6. Basic model theory

We survey important notions and results in model theory.

Isomorphisms

Theorem 6.1. *Suppose that h is an isomorphism from $\overline{A}$ onto $\overline{B}$, where these are $\mathscr{L}$-structures. Suppose that $a \in {}^{\omega}A$, σ is a term, and φ is any formula. Then $h(\sigma^{\overline{A}}(a)) = \sigma^{\overline{B}}(h \circ a)$. And $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[h \circ a]$. Finally, $\overline{A}$ and $\overline{B}$ are elementarily equivalent.*

Proof. We prove the first statement by induction on σ. For σ a variable v_i we have $h(v_i^{\overline{A}}(a)) = h(a_i) = (h \circ a)_i = v_i^{\overline{B}}(h \circ a)$. For k an individual constant, $h(k^{\overline{A}}(a)) = h(k^{\overline{A}}) = k^{\overline{B}} = k^{\overline{B}}(h \circ a)$. The inductive step, with $\tau = \mathbf{F}\sigma_0 \ldots \sigma_{m-1}$:

$$\begin{aligned}
h(\tau^{\overline{A}}(a)) &= h(\mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))) \\
&= \mathbf{F}^{\overline{B}}(h(\sigma_0^{\overline{A}}(a)), \ldots, h(\sigma^{\overline{A}}(a))) \\
&= \mathbf{F}^{\overline{B}}(\sigma_0^{\overline{B}}(h \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(h \circ a)) \\
&= \tau^{\overline{B}}(h \circ a),
\end{aligned}$$

as desired.

We prove the second statement by induction on φ. The atomic cases:

$$\begin{aligned}
\overline{A} \models (\sigma = \tau)[a] \text{ iff } & \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \\
\text{iff } & h(\sigma^{\overline{A}}(a)) = h(\tau^{\overline{A}}(a)) \\
\text{iff } & \sigma^{\overline{B}}(h \circ a) = \tau^{\overline{B}}(h \circ a) \\
\text{iff } & \overline{B} \models (\sigma = \tau)[h \circ a];
\end{aligned}$$

$$\begin{aligned}
\overline{A} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}[a] \text{ iff } & (\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \in \mathbf{R}^{\overline{A}} \\
\text{iff } & (h(\sigma_0^{\overline{A}}(a)), \ldots, h(\sigma_{m-1}^{\overline{A}}(a))) \in \mathbf{R}^{\overline{B}} \\
\text{iff } & (\sigma_0^{\overline{B}}(h \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(h \circ a)) \in \mathbf{R}^{\overline{B}} \\
\text{iff } & \overline{B} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}[h \circ a].
\end{aligned}$$

The induction steps for $\neg$ and $\wedge$:

$$\begin{aligned}
\overline{A} \models \neg\psi[a] \text{ iff } & \text{not}(\overline{A} \models \psi[a]) \\
\text{iff } & \text{not}(\overline{B} \models \psi[h \circ a]) \\
\text{iff } & \overline{B} \models \neg\psi[h \circ a];
\end{aligned}$$

$$\begin{aligned}
\overline{A} \models (\psi \wedge \chi)[a] \text{ iff } & \overline{A} \models \psi[a]) \text{ and } \overline{A} \models \chi[a] \\
\text{iff } & (\overline{B} \models \psi[h \circ a]) \text{ and } \overline{B} \models \chi[h \circ a] \\
\text{iff } & \overline{B} \models (\psi \wedge \chi)[h \circ a].
\end{aligned}$$

Finally, for the $\exists$ induction step, by symmetry we go one direction only. Suppose that $\overline{A} \models \exists v_i \psi[a]$. Choose $u \in A$ such that $\overline{A} \models \psi[a^i_u]$. Then by the inductive hypothesis, $\overline{B} \models \psi[h \circ a^i_u]$. Now $h \circ a^i_u = (h \circ a)^i_{h(u)}$, so $\overline{B} \models \varphi[h \circ a]$.

The final statement of the theorem now follows in a clear fashion. □

Although this theorem is simple, it is important and useful. An example of a non-obvious use of it is to check this little fact: in the language with equality alone, with any nonempty set A as a structure to look at, the only definable subsets of A are $\emptyset$ and A. They are defined by the following formulas $\varphi(x)$: $x \neq x$ and $x = x$. Suppose that $\emptyset \subset X \subset A$ and $X = \{a \in A : A \models \psi[a]\}$, where ψ is a formula having only x free. Choose $a \in X$ and $b \in A \backslash X$. Let h be the transposition (a, b), as a permutation of A. (Here (a, b) is not the ordered pair (a, b).) Thus $A \models \psi[a]$, so by the theorem, $A \models \psi[b]$, so $b \in X$, contradiction.

Note that a theory is complete iff any two of its models are elementarily equivalent.

Elimination of quantifiers

The elimination of quantifier method is best understood by going through a simple example. Roughly speaking, a theory admits elimination of quantifiers if every formula in the theory is equivalent in the theory to a formula built up from simple sentences and quantifier-free formulas using only sentential connectives. In the example we give, only a very simple sentence is involved in these building blocks. The proof is rather direct, and for more complicated theories indirect methods are usually easier. We return to this later in this chapter.

Theorem 6.2. *In the theory $(\mathscr{L}, \Gamma)$ of dense linear order without first or last elements, for every formula φ there is a formula ψ built up from atomic formulas and the sentence $\exists v_0(v_0 = v_0)$ using $\neg$ and $\wedge$, with the same free variables as φ, such that $\Gamma \models \varphi \leftrightarrow \psi$.*

Proof. To clarify what this theory is: the language $\mathscr{L}$ has only one non-logical constant, a binary relation symbol $<$. Γ consists of the following sentences:

$$\neg\exists v_0(v_0 < v_0);$$
$$\forall v_0 \forall v_1 \forall v_2(v_0 < v_1 \wedge v_1 < v_2 \rightarrow v_0 < v_2);$$
$$\forall v_0 \forall v_1(v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0);$$
$$\forall v_0 \forall v_1[v_0 < v_1 \rightarrow \exists v_2(v_0 < v_2 \wedge v_2 < v_1)];$$
$$\forall v_0 \exists v_1(v_1 < v_0);$$
$$\forall v_0 \exists v_1(v_0 < v_1).$$

Now to prove the theorem, we proceed by induction on φ. The atomic case, and the induction steps involving $\neg$ and $\wedge$ are obvious. Now assume that we know the result for φ', and φ is $\exists x\varphi'$. Thus there is a quantifier-free formula ψ' with the same free variables as φ' such that $\Gamma \models \varphi' \leftrightarrow \psi'$. Hence $\Gamma \models \exists x\varphi' \leftrightarrow \exists x\psi'$. So we have arrived at this conclusion:

(*) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is quantifier-free.

Now let θ be a formula in disjunctive normal form with the same free variables as χ such that $\models \chi \leftrightarrow \theta$. So $\models \exists x\chi \leftrightarrow \exists x\theta$. Using the universally valid formula $\exists x(\mu \vee \rho) \leftrightarrow \exists x\mu \vee \exists x\rho$, we thus see:

(**) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of atomic formulas and their negations.

