8. Morley rank

Let $\overline{M}$ be an $\mathscr{L}$-structure and $\varphi(\overline{v})$ a formula of $\mathscr{L}_M$. We define $\mathrm{RM}(\overline{M},\varphi,\alpha)\in\{0,1\}$ for every ordinal α by recursion on α.

- $\mathrm{RM}(\overline{M},\varphi,0)=1$ iff $\varphi(\overline{M})$ is nonempty.
- For α limit, $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ iff $\mathrm{RM}(\overline{M},\varphi,\beta)=1$ for all $\beta<\alpha$.
- $\mathrm{RM}(\overline{M},\varphi,\alpha+1)=1$ iff there exist formulas $\psi_0(\overline{v}),\psi_1(\overline{v}),\ldots$ such that $\psi_0(\overline{M}),\psi_1(\overline{M}),\ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}(\overline{M},\psi_i,\alpha)=1$ for every $i\in\omega$.

Proposition 8.1. *If* $\overline{M}\models\varphi\to\psi$ *and* $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$, *then* $\mathrm{RM}(\overline{M},\psi,\alpha)=1$.

Proof. Induction on α. □

Proposition 8.2. *If* $\mathrm{RM}(\overline{M},\varphi,\alpha)=0$, *then* $\mathrm{RM}(\overline{M},\varphi,\beta)=0$ *for all* $\beta\geq\alpha$.

Proof. Induction on β. □

Now we define the *Morley rank* $\mathrm{RM}^{\overline{M}}(\varphi)$ of φ in $\overline{M}$ as follows. If $\varphi(\overline{M})=\emptyset$, then $\mathrm{RM}^{\overline{M}}(\varphi)=-1$. If α is minimum such that $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ and $\mathrm{RM}(\overline{M},\varphi,\alpha+1)=0$, then $\mathrm{RM}^{\overline{M}}(\varphi)=\alpha$. If $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ for all α, then $\mathrm{RM}^{\overline{M}}(\varphi)=\infty$.

We will also define the Morley rank of structures below.

Proposition 8.3. $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ *iff* $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha$.

Proof. Suppose that $\mathrm{RM}^{\overline{M}}(\varphi)=\beta<\alpha$. Then $\mathrm{RM}(\overline{M},\varphi,\beta+1)=0$, and so by Proposition 8.2, $\mathrm{RM}(\overline{M},\varphi,\alpha)=0$.

Suppose that $\mathrm{RM}^{\overline{M}}(\varphi)=\beta\geq\alpha$. Then $\mathrm{RM}(\overline{M},\varphi,\beta)=1$, and so by Proposition 8.2, also $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$. □

By Proposition 8.3, the definition of Morley rank can be reformulated as follows.

- $\mathrm{RM}^{\overline{M}}(\varphi)=-1$ if $\varphi(\overline{M})=\emptyset$.
- $\mathrm{RM}^{\overline{M}}(\varphi)\geq 0$ if $\varphi(\overline{M})\neq\emptyset$.
- For α limit, $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha$ iff $\mathrm{RM}^{\overline{M}}(\varphi)\geq\beta$ for every $\beta<\alpha$.
- $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha+1$ iff there exist formulas $\psi_0(\overline{v}),\psi_1(\overline{v}),\ldots$ such that $\psi_0(\overline{M}),\psi_1(\overline{M}),\ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}^{\overline{M}}(\psi_i)\geq\alpha$ for every $i\in\omega$.

Proposition 8.4. *Suppose that* $\overline{M}$ *is* ω*-saturated,* $\varphi(\overline{v},\overline{w})$ *is a formula,* $\overline{a},\overline{b}\in M$, *and* $\mathrm{tp}^{\overline{M}}(\overline{a})=\mathrm{tp}^{\overline{M}}(\overline{b})$. *Then* $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a}))=\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b}))$.

Proof. We prove by induction on α that if $\varphi(\overline{v},\overline{w})$ is any formula, $\overline{a},\overline{b}\in M$, and $\mathrm{tp}^{\overline{M}}(\overline{a})=\mathrm{tp}^{\overline{M}}(\overline{b})$, then $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a}))\geq\alpha$ iff $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b}))\geq\alpha$. To begin with,

$$\begin{aligned}\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a}))\geq 0\quad&\text{iff}\quad\varphi(\overline{M},\overline{a})\neq\emptyset\\&\text{iff}\quad\text{there is a }\overline{c}\text{ such that }\overline{M}\models\varphi[\overline{c},\overline{a}]\end{aligned}$$

$$\begin{aligned} &\text{iff} \quad \exists \overline{v}\varphi(\overline{v},\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a}) \\ &\text{iff} \quad \exists \overline{v}\varphi(\overline{v},\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{b}) \\ &\text{iff} \quad \varphi(\overline{M},\overline{b}) \neq \emptyset \\ &\text{iff} \quad \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq 0. \end{aligned}$$

