3. Sentential logic

Here we discuss some more components of our final first-order logic: the logic surrounding words like "not", "and", etc. The language here is simpler than what we have dealt with so far. We have only the following symbols:

n, a symbol for negation.

a, a symbol for conjunction ("and").

Symbols $S_0, S_2, \ldots$, called *sentential variables.*

Symb is the collection of all of these symbols. An *expression* is a finite nonempty sequence of members of Symb. We define operations $\neg$ and $\wedge$ on the set of expressions:

$$\neg\varphi = \langle \mathbf{n} \rangle^\frown \varphi; \quad \varphi \wedge \psi = \langle \mathbf{a} \rangle^\frown \varphi^\frown \psi.$$

The collection of *sentential formulas* is the smallest collection C of expressions such that $\langle S_i \rangle$ is in C for each $i \in \omega$, and C is closed under the operations $\neg$, $\wedge$. Frequently we write S_i instead of $\langle S_i \rangle$.

In analogy to Proposition 2.1 we have:

Proposition 3.1. *(i) No proper initial segment of a sentential formula is a formula.*
(ii) If φ is a sentential formula, then exactly one of the following holds:
(a) φ is a sentential variable.
(b) φ is $\neg\psi$ for some sentential formula ψ.
(c) φ is $\psi \wedge \chi$ for some sentential formulas ψ, χ.
(iii) If φ and ψ are sentential formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.
(iv) If $\varphi, \psi, \varphi', \psi'$ are sentential formulas and $\varphi \wedge \psi = \varphi' \wedge \psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$. $\square$

Now we define satisfaction and truth for this special language. A *sentential assignment* is a function mapping ω into $\{0, 1\}$. We think of 0 as "false" and 1 as "true". Then the value of an arbitrary formula φ under a sentential assignment f is denoted by $\varphi[f]$ and is defined as follows:

$$\begin{aligned} S_i[f] &= f(i); \\ (\neg\varphi)[f] &= 1 - \varphi[f]; \\ (\varphi \wedge \psi)[f] &= \varphi[f] \cdot \psi[f]. \end{aligned}$$

We say that f *satisfies* φ, or that φ is *true* under f iff $\varphi[f] = 1$. A sentential formula is a *tautology* iff it is true under every assignment.

We introduce some further logical notions:

$$\begin{aligned} \varphi \to \psi &= \neg(\varphi \wedge \neg\psi); \\ \varphi \vee \psi &= \neg(\neg\varphi \wedge \neg\psi); \\ \varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi). \end{aligned}$$

Here are some common tautologies:

(T1) $S_0 \to S_0$.
(T2) $S_0 \leftrightarrow \neg\neg S_0$.
(T3) $(S_0 \to \neg S_0) \to \neg S_0$.
(T4) $(S_0 \to \neg S_1) \to (S_1 \to \neg S_0)$.
(T5) $S_0 \to (\neg S_0 \to S_1)$.
(T6) $(S_0 \to S_1) \to [(S_1 \to S_2) \to (S_0 \to S_2)]$.
(T7) $[S_0 \to (S_1 \to S_2)] \to [(S_0 \to S_1) \to (S_0 \to S_2)]$.
(T8) $(S_0 \wedge S_1) \to (S_1 \wedge S_0)$.
(T9) $(S_0 \wedge S_1) \to S_0$.
(T10) $(S_0 \wedge S_1) \to S_1$.
(T11) $S_0 \to [S_1 \to (S_0 \wedge S_1)]$.
(T12) $S_0 \to (S_0 \vee S_1)$.
(T13) $S_1 \to (S_0 \vee S_1)$.
(T14) $(S_0 \to S_2) \to [(S_1 \to S_2) \to ((S_0 \vee S_1) \to S_2)]$.
(T15) $\neg(S_0 \wedge S_1) \leftrightarrow (\neg S_0 \vee \neg S_1)$.
(T16) $S_0 \wedge S_1 \leftrightarrow \neg(\neg S_0 \vee \neg S_1)$.
(T17) $\neg(S_0 \vee S_1) \leftrightarrow (\neg S_0 \wedge \neg S_1)$.
(T18) $[S_0 \vee (S_1 \vee S_2)] \leftrightarrow [(S_0 \vee S_1) \vee S_2]$.
(T19) $[S_0 \wedge (S_1 \wedge S_2)] \leftrightarrow [(S_0 \wedge S_1) \wedge S_2]$.
(T20) $[S_0 \wedge (S_1 \vee S_2)] \leftrightarrow [(S_0 \wedge S_1) \vee (S_0 \wedge S_2)]$.
(T21) $[S_0 \vee (S_1 \wedge S_2)] \leftrightarrow [(S_0 \vee S_1) \wedge (S_0 \vee S_2)]$.
(T22) $S_0 \wedge S_1 \leftrightarrow \neg(S_0 \to \neg S_1)$l.