Now we can make several additional simplifications. By the universally valid formula $\models \exists x(\mu \wedge \rho) \leftrightarrow \exists x\mu \wedge \rho$ if x does not occur in ρ, we may assume that each conjunct of χ actually involves x. We may assume that in χ no formula is repeated, and no formula is found along with its negation (if the latter happens, φ is actually equivalent to $\neg\exists v_0(v_0 = v_0)$). The formulas $x = x$ and $\neg(x = x)$ may be eliminated. The formula $y = x$ can be replaced by $x = y$. If $x = y$ is one of the conjuncts of χ, then $\models \exists x\chi \leftrightarrow \theta$, where θ is obtained from χ by replacing each occurrence of x by y. And we may rearrange the conjuncts. So we arrive at:

(***) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of the following form:

$$(****)\qquad \begin{aligned} &\neg(x = y_1) \wedge \ldots \wedge \neg(x = y_k) \wedge \\ &x < z_1 \wedge \ldots \wedge x < z_l \wedge \\ &w_1 < x \wedge \ldots \wedge w_m < x \wedge \\ &\neg(x < u_1) \wedge \ldots \wedge \neg(x < u_n) \wedge \\ &\neg(s_1 < x) \wedge \ldots \wedge \neg(s_p < x). \end{aligned}$$

Here some of the integers k, l, m, n, p can be zero; the variables $y_1, \ldots, y_k$ are all distinct among themselves and different from x. Similarly for the z_i's and w_i's. The variables u_i are distinct among themselves, but one of them might equal x. Similarly for the variables s_i.

What we have described so far is a procedure common to most applications of the elimination of quantifiers method. In particular, we have not used the order axioms yet.

Now we can use those axioms: since $\Gamma \models \forall v_0 \neg(v_0 < v_0)$, we may assume that none of the u_i's or s_i's are equal to x. Also, $\Gamma \models (\neg(v_0 < v_1) \leftrightarrow v_0 = v_1 \vee v_1 < v_0)$; so using this, distributive laws, and the way equality was eliminated above, we may assume that $n = 0 = p$. Furthermore, $\Gamma \models (\neg(v_0 = v_1) \leftrightarrow v_0 < v_1 \vee v_1 < v_0$. So again using distributive laws we may assume that $k = 0$. Thus we have arrived at

(*****) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of the following form:

$$x < z_1 \wedge \ldots \wedge x < z_l \wedge w_1 < x \wedge \ldots \wedge w_m < x.$$

If $l = 0$, this is equivalent under Γ to $\exists v_0(v_0 = v_0)$ by the "no last element" sentence. If $m = 0$, it is equivalent under Γ to $\exists v_0(v_0 = v_0)$ by the "no first element" sentence. If $l \neq 0 \neq m$, it is equivalent under Γ to

$$z_1 < w_1 \wedge \ldots \wedge z_1 < w_m \wedge$$

$$z_2 < w_1 \wedge \ldots \wedge z_2 < w_m \wedge$$
$$\ldots$$
$$z_l < w_1 \wedge \ldots \wedge z_l < w_m,$$

by the "denseness" sentence. □

Corollary 6.3. *The only definable subsets of* $\mathbb{Q}$ *in the structure* $(\mathbb{Q}, <)$ *are* $\emptyset$ *and* $\mathbb{Q}$.

Proof. Suppose that A is a definable subset of $\mathbb{Q}$ in $(\mathbb{Q}, <)$. Thus there is a formula φ with only v_0 as a possible free variable such that $A = \{a \in \mathbb{Q} : (\mathbb{Q}, <) \models \varphi[a]\}$. By Theorem 6.2, let ψ be a formula built up from $\exists v_0(v_0 = v_0)$ and atomic formulas using only $\neg$ and $\wedge$, and with only v_0 as a possible free variable such that $\Gamma \models \varphi \leftrightarrow \psi$. This means that up to logical equivalence ψ is either $\exists v_0(v_0 = v_0)$ or $\neg\exists v_0(v_0 = v_0)$. So $A = \emptyset$ or $A = \mathbb{Q}$. □

Ehrenfeucht-Fraissé games

We explain now a game-theoretic characterization of elementary equivalence, due independently to Ehrenfeucht and Fraissé. Actually Fraissé's method does not explicitly involve games, while Ehrenfeucht's does; but they are really easily seen to be equivalent. We restrict ourselves to the game formulation.

The formulation of the game depends on the notion of a partial isomorphism. Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures. A *partial isomorphism* between $\overline{A}$ and $\overline{B}$ is a function f mapping a subset of A into B such that for any $m \in \omega$, any atomic formula φ with free variables among $v_0, \ldots, v_{m-1}$, and any $a \in {}^m\mathrm{dmn}(f)$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$.

Now we describe the $(m, \overline{A}, \overline{B})$-elementary game, where m is a positive integer and $\overline{A}$ and $\overline{B}$ are $\mathscr{L}$-structures. The game takes place between two players, ISO and NON-ISO, and each player gets m moves. For ease of reference we think of ISO as feminine, NON-ISO as maxculine. NON-ISO moves first, and the two players take turns through m plays, numbered $0, \ldots, m-1$. At his i-th turn, NON-ISO picks $\varepsilon \in \{0, 1\}$, and if $\varepsilon = 0$ he picks an element $a_i \in A$, while if $\varepsilon = 1$ he picks an element $b_i \in B$. Then ISO uses the ε that NON-ISO chose, and she picks $b_i \in B$ if $\varepsilon = 0$, and $a_i \in A$ if $\varepsilon = 1$ (the opposite thing from NON-ISO). Thus each play in the game produces a pair (a_i, b_i). At the end of the game, ISO wins if $\{(a_i, b_i) : i < m\}$ is a partial isomorphism, otherwise NON-ISO wins. A *winning strategy* for ISO is a rule that unambiguously tells her how to play at each step, given what has happened up to that point. More precisely, let us define a *state* to be a quadruple (a, b, ε, c) such that, for some $i < m - 1$ we have $a \in {}^iA$, $b \in {}^iB$, $\varepsilon \in \{0, 1\}$, and $c \in A$ if $\varepsilon = 0$, $c \in B$ if $\varepsilon = 1$. Then a *strategy* for ISO is a function F defined on all states such that $F(a, b, \varepsilon, c) \in B$ if $\varepsilon = 0$ and $F(a, b, \varepsilon, c) \in A$ if $\varepsilon = 1$. A *complete play* of the game is a triple $(\varepsilon, a, b) \in {}^m2 \times {}^mA \times {}^mB$. Such a complete play is *played according to the strategy F for ISO* provided that, for each $i < m - 1$, either

$\varepsilon_i = 0$ and $b_i = F(a \restriction i, b \restriction i, 0, a_i)$ or
$\varepsilon_i = 1$ and $a_i = F(a \restriction i, b \restriction i, 1, b_i)$.

Finally, we say that F is a *winning strategy* for ISO provided that ISO wins whenever she plays according to F.

Theorem 6.4. (Ehrenfeucht) *Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures, and let m be a positive integer. Suppose that* ISO *has a winning strategy for the $(m, \overline{A}, \overline{B})$-elementary game. Then $\overline{A} \models \varphi$ iff $\overline{B} \models \varphi$, for every sentence φ in prenex normal form with at most m initial quantifiers.*

Proof. Let F be a winning strategy for ISO for the $(m, \overline{A}, \overline{B})$-elementary game. Let φ be a sentence $Q_0 v_0 \ldots Q_{m-1} v_{m-1} \psi$ where ψ is quantifier-free and each Q_i is $\exists$ or $\forall$. If a sentence in prenex normal form has fewer than m initial quantifiers or some are repeated, then the outer most of the repeated ones can be deleted; then we can put at the beginning extra quantifiers on variables not appearing in the sentence to make the total equal to m. And if the variables are not $v_0, \ldots, v_{m-1}$ we can rename bound variables to make them $v_0, \ldots, v_{m-1}$ in that order. Thus it suffices to treat the case indicated.