If α is limit and we know the result for all $\beta < \alpha$, then

$$\begin{aligned} \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \alpha \quad &\text{iff} \quad \text{for all } \beta < \alpha,\ \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \beta \\ &\text{iff} \quad \text{for all } \beta < \alpha,\ \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \beta \\ &\text{iff} \quad \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \alpha. \end{aligned}$$

Now suppose that the equivalence is true for α, and $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \alpha+1$; we prove that $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \alpha+1$. By symmetry this is all that is required. Now there are $\mathscr{L}_M$-formulas $\psi_0, \psi_1, \ldots$ such that $\langle \psi_i(\overline{M}) : i < \omega \rangle$ is a system of pairwise disjoint subsets of $\varphi(\overline{M},\overline{a})$ and $\mathrm{RM}(\psi_i) \geq \alpha$ for all $i < \omega$. For each $i < \omega$ there is a sequence $\overline{c}_i$ of elements of M such that ψ_i is $\psi_i(\overline{v},\overline{c}_i)$. We now define $\overline{d}_0, \overline{d}_1, \ldots$. Suppose that $\overline{d}_0, \ldots, \overline{d}_m$ have been defined so that

$$(*) \qquad \mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_m) = \mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_m).$$

Let

$$\Delta = \{\chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) : \overline{M} \models \chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1})\}.$$

Suppose that Δ' is a finite subset of Δ. Then

$$\overline{M} \models \bigwedge\{\chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\},$$

so

$$\overline{M} \models \exists \overline{w} \bigwedge\{\chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_m,\overline{w}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\}.$$

It follows that the formula

$$\exists \overline{w} \bigwedge\{\chi(\overline{v},\overline{u}_0,\ldots,\overline{u}_m,\overline{w}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\}$$

is in $\mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_m)$, and hence by $(*)$ it is in $\mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_m)$. Thus $\overline{M} \models \exists \overline{w} \bigwedge \Delta'$. Now since $\overline{M}$ is ω-saturated, there is a $\overline{d}_{m+1}$ in M such that $\overline{M} \models \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_{m+1})$ for each formula $\chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w})$ in Δ. It follows that

$$\mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1}) = \mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_{m+1}).$$

This finishes the definition of $\overline{d}_0, \overline{d}_1, \ldots$. So we have

$$\mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\overline{c}_1,\ldots) = \mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\overline{d}_1,\ldots).$$

Now for $i \neq j$ we have $\psi_i(\overline{M}, \overline{c}_i) \cap \psi_i(\overline{M}, \overline{c}_j) = \emptyset$. Hence

$$\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \wedge \psi_j(\overline{v}, \overline{w}_j)] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \overline{c}_1, \ldots),$$

and hence

$$\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \wedge \psi_j(\overline{v}, \overline{w}_j)] \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{d}_0, \overline{d}_1, \ldots),$$

which means that $\psi_i(\overline{M}, \overline{d}_i) \cap \psi_j(\overline{M}, \overline{d}_j) = \emptyset$. Also, $\psi_i(\overline{M}, \overline{c_i}) \subseteq \varphi(\overline{M}, \overline{a})$, so

$$\forall\overline{v}[\psi_i(\overline{v}, \overline{w}) \rightarrow \varphi(\overline{M}, \overline{x})] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \overline{c}_1, \ldots),$$

and so

$$\forall\overline{v}[\psi_i(\overline{v}, \overline{w}) \rightarrow \varphi(\overline{M}, \overline{x})] \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{d}_0, \overline{d}_1, \ldots),$$

from which it follows that $\psi_i(\overline{M}, \overline{d}_i) \subseteq \varphi(\overline{M}, \overline{b})$. From (∗∗) the inductive hypothesis gives $\mathrm{RM}(\psi_i(\overline{v}, \overline{d}_i)) \geq \alpha$. So $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v}, \overline{b})) \geq \alpha + 1$.

Proposition 8.5. *Suppose that $\overline{M}$ and $\overline{N}$ are ω-saturated models of T and $\overline{M} \preceq \overline{N}$. Then $\mathrm{RM}^{\overline{M}}(\varphi) = \mathrm{RM}^{\overline{N}}(\varphi)$ for any $\mathscr{L}_M$-formula φ.*

Proof. We prove by induction on α that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha$ iff $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha$. For $\alpha = 0$,

$$\begin{aligned}
\mathrm{RM}^{\overline{M}}(\varphi) \geq 0 \quad & \text{iff} \quad \varphi(\overline{M}) \neq \emptyset \\
& \text{iff} \quad \overline{M} \models \exists\overline{v}\varphi(\overline{v}) \\
& \text{iff} \quad \overline{N} \models \exists\overline{v}\varphi(\overline{v}) \\
& \text{iff} \quad \varphi(\overline{N}) \neq \emptyset \\
& \text{iff} \quad \mathrm{RM}^{\overline{N}}(\varphi) \geq 0.
\end{aligned}$$

