseteq D$.*

Proof. Let $\mathscr{B}$ be the collection of all families $\mathscr{C}$ of subsets of I such that $\mathscr{A} \subseteq \mathscr{C}$ and $\mathscr{C}$ has fip. We partially order $\mathscr{B}$ by inclusion. Then the hypothesis of Zorn's lemma holds for $\mathscr{B}$. In fact, suppose that $\mathscr{D}$ is a collection of members of $\mathscr{B}$ linearly ordered by inclusion. Let $\mathscr{E} = \bigcup \mathscr{D}$. We claim that $\mathscr{E}$ has fip. For, suppose that F is a finite subset of $\mathscr{E}$. For each $X \in F$ choose $\mathscr{F}_X \in \mathscr{E}$ such that $X \in \mathscr{F}_X$. Since $\mathscr{E}$ is linearly ordered by inclusion, there is a $Z \in F$ such that $\mathscr{F}_X \subseteq \mathscr{F}_Z$ for all $X \in F$. Then, since $\mathscr{F}_X$ has fip, we get $\bigcap F \neq \emptyset$. This verifies that the hypothesis of Zorn's lemma holds. Hence we can let D be a maximal member of $\mathscr{B}$. So, D has fip, and is maximal with this property. By Theorem 1.13, D is as desired. $\square$

This ends our set-theoretic interlude concerning ultrafilters.

Theorem 1.15. *Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures and D is a ultrafilter on I. For brevity let $B = \prod_{i \in I} A_i$. Define $\equiv_D$, a relation on B, by*

$$f \equiv_D g \quad \textit{iff} \quad \{i \in I : f_i = g_i\} \in D.$$

Then $\equiv_D$ is an equivalence relation on B, and it satisfies the conditions for a congruence relation on B as far as individual constants and function symbols are concerned. Moreover, if R is an n-ary relation symbol, $a, b \in {}^nB$ and $a_j \equiv_D b_j$ for all $j < n$, then

$$\{i \in I : \mathrm{pr}_i \circ a \in R^{A_i}\} \in D \quad \text{iff} \quad \{i \in I : \mathrm{pr}_i \circ b \in R^{A_i}\} \in D.$$

Proof. Clearly $\equiv_D$ is reflexive and symmetric. For transitivity, suppose that $a \equiv_D b \equiv_D c$. Now

$$\{i \in I : a_i = b_i\} \cap \{i \in I : b_i = c_i\} \subseteq \{i \in I : a_i = c_i\},$$

and the left side is in D; hence also the right side is in D, and it follows that $a \equiv_D c$. So $\equiv_D$ is an equivalence relation on B.

Suppose that F is an m-ary function symbol and $f_j, f'_j \in B$ for $j < m$, with $f_j \equiv_D f'_j$ for all $j < m$. Let $g = F^{\overline{B}}(f_0, \ldots, f_{m-1})$ and $g' = F^{\overline{B}}(f'_0, \ldots, f'_{m-1})$. Then

$$\bigcap_{j<m} \{i \in I : f_j(i) = f'_j(i)\} \subseteq \{i \in I : g(i) = g'(i)\},$$

and the first set is in D, so that the second set is also, proving that $g \equiv_D g'$.

Now assume that R is an n-ary relation symbol, $a, b \in {}^nB$ and $a_j \equiv_D b_j$ for all $i < n$. Then

$$\begin{aligned}&\bigcap_{j<n} \{i \in I : a_j(i) = b_j(i)\} \cap \{i \in I : \mathrm{pr}_i \circ a \in R^{A_i}\}\\ &\quad= \bigcap_{j<n} \{i \in I : a_j(i) = b_j(i)\} \cap \{i \in I : \mathrm{pr}_i \circ b \in R^{A_i}\},\end{aligned}$$

and the desired conclusion follows. □

This theorem justifies the following definition. We make the collection $C \stackrel{\text{def}}{=} \prod_{i\in I} A_i/{\equiv_D}$ of equivalence classes under $\equiv_D$ into a structure similar to the A_i's as follows. Let $B = \prod_{i\in I} A_i$.

$$\begin{aligned}F^{\overline{C}}([a_0], \ldots, [a_{m-1}]) &= [F^{\overline{B}}(a_0, \ldots, a_{m-1})];\\ ([a'_0], \ldots, [a'_{n-1}]) \in R^{\overline{C}} \quad &\text{iff} \quad \{i \in I : (a'_0(i), \ldots, a'_{n-1}) \in R^{A_i}\} \in D;\\ k^{\overline{C}} &= [k^{\overline{B}}],\end{aligned}$$

where F is an m-ary function symbol, R is an n-ary relation symbol, k is a n individual constant, and each a_i and a'_i is in B. Note that $[f]$ denotes the equivalence class of f under $\equiv_F$.

