Lemma 18.1. *If $|A|^{+3} < \min(A)$, then* $\mathrm{pcf}(A)$ *is a progressive interval of regular cardinals, and* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. $|\mathrm{pcf}(A)| \leq |A|^{+3}$ by 17.1, so $\mathrm{pcf}(A)$ is progressive. Hence $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$ by 10.10. □

Lemma 18.2. *Let $|A|^{+3} < \min(A)$, with A a progressive interval of regular cardinals. Then* $\mathrm{pcf} \restriction \mathscr{P}(\mathrm{pcf}(A))$ *is a topological closure operator.*

Proof. By 9.1 and 18.1 □

Theorem 18.3. *Suppose that $|A|^{+3} < \min(A)$, with A a progressive interval of regular cardinals. Then* $\mathrm{pcf}(A)$ *has a transitive system of generators $\langle b_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that* $\mathrm{pcf}(b_\lambda) = b_\lambda$ *for all $\lambda \in \mathrm{pcf}(A)$.*

Proof. By 13.6, let $f = \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be such that f^λ is a minimally obedient universal sequence for λ, for each $\lambda \in \mathrm{pcf}(A)$. Let $\kappa = |A|^+$ and choose N, Ψ such that $H_1(A, \kappa, N, \Psi)$. Then by 15.5 we get a transitive system $\langle b_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ of generators for $\mathrm{pcf}(A)$.

Now by induction on $\lambda \in \mathrm{pcf}(A)$ we define a new generator b^*_λ for λ over $\mathrm{pcf}(A)$ such that $\mathrm{pcf}(b^*_\lambda) = b^*_\lambda$. For $\lambda = \min(\mathrm{pcf}(A))$, let $b^*_\lambda = \{\lambda\}$. Suppose that we have defined b^*_θ for all $\theta < \lambda$, so that:

(1) b_θ is a θ-generator,

(2) $\mathrm{pcf}(b^*_\theta) = b^*_\theta$, and

(3) For all $\rho \in b^*_\theta$, also $b^*_\rho \subseteq b^*_\theta$.

Now $b_\lambda \in J_{\leq\lambda}[\mathrm{pcf}(A)]$, so $\mathrm{pcf}(b_\lambda) \subseteq \lambda^+$. Applying 11.13 to $\mathrm{pcf}(A)$ in place of A and $\mathrm{pcf}(b_\lambda)$ in place of X, we get a finite subset F of $\mathrm{pcf}(b_\lambda) \cap \lambda$ such that $\mathrm{pcf}(b_\lambda) \subseteq \bigcup_{\mu\in F} b^*_\mu \cup b_\lambda$. Let $b^*_\lambda = \bigcup_{\mu\in F} b^*_\mu \cup b_\lambda$. Clearly $\mathrm{pcf}(b^*_\lambda) \subseteq \lambda^+$. Now

$$b_\lambda = \left(b_\lambda \cap \bigcup_{\mu\in F} b^*_\mu\right) \cup \left(b^*_\lambda \backslash \bigcup_{\mu\in F} b^*_\mu\right),$$

so $b_\lambda \in J_{<\lambda}[\mathrm{pcf}(A)] + b^*_\lambda$. It follows that b^*_λ is a generator for λ, proving (1) for λ. For (2), note that

$$\begin{aligned}
\mathrm{pcf}(b^*_\lambda) &= \mathrm{pcf}\left(\bigcup_{\mu\in F} b^*_\mu \cup b_\lambda\right) \\
&= \bigcup_{\mu\in F} \mathrm{pcf}(b^*_\mu) \cup \mathrm{pcf}(b_\lambda) \\
&= \bigcup_{\mu\in F} b^*_\mu \cup b_\lambda \cup \mathrm{pcf}(b_\lambda) \\
&= \bigcup_{\mu\in F} b^*_\mu \cup b_\lambda \\
&= b^*_\lambda.
\end{aligned}$$

So (2) holds for λ.

Finally, we prove by induction on ρ that if $\rho \in b^*_\lambda$ then $b^*_\rho \subseteq b^*_\lambda$. Assume that this implication is true for all $\nu < \rho$. Now suppose that $\rho \in b^*_\lambda$. If $\rho \in b^*_\mu$ for some $\mu \in G$, then $b^*_\rho \subseteq b^*_\mu \subseteq b^*_\lambda$ by the inductive hypothesis on λ. Suppose that $\rho \in b_\lambda$. Then $b_\rho \subseteq b_\lambda \subseteq b^*_\lambda$. Now by construction, there is a finite $G \subseteq \mathrm{pcf}(b_\rho) \cap \rho$ such that $b^*_\rho = \bigcup_{\nu \in G} b^*_\nu \cup b_\rho$. For each $\nu \in G$ we have $\nu \in \mathrm{pcf}(b_\rho) \subseteq \mathrm{pcf}(b_\lambda) \subseteq b^*_\lambda$, and so by the inductive hypothesis on ρ, $b^*_\nu \subseteq b^*_\lambda$. It follows that $b^*_\rho \subseteq b^*_\lambda$, as desired in (3) for λ. □

Theorem 18.4. *Assume that A is a progressive interval of regular cardinals and $|A|^{+3} < \min(A)$. Let $\langle b_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ be a transitive system of pcf-closed generators for $\mathrm{pcf}(A)$, as in 18.3, with $b_{\max(\mathrm{pcf}(A))} = \mathrm{pcf}(A)$.*

Then the topological space $\mathrm{pcf}(A)$, with topology given in Lemma 18.2, is compact Hausdorff and totally disconnected (i.e., zero-dimensional). Moreover, each b_λ is clopen, and $\{b_\lambda : \lambda \in \mathrm{pcf}(A)\}$ generates the Boolean algebra of clopen subsets of $\mathrm{pcf}(A)$. This Boolean algebra is superatomic.

