7. Morley's theorem

This chapter is devoted to the proof of Morley's theorem, which says that in a countable language, if Γ is a theory with only infinite models and Γ is κ-categorical for some uncountable cardinal κ, then it is κ-categorical for every uncountable cardinal κ. In the course of developing the proof we will introduce several new model-theoretic concepts. We follow Marker, **Model theory, an introduction.**

Unless otherwise mentioned, T is a complete theory in a countable language having only infinite models.

Some useful notation is as follows. If $\overline{M}$ is a structure and $\varphi(\overline{v})$ is a formula with parameters in M, with $\overline{v}$ of length m, then by $\varphi(\overline{M})$ we mean the set $\{\overline{a} \in {}^mM : \overline{M} \models \varphi(\overline{a})\}$. Here we make a slight abuse of notation, in that we write $\overline{M} \models \varphi(\overline{a})$ when we should write something like $(\overline{M}, \overline{b}) \models \varphi(\overline{v}, \overline{w})[\overline{a}, \overline{b}]$, where $\overline{b}$ is the sequence of parameters in φ. Similar abuses will take place later without comment.

Let $\overline{M}$ be an $\mathscr{L}$-structure, and suppose that $A \cup \{b\} \subseteq M$. We say that *b is algebraic over A* iff there is a formula $\varphi(x, \overline{a})$ with $\overline{a} \in A$ such that $\varphi(\overline{M}, \overline{a})$ is finite and $\overline{M} \models \varphi(b, \overline{a})$. Now if $A \subseteq D \subseteq M$ we define

$$\mathrm{acl}_D(A) = \{b \in D : b \text{ is algebraic over } A\}.$$

Lemma 7.1. *(i) $A \subseteq \mathrm{acl}_D(A)$.*
(ii) $\mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$.
(iii) If $A \subseteq B$ then $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(B)$.
(iv) If $a \in \mathrm{acl}_D(A)$, then $a \in \mathrm{acl}_D(A_0)$ for some finite $A_0 \subseteq A$.

Proof. (i): For any $a \in A$, let $\varphi(x, a)$ be the formula $x = a$.

(ii): Suppose that $b \in \mathrm{acl}_D(\mathrm{acl}_D(A))$. Accordingly, choose $\varphi(v, \overline{w})$ and $\overline{a} \in \mathrm{acl}_D(A)$ such that

$$\overline{M} \models \varphi(b, \overline{a}) \text{ and } \{y \in M : \overline{M} \models \varphi(y, \overline{a})\} \text{ is finite.}$$

Say $\overline{a}$ has length n. Then for all $i < n$ we get $\psi_i(v, \overline{u}^i)$ and $\overline{c}^i \in A$ such that

$$\overline{M} \models \psi_i(a_i, \overline{c}^i) \text{ and } \{y \in M : \overline{M} \models \psi_i(y, \overline{c}^i)\} \text{ is finite.}$$

Let $k = |\{y \in M : \overline{M} \models \varphi(y, \overline{a})\}|$. Now let $\chi(v, \overline{u}^0, \ldots, \overline{u}^{n-1})$ be the formula

$$\exists v_0 \ldots v_{n-1} \left[\bigwedge_{i<n} \psi_i(v_i, \overline{u}^i) \wedge \varphi(v, v_0, \ldots, v_{n-1}) \wedge (\exists! k) v_j \varphi(v_j, v_0, \ldots, v_{n-1}) \right].$$

Here $(\exists! k)v_j \ldots$ abbreviates "there are exactly k v_j such that $\ldots$", which is easy to express in our language.

Now we want to show that $\overline{M} \models \chi(b, \overline{c}^0, \ldots, \overline{c}^{n-1})$. For any $i < n$ we have $\overline{M} \models \psi_i(a_i, \overline{c}^i)$, and so

$$\overline{M} \models \bigwedge_{i<n} \psi_i(a_i, \overline{c}^i) \wedge \varphi(b, \overline{a}) \wedge (\exists! k) v_j \varphi(v_j, \overline{a}).$$

Hence $\overline{M} \models \chi(b, \overline{c}^0, \ldots, \overline{c}^{n-1})$.

Next we want to show that $\{y \in M : \overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$ is finite. Let

$$K = \prod_{i<m} \{y \in M : \overline{M} \models \psi_i(y, \overline{c}^i)\}.$$

Thus K is finite. Suppose that $\overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$. Choose $\overline{e}$ such that

$$\overline{M} \models \bigwedge_{i<n} \psi_i(e_i, \overline{c}^i) \wedge \varphi(y, \overline{e}) \wedge (\exists! k) v_j \varphi(v_j, \overline{e}).$$

Then $\overline{e} \in K$. Hence there are at most $|K| \cdot k$ elements y such that $\overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$.

(iii) and (iv) are clear. $\square$

If $\overline{M}$ is a structure, m is a positive integer, and $D \subseteq {}^m M$, then we say that D is *definable with parameters* iff there is a formula $\varphi(\overline{v})$ in $\mathscr{L}_M$ with $\overline{v}$ of length m such that $D = \{a \in {}^m M : \overline{M}_M \models \varphi(a)\}$.

A subset D of M^n is *minimal* in $\overline{M}$ iff D is infinite, and for any set $Y \subseteq D$ definable with parameters, either Y is finite or $D \backslash Y$ is finite. In case $\varphi(\overline{v}, \overline{a})$ defines D, we also say that φ is minimal.

Lemma 7.2. *Suppose that $D \subseteq M$ is definable and minimal in $\overline{M}$ and $A \cup \{a, b\} \subseteq D$. Suppose that $a \in \mathrm{acl}_D(A \cup \{b\}) \backslash \mathrm{acl}_D(A)$. Then $b \in \mathrm{acl}_D(A \cup \{a\})$.*

Proof. Assume the hypotheses. Thus there is a formula $\varphi(a, b)$ with additional parameters from A, and a positive integer n, such that $\overline{M} \models \varphi(a, b)$ and $|\{x \in D : \overline{M} \models \varphi(x, b)\}| = n$. Let $\psi(w)$ be the formula with parameters from A asserting that $|\{x \in D : \varphi(x, w)\}| = n$. If $\psi(w)$ defines a finite subset of D, then $b \in \mathrm{acl}_D(A)$. Hence $A \cup \{b\} \subseteq \mathrm{acl}_D(A)$, hence by Lemma 7.1 $a \in \mathrm{acl}_D(A \cup \{b\}) \subseteq \mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$, contradiction. It follows that $\psi(w)$ defines a cofinite subset of D.

If $\{y \in D : \overline{M} \models \varphi(a, y) \wedge \psi(y)\}$ is finite then since b is in this set we get $b \in \mathrm{acl}_D(A \cup \{a\})$, as desired. Thus we may assume that $\{y \in D : \overline{M} \models \varphi(a, y) \wedge \psi(y)\}$ is cofinite in D; say that its complement has size l. Let $\chi(x)$ be the formula expressing that

$$|D \backslash \{y \in D : \varphi(x, y) \wedge \psi(y)\}| = l.$$

since $\overline{M} \models \chi(a)$, our assumption that $a \notin \mathrm{acl}_D(A)$ implies that $\chi(\overline{M})$ is cofinite. Let $a_0, \ldots, a_n$ be distinct members of $\chi(\overline{M})$. Then for each $i \leq n$ the set $B_i \stackrel{\text{def}}{=} \{y \in D : \overline{M} \models \varphi(a_i, y) \wedge \psi(y)\}$ is cofinite. Let $c \in \bigcap_{i \leq n} B_i$. Thus $\varphi(a_i, c)$ for each $i \leq n$, so $|\{x \in D : \overline{M} \models \varphi(x, c)\}| \geq n + 1$, contradicting the choice of $\psi(c)$. $\square$

Suppose that $D \subseteq M^n$. We say that D is *strongly minimal* in $\overline{M}$ iff D is minimal in any elementary extension of $\overline{M}$. Similarly for a formula φ.

Given $A \subseteq D$, we call A *independent* iff $\forall a \in A[a \notin \mathrm{acl}_D(A \backslash \{a\})]$. For $C \subseteq D$ we say that A is *independent over* C iff $\forall a \in A[a \notin \mathrm{acl}_D(C \cup (A \backslash \{a\}))$. Note then that $A \cap C = \emptyset$.

For $\overline{a}$ a sequence of elements of M and $A \subseteq M$ we define

$$\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \{\varphi(\overline{v}) : \varphi \text{ is a formula with parameters from } A \text{ and } \overline{M} \models \varphi(\overline{a})\}.$$

Note that if $\overline{a}$ is the empty sequence, then $\mathrm{tp}(\overline{a}/A)$ is simply the set of all sentences with parameters from A that hold in $\overline{M}$. If A is empty, we just omit it.

Lemma 7.3. *Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:*
(i) $A = \emptyset$.
(ii) $A \subseteq \overline{M}_0 \prec \overline{M}, \overline{N}$.
Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from A, $n \in \omega$, $a \in {}^n\varphi(\overline{M})$, $\mathrm{rng}(\overline{a})$ is independent over A, and $b \in {}^n\varphi(\overline{N})$, $\mathrm{rng}(\overline{b})$ is independent over A. Then $\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \mathrm{tp}^{\overline{N}}(\overline{b}/A)$.

Proof. Induction on n. For $n = 0$ the conclusion is clear if (i) holds, since $\overline{M} \equiv \overline{N}$. The conditions in (ii) also clearly give the conclusion.

Now assume the result for n, and suppose that $a \in {}^{n+1}\varphi(\overline{M})$, $\mathrm{rng}(\overline{a})$ is independent over A, $b \in {}^{n+1}\varphi(\overline{N})$, and $\mathrm{rng}(\overline{b})$ is independent over A. So by the inductive hypothesis,

$$\mathrm{tp}^{\overline{M}}((\overline{a} \restriction n)/A) = \mathrm{tp}^{\overline{N}}((\overline{b} \restriction n)/A). \tag{1}$$

Let $\psi(\overline{v})$ be a formula with parameters from A such that $\overline{M} \models \psi(\overline{a})$. Now $a_n \in \varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ and $a_n \notin \mathrm{acl}_D(A \cup \{a_0, \ldots, a_{n-1}\})$, so $\varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ is infinite. Since φ is strongly minimal, this set is actually cofinite in $\varphi(\overline{M})$. So there is an integer m such that

$$\overline{M} \models |\{v : \varphi(v) \wedge \neg\psi(a_0, \ldots, a_{n-1}, v\}| = m.$$

Thus the formula $\chi(\overline{w})$ expressing that

$$|\{v : \varphi(v) \wedge \neg\psi(w_0, \ldots, w_{n-1}, v\}| = m$$

is in $\mathrm{tp}^{\overline{M}}((\overline{a} \restriction n)/A)$, and hence by (1) we get

$$\overline{N} \models |\{v : \varphi(v) \wedge \neg\psi(b_0, \ldots, b_{n-1}, v\}| = m.$$

Since $b_n \notin \mathrm{acl}_D(A \cup \{b_0, \ldots, b_{n-1}\})$, it follows that $\overline{N} \models \psi(\overline{b})$, as desired. $\square$

If X is an infinite subset of M, then X *is an indiscernible set* over $\overline{M}$ iff for any formula $\varphi(\overline{v})$ and any two sequences $\overline{x}, \overline{y}$ of distinct elements of X we have $\overline{M} \models \varphi(\overline{x}) \leftrightarrow \varphi(\overline{y})$.

Corollary 7.4. *Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:*
(i) $A = \emptyset$.
(ii) $A \subseteq \overline{M}_0 \prec \overline{M}, \overline{N}$.
Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from A, and B and C are infinite subsets of $\varphi(\overline{M})$ each independent over A. Then B and C are sets of indiscernibles over $\overline{M}$, and for any $n \in \omega$ and one-one sequences $\overline{b} \in {}^nB$ and $\overline{c} \in {}^nC$ we have $\mathrm{tp}(\overline{b}/A) = \mathrm{tp}(\overline{c}/A)$. $\square$

If $Y \subseteq D$, we say that $A \subseteq Y$ is a *basis* for Y iff A is independent and $\mathrm{acl}_D(A) = \mathrm{acl}_D(Y)$.

