1. Basic notions

Last change: Sept 4, 2009

Double orders

We introduce some terminology about orderings on a class P, some of it new.

• R is a *relation on* a class P iff $R \subseteq P \times P$

• R is *irreflexive* iff there is no x such that $(x, x) \in R$.

• R is *transitive* iff for all x, y, z, if $(x, y), (y, z) \in R$ then $(x, z) \in R$.

• R is a *partial order* iff it is irreflexive and transitive.

We usually use symbols $<$ or $\prec$ or something like that for partial orders, and write $x < y$ rather than $(x, y) \in <$.

• (P, R) is a *partially ordered system* iff R is a partial order on P.

Here it is possible that P and R are proper classes; then we cannot form the ordered pair (P, R). So, more rigorously, "(P, R) is a partially ordered system" is just the obvious formula with two free variables P, R, saying that R is a partial order on P.

Note that if (P, R) is a partially ordered system and $P \subseteq B$, then (B, R) is a partially ordered system.

Frequently we simply say that P is a partially ordered class if the relation R is clear from the context. Similarly for other notions introduced below.

• R is *reflexive on* a class P iff $R \subseteq P \times P$ and $(x, x) \in R$ for all $x \in P$.

• R is *antisymmetric* iff for all x, y, if $(x, y), (y, x) \in R$ then $x = y$.

• R is a *quasi-order* on a class P iff R is transitive and reflexive on P.

We do not assume antisymmetry here. We usually use symbols $\leq$ or $\preceq$ or something like that for quasi-orders.

• (P, R) is a *quasi-ordered system* iff R is a quasiorder on P.

• $(P, <, \preceq)$ is a *double ordered system* iff $(P, <)$ is a partially ordered system, $(P, \preceq)$ is a quasi-ordered system, and the following conditions hold:

(1) If $p < q$ or $p = q$, then $p \preceq q$.

(2) If $p < q \preceq r$ or $p \preceq q < r$, then $p < r$.

(3) For all $p \in P$ there is a $q \in P$ such that $p < q$.

To give our main examples of double orders, we need to recall the notions of ideals and filters.

Ideals and filters

A *filter* on A is a collection F of subsets of A with the following properties:

(4) $A \in F$.
(5) If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.
(6) If $X, Y \in F$ then $X \cap Y \in F$.

A filter F is *proper* iff $F \neq \mathscr{P}(A)$.

An *ideal* on A is a collection I of subsets of A such that the following conditions hold:

(7) $\emptyset \in I$
(8) If $X \subseteq Y \in I$ then $X \in I$.
(9) If $X, Y \in I$ then $X \cup Y \in I$.

An ideal I is *proper* iff $I \neq \mathscr{P}(A)$. A subset X of A is *I-positive* iff $X \notin I$. Let $I^+ = \{X \subseteq A : X \notin I\}$.

If F is a filter on A, let $F^{\mathrm{id}} = \{X \subseteq A : A \backslash X \in F\}$. Then F^{id} is an ideal on A. If I is an ideal on A, let $I^{\mathrm{fi}} = \{X \subseteq A : A \backslash X \in I\}$. Then I^{fi} is a filter on A. If F is a filter on A, then $(F^{\mathrm{id}})^{\mathrm{fi}} = F$. If I is an ideal on A, then $(I^{\mathrm{fi}})^{\mathrm{id}} = I$.

Note that if F is a filter, then a set $X \subseteq A$ is F^{id}-positive iff $A \backslash X \notin F$, and $(F^{\mathrm{id}})^+ = \{X \subseteq A : (A \backslash X) \notin F\}$.

An *ultrafilter* on A is a maximal proper filter on A.

Proposition 1.1. *Let F be a proper filter on A. Then F is an ultrafilter on A iff for every $X \subseteq A$, either $X \in F$ or $A \backslash X \in F$.*

Proof. $\Rightarrow$: Suppose that F is an ultrafilter on A, $X \subseteq A$, and $X \notin F$. Then $G \stackrel{\text{def}}{=} \{Y \subseteq A : Z \cap X \subseteq Y \text{ for some } Z \in F\}$ is clearly a filter on A containing F, and $G \neq F$ since $X \in G \backslash F$. So $G = \mathscr{P}(A)$. Thus there is a $Z \in F$ such that $Z \cap X \subseteq \emptyset$, so $Z \subseteq A \backslash X$. It follows that $A \backslash X \in F$.

$\Leftarrow$: clear. □

If I is an ideal on A and $X \subseteq A$, then $I + X$ is the smallest ideal containing $I \cup \{X\}$.

Proposition 1.2. *Suppose that I is an ideal on A and $B, X \subseteq A$. Then the following conditions are equivalent:*

(i) $X \in I + B$.
(ii) There is a $Y \in I$ such that $X \subseteq Y \cup B$.
(iii) $X \backslash B \in I$.

Proof. Clearly (ii)$\Rightarrow$(i). The set

$$\{Z \subseteq A : \exists Y \in I[Z \subseteq Y \cup B]\}$$

is clearly an ideal containing $I \cup \{B\}$, so (i)$\Rightarrow$(ii). If Y is as in (ii), then $X \backslash B \subseteq Y$, and hence $X \backslash B \in I$; so (ii)$\Rightarrow$(iii). If $X \backslash B \in I$, then $X \subseteq (X \backslash B) \cup B$, so X satisfies the condition of (ii). So (iii)$\Rightarrow$(ii). □

If I an ideal on A and for $X, Y \subseteq A$ we define $X \subseteq_I Y$ iff $X \backslash Y \in I$. For a filter F on A, $X \subseteq_F Y$ mean that $(A \backslash X) \cup Y \in F$. Also, $X \subset_I Y$ means that $X \subseteq_I Y$ while $Y \not\subseteq_I X$.

Ordinal-valued functions

Let A be any set. We will deal frequently with the class ${}^A\mathrm{Ord}$ of all functions mapping a set A into the class Ord of all ordinals. If A is nonempty, then ${}^A\mathrm{Ord}$ is a proper class.

Suppose that F is a filter on a set A and $R \subseteq \mathrm{Ord} \times \mathrm{Ord}$. Then for functions $f, g \in {}^A\mathrm{Ord}$ we define

$$f\,R_F\,g \quad \text{iff} \quad \{a \in A : f(a)\,R\,g(a)\} \in F.$$

The most important cases of this notion that we will deal with are $f <_F g$, $f \le_F g$, and and $f =_F g$. Thus

$$\begin{aligned} f <_F g \quad &\text{iff} \quad \{a \in A : f(a) < g(a)\} \in F; \\ f \le_F g \quad &\text{iff} \quad \{a \in A : f(a) \le g(a)\} \in F; \\ f =_F g \quad &\text{iff} \quad \{a \in A : f(a) = g(a)\} \in F. \end{aligned}$$

Note that each of these is a proper class relation. If I is an ideal, then by $R_I, <_I, \le_I, =_I$ we mean $R_F, <_F, \le_F, =_F$ with $F = I^{\mathrm{fi}}$.

