9. Interpolation

Lemma 9.1. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are languages and $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\overline{B}$ is an $\mathscr{L}_1$-structure and $\overline{C}$ is an $\mathscr{L}_2$-structure. Suppose that $\overline{a}$ is a sequence of elements of B, $\overline{c}$ is a sequence of elements of C, and $(\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \equiv (\overline{C} \upharpoonright \mathscr{L}, \overline{c})$.*

Then there exist an elementary extension $\overline{D}$ of $\overline{B}$ and an elementary embedding g of $\overline{C} \upharpoonright \mathscr{L}$ into $\overline{D} \upharpoonright \mathscr{L}$ such that $g \circ \overline{c} = \overline{a}$.

Note that the sequences $\overline{a}$ and $\overline{c}$ can both be empty, or both infinite; but they are of the same length.

Proof. This proof is patterned after that of Proposition 8.6. Our first goal is to obtain an isomorphic copy $\overline{C}'$ of $\overline{C}$ so that $B \cap C' = \mathrm{rng}(\overline{a})$. Let Q be a set such that $Q \cap B = \emptyset$ and $|Q| = |C \backslash \mathrm{rng}(\overline{c})|$. Let f be a bijection from $C \backslash \mathrm{rng}(\overline{c})$ onto Q. Define $C' = \mathrm{rng}(\overline{a}) \cup Q$. Note that $B \cap C' = \mathrm{rng}(\overline{a})$. Define $f' : C \to C'$ by setting, for any $d \in C$,

$$f'(d) = \begin{cases} a_i & \text{if } d = c_i, \\ f(d) & \text{if } d \in C \backslash \mathrm{rng}(\overline{c}). \end{cases}$$

We now define a structure on C'. If R is an m-ary relation symbol of $\mathscr{L}_2$, let

$$R^{\overline{C}'} = \{d \in {}^m C' : (f')^{-1} \circ c \in R^{\overline{C}}\},$$

while if F is an m-ary function symbol of $\mathscr{L}_2$ and $d \in {}^m C'$ define

$$F^{\overline{C}'}(d) = f'(F^{\overline{C}}((f')^{-1} \circ d)).$$

Clearly f' is an isomorphism from $\overline{C}$ onto $\overline{C}'$. Moreover, $(\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \equiv (\overline{C}' \upharpoonright \mathscr{L}, \overline{a})$. In fact, for any sentence $\varphi(\overline{a})$ of $\mathscr{L}$ we have

$$\begin{aligned} (\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \models \varphi(\overline{a}) \quad & \text{iff} \quad (\overline{C} \upharpoonright \mathscr{L}, \overline{c}) \models \varphi(\overline{c}) \\ & \text{iff} \quad (\overline{C}' \upharpoonright \mathscr{L}, \overline{a}) \models \varphi(\overline{a}). \end{aligned}$$

Now we claim that $\mathrm{Eldiag}(\overline{B}) \cup \mathrm{Eldiag}(\overline{C'} \upharpoonright \mathscr{L})$ has a model. Here the same constants are used in $\mathrm{Eldiag}(\overline{B})$ and $\mathrm{Eldiag}(\overline{C'} \upharpoonright \mathscr{L})$ for the members of $\mathrm{rng}(\overline{a})$. If not, by the compactness theorem some finite subset fails to have a model. Say Δ_0 is a finite subset of $\mathrm{Eldiag}(\overline{B})$ and Δ_1 is a finite subset of $\mathrm{Eldiag}(\overline{C}' \upharpoonright \mathscr{L})$ such that $\Delta_0 \cup \Delta_1$ does not have a model. Then $\bigwedge \Delta_1$ has the form $\psi(c_{a(i_0)}, \ldots c_{a(i_{n-1})}, c_{d(0)}, \ldots, c_{d(m-1)})$ with each $d(i)$ in $C' \backslash \mathrm{rng}(a)$. Thus

$$\Delta_0 \models \neg\psi(c_{a(i_0)}, \ldots c_{a(i_{n-1})}, c_{d(0)}, \ldots, c_{d(m-1)}).$$