Lemma 7.5. *Assume that $D \subseteq M$ is minimal in $\overline{M}$.*

(i) A union of a chain of independent sets over a set $C \subseteq D$ is again independent over C. (Hence we can apply Zorn's lemma in this context.)

(ii) For any $Y \subseteq D$, any maximal independent subset subset of Y is a basis for Y.

Proof. (i): Let $\mathscr{A}$ be a chain of independent sets over C. Suppose that $a \in \mathrm{acl}_D(C \cup (\bigcup \mathscr{A} \backslash \{a\}))$. Thus a is algebraic over $C \cup (\bigcup \mathscr{A} \backslash \{a\})$, and so there is a formula $\varphi(x, \overline{c}, \overline{b})$ with $\overline{c} \in C$ and $\overline{b} \in \bigcup \mathscr{A} \backslash \{a\}$ such that $\varphi(\overline{M}, \overline{c}, \overline{b})$ is finite and $\overline{M} \models \varphi(a, \overline{c}, \overline{b})$. Then there is an $X \in \mathscr{A}$ such that $\overline{b} \in X$, so that $a \in \mathrm{acl}_D(C \cup (X \backslash \{a\}))$, contradiction.

(ii): Suppose that A is a maximal independent subset of Y. Obviously $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(Y)$. Suppose that $a \in \mathrm{acl}_D(Y) \backslash \mathrm{acl}_D(A)$. Let $\varphi(x, \overline{y})$ be a formula with $\overline{y} \in Y$ such that $\varphi(\overline{M}, \overline{y})$ is finite and $\overline{M} \models \varphi(a, \overline{y})$. If each $y_i \in \mathrm{acl}_D(A)$, then $a \in \mathrm{acl}_D(\mathrm{rng}(\overline{y})) \subseteq \mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$, contradiction. So there is an i such that $y_i \notin \mathrm{acl}_D(A)$. If $b \in A$ and $b \in \mathrm{acl}_D(\{y_i\} \cup (A \backslash \{b\}))$, then $b \notin \mathrm{acl}_D(A) \backslash \{b\})$ by independence, and so $y_i \in \mathrm{acl}_D(A)$ by Lemma 7.2, contradiction. Hence $A \cup \{y_i\}$ is independent, contradiction. □

Lemma 7.6. *Let D be strongly minimal over $\overline{M}$. Then:*

(i) Let $A, B \subseteq D$ be independent with $A \subseteq \mathrm{acl}_D(B)$. Then:

(a) Suppose that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cup B_0$ is a basis for $\mathrm{acl}_D(B)$, and $a \in A \backslash A_0$. Then there is a $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is a basis for $\mathrm{acl}_D(B)$.

(b) $|A| \leq |B|$.

(ii) If A and B are bases for $Y \subseteq D$, then $|A| = |B|$.

Proof. (i): Assume the hypotheses. (a): Assume the hypotheses. Then $a \in A \subseteq \mathrm{acl}_D(B) = \mathrm{acl}_D(A_0 \cup B_0)$, so by Lemma 7.1(iv) there is a finite $X \subseteq A_0 \cup B_0$ such that $a \in \mathrm{acl}_D(X)$. Let $C \subseteq B_0$ be of smallest size such that $a \in \mathrm{acl}_D(A_0 \cup C)$. Thus C is finite, and $A_0 \cap C = \emptyset$ by the minimality of C. Since A is independent and $a \notin A_0$, we have $C \neq \emptyset$. Fix $b \in C$. Now $a \in \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}) \cup \{b\}) \backslash \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}))$, so by Lemma 7.2, $b \in \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}) \cup \{a\})$. Hence $A_0 \cup B_0 \subseteq \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}))$, and hence

$$\begin{aligned}\mathrm{acl}_D(B) &= \mathrm{acl}_D(A_0 \cup B_0) \\ &\subseteq \mathrm{acl}_D(\mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}))) \\ &= \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\})) \\ &\subseteq \mathrm{acl}_D(B).\end{aligned}$$

Thus $\mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\})) = \mathrm{acl}_D(B)$. We claim that $X \stackrel{\mathrm{def}}{=} A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is independent. For, suppose that $x \in X$ and $x \in \mathrm{acl}_D(X \backslash \{x\})$.

Case 1. $x = a$. Thus $a \in \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, hence $A_0 \cup \{a\} \cup (B_0 \backslash \{b\}) \subseteq \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, hence $b \in \mathrm{acl}_D(B) = \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}) \subseteq \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, contradicting the fact that $A_0 \cup B_0$ is independent. (Recall that $A_0 \cap C = \emptyset$, hence $b \notin A_0$.)

Case 2. $x \neq a$. Now $X \backslash \{x\} = \{a\} \cup (A_0 \cup (B_0 \backslash \{b\})) \backslash \{x\})$ and $x \notin \mathrm{acl}_D((A_0 \cup (B_0 \backslash \{b\})) \backslash \{x\})$ by the independence of $A_0 \cup B_0$. So by Lemma 2 we get $a \in \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\}))$, i.e., Case 1, contradiction.

(b): *Case 1.* B is finite; say $|B| = n$. Suppose that $a_0, \ldots, a_n$ are distinct elements of A. We now define distinct elements b_i of B for $i < n$ by recursion. Suppose they have been defined for all $j < i$, where $0 \leq i < n-1$, so that $\{a_j : j < i\} \cup (B \backslash \{b_j : j < i\}$ is a basis for $\mathrm{acl}_D(B)$. Since $a_i \in A\backslash\{a_j : j < i\}$, we can apply (a) to obtain b_i such that $\{a_j : j \leq i\} \cup (B\backslash\{b_j : j \leq i\}$ is a basis for $\mathrm{acl}_D(B)$.

It follows that $\{a_j : j < n\}$ is a basis for $\mathrm{acl}_D(B)$. Hence $a_n \in \mathrm{acl}_D(\{a_j : j < n\})$, contradicting A independent.

Thus we must have $|A| \leq |B|$.

Case 2. B is infinite. By Case 1, $|A \cap \mathrm{acl}(B_0)| \leq |B_0|$ for each finite subset B_0 of B. Now

$$A \subseteq \bigcup_{\substack{B_0 \subseteq B,\\ B_0 \text{ finite}}} (A \cap \mathrm{acl}(B_0),]$$

so clearly $|A| \leq |B|$.

(ii) follows from (i)(b). □

For D strongly minimal, the *dimension* of D, $\dim(D)$, is the cardinality of a basis for D.

Lemma 7.7. *If D is strongly minimal and uncountable, then* $\dim(D) = |D|$.

Proof. Since the language is countable, also $\mathrm{acl}(B)$ is countable for every finite subset B of D. If $X \subseteq D$ and $|X| < |D|$, then

$$|\mathrm{acl}(X)| \leq \left| \bigcup_{\substack{B \subseteq X,\\ B \text{ finite}}} \mathrm{acl}(B) \right| \leq |X| \cdot \omega < |D|. \qquad \square$$

Let $\overline{a}$ be a sequence of elements of M and $A \subseteq M$. We say that $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is *isolated* if there is a formula $\varphi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ such that for every formula $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ we have $\overline{M} \models \forall\overline{v}[\varphi(\overline{v}) \to \chi(\overline{v})]$.

Lemma 7.8. *If $A \cup \{b\} \subseteq M$ and b is algebraic over A, then $\mathrm{tp}^{\overline{M}}(b/A)$ is isolated.*

Proof. Let $\overline{a} \in A$ and $\varphi(v, \overline{a})$ be such that $\overline{M} \models \varphi(b, \overline{a})$ and $\{y \in M : \overline{M} \models \varphi(y, \overline{a})\}$ is finite. Let

$$\begin{aligned} B = \{d \in M : \overline{M} \models \varphi(d, \overline{a}) &\text{ and there exist a formula } \psi(v, \overline{c}) \text{ with } \overline{c} \in A \\ &\text{such that } \overline{M} \models \psi(b, \overline{c}) \text{ and } \overline{M} \models \neg\psi(d, \overline{c})\}. \end{aligned}$$

Note that B is finite. For each $d \in B$, choose ψ_d and $\overline{c}_d$ as indicated. Let $\varphi'(v, \overline{e})$ be the formula

$$\varphi(v, \overline{a}) \wedge \bigwedge_{d \in B} \psi_d(v, \overline{c}_d).$$

Thus $\overline{M} \models \varphi'(b, \overline{e})$, and so $\varphi'(v, \overline{e}) \in \mathrm{tp}^{\overline{M}}(b/A)$. Now suppose that $\chi(v, \overline{u}) \in \mathrm{tp}^{\overline{M}}(b/A)$, but there is a $d \in M$ such that $\overline{M} \models \varphi'(d, \overline{e}) \wedge \neg\chi(d, \overline{u})$; we want to get a contradiction. We have $\overline{M} \models \varphi(d, \overline{a})$, so it follows that $d \in B$, hence $\overline{M} \models \neg\psi_d(d, \overline{c})$; but this contradicts $\overline{M} \models \varphi'(d, \overline{e})$. □

Lemma 7.9. *Suppose that $\overline{M}, \overline{N} \models T$, $\varphi(v)$ is strongly minimal, and* $\dim(\varphi(\overline{M})) = \dim(\varphi(\overline{N}))$. *Then there is a bijection $f : \varphi(\overline{M}) \to \varphi(\overline{N})$ such that for every formula $\psi(\overline{w})$ and every $\overline{a} \in \varphi(\overline{M})$, $\overline{M} \models \psi(\overline{a})$ iff $\overline{N} \models \psi(f \circ \overline{a})$.*

Proof. Assume the hypotheses. Let B be a base for $\varphi(\overline{M})$, and let C be a base for $\varphi(\overline{N})$. Thus $|B| = |C|$, and we let $h : B \to C$ be a bijection. Let

$$\begin{aligned} I = \{g : g : B' \to C' \text{ is a surjection, } B \subseteq B' \subseteq \varphi(\overline{M}), C \subseteq C' \subseteq \varphi(\overline{N}) \text{ and} \\ \forall \chi \forall \overline{a} \in B'[\overline{M} \models \chi(\overline{a}) \leftrightarrow \overline{N} \models \chi(g \circ \overline{a})]\}. \end{aligned}$$

Note that every $g \in I$ is injective; consider the formula $x \neq y$. Now $h \in I$, since for any $\chi(\overline{w})$ and any $\overline{a} \in B$,

$$\begin{aligned} \overline{M} \models \chi(\overline{a}) \quad &\text{iff} \quad \chi(\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a}) \\ &\text{iff} \quad \chi(\overline{w}) \in \mathrm{tp}^{\overline{N}}(h \circ \overline{a}) \quad \text{by Corollary 7.4} \\ &\text{iff} \quad \overline{N} \models \chi(h \circ \overline{a}). \end{aligned}$$

Clearly we can apply Zorn's lemma to I and obtain a maximal member g of it, with associated sets B', C'. We claim that $\mathrm{dmn}(g) = \varphi(\overline{M})$ and $\mathrm{rng}(g) = \varphi(\overline{N})$. By symmetry we prove only that $\mathrm{dmn}(g) = \varphi(\overline{M})$. In fact, suppose that this is not true. Let $b \in \varphi(\overline{M}) \backslash B'$. Since $\mathrm{acl}(B) = \varphi(\overline{M})$, we also have $\mathrm{acl}(B') = \varphi(\overline{M})$, and so $b \in \mathrm{acl}(B')$. Hence by Lemma 7.8 let $\psi(v, \overline{c}) \in \mathrm{tp}^{\overline{M}}(b/B')$ isolate $\mathrm{tp}^{\overline{M}}(b/B')$, where $\overline{c} \in B'$. Now $\overline{M} \models \exists x \psi(x, \overline{c})$, so from $g \in I$ we get $\overline{N} \models \exists x \psi(x, g \circ \overline{c})$. Say $\overline{N} \models \psi(d, g \circ \overline{c})$. Extend g to $g' : B' \cup \{b\} \to C' \cup \{d\}$ by setting $g'(b) = d$. So g' is a surjection from $B' \cup \{b\}$ to $C' \cup \{d\}$. Now take any formula $\chi(v, \overline{w})$ and any $\overline{e} \in B'$. Then

$$\begin{aligned} \overline{M} \models \chi(b, \overline{e}) \quad &\Rightarrow \quad \chi(v, \overline{e}) \in \mathrm{tp}^{\overline{M}}(b) \\ &\Rightarrow \quad \overline{M} \models \forall v[\psi(v, \overline{c}) \to \chi(v, \overline{e}) \\ &\Rightarrow \quad \overline{N} \models \forall v[\psi(v, g \circ \overline{c}) \to \chi(v, g \circ \overline{e}) \\ &\Rightarrow \quad \overline{N} \models \chi(d, g \circ \overline{e}); \end{aligned}$$

this shows that $g' \in I$, contradiction. $\square$

A theory T is *strongly minimal* iff the formula $v = v$ is strongly minimal for each model $\overline{M}$ of T.