Proposition 1.3. *If I is an ideal on A and $f, g \in {}^A\mathrm{Ord}$, then*
(i) $f <_I g$ iff $\{a \in A : f(a) \ge g(a)\} \in I$;
(ii) $f \le_I g$ iff $\{a \in A : f(a) > g(a)\} \in I$. $\square$

Note that if $F = \{A\}$ (or $I = \{\emptyset\}$), then $f \le_F g$ or $f \le_I g$ means that $f(i) R g(i)$ for all $i \in A$. Then we drop the subscript ${}_F$ or ${}_I$. In particular,

$$\begin{aligned} f < g \quad &\text{iff} \quad \forall i \in A[f(i) < g(i)]; \\ f \le g \quad &\text{iff} \quad \forall i \in A[f(i) \le g(i)]. \end{aligned}$$

Also note that $f =_F g$ really means $f = g$ for $F = \{A\}$.

The following trivial proposition is nevertheless important in what follows.

Proposition 1.4. *Let F be a proper filter on A. Then*
(i) $({}^A\mathrm{Ord}, <_F, \le_F)$ is a double order.
(ii) $<_F$ is irreflexive and transitive.
(iii) $\le_F$ is reflexive on ${}^A\mathrm{Ord}$, and it is transitive.
(iv) $f \le_F g <_F h$ implies that $f <_F h$.
(v) $f <_F g \le_F h$ implies that $f <_F h$.
(vi) $f <_F g$ or $f =_F g$ implies $f \le_F g$.
(vii) If $f =_F g$, then $f \le_F g$.
(viii) If $f \le_F g \le_F f$, then $f =_F g$.
(ix) If $f =_F g$, then the following conditions hold:
(a) $f <_F h$ iff $g <_F h$,
(b) $f \le_F h$ iff $g \le_F h$,
(c) $h <_F f$ iff $h <_F g$, and
(d) $h \le_F f$ iff $h \le_F g$. $\square$

Some care must be taken in working with these notions. The following examples illustrate this.

(10) An example with $f \leq_F g$ but neither $f <_F g$ nor $f =_F g$ nor $f = g$: Let $A = \omega$, $F = \{A\}$, and define $f, g \in {}^\omega\omega$ by setting $f(n) = n$ for all n and

$$g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

(11) An example where $f =_F g$ but neither $f <_F g$ nor $f = g$: Let $A = \omega$ and let F consist of all subsets of ω that contain all even natural numbers. Define f and g by

$$f(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd;} \end{cases} \quad g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

If $f, g \in {}^A\text{Ord}$ and $R \subseteq \text{Ord} \times \text{Ord}$, we let $R(f, g) = \{a \in A : (f(a), g(a)) \in R$. Thus $<(f, g) = \{a \in A : f(a) < g(a)\}$.

Restricted ordinal-valued functions

Let Limord be the class of all limit ordinals.

Proposition 1.5. *Suppose that $P \in {}^A\text{Limord}$ and F is a filter on A. Then*

$$\left(\prod_{a \in A} P_a, <_F, \leq_F\right)$$

is a double order. □

Here there is a slight abuse of notation, since by $<_F$ and $\leq_F$ we mean the restrictions of the class notion defined above to the set $\prod_{a \in A} P_a$.

These double orders $\left(\prod_{a \in A} P_a, <_F, \leq_F\right)$, consisting of restricted ordinal-valued functions, are one of the main topics of these notes.

Proposition 1.6. *If F is a proper filter on A, $g, h \in {}^A\text{Ord}$, $h(a) > 0$ for all $a \in A$, and $g <_F h$, then there is a $k \in \prod_{a \in A} h(a)$ such that $g =_F k$.*

Proof. For any $a \in A$ let

$$k(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $k \in \prod_{a \in A} h(a)$. Moreover,

$$\{a \in A : g(a) = k(a) \supseteq \{a \in A : g(a) < h(a)\} \in F,$$

so $g =_F k$. □

Reduced products

Let $\langle (P_a, <_a, \preceq_a) : a \in A\rangle$ be a system of double ordered systems, and let F be a filter on A. Clearly $=_F$ (restricted to $\prod_{a\in A} P_a$) is an equivalence relation on $\prod_{a\in A} P_a$. We let $[f]_F$, or simply $[f]$, denote the equivalence class of f, and we let $\prod_{a\in A} P_a/F$ be the collection of all equivalence classes. Furthermore, we define

$$\begin{aligned}
[f] < [g] \quad &\text{iff} \quad \exists f' \in [f]\exists g' \in [g][\{a \in A : f'(a) <_a g'(a)\} \in F];\\
[f] \preceq [g] \quad &\text{iff} \quad \exists f' \in [f]\exists g' \in [g][\{a \in A : f'(a) \preceq_a g'(a)\} \in F].
\end{aligned}$$

The triple $(\prod_{a\in A} P_a, <, \preceq)$ is called a *reduced product* of $\langle P_a : a \in A\rangle$; if F is an ultrafilter, it is called an *ultraproduct* of $\langle P_a : a \in A\rangle$.

Proposition 1.7. *Suppose that $\langle P_a : a \in A\rangle$ is a system of double orders, F is a filter on A, and $f, g \in \prod_{a\in A} P_a$. Then*

(i)
$$[f] < [g] \quad \text{iff} \quad \{a \in A : f(a) <_a g(a)\} \in F;$$

(ii)
$$[f] \preceq [g] \quad \text{iff} \quad \{a \in A : f(a) \preceq_a g(a)\} \in F.$$

Proof. We only prove (i), as the proof of (ii) is similar. $\Leftarrow$ is obvious. Now suppose that $[f] < [g]$. Then there exist $f' \in [f]$ and $g' \in [g]$ such that $\{a \in A : f'(a) <_a g'(a)\} \in F$. Thus $\{a \in A : f'(a) = f(a)\} \in F$ and $\{a \in A : g'(a) = g(a)\} \in F$. Note that

$$\begin{aligned}
\{a \in A : f'(a) <_a g'(a)\} &\cap \{a \in A : f'(a) = f(a)\}\\
&\cap \{a \in A : g'(a) = g(a)\} \subseteq \{a \in A : f(a) <_a g(a)\}.
\end{aligned}$$

It follows that $\{a \in A : f(a) <_a g(a)\} \in F$. $\square$

Proposition 1.8. *Suppose that $\langle P_a : a \in A\rangle$ is a system of double orders and F is a proper filter on A. Then $(\prod_{a\in A} P_a/F, <, \preceq)$ is a double order.*

Proof. The proof is routine. We show that $<$ is irreflexive as an illustration. For any $f \in \prod_{a\in A} P_a$ we have $\{a \in A : f(a) < f(a)\} = \emptyset \notin F$, and so $[f] \not< [f]$. $\square$

More on double orders

- $(P, <, \preceq)$ is a *simple double ordered system* iff $(P, <, \leq)$ is a double ordered system and for all $a, b \in P$, $a \leq b$ iff ($a < b$ or $a = b$).

Simple double orders are essentially given by their partial orders, or by their quasi-orders.