Now the constants $c_{d(i)}$ do not occur in the formulas of Δ_0. Hence, replacing each $c_{b(i)}$ by a new variable w_i we get

$$\Delta_0 \models \forall \overline{w} \neg\psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

Since $\overline{B}_B$ is a model of Δ_0, we get

$$\overline{B}_B \models \forall \overline{w} \neg \psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}),$$

hence

$$\overline{B}_{\mathrm{rng}(a)} \models \forall \overline{w} \neg \psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

Hence by the above,

$$\overline{C}'_{\mathrm{rng}(a)} \models \forall \overline{w} \neg \psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

But this is impossible.

Hence $\mathrm{Eldiag}(\overline{B}) \cup \mathrm{Eldiag}(\overline{C'} \restriction \mathscr{L})$ has a model, say $(\overline{D}, h(b), k(c))_{b \in B, c \in C'}$, where $h(a_i) = k(a_i)$ for all i. By the elementary diagram lemma, h is an elementary embedding of $\overline{B}$ into $\overline{D}$ and k is an elementary embedding of $\overline{C}' \restriction \mathscr{L}$ into $\overline{D} \restriction \mathscr{L}$. Now let $\overline{D}'$ be an elementary extension of $\overline{B}$ and l an isomorphism of $\overline{D}$ with $\overline{D}'$ such that $l \circ h$ is the identity on B. Now $l \circ k \circ f'$ is an elementary embedding of $\overline{C} \restriction \mathscr{L}$ into $\overline{D}' \restriction \mathscr{L}$, and

$$l \circ k \circ f' \circ \overline{c} = l \circ k \circ \overline{a} = l \circ h \circ \overline{a} = \overline{a}. \qquad \square$$

Theorem 9.2. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are languages and $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\overline{B}$ is an $\mathscr{L}_1$-structure and $\overline{C}$ is an $\mathscr{L}_2$-structure. Suppose that $\overline{a}$ is a sequence of elements of $B \cap C$, and $(\overline{B} \restriction \mathscr{L}, \overline{a}) \equiv (\overline{C} \restriction \mathscr{L}, \overline{a})$.*

Then there exist an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure $\overline{D}$ and a function g such that $\overline{B} \preceq (\overline{D} \restriction \mathscr{L}_1)$ and g is an elementary embedding of $\overline{C}$ into $\overline{D} \restriction \mathscr{L}_2$ such that $g \circ \overline{a} = \overline{a}$.

Proof. Define $B_0 = B$ and $C_0 = C$. We apply Lemma 9.1 to get $B_0 \preceq B_1$ and an elementary embedding $g_0 : C \restriction \mathscr{L} \to B_1 \restriction \mathscr{L}$ such that $g_0 \circ \overline{a} = \overline{a}$. Let $\overline{c}$ enumerate C_0. Then we have $(C_0 \restriction \mathscr{L}, \overline{c}) \equiv (B_1 \restriction \mathscr{L}, g_0 \circ \overline{c})$, so we apply Lemma 9.1 to get $C_0 \preceq C_1$ and $h_1 : B_1 \restriction \mathscr{L} \to C_1 \restriction \mathscr{L}$ such that $h_1 \circ g_0 \circ \overline{c} = \overline{c}$. This means that $h_1 \circ g$ is the identity on C_0. Thus so far we have the following diagram:

$$\begin{tikzcd}
& B_0 \arrow[rr, "\preceq"] & & B_1 \arrow[dr, "h_1"] & \\
\overline{a} \arrow[ur, "\subseteq"] \arrow[rr, "\subseteq"'] & & C_0 \arrow[ur, "g_0"] \arrow[rr, "\preceq"'] & & C_1
\end{tikzcd}$$