For each infinite cardinal κ, $I(T, \kappa)$ is the number of nonisomorphic models of T of size κ.

Theorem 7.10. *Suppose that T is strongly minimal.*
(i) If $\overline{M}, \overline{N} \models T$, then $\overline{M} \cong \overline{N}$ iff $\dim(\overline{M}) = \dim(\overline{N})$.
(ii) T is κ-categorical for each uncountable cardinal κ.
(iii) $I(T, \omega) \leq \omega$.

Proof. (i) is immediate from Lemma 7.9. (ii) follows from (i) by Lemma 7.7. (iii) follows from (i) since $\dim(\overline{M}) \leq \omega$ for any countable model $\overline{M}$ of T. $\square$

A set Γ of formulas is *finitely satisfiable* in $\overline{M}$ iff for every finite subset Δ of Γ there is an $a \in {}^{\omega}M$ such that $\overline{M} \models \varphi[a]$ for all $\varphi \in \Delta$. For any model $\overline{M}$ of T, any subset A of M, and any positive integer n, an *n-type* over $\overline{M}$ is a set of formulas with free variables among $v_0, \ldots, v_{n-1}$ and with parameters from A which is finitely satisfiable over $\overline{M}$. It is a *complete* n-type iff for any formula φ with free variables among $v_0, \ldots, v_{n-1}$ and parameters from A, either φ or $\neg\varphi$ is a member of it. We let $S_n^{\overline{M}}(A)$ be the set of all complete n-types over A with respect to $\overline{M}$. Note that $|S_n^{\overline{M}}(A)| \le 2^{\max(\omega, \mathrm{A})}$. T is *κ-stable* iff for every $\overline{M} \models T$, every $A \subseteq M$ of size κ, and every positive integer n we have $|S_n^{\overline{M}}(A)| = \kappa$.

Lemma 7.11. *If T is ω-stable and $\overline{M} \models T$, then there is a minimal formula for $\overline{M}$.*

Proof. Suppose not. We define formulas φ_f for each $f \in {}^{<\omega}2$ by induction on $\mathrm{dmn}(f)$. Let $\varphi_\emptyset$ be the formula $v = v$. Now suppose that φ_f has been defined so that $\varphi_v(\overline{M})$ is infinite. Since φ_f is not minimal, there is a formula ψ with parameters such that $\varphi_f(\overline{M}) \cap \psi(\overline{M})$ and $\varphi_f(\overline{M}) \wedge \neg\psi(\overline{M})$ are infinite. We let $\varphi_{f^\frown\langle 0\rangle}$ be $\varphi_f \wedge \psi$ and $\varphi_{f^\frown\langle 1\rangle}$ be $\varphi_f \wedge \neg\psi$.

Let A be the set of all parameters appearing in any formula φ_f for $f \in {}^{<\omega}2$. So A is countable. For each $f \in {}^{\omega}2$ the set

$$\{\varphi_{f\restriction n} : n \in \omega\}$$

is finitely satisfiable in $\overline{M}$ and hence is contained in a complete type t_f over $\overline{M}$. This gives 2^ω complete types over A, contradicting ω-stability. $\square$

Lemma 7.12. *If $\overline{M}$ is ω-saturated and $\varphi(\overline{v}, \overline{a})$ is a minimal formula in $\overline{M}$, then $\varphi(\overline{v}, \overline{a})$ is a strongly minimal.*

Proof. Suppose not. Let $\overline{M} \prec \overline{N}$ with $\psi({}^n\overline{N}, \overline{b})$ an infinite and coinfinite subset of $\varphi({}^n\overline{N}, \overline{a})$, where $\overline{b} \in N$. Then $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a})$ is a complete type in $\overline{N}$, hence it is finitely satisfiable in $\overline{N}$, so it is finitely satisfiable in $\overline{M}$. Thus it is a complete type in $\overline{M}$ over $\overline{a}$. So by the ω-saturation of $\overline{M}$, it is satisfiable in $\overline{M}$, say by $\overline{b}'$. Thus $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b}'/\overline{a})$. Now for any positive integer p,

$$\overline{N} \models \exists_{\ge p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{b})],$$

hence

$$\exists_{\ge p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{w})] \in \mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}),$$

hence

$$\exists_{\ge p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{w})] \in \mathrm{tp}^{\overline{M}}(\overline{b}'/\overline{a}),$$

$$\overline{M} \models \exists_{\ge p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{b}')].$$

It follows that $\varphi({}^n\overline{M}) \cap \psi({}^n\overline{M})$ is infinite. Similarly, $\varphi({}^n\overline{M}) \cap \neg\psi({}^n\overline{M})$ is infinite, contradiction. $\square$

A *Vaughtian pair* for T is a pair $(\overline{M}, \overline{N})$ of models of T such that there is a formula $\varphi(\overline{v})$ such that $\overline{M} \prec \overline{N}$, $M \ne N$, $\varphi(\overline{M})$ is infinite, and $\varphi(\overline{M}) = \varphi(\overline{N})$.

Lemma 7.13. *Suppose that T does not have any Vaughtian pairs, $\overline{M} \models T$, and $\varphi(\overline{v}, \overline{w})$ is a formula with parameters from M, with $\overline{v}$ of length m and $\overline{w}$ of length k. Then there is a natural number n such that for all $\overline{a} \in M$, if $|\varphi(\overline{M}, \overline{a})| > n$, then $\varphi(\overline{M}, \overline{a})$ is infinite.*

Proof. Suppose not. For each $n \in \omega$ let $\overline{a}_n \in M$ be such that $\varphi(\overline{M}, \overline{a}_n)$ is finite, but of size $> n$.

Adjoin to the language a new one-place relation symbol U. Let Γ be the set of formulas of the following four types:

(1) $\forall \overline{x} \left[\bigwedge_{i<p} Ux_i \rightarrow [\psi \leftrightarrow \psi^U]\right]$, for each formula ψ with free variables among $\overline{x}$, where $\overline{x} = \langle x_i : i < p\rangle$, and ψ^U indicates relativization of quantifiers to U.

(2) $\exists x \neg Ux$.

(3) $\exists_{\geq s} \overline{v} \varphi(\overline{v}, \overline{w})$ for each $s \in \omega$.

(4) $\varphi(\overline{v}, \overline{w}) \rightarrow \bigwedge_{i<k} Uw_i$.

Now let $\overline{N}$ be a proper elementary extension of $\overline{M}$. For each $n \in \omega$ we have $\varphi(\overline{M}, \overline{a}_n) = \varphi(\overline{N}, \overline{a}_n)$, since $\varphi(\overline{M}, \overline{a}_n)$ is finite. Each finite subset of Γ is satisfiable in the structure $(\overline{N}, M)$. Hence by the compactness theorem we get an elementary extension $(\overline{N}', M')$ of $(\overline{N}, M)$ such that Γ is realizable in $(\overline{N}', M')$, say by $\overline{a}$. Let $\overline{M}'$ be the structure with universe M'. Then by (1), $\overline{N}'$ is an elementary extension of $\overline{M}'$, and it is a proper extension by (2). By (3), $\varphi(\overline{N}', \overline{a})$ is infinite, and by (4) we have $\varphi(\overline{N}', \overline{a}) \subseteq M'$, hence $\varphi(\overline{N}', \overline{a}) = \varphi(\overline{M}', \overline{a})$ by elementarity. Thus $(\overline{M}', \overline{N}')$ is a Vaughtian pair, contradiction. $\square$

Lemma 7.14. *If T has no Vaughtian pairs, then for every $\overline{M} \models T$ and every formula φ with parameters from $\overline{M}$, if φ is minimal for $\overline{M}$ then it is strongly minimal for $\overline{M}$.*

Proof. Suppose not. Let φ be $\varphi(\overline{v})$, with parameters from M. Then there is an elementary extension $\overline{N}$ of $\overline{M}$ and a formula $\psi(\overline{v}, \overline{b})$ with $\overline{b} \in N$ such that $\varphi(\overline{N}) \cap \psi(\overline{N}, \overline{b})$ and $\varphi(\overline{N}) \cap \neg\psi(\overline{N}, \overline{b})$ are infinite. By Lemma 7.13 applied twice, let $n \in \omega$ be such that for all $\overline{a} \in M$,

$$|\varphi(\overline{M}) \cap \psi(\overline{M}, \overline{a})| > n \rightarrow \varphi(\overline{M}) \cap \psi(\overline{M}, \overline{a}) \text{ is infinite, and}$$
$$|\varphi(\overline{M}) \cap \neg\psi(\overline{M}, \overline{a})| > n \rightarrow \varphi(\overline{M}) \cap \neg\psi(\overline{M}, \overline{a}) \text{ is infinite.}$$

Thus by the minimality of φ,

$$\overline{M} \models \forall \overline{w}[|\varphi(\overline{M}) \cap \psi(\overline{M}, \overline{w})| \leq n \vee |\varphi(\overline{M}) \cap \neg\psi(\overline{M}, \overline{w})| \leq n].$$

So this also holds in $\overline{N}$, and it follows that $\varphi(\overline{N}) \cap \psi(\overline{N}, \overline{b})$ is finite or $\psi(\overline{N}) \cap \neg\psi(\overline{N}, \overline{b})$ is finite, contradiction. $\square$

Corollary 7.15. *If T is ω-stable and has no Vaughtian pairs, then for every $\overline{M} \models T$ there is a strongly minimal formula over $\overline{M}$.* $\square$

Corollary 7.16. *If T has no Vaughtian pairs, $\overline{M} \models T$, and $\varphi(\overline{v})$ is a formula with parameters from M, and if $\varphi(\overline{M})$ is infinite, then no proper elementary submodel of $\overline{M}$ contains both $\varphi(\overline{M})$ and the parameters of $\varphi(\overline{v})$.*

Proof. Suppose that $\overline{N}$ is a proper elementary submodel of $\overline{M}$ which contains both $\varphi(\overline{M})$ and the parameters of $\varphi(\overline{v})$. Then for any $\overline{a} \in N$, $\overline{N} \models \varphi(\overline{a})$ implies that $\overline{M} \models \varphi(\overline{a})$ by elementarity. Conversely, if $\overline{M} \models \varphi(\overline{a})$ with $\overline{a} \in M$, then $\overline{a} \in N$ by assumption, so $\overline{N} \models \varphi(\overline{a})$ by elementarity. Thus $\varphi(\overline{M}) = \varphi(\overline{N})$. So $(\overline{M}, \overline{N})$ is a Vaughtian pair, contradiction. □

Lemma 7.17. *Suppose that T is ω-stable, $\overline{M} \models T$, $A \subseteq M$, $\varphi(\overline{v})$ is a formula with parameters from A, and $\overline{M} \models \exists\overline{v}\varphi(\overline{v})$. Then there is an $\overline{a} \in M$ such that $\varphi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ and $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Suppose that this does not hold. We construct formulas ψ_f for each $f \in {}^{<\omega}2$. Let $\psi_\emptyset = \varphi$. Suppose that we have constructed $\psi_f(\overline{v})$, a formula with parameters from A, so that

(*) $\overline{M} \models \exists\overline{v}\psi_f(\overline{v})$, and for all $\overline{a} \in M$, if $\varphi_f(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is not isolated.