For α limit, assuming the equivalence for all $\beta < \alpha$,

$$\begin{aligned}
\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha \quad & \text{iff} \quad \text{for all } \beta < \alpha[\mathrm{RM}^{\overline{M}}(\varphi) \geq \beta] \\
& \text{iff} \quad \text{for all } \beta < \alpha[\mathrm{RM}^{\overline{N}}(\varphi) \geq \beta] \\
& \text{iff} \quad \mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha.
\end{aligned}$$

Now assume the equivalence for α. Suppose that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha+1$. Then there are formulas $\psi_0(\overline{v}), \psi_1(\overline{v}), \ldots$ of $\mathscr{L}_M$ such that $\psi_0(\overline{M}), \psi_1(\overline{M}), \ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}^{\overline{M}}(\psi_i) \geq \alpha$ for all $i < m$. By the inductive hypothesis, $\mathrm{RM}^{\overline{N}}(\psi_i) \geq \alpha$ for all $i < m$. For distinct $i, j < \omega$ we have $\overline{M} \models \neg\exists\overline{v}(\psi_i(\overline{v}) \wedge \psi_j(\overline{v}))$, and hence $\overline{N} \models \neg\exists\overline{v}(\psi_i(\overline{v}) \wedge \psi_j(\overline{v}))$. So $\psi_0(\overline{N}), \psi_1(\overline{N}), \ldots$ are pairwise disjoint. Also, $\overline{M} \models \forall\overline{v}[\psi_i(\overline{v}) \rightarrow \varphi(\overline{v})]$ for each $i < \omega$, so $\overline{N} \models \forall\overline{v}[\psi_i(\overline{v}) \rightarrow \varphi(\overline{v})]$. Hence each $\psi_i(\overline{N})$ is a subset of $\varphi(\overline{N})$. It follows that $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha + 1$.

Suppose now that $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha + 1$. Then there are formulas $\psi_0(\overline{v}), \psi_1(\overline{v}), \ldots$ of $\mathscr{L}_N$ such that $\psi_0(\overline{N}), \psi_1(\overline{N}), \ldots$ are pairwise disjoint subsets of $\varphi(\overline{N})$ and $\mathrm{RM}^{\overline{N}}(\psi_i) \geq \alpha$ for all $i < m$. Write $\varphi(\overline{v}) = \varphi(\overline{v}, \overline{a})$ with $\overline{a} \in M$ and $\psi_i(\overline{v}) = \psi_i(\overline{v}, \overline{b}_i)$ with $\overline{b}_i \in N$. We now define $\overline{c}_i$ in M for $i < \omega$ so that

$$(*) \qquad \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots, \overline{b}_{m-1})$$

for every $m \in \omega$. Note that $(*)$ holds for $m = 0$ since $\overline{M} \preceq \overline{N}$. Suppose now that $(*)$ holds for m. Let

$$\Delta = \{\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) : \overline{N} \models \chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m)\}.$$

Suppose that Δ' is a finite subset of Δ. Then

$$\overline{N} \models \bigwedge\{\chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\},$$

so

$$\overline{N} \models \exists \overline{w} \bigwedge\{\chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_{m-1}, \overline{w}) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\}.$$

Hence by $(*)$ for m we get

$$\overline{M} \models \exists \overline{w} \bigwedge\{\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\}.$$

Since $\overline{M}$ is ω-saturated, there is a $\overline{c}_m$ in M such that $\overline{M} \models \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{c}_m)$ for each formula $\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}$ in Δ. It follows that $(*)$ holds for $m + 1$.

Now suppose that $i, j \in \omega$ with $i \neq j$. Then $\overline{N} \models \neg\exists\overline{v}[\psi_i(\overline{v}, \overline{b_i}) \wedge \psi_j(\overline{v}, \overline{b_j})$, so that $\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w_i}) \wedge \psi_j(\overline{v}, \overline{w_j}) \in \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots)$. Hence by $(*)$, $\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w_i}) \wedge \psi_j(\overline{v}, \overline{w_j}) \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots)$. Hence $\psi_i(\overline{M}, \overline{c_i}) \cap \psi_i(\overline{M}, \overline{c}_j) = \emptyset$. Also, $\overline{N} \models \forall\overline{v}[\psi_i(\overline{v}, \overline{b}_i) \rightarrow \varphi(\overline{v}, \overline{a})]$, so $\forall\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \rightarrow \varphi(\overline{v}, \overline{a})] \in \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots)$. Hence by $(*)$, $\forall\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \rightarrow \varphi(\overline{v}, \overline{a})] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots)$. Hence $\psi_i(\overline{M}, \overline{c}_i) \subseteq \varphi(\overline{M}, \overline{a})$. Now since $\overline{M} \preceq \overline{N}$, we have

$$\mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m)$$

for each $m \in \omega$, and hence by $(*)$, also $\mathrm{tp}^{\overline{N}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m)$ for each $m \in \omega$. Hence by Proposition 8.4 we get $\mathrm{RM}^{\overline{N}}(\psi_i(\overline{v}, \overline{c}_i) \geq \alpha$, and so by the inductive hypothesis $\mathrm{RM}^{\overline{N}}(\psi_i(\overline{v}, \overline{c}_i) \geq \alpha$ for each $i < \omega$. It follows that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha + 1$. $\square$