Now suppose that C_i, B_{i+1}, C_{i+1}, g_i, and h_{i+1} have been defined so that B_{i+1} is an $\mathscr{L}_1$-structure, $C_i \preceq C_{i+1}$ are $\mathscr{L}_2$-structures, $g_i : C_i \restriction \mathscr{L} \to B_{i+1} \restriction \mathscr{L}$ is an elementary embedding, $h_{i+1} : B_{i+1} \restriction \mathscr{L} \to C_{i+1} \restriction \mathscr{L}$ is an elementary embedding, and $h_{i+1} \circ g_i$ is the identity on C_i. Let $\overline{b}$ enumerate B_{i+1}. Then $(B_{i+1} \restriction \mathscr{L}, \overline{b}) \equiv (C_{i+1} \restriction \mathscr{L}, h_{i+1} \circ \overline{b})$. So we can apply Lemma 9.1 and get $B_{i+1} \preceq B_{i+2}$ and an elementary embedding g_{i+1} of $C_{i+1} \restriction \mathscr{L}$ into $B_{i+2} \restriction \mathscr{L}$ such that $g_{i+1} \circ h_{i+1} \circ \overline{b} = \overline{b}$. Thus $g_{i+1} \circ h_{i+1}$ is the identity on B_{i+1}. Then let $\overline{d}$ enumerate C_{i+1}. So we have $(C_{i+1} \restriction \mathscr{L}, \overline{d}) \equiv (B_{i+2} \restriction \mathscr{L}, g_{i+1} \circ \overline{d})$, so by Lemma 9.1 we get $C_{i+1} \preceq C_{i+2}$ and an elementary embedding h_{i+2} of $B_{i+2} \restriction \mathscr{L}$ into $C_{i+2} \restriction \mathscr{L}$ such that $h_{i+2} \circ g_{i+1}$ is the identity on C_{i+1}.

This completes the inductive definition; we have

$$\begin{tikzcd}
 & B_0 \arrow[rr, "\preceq"] & & B_1 \arrow[rr, "\preceq"] \arrow[dr, "h_1"] & & B_2 \arrow[rr, "\preceq"] \arrow[dr, "h_2"] & & B_3 & \cdots \\
\overline{a} \arrow[ur, "\subseteq"] \arrow[rr, "\subseteq"'] & & C_0 \arrow[ur, "g_0"] \arrow[rr, "\preceq"'] & & C_1 \arrow[ur, "g_1"] \arrow[rr, "\preceq"'] & & C_2 \arrow[ur, "g_2"] & \cdots
\end{tikzcd}$$

We claim that $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots$. For, if $c \in C_i$, then $g_{i+1}(c) = g_{i+1}(h_{i+1}(g_i(c))) = g_i(c)$. Also, $h_1 \subseteq h_2 \subseteq h_3 \subseteq \cdots$. For, if $b \in B_i$, then $h_{i+1}(b) = h_{i+1}(g_i(h_i(b))) = h_i(b)$.

Let $B_\omega = \bigcup_{i<\omega} B_i$ (an $\mathscr{L}_1$-structure) and $C_\omega = \bigcup_{i<\omega} C_i$ (an $\mathscr{L}_2$-structure). Let $k = \bigcup_{i<\omega} g_i$. Then k is an embedding of $C_\omega \upharpoonright \mathscr{L}$ into $B_\omega \upharpoonright \mathscr{L}$. Actually k is onto; for, given $b \in B_\omega$, say $b \in B_i$. Then $k(h_i(b)) = g_i(h_i(b) = b$. Thus k is an isomorphism of $C_\omega \upharpoonright \mathscr{L}$ onto $B_\omega \upharpoonright \mathscr{L}$. Now we expand B_ω to an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure D by defining, for any symbol S in $\mathscr{L}_2 \backslash \mathscr{L}_1$ $S^D = k(S^{C_\omega})$ (in the natural sense). Thus $B \preceq B_\omega = D \upharpoonright \mathscr{L}_1$. We claim that $g_0 : C \to D \upharpoonright \mathscr{L}_2$ is an elementary embedding. Since $C \preceq C_\omega$, it suffices to show that k is an isomorphism of C_ω onto $D \upharpoonright \mathscr{L}_2$. This is clear by the definition above. $\square$

If $\mathscr{L}$ and $\mathscr{L}'$ are languages with $\mathscr{L} \subseteq \mathscr{L}'$, and T is a theory in $\mathscr{L}'$, then we denote by $T_{\mathscr{L}}$ the set of all sentences φ of $\mathscr{L}$ such that $T \models \varphi$.