This is true for $f = \emptyset$ by assumption. We claim

(**) There is a formula $\chi(\overline{v})$ with parameters from A such that $\overline{M} \models \exists\overline{v}[\psi_f(\overline{v}) \wedge \chi(\overline{v})]$ and $\overline{M} \models \exists\overline{v}[\psi_f(\overline{v}) \wedge \neg\chi(\overline{v})]$.

Suppose not. Take any $\overline{a}$ such that $\overline{M} \models \psi_f(\overline{a})$. Suppose that $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$. Now by (**) failing we have

$\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \chi(\overline{v})]$ or $\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \neg\chi(\overline{v})]$.

But $\overline{M} \models \chi(\overline{a})$ and $\overline{M} \models \psi_f(\overline{a})$, so it follows that $\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \chi(\overline{v})]$. This proves that $\psi_f(\overline{v})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, contradiction. Hence (**) holds. We take such a formula $\chi(\overline{v})$ and define $\psi_{f^\frown\langle 0\rangle}$ to be $\psi_f(\overline{v}) \wedge \chi(\overline{v})$ and $\psi_{f^\frown\langle 1\rangle}$ to be $\psi_f(\overline{v}) \wedge \neg\chi(\overline{v})$. This finishes the construction.

But this clearly gives 2^ω types over A, contradicting ω-stability. □

If $\overline{M}, \overline{N}$ are structures, $A \subseteq M$, and $f : A \rightarrow N$, we say that f is *partial elementary* iff for every formula $\varphi(\overline{v})$ without parameters and every $\overline{a} \in A$, $\overline{M} \models \varphi(\overline{a})$ iff $\overline{N} \models \varphi(f \circ \overline{a})$.

$\overline{M}$ is a *prime* model of T iff $\overline{M}$ can be elementarily embedded in every model of T. If $\overline{M} \models T$ and $A \subseteq M$, we say that $\overline{M}$ is *prime over A for T* iff for every model $\overline{N}$ of T, every partial elementary $f : A \rightarrow N$ can be extended to an elementary $f^+ : \overline{M} \rightarrow \overline{N}$.

Lemma 7.18. *If $\overline{a} \in {}^mM$, $\overline{b} \in {}^nM$, $A \subseteq M$, and $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$ is isolated, then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Let $\varphi(\overline{v}, \overline{w})$, a formula with parameters in A, isolate $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. We claim that $\exists\overline{w}\varphi(\overline{v}, \overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$. First, $\overline{M} \models \varphi(\overline{a}, \overline{b})$, so $\overline{M} \models \exists\overline{w}\varphi(\overline{a}, \overline{w})$. Second, suppose that $\overline{M} \models \chi(\overline{a})$, where χ has parameters in A. Then $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$, so $\overline{M} \models \forall\overline{v}\forall\overline{w}[\varphi(\overline{v}, \overline{w}) \rightarrow \chi(\overline{v})]$. Hence $\overline{M} \models \forall\overline{v}[\exists\overline{w}\varphi(\overline{v}, \overline{w}) \rightarrow \chi(\overline{v}]$ by elementary logic. □

Lemma 7.19. *Suppose that $A \subseteq B \subseteq M$, and $\overline{M} \models T$. Suppose that every $\overline{b} \in B$ realizes an isolated type over A, and suppose that $\mathrm{tp}^{\overline{M}}(\overline{a}/B)$ is isolated. Then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Suppose that $\varphi(\overline{v}, \overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/B)$, where $\overline{b} \in B$ are the parameters of φ. By hypothesis, let $\theta(\overline{w})$ isolate $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$. We claim that $\varphi(\overline{v}, \overline{w}) \wedge \theta(\overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. For, $\overline{M} \models \varphi(\overline{a}, \overline{b})$ and $\overline{M} \models \theta(\overline{b})$, so $\overline{M} \models \varphi(\overline{a}, \overline{b}) \wedge \theta(\overline{b})$. Now suppose that $\overline{M} \models \chi(\overline{a}, \overline{b})$. Hence $\overline{M} \models \forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})$. Hence the formula

$$\forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})$$

is in $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$, and it follows that

$$\overline{M} \models \forall\overline{w}[\theta(\overline{w}) \rightarrow \forall\overline{v}[\varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})].$$

Hence by elementary logic,

$$\overline{M} \models \forall\overline{w}\forall\overline{v}[\theta(\overline{w}) \wedge \varphi(\overline{v}, \overline{b}) \rightarrow \chi(\overline{v}, \overline{b})].$$

So we have shown that $\varphi(\overline{v}, \overline{w}) \wedge \theta(\overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. Now by Lemma 7.18 it follows that $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated. $\square$

Theorem 7.20. *Let T be ω-stable. Suppose that $\overline{M} \models T$ and $A \subseteq M$. Then there is an $\overline{M}_0 \preceq \overline{M}$ which is prime over A for T, and is such that every element of M_0 realizes an isolated type over A with respect to $\overline{M}_0$.*

Proof. We define a sequence $\langle A_\alpha : \alpha \leq \delta\rangle$ by recursion, where δ is also defined in the construction. Let $A_0 = A$. If α is a limit ordinal and A_β has been defined for all $\beta < \alpha$, then we let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now suppose that A_α has been defined. If no element of $M\backslash A_\alpha$ realizes an isolated type over A_α (in particular, if $M = A_\alpha$), we stop and let $\delta = \alpha$. Otherwise we pick an element $a_\alpha \in M\backslash A_\alpha$ realizing an isolated type over A_α and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

(1) A_δ is closed under the fundamental functions of $\overline{M}$.

In fact, suppose that $\mathbf{F}$ is an m-ary function symbol and $\overline{a} \in {}^m A_\delta$. Now $\mathrm{tp}^{\overline{M}}(\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta)$ is isolated over A_δ. For, suppose that $\varphi(v) \in \mathrm{tp}^{\overline{M}}(\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta)$. Thus $\overline{M} \models \varphi(\mathbf{F}^{\overline{M}}(\overline{a}))$, and so $\overline{M} \models \forall v[\mathbf{F}^{\overline{M}}(\overline{a}) = v \rightarrow \psi(v)]$, so that $\mathbf{F}^{\overline{M}}(\overline{a}) = v$ isolates $\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta$. It follows that $\mathbf{F}^{\overline{M}}(\overline{a}) \in A_\delta$.

Let $\overline{M}_0$ be the substructure of $\overline{M}$ with universe A_δ.

(2) $\overline{M}_0 \preceq \overline{M}$.

We apply Tarski's lemma. Suppose that $\varphi(v, \overline{a})$ is a formula with parameters $\overline{a} \in A_\delta$, and $\overline{M} \models \exists v\varphi(v, \overline{a})$. By Lemma 7.17, choose $b \in M$ such that $\varphi(v, \overline{a}) \in \mathrm{tp}^{\overline{M}}(b/\overline{a})$ and $\mathrm{tp}^{\overline{M}}(b/\overline{a})$ is isolated. By construction we have $b \in A_\delta$, as desired.

Now suppose that $\overline{N} \models T$ and $f : A \to \overline{N}$ is partial elementary. We now define $f_0 \subseteq \cdots \subseteq f_\delta$ by recursion so that $f_\alpha : A_\alpha \to \overline{N}$ is partial elementary. Let $f_0 = f$. If $\alpha \leq \delta$ is a limit ordinal and f_β has been defined for all $\beta < \alpha$, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Clearly f_α is partial elementary. Now suppose that f_α has been defined, where $\alpha < \delta$, with $f_\alpha : A_\alpha \to \overline{N}$ partial elementary. Then by construction, $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, where $a_\alpha \in M \backslash A_\alpha$ and $\mathrm{tp}^{\overline{M}}(a_\alpha/A_\alpha)$ is isolated. Let $\varphi(v, \overline{b})$ be a formula with parameters $\overline{b} \in A_\alpha$ which isolates $\mathrm{tp}^{\overline{M}}(a_\alpha/A_\alpha)$. Thus the following conditions hold:

(3) $\overline{M} \models \varphi(a_\alpha, \overline{b})$.

(4) For every formula $\chi(v, \overline{c})$ with parameters $\overline{c} \in A_\alpha$, if $\overline{M} \models \chi(a_\alpha, \overline{c})$ then $\overline{M} \models \forall v[\varphi(v, \overline{b}) \to \chi(v, \overline{c})]$.

Now by (3) we have $\overline{M} \models \exists v \varphi(v, \overline{b})$, so by the assumption that f_α is partial elementary we have $\mathbb{N} \models \exists v \varphi(v, f_\alpha \circ \overline{b})$. Choose $d \in N$ so that $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$. Let $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, d)\}$. To show that $f_{\alpha+1}$ is partial elementary, suppose that $\chi(v, \overline{c})$ is a formula with parameters $\overline{c} \in A_\alpha$, and $\overline{M} \models \chi(a_\alpha, \overline{c})$. So by (4) we have $\overline{M} \models \forall v[\varphi(v, \overline{b}) \to \chi(v, \overline{c})]$, hence $\overline{N} \models \forall v[\varphi(v, f \circ \overline{b}) \to \chi(v, f_\alpha \circ \overline{c})]$. Now $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$, so $\overline{N} \models \chi(d, f_\alpha \circ \overline{c})$. Hence $f_{\alpha+1}$ is partial elementary.

This finishes the construction of the f_α's. In particular, f_δ is an elementary mapping of $\overline{M}_0$ into $\overline{N}$, as desired.

It remains to show that every element of M_0 realizes an isolated type over A with respect to $\overline{M}_0$. We prove by induction on α that every element of A_α realizes an isolated type over A with respect to $\overline{M}$, for each $\alpha \leq \delta$. This is true for $\alpha = 0$, since any element $a \in A$ is isolated over A by the formula $v = a$. The inductive step to a limit ordinal α is obvious. Now suppose that $b \in A_{\alpha+1}$. Then b is isolated over A_α by construction, so b is isolated over A by the inductive hypothesis and Lemma 7.19.

Clearly being isolated over A with respect to $\overline{M}$ implies isolated over A with respect to $\overline{M}_0$. $\square$

Corollary 7.21. *If T is ω-stable, then it has a prime model.*

Proof. Take $A = \emptyset$ in Theorem 7.20. $\square$

Corollary 7.22. *If T is ω-stable and has no Vaughtian pairs, and if $\varphi(\overline{v})$ is a formula with parameters in M such that $\varphi(\overline{M})$ is infinite, then $\overline{M}$ is prime over $\varphi(\overline{M})$.*

Proof. By Theorem 7.20 there is an $\overline{N} \preceq \overline{M}$ which is prime over $\varphi(\overline{M})$. Since $\varphi(\overline{M}) \subseteq N$, we have $\varphi(\overline{N}) = \varphi(\overline{M})$. Since T has no Vaughtian pairs, it follows that $\overline{N} = \overline{M}$. $\square$

Theorem 7.23. *If T is ω-stable and has no Vaughtian pairs, then T is κ-categorical for every uncountable cardinal κ.*

Proof. Assume the hypotheses, with κ uncountable. Suppose that $\overline{M}, \overline{N} \models T$ with $|M| = |N| = \kappa$. Let $\overline{M}_0$ be a prime model of T by Corollary 7.21. Wlog $\overline{M}_0 \preceq \overline{M}, \overline{N}$. By Corollary 7.15 let $\varphi(\overline{v})$ be strongly minimal over $\overline{M}_0$.

(1) $|\varphi(\overline{M})| = |\varphi(\overline{N})| = \kappa$.

For, suppose that $|\varphi(\overline{M})| < \kappa$. By the downward Löwenheim-Skolem theorem, let $\overline{P}$ be an elementary substructure of $\overline{M}$ containing both $\varphi(\overline{M})$ and the parameters of φ, with $|P| < \kappa$. This contradicts Corollary 7.16. Hence $|\varphi(\overline{M})| = \kappa$. By symmetry, $|\varphi(\overline{N})| = \kappa$.