Proposition 8.6. (amalgamation) *Suppose that $\overline{M}$, $\overline{N}_0$, and $\overline{N}_1$ are structures, and $f : M \rightarrow N_0$ and $g : M \rightarrow N_1$ are elementary embeddings. Then there exist a structure $\overline{P}$ and elementary embeddings $h : \overline{N}_0 \rightarrow \overline{P}$ and $k : \overline{N}_1 \rightarrow \overline{P}$ such that $h \circ f = k \circ g$; so the following diagram commutes:*

$$
\begin{tikzcd}
 & \overline{N}_0 \arrow[dr, "h"] & \\
\overline{M} \arrow[ur, "f"] \arrow[dr, "g"'] & & \overline{P} \\
 & \overline{N}_1 \arrow[ur, "k"'] &
\end{tikzcd}
$$

Proof. Our first goal is to obtain isomorphic copies $\overline{N}_0'$ and $\overline{N}_1'$ of $\overline{N}_0$ and $\overline{N}_1$ such that $\overline{M} \preceq \overline{N}_0'$, $\overline{M} \preceq \overline{N}_1'$, and $N_0' \cap N_1' = M$.

Let Q_0 and Q_1 be sets such that $Q_0 \cap M = \emptyset = Q_1 \cap M = Q_0 \cap Q_1$, $|Q_0| = |N_0 \backslash f[M]|$, and $|Q_1| = |N_1 \backslash g[M]|$. Let $f' : N_0 \backslash f[M] \to Q_0$ and $g' : N_1 \backslash g[M] \to Q_1$ be bijections. Let $N_0' = M \cup Q_0$ and $N_1' = M \cup Q_1$. Note that $N_0' \cap N_1' = M$. Define $f'' : N_0 \to N_0'$ by setting, for any $a \in N_0$,

$$f''(a) = \begin{cases} f^{-1}(a) & \text{if } a \in f[M], \\ f'(a) & \text{if } a \in N_0 \backslash f[M], \end{cases}$$

and define $g'' : N_1 \to N_1'$ by setting, for any $a \in N_1$,

$$g''(a) = \begin{cases} g^{-1}(a) & \text{if } a \in g[M], \\ g'(a) & \text{if } a \in N_1 \backslash g[M]. \end{cases}$$

Clearly f'' and g'' are bijections.

We now define structures on N_0' and N_1'. If R is an m-ary relation symbol, then

$$R^{\overline{N}_0'} = \{a \in {}^m N_0' : (f'')^{-1} \circ a \in R^{\overline{N}_0}\};$$
$$R^{\overline{N}_1'} = \{a \in {}^m N_1' : (g'')^{-1} \circ a \in R^{\overline{N}_1}\}.$$

If F is an m-ary function symbol, then

$$F^{\overline{N}_0'}(a) = f''(F^{\overline{N}_0}((f'')^{-1} \circ a)) \quad \text{for any } a \in {}^m N_0';$$
$$F^{\overline{N}_1'}(a) = g''(F^{\overline{N}_1}((g'')^{-1} \circ a)) \quad \text{for any } a \in {}^m N_1'.$$

Then it is easy to check that f'' is an isomorphism from $\overline{N}_0$ onto $\overline{N}_0'$ and g'' is an isomorphism from $\overline{N}_1$ onto $\overline{N}_1'$. Now take any formula φ and any $a \in {}^\omega M$. Then

$$\begin{aligned}
\overline{M} \models \varphi[a] \quad & \text{iff} \quad \overline{N}_0 \models \varphi[f \circ a] \\
& \text{iff} \quad \overline{N}_0' \models \varphi[f'' \circ f \circ a] \\
& \text{iff} \quad \overline{N}_0' \models \varphi[f^{-1} \circ f \circ a] \\
& \text{iff} \quad \overline{N}_0' \models \varphi[a].
\end{aligned}$$

Thus $\overline{M} \preceq \overline{N}'_0$. Similarly $\overline{M} \preceq \overline{N}'_1$.

