# Math 6000 HW 7

**Due Friday, March 20**
Submissions will also be accepted Monday, April 2 (after spring break).

**Exercises:** (7.1.1), **7.2.1**, **7.3.2**, **7.4.1**, **7.4.2**, **7.5.1**, **7.5.2**, **7.5.3**

To pass this assignment, submit well-written, complete, correct solutions to at least **four** exercises from among the non-parenthesized numbers above.

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(7.1.1) Verify that the positive diagram of a structure is logically equivalent to a subset of all its *unnested* atomic formulas.

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**7.2.1.** Define recursively a rank function $r$ from a finite partial ordering $\mathcal{A} = (A, \prec)$ to $\mathbb{N}$ by setting $r(a) = 0$ for every $\prec$-minimal element $a$ in $A$, and setting $r(b) = 1$ for every $\prec$-minimal element $b$ in $A \setminus \{a \in A : r(a) = 0\}$, and so forth.

Use this to extend $\prec$ to a linear order on $A$.

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**7.3.2.** Find the four isomorphism types of countable dense linear orderings (and justify your answer).

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**7.4.1.** Given orderings $\mathcal{X}$ and $\mathcal{Y}$, one defines their **sum** $\mathcal{X} + \mathcal{Y}$, to be **disjoint union** of $X$ and $Y$ equipped with the order extending the orders of $\mathcal{X}$ and $\mathcal{Y}$ defined by $x < y$ for all $x \in X$ and $y \in Y$.

Prove that the sum of two well-orderings again is a well-ordering.

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**7.4.2.** Show that the class of well-orderings is not axiomatizable (as a class of $L_<$-structures).

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**7.5.1.** Given ordinals $\alpha$ and $\beta$, their sum $\alpha + \beta$ is defined to be the uniquely determined ordinai which is isomorphic (as an ordering) to the sum of the ordering $\alpha$ and $\beta$ (in the sense of Exercise 7.4.1).

( a ) Show that the addition on $\omega$ coincides with the usual addition of natural numbers.

( b ) Prove that the successor of $\alpha$ is the sum of $\alpha$ and 1 (this justifying the notation $\alpha + 1$ for the successor of $\alpha$).

( c ) Verify $1 + \alpha = \alpha$ for all infinite $\alpha \in \mathbf{On}$ (and that addition of ordinals is not commutative).

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**7.5.2.** Let $X$ be a subset of **On**. Prove that $\bigcup X$ is the supremum of $X$ in $(\mathbf{On}, <)$.

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**7.5.3.** Show that the complete $L_{<}$-theory of any infinite ordinai number has a non-well-founded model, i.e., a model that is not well-ordered.