Proposition 1.9. *Let $(P, <)$ be a partially ordered system. Define $a \leq b$ iff ($a < b$ or $a = b$). Then $(P, <, \leq)$ is a simple double ordered system. Moreover, $\leq$ is antisymmetric.*

Let $(P, \leq)$ be a quasi-ordered system for which $\leq$ is antisymmetric. Define $a < b$ iff ($a \leq b$ and $a \neq b$). Then $(P, <, \leq)$ is a simple double ordered system.

The processes described here are inverses of each other. $\square$

Elements p, q of a partial order P are *comparable* iff $p > q$, $q < p$, or $p = q$. A *linear order* is a partial order $(P, <)$ in which any two elements are comparable. A *linear double order* is a double order $(P, <, \preceq)$ in which $(P, <)$ is a linear order.

Proposition 1.10. *Every linear double order is simple.*

Proof. Assume that $p \preceq q$ and $p \neq q$. If $q < p$, then $p < p$, contradiction. So $p < q$.

Conversely, $p < q$ or $p = q$ implies $p \leq q$ without even assuming that the double order is linear. □

Proposition 1.11. *Suppose that $\langle P_a : a \in A\rangle$ is a system of linear double orders and F is an ultrafilter on A. Then $(\prod_{a\in A} P_a/F, <)$ is a linear double order.*

Proof. For any $f, g \in \prod_{a\in A} P_a$ we have

$$A = \{a \in A : f(a) < g(a)\} \cup \{a \in A : f(a) = g(a)\} \cup \{a \in A : g(a) < f(a)\}.$$

Since $A \in F$, one of these three sets is also in F; otherwise the empty set, which is the intersection of their complements, would be in F. It follows that $[f] < [g]$, $[f] = [g]$, or $[g] < [f]$. □

Proposition 1.12. *Suppose that $\langle P_a : a \in A\rangle$ is a system of double orders, F is a proper filter on A, and $B \in F$. Then*

(i) $F \cap \mathscr{P}(B)$ is a proper filter on B

(ii) If F is an ultrafilter on A, then $F \cap \mathscr{P}(B)$ is an ultrafilter on B.

(iii) $(\prod_{a\in A} P_a/F, <, \preceq)$ is isomorphic to $(\prod_{b\in B} P_b/(F \cap \mathscr{P}(B)), <, \preceq)$.

Proof. (i) and (ii) are straightfoward. For (iii), it is easy to see that there is a function φ such that $\varphi([f]) = [f \restriction B]$ for all $f \in \prod_{a\in A} P_a$, and this function is the desired isomorphism. □

• A subset $X \subseteq P$ is *cofinal* iff $\forall p \in P \exists q \in X(p \preceq q)$. This is equivalent to saying that X is cofinal in P iff $\forall p \in P \exists q \in X(p < q)$.

• Since clearly P itself is cofinal in P, we can make the basic definition of the cofinality $\mathrm{cf}(P)$ of P:

$$\mathrm{cf}(P) = \min\{|X| : X \text{ is cofinal in } P\}.$$

Note that it is possible for $\mathrm{cf}(P)$ to be singular, unlike the situation for cofinality of ordinals, or linear orders. Here is an example with P of the main form we are considering: restricted ordinal valued functions. Take $P = \prod_{i\in\omega} \aleph_i$, $A = \omega$, $F = \{\omega\}$. Then $\mathrm{cf}(P, <_F, \leq_F) = \aleph_\omega$.

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *$<$-increasing* iff $\forall \xi, \eta < \lambda(\xi < \eta \to p_\xi < p_\eta)$. Similarly for *$\preceq$-increasing*. These notions make sense even if P is a proper class.

• We say that P *has true cofinality* iff P has a linearly ordered subset which is cofinal.

Note that not every P has true cofinality, even among the double orderings $(P, <_F, \preceq_F)$ with $P \subseteq {}^A\mathrm{Ord}$ and F a filter on A. For example, take $P = \prod_{i\in\omega} \aleph_i$, $A = \omega$, $F = \{\omega\}$.

Proposition 1.13. *Suppose that $\langle p_\alpha : \alpha < \lambda\rangle$ is strictly increasing in the sense of $<$, is cofinal in P, and λ is regular. Then P has true cofinality, and its cofinality is λ.*

Proof. Obviously P has true cofinality. If X is a subset of P of size less than λ, for each $q \in X$ choose $\alpha_q < \lambda$ such that $q < p_{\alpha_q}$. Let $\beta = \sup_{q\in X} \alpha_q$. Then $\beta < \lambda$ since λ is regular. For any $q \in X$ we have $q < p_\beta$. This argument shows that $\mathrm{cf}(P) = \lambda$. □

Proposition 1.14. *Suppose that P has true cofinality. Then:*
(i) $\mathrm{cf}(P)$ *is regular.*
(ii) $\mathrm{cf}(P)$ *is the least size of a linearly ordered subset which is cofinal in* (P).
(iii) There is a $<$-increasing, cofinal sequence in P of length $\mathrm{cf}(P)$.

Proof. Let X be a linearly ordered subset of P which is cofinal in P, and let $\{y_\alpha : \alpha < \mathrm{cf}(P)\}$ be a subset of P which is cofinal in P; we do not assume that $\langle y_\alpha : \alpha < \mathrm{cf}(P)\rangle$ is $<$- or $\preceq$-increasing.

(iii): We define a sequence $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ by recursion. Let x_0 be any element of X. If x_α has been defined, let $x_{\alpha+1} \in X$ be such that $x_\alpha, y_\alpha < x_{\alpha+1}$; it exists since X is cofinal. Now suppose that $\alpha < \mathrm{cf}(P)$ is limit and x_β has been defined for all $\beta < \alpha$. Then $\{x_\beta : \beta < \alpha\}$ is not cofinal in P, so there is a $z \in P$ such that $z \not< x_\beta$ for all $\beta < \alpha$. Choose $x_\alpha \in X$ so that $z < x_\alpha$. Since X is linearly ordered, we must then have $x_\beta < x_\alpha$ for all $\beta < \alpha$. This finishes the construction. Since $y_\alpha < x_{\alpha+1}$ for all $\alpha < \mathrm{cf}(P)$, it follows that $\{x_\alpha : \xi < \mathrm{cf}(P)\}$ is cofinal in P. So (iii) holds.

(i): Suppose that $\mathrm{cf}(P)$ is singular, and let $\langle \beta_\xi : \xi < \mathrm{cf}(\mathrm{cf}(P))\rangle$ be a strictly increasing sequence cofinal in $\mathrm{cf}(P)$. With $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ as in (iii), it is then clear that $\{x_{\beta_\xi} : \xi < \mathrm{cf}(\mathrm{cf}(P))\}$ is cofinal in P, contradiction (since $\mathrm{cf}(\mathrm{cf}(P)) < \mathrm{cf}(P)$ because $\mathrm{cf}(P)$ is singular).

(ii): By (iii), there is a linearly ordered subset of P of size $\mathrm{cf}(P)$ which is cofinal in P; by the definition of cofinality, there cannot be one of smaller size. □

For P with true cofinality, the cardinal $\mathrm{cf}(P)$ is called the *true cofinality* of P, and is denoted by $\mathrm{tcf}(P)$. We write

$$\mathrm{tcf}(P) = \lambda$$

to mean that P has true cofinality, and it is equal to λ.