Now we claim that $\text{Eldiag}(\overline{N}'_0) \cup \text{Eldiag}(\overline{N}'_1)$ has a model. If not, by the compactness theorem some finite subset fails to have a model. Say Δ_0 is a finite subset of $\text{Eldiag}(\overline{N}'_0)$, Δ_1 is a finite subset of $\text{Eldiag}(\overline{N}'_1)$, and $\Delta_0 \cup \Delta_1$ does not have a model. Then $\bigwedge \Delta_1$ has the form $\psi(c_{a(0)}, \ldots, c_{a(m-1)}, c_{b(0)}, \ldots, c_{b(n-1)})$ with each $a(i) \in M$ and each $d(i) \in N'_1 \backslash M$. Thus

$$\Delta_0 \models \neg\psi(c_{a(0)}, \ldots, c_{a(m-1)}, c_{b(0)}, \ldots, c_{b(n-1)}).$$

Replacing $c_{b(i)}$ by a variable w_i, we get

$$\Delta_0 \models \forall \overline{u} \neg\psi(c_{a(0)}, \ldots, c_{a(m-1)}, \overline{u}).$$

Now $(\overline{N}'_0)_{N'_0}$ is a model of Δ_0, hence $\overline{N}'_0 \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$, hence $\overline{M} \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$, hence $\overline{N}'_1 \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$. But this is impossible. Hence we have shown that $\text{Eldiag}(\overline{N}'_0) \cup \text{Eldiag}(\overline{N}'_1)$ has a model. Such a model has the form $(\overline{P}, h(s), k(t))_{s \in N'_0, t \in N'_1}$, where $h(a) = k(a)$ for all $a \in M$. By the elementary diagram lemma 6.15, h is an elementary embedding of $\overline{N}'_0$ into $\overline{P}$, and k is an elementary embedding of $\overline{N}'_1$ into $\overline{P}$.

Hence $h \circ f''$ is an elementary embedding of $\overline{N}_0$ into $\overline{P}$, $k \circ g''$ is an elementary embedding of $\overline{N}_1$ into $\overline{P}$, and for any $a \in M$,

$$h(f''(f(a))) = h(f^{-1}(f(a))) = h(a) = k(a) = k(g^{-1}(g(a))) = k(g''(g(a))). \qquad \square$$

Corollary 8.7. *Suppose that $\overline{M}$ is an $\mathscr{L}$-structure and $\overline{N}_0$ and $\overline{N}_1$ are ω-saturated elementary extensions of $\overline{M}$. Then for any formula φ of $\mathscr{L}_M$,* $\text{RM}^{\overline{N}_0}(\varphi) = \text{RM}^{\overline{N}_1}(\varphi)$.

Proof. By Proposition 8.6 let $\overline{N}_2$ be f and g be elementary embeddings of $\overline{N}_0$ and $\overline{N}_1$ into a structure $\overline{N}_2$. Let $\overline{N}_3$ be an ω-saturated elementary extension of $\overline{N}_2$. Then by Proposition 8.5, $\text{RM}^{\overline{N}_0}(\varphi) = \text{RM}^{\overline{N}_3}(\varphi) = \text{RM}^{\overline{N}_1}(\varphi)$. $\square$

Proposition 8.8. *If $\varphi(\overline{v})$ and $\psi(\overline{v})$ are formulas of $\mathscr{L}_M$ and $\overline{M} \models \forall \overline{v}[\varphi \to \psi]$, then* $\text{RM}(\varphi) \leq \text{RM}(\psi)$.

Proof. We prove by induction on α that $\text{RM}(\varphi) \geq \alpha$ implies that $\text{RM}(\psi) \geq \alpha$. For $\alpha = 0$, if $\text{RM}(\varphi) \geq 0$, then $\varphi(\overline{M}) \neq \emptyset$; hence $\text{RM}(\psi) \neq \emptyset$ and so $\text{RM}(\psi) \geq 0$. Suppose that α is a limit ordinal and we know the implication for all $\beta < \alpha$. Suppose that $\text{RM}(\varphi) \geq \alpha$. Then $\forall \beta < \alpha[\text{RM}(\varphi) \geq \beta]$, hence $\forall \beta < \alpha[\text{RM}(\psi) \geq \beta]$, hence $\text{RM}(\psi) \geq \alpha$. Now suppose that we know the implication for α, and $\text{RM}(\varphi) \geq \alpha + 1$. Then there are pairwise disjoint subsets $\psi_i(\overline{M})$ of $\varphi(\overline{M})$ such that $\text{RM}(\psi_i) \geq \alpha$. Since also $\psi_i(\overline{M}) \subseteq \psi(\overline{M})$, it follows that $\text{RM}(\psi) \geq \alpha + 1$. $\square$

Proposition 8.9. $\text{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) = \max(\text{RM}(\varphi(\overline{v})), \text{RM}(\psi(\overline{v}))$.

