HW 3

Due Friday Feb 9

EXERCISES
Sec 3.1: 3.1.1, **3.1.2**, (3.1.3), (3.1.4), **3.1.5**,
Sec 3.2: (3.2.1), (3.2.2), **3.2.3**,
Sec 3.3: 3.3.1, (3.3.2), (3.3.3), (3.3.4), (3.3.5), **3.3.6**,
Sec 3.4: 3.4.1, (3.4.2), (3.4.3), **3.4.4**, **3.4.5**.

(Solutions to numbers in parentheses should not be turned in.)

Section 3.1.

3.1.1. Prove the coincidence lemma for satisfiability (from the remark before Lemma 3.1.1).

3.1.2. Prove the following equivalences for all sentences φ and ψ and structures $\mathcal{M}$ (of matching signature):
(i) $\mathcal{M} \models \varphi \vee \psi$ if and only if, $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$;
(ii) $\mathcal{M} \models \varphi \to \psi$ if and only if, *if* $\mathcal{M} \models \varphi$ *then* $\mathcal{M} \models \psi$;
(iii) $\mathcal{M} \models \varphi \leftrightarrow \psi$ if and only if, $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$;
(iv) $\mathcal{M} \models \forall x \varphi$ if and only if, *for all* $a \in M$, $\mathcal{M} \models \varphi(a)$.

3.1.3. (not required)
Show that the sentences $\exists x(x = x)$ and $\forall x(x = x)$ are valid. Which of the two sentences is true in empty structures

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3.1.4. (not required)
For any natural number n find an $L_{=}$-structure that is true in an L-structure precisely if it has cardinality n.

3.1.5. Let $\varphi = \varphi(z)$ be an L-formula, and $t = t(\mathbf{x})$ an L-term, $\mathcal{M}$ an L-structure, and $\mathbf{a}$ a tuple of M (of appropriate length). Prove the so called *Subsititution Lemma*, which says that $\mathbf{a}$ satisfies the formula $\varphi(t(\mathbf{x}))$ in $\mathcal{M}$ if and only if $t^{\mathcal{M}}(\mathbf{a})$ satisfies $\varphi(z)$. More precisely, letting ψ denote the formula $[t/z]\varphi := \varphi_z(t)$ (and hence $\psi = \psi(\mathbf{x}) = \varphi_z(t(\mathbf{x}))$), we have $\mathcal{M} \models \varphi(t^{\mathcal{M}}(\mathbf{a}))$ iff $\mathcal{M} \models \psi(\mathbf{a})$.

Section 3.2.

3.2.1. (not required)
Show that every finite subset of a structure $\mathcal{M}$ is parametrically definable in $\mathcal{M}$. Derive the same for every **cofinite** subset of $\mathcal{M}$ (i.e., a subset $X \subseteq M$ whose complement $M - X$ is finite.)

3.2.2. (not required)
Consider a given set M as an $L_=$-structure $\mathcal{M}$. Describe all sets defined (with parameters) by atomic $L_=$-formulas in $\mathcal{M}$.

3.2.3 Let L be a language of signature $\sigma = (\mathbf{C}, \mathbf{F}, \mathbf{R}, \sigma')$. For all $x \in \mathbf{C}$, choose a new unary relation symbol (a *new predicate*) P_c, and, for all $f \in \mathbf{F}$ with $\sigma'(f) = n$, a new $(n+1)$-place relation symbol R_f.

Set $\mathbf{R}^* = \mathbf{R} \cup \{P_c : c \in \mathbf{C}\} \cup \{R_f : f \in \mathbf{F}\}$ and let L^* be the language with non-logical symbols $\mathbf{R}^*$. Given an L-structure $\mathcal{M}$, let $\mathcal{M}^*$ be the L^* -structure with the same underlying universe M such that

- $R^{\mathcal{M}} = R^{\mathcal{M}^*}$, for all $R \in \mathbf{R}$,
- $\mathcal{M}^* \models P_c(d)$ iff $c^{\mathcal{M}} = d$, for all $c \in \mathbf{C}$,
- $\mathcal{M}^* \models R_f(\mathbf{a}, b)$ iff $f^{\mathcal{M}}(\mathbf{a}) = b$, for all $f \in \mathbf{F}$.

Prove that $\mathcal{M}$ and $\mathcal{M}^*$ have the same definable sets.

Section 3.3.

3.3.1. (The Deduction Theorem)
Given L-sentences φ and ψ and a set Σ of L-sentences, show that $\Sigma \models \varphi \to \psi$ if and only if $\Sigma \cup \{\varphi\} \models \psi$.