• P is λ-*directed* iff for any subset Q of P such that $|Q| < \lambda$ there is a $p \in P$ such that $q < p$ for all $q \in Q$.

Proposition 1.15. (Pouzet) *Suppose that P is a double order. Then for any infinite cardinal λ the following conditions are equivalent:*
(i) $\mathrm{tcf}(P) = \lambda$
(ii) P has a cofinal subset of size λ, and P is λ-directed.

Proof. (i)⇒(ii) is clear, remembering that λ is regular. Now assume (ii), and let X be a cofinal subset of P of size λ.

First we show that λ is regular. Suppose that it is singular. Write $X = \bigcup_{\alpha<\mathrm{cf}(\lambda)} Y_\alpha$ with $|Y_\alpha| < \lambda$ for each $\alpha < \mathrm{cf}(\lambda)$. Let p_α be an upper bound for Y_α for each $\alpha < \mathrm{cf}(\lambda)$, and let q be an upper bound for $\{p_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. Choose r such that $q < r$. Then choose $s \in X$ with $r < s$. Say $s \in Y_\alpha$. Then $s < p_\alpha < q < r < s$, contradiction.

So, λ is regular. Let $X = \{r_\alpha : \alpha < \lambda\}$. Now we define a sequence $\langle p_\alpha : \alpha < \lambda\rangle$ by recursion. Having defined p_β for all $\beta < \alpha$, by (ii) let p_α be such that $p_\beta < p_\alpha$ for all $\beta < \alpha$, and $r_\beta < p_\alpha$ for all $\beta < \alpha$. Clearly this sequence shows that $\mathrm{tcf}(P) = \lambda$. $\square$

Proposition 1.16. *If G is a cofinal subset of P, then $\mathrm{cf}(P) = \mathrm{cf}(G)$. Moreover, $\mathrm{tcf}(P) = \mathrm{tcf}(G)$, in the sense that if one of them exists then so does the other, and they are equal.* (That is what we mean in the future too when we assert the equality of true cofinalities.)

Proof. Let H be a cofinal subset of P of size $\mathrm{cf}(P)$. For each $p \in H$ choose $q_p \in G$ such that $p < q_p$. Then $\{q_p : p \in H\}$ is cofinal in G. In fact, if $r \in G$, choose $p \in H$ such that $r < p$. Then $r < q_p$, as desired. This shows that $\mathrm{cf}(G) \leq \mathrm{cf}(P)$.

Now suppose that K is a cofinal subset of G. Then it is also cofinal in P. For, if $p \in P$ choose $q \in G$ such that $p < q$, and then choose $r \in K$ such that $q < r$. So $p < r$, as desired. This shows the other inequality.

For the true cofinality, we apply Proposition 1.15. So suppose that P has true cofinality λ. By Proposition 1.15 and the first part of this proof, G has a cofinal subset of size λ, since cofinality is the same as true cofinality when the latter exists. Now suppose that $X \subseteq G$ is of size $< \lambda$. Choose an upper bound p for it in P. Then choose $q \in G$ such that $p < q$. So q is an upper bound for X, as desired. Thus by Proposition 1.15 we have $\mathrm{tcf}(G) = \lambda$.

The other implication, that the existence of $\mathrm{tcf}(G)$ implies that of $\mathrm{tcf}(P)$ and their equality, is even easier, since a sequence cofinal in G is also cofinal in P. $\square$

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *persistently cofinal* iff

$$\forall h \in P \exists \xi_0 < \lambda \forall \xi (\xi_0 \leq \xi < \lambda \Rightarrow h < p_\xi).$$

Proposition 1.17. *(i) If $\langle p_\xi : \xi < \lambda\rangle$ is $<$-increasing and cofinal in P, then it is persistently cofinal.*

(ii) If $\langle p_\xi : \xi < \lambda\rangle$ and $\langle p'_\xi : \xi < \lambda\rangle$ are two sequences of members of P, $\langle p_\xi : \xi < \lambda\rangle$ is persistently cofinal in P, and $p_\xi \preceq p'_\xi$ for all $\xi < \lambda$, then also $\langle p'_\xi : \xi < \lambda\rangle$ is persistently cofinal in P. $\square$

• An *upper bound* for P is an element $p \in {}^A\mathrm{Ord}$ such that $q \preceq p$ for all $q \in P$.

• A *least upper bound* for P is an upper bound a for P such that $a \leq_P a'$ for every upper bound a' for P. So if a and b are least upper bounds for P, then $a \leq_P b \leq_P a$, but it is not necessarily true that $a = b$.

In the case of double orders obtained from ordinal-valued functions we have the following equivalent definition of least upper bounds:

Proposition 1.18. *Suppose that F is a proper filter on A, $P \subseteq {}^A\mathrm{Ord}$, and $f \in {}^A\mathrm{Ord}$. Then the following conditions are equivalent.*

(i) f is a least upper bound of P under $<_F$.

(ii) f is an upper bound of P under $<_F$, and for any $f' \in {}^A\mathrm{Ord}$, if f' is an upper bound of P under $<_F$ and $f' \leq_F f$, then $f =_F f'$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Hence $f \leq_F f'$, so $f =_F f'$ by Proposition 1.4(viii).

(ii)⇒(i): Assume (ii), and suppose that $g \in {}^A\mathrm{Ord}$ is an upper bound for P. For each $a \in A$ let $g'(a) = \min(f(a), g(a))$. Then $\{a \in A : g'(a) \leq f(a)\} = A \in F$, so $g' \leq_F f$. If $h \in F$, then

$$\{a \in A : h(a) \leq g'(a)\} \supseteq \{a \in A : h(a) \leq f(a)\} \cap \{a \in A : h(a) \leq g(a)\} \in F,$$

so $h \leq_F g'$. Thus g' is an upper bound for P. It now follows from (ii) that $f =_F g'$. Clearly $g' \leq_F g$, so $f \leq_F g$ by Proposition 1.4(ix). □

Now we come to an ordering notion which is basic for pcf theory.

- An *exact upper bound* for P is a least upper bound a for P such that P is cofinal in $\{p \in P : p < a\}$.

Note that under the hypothesis here, $a \notin P$.

In general, there are sets which have least upper bounds but no exact upper bounds. For example, define $f_{i\xi}$ for each $i \in \omega$ and $\xi \in \aleph_i$ by letting $\mathrm{dmn}(f_{i\xi}) = \omega$ and for each $j \in \omega$ let

$$f_{i\xi}(j) = \begin{cases} \xi & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $P = \{f_{i\xi} : i \in \omega,\ \xi \in \aleph_i\}$. Let $F = \{\omega\}$, and let $g(i) = \aleph_i$ for all $i \in \omega$. Then g is a l.u.b. for P, but it is not an exact upper bound.

In the next proposition we see that in the definition of exact upper bound we can weaken the least upper bound condition, under a mild restriction on the set in question.