Proof. It suffices to show that $\text{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$ iff $\text{RM}(\varphi(\overline{v})) \geq \alpha$ or $\text{RM}(\psi(\overline{v})) \geq \alpha$ for every ordinal α, by induction. For $\alpha = 0$,

$$\begin{aligned} \text{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq 0 \quad &\text{iff} \quad \varphi(\overline{M}) \neq \emptyset \text{ or } \psi(\overline{M}) \neq \emptyset \\ &\text{iff} \quad \text{RM}(\varphi(\overline{v})) \geq 0 \text{ or } \text{RM}(\psi(\overline{v})) \geq 0. \end{aligned}$$

For α limit, assume the result for any $\beta < \alpha$. Suppose that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$. Then for any $\beta < \alpha$ we have $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \beta$, and hence by the inductive hypothesis, $\mathrm{RM}(\varphi(\overline{v})) \geq \beta$ or $\mathrm{RM}(\psi(\overline{v})) \geq \beta$. If $\mathrm{RM}(\varphi(\overline{v})) \geq \beta$ for all $\beta < \alpha$, then $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha$. If $\mathrm{RM}(\varphi(\overline{v})) < \beta$ for some $\beta < \alpha$, then $\mathrm{RM}(\psi(\overline{v})) \geq \beta$ for all $\beta < \alpha$ and so $\mathrm{RM}(\psi(\overline{v})) \geq \alpha$. Thus $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$ implies that $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha$ or $\mathrm{RM}(\psi(\overline{v})) \geq \alpha$. The converse holds by Proposition 8.8.

Now assume the result for α and suppose that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha + 1$. Choose formulas $\chi_i(\overline{v})$ for $i \in \omega$ such that the sets $\chi_i(\overline{M})$ are pairwise disjoint, contained in $\varphi(\overline{M}) \cup \psi(\overline{M})$, and each of Morley rank $\geq \alpha$. Suppose that $\mathrm{RM}(\varphi(\overline{v})) < \alpha + 1$ and $\mathrm{RM}(\psi(\overline{v})) < \alpha + 1$. Now the sets $\chi_i(\overline{M}) \wedge \varphi(\overline{M})$ are pairwise disjoint and contained in $\varphi(\overline{M})$. It follows that there is an $m \in \omega$ such that $\mathrm{RM}(\chi_i(\overline{v}) \wedge \varphi(\overline{v})) < \alpha$ for all $i \geq m$. Similarly, there is an $n \in \omega$ such that $\mathrm{RM}(\chi_i(\overline{v}) \wedge \psi(\overline{v})) < \alpha$ for all $i \geq n$. Let $p = \max(m, n)$. Then by the inductive hypothesis, $\mathrm{RM}(\chi_p(\overline{v}) < \alpha$, contradiction. This shows that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha + 1$ implies that $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha + 1$ or $\mathrm{RM}(\psi(\overline{v})) \geq \alpha + 1$. The converse holds by Proposition 8.8. □

Proposition 8.10. *If $\varphi(\overline{M}) \neq \emptyset$, then $\mathrm{RM}(\varphi(\overline{v}) = 0$ iff $\varphi(\overline{M})$ is finite.*

Proof. $\Rightarrow$: suppose that $\varphi(\overline{M})$ is infinite, and choose distinct $\overline{a}_i \in M$ such that $\overline{M} \models \varphi[\overline{a}_i)$ for all $i \in \omega$. Then let ψ_i be the formula $\overline{v} = \overline{a}_i$ for each $i \in \omega$. Then the sets $\psi_i(\overline{M})$ are pairwise disjoint, nonempty, and are subsets of $\varphi(\overline{M})$. Since $\mathrm{RM}(\psi_i(\overline{v})) \geq 0$ for each $i \in \omega$, it follows that $\mathrm{RM}(\varphi(\overline{v}) \geq 1$.

$\Leftarrow$: If $\varphi(\overline{M})$ is finite, then there do not exist infinitely many pairwise disjoint nonempty subsets of it. So $\mathrm{RM}(\varphi(\overline{v})) \leq 0$. Since $\varphi(\overline{M}) \neq \emptyset$, actually $\mathrm{RM}(\varphi(\overline{v})) = 0$. □

Proposition 8.11. *Suppose that $\overline{M}$ is a structure, φ is an $\mathscr{L}_M$-formula, and $\mathrm{RM}^{\overline{M}}(\varphi)$ $= \alpha$ for some ordinal α. Then there is a positive integer d such that if $\psi_1, \ldots, \psi_m$ are $\mathscr{L}_M$-formulas such that $\psi_1(\overline{M}), \ldots, \psi_m(\overline{M})$ are pairwise disjoint subsets of $\varphi(\overline{M})$ each of Morley rank α, then $m \leq d$.*

Proof. By recursion we construct subsets T_i of ${}^{<\omega}2$ and formulas φ_σ for $\sigma \in T_i$. Let $T_0 = \{\emptyset\}$ and $\varphi_\emptyset = \varphi$. Suppose T_i has been defined along with the formulas φ_σ for $\sigma \in T_i$. Take any $\sigma \in T_i$. If there is a formula ψ in $\mathscr{L}_M$ such that $\mathrm{RM}^{\overline{M}}(\varphi_\sigma \wedge \psi) = \mathrm{RM}^{\overline{M}}(\varphi_\sigma \wedge \neg\psi) = \alpha$, we take such a formula ψ, put $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ in T_{i+1}, and define $\varphi_{\sigma^\frown\langle 0\rangle} = \varphi_\sigma \wedge \psi$ and $\varphi_{\sigma^\frown\langle 1\rangle} = \varphi_\sigma \wedge \neg\psi$. If such a formula ψ does not exist, we put σ in T_{i+1}.