We prove the following statement by downward induction, from $k = m$ to $k = 0$:

(1) For every $k \leq m$ and every complete play (ε, a, b) of the game according to F,

$$\overline{A} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[a \restriction k] \quad \text{iff} \quad \overline{B} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[b \restriction k].$$

The case $k = m$ is given by the assumption that ISO wins the game. Now suppose that (1) holds for $k + 1$; we prove it for k. So, assume the hypothesis of (1) for k. Because of the symmetry involved, we now take only the case $Q_k = \forall$, and prove only $\Rightarrow$. Thus assume that $\overline{A} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[a \restriction k]$. Let y be any element of B; we want to show that $\overline{B} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[b_0, \ldots, b_{k-1}, y]$. We now consider the following complete play of the game according to F. The players play just as in (ε, a, b) through stage k, thus producing $a_0, \ldots, a_{k-1}$ and $b_0, \ldots b_{k-1}$. Now NON-ISO chooses 1 and y. So ISO chooses (according to F) $x \in A$. Then let the play continue according to F, with NON-ISO choosing what he pleases, producing a complete play (ε', a', b'). . Since we are assuming

$$\overline{A} \models \forall v_k Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[a \restriction k],$$

we get

$$\overline{A} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[a_0, \ldots, a_{k-1}, x].$$

Then the inductive hypothesis yields

$$\overline{B} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[b_0, \ldots, b_{k-1}, y],$$

as desired. □

Corollary 6.5. (Ehrenfeucht) *If* ISO *has a winning strategy for the $(m, \overline{A}, \overline{B})$-elementary game for every positive integer m, then $\overline{A}$ and $\overline{B}$ are elementarily equivalent.*

We give one application of elementary games.

Proposition 6.6. *Let Γ be the theory of an infinite equivalence relation with infinitely many equivalence classes, each of which is infinite. Then Γ is complete. More precisely,* *the language has only one non-logical constant, a binary relation symbol E, and Γ consists of the following sentences:*

$\forall x[xEx]$

E is symmetric and transitive

$$\forall v_0 \ldots \forall v_{n-1} \left[\bigwedge_{i<j<n} \neg(v_i E v_j) \rightarrow \exists v_n \left[\bigwedge_{i<n} \neg(v_i E v_n) \right] \right]$$

for each positive integer n

$$\forall v_0 \ldots \forall v_{n-1} \left[\bigwedge_{i<j<n} \neg(v_i = v_j) \wedge \bigwedge_{i<j<n} v_i E v_j \rightarrow \exists v_n \left[v_0 E v_n \wedge \bigwedge_{i<n} \neg(v_i = v_n) \right] \right]$$

for each positive integer n.

Proof. Assume that $\overline{A}$ and $\overline{B}$ are models of Γ and m is a positive integer. The strategy of ISO is as follows. Suppose that we are at the i-th turn and NON-ISO chooses 0 and an element $a \in A$. The move of ISO depends on the following possibilities. If the turns so far have not produced a partial isomorphism, then ISO selects any element of B. Suppose that the turn so far have produced a partial isomorphism f.

Case 1. No element of A equivalent to a has been selected yet. Then ISO picks an element of B not equivalent to any element selected so far.

Case 2. There is an element $a' \in A$ which has already been selected which is equivalent to a, while a itself has not been previously selected. Then ISO picks an element of B equivalent to $f(a')$ which has not yet been selected.

Case 3. a has already been selected. Then ISO picks $f(a)$.

If NON-ISO choose 1 and an element of B, ISO does a similar thing, interchanging the roles of A and B.

Clearly this produces a partial isomorphism. $\square$

Diagrams

A language $\mathscr{L}'$ is an *expansion* of a language $\mathscr{L}$ iff $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding new non-logical symbols. Given an $\mathscr{L}$-structure $\overline{A}$, an *expansion* of $\overline{A}$ to $\mathscr{L}'$ is a structure obtained from $\mathscr{A}$ by adjoining interpretations of the new symbols.

Going the other way, $\mathscr{L}$ is called a *reduct* of $\mathscr{L}'$ and $\overline{A}$ is called a reduct of $\overline{B}$.

Right now we are interested in the case of adding new individual constants. Let $\mathscr{L}$ be any language, and let A be any set. Then $\mathscr{L}_A$ is the expansion of $\mathscr{L}$ by adding new individual constants c_a for $a \in A$. Given any $\mathscr{L}$-structure $\overline{B}$ and a system $\langle b_a : a \in A \rangle$ of elements of B, by $\overline{B}_b = (\overline{B}, b_a)_{a \in A}$ we mean the expansion of $\overline{B}$ to an $\mathscr{L}_A$-structure such that $c_a^{\overline{B}_b} = b_a$ for all $a \in A$. As a special case we have the expansion $(\overline{A}, a)_{a \in A}$ of $\overline{A}$ itself, which we denote by $\overline{A}_A$ instead of $\overline{A}_{\mathrm{Id}\restriction A}$.

Let $\mathscr{L}$ be a language and $\overline{A}$ an $\mathscr{L}$-structure. The *diagram* of $\overline{A}$, denoted by $\mathrm{Diag}(\overline{A})$, is the set of all atomic sentences and negations of atomic sentences which hold in the structure $\overline{A}_A$. This takes place in the language $\mathscr{L}_A$.

Lemma 6.7. *Assume that σ is a term in $\mathscr{L}$, $e \in {}^{\omega}A$, σ_e is obtained from σ by replacing each variable v_i by $c_{e(i)}$ (the individual constant of $\mathscr{L}_A$ associated with $e(i)$), $\overline{D}$ is an $\mathscr{L}$-structure, and $d \in {}^{A}D$.*

Then $\sigma^{\overline{D}}(d \circ e) = \sigma_e^{\overline{D}_d}$.

Proof. This is immediate from the proof of Lemma 4.8. □

Theorem 6.8. (Diagram lemma) *For any $\mathscr{L}$-structures $\overline{A}$ and $\overline{B}$, and any $f : A \to B$ the following conditions are equivalent:*

(i) f is an isomorphism from $\overline{A}$ into $\overline{B}$.

(ii) $(\overline{B}, f(a))_{a\in A}$ is a model of $\mathrm{Diag}(\overline{A})$.

Proof. (i)⇒(ii): Let f be an isomorphism from $\overline{A}$ into $\overline{B}$.

Suppose that σ and τ are variable-free terms of $\mathscr{L}_A$ and $\overline{A}_A \models \sigma = \tau$. Thus $\sigma^{\overline{A}_A} = \tau^{\overline{A}_A}$. Now there is a sequence $e \in {}^{\omega}A$ and terms ρ, ξ such that $\sigma = \rho_e$ and $\tau = \xi_e$. Thus $\rho_e^{\overline{A}_A} = \xi_e^{\overline{A}_A}$. By Lemma 6.7 we then get $\rho^{\overline{A}}(e) = \xi^{\overline{A}}(e)$. (The "$d$" in Lemma 6.7 is the identity.) By Proposition 2.4(i) we then get $\rho^{\overline{B}}(f \circ e) = \xi^{\overline{B}}(f \circ e)$. Then by Lemma 6.7 again we have $\sigma^{\overline{B}_f} = \rho_e^{\overline{B}_f} = \rho^{\overline{B}}(f \circ e) = \xi^{\overline{B}}(f \circ e) = \xi_e^{\overline{B}_f} = \tau^{\overline{B}_f}$. Hence $\overline{B}_f \models \sigma = \tau$.