The structure with universe C so defined is called the *ultraproduct* of $\langle \overline{A}_i : i \in I\rangle$ and is denoted by $\prod_{i\in I} \overline{A}_i/D$.

Theorem 1.16. *Suppose that K is a nonempty collection of subuniverses of a structure A, directed by $\subseteq$. Then there is an ultrafilter D on K such that $\bigcup K$ can be isomorphically embedded in $\prod_{B\in K} B/{\equiv_D}$.*

Proof. For brevity let $M = \prod_{B\in K} B$; $\overline{M}$ is the corresponding structure. For each $B \in K$ let $L_B = \{C \in K : B \subseteq C\}$.

(1) $\{L_B : B \in K\}$ has fip.

For, let $X \in [K]^{<\omega}$. Choose $C \in K$ such that $B \subseteq C$ for all $B \in X$. Then $C \in \bigcap_{B\in X} L_B$.

By (1), let D be an ultrafilter on K containing $\{L_B : B \in K\}$. Let $\overline{N} = \overline{M}/{\equiv_D}$. Fix $a_B \in B$ for every $B \in K$. We define $f : \bigcup K \to M$ by setting, for any $b \in \bigcup K$ and $B \in K$,

$$(f(b))_B = \begin{cases} b & \text{if } b \in B,\\ a_B & \text{otherwise.}\end{cases}$$

Then let $g(b) = [f(b)]$ for every $b \in \bigcup K$. So $g : \bigcup K \to N$.

Now to show that g is a homomorphism, first suppose that $\mathbf{F}$ is an m-ary function symbol and $b_0, \ldots, b_{m-1} \in \bigcup K$.

(2) $[f(\mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}))] = [\mathbf{F}^{\overline{M}}(f(b_0), \ldots, f(b_{m-1}))]$. ‘

In fact, choose $B \in K$ such that $b_0, \ldots, b_{m-1} \in B$. Then for any $C \in L_B$ we have $\mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}) \in C$, and

$$\begin{aligned}(f((\mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}))_C &= \mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}) \\ &= \mathbf{F}^{\overline{A}}((f(b_0))_C, \ldots, (f(b_{m-1}))_C) \\ &= (\mathbf{F}^{\overline{M}}(f(b_0), \ldots, f(b_{m-1})))_C,\end{aligned}$$

and (2) follows.

Now we calculate:

$$\begin{aligned}\mathbf{F}^{\overline{N}}(g(b_0), \ldots, g(b_{m-1})) &= \mathbf{F}^{\overline{N}}([f(b_0)], \ldots, [f(b_{m-1})]) \\ &= [\mathbf{F}^{\overline{M}}(f(b_0), \ldots, f(b_{m-1}))] \\ &= [f(\mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1}))] \quad \text{by (2)} \\ &= g(\mathbf{F}^{\overline{A}}(b_0, \ldots, b_{m-1})) \\ &= g(\mathbf{F}^{\overline{\bigcup K}}(b_0, \ldots, b_{m-1})).\end{aligned}$$

Next, let $\mathbf{R}$ be an n-ary relation symbol and $b_0, \ldots, b_{n-1} \in \bigcup K$. Choose $B \in K$ such that $b_0, \ldots, b_{n-1} \in B$. Then

$$\begin{aligned}\langle b_0, \ldots, b_{n-1}\rangle \in \mathbf{R}^{\overline{\bigcup K}} \quad &\text{iff} \quad \langle b_0, \ldots, b_{n-1}\rangle \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad \langle (f(b_0))_B, \ldots, (f(b_{n-1}))_B\rangle \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad L_B \cap \{C \in K : \langle (f(b_0))_C, \ldots, (f(b_{n-1}))_C\rangle \in \mathbf{R}^{\overline{C}}\} \in D \\ &\text{iff} \quad \{C \in K : \langle (f(b_0))_C, \ldots, (f(b_{n-1}))_C\rangle \in \mathbf{R}^{\overline{C}}\} \in D \\ &\text{iff} \quad \langle [f(b_0)], \ldots, [f(b_{n-1})]\rangle \in \mathbf{R}^N \\ &\text{iff} \quad \langle g(b_0), \ldots, g(b_{n-1})\rangle \in \mathbf{R}^N.\end{aligned}$$

For $\mathbf{k}$ an individual constant,

$$g(\mathbf{k}^{\overline{\bigcup K}}) = [f(\mathbf{k}^{\overline{A}})] = [\mathbf{k}^{\overline{M}}] = \mathbf{k}^{\overline{N}}.$$

It remains only to show that g is one-one. Suppose that $b, c \in \bigcup K$ with $b \neq c$. Choose $B \in K$ such that $b, c \in B$. Then for any $C \in L_B$ we have $(f(b))_C = b \neq c = (f(c))_C$. So $L_B \subseteq \{C \in K : (f(b))_C \neq (f(c))_C\}$, so that $\{C \in K : (f(b))_C \neq (f(c))_C\} \in D$. Hence $f(b) \not\equiv_D f(c)$, and so $g(b) = [f(b)] \neq [f(c)] = g(c)$. □

Corollary 1.17. *Any structure can be isomorphically embedded in an ultraproduct of its finitely generated substructures.* □

Exc. 1.1. Let $\mathscr{L}$ be a language with no individual constants. Define an $\mathscr{L}$-structure $\overline{A}$ and subuniverses B, C of $\overline{A}$ such that $B \cap C = \emptyset$.