Proposition 1.19. *Suppose F is a filter on A, P is a nonempty set of functions in ${}^A\mathrm{Ord}$, and $\forall f \in P \exists f' \in P[f <_F f']$. Suppose that h is an upper bound of P under $<_F$, and for all $g \in {}^A\mathrm{Ord}$, if $g <_F h$ then there is an $f \in P$ such that $g <_F f$. Then h is an exact upper bound for P.*

Proof. First note that $\{a \in A : h(a) \neq 0\} \in F$. In fact, choose $f \in P$. Then $f <_F h$, and so $\{a \in A : h(a) \neq 0\} \supseteq \{a \in A : f(a) < h(a)\} \in F$, as desired.

Now we show that h is a least upper bound for F. Let k be any upper bound. Let

$$l(a) = \begin{cases} k(a) & \text{if } k(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{a \in A : l(a) < h(a)\} \supseteq \{a \in A : h(a) \neq 0\}$, it follows by the above that $\{a \in A : l(a) < h(a)\} \in F$, and so $l <_F h$. So by assumption, choose $f \in P$ such that $l <_F f$. Now $f \leq_F k$, so $l <_F k$ and hence

$$\{a \in A : h(a) \leq k(a)\} \supseteq \{a \in A : l(a) < k(a)\} \in F,$$

so $h \leq_F k$, as desired.

For the other property in the definition of exact upper bound, suppose that $g <_I h$. Then by assumption there is an $f \in P$ such that $g <_I f$, as desired. □

Corollary 1.20. *If $h \in {}^A\mathrm{Ord}$ takes only limit ordinal values and $P \subseteq \prod_{a \in A} h(a)$, then h is an exact upper bound of P with respect to a filter F on A iff P is cofinal in $\prod_{a \in A} h(a)$.* □

The next proposition shows that exact upper bounds restrict to certain subset of A.

Proposition 1.21. *Suppose that P is a nonempty subset of ${}^A\mathrm{Ord}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$. Also, suppose that $A_0 \notin I$. Then:*

(i) $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A_0)$ is a proper ideal on A_0.
(ii) For any $f, f' \in {}^A\mathrm{Ord}$, if $f <_I f'$ then $f \restriction A_0 <_J f' \restriction A_0$.
(iii) $h \restriction A_0$ is an exact upper bound for $\{f \restriction A_0 : f \in F\}$.

Proof. Let $F = I^{\text{fi}}$.

(i) is clear. Note that $J^{\text{fi}} = I^{\text{fi}} \cap \mathscr{P}(A_0)$. Now assume the hypotheses of (ii). Then

$$\{a \in A_0 : f(a) < f'(a)\} = A_0 \cap \{a \in A : f(a) < f'(a)\},$$

and so $f \restriction A_0 <_J f' \restriction A_0$.

For (iii), by (ii) we see that $h \restriction A_0$ is an upper bound for $\{f \restriction A_0 : f \in P\}$. To see that it is an exact upper bound, we will apply Proposition 1.19. So, suppose that $k <_J h \restriction A_0$. Fix $f \in P$. Now define $g \in {}^A\mathrm{Ord}$ by setting

$$g(a) = \begin{cases} f(a) & \text{if } a \in A \backslash A_0, \\ k(a) & \text{if } a \in A_0. \end{cases}$$

Then

$$\{a \in A : g(a) < h(a)\} \supseteq \{a \in A : f(a) < h(a)\} \cap \{a \in A_0 : k(a) < h(a) \in F,$$

so $g <_I h$. Hence there is an $l \in F$ such that $g <_I l$. Hence

$$\{a \in A_0 : k(a) < l(a)\} \supseteq A_0 \cap \{a \in A : g(a) < l(a)\},$$

and $\{a \in A : g(a) < l(a)\} \in F$, so $k <_J l$, as desired. □

Next, increasing the ideal maintains exact upper bounds:

Proposition 1.22. *Suppose that P is a nonempty subset of ${}^A\mathrm{Ord}$, I is a proper ideal on A, h is an exact upper bound for P with respect to I, and $\forall f \in P \exists f' \in P(f <_I f')$.*

Let J be a proper ideal on A such that $I \subseteq J$. Then h is an exact upper bound for P with respect to J.

Proof. We will apply Proposition 1.19. Let $F = I^{\text{fi}}$ and $G = J^{\text{fi}}$. So $F \subseteq G$. Note that h is clearly an upper bound for F with respect to J. Now suppose that $g <_J h$. Fix $f \in P$. Define g' by

$$g'(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ f(a) & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) < h(a)\} \supseteq \{a \in A : f(a) < h(a)\} \in F$, since $f <_I h$. So $g' <_I h$. Hence by the exactness of h there is a $k \in F$ such that $g' <_I k$. So

$$\{a : g(a) < k(a)\} \supseteq \{a \in A : g'(a) < k(a)\} \cap \{a \in A : g(a) < h(a)\},$$

and this intersection is in G since the first set is in F and the second one is in G. Hence $g <_J k$, as desired. □

The next proposition indicates that if we take the case of $P \subseteq \prod_{a\in A} h(a)$ for some function h, then definitions and theorems about $(P, <_F)$ are equivalent to ones about the reduced product $\prod_{a\in A} h(a)/F$.

Proposition 1.23. *Suppose that $h \in {}^A\mathrm{Ord}$, and h takes only limit ordinal values. Then*

(i) If $X \subseteq \prod_{a\in A} h(a)$, then X is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$ iff $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(ii) $\mathrm{cf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{cf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iii) $\mathrm{tcf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{tcf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iv) If $X \subseteq \prod_{a\in A} h(a)$ and $f \in \prod_{a\in A} h(a)$, then f is an exact upper bound for X iff $[f]$ is an exact upper bound for $\{[g] : g \in X\}$.

Proof. (i) is immediate from Proposition 1.7. For (ii), if X is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$, then clearly $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$, by Proposition 1.7 again; so $\geq$ holds. Now suppose that $\{[f] : f \in Y\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$. Given $g \in \prod_{a\in A} h(a)$, choose $f \in Y$ such that $[g] <_I [f]$. Then $g <_I f$. So Y is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$, and $\leq$ holds.

(iii) and (iv) are proved similarly. □

The following obvious proposition will be useful.

Proposition 1.24. *Suppose that $F \cup \{f, g\} \subseteq {}^A\mathrm{Ord}$, I is an ideal on A, and $f =_I g$. Suppose that f is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$. Then also g is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$, respectively.* □

Here is our simplest existence theorem for exact upper bounds.

- If X is a collection of members of ${}^A\mathrm{Ord}$, then $\sup X \in {}^A\mathrm{Ord}$ is defined by

$$(\sup X)(a) = \sup\{f(a) : f \in X\}.$$

Proposition 1.25. *Suppose that $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda\rangle$ is an increasing sequence of members of ${}^A Ord$ in the partial ordering $<$ of everywhere dominance. (That is, $f < g$ iff $f(a) < g(a)$ for all $a \in A$.) Then $\sup(\mathrm{rng}(f))$ is an exact upper bound for f, and $\mathrm{cf}((\sup(\mathrm{rng}(f)))(a)) = \lambda$ for every $a \in A$.*

Proof. Again we apply Proposition 1.19. For brevity let $h = \sup(\mathrm{rng}(f))$. Then clearly h is an upper bound for f. Now suppose that $k \in {}^A\mathrm{Ord}$ and $k < h$. Then for every $a \in A$ we have $k(a) < h(a)$, and hence there is a $\xi_a < \lambda$ such that $k(a) < f_{\xi_a}(a)$. Let $\eta = \sup_{a \in A} \xi_a$. So $\eta < \lambda$ since λ is regular and greater than $|A|$. Clearly $k < f_\eta$, as desired. $\square$

We now prove some results which show that under a weak hypothesis we can restrict attention to $\prod A$ for A a nonempty set of infinite regular cardinals instead of $\prod_{a \in A} h(a)$, as far as cofinality notions are concerned. Here $\prod A$ consists of all choice functions f with domain A; $f(a) \in a$ for all $a \in A$.