Almost exactly the same proof shows that $\overline{A}_A \models \neg(\sigma = \tau)$ implies that $\overline{B}_f \models \neg(\sigma = \tau)$.

Next, suppose that R is an m-ary relation symbol, $\sigma_0, \ldots, \sigma_{m-1}$ are variable-free terms of $\mathscr{L}_A$, and $\overline{A}_A \models R\sigma_0 \ldots \sigma_{m-1}$. Thus $\langle \sigma_0^{\overline{A}_A}, \ldots, \sigma_{m-1}^{\overline{A}_A} \rangle \in R^{\overline{A}_A}$. Now there is a sequence $e \in {}^{\omega}A$ and terms ρ_i for $i < m$ such that $\rho_{ie} = \sigma_i$ for each $i < m$. Thus $\langle \rho_{0e}^{\overline{A}_A}, \ldots, \rho_{(m-1)e}^{\overline{A}_A} \rangle \in R^{\overline{A}_A}$. By Lemma 6.7 we then get $\langle \rho_0^{\overline{A}}(e), \ldots, \rho_{m-1}^{\overline{A}}(e) \rangle \in R^{\overline{A}_A}$. Hence $\langle f(\rho_0^{\overline{A}}(e)), \ldots, f(\rho_{m-1}^{\overline{A}}(e)) \rangle \in R^{\overline{B}}$. By Proposition 2.4(i), $f(\rho_i^{\overline{A}}(e)) = \rho_i^{\overline{B}}(f \circ e)$ for each $i < m$. By Lemma 6.7 again, $\rho_i^{\overline{B}}(f \circ e) = \rho_{ie}^{\overline{B}_f}$ for each $i < m$. Thus $\langle \rho_{0e}^{\overline{B}_f}, \ldots, \rho_{(m-1)e}^{\overline{B}_f} \rangle \in R^{\overline{B}_f}$. Since $\rho_{ie} = \sigma_i$ for each $i < m$, we infer that $\overline{B}_f \models R\sigma_0 \ldots \sigma_{m-1}$.

The same steps work for $\neg R\sigma_0 \ldots \sigma_{m-1}$.

(ii)⇒(i): Assume (ii). Assume that $\overline{B}_f$ is a model of the diagram of $\overline{A}$. We claim that f is the desired isomorphism. To show that f is one-to-one, suppose that $x, y \in A$ and $f(x) = f(y)$. Then $\overline{B}_f \models c_x = c_y$. Hence $\overline{A}_A \models c_x = c_y$, as otherwise $\neg(c_x = c_y) \in \mathrm{Diag}(\overline{A})$, contradicting the fact that $\overline{B}_f$ is a model of the diagram. Hence $x = y$. So f is one-one.

For k an individual constant, the sentence $c_{k^{\overline{A}}} = k$ is in $\mathrm{Diag}(\overline{A})$, hence $\overline{B}_f \models c_{k^{\overline{A}}} = k$, so that $f(k^{\overline{A}}) = c_{k^{\overline{A}}}^{\overline{B}_f} = k^{\overline{B}_f} = k^{\overline{B}}$.

Next, suppose that F is an m-ary operation symbol and $e \in {}^{m}A$; we want to show that $f(F^{\overline{A}}(e_0, \ldots, e_{m-1})) = F^{\overline{B}}(f(e_0), \ldots, f(e_{m-1}))$. Let $u = F^{\overline{A}}(e_0, \ldots, e_{m-1})$. Then $Fc_{e_0} \ldots c_{e_{m-1}} = c_u$ is in the diagram, and so $\overline{B}_f$ is a model of it. Hence $(Fc_{e_0} \ldots c_{m-1})^{\overline{B}_f} =$

$c_u^{\overline{B}_f}$, so

$$F^{\overline{B}_f}(f(e_0), \ldots, f(e_{m-1})) = f(u) = f(F^{\overline{A}}(e_0, \ldots, e_{m-1})),$$

as desired.

Finally, suppose that R is an m-ary relation symbol, $e \in {}^mA$, and $e \in R^{\overline{A}}$. Then $Rc_{e_0} \ldots c_{e_{m-1}}$ is in the diagram, and so it holds in $\overline{B}_f$. Hence $\langle f(e_0), \ldots, f(e_{m-1})\rangle \in R^{\overline{B}_f}$. Similarly starting with $e \notin R^{\overline{A}}$. □

For the next theorem, recall the notion of finite generation from just before Theorem 1.11.

Theorem 6.9. (Henkin) *Let Γ be a set of sentences in a language $\mathscr{L}$, and let $\overline{A}$ be an $\mathscr{L}$-structure. Then $\overline{A}$ can be isomorphically embedded in some model of Γ iff every finitely generated substructure of $\overline{A}$ can be so embedded.*

Proof. $\Rightarrow$ is trivial, so assume that every finitely generated substructure of $\overline{A}$ can be isomorphically embedded in a model of Γ; we want to show that $\overline{A}$ itself can be so embedded. By the diagram lemma it suffices to show that the set $\Gamma \cup \mathrm{Diag}(A)$ has a model. By the compactness theorem it suffices to take finite subsets Δ of Γ and Θ of $\mathrm{Diag}(\overline{A})$ and show that $\Delta \cup \Theta$ has a model. Let $B = \{a \in A : c_a$ occurs in some sentence of Θ; so B is a finite subset of A. Hence there is a finitely generated substructure $\overline{C}$ of $\overline{A}$ such that $B \subseteq C$. Let $f : \overline{C} \to \overline{D}$ be an isomorphism from $\overline{C}$ into a model $\overline{D}$ of Γ. Choose $g : A \to D$ extending f. We claim that $\overline{D}_g$ is a model of Θ. To prove this, take any $\varphi \in \Theta$.

First suppose that φ is $\sigma = \tau$ with σ and τ variable-free terms of $\mathscr{L}_A$. Then there are terms ρ, ξ of $\mathscr{L}$ and $e \in {}^\omega B$ such that $\rho_e = \sigma$ and $\xi_e = \tau$. Note that $f \circ e = g \circ e$. Then

$$\begin{aligned}\sigma^{\overline{D}_g} &= \rho_e^{\overline{D}_g} = \rho^{\overline{D}}(g \circ e) = \rho^{\overline{D}}(f \circ e) = f(\rho^{\overline{C}}(e)) = f(\rho^{\overline{A}}(e)) = f(\rho_e^{\overline{A}_A}) = f(\sigma^{\overline{A}_A}) \\ &= f(\tau^{\overline{A}_A}) = f(\xi_e^{\overline{A}_A}) = f(\xi^{\overline{A}}(e) = f(\xi^{\overline{C}}(e)) = \xi^{\overline{D}}(f \circ e) = \xi^{\overline{D}}(g \circ e) = \xi_e^{\overline{D}_g} = \tau^{\overline{D}_g}.\end{aligned}$$

Hence $\overline{D}_g \models \sigma = \tau$.

The case when φ is $\neg(\sigma = \tau)$ is treated similarly.

Second, suppose that φ is $R\sigma_0 \ldots \sigma_{m-1}$ with each σ_i a variable-free term of $\mathscr{L}_A$. Then there are terms ρ_i of $\mathscr{L}$ for $i < m$ and $e \in {}^\omega B$ such that $\rho_{ie} = \sigma_i$ for all $i < m$. Again, $f \circ e = g \circ e$.