By induction we have:

(1) for every $i \in \omega$, $\bigcup_{\sigma \in T_i} \varphi_\sigma(\overline{M}) = \varphi(\overline{M})$.

Now let $T = \bigcup_{i \in \omega} T_i$. We claim that T is finite. Suppose not. Then by König's tree lemma, there is an increasing sequence $\langle \sigma_i : i \in \omega\rangle$ of members of T. Let $\chi_i = \varphi_{\sigma_i} \wedge \neg\varphi_{\sigma_{i+1}}$ for all $i \in \omega$. Then $\langle \chi_i(\overline{M}) : i \in \omega\rangle$ is a system of pairwise disjoint subsets of $\varphi(\overline{M})$ and each χ_i has Morley rank α, contradicting $\mathrm{RM}(\varphi) = \alpha$.

Since T is finite, there is an $i \in \omega$ such that $T_i = T_j$ for all $j \geq i$. Let $\langle \psi_1, \ldots \psi_d\rangle$ enumerate T_i. By (1), $\langle \psi_i(\overline{M}) : i = 1, \ldots d\rangle$ is a partition of $\varphi(\overline{M})$.

Now suppose that $\theta_1, \ldots, \theta_m$ is a sequence of $\mathscr{L}_M$-formulas each of rank α such that $\langle \theta_1(\overline{M}), \ldots \theta_m(\overline{M}) : 1 \leq i \leq m\rangle$ is a sequence of pairwise disjoint subsets of $\varphi(\overline{M})$. We claim that $m \leq d$ (as desired). Suppose that $m > d$. Now for each $i \leq d$ there is at most one $j \leq m$ such that $\mathrm{RM}(\psi_i \wedge \theta_j) = \alpha$. Hence there is a $j \leq m$ such that $\mathrm{RM}(\psi_i \wedge \theta_j) < \alpha$ for all $i \leq d$. But $\theta_j(\overline{M})$ is a subset of $\varphi(\overline{M})$, and so by (1), $\overline{M} \models \theta_j \leftrightarrow \bigvee_{i \leq d}(\psi_i \wedge \theta_j)$. This contradicts Proposition 8.9. □

The smallest d satisfying the conditions of Proposition 8.11 is called the *Morley degree* of φ in $\overline{M}$, and is denoted by $\deg_{\overline{M}}(\varphi)$.

Corollary 8.12. *A formula φ is minimal over $\overline{M}$ iff* $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$.

Proof. $\Rightarrow$: Let φ be minimal. Then $\varphi(\overline{M})$ is infinite, hence nonempty, so $\mathrm{RM}^{\overline{M}}(\varphi) \geq 0$; by Proposition 8.10, $\mathrm{RM}^{\overline{M}}(\varphi) \geq 1$. Now $\varphi(\overline{M})$ cannot be partitioned into two infinite definable subsets, so $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$.

$\Leftarrow$: Assume that $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$. Then by Proposition 8.10, $\varphi(\overline{M})$ is infinite. Since $\deg_{\overline{M}}(\varphi) = 1$, it cannot be partitioned into two infinite definable subsets. So φ is minimal. □

The *Morley rank* of a theory T is the rank of the formula $v = v$ in any ω-saturated model of T.

The theory T is *totally transcendental* iff its rank is less than ∞.

If $\overline{M}$ is ω-saturated, a formula $\varphi(\overline{v}, \overline{w})$ has the *order property* over $\overline{M}$ iff there are sequences $\langle \overline{a}_i : i < \omega\rangle$ and $\langle \overline{b}_i : i < \omega\rangle$ such that for all $i, j \in \omega$, $\overline{M} \models \varphi(\overline{a}_i, \overline{b}_j)$ iff $i < j$.

Proposition 8.13. *If T is totally transcendental and $\overline{M}$ is an ω-saturated model of T, then no formula has the order property over $\overline{M}$.*

Proof. Suppose to the contrary that T is totally transcendental and $\overline{M}$ is an ω-saturated model of T with a formula $\varphi(\overline{v}, \overline{w})$ having the order property over $\overline{M}$, say with sequences $\langle \overline{a}_i : i < \omega\rangle$ and $\langle \overline{b}_i : i < \omega\rangle$ such that for all $i, j \in \omega$, $\overline{M} \models \varphi(\overline{a}_i, \overline{b}_j)$ iff $i < j$. Adjoin new individual constants $\overline{c}_q$ and $\overline{d}_q$ for $q \in \mathbb{Q}$ and consider the following set of sentences in the expanded language:

$$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{c}_q, \overline{d}_r) : q < r\} \cup \{\neg\varphi(\overline{c}_q, \overline{d}_r) : r \leq q\}.$$