Lemma 9.3. *Let $\mathscr{L}$ and $\mathscr{L}'$ be languages with $\mathscr{L} \subseteq \mathscr{L}'$, and let T be a theory in $\mathscr{L}'$. Let $\overline{M}$ be an $\mathscr{L}$-structure. Then $\overline{M} \models T_{\mathscr{L}}$ iff there is a model $\overline{N}$ of T such that $\overline{M} \preceq \overline{N} \upharpoonright \mathscr{L}$.*

Proof. $\Rightarrow$: Assume that $\overline{M} \models T_{\mathscr{L}}$. It suffices to show that $S \stackrel{\text{def}}{=} T \cup \text{Eldiag}(\overline{M})$ has a model. To apply the compactness theorem, suppose that there is a finite subset of it with no model. This finite subset has the form $\Delta_0 \cup \Delta_1$ with Δ_0 a finite subset of T and Δ_1 a finite subset of Eldiag$(\overline{M})$. This yields $T \models \neg \bigwedge \Delta_1$. Replacing the diagram constants in Δ_1 by variables, we obtain a formula $\varphi(\overline{w})$ such that $T \models \forall \overline{w} \neg \varphi(\overline{w})$, with $\overline{M} \models \exists \overline{w} \varphi(\overline{w})$. Then $\forall \overline{w} \neg \varphi(\overline{w}) \in T_{\mathscr{L}}$, hence $\overline{M} \models \forall \overline{w} \neg \varphi(\overline{w})$, contradiction.

$\Leftarrow$: obvious. $\square$

Theorem 9.4. *Let $\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$ be languages with $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that T_1 and T_2 are theories in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, and $T_1 \cup T_2$ does not have a model. Then there is a sentence φ of $\mathscr{L}$ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$.*

Proof. By the compactness theorem it suffices to show that $(T_1)_{\mathscr{L}} \cup T_2$ does not have a model. Suppose that $\overline{M}$ is a model of $(T_1)_{\mathscr{L}} \cup T_2$. By Lemma 9.3 let $\overline{N}$ be a model of T_1 such that $\overline{M} \upharpoonright \mathscr{L} \preceq \overline{N} \upharpoonright \mathscr{L}$. Then $\overline{M} \upharpoonright \mathscr{L} \equiv \overline{N} \upharpoonright \mathscr{L}$, so by Theorem 9.2 there exist an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure $\overline{D}$ and a function g such that $\overline{N} \preceq \overline{D} \upharpoonright \mathscr{L}_1$ and g is an elementary embedding of $\overline{M}$ into $\overline{D} \upharpoonright \mathscr{L}_2$. Since $\overline{N} \models T_1$ and $\overline{N} \preceq \overline{D} \upharpoonright \mathscr{L}_1$, it follows that $\overline{D} \models T_1$. Since $\overline{M} \models T_2$ and g is an elementary embedding of $\overline{M}$ into $\overline{D} \upharpoonright \mathscr{L}_2$, it follows that $\overline{D} \models T_2$. So $\overline{D}$ is a model of $T_1 \cup T_2$, contradiction. $\square$

Corollary 9.5. (Craig's interpolation theorem) *If φ and ψ are sentences and $\models \varphi \to \psi$, then there is a sentence χ such that $\models \varphi \to \chi$, $\models \chi \to \psi$, and the non-logical symbols that occur in χ occur in both φ and ψ.*