We have $\overline{A}_A \models R\sigma_0 \ldots \sigma_{m-1}$. Thus

$$\langle \sigma_0^{\overline{A}_A}, \ldots, \sigma_{m-1}^{\overline{A}_A}\rangle \in R^{\overline{A}}.$$

So

$$\langle \rho_{0e}^{\overline{A}_A}, \ldots, \rho_{(m-1)e}^{\overline{A}_A}\rangle \in R^{\overline{A}}.$$

Hence Lemma 6.7 yields

$$\langle \rho_0^{\overline{A}}(e), \ldots, \rho_{m-1}^{\overline{A}}(e)\rangle \in R^{\overline{A}}.$$

Now an easy induction shows that $\eta^{\overline{A}}(e) = \eta^{\overline{C}}(e)$ for any term η. Hence

$$\langle \rho_0^{\overline{C}}(e), \ldots, \rho_{m-1}^{\overline{C}}(e)\rangle \in R^{\overline{C}}.$$

Now the fact that f is an isomorphism gives

$$\langle f(\rho_0^{\overline{C}}(e)), \ldots, f(\rho_{m-1}^{\overline{C}}(e))\rangle \in R^{\overline{D}}.$$

By the argument in the equality case, $f(\rho_i^{\overline{C}}(e)) = \sigma_i^{\overline{D}_g}$ for each $i < m$. Hence

$$\langle \sigma_0^{\overline{D}_g}, \ldots \sigma_{m-1}^{\overline{D}_g}\rangle \in R^{\overline{D}}.$$

Hence $\overline{D}_g \models R\sigma_0 \ldots \sigma_{m-1}$.

The case $\neg R\sigma_0 \ldots \sigma_{m-1}$ is treated similarly. $\square$

Elementary extensions

Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures. An *elementary embedding* of $\overline{A}$ into $\overline{B}$ is a function $f : A \to B$ satisfying the following condition:

($*$) For every formula φ and every $a \in {}^{\omega}A$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$.

In case f is the identity on A we say that $\overline{A}$ is an *elementary substructure* of $\overline{B}$ and $\overline{B}$ is an *elementary extension* of $\overline{A}$, and write $\overline{A} \preceq \overline{B}$. Then the condition ($*$) takes the form

($**$) $A \subseteq B$, and for every formula α and every $a \in {}^{\omega}A$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$.

Proposition 6.10. *If f is an elementary embedding of $\overline{A}$ into $\overline{B}$, then f is an isomorphism from $\overline{A}$ onto a substructure $\overline{C}$ of $\overline{B}$, and $\overline{C}$ is an elementary substructure of $\overline{B}$.*

In particular, if $\overline{A} \preceq \overline{B}$, then $\overline{A}$ is a substructure of $\overline{B}$.

Proof. First we show that f is one-one. Suppose that x and y are distinct members of A. Let φ be the formula $\neg(v_0 = v_1)$. Then $\overline{A} \models \varphi[x, y]$, hence $\overline{B} \models \varphi[f(x), f(y)]$, so $f(x) \neq f(y)$. Thus f is one-one.

Let $C = \text{rng}(f)$. Suppose that k is an individual constant. Let φ be the formula $v_0 = k$. Then $\overline{A} \models \varphi[k^{\overline{A}}, k]$, and so $\overline{B} \models \varphi[f(k^{\overline{A}}), k]$, i.e., $f(k^{\overline{A}}) = k^{\overline{B}}$. Thus $k^{\overline{B}} \in C$, and f preserves k. Suppose that F is an m-ary operation symbol and $a \in {}^{m}A$. Let $c = F^{\overline{A}}(a)$. Let φ be the formula $Fv_0 \ldots v_{m-1} = v_m$. Then $\overline{A} \models \varphi[a_0, \ldots, a_{m-1}, c]$, and so $\overline{B} \models \varphi[f(a_0), \ldots, f(a_{m-1}), f(c)]$, i.e. $F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1}) = f(c) = f(F^{\overline{A}}(a_0, \ldots, a_{m-1})$. Hence C is closed under $F^{\overline{B}}$, and f preserves F.

Now we have shown that C is a subuniverse of $\overline{B}$; and we have also checked the isomorphism properties of f for individual constants and for operation symbols. Now suppose that R is an m-ary relation symbol and $a \in {}^{m}A$. Let φ be the formula $Rv_0 \ldots v_{m-1}$. Suppose that $a \in R^{\overline{A}}$. Then $\overline{A} \models \varphi[a_0, \ldots, a_{m-1}]$. Hence $\overline{B} \models \varphi[f(a_0), \ldots, f(a_{m-1})]$, i.e., $f \circ a \in R^{\overline{B}}$. The converse is similar. So f is an isomorphism from $\overline{A}$ onto $\overline{C}$.

To show that $\overline{C}$ is an elementary substructure of $\overline{B}$, suppose that $c \in {}^{\omega}C$ and φ is any formula. We can write $c = f \circ a$ for some $a \in {}^{\omega}A$. Hence $\overline{C} \models \varphi[c]$ iff $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$, as desired. $\square$

Since the condition for elementary extension can be applied to sentences, it follows that $\overline{A} \preceq \overline{B}$ implies that $\overline{A}$ and $\overline{B}$ are elementarily equivalent. But $\overline{A} \preceq \overline{B}$ means more than just $\overline{A} \leq \overline{B}$ and $\overline{A}$ elementarily equivalent to $\overline{B}$. For example, let $\overline{A}$ be $\omega\backslash 1$ under its usual order, and $\overline{B}$ be ω under its usual order. Then $\overline{A} \leq \overline{B}$ and they are elementarily equivalent since they are isomorphic; see Theorem 6.1. But $\overline{A}$ is not an elementary substructure of $\overline{B}$. In fact, let φ be the formula $\forall v_1[v_0 < v_1 \vee v_0 = v_1]$. Then $\overline{A} \models \varphi[1]$, but $\overline{B} \not\models \varphi[1]$ since $0 < 1$.

Lemma 6.11. (Tarski) *Suppose that $\overline{A}$ is a substructure of $\overline{B}$. Then the following conditions are equivalent:*

(i) $\overline{A} \preceq \overline{B}$.

(ii) For every formula φ, every $i < \omega$, and every $x \in {}^\omega A$, if there is a $b \in B$ such that $\overline{B} \models \varphi[x^i_b]$, then there is an $a \in A$ such that $\overline{B} \models \varphi[x^i_a]$.

Proof. (i)⇒(ii): Assume that $\overline{A} \preceq \overline{B}$, φ is a formula, $i < \omega$, $x \in {}^\omega A$, $b \in B$, and $\overline{B} \models \varphi[x^i_b]$. Then $\overline{B} \models \exists v_i\varphi[a]$, so $\overline{A} \models \exists v_i\varphi[a]$. Choose $a \in A$ such that $\overline{A} \models \varphi[x^i_a]$. Then also $\overline{B} \models \varphi[x^i_a]$.

(ii)⇒(i): Assume (ii). Now we prove by induction on ψ that for any $a \in {}^\omega A$ we have $\overline{A} \models \psi[a]$ iff $\overline{B} \models \psi[a]$. This is true for ψ atomic, using Proposition 2.3. The induction steps involving $\neg$ and $\wedge$ are clear. Now suppose that ψ is $\exists v_i\varphi$.

Suppose that $\overline{A} \models \psi[a]$. Choose $u \in A$ such that $\overline{A} \models \varphi[a^i_u]$. Then $\overline{B} \models \varphi[a^i_u]$ by the inductive hypothesis. Hence $\overline{B} \models \psi[a]$.