3.3.2. (not required)
Prove that a finite conjunction of existential (resp. universal) formulas is logically equivalent to an existential (resp. universal) formula. Prove the same for finite disjunctions. In which of the four statements you proved can you not in general simply exchange the quantifiers with the logical connectives?

3.3.3. (not required)
Verify the assertions made in all of the remarks in Section 3.3, in particular, prove the theorems on normal forms.

3.3.4. (not required)
Let Σ be a set of L-sentences, and $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$ sets of L-formulas in the free variables $\mathbf{x} = (x_0, \ldots, x_{n-1})$. Then Φ and Ψ are Σ-equivalent iff for some (every) expansion $L(\mathbf{c})$ by an n-tuple of new constants $\mathbf{c}$, the sets of $L(\mathbf{c})$-sentences $\Phi(\mathbf{c})$ and $\Psi(\mathbf{c})$ are Σ-equivalent.

3.3.5. (not required)
Show that adding dummy variables preserves logical equivalence.

3.3.6. Prove by induction on the complexity of formlas that every formula is equivalent to a formula obtained from *unnested* atomic formulas using $\neg$, $\wedge$, and $\exists$.

Section 3.4.

3.4.1. Prove that isomorphic structures have the same L-theory and hence, if $\mathcal{M}$ is isomorphic to a structure from an axiomatizable class $\mathbf{K}$, then $\mathcal{M}$ itself is in $\mathbf{K}$.

3.4.2. (not required)
Axiomatize the class of all vector spaces over a field K (in the language of Exercise 1.2.1).

3.4.3. (not required) (In the notation of Exercise 3.2.3.)
Let $\Sigma = \{\exists^{=1} x\, P_c(x) : c \in \mathbf{C}\} \cup \{\forall \mathbf{x}\, \exists^{=1} y\, R_f(\mathbf{x}, y) : f \in \mathbf{F}\} \subseteq L_0^*$. Prove that the class of structures satisfying Σ is $\{\mathcal{M}^* : \mathcal{M} \text{ is an } L\text{-structure}\}$, hence $\operatorname{Mod}\Sigma = \{\mathcal{M}^* : \mathcal{M} \text{ is a nonempty } L\text{-structure}\}$.

In the next few exercises involve the notion of “Galois correspondence,” which we now review. If X is a set (class, resp.) then the set (class) of all subsets (subclasses) of X is denoted by $\mathfrak{P}(X)$ and called the “powerset” of X.

Let X and Y be two sets (or classes). A **Galois correspondence** between X and Y is a pair of maps $\alpha\colon \mathfrak{P}(X) \to \mathfrak{P}(Y)$ and $\beta\colon \mathfrak{P}(Y) \to \mathfrak{P}(X)$ satisfying the following properties:

- if $X_0 \subseteq X_1 \subseteq X$ then $\alpha(X_0) \supseteq \alpha(X_1)$;
- if $Y_0 \subseteq Y_1 \subseteq Y$ then $\beta(Y_0) \supseteq \beta(Y_1)$;
- $A \subseteq \beta\alpha(A)$ for all $A \subseteq X$;
- $B \subseteq \alpha\beta(B)$ for all $B \subseteq Y$.

Extra Exercise. Prove that the map $\beta\alpha\colon \mathfrak{P}(X) \to \mathfrak{P}(X)$ is a **closure operator** on X. That is, for all $A \subseteq A' \subseteq X$, we have:

- $A \subseteq \beta\alpha(A)$ ($\beta\alpha$ is "extensive")
- $\beta\alpha(A) \subseteq \beta\alpha(A')$ ($\beta\alpha$ is "monotone")
- $\beta\alpha\beta\alpha(A) = \beta\alpha(A)$ ($\beta\alpha$ is "idempotent")

3.4.4. Prove that $\Sigma \mapsto \mathrm{Mod}_L\,\Sigma$ and $\mathbf{K} \mapsto \mathrm{Th}_L\,\mathbf{K}$ defines a Galois correspondence between L_0 (the set of all L-sentence) and $\mathrm{Mod}_L\,\emptyset$ (the class of all nonempty L-structures). Derive that, given theories S and T in the same laguage, we have $S = T$ if and only if $\mathrm{Mod}\,S = \mathrm{Mod}\,T$.

3.4.5. Show that, under the Galois correspondence of the preceding exercise, the deductively closed sets of L-sentences (including the inconsistent one) are exactly the closed subsets of L_0, while the axiomatizable class of nonempty L-structures (including the empty class) are exactly the closed subsets of $\mathrm{Mod}_L\,\emptyset$.