Clearly every finite subset of this set has a model, so the whole set has a model. This gives an elementary extension $\overline{N}$ of $\overline{M}$ such that there are systems $\langle \overline{s}_q : q \in \mathbb{Q}\rangle$ and $\langle \overline{t}_q : q \in \mathbb{Q}\rangle$ of elements of N such that for all $q, r \in \mathbb{Q}$, $\overline{N} \models \varphi(\overline{s}_q, \overline{t}_r)$ iff $q < r$. Let $\overline{P}$ be and ω-saturated elementary extension of $\overline{N}$. Now for any $r \in \mathbb{Q}$, the set $\{q \in \mathbb{Q} : \overline{P} \models \varphi(\overline{s}_q, \overline{t}_r)\}$ is an infinite convex set. Let $\psi(\overline{v})$ be a formula of smallest Morley rank and degree such that $\{q \in \mathbb{Q} : \overline{P} \models \psi(\overline{s}_q)\}$ is infinite and convex. Choose r in the interior of this set. Let $\psi_0(\overline{v})$ be $\psi(\overline{v}) \wedge \varphi(\overline{v}\overline{t}_r)$ and let $\psi_1(\overline{v})$ be $\psi(\overline{v}) \wedge \neg\varphi(\overline{v}\overline{t}_r)$. Each set $\{q \in \mathbb{Q} : \overline{P} \models \psi_i(\overline{s}_q)$ is infinite and closed downwards. By Proposition 8.8, each ψ_i has Morley rank $\leq$ that of ψ; so $\mathrm{RM}^{\overline{P}}(\psi_i) = \mathrm{RM}^{\overline{P}}(\psi)$. But clearly both ψ_0 and ψ_1 have degree less than that of ψ, contradiction.

If p is an n-type over $A \subseteq M$ then we define $\mathrm{RM}(p)$ to be $\min\{\mathrm{RM}(\varphi) : \varphi \in p\}$. We let φ_p be a formula such that $\mathrm{RM}(p) = \mathrm{RM}(\varphi_p)$ and also with $\deg(\varphi_p)$ minimum among all formulas ψ such that $\mathrm{RM}(p) = \mathrm{RM}(\psi)$.

Lemma 8.14. *If $p, q \in S_n(A)$, $\mathrm{RM}(p), \mathrm{RM}(q) < \infty$, and $p \neq q$, then $\varphi_p \neq \varphi_q$.*

Proof. Let $\psi \in p \backslash q$. Then $\varphi_p \wedge \psi \in p$. So $\mathrm{RM}(\varphi_p \wedge \psi) \leq \mathrm{RM}(\varphi_p)$, so by the minimality of $\mathrm{RM}(\varphi_p)$ we get $\mathrm{RM}(\varphi_p \wedge \psi = \mathrm{RM}(\varphi_p)$. Similarly, $\mathrm{RM}(\varphi_q \wedge \neg\psi = \mathrm{RM}(\varphi_q)$. If $\varphi_p = \varphi_q$, then $\mathrm{RM}(\varphi_p \wedge \psi) = \mathrm{RM}(\varphi_p \wedge \neg\psi) = \mathrm{RM}(\varphi_p)$, and so $\deg(\varphi_p \wedge \psi) < \deg(\varphi_p)$, contradiction. □

Theorem 8.15. *If T is a theory in a countable language, then T is totally transcendental iff T is ω-stable.*

Proof. $\Rightarrow$: Assume that T is totally transcendental. Let $\overline{M}$ be a model of T, and let $A \subseteq M$ be countable. For each $p \in S_n(A)$ we have $\mathrm{RM}^{\overline{M}}(p) < \infty$, so φ_p exists. By Lemma 8.14 there are only countably many possible formulas φ_p, so $|S_n(A)| \leq \omega$.

$\Leftarrow$: Suppose that T is ω-stable but T is not totally transcendental. Thus there is an ω-saturated model $\overline{M}$ of T such that $\mathrm{RM}^{\overline{M}}(v = v) = \infty$. Let $\beta = \sup\{\mathrm{RM}(\psi) : \psi$ is an $\mathscr{L}_M$-formula and $\mathrm{RM}(\psi) < \infty\}$. We now define formulas φ_f for $f \in {}^{<\omega}2$. Let $\varphi_\emptyset$ be $v = v$. Suppose that φ_f has been defined so that $\mathrm{RM}(\varphi_f) = \infty$. Then there is a formula ψ such that $\mathrm{RM}(\varphi_f \wedge \psi) \geq \beta + 1$ and $\mathrm{RM}(\varphi_f \wedge \neg\psi) \geq \beta + 1$. Hence $\mathrm{RM}(\varphi_f \wedge \psi) = \infty$ and $\mathrm{RM}(\varphi_f \wedge \psi) = \infty$. let $\varphi_{f\frown\langle 0\rangle}$ be $\varphi_f \wedge \psi$ and $\varphi_{f\frown\langle 1\rangle}$ be $\varphi_f \wedge \neg\psi$. This completes the construction, and shows that T is not ω-stable, contradiction. □