Suppose that $\overline{B} \models \psi[a]$. By (ii), choose $u \in A$ such that $\overline{B} \models \varphi[a^i_u]$. Then $\overline{A} \models \varphi[a^i_u]$ by the inductive hypothesis. Hence $\overline{A} \models \psi[a]$. □

Proposition 6.12. *If $\overline{A}$ is an elementary substructure of $\overline{B}$, and $\overline{C}$ is an elementary substructure of $\overline{B}$ containing A, then $\overline{A} \preceq \overline{C}$.*

Proof. For any formula φ and any $a \in {}^\omega A$ we have $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$ iff $\overline{C} \models \varphi[a]$. □

For the next fact, see the definition of unions of structures given in Chapter 1.

Proposition 6.13. *Suppose that $\langle A_i : i \in I\rangle$ is a system of structures, $(I, \leq)$ is a directed set, and $A_i \preceq A_j$ if $i, j \in I$ with $i \leq j$. Then $A_i \preceq \bigcup_{j\in I} A_j$.*

Proof. Let $\overline{B} = \bigcup_{i\in I} \overline{A}_i$. We prove the following by induction on φ;

$(*)$ For every $i \in I$, every $m \in \omega$ such that $j < m$ for every variable v_j occurring in φ, and every $a \in {}^m A_i$, $\overline{A}_i \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$.

The atomic case, and the inductive steps for $\neg$ and $\wedge$, are clear. Now suppose that φ is $\exists v_k\psi$. If $\overline{A}_i \models \varphi[a]$, the inductive hypothesis easily gives $\overline{B} \models \varphi[a]$. Now suppose that $\overline{B} \models \varphi[a]$. Choose $b \in B$ such that $\overline{B} \models \psi[a^k_b]$. Then there is a $j \in I$ such that $i < j$ and $a^k_b \in A_j$. Hence $\overline{A}_j \models \psi[a^k_b]$ by the inductive hypothesis. So $\overline{A}_j \models \varphi[a]$, and $\overline{A}_i \models \varphi[a]$ since $\overline{A}_i \preceq \overline{A}_j$. □

By the *elementary diagram* of $\overline{A}$, denoted by Eldiag($\overline{A}$), we mean the set of all sentences in $\mathscr{A}_A$ which hold in $\overline{A}_A$.

Lemma 6.14. *Let $\overline{A}$ be an $\mathscr{L}$-structure, $e \in {}^\omega A$, φ an $\mathscr{L}$-formula, and let φ_e be obtained from φ by replacing every free occurrence of v_i by $c_{e(i)}$, the constant corresponding to $e(i)$ in the language $\mathscr{L}_A$. Then $\overline{A}_A \models \varphi_e$ iff $\overline{A} \models \varphi[e]$.*

Proof. Immediate from Lemma 4.8. □

Theorem 6.15. (Elementary diagram theorem) *Let $\overline{A}$ and $\overline{B}$ be similar structures and $f : A \to B$. Then the following conditions are equivalent:*

(i) f is an elementary embedding of $\overline{A}$ into $\overline{B}$.

(ii) $(\overline{B}, f(a))_{a\in A}$ is a model of $\mathrm{Eldiag}(\overline{A})$.

Proof. (i)⇒(ii): Let f be an elementary embedding of $\overline{A}$ into $\overline{B}$. We claim that $\overline{B}_f$ is a model of $\mathrm{Eldiag}(\overline{A})$. For, let φ be a sentence of $\mathscr{L}_A$ which holds in $\overline{A}_A$. Then there exist a formula ψ in $\mathscr{L}$ and an $a \in {}^{\omega}A$ such that φ is obtained from ψ by replacing each free occurrence of v_i in ψ by c_{a_i}, the constant of $\mathscr{L}_A$ corresponding to a_i. Then by Lemma 6.14 we have $\overline{A} \models \psi[a]$. Hence $\overline{B} \models \psi[f \circ a]$. Then by Lemma 6.14 again we have $\overline{B}_f \models \varphi$, as desired.

(ii)⇒(i): Assume that $\overline{B}_f$ is a model of $\mathrm{Eldiag}(\overline{A})$. We claim that f is an elementary embedding of $\overline{A}$ into $\overline{B}$. By the proof of Theorem 6.8, f is an isomorphism from $\overline{A}$ into $\overline{B}$. Now suppose that φ is a formula of $\mathscr{L}$ and $a \in {}^{\omega}A$. Suppose that $\overline{A} \models \varphi[a]$. Then by Lemma 6.14, $\overline{A}_A \models \psi$, where ψ is obtained from φ by replacing each free occurrence of v_i by $c_{a(i)}$, for all $i < \omega$. Hence ψ is in $\mathrm{Eldiag}(\overline{A})$, so $\overline{B}_f$ is a model of it. By Lemma 6.14, $\overline{B} \models \varphi[f \circ a]$. The converse holds by considering $\neg\varphi$. □

Löwenheim-Skolem theorems

The theorems are existence statements concerning starting with one structure and finding smaller elementary substructures, or larger elementary extensions. To start with we need a result from elementary set theory describing the closure of a set under finitary partial operations.

Given a set A, a *partial operation* on A is a function f which, for some positive integer m, maps a subset of ${}^{m}A$ into A. The integer m is uniquely determined by f if $f \neq \emptyset$, and we denoter it by $\rho(f)$. We say that a subset X of A is *closed* under a nonempty partial function f iff for every a which is in the domain of f and is a member of ${}^{\rho(f)}X$ we have $f(a) \in X$. If $\mathscr{F}$ is a collection of partial functions on A and $X \subseteq A$, then $\mathrm{cl}(X, \mathscr{F})$ is the intersection of all subsets of A which contain X are are closed under each member of $\mathscr{F}$.

Theorem 6.16. *Let A be any set, and $\mathscr{F}$ a collection of partial finitary operations on A. Suppose that $X \subseteq A$.*

(i) Define $Y_0 = X$ and for any $i \in \omega$ let

$$Y_{i+1} = Y_i \cup \{f(a) : \emptyset \neq f \in \mathscr{F}, a \in \mathrm{dmn}(f) \cap {}^{\rho(f)}Y_i\}.$$

Then $\mathrm{cl}(X, \mathscr{F}) = \bigcup_{i\in\omega} Y_i$.

(ii) $\mathrm{cl}(X, \mathscr{F}) \leq \max(|\mathscr{F}|, |X|, \omega)$.

Proof. By induction, $Y_i \subseteq \mathrm{cl}(X, \mathscr{F})$ for every $i < \omega$. Hence $\bigcup_{i\in\omega} Y_i \subseteq \mathrm{cl}(X, \mathscr{F})$. For the other inclusion it suffices to show that $X \subseteq \bigcup_{i\in\omega} Y_i$ and $\bigcup_{i\in\omega} Y_i$ is closed under each $f \in \mathscr{F}$. Since $Y_0 = X$, it is obvious that $X \subseteq \bigcup_{i\in\omega} Y_i$. Now suppose that $f \in \mathscr{F}$ and a is in the domain of f and $\mathrm{rng}(a) \subseteq \bigcup_{i\in\omega} Y_i$. Clearly there is an $i < \omega$ such that $\mathrm{rng}(a) \subseteq Y_i$. Then $f(a) \in Y_{i+1}$, as desired. This proves (i).