By Lemma 7.7, $\dim(\varphi(\overline{M})) = \dim(\varphi(\overline{N}))$, an hence there is a bijection $f : \varphi(\overline{M}) \to \varphi(\overline{N})$ which is a partial elementary embedding of $\varphi(\overline{M})$ into $\overline{N}$, by Lemma 7.9. By Corollary 7.22, $\overline{M}$ is prime over $\varphi(\overline{M})$, and hence f can be extended to an elementary embedding of $\overline{M}$ into $\overline{N}$. By Corollary 7.16, f maps onto $\overline{N}$. □

We now give some results of a general nature before turning to the converse of Theorem 7.23. We will use Ramsey's theorem from set theory, and we begin with a proof of it.

Ramsey's Theorem. *Suppose that M is an infinite set, n and r are positive integers, and $f : [M]^n \to r$. (r is considered as equal to $\{0, \ldots, r-1\}$.) Then there exist an $i < r$ and an infinite $N \subseteq M$ such that $f(a) = i$ for all $a \in [N]^n$.*

Proof. We may assume that $M = \omega$. We proceed by induction on n. First suppose that $n = 1$. Thus $f : [\omega]^1 \to r$, so $\omega = \bigcup_{i \in r}\{j \in \omega : f(\{j\}) = i\}$. It follows that there is an $i \in r$ such that $N \stackrel{\text{def}}{=} \{j \in \omega : f(\{j\}) = i\}$ is infinite, as desired.

Now assume that the theorem holds for $n \geq 1$, and suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \backslash \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H^S_m \subseteq S \backslash \{m\}$ such that g_m is constant on $[H^S_m]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each m_i will be in ω, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H^{S_i}_{m_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r}\{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \stackrel{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j.$$ □

Now we return to model theory. Let $(I, <)$ be a linear order, $\overline{M}$ a structure, and $\langle a_i : i \in I\rangle$ a system of distinct elements of M. We say that $\langle a_i : i \in I\rangle$ is a *system of order indiscernibles for* $\overline{M}$ iff for every formula $\varphi(w_1, \ldots, w_m)$ with free variables among the distinct variables $w_1, \ldots, w_m$ and all sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ of elements of I we have

$$\overline{M} \models \varphi(a_{i_1}, \ldots, a_{i_m}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_m}).$$

Theorem 7.24. *Let T be a theory with infinite models, and let $(I, <_I)$ be an infinite linear order. Then T has a model with a system $\langle a_i : i \in I\rangle$ of order indiscernibles.*

Proof. We will work with the standard sequence $v_1, v_2, \ldots$ of variables; all variables are assumed to be among these. Adjoin to the language a system $\langle c_i : i \in I\rangle$ of distinct new individual constants. Let Γ be the union of the following set of sentences:

(1) T;

(2) $c_i \neq c_j$ for $i \neq j$.

(3) $\varphi(c_{i_1}, \ldots, c_{i_p}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_p})$ for every formula $\varphi(v_1, \ldots, v_p)$ with free variables exactly the variables $v_1, \ldots, v_p$ and all sequences $i_1 <_I \cdots <_I i_p$ and $j_1 <_I \cdots <_I j_p$ of elements of I.

We claim that every finite subset of Γ has a model. So, suppose that $\Delta \subseteq \Gamma$ is finite. Let I_0 be the set of all $i \in I$ such that c_i occurs in one of the formulas in Δ. Let $\varphi_1, \ldots, \varphi_m$ be all of the formulas occuring in the third part of Δ as above, and for each $k \in [1, m]$ let p_k be the "p" involved. Let $n = \max\{p_k : 1 \leq k \leq n\}$. Let $\overline{M}$ be an infinite model of T, and fix any linear order $<_M$ of M. We now define $F : [M]^n \to \mathscr{P}(m)$ as follows. Given $A \in [M]^n$ with $A = \{a_1, \ldots, a_n\}$, $a_1 <_M \cdots <_M a_n$, let

$$F(A) = \{k : \overline{M} \models \varphi_k[a_1, \ldots, a_n]\}.$$

By Ramsey's theorem let $X \in [M]^\omega$ and $\eta \in \mathscr{P}(m)$ be such that $F(A) = \eta$ for all $A \in [X]^n$. Let $I_0 = \{s_0, \ldots, s_{m-1}\}$ with $s_0 <_I \cdots <_I s_{m-1}$. Let $x_0 <_M \cdots <_M x_{m+n-1}$ be elements of X. Define $a_{s_k} = x_k$ for all $k < m$. Thus for any $i, j \in I_0$ we have $i <_I j$ iff $a_i <_M a_j$. Now $(\overline{M}, a_i)_{i \in I_0}$ is a model of Δ. In fact, this is clear for the first two kinds of sentences above. Now take one of the third sort:

$\varphi_k(c_{i_1}, \ldots, c_{i_{p_k}}) \leftrightarrow \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}})$ where $\varphi_k(v_1, \ldots, v_{p_k})$ is a formula with free variables exactly the variables $v_1, \ldots, v_{p_k}$ and with sequences $i_1 <_I \cdots <_I i_{p_k}$ and $j_1 <_I \cdots <_I j_{p_k}$ of elements of I_0. Using the additional n elements of X mentioned above, extend $a_{i_1}, \ldots, a_{i_{p_k}}$ to a sequence $\overline{b} \in {}^nX$ strictly increasing in the sense of $<_M$, and extend $a_{j_1}, \ldots, a_{j_{p_k}}$ to a sequence $\overline{c} \in {}^nX$ strictly increasing in the sense of $<_M$. Then

$$\begin{aligned}
(\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{i_1}, \ldots, c_{i_{p_k}}) \quad & \text{iff} \quad \overline{M} \models \varphi_k[\overline{b}] \\
& \text{iff} \quad k \in F(\mathrm{rng}(\overline{b})) \\
& \text{iff} \quad k \in \eta \\
& \text{iff} \quad k \in F(\mathrm{rng}(\overline{c})) \\
& \text{iff} \quad \overline{M} \models \varphi_k[\overline{c}] \\
& \text{iff} \quad (\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}}).
\end{aligned}$$

This finishes the proof that $(\overline{M}, a_i)_{i\in I_0}$ is a model of Δ.

Hence by the compactness theorem, let $(\overline{N}, d_i)_{i\in I}$ be a model of Γ. We claim that $\overline{N}$ is as desired. For, suppose that $\varphi(\overline{w})$ is a formula with every free variable occurring in the sequence $\overline{w}$ of distinct variables, $\overline{w} = \langle w_1, \ldots, w_q\rangle$, and $i_1 <_I \cdots <_I i_q$, $j_1 <_I \cdots <_I j_q$. Let the variables actually occurring free in φ be $w_{s(1)}, \ldots, w_{s(r)}$, with $1 \leq s(1) < \cdots < s(r) \leq q$. Let φ' be obtained from φ by replacing $w_{s(1)}, \ldots, w_{s(r)}$ by $v_1, \ldots, v_r$ respectively, after changing bound variables to avoid clashes. Then φ' is a formula with exactly the free variables $v_1, \ldots, v_r$. Moreover, $i_{s(1)} <_I \cdots <_I i_{s(r)}$ and $j_{s(1)} <_I \cdots <_I j_{s(r)}$. Hence

$$(\overline{N}, d_i)_{i\in I} \models \varphi'(c_{i_{s(1)}}, \ldots, c_{i_{s(r)}}) \leftrightarrow \varphi'(c_{j_{s(1)}}, \ldots, c_{j_{s(r)}}).$$

It follows that

$$\begin{aligned}
&\overline{N} \models \varphi'(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \leftrightarrow \varphi'(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}});\\
&\overline{N} \models \varphi(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \leftrightarrow \varphi(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}});\\
&\overline{N} \models \varphi(d_{i_1}, \ldots, d_{i_q}) \leftrightarrow \varphi(d_{j_1}, \ldots, d_{j_q}).
\end{aligned}$$

□

A theory T in a language $\mathscr{L}$ *has built-in Skolem functions* iff for every positive integer n, every system $v, w_1, \ldots, w_n$ of distinct variables, and every formula $\varphi(v, w_1, \ldots, w_n)$ without parameters whose free variables are among $v, w_1, \ldots, w_n$, there is an m-ary function symbol f such that

$$T \models \forall\overline{w}[\exists v\varphi(v, \overline{w}) \to \varphi(f(\overline{w}), \overline{w})].$$

Theorem 7.25. *Let T be a theory in a language $\mathscr{L}$. Then there exist a language $\mathscr{L}^* \supseteq \mathscr{L}$ and a theory $T^* \supseteq T$ in $\mathscr{L}^*$ such that:*

(i) T^* has built-in Skolem functions.

(ii) Each model of T can be expanded to a model of T^*.

(iii) $|\mathscr{L}^*| = |\mathscr{L}| + \omega$.

Proof. Fix $c \in M$ We define $\mathscr{L}_0, \mathscr{L}_1, \ldots$ and $T_0, T_1, \ldots$ by recursion. Let $\mathscr{L}_0 = \mathscr{L}$ and $T_0 = T$. Having defined $\mathscr{L}_m$ and T_m, for each formula $\varphi(v, w_1, \ldots, w_n)$ as in the above definition, introduce an n-ary function symbol f_φ, and add the following sentence to T_m:

$$\forall\overline{w}[\exists v\varphi(v, \overline{w}) \to \varphi(f_\varphi(\overline{w}), \overline{w})].$$

This finishes the construction. Let $\mathscr{L}^* = \bigcup_{m\in\omega} \mathscr{L}_m$ and $T^* = \bigcup_{m\in\omega} T_m$. The desired conditions are easy to check. □

Theorem 7.26. *Let $\mathscr{L}$ be countable and let T be an $\mathscr{L}$-theory with an infinite model. Suppose that κ is an infinite cardinal. Then there is a model $\overline{M}$ of T of size κ such that for every $A \subseteq M$ and every positive integer n, $\overline{M}$ realizes at most $|A| + \omega$ n-types over A.*

Proof. By Theorem 7.24, let $\overline{N}$ be a model of T with a system $\langle a_\alpha : \alpha < \kappa\rangle$ of order indiscernibles with respect to $(\kappa, <)$. Let $I = \{a_\alpha : \alpha < \kappa\}$. Let $\mathscr{L}^*$ and T^* be as in Theorem 7.25. Let M be the closure under all of the functions of $\overline{N}^*$ of I. Then M is the universe of some substructure $\overline{M}^*$ of $\overline{N}^*$. Let $\overline{M}$ be the reduct of $\overline{M}^*$ to the language $\mathscr{L}$.