Exc. 1.2. Carry out the "easy induction" at the beginning of the proof of Proposition 1.2.

Exc. 1.3. If X and Y are nonempty subsets of the universe A of an algebra $\overline{A}$, then $\langle X \cup \langle Y \rangle \rangle = \langle X \cup Y \rangle$.

Exc. 1.4. If K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$, then $\langle \bigcup K \rangle = \langle \bigcup_{X \in K} \langle X \rangle \rangle$.

Exc. 1.5. Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{A}$. Show that $f[C]$ is a subuniverse of $\overline{B}$.

Exc. 1.6. Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{B}$. Show that $f^{-1}[C]$ is a subuniverse of $\overline{A}$.

Exc. 1.7. If X generates $\overline{A}$, f and g are homomorphisms from $\overline{A}$ into $\overline{B}$, and $f \restriction X = g \restriction X$, then $f = g$.

Exc. 1.8. If f is a homomorphism from $\overline{A}$ into $\overline{B}$ and X is a nonempty subset of A, then $f[\langle X \rangle] = \langle f[X] \rangle$.

Exc. 1.9. If $\overline{A}$ is a substructure of $\overline{B}$ and $\equiv$ is a congruence relation on $\overline{B}$, then $\equiv \cap (A \times A)$ is a congruence relation on $\overline{A}$.

Exc. 1.10. Suppose that R is a congruence relation on $\overline{A}$, and S is a congruence relation on $\overline{A}/R$. Define $T = \{(a_0, a_1) \in A \times A : ([a_0]_R, [a_1]_R) \in S\}$. Show that T is a congruence relation on A and $R \subseteq T$.

Exc. 1.11. (Continuing exercise 1.10) Show that the procedure of exercise 1.10 establishes a one-one order-preserving correspondence between congruence relations on A/R and those congruence relations on A with include R.

Exc. 1.12. Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures, and $\overline{B}$ is another structure similar to them. Suppose that f_i is a homomorphism from $\overline{B}$ into $\overline{A}_i$ for each $i \in I$. Show that there is a homomorphism g from $\overline{B}$ into $\prod_{i \in I} \overline{A}_i$ such that $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$.

Exc. 1.13. Show that a product of partial orderings is a partial ordering.

Exc. 1.14. A partial ordering $(A, <)$ is a *linear ordering* iff for any two distinct $x, y \in A$ we have $x < y$ or $y < x$. Give an example of two linear orderings whose product is not a linear ordering.

Exc. 1.15. Give an example of two ordered fields whose product is not even a field.

Exc. 1.16. Let F be a proper filter on a set I. Show that F is an ultrafilter iff for all $a, b \subseteq I$, if $a \cup b \in F$ then $a \in F$ or $b \in F$.

Exc. 1.17. Show that any ultraproduct of linear orderings is a linear ordering.

Exc. 1.18. Suppose that I is a nonempty set, and $\langle J_i : i \in I\rangle$ is a system of nonempty sets. Also suppose that F_i is an ultrafilter on J_i for each $i \in I$, and G is an ultrafilter on I. Let $K = \{(i, j) : i \in I, j \in J_i\}$, and define

$$H = \{X \subseteq K : \{i \in I : \{j \in J_i : (i, j) \in X\} \in F_i\} \in G\}.$$

Show that H is an ultrafilter on K.

Exc. 1.19. Under the notation of exercise 1.18, show that there is an isomorphism f of the structure $\prod_{i\in I}(\prod_{j\in J_i} \overline{A_{ij}}/F_i)/G)$ onto $\prod_{(i,j)\in K} \overline{A_{ij}}/H$ such that:

$$\forall r \in \prod_{i\in I}\left(\prod_{j\in J_i} \overline{A_{ij}}/F_i)/G\right)\left[\left[r = [s]_G \text{ with } s \in \prod_{i\in I}\left(\prod_{j\in J_i} \overline{A_{ij}}/F_i\right)\right]\right.$$
$$\left.\text{and } \forall i \in I\left[s_i = [t_i]_{F_i} \text{ with } t_i \in \prod_{j\in J_i} \overline{A_{ij}}\right] \text{ implies that } f(r) = [\langle t_i(j) : (i, j) \in K\rangle]_H\right].$$