For (ii), it suffices to show that $|Y_i| \le \max(|\mathscr{F}|, |X|, \omega)$ for each $i \in \omega$. This is clear, by induction on i. $\square$

Theorem 6.17. (Downward Löwenheim-Skolem theorem) *Suppose that $\overline{B}$ is an infinite $\mathscr{L}$-structure, $X \subseteq B$, κ is a cardinal, $|X| \le \kappa \le |B|$, and $\mathscr{L}$ has at most κ non-logical symbols.*

Then $\overline{B}$ has an elementary substructure $\overline{A}$ such that $X \subseteq A$ and $|A| = \kappa$.

Proof. Let $<$ well-order B. Let F consist of all triples (φ, i, m) such that φ is a formula with free variables among $v_0, \ldots, v_{m-1}$ and $i < m$. With each $(\varphi, i, m) \in F$ we associate a partial m-ary function $f_{(\varphi,i,m)}$ on B. The domain of $f_{(\varphi,i,m)}$ is $\{a \in {}^mB : \exists u \in B[\overline{B} \models \varphi[a^i_u]]\}$, and its value at any such is the $<$-least u such that $\overline{B} \models \varphi[a^i_u]$. Let $\mathscr{F} = \{f_{(\varphi,i,m)} : (\varphi, i, m) \in F$. Let Y be a subset of B containing X and of size κ. Finally, set $A = \mathrm{cl}(Y, \mathscr{F})$. Clearly $X \subseteq A$ and $|A| = \kappa$.

First we prove that A is a subuniverse of $\overline{B}$. Let k be an individual constant of $\mathscr{L}$. Let φ be the formula $v_0 = k$. Note that $\mathrm{dmn}(f_{(\varphi,0,1)}) = \{a \in {}^1B : \exists u \in B[\overline{B} \models \varphi[a^0_u]]\} = {}^1B$, and for any $a \in B[f_{(\varphi,0,1)}(a) = k^{\overline{B}}$. Thus $k^{\overline{B}} \in A$.

Next let F be an m-ary operation symbol, and suppose that $a \in {}^mA$. Let φ be the formula $Fv_0, \ldots, v_{m-1} = v_m$. Then $\mathrm{dmn}(f_{(\varphi,m,m+1)}) = \{b \in {}^{m+1}B : \exists u \in B[\overline{B} \models \varphi[b^m_u]]\} = {}^{m+1}B$. It follows that

$$f_{(\varphi,m,m+1)}(a_0, \ldots, a_{m-1}, a_0) = F^{\overline{B}}(a_0, \ldots, a_{m-1}) \in A.$$

So A is a subuniverse of $\overline{B}$ and we let $\overline{A}$ be the associated substructure.

To show that $\overline{A} \preceq \overline{B}$ we will apply Tarski's lemma 6.11. To this end, let φ be a formula, $i < \omega$, $x \in {}^\omega A$, and suppose that there is a $b \in B$ such that $\overline{B} \models \varphi[x^i_b]$. Choose n so that $j < n$ for every variable v_j that occurs free in φ, and also so that $i < n$. Let $u = f_{(\varphi,i,n)}(x \restriction n)$. Then by definition, $\overline{B} \models \varphi[(x \restriction n)^i_u]$. Also, $u \in A$. Hence $\overline{B} \models \varphi[x^i_u]$. This verifies (ii) in Tarski's lemma. Hence $\overline{A} \preceq \overline{B}$. $\square$

A somewhat philosophical application of the downward Löwenheim-Skolem theorem is *Skolem's paradox*: if there is a model of the axioms for set theory, then there is a countable model, even though from the axioms one can prove the existence of uncountable sets. The "solution" of the paradox is that the model only "thinks" that the sets are uncountable; they are really countable. This is connected to the notion of absoluteness for models of set theory.

Theorem 6.18. (Upward Löwenheim-Skolem theorem) *If $\overline{A}$ is an $\mathscr{L}$-structure, $|A| \ge \omega$, and κ is a cardinal such that $|A| \le \kappa$ and $\mathscr{L}$ has at most κ non-logical constants, then $\overline{A}$ has an elementary extension $\overline{B}$ of size κ. In case $|A| = \kappa$, we can insist that $A \ne B$.*

Proof. Expand $\mathscr{L}_A$ to a language $\mathscr{L}'$ by adding κ many new individual constants $\langle k_\alpha : \alpha < \kappa \rangle$. By the elementary diagram theorem 6.15 it suffices to show that the following set has a model:

$$\mathrm{Eldiag}(\overline{A}) \cup \{\neg(k_\alpha = k_\beta) : \alpha < \beta < \kappa\} \cup \{\neg(c_a = k_\alpha) : a \in A, \alpha < \kappa\}.$$

To apply the compactness theorem, take a finite subset Γ of this set. We can write $\Gamma = \Delta_0 \cup \Delta_1 \cup \Delta_2$, where $\Delta_0, \Delta_1, \Delta_2$ are finite subsets of $\mathrm{Eldiag}(\overline{A})$, $\{\neg(k_\alpha = k_\beta) : \alpha < \beta < \kappa\}$ and $\{\neg(c_a = k_\alpha) : a \in A, \alpha < \kappa\}$ respectively. We consider the following $\mathscr{L}'$-structure: $(\overline{A}, a, d_\alpha)_{a\in A,\alpha<\kappa}$, where $d \in {}^\kappa A$ is such that, if Θ is the set of all $\alpha < \kappa$ such that k_α occurs in some formula of $\Delta_1 \cup \Delta_2$, then $d \restriction \Theta$ is one-one and for all $\alpha \in \Theta$, d_α is different from all $a \in A$ such that c_a occurs in some formula of Δ_2. Clearly this gives a model of Γ.

So by the compactness theorem we get a model $(\overline{B}, b_a, d_\alpha)_{a\in A,\alpha<\kappa}$ of Γ. By the elementary diagram theorem 6.15, there is an elementary embedding f of $\overline{A}$ into $\overline{B}$. Clearly $|B| \geq \kappa$, and $f[A] \neq B$. We may assume that $\overline{A} \preceq \overline{B}$. Take any $b \in B\backslash A$. By the downward Löwenheim-Skolem theorem, $\overline{B}$ has an elementary substructure $\overline{C}$ of size κ with $A \cup \{b\} \subseteq C$. Clearly $\overline{A} \preceq \overline{C}$ and $\overline{C}$ is as desired. $\square$

The Löwenheim-Skolem theorems can be used to prove our first major theorem concerning complete theories–Vaught's criterion. For any cardinal κ, we call a theory (weakly) κ-*categorical* if any two models of Γ of size κ are (elementarily equivalent) isomorphic. For κ finite, this notion is trivial:

Proposition 6.19. *If κ is a finite cardinal and Γ is a theory with a model of size κ, then the following conditions are equivalent.*

(i) Γ is κ-categorical.

(ii) Γ is weakly κ-categorical.

If Γ is complete, then all models of Γ have size κ, and both of these conditions hold.

Proof. By Theorem 6.1, (i) implies (ii). Now suppose that (ii) holds, and $\overline{A}$ and $\overline{B}$ are models of Γ; we want to show that they are isomorphic. First suppose that our language $\mathscr{L}$ has only finitely many non-logical symbols. Then $\mathrm{Diag}(\overline{A})$ is finite. Let f be a bijection from A onto some positive integer m, and form a formula φ by replacing each constant c_a in $\bigwedge \mathrm{Diag}(\overline{A})$ corresponding to an element $a \in A$ by $v_{f(a)}$. Then the sentence

$$\exists v_0 \ldots \exists v_{m-1} \left[\varphi \wedge \forall v_m \bigvee_{i<m} (v_i = v_m)\right]$$

holds in $\overline{A}$, and hence also in $\overline{B}$. Hence there is a $g \in {}^A B$ such that $\overline{B}_g$ is a model of $\mathrm{Diag}(\overline{A})$, and with g a surjection. By the proof of the diagram lemma 6.8, g is an isomorphism from $\overline{A}$ onto $\overline{B}$.