Proof. Assume that φ and ψ are sentences and $\models \varphi \to \psi$. Let $\mathscr{L}_1$ consist of all of the non-logical symbols occurring in φ, and let $\mathscr{L}_2$ consist of all of the non-logical symbols occurring in ψ. Let $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Let $T_1 = \{\theta : \theta \text{ is a sentence of } \mathscr{L}_1 \text{ and } \models \varphi \to \theta\}$ and let $T_2 = \{\theta : \theta \text{ is a sentence of } \mathscr{L}_2 \text{ and } \models \neg\psi \to \theta\}$. Then $T_1 \cup T_2$ does not have a model. Hence by Theorem 9.4 there is a sentence θ of $\mathscr{L}$ such that $T_1 \models \theta$ and $T_2 \models \neg\theta$. Hence $\varphi \to \theta$ and $\theta \to \psi$. □

Corollary 9.6. (Robinson's consistency theorem) *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be languages, and let T_1 and T_2 be theories in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, both of which have models. Let $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\{\varphi : \varphi$ is a sentence of $\mathscr{L}$ and $T_1 \models \varphi$ and $T_2 \models \varphi\}$ is a complete theory in $\mathscr{L}$. Then $T_1 \cup T_2$ has a model.*

Proof. Suppose not. Then by Theorem 9.4 there is a sentence φ of $\mathscr{L}$ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$. This contradicts the completeness of the above theory. □

Proposition 9.7. (Padoa's method) *Let $\mathscr{L}$ be a language and let S be a non-logical symbol of $\mathscr{L}$, and let T be a theory in $\mathscr{L}$. Suppose that $\overline{M}$ and $\overline{N}$ are models of T such that $\overline{M} \upharpoonright (\mathscr{L} - S) = \overline{N} \upharpoonright (\mathscr{L} - S)$ while $S^{\overline{M}} \neq S^{\overline{N}}$.*

Then S is not definable in $\mathscr{L}$ under T. That is, if S is an m-ary relation symbol then there does not exist a formula $\varphi(v_0, \ldots, v_{m-1})$ of $\mathscr{L} - S$ such that $T \models \forall\overline{v}[\varphi \leftrightarrow S\overline{v}]$; and similarly for function symbols and individual constants.

Proof. Assume the hypotheses, but suppose that such a formula φ exists. Then for $\overline{b}$ in M we have

$$\begin{aligned} \overline{M} \models S\overline{b} \quad &\text{iff} \quad \overline{M} \models \varphi[\overline{b}] \\ &\text{iff} \quad \overline{M} \upharpoonright (\mathscr{L} - S) \models \varphi[\overline{b}] \\ &\text{iff} \quad \overline{N} \upharpoonright (\mathscr{L} - S) \models \varphi[\overline{b}] \\ &\text{iff} \quad \overline{N} \models \varphi[\overline{b}]. \end{aligned}$$

Thus $S^{\overline{M}} = S^{\overline{N}}$, contradiction. □

Theorem 9.8. *Let $\mathscr{L}$ and $\mathscr{L}^+$ be languages with $\mathscr{L} \subseteq \mathscr{L}^+$. Let T be a theory in $\mathscr{L}^+$ and $\varphi(\overline{x})$ a formula of $\mathscr{L}^+$. Then the following are equivalent:*

(i) If $\overline{A}$ and $\overline{B}$ are models of T and $\overline{A} \upharpoonright \mathscr{L} = \overline{B} \upharpoonright \mathscr{L}$, then for all tuples $\overline{a}$ in A, $\overline{A} \models \varphi[\overline{a}]$ iff $\overline{B} \models \varphi[\overline{a}]$.

(ii) There is a formula $\psi(\overline{x})$ of $\mathscr{L}$ such that $T \models \forall\overline{x}[\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})]$.

Proof. Clearly (ii)⇒(i). Now assume (i). Let

$$\Phi = T \cup \{\psi(\overline{c}) : \psi(\overline{x}) \text{ is a formula of } \mathscr{L} \text{ and } T \cup \varphi(\overline{c}) \models \psi(\overline{c})\}.$$

Clearly it suffices to show that any model $(\overline{A}, \overline{a})$ of Φ is a model of $\varphi(\overline{c})$.