Proposition 1.26. *Suppose that $h \in {}^A\mathrm{Ord}$ and $h(a)$ is a limit ordinal for every $a \in A$. For each $a \in A$, let $S(a) \subseteq h(a)$ be cofinal in $h(a)$ with order type $\mathrm{cf}(h(a))$. Suppose that I is a proper ideal on A. Then*

(i) $\mathrm{cf}(\prod_{a \in A} h(a), <_I, \leq_I) = \mathrm{cf}(\prod_{a \in A} S(a), <_I, \leq_I)$ *and*

(ii) $\mathrm{tcf}(\prod_{a \in A} h(a), <_I, \leq_I) = \mathrm{tcf}(\prod_{a \in A} S(a), <_I, \leq_I)$.

Proof. For each $f \in \prod h$ define $g_f \in \prod_{a \in A} S(a)$ by setting

$$g_f(a) = \text{ least } \alpha \in S(a) \text{ such that } f(a) \leq \alpha.$$

We prove (i): suppose that $X \subseteq \prod_{a \in A} h(a)$ and X is cofinal in $(\prod_{a \in A} h(a), <_I, \leq_I)$; we show that $\{g_f : f \in X\}$ is cofinal in $\mathrm{cf}(\prod_{a \in A} S(a), <_I, \leq_I)$, and this will prove $\geq$. So, let $k \in \prod_{a \in A} S(a)$. Thus $k \in \prod_{a \in A} h(a)$, so there is an $f \in X$ such that $k <_I f$. Since $f \leq g_f$ (everywhere), it follows that $k <_I g_f$, as desired. Conversely, suppose that $Y \subseteq \prod_{a \in A} S(a)$ and Y is cofinal in $(\prod_{a \in A} S(a), <_I)$; we show that also Y is cofinal in $\prod_{a \in A} h(a)$, and this will prove $\leq$ of the claim. Let $f \in \prod_{a \in A} h(a)$. Then $f \leq g_f$ (everywhere), and there is a $k \in Y$ such that $g_f <_I k$; so $f <_I k$, as desired.

This finishes the proof of (i).

For (ii), first suppose that $\mathrm{tcf}(\prod_{a \in A} h(a), <_I, \leq_I)$ exists; call it λ. Thus λ is an infinite regular cardinal. Let $\langle f_i : i < \lambda \rangle$ be a $<_I$-increasing cofinal sequence in $\prod h$. We claim that $g_{f_i} \leq_I g_{f_j}$ if $i < j < \lambda$. In fact, if $a \in A$ and $f_i(a) < f_j(a)$, then $f_i(a) < f_j(a) \leq g_{f_j}(a) \in S(a)$, and so by the definition of g_{f_i} we get $g_{f_i}(a) \leq g_{f_j}(a)$. This implies that $g_{f_i} \leq_I g_{f_j}$. Now $\mathrm{cf}(\prod_{a \in A} h(a), <_I, \leq_I) = \lambda$, so for any $B \in [\lambda]^{<\lambda}$ there is a $j < \lambda$ such that $\forall i \in B[g_{f_i} <_I f_j \leq g_{f_j}]$. It follows that we can take a subsequence of $\langle g_{f_i} : i < \lambda \rangle$ which is strictly increasing modulo I; it is also clearly cofinal, and hence $\lambda = \mathrm{tcf}(\prod_{a \in A} S(a), <_I, \leq_I)$ by Proposition 2.16.

Conversely, suppose that $\mathrm{tcf}(\prod_{a \in A} S(a), <_I, \leq_I)$ exists; call it λ. Let $\langle f_i : i < \lambda \rangle$ be a $<_I$-increasing cofinal sequence in $\prod_{a \in A} S(a)$. Then it is also a sequence showing that $\mathrm{tcf}(\prod_{a \in A} h(a), <_I)$ exists and equals $\mathrm{tcf}(\prod_{a \in A} S(a), <_I, \leq_I)$. $\square$

Proposition 1.27. *Suppose that $\langle L_a : a \in A \rangle$ and $\langle M_a : a \in A \rangle$ are systems of double ordered sets such that each L_a and M_a has no last element under $<$. Suppose that L_a is isomorphic to M_a for all $a \in A$. Let I be any ideal on A. Then*

$$\left(\prod_{a \in A} L_a, <_I, \leq_I \right) \cong \left(\prod_{a \in A} M_a, <_I, \leq_I \right). \qquad \square$$

Putting the last two propositions together, we see that to determine cofinality and true cofinality of $(\prod h, <_I, \leq_I)$, where $h \in {}^A\mathrm{Ord}$ and $h(a)$ is a limit ordinal for all $a \in A$, it suffices to take the case in which each $h(a)$ is an infinite regular cardinal. (One passes from $h(a)$ to $S(a)$ and then to $\mathrm{cf}(h(a))$.)

We can still make a further reduction, given in the following useful lemma.

Lemma 1.28. (Rudin-Keisler) *Suppose that c maps the set A into the class of regular cardinals, and $B = \{c(a) : a \in A\}$ is its range. For any filter F over A, define its Rudin-Keisler projection G on B by*

$$X \in G \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}[X] \in F.$$

Then G is a filter on B, and there is an isomorphism h of $\prod B/G$ into $\prod_{a\in A} c(a)/F$ such that for any $e \in \prod B$ we have $h(e/G) = \langle e(c(a)) : a \in A\rangle/F$.

If $|A| < \min(B)$, then the range of h is cofinal in $\prod_{a\in A} c(a)/F$, and we have
(i) $\mathrm{cf}(\prod B/G) = \mathrm{cf}(\prod_{a\in A} c(a)/F$ *and*
(ii) $\mathrm{tcf}(\prod B/G) = \mathrm{tcf}(\prod_{a\in A} c(a)/F)$.

Proof. Clearly G is a filter. Next, for any $e \in \prod B$ let $\overline{e} = \langle e(c(a)) : a \in A\rangle$. Then for any $e_1, e_2 \in \prod B$ we have

$$\begin{aligned} e_1 =_G e_2 \quad &\text{iff} \quad \{b \in B : e_1(b) = e_2(b)\} \in G \\ &\text{iff} \quad c^{-1}[\{b \in B : e_1(b) = e_2(b)\}] \in F \\ &\text{iff} \quad \{a \in A : e_1(c(a)) = e_2(c(a))\} \in F \\ &\text{iff} \quad \overline{e_1} =_F \overline{e_2}. \end{aligned}$$

This shows that h exists as indicated and is one-one. Similarly, h preserves $<_F$ and $\leq_F$ in each direction. So the first part of the lemma holds.