So $\overline{M} \models T$, and $|M| = \kappa$. Suppose that $A \subseteq M$. For each $b \in M$ we can write $b = t_b(x_b)$, where t_b is a term and x_b is a strictly increasing sequence $\langle x_b(0), \ldots, x_b(m_b - 1)\rangle$ of elements of I. Let $X = \{y \in I : y = x_b(u)$ for some $b \in A$ and $u < m_b\}$. Now for any $c \in {}^nM$ we define (with $c = \langle c(i) : i < n\rangle$)

$$
\begin{aligned}
L_c &= \langle \tau_{c(i)} : i < n\rangle;\\
N_c &= \{(i,j,u,v) : i < j < n, u < m_{c(i)}, v < m_{c(j)}\};\\
\text{for } (i,j,u,v) \in N_c,\ F_c(i,j,u,v) &= \begin{cases} 0 & \text{if } x_{c(i)}(u) < x_{c(j)}(v),\\ 1 & \text{if } x_{c(i)}(u) = x_{c(j)}(v),\\ 2 & \text{if } x_{c(i)}(u) > x_{c(j)}(v);\end{cases}\\
P_c &= \{(i,u,y) : i < n, u < m_i, y \in X\};\\
\text{for } (i,u,y) \in P_c,\ G_c(i,u,y) &= \begin{cases} 0 & \text{if } x_{c(i)}(u) = y,\\ 1 & \text{if } x_{c(i)}(u) < y,\\ 2 & \text{if } x_{c(i)}(u) > y;\end{cases}\\
T(c) &= \langle L_c, F_c, G_c\rangle.
\end{aligned}
$$

Now we claim that if $c, d \in {}^nM$ and $T(c) = T(d)$, then $\mathrm{tp}^{\overline{M}}(c/A) = \mathrm{tp}^{\overline{M}}(d/A)$. For, assume that $T(c) = T(d)$, and let $\varphi(\overline{v}, a)$ be given, with $a \in {}^lA$. Let

$$
\begin{aligned}
Y_c &= \{x_{c(i)}(u) : i < n, u < m_i\} \cup \{x_{a(i)}(u) : i < l, u < m_i\};\\
Y_d &= \{x_{d(i)}(u) : i < n, u < m_i\} \cup \{x_{a(i)}(u) : i < l, u < m_i\}.
\end{aligned}
$$

Clearly $|Y_c| = |Y_d|$. Let $\langle z^c_i : i < e\rangle$ and $\langle z^d_i : i < e\rangle$ enumerate Y_c and Y_d respectively, in the order $<_I$. Let $\langle w_i : i < e\rangle$ be a sequence of new variables. Say $x_{c(i)}(u) = z^c_{k(i,u)}$ and $x_{a(i)}(u) = z^c_{l(i,u)}$. Then by $T(c) = T(d)$ we have $x_{d(i)}(u) = z^d_{k(i,u)}$ and $x_{a(i)}(u) = z^d_{l(i,u)}$. Let φ' be the formula

$$\varphi(\langle t_{c(i)}(w_{k(i,0)}, \ldots, w_{k(i,m_i-1)}) : i < n\rangle, \langle t_{a(i)}(w_{l(i,0)}, \ldots, w_{l(i,m_i-1)}) : i < l\rangle).$$

Then

$$
\begin{aligned}
\overline{M} \models \varphi(c,a) \quad &\text{iff} \quad \overline{M} \models \varphi'(z^c)\\
&\text{iff} \quad \overline{M} \models \varphi'(z^d)\\
&\text{iff} \quad \overline{M} \models \varphi(d,a).
\end{aligned}
$$

This proves our claim. Now clearly there are at most $|A| + \omega$ choices for $T(c)$, so the conclusion of the theorem follows. $\square$

Now we again make the standing assumption that T is a complete theory in a countable language with only infinite models.

Theorem 7.27. *If T is κ-categorical for some uncountable κ, then T is ω-stable.*

Proof. Suppose that T is not ω-stable. Then there is a model $\overline{M}$ of T, a countable subset A of M, and a positive integer n, such that $|S^{\overline{M}}_n(A)| > \omega$. Let $\overline{M}'$ be a countable elementary submodel of $\overline{M}$ containing A. Then $\overline{M}' \models T$ and $|S_n^{\overline{M}'}(A)| > \omega$. Hence $\overline{M}'$ has an elementary extension $\overline{N}_0$ of size κ which realises uncountably many n-types over A. By Theorem 7.26 there is a model $\overline{N}_1$ of T such that for every countable $B \subseteq N_1$, $\overline{N}_1$ realizes only countably many n types over B. Hence $\overline{N}_0$ and $\overline{N}_1$ are not isomorphic. $\square$

If $\overline{M}$ is an infinite structure and κ is an infinite cardinal, we say that $\overline{M}$ is *κ-homogeneous* iff for every $A \in [M]^{<\kappa}$, every partial elementary map $f : A \to \overline{M}$, and every $a \in M$, there is a partial elementary map $f^+ : A \cup \{a\} \to \overline{M}$ which extends f. We say that $\overline{M}$ is homogeneous iff it is $|M|$-homogeneous.

Lemma 7.28. *Suppose that $\overline{M}$ and $\overline{N}$ are $\mathscr{L}$ structures, n is a positive integer, $a \in {}^nM$, and $b \in {}^nN$. Then the following conditions are equivalent:*

(i) $\mathrm{tp}^{\overline{M}}(a) = \mathrm{tp}^{\overline{N}}(b)$.

(ii) There is a partial elementary map $f : \mathrm{rng}(a) \to N$ such that $b = f \circ a$.

Proof. (i)⇒(ii): Assume (i). Define $f(a_i) = b_i$ for all $i < n$. f is well defined, since $a_i = a_j$ implies that $v_i = v_j \in \mathrm{tp}^{\overline{M}}(a) = \mathrm{tp}^{\overline{N}}(b)$, hence $b_i = b_j$. Clearly f is partial elementary.

(ii)⇒(i): clear. $\square$

Lemma 7.29. *Suppose that κ is an infinite cardinal, $\overline{M}$ is κ-homogeneous, n is a positive integer, $\overline{a}, b \in {}^nM$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(b)$, and $c \in M$. Then there is a $d \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.*

Proof. This is immediate from Lemma 7.28. $\square$

Lemma 7.30. *The following are equivalent:*

(i) $\overline{M}$ is ω-homogeneous.

(ii) For every positive integer n, all $\overline{a}, b \in {}^nM$, and all $c \in M$, if $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(b)$, then there is a $d \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). So by Lemma 7.28 there is an elementary map $f : \mathrm{rng}(\overline{a}) \to M$ such that $b = f \circ \overline{a}$. By (i), extend f to an elementary map $f^+ : \mathrm{rng}(\overline{a}) \cup \{c\} \to M$. Let $d = f(c)$. Then by Lemma 7.28 again, $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.

(ii)⇒(i): Assume (ii) and suppose that $f : A \to M$ is partial elementary, where A is a finite subset of M, and suppose that $c \in M$. Say $\mathrm{rng}(\overline{a}) = A$. By Lemma 7.28 we have $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(f \circ \overline{a})$. Hence by (ii) choose $d \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}((f \circ \overline{a})^\frown\langle d\rangle)$. By Lemma 28 we get a partial elementary map g such that $(f \circ \overline{a})^\frown\langle d\rangle = g \circ (\overline{a}^\frown\langle c\rangle)$. Thus g extends f and $g(c) = d$, as desired. $\square$

Theorem 7.31. *If $\overline{M}$ and $\overline{N}$ are countable homogeneous models of T and for each positive integer n they realize the same n-types, then they are isomorphic.*

Proof. Let $a_0, a_1, \ldots$ enumerate M and $b_0, b_1, \ldots$ enumerate N. We now define by recursion partial elementary maps $f_0, f_1, \ldots$ from subsets of M into $\overline{N}$. Let $f = \emptyset$; so it is partial elementary into $\overline{N}$ because T is complete. Now suppose that a partial elementary map f_s has been defined from a finite subset of M into $\overline{N}$. Let $\overline{c}$ be a sequence enumerating the domain of f.

Case 1. s is even, say $s = 2i$. By hypothesis, let $d, e \in N$ such that $\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle a_i\rangle) = \mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle)$. Hence $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{N}}(d)$. Also, by Lemma 7.28, $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{N}}(f_s \circ \overline{c})$. So $\mathrm{tp}^{\overline{N}}(d) = \mathrm{tp}^{\overline{N}}(f_s \circ \overline{c})$. Since $\overline{N}$ is homogeneous, by Lemma 7.29 there is a $u \in N$ such that $\mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f_s \circ \overline{c}) \circ \langle u\rangle)$. Let $f_{s+1} = f_s \cup \{(a_i, u)\}$. Then

$$\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle a_i\rangle) = \mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f_s \circ \overline{c})\circ\rangle u\rangle) = \mathrm{tp}^{\overline{N}}(f_{s+1} \circ (\overline{c}^\frown\langle a_i\rangle)),$$

so by Lemma 7.28 f_{s+1} is partial elementary.

Case 2. s is odd, say $s = 2i + 1$. This is treated similarly. Choose $d, e \in M$ such that $\mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f \circ \overline{c})^\frown\langle b_i\rangle)$. Hence $\mathrm{tp}^{\overline{M}}(d) = \mathrm{tp}^{\overline{N}}(f \circ \overline{c})$. Also, by Lemma 7.28 $\mathrm{tp}^{\overline{M}})(\overline{c}) = \mathrm{tp}^{\overline{N}}(f \circ \overline{c})$. So $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{M}}(d)$. Since $\overline{M}$ is homogeneous, by Lemma 7.27 there is a $u \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle u\rangle) = \mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle)$. Now if there is an i such that $c_i = u$, then $d_i = e$, hence $f(c_i) = b_i$. Hence $f_{\sigma+1} \overset{\text{def}}{=} f_s \cup \{(u, b_i)\}$ is a function. Also,

$$\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle u\rangle) = \mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f \circ \overline{c})^\frown\langle b_i\rangle) = \mathrm{tp}^{\overline{N}}(f_{s+1} \circ (\overline{c}^\frown\langle u\rangle),$$

so by Lemma 7.28 f_{s+1} is partial elementary.

Clearly $\bigcup_{s\in\omega} f_s$ is as desired. □

We consider an expansion $\overline{L}_U$ of our language $\overline{L}$ obtained by adjoining a one-place relation symbol U. For each formula $\varphi(v_0, \ldots, v_{n-1})$ of $\mathscr{L}$ we associate a formula $\varphi^U(v_0, \ldots, v_{n-1})$ of $\mathscr{L}_U$, as follows:

If φ is atomic, then φ^U is $Uv_0 \wedge \ldots \wedge Uv_{n-1} \wedge \varphi$.
$(\neg\psi)^U = \neg\psi^U$.
$(\psi \wedge \chi)^U = \psi^U \wedge \chi^U$.
$(\exists w\psi)^U = \exists w[Uw \wedge \psi^U]$.

Proposition 7.32. *If $\overline{M}$ is a substructure of $\overline{N}$, $\varphi(v_0, \ldots, v_{n-1})$ is a formula of $\mathscr{L}$, and $a \in {}^nM$, then $\overline{M} \models \varphi(\overline{a})$ iff $(\overline{N}, U) \models \varphi^U(\overline{a})$.*

Proof. An easy induction on φ. □

Theorem 7.33. *If there is a Vaughtian pair $(\overline{M}, \overline{N})$, then there is one in which N is countable.*

Proof. Let φ be a formula such that $\varphi(\overline{M})$ is infinite and $\varphi(\overline{M}) = \varphi(\overline{N})$. Let $\overline{a}$ be the parameters from M occurring in φ. We consider the structure $(\overline{N}, M)$ in the language $\mathscr{L}_U$. Let $(\overline{N}_0, M_0)$ be a countable elementary substructure of $(\overline{N}, M)$ such that $\overline{a} \in M_0$. Among the sentences holding in $(\overline{N}, M)$ are those asserting that M is closed under the fundamental function of $\overline{N}$. Hence M_0 is closed under the fundamental functions of $\overline{N}_0$, and hence M_0 is the universe of a substructure $\overline{M}_0$ of $\overline{N}_0$. For any formula $\psi(b)$ with

$b \in M_0$ we have, using Proposition 7.32,

$$\begin{aligned}\overline{M}_0 \models \psi(b) \quad &\text{iff} \quad (\overline{N}_0, M_0) \models \psi^U(b)\\ &\text{iff} \quad (\overline{N}, M) \models \psi^U(b)\\ &\text{iff} \quad \overline{M} \models \psi(b)\\ &\text{iff} \quad \overline{N} \models \psi(b)\\ &\text{iff} \quad \overline{N}_0 \models \psi(b).\end{aligned}$$

Thus $\overline{M}_0 \preceq \overline{N}_0$. Moreover, the sentence $\exists x \neg Ux$ holds in $(\overline{N}, M)$, hence also in $(\overline{N}_0, M_0)$, so that $\overline{M}_0 \neq \overline{N}_0$.

