HW 1

Exercises: 1.2.1, 1.3.1, 1.3.2, (1.3.3), (1.4.1), (1.5.1), 1.6.1, 1.6.2.

Exercises in parentheses are not required. (Please do not submit solutions to these exercises.)

1.2.1. Find a signature appropriate for the description of vector spaces over a give field $\mathcal{K}$.

1.3.1. Given $X \subseteq M$, let $\operatorname{Aut}_{\{X\}} \mathcal{M}$ be the set $\{h \in \operatorname{Aut} \mathcal{M} \colon h[X] = X\}$. Show that $\operatorname{Aut}_X \mathcal{M}$ is a normal subgroup of $\operatorname{Aut}_{\{X\}} \mathcal{M}$. What happens if, instead of $h[X] = X$, we require only $h[X] \subseteq X$?

1.3.2. Find a structure with a bijective endomorphism that is not an automorphism.

(1.3.3) Find an infinite structure $\mathcal{M}$ with a trivial automorphism group, i.e., $\operatorname{Aut} \mathcal{M} = \{\mathrm{id}_M\}$.

(1.4.1) Describe the difference between substructures of $\mathbb{Z}$ according to whether $\mathbb{Z}$ is considered in the signature $(0; +)$ or in the signature $(0; +, -)$.

(1.5.1) Given a signature σ, find a signature $\sigma_1 \supseteq \sigma$ such that all σ-sturctures $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{N} \leq \mathcal{M}$ have expansions $\mathcal{M}'$ and $\mathcal{N}'$ to σ_1 such that $\mathcal{N}' \subseteq \mathcal{M}'$ and $\operatorname{Aut} \mathcal{M}' = \operatorname{Aut}_{\{N\}} \mathcal{M}$

1.6.1. Show that $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$ is uncountable as soon as no M_i is empty and infinitely many of the M_i have at least two elements.

1.6.2. Find an embedding $e \colon \mathcal{M} \to \mathcal{M}^I$ such that $p_i e = \mathrm{id}_M$ for all $i \in I$.

Consider the conjecture below (which was inspired by questions asked by Jared Louchs and Connor Meredith).

Suppose we are given two signatures of the following form

$$\sigma_0 = (\emptyset, \emptyset, \{r\}, \sigma_0') \quad \text{and} \quad \sigma_1 = (\emptyset, \{f\}, \emptyset, \sigma_1')$$

such that $\sigma_0'(r) = \sigma_1'(f) + 1$. To ease notation below, let's take $\sigma_1'(f) = n$.

(σ_0 might be called a "purely relational" signature, since it has only relation symbols; similarly, σ_1 is "purely algebraic".)

Now, suppose we have structures $\mathbf{P} = (S, r^{\mathbf{P}})$ and $\mathbf{L} = (S, f^{\mathbf{L}})$ of types σ_0 and σ_1 respectively, defined on the same universe, S.

Let's make up a new definition that might be useful in this situation. Let us say that $\mathbf{P}$ is a **proxy** for $\mathbf{L}$ if each of $r^{\mathbf{P}}$ and $f^{\mathbf{L}}$ can be defined in terms the other as follows: for all $x_0, x_1, \dots, x_{n-1}, y \in S$,

$$f^{\mathbf{L}}(x_0, x_1, \dots, x_{n-1}) = y \quad \Leftrightarrow \quad r^{\mathbf{P}}(x_0, x_1, \dots, x_{n-1}, y).$$

EXERCISE. Try to prove, or disprove by counterexample, the following

Conjecture: If $\mathbf{P} = (S, r^{\mathbf{P}})$ is a **proxy** for $\mathbf{L} = (S, f^{\mathbf{L}})$, and if $\mathbf{Q} = (T, r^{\mathbf{Q}})$ is a **proxy** for $\mathbf{M} = (T, f^{\mathbf{M}})$, then a relational homomorphism $h : \mathbf{P} \to \mathbf{Q}$ is *strong* if and only if the map $h : S \to T$ is an algebraic homomorphism from $\mathbf{L}$ to $\mathbf{M}$.

Further food for thought

Do you think the notion of "proxy" defined above captures the sense in which the poset we saw in class can be represeted as a semilattice? If not, think about alternative definitions of "equivalent" structures of different signatures. This is an open-ended question that may help you get better aquainted with structures and their signatures, but it is not a central theme of the course. In other words, it's something to think about over dinner.