Now suppose that $|A| < \min(B)$. Let H be the range of h. By Proposition 1.16, (i) and (ii) follow from H being cofinal in $\prod_{a\in A} c(a)/F$. Let $g \in \prod_{a\in A} c(a)$. Define $e \in \prod B$ by setting, for any $b \in B$,

$$e(b) = \sup\{g(a) : a \in A \text{ and } c(a) = b\}.$$

The additional supposition implies that $e \in \prod B$. Now note that $\{a \in A : g(a) \leq e(c(a))\} = A \in F$, so that $g/F \leq h(e/G)$, as desired. $\square$

According to these last propositions, the calculation of true cofinalities for double orders of the form $(\prod_{a\in A} h(a), <_I, \leq_I)$, with $h \in {}^A\mathrm{Ord}$ and $h(a)$ a limit ordinal for every $a \in A$, and with $|A| < \min(\{\mathrm{cf}(h(a) : a \in A)$, reduces to the calculation of true cofinalities of double orders of the form $(\prod B, <_J, \leq_J)$ with B a set of regular cardinals with $|B| < \min(B)$.

Lemma 1.29. *If $(P_i, <_i, \leq_i)$ is a double order with true cofinality λ_i for each $i \in I$ and D is an ultrafilter on I, then* $\mathrm{tcf}(\prod_{i\in I} \lambda_i/D) = \mathrm{tcf}(\prod_{i\in I} P_i/D)$.

Proof. Note that $\prod_{i\in I} \lambda_i/D$ is a linear order, and so its true cofinality μ exists and equals its cofinality. So the lemma is asserting that the ultraproduct $\prod_{i\in I} P_i/D$ has μ as true cofinality.

Let $\langle g_\xi : \xi < \mu \rangle$ be a sequence of members of $\prod_{i \in I} \lambda_i$ such that $\langle g_\xi / D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} \lambda_i / D$. For each $i \in I$ let $\langle f_{\xi,i} : \xi < \lambda_i \rangle$ be strictly increasing and cofinal in $(P_i, <_i)$. For each $\xi < \mu$ define $h_\xi \in \prod_{i \in I} P_i$ by setting $h_\xi(i) = f_{g_\xi(i),i}$. We claim that $\langle h_\xi / D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} P_i / D$ (as desired).

To prove this, first suppose that $\xi < \eta < \mu$. Then

$$\{i \in I : h_\xi(i) < h_\eta(i)\} = \{i \in I : f_{g_\xi(i),i} <_i f_{g_\eta(i),i}\} = \{i \in I : g_\xi(i) < g_\eta(i)\} \in D;$$

so $h_\xi / D < h_\eta / D$.

Now suppose that $k \in \prod_{i \in I} P_i$; we want to find $\xi < \mu$ such that $k/D < h_\xi / D$. Define $l \in \prod_{i \in I} \lambda_i$ by letting $l(i)$ be the least $\xi < \mu$ such that $k(i) < f_{\xi,i}$. Choose $\xi < \mu$ such that $l/D < g_\xi / D$. Now if $l(i) < g_\xi(i)$, then $k(i) < f_{l(i),i} <_i f_{g_\xi(i),i} = h_\xi(i)$. So $k/D < h_\xi / D$. □

If A is a set of limit ordinals, we denote $\prod_{a \in A} a$ by $\prod A$.

Now suppose that A is a set of regular cardinals. We define

$$\mathrm{pcf}(A) = \left\{ \mathrm{cf}\left(\prod A/D \right) : D \text{ is an ultrafilter on } A \right\}.$$

By definition, $\mathrm{pcf}(\emptyset) = \emptyset$.

This is the main notion studied in these notes. We begin with a very easy proposition as an introduction. It is basic for further results, which will come after some combinatorial preliminaries.

Proposition 1.30. *Let A and B be sets of regular cardinals.*
(i) $A \subseteq \mathrm{pcf}(A)$.
(ii) If $A \subseteq B$, *then* $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(B)$.
(iii) $\mathrm{pcf}(A \cup B) = \mathrm{pcf}(A) \cup \mathrm{pcf}(B)$.
(iv) If $B \subseteq A$, *then* $\mathrm{pcf}(A) \backslash \mathrm{pcf}(B) \subseteq \mathrm{pcf}(A \backslash B)$.
(v) If A is finite, then $\mathrm{pcf}(A) = A$.
(vi) If $B \subseteq A$, B is finite, and A is infinite, then $\mathrm{pcf}(A) = \mathrm{pcf}(A \backslash B) \cup B$.
(vii) $\min(A) = \min(\mathrm{pcf}(A))$.
(viii) If A is infinite, then the first ω members of A are the same as the first ω members of $\mathrm{pcf}(A)$.

Proof. (i): For each $a \in A$, the principal ultrafilter with $\{a\}$ as a member shows that $a \in \mathrm{pcf}(A)$.

(ii): Any ultrafilter F on A can be extended to an ultrafilter G on B. The mapping $[f] \mapsto [f]$ is easily seen to be an isomorphism of $\prod A/F$ onto $\prod B/G$. Note here that $[f]$ is used in two senses, one for an element of $\prod A/F$, where each member of $[f]$ is in $\prod A$, and the other for an element of $\prod B/G$, with $[f]$ having members in the larger set $\prod B$.

(iii): $\supseteq$ holds by (ii). Now suppose that D is an ultrafilter on $A \cup B$. Then $A \in D$ or $B \in D$, and this proves $\subseteq$.

(iv): Suppose that $B \subseteq A$ and $\lambda \in \mathrm{pcf}(A) \backslash \mathrm{pcf}(B)$. Let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $B \notin D$, as otherwise $\lambda \in \mathrm{pcf}(B)$. So $A \backslash B \in D$, and so $\lambda \in \mathrm{pcf}(A \backslash B)$.

(v): If A is finite, then every ultrafilter on A is principal.

(vi): We have

$$\begin{aligned}\mathrm{pcf}(A) &= \mathrm{pcf}(A\backslash B) \cup \mathrm{pcf}(B) \quad \text{by (iii)}\\ &= \mathrm{pcf}(A\backslash B) \cup B \quad \text{by (v)}\end{aligned}$$

(vii): Let $a = \min(A)$. Thus $a \in \mathrm{pcf}(A)$ by (i). Suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda < a$; we want to get a contradiction. Say $\langle [g_\xi] : \xi < \lambda\rangle$ is strictly increasing and cofinal in $\prod A/D$. Now define $h \in \prod A$ as follows: for any $b \in A$, $h(b) = \sup\{g_\xi(b) + 1 : \xi < \lambda\}$. Thus $[g_\xi] < [h]$ for all $\xi < \lambda$, contradiction.