Clearly $\varphi(\overline{M}_0)$ is infinite and $\varphi(\overline{M}_0) = \varphi(\overline{N}_0)$. □

Lemma 7.34. *Suppose that $\overline{M} \preceq \overline{N}$ and in the language $\mathscr{L}_U$ we have $(\overline{N}, M) \preceq (\overline{N}', M')$. Then M' is the universe of a structure $\overline{M}'$, and $\overline{M} \preceq \overline{M}' \preceq \overline{N}'$.*

Proof. Clearly $M \subseteq M'$, and M' is closed under the fundamental functions of $\overline{N}'$, and hence is the universe of a structure $\overline{M}'$. If φ is a formula and $\overline{a} \in M$, then by Proposition 7.32,

$$\overline{M} \models \varphi(\overline{a}) \quad \text{iff} \quad (\overline{N}, M) \models \varphi^U(\overline{a}) \quad \text{iff} \quad (\overline{N}', M') \models \varphi^U(\overline{a}) \quad \text{iff} \quad \overline{M}' \models \varphi(\overline{a}).$$

Thus $\overline{M} \preceq \overline{M}'$.

Next we claim that

$$(1) \qquad (\overline{N}, M) \models \forall \overline{v}[Uv_0 \wedge \ldots \wedge Uv_{n-1} \rightarrow (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

In fact, suppose that $\overline{a} \in M$ is given. Then

$$(\overline{N}, M) \models \varphi(\overline{a}) \quad \text{iff} \quad \overline{N} \models \varphi(\overline{a}) \quad \text{iff} \quad \overline{M} \models \varphi(\overline{a}) \quad \text{iff} \quad (\overline{N}, M) \models \varphi^U(\overline{a}).$$

This proves (1). Hence we also get

$$(2) \qquad (\overline{N}', M') \models \forall \overline{v}[Uv_0 \wedge \ldots \wedge Uv_{n-1} \rightarrow (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

Now let $b \in M'$. Then using (2),

$$\overline{M}' \models \varphi(b) \quad \text{iff} \quad (\overline{N}', M') \models \varphi^U(b) \quad \text{iff} \quad (\overline{N}', M') \models \varphi(b) \quad \text{iff} \quad \overline{N}' \models \varphi(b). \qquad \square$$

Lemma 7.35. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ countable, $\overline{a} \in M$, $\overline{b} \in N$.*

Then there exist countable $M', \overline{N}'$ and $\overline{c}$ such that $(\overline{N}, M) \prec (\overline{N}', M')$ and $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}'}(\overline{c}/\overline{a})$.

Proof. Say $\overline{b}$ is of length n. In $\mathscr{L}_U$ let $\Gamma(\overline{v})$ be the following set of formulas:

Eldiag$(\overline{N}, M)$
$\{\bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v}, \overline{a}) : \overline{N} \models \varphi(\overline{b}, \overline{a})\}$.

If $\varphi_0, \ldots, \varphi_{m-1}$ are such that $\overline{N} \models \varphi_i(\overline{b}, \overline{a})$ for all $i < m$, then $\overline{N} \models \exists \overline{v} \bigwedge_{i<m} \varphi_i(\overline{v}, \overline{a})$, hence $\overline{M} \models \exists \overline{v} \bigwedge_{i<m} \varphi_i(\overline{v}, \overline{a})$, hence by Proposition 7.32,

$$(\overline{N}, M) \models \exists \overline{v} \left(\bigwedge_{i<n} U v_i \wedge \bigwedge_{i<m} \varphi_i^U(\overline{v}, \overline{a}) \right).$$

This shows that every finite subset of $\Gamma(\overline{v})$ is satisfiable. Hence there exist a countable $(\overline{N}', M')$ and $\overline{c} \in M'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$ and $(\overline{N}', M') \models \varphi^U(\overline{c}, \overline{a})$ whenever $\overline{N} \models \varphi(\overline{b}, \overline{a})$. If $\overline{N} \models \varphi(\overline{b}, \overline{a})$, then $\overline{M}' \models \varphi(\overline{c}, \overline{a})$. □

Corollary 7.36. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^*, \overline{N}^*$ such that $(\overline{N}, M) \preceq (\overline{N}^*, M^*)$, and for every $\overline{a} \in M$ and every $\overline{b} \in N$ there is a $\overline{c} \in M^*$ such that $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}^*}(\overline{c}/\overline{a})$.*

Proof. Iterate Lemma 7.35. □

Lemma 7.37. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $\overline{a}, \overline{b}, c \in N$, $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, $d \in N^\star$ and $\mathrm{tp}^{\overline{N}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{N}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Apply the compactness theorem to the set

Eldiag$(\overline{N}, M)$
$\{\varphi(\overline{b}, u) : \overline{N} \models \varphi(\overline{a}, c)\}$ (u a new constant) □

Corollary 7.38. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, and for all $\overline{a}, \overline{b}, c \in N$, if $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$, then there is a $d \in N^\star$ such that $\mathrm{tp}^{\overline{N}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{N}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Iterate Lemma 7.37. □

Lemma 7.37a. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $\overline{a}, \overline{b}, c \in M$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b})$. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, $d \in M^\star$ and $\mathrm{tp}^{\overline{M}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Apply the compactness theorem to the set

Eldiag$(\overline{N}, M)$
$\{Uu \wedge \varphi^U(\overline{b}, u) : \overline{M} \models \varphi(\overline{a}, c)\}$ (u a new constant) □

Corollary 7.38a. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, and for all $\overline{a}, \overline{b}, c \in M$, if $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$, then there is a $d \in M^\star$ such that $\mathrm{tp}^{\overline{M}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Iterate Lemma 7.37a. □

Lemma 7.39. *Suppose that $\overline{M} \prec \overline{N}$ (so $M \neq N$), and $\overline{N}$ is countable. Then there exist countable $\overline{M}', \overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, $\overline{N}'$ and $\overline{M}'$ are homogeneous and they realize the same n-types for all positive integers n. Moreover, they are isomorphic.*

Proof. We define an elementary chain $\langle(\overline{P}_i, Q_i) : i \in \omega\rangle$ by recursion. Let $\overline{P}_0 = \overline{N}$ and $Q_0 = M$. Suppose that $(\overline{P}_{3i}, Q_{3i})$ has been defined. Apply Corollary 7.36 to get an elementary extension $(\overline{P}_{3i+1}, Q_{3i+1})$ of $(\overline{P}_{3i}, Q_{3i})$ such that every type realized in $\overline{P}_{3i}$ is realized in $\overline{Q}_{3i+1}$. Note that these types are realized in $\overline{P}_{3i+1}$. Next, apply Corollary 7.38a to obtain an elementary extension $(\overline{P}_{3i+2}, Q_{3i+2})$ of $(\overline{P}_{3i+1}, Q_{3i+1})$ such that for all $\overline{a}, b, c \in Q_{3i+1}$, if $\mathrm{tp}^{\overline{Q}_{3i+1}}(\overline{a}) = \mathrm{tp}^{\overline{Q}_{3i+1}}(b)$, then there is a $d \in Q_{3i+2}$ such that $\mathrm{tp}^{\overline{Q}_{3i+1}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{Q}_{3i+2}}(b^\frown\langle d\rangle)$. Finally, apply Corollary 7.38 to obtain an elementary extension $(\overline{P}_{3i+3}, Q_{3i+3})$ of $(\overline{P}_{3i+2}, Q_{3i+2})$ such that for all $\overline{a}, b, c \in P_{3i+2}$, $\mathrm{tp}^{\overline{P}_{3i+2}}(\overline{a}) = \mathrm{tp}^{\overline{P}_{3i+2}}(b)$ implies that $\mathrm{tp}^{\overline{P}_{3i+3}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{P}_{3i+3}}(b^\frown\langle d\rangle)$ for some $d \in P_{3i+3}$.

This finishes the construction. Let $\overline{N}' = \bigcup_{i\in\omega} \overline{P}_i$ and $M' = \bigcup_{i\in\omega} Q_i$. The desired conclusion is clear, using Theorem 7.31 for the last statement. □

Suppose that $\omega \le \lambda < \kappa$. We say that T *has a* (κ, λ)*-model* iff there exist an $\overline{M} \models T$ and a formula $\varphi(\overline{v})$ such that $|M| = \kappa$ and $|\varphi(\overline{M})| = \lambda$.

Lemma 7.40. *If $\omega \le \lambda < \kappa$ and T has a (κ, λ)-model, then T has a Vaughtian pair.*

Proof. Let $\overline{N}$ be a (κ, λ)-model, with associated formula $\varphi(\overline{v})$. By the downward Löwenheim-Skolem theorem, let $\overline{M}$ be an elementary substructure of $\overline{N}$ of size λ such that $\varphi(\overline{N}) \subseteq M$. Clearly then $\varphi(\overline{N}) = \varphi(\overline{M})$, so that $(\overline{M}, \overline{N})$ is a Vaughtian pair. □

Theorem 7.41. *If T has a Vaughtian pair, then T has an $(\aleph_1, \aleph_0)$-model.*

Proof. Assume that T has a Vaughtian pair. By Lemma 7.33 we may assume that $(\overline{M}, \overline{N})$ is a Vaughtian pair with N countable. Say $\varphi(\overline{M}) = \varphi(\overline{N})$ is infinite. Also, $M \neq N$. By Lemma 7.39 there are countable $\overline{M}', \overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, $\overline{N}'$ and $\overline{M}'$ are homogeneous, the realize the same n-types for every positive integer n, and they are isomorphic. Still $M' \neq N'$. Now $(\overline{N}, M) \models \forall\overline{v}[\varphi(\overline{v}) \leftrightarrow \bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v})]$, so also $(\overline{N}', M') \models \forall\overline{v}[\varphi(\overline{v}) \leftrightarrow \bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v})]$, and this implies that $\varphi(\overline{M}') = \varphi(\overline{N}')$.

We now define by recursion a sequence $\langle\overline{P}_\alpha : \alpha < \omega_1\rangle$ of models. Let $\overline{P}_0 = \overline{N}'$. Now suppose that $\overline{P}_\alpha$ has been defined so that $\overline{P}_\alpha \cong \overline{N}'$. Then also $\overline{P}_\alpha \cong \overline{M}'$, so P_α has an elementary extension $\overline{P}_{\alpha+1}$ such that $(\overline{N}', M') \cong (\overline{P}_{\alpha+1}, P_\alpha)$. To see this, let g be an isomorphism from $\overline{P}_\alpha$ onto $\overline{M}'$, and let Q be a set such that $Q \cap (N'\backslash M') = Q \cap P_\alpha = \emptyset$ and $|Q| = |N'\backslash M'$. Let $P_{\alpha+1} = P_\alpha \cup Q$, and let $f : P_{\alpha+1} \to N'$ be a bijection such that $f \restriction P_\alpha = g$ while $f \restriction Q$ is a bijection from Q onto $N'\backslash M'$. We can make $P_{\alpha+1}$ into a structure so that f is an isomorphism from $\overline{P}_{\alpha+1}$ onto $\overline{N}'$. Then $\overline{P}_\alpha$ is an elementary substructure of $\overline{P}_{\alpha+1}$, since for $a \in {}^\omega P_\alpha$ we have

$$\overline{P}_\alpha \models \varphi[a] \quad \text{iff} \quad \overline{M}' \models \varphi[g \circ a] \quad \text{iff} \quad \overline{N}' \models \varphi[g \circ a] \quad \text{iff} \quad \overline{P}_{a+1} \models \varphi[a].$$

For α limit, let $\overline{P}_\alpha = \bigcup_{\beta<\alpha} \overline{P}_\beta$. Since then $\overline{P}_\alpha$ is the union of models isomorphic to $\overline{N}'$, it is clearly homogeneous and realizes the same types as $\overline{N}'$. Hence it is isomorphic to $\overline{N}'$. This finishes the construction.

Let $\overline{P}_{\omega_1} = \bigcup_{\alpha<\omega_1} \overline{P}_\alpha$. Then $|P_{\omega_1}| = \omega_1$. Now by induction we have $\varphi(\overline{P}_\alpha) = \varphi(\overline{M}')$ for all $\alpha \le \omega_1$. Hence $|\varphi(\overline{P}_{\omega_1})| = \omega$. □

Lemma 7.42. *Suppose that T is ω-stable, $\overline{M} \models T$, and $|M| \geq \aleph_1$. Then $\overline{M}$ has a proper elementary extension $\overline{N}$ such that for every finite sequence $\overline{w}$ of variables and every $\Gamma(\overline{w})$ of formulas with free variables among $\overline{w}$ and with parameters from M and with $\Gamma(\overline{w})$ countable, if $\Gamma(\overline{w})$ is realized in $\overline{N}$, then it is also realized in $\overline{M}$.*

Proof. First we claim

(1) There is a formula $\varphi(v)$ with parameters from M such that $|\varphi(\overline{M})| \geq \aleph_1$, and for every formula $\psi(v)$ with parameters from M, either $|\varphi(\overline{M}) \cap \psi(\overline{M})| < \aleph_1$ or $|\varphi(\overline{M}) \cap \neg\psi(\overline{M})| < \aleph_1$.