Proof. We prove this in several steps.

(1) If $\lambda \in \mathrm{pcf}(A)$, then $\lambda = \max b_\lambda$.

This is true since $\lambda = \max(\mathrm{pcf}(b_\lambda)) = \max b_\lambda$ by 11.9(vii).

(2) b_λ is clopen for each $\lambda \in \mathrm{pcf}(A)$.

In fact, it suffices to show that $\mathrm{pcf}(\mathrm{pcf}(A)\backslash b_\lambda) \subseteq \mathrm{pcf}(A)\backslash b_\lambda$. So, suppose that $\mu \in \mathrm{pcf}(\mathrm{pcf}(A)\backslash b_\lambda)$, but suppose also that $\mu \in b_\lambda$. Then $b_\mu \subseteq b_\lambda$. Since $\mu \in \mathrm{pcf}(\mathrm{pcf}(A)\backslash b_\lambda)$, let D be an ultrafilter on $\mathrm{pcf}(A)$ such that $\mathrm{pcf}(A)\backslash b_\lambda \in D$ and $\mu = \mathrm{cf}(\prod \mathrm{pcf}(A)/D)$. Since $b_\mu \subseteq b_\lambda$, we have $\mathrm{pcf}(A)\backslash b_\mu \in D$, in contradiction with 11.9(ii). so (2) holds.

(3) The topology is Hausdorff.

For, suppose that λ, μ are distinct elements of $\mathrm{pcf}(A)$. Say $\mu < \lambda$. By (1), $\lambda \in \mathrm{pcf}(A)\backslash b_\mu$. Thus (1) and (2) imply that b_μ and $\mathrm{pcf}(A)\backslash b_\mu$ are disjoint open neighborhoods of μ, λ respectively.

Now let B be the set of all finite intersections of members of $\{b_\lambda : \lambda \in \mathrm{pcf}(A)\}$ and their complements.

(4) The nonzero members of B form a base for the topology on $\mathrm{pcf}(A)$.

For, suppose that $U \subseteq \mathrm{pcf}(A)$ is open and $\lambda \in U$. We claim that the following set does *not* have fip:

$$(*) \qquad \{b_\mu : \lambda \in b_\mu\} \cup \{\mathrm{pcf}(A)\backslash b_\mu : \lambda \notin b_\mu\} \cup \{\mathrm{pcf}(A)\backslash U\}.$$

For, suppose it does have fip; extend it to an ultrafilter D on $\mathrm{pcf}(A)$, and let $\mu = \mathrm{cf}(\prod \mathrm{pcf}(A)/D)$. By 11.9(i), $b_\mu \in D$. Hence by the definition of D we get $\lambda \in b_\mu$. Since $\mu = \max(b_\mu)$ by (1), it follows that $\lambda \leq \mu$. Now $\lambda = \max b_\lambda$ by (1), so $\lambda \in b_\lambda$, and so $b_\lambda \in D$. Now $b_\lambda \in J_{\leq\lambda}[\mathrm{pcf}(A)]$, so $\mathrm{pcf}(b_\lambda) \subseteq \lambda^+$. Since $b_\lambda \in D$, it follows that $\mu \leq \lambda$. So $\lambda = \mu$. But also $\mathrm{pcf}(A)\backslash U \in D$, so $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A)\backslash U) = \mathrm{pcf}(A)\backslash U$ since U is open, contradiction. So (4) holds.

(5) pcf(A) is compact.

Let $\mathscr{A}$ be an open cover of pcf(A). We prove by induction on $\lambda \in \mathrm{pcf}(A)$ that b_λ is covered by a finite subset of $\mathscr{A}$. Since $\mathrm{pcf}(A) = b_{\max(\mathrm{pcf}(A))}$, this will prove (5). So suppose that this is true for all $\mu < \lambda$. Choose $U \in \mathscr{A}$ such that $\lambda \in U$. Then by (4), let c be a member of B such that $\lambda \in c \subseteq U$. Now we apply 11.13 to $\mathrm{pcf}(A), b_\lambda \backslash c$ in place of A, X. Using the fact that $b_\lambda \backslash c$ is closed, we get a finite subset N of $b_\lambda \backslash c$ such that $b_\lambda \backslash c \subseteq \bigcup_{\mu \in N} b_\mu$. Now $\lambda \in c$, so $\lambda \notin b_\lambda \backslash c$. also, $b_\lambda = \mathrm{pcf}(b_\lambda) \subseteq \lambda^+$, so $b_\lambda \backslash c \subseteq \lambda$. Hence by the inductive hypothesis, for each $\mu \in N$ there is a finite subset $\mathscr{A}_\mu$ of $\mathscr{A}$ such that $b_\mu \subseteq \bigcup \mathscr{A}_\mu$. Hence

$$b_\lambda \subseteq U \cup \bigcup_{\mu \in N} \bigcup \mathscr{A}_\mu,$$

finishing the inductive proof.

Thus we have now proved the first part of the theorem: pcf(A) is a compact totally disconnected Hausdorff space. Moreover, by (4) each member of B is clopen. If U is any clopen set, then it is compact, and so by (4) it is a finite union of members of B. This shows that the Boolean algebra of clopen subsets of pcf(A) is generated by $\{b_\lambda : \lambda \in \mathrm{pcf}(A)\}$.

It remains only to show that this Boolean algebra is superatomic. By duality, it suffices to show that any nonempty closed subset F has an isolated point. Let λ be the least member of F. Then λ is the greatest element of b_λ, so b_λ is an open set such that $b_\lambda \cap F = \{\lambda\}$, as desired. $\square$