(viii): Let $\kappa_0 < \kappa_1 < \cdots$ be the first ω elements of A. Suppose that $\lambda \in \mathrm{pcf}(A)\backslash A$ and $\lambda < \kappa_m$ for some $m \in \omega$; we want to get a contradiction. By (vi) we have

$$\mathrm{pcf}(A) = \mathrm{pcf}(A\backslash\{\kappa_0, \ldots, \kappa_m\}) \cup \{\kappa_0, \ldots, \kappa_m\},$$

and it follows that $\lambda \in \mathrm{pcf}(A\backslash\{\kappa_0, \ldots, \kappa_m\})$. Hence $\kappa_{m+1} \leq \lambda$ by (vii), contradiction. □

A set A is *progressive* iff A is an infinite set of regular cardinals and $|A| < \min(A)$.

If $\alpha < \beta$ are ordinals, then $(\alpha, \beta)_{\mathrm{reg}}$ is the set of all regular cardinals κ such that $\alpha < \kappa < \beta$. Similarly for $[\alpha, \beta)_{\mathrm{reg}}$, etc. All such sets are called *intervals of regular cardinals.*

Proposition 1.31. *Assume that A is a progressive set, then*

(i) Every infinite subset of A is progressive.

(ii) If α is an ordinal and $A \cap \alpha$ is unbounded in α, then α is a singular cardinal.

(iii) If A is an interval of regular cardinals, then A does not have any weak inaccessible as a member, except possibly its first element.

(iv) If A is an infinite interval of regular cardinals, then there is a singular cardinal λ such that $A \cap \lambda$ is unbounded in λ and $A\backslash\lambda$ is finite.

Proof. (i): Obvious.

(ii): Obviously α is a cardinal. Now $A\cap\alpha$ is cofinal in α and $|A\cap\alpha| \leq |A| < \min(A) < \alpha$. Hence α is singular.

(iii): If $\kappa \in A$, then by (ii), $A \cap \kappa$ cannot be unbounded in κ; hence κ is a successor cardinal, or is the first element of A.

(iv) Let $\sup(A) = \aleph_{\alpha+n}$ with α a limit ordinal. Since A is an infinite interval of regular cardinals, it follows that $A \cap \aleph_\alpha$ is unbounded in $\aleph_\alpha$, and hence by (ii), $\aleph_\alpha$ is singular. Hence the desired conclusion follows. □

The following results about cofinality will be useful close to the end of these notes.

Proposition 1.32. *If $\kappa \leq \mu$ are infinite cardinals, then*

$$(*) \qquad |[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa.$$

Proof. Let $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$, and let $\langle Y_i : i < \lambda\rangle$ be an enumeration of a cofinal subset of $\mathrm{cf}([\mu]^\kappa, \subseteq)$. For each $i < \lambda$ let f_i be a bijection from Y_i to κ. Now the inequality $\geq$ in $(*)$ is clear. For the other direction, we define an injection g of $[\mu]^\kappa$ into $\lambda \times \mathscr{P}(\kappa)$, as follows.

Given $E \in [\mu]^\kappa$, let $i < \lambda$ be minimum such that $E \subseteq Y_i$, and define $g(E) = (i, f_i[E])$. Clearly g is one-one. $\square$

Proposition 1.33. *(i) If $\kappa_1 < \kappa_2 \leq \mu$, then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \leq \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

(ii) $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$.
(iii) If $\kappa^+ \leq \mu$, then $\mathrm{cf}([\mu]^\kappa, \subseteq) \leq \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+$.
(iv) If $\kappa \leq \mu_1 < \mu_2$, then $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \leq \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.
(v) If $\kappa \leq \mu$, then $\mathrm{cf}([\mu^+]^\kappa, \subseteq) \leq \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot \mu^+$.
(vi) $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$, *while for $m \in \omega \backslash 1$,* $\mathrm{cf}([\aleph_m]^{\aleph_0}) = \aleph_m$.
(vii) $\mathrm{cf}([\mu]^{\leq\kappa}, \subseteq) = \mathrm{cf}([\mu]^\kappa, \subseteq)$.

Proof. (i): Let $M \subseteq [\mu]^{\kappa_2}$ be cofinal in $([\mu]^{\kappa_2}, \subseteq)$ of size $\mathrm{cf}([\mu]^{\kappa_2}, \subseteq)$, and let $N \subseteq ([\kappa_2]^{\kappa_1}, \subseteq)$ be cofinal in $([\kappa_2]^{\kappa_1}, \subseteq)$ of size $\mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq)$. For each $X \in M$ let $f_X : \kappa_2 \to X$ be a bijection. It suffices now to show that $\{f_X[Y] : X \in M, Y \in N\}$ is cofinal in $([\mu]^{\kappa_1}, \subseteq)$. Suppose that $W \in [\mu]^{\kappa_1}$. Choose $X \in M$ such that $W \subseteq X$. Then $f_X^{-1}[W] \in [\kappa_2]^{\kappa_1}$, so there is a $Y \in N$ such that $f_X^{-1}[W] \subseteq Y$. Then $W \subseteq f_X[Y]$, as desired.

(ii): The set $\{\gamma < \kappa^+ : |\gamma \backslash \kappa| = \kappa\}$ is clearly cofinal in $([\kappa^+]^\kappa$. If M is a nonempty subset of $[\kappa^+]^\kappa$ of size less than κ^+, then $|\bigcup M| = \kappa$, and $(\bigcup M) + 1$ is a member of $[\kappa^+]^\kappa$ not covered by any member of M. So (ii) holds.

(iii): Immediate from (i) and (ii).

(iv): Let $M \subseteq [\mu_2]^\kappa$ be cofinal of size $\mathrm{cf}([\mu_2]^\kappa, \subseteq)$. Let $N = \{X \cap \mu_1 : X \in M\} \backslash [\mu_1]^{<\kappa}$. It suffices to show that N is cofinal in $\mathrm{cf}([\mu_1]^\kappa, \subseteq)$. Suppose that $X \in [\mu_1]^\kappa$. Then also $X \in [\mu_2]^\kappa$, so we can choose $Y \in M$ such that $X \subseteq Y$. Clearly $X \subseteq Y \cap \mu_1 \in N$, as desired.

(v): For each $\gamma \in [\mu, \mu^+)$ let f_γ be a bijection from γ to μ. Let $E \subseteq [\mu]^\kappa$ be cofinal in $([\mu]^\kappa, \subseteq)$ and of size $\mathrm{cf}([\mu]^\kappa, \subseteq)$. It suffices to show that $\{f_\gamma^{-1}[X] : \gamma \in [\mu, \mu^+), X \in E\}$ is cofinal in $([\mu^+]^\kappa, \subseteq)$. So, take any $Y \in [\mu^+]^\kappa$. Choose $\gamma \in [\mu, \mu^+)$ such that $Y \subseteq \gamma$. Then $f_\gamma[Y] \in [\mu]^\kappa$, so we can choose $X \in E$ such that $f_\gamma[Y] \subseteq X$. Then $Y \subseteq f_\gamma^{-1}[X]$, as desired.

(vi): Clearly $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$. By induction it is clear from (v) that $\mathrm{cf}([\aleph_m]^{\aleph_0}) \leq \aleph_m$. For $m > 0$ equality must hold, since if $X \subseteq [\aleph_m]^{\aleph_0}$ and $|X| < \aleph_m$, then $\bigcup X < \aleph_m$, and no denumerable subset of $\aleph_m \backslash \bigcup X$ is contained in a member of X.

(vii): Clear. $\square$