Suppose not. Then it is easy to define formulas φ_f for $f \in {}^{<\omega}2$ such that the following conditions hold for each f:

(2) $\varphi_\emptyset$ is the formula $v = v$.

(3) $|\varphi_f(\overline{M})| \geq \aleph_1$.

(4) $\varphi_{f\frown\langle 0\rangle}(\overline{M}) \cap \varphi_{f\frown\langle 1\rangle}(\overline{M}) = \emptyset$.

This gives 2^ω types over M, contradicting the ω-stability of T. So (1) holds.

Choose $\varphi(v)$ as in (1), and let

$$p = \{\psi(v) : \psi(v) \text{ is a formula with parameters from } M, \text{ and } |\varphi(\overline{M}) \cap \psi(\overline{M})| \geq \aleph_1\}.$$

Note that $\varphi(\overline{M}) \cap \psi(\overline{M})$ is a co-countable subset of $\varphi(\overline{M})$, and an intersection of countably many co-countable subset of a set is still co-countable. Hence

(5) p is finitely satisfiable.

From (1) it also follows that p is a complete type.

Let $\overline{M}'$ be a proper elementary extension of $\overline{M}$ containing an element c which realizes p, and choose $d \in M' \backslash M$. Now we apply Theorem 20 to get an elementary substructure $\overline{N}$ of $\overline{M}'$ which is prime over $M \cup \{c, d\}$ for T and is such that every finite sequence of elements of N realizes an isolated type over $M \cup \{c, d\}$. Thus $M \cup \{c, d\} \subseteq N$, so clearly $\overline{M} \prec \overline{N}$. Now suppose that $\Gamma(\overline{w})$ is a set of formulas with free variables among $\overline{w}$, with parameters from M and $\Gamma(\overline{w})$ is countable, and such that it is realized in $\overline{N}$, say by b. Let $\theta(\overline{w}, v)$ be a formula which isolates $\mathrm{tp}^{\overline{N}}(b/M \cup \{c\})$.

(6) $\exists\overline{w}\theta(\overline{w}, v) \in p$.

In fact, otherwise $\neg\exists\overline{w}\theta(\overline{w}, v) \in p$, hence $\overline{M}' \models \neg\exists\overline{w}\theta(\overline{w}, c)$, hence $\overline{N} \models \neg\exists\overline{w}\theta(\overline{w}, c)$. This contradicts $\overline{N} \models \theta(b, c)$.

(7) $\forall\overline{w}[\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})] \in p$ for every $\gamma(\overline{w}) \in \Gamma(\overline{w})$.

For, otherwise $\exists\overline{w}[\theta(\overline{w}, v) \wedge \neg\gamma(\overline{w}] \in p$, hence $\overline{M}' \models \exists\overline{w}[\theta(\overline{w}, c) \wedge \neg\gamma(\overline{w}]$, hence $\overline{N} \models \exists\overline{w}[\theta(\overline{w}, c) \wedge \neg\gamma(\overline{w}]$, contradicting $\overline{N} \models \varphi(b, c) \wedge \gamma(b)$.

Now let

$$\Delta = \{\exists \overline{w}\theta(\overline{w}, v)\} \cup \{\forall \overline{w}[\theta(\overline{w}, v) \to \gamma(\overline{w})] : \gamma(\overline{w}) \in \Gamma(\overline{w})\}.$$

If $\delta(v) \in \Delta$, then $\delta(v) \in p$, and so $|\varphi(\overline{M})\backslash\delta(\overline{M})| < \aleph_1$. It follows that $\bigcap_{\delta(v)\in\Delta} \delta(\overline{M}) \neq \emptyset$, i.e. there is a $c' \in M$ such that $\overline{M} \models \delta(c)$ for every $\delta(v) \in \Gamma(v)$. In particular, $\overline{M} \models \exists \overline{w}\theta(\overline{w}, c')$, so we can choose $b' \in M$ such that $\overline{M} \models \theta(b', c)$. Now for each $\gamma(\overline{w}) \in \Gamma(\overline{w})$ the formula $\forall \overline{w}[\theta(\overline{w}, v) \to \gamma(\overline{w})]$ is in Δ, so it follows that $\overline{M} \models \gamma(b')$. □

Theorem 7.43. *Suppose that T is ω-stable and has an $(\aleph_1, \aleph_0)$-model. Then for any $\kappa > \aleph_1$ it has a $(\kappa, \aleph_0)$-model.*

Proof. Let $\overline{M} \models T$ with $|M| = \aleph_1$, and let $\varphi(\overline{v})$ be a formula with $|\varphi(\overline{M})| = A_0$. We now construct an elementary chain $\langle \overline{N}_\alpha : \alpha < \kappa\rangle$ by recursion. Let $\overline{N}_0 = \overline{M}$. Now suppose that $\overline{N}_\alpha$ has been defined so that $\varphi(\overline{M}) = \varphi(\overline{N}_\alpha)$. We apply Lemma 7.42 to obtain a proper elementary extension $\overline{N}_{\alpha+1}$ of $\overline{N}_\alpha$ such that if $G(\overline{w})$ is a countable type over M realized in $N_{\alpha+1}$, then it is realized in $\overline{N}_\alpha$. Let

$$\Gamma_\alpha(\overline{v}) = \{\varphi(\overline{v})\} \cup \{\overline{v} \neq \overline{a} : \overline{a} \in M \text{ and } \overline{M} \models \varphi(\overline{a})\}$$

Thus Γ_α is a countable type over $\overline{M}$, but it is not realized in $\overline{N}_\alpha$. Hence it is not realized in $\overline{N}_{\alpha+1}$. It follows that $\varphi(\overline{N}_{\alpha+1}) = \varphi(\overline{M})$.

For α limit we let $\overline{N}_\alpha = \bigcup_{\beta<\alpha} \overline{N}_\beta$. Clearly still $\varphi(\overline{N}_\alpha) = \varphi(\overline{M})$.

Finally, $\bigcup_{\alpha<\kappa} \overline{N}_\alpha$ is as desired. □

Theorem 7.44. *If $\overline{M}$ is an infinite structure and κ is a cardinal $\geq |M|$, then $\overline{M}$ has an elementary extension $\overline{N}$ of cardinality κ such that for every formula $\varphi(\overline{v})$ with parameters from N, if $\varphi(\overline{N})$ is infinite then $|\varphi(\overline{N})| = \kappa$.*

Proof. For each formula $\varphi(\overline{v})$ adjoin κ many tuples of new constants of the length of $\overline{v}$, and apply the compactness theorem to the set consisting of Eldiag$(\overline{M})$ together with sentences saying, for each $\varphi(\overline{v})$ such that $\varphi(\overline{M})$ is infinite, that the κ many tuples for this formula are all distinct and statisfy φ. □

Theorem 7.45. *Suppose that κ is uncountable and T is κ-categorical. Then T has no Vaughtian pairs.*

Proof. Assume the hypothesis. By Theorem 7.27, T is ω-stable. Suppose that there is a Vaughtian pair. Then by Theorem 7.41 T has an $(\aleph_1, \aleph_0)$-model, and then by Theorem 7.43 it has a $(\kappa, \aleph_0)$-model $\overline{M}$. So $|M| = \kappa$ and $|\varphi(\overline{M})| = \aleph_0$ for some formula $\varphi(\overline{v})$. By Theorem 7.44, there is a model $\overline{N}$ of T in which $|\varphi(\overline{N})| = |N| = \kappa$. This contradicts κ-categoricity.

Theorem 7.46. (Baldwin, Lachlan) *Let κ be uncountable. Then the following conditions are equivalent:*

(i) T is κ-categorical

(ii) T is ω-stable and has no Vaughtian pairs.

Proof. (i)⇒(ii): Theorems 7.27 and 7.45.

(ii)⇒(i): Theorem 7.23. □

Theorem 7.47. (Morley) *T is κ categorical for some uncountable κ iff it is κ-categorical for every uncountable κ.* □

EXERCISES

Exc. 7.1. Let $\overline{M}$ be a field, A a subfield, and $a \in M$. Suppose that a is algebraic over A in the usual sense of field theory. Show that a is algebraic over A in the model-theoretic sense.

Exc. 7.2. Let $\overline{M} = (\omega, <)$. Show that every element of ω is algebraic over $\emptyset$.

Exc. 7.3. Let $\overline{A} = ([\omega]^2, R)$, where

$$R = \{(a,b) : a, b \in [\omega]^2,\ a \neq b \text{ and } a \cap b \neq \emptyset\}.$$

(i) Show that $\{a \in [\omega]^2 : (a, \{0,1\}) \in R\}$ is neither finite nor cofinite.

(ii) Infer from (i) that $[\omega]^2$ is not minimal.

(iii) If f is a permutation of ω, define $f^+ : [\omega]^2 \to [\omega]^2$ by setting $f^+(a) = f[a]$ for any $a \in [\omega]^2$. Show that f^+ is an automorphism of $\overline{A}$.

(iv) Let $X = \{a \in [\omega]^2 : 0 \in a \text{ and } a \cap \{1,2\} = \emptyset\}$. Show that X is definable in $\overline{A}$ with parameters.

(v) Show that X is minimal.

Exc. 7.4. Let V be an infinite vector space over a finite field F. We consider V as a structure $(V, +, f_a)_{a\in F}$, where $f_a(v) = av$ for any $v \in V$ and $a \in F$. Show that V is minimal.

Exc. 7.5. (continuing exc. 7.4) Prove that for any subset A of V, $\text{acl}(A) = \text{span}(A)$.

Exc. 7.6. (continuing excs. 7.4, 7.5) By exercise 7.4 and Lemma 7.2, the following holds in $\overline{V}$: if $a \in \text{span}(A \cup \{b\})\backslash\text{span}(A)$, then $b \in \text{span}(A \cup \{a\})$. Prove this statement using ordinary linear algebra.

Exc. 7.7. Give an example of a set Γ of sentences and two sentences φ and ψ, such that $\Gamma \models \varphi$ iff $\Gamma \models \psi$, but $\Gamma \not\models (\varphi \leftrightarrow \psi)$.

Exc. 7.8. Show that for Γ a set of sentences and for sentences φ, ψ, if $\Gamma \models \varphi \leftrightarrow \psi$ then $\Gamma \models \varphi$ iff $\Gamma \models \psi$.

Exc. 7.9. Prove that the following two conditions are equivalent:

(i) $\overline{M} \models \varphi[a]$ iff $\overline{M} \models \psi[a]$.

(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)[a]$.

Exc. 7.10. Prove that the following two conditions are equivalent, for any sentences φ, ψ:

(i) $\overline{M} \models \varphi$ iff $\overline{M} \models \psi$.

(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)$.

Exc. 7.11. In the language with no non-logical symbols, show that ω is an indiscernible set in ω.

Exc. 7.12. (Continuing exercises 7.4, 7.5, 7.6) Let $A = \{w_1, w_2\}$, two members of V, and let $b = w_1$. Thus $b \in \text{span}(A)$. According to Lemma 7.8, $\text{tp}^{\overline{V}}(b/A)$ is isolated. Give a formula $\varphi(v_0, \overline{a})$ with $\overline{a} \in A$ which isolates $\text{tp}^{\overline{V}}(b/A)$.

Exc. 7.13. Suppose that $\overline{M}$ is an infinite structure, $\varphi(v_0)$ is a formula with at most v_0 free, and $\varphi(\overline{M})$ is infinite. Show that $\overline{M}$ has a proper elementary extension $\overline{N}$ such that $(\overline{M}, \overline{N})$ is not a Vaughtian pair.