Some of these tautologies show that we could have selected different primitive notions for the sentential part of first-order logic. Thus:

$\neg$ and $\vee$ suffice, by (T16).

$\neg$ and $\to$ suffice, by (T22).

We now define general conjunctions and disjunctions:

$$\bigwedge_{i\le 0} \varphi_i = \varphi_0;$$

$$\bigwedge_{i\le m+1} \varphi_i = \left(\bigwedge_{i\le m} \varphi_i\right) \wedge \varphi_{m+1};$$

$$\bigvee_{i\le 0} \varphi_i = \varphi_0;$$

$$\bigvee_{i\le m+1} \varphi_i = \left(\bigvee_{i\le m} \varphi_i\right) \vee \varphi_{m+1}.$$

We also might write $\varphi_0 \wedge \ldots \wedge \varphi_m$ in place of $\bigwedge_{i \leq m} \varphi_i$; similarly for $\bigvee$. Sometimes we will not explicitly give an order; for example we might write $\bigvee_{i \in I} \varphi_i$. In such a case, any order should be ok.

For any sentential formula φ, let $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$.

Lemma 3.2. *If f and g are sentential assignments which agree on every i such that S_i occurs in φ, then $\varphi[f] = \varphi[g]$.*

Proof. By induction on φ. □

Theorem 3.3. (Disjunctive normal form) *If φ is a sentential formula which is true under some sentential assignment, and if every sentential variable S_i occurring in φ has $i < m$, then there is a nonempty set $M \subseteq {}^m2$ such that the following formula is a tautology:*

$$\varphi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} S_i^{\varepsilon(i)}.$$

Proof. Let

$$M = \{\varepsilon \in {}^m2 : \varphi[f] = 1 \text{ for some } f \supseteq \varepsilon\}.$$

Note that M is nonempty, since φ is true under some assignment. Now take any sentential assignment f. Note that

$$\left(\bigwedge_{i<m} S_i^{f(i)}\right)[f] = 1.$$

Hence the right side of the formula in the Theorem is true under f iff $f \restriction m \in M$, and this is true by Lemma 3.2 iff $\varphi[f] = 1$. □

EXERCISES

Exc. 3.1. Define $\varphi|\psi = \neg\varphi \wedge \neg\psi$. (The Sheffer stroke.). Show that $\neg$ and $\wedge$ can be defined in terms of $|$.

Exc. 3.2. A formula φ involving only $S_0, \ldots, S_m$ determines a function $t_\varphi : {}^{m+1}2 \to 2$ defined by $t_\varphi(x) = \varphi[x]$ for any $x \in {}^{m+1}2$. Show that any member of $\bigcup_{0<m<\omega} {}^{({}^m2)}2$ can be obtained in this way.

Exc. 3.3. Show that the following formula is a tautology:

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)]$$

(This formula can be used as a single axiom in an axiomatic development of sentential logic.)