(*) There is a model $(\overline{B}, \overline{a})$ of $T \cup \{\varphi(\overline{c})\}$ such that $(\overline{A} \upharpoonright \mathscr{L}, \overline{a}) \preceq (\overline{B} \upharpoonright \mathscr{L}, \overline{a})$.

To prove this, it suffices to show that the set

$$T \cup \text{eldiag}(\overline{A} \upharpoonright \mathscr{L}) \cup \{\varphi(\overline{c})\}$$

has a model, where in $\operatorname{eldiag}(\overline{A} \restriction \mathscr{L})$ the tuple $\overline{c}$ corresponds to $\overline{a}$. Suppose not. Then we can write

$$T \vdash \varphi(\overline{c}) \to \forall \overline{y} \neg \psi(\overline{c}, \overline{y}),$$

where $(\overline{A} \restriction \mathscr{L}, \overline{a}) \models \exists \overline{y} \psi(\overline{c}, \overline{y})$ and ψ is an $\mathscr{L}$-formula. But this means that $\forall \overline{y} \neg \psi(\overline{c}, \overline{y})$ is in Φ, contradiction. Thus (*) holds.

Let $\mathscr{L}'$ be obtained from $\mathscr{L}^+$ by replacing each symbol S in $\mathscr{L}^+ \backslash \mathscr{L}$ by a new symbol S' of the same kind. Thus $\mathscr{L}' \cap \mathscr{L}^+ = \mathscr{L}$. With each $\mathscr{L}^+$-structure $\overline{C}$ let $\overline{C}'$ be the $\mathscr{L}'$-structure such that $S^{\overline{C}'} = S^{\overline{C}}$ if S is in $\mathscr{L}$, and $(S')^{\overline{C}'} = S^{\overline{C}}$ for S a symbol of $\mathscr{L}^+ \backslash \mathscr{L}$. With each formula φ of $\mathscr{L}^+$, let φ' be obtained from φ by a similar replacement. Clearly $\overline{C} \models \varphi(\overline{b})$ iff $\overline{C}' \models \varphi'(\overline{b})$ for any tuple $\overline{b}$.

Now we apply Theorem 9.2 to $\overline{A}'$ and $\overline{B}$. We obtain an $(\mathscr{L}' \cup \mathscr{L}^+)$-structure $\overline{D}$ such that $\overline{A}' \preceq \overline{D} \restriction \mathscr{L}'$ and an elementary embedding $g : \overline{B} \to \overline{D} \restriction \mathscr{L}^+$. We can write $\overline{D} \restriction \mathscr{L}' = \overline{E}'$ for some $\mathscr{L}^+$-structure $\overline{E}$. Then $\overline{D} \restriction \mathscr{L}^+$ and $\overline{E}$ are models of T and $\overline{D} \restriction \mathscr{L} = \overline{E} \restriction \mathscr{L}$. Since $\overline{B} \models \varphi(\overline{a})$, it follows from (i) that $\overline{E} \models \varphi(\overline{a})$. Clearly $\overline{A} \preceq \overline{E}$, so $\overline{A} \models \varphi(\overline{a})$, as desired. $\square$

Corollary 9.9. (Beth's definability theorem) *Let $\mathscr{L}$ and $\mathscr{L}^+$ be languages with $\mathscr{L} \subseteq \mathscr{L}^+$. Let T be a theory in $\mathscr{L}^+$ and S a nonlogical symbol of $\mathscr{L}^+$. Then the following are equivalent:*

(i) If $\overline{A}$ and $\overline{B}$ are models of T and $\overline{A} \restriction \mathscr{L} = \overline{B} \restriction \mathscr{L}$, then $S^{\overline{A}} = S^{\overline{B}}$.

(ii) There is a formula $\psi(\overline{x})$ of $\mathscr{L}$ such that $T \models \forall \overline{x}[S(\overline{x}) \leftrightarrow \psi(\overline{x})]$. $\square$