Second, suppose that $\mathscr{L}$ has infinitely many non-logical symbols. A *finite reduct* of $\mathscr{L}$ is a reduct of $\mathscr{L}$ with only finitely many non-logical symbols. Similarly for a finite reduct of $\overline{A}$ or of $\overline{B}$. Now for each finite reduct $\overline{A}_F$ of $\overline{A}$, given by a finite set F of non-logical symbols of $\mathscr{L}$, let I_F be the set of all isomorphisms from $\overline{A}_F$ onto $\overline{B}_F$. Note that if $F \subseteq G$, both being finite sets of non-logical symbols of $\mathscr{L}$, then $I_G \subseteq I_F$. Let J be a set I_F of smallest size. Then each member of J is an isomorphism from $\overline{A}$ onto $\overline{B}$. For, if S is any non-logical constant of $\mathscr{L}$, let $G = F \cup \{S\}$. Then $I_G \subseteq I_F$, hence $I_G = I_F = L$, and so each member of L preserves S.

This finishes the proof of equivalence of (i) and (ii).

Now suppose that Γ is complete. Since Γ has a model $\overline{A}$ of size κ, the sentence

$$\exists v_0 \ldots \exists v_{\kappa-1} \left[\bigwedge_{i<j<\kappa} \neg(v_i = v_j) \wedge \forall v_\kappa \left[\bigvee_{i<\kappa} (v_i = v_\kappa) \right] \right]$$

holds in $\overline{A}$, hence in all models of Γ since it is complete. So all models of Γ have size κ. Clearly (i) and (ii) hold. □

Theorem 6.20. (Vaught's Criterion) *Let Γ be a theory in a language $\mathscr{L}$ with only infinite models, and let κ be a cardinal $\geq$ the number of non-logical symbols of $\mathscr{L}$. Suppose that Γ is weakly κ-categorical. Then Γ is complete.*

Let $\overline{A}$ and $\overline{B}$ be models of Γ; we want to show that they are elementarily equivalent. We can apply the uppward or downward Löwenheim-Skolem theorem to get a model $\overline{C}$ elementarily equivalent to $\overline{A}$ and of size κ. Similarly with $\overline{B}$, so the conclusion follows. □

Some typical applications of Vaught's criterion are:

- The theory of dense linear orders with no end points. ($\aleph_0$-categorical)
- The theory of atomless Boolean algebras. ($\aleph_0$-categorical)
- The theory of algebraically closed fields of a given characteristic. ($\aleph_1$-categorical)

κ-saturated structures

We generalize a procedure and notation described in the subsection on diagrams above. Given an $\mathscr{L}$-structure $\overline{A}$ and a subset X of A, let $\mathscr{L}_X$ be the expansion of $\mathscr{L}$ obtained by adding new individual constants c_a for each $a \in X$. If $\overline{B}$ is an $\mathscr{L}$-structure and $b \in {}^X B$, then $\overline{B}_b$ is the expansion of $\overline{B}$ to an $\mathscr{L}_X$ structure in which the denotation of c_a is b_a for each $a \in X$.

Now let κ be an infinite cardinal. An $\mathscr{L}$-structure $\overline{A}$ is *κ-saturated* iff for every $X \in [A]^{<\kappa}$, every $i < \omega$, and every set Γ of formulas of $\mathscr{L}_X$ each with only v_i possibly free, the following condition (c) implies condition (x):

(c) For every finite $\Theta \subseteq \Gamma$ there is a $b \in A$ such that $\overline{A}_X \models \varphi[b]$ for all $\varphi \in \Theta$.

(x) There is a $b \in A$ such that $\overline{A}_X \models \varphi[b]$ for all $\varphi \in \Gamma$.

Here "(c)" stands for "consistent" and "(x)" stands for "exists". By the compactness theorem, a structure with a consistent set of formulas has an elementary extension with an element satisfying all of the formulas. A κ-saturated structure already itself has such an element. κ-saturated structures are very useful in model theory. The main result we present here is an existence theorem for them.

κ-saturation is defined in terms of a single element satisfying a set of formulas. What about two or three or more elements? According to the next proposition, this generalization is not necessary.

Proposition 6.21. *Suppose that κ is an infinite cardinal and $\overline{A}$ is an $\mathscr{L}$-structure. Then the following conditions are equivalent:*

(i) $\overline{A}$ is κ-saturated.

(ii) For each $X \in [A]^{<\kappa}$ and for every set Γ of formulas of $\mathscr{L}_X$, if for each finite $\Theta \subseteq \Gamma$ there is an $a \in {}^\omega A$ such that $\forall\varphi \in \Theta[\overline{A}_X \models \varphi[a]]$, then there is an $a \in {}^\omega A$ such that $\forall\varphi \in \Gamma[\overline{A}_X \models \varphi[]]$.

Proof. Obviously (ii) implies (i). Now suppose that (i) holds. Let Δ be the closure of Γ under $\wedge$. For each $\varphi \in \Delta$ choose a positive integer $m(\varphi)$ such that for every $i \in \omega$, if v_i occurs free in φ, then $i < m(\varphi)$. Now in addition to the individual constants c_a, $a \in X$ used to form $\mathscr{L}_X$, we introduce more individual constants k_i for $i < \omega$.

Now we define $a \in {}^\omega A$ by recursion. Suppose that $i < \omega$ and $a \restriction i$ has been defined so that

$$(1) \qquad (\overline{A}, b, a_j)_{b\in X, j<i} \models \exists v_i \ldots \exists v_{m(\varphi)-1}\varphi(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, \ldots, v_{m(\varphi)-1}) \\ \text{for every } \varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta.$$

Clearly (1) holds for $i = 0$. Let Ω be the set of all formulas

$$(2) \qquad \exists v_{i+1} \ldots \exists v_{m(\varphi)-1}\varphi(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, v_{i+1}, \ldots, v_{m(\varphi)-1})$$

with $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta$. These are formulas in $\mathscr{L}_Y$, where $Y = X \cup \{a_j : j < i\}$, and each of them has at most v_i free. Note that $|Y| < \kappa$. Now if Ω' is a finite subset of Ω, then there is a finite subset Δ' of Δ such that Ω' consists of all formulas (2) such that $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta'$. Now $\bigwedge\Delta' \in \Delta$, so by the inductive assumption the formula

$$\exists v_i \ldots \exists v_{m(\varphi)-1} \bigwedge \Delta'(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, \ldots v_{m(\varphi)-1})$$

holds in $(\overline{A}, b, a_j)_{b\in X, j<i}$. Hence there is a $u \in A$ such that

$$(\overline{A}, b, a_j)_{b\in X, j<i} \models \exists v_{i+1} \ldots \exists v_{m(\varphi)-1} \bigwedge \Delta'(k_{a_0}, \ldots, k_{a_i}, v_i, \ldots v_{m(\varphi)-1})[u].$$

Now it follows from (i) that there is an $a_i \in A$ such that

$$(\overline{A}, b, a_j)_{b\in X, j\le i} \models \exists v_{i+1} \ldots \exists v_{m(\varphi)-1}\varphi(k_{a_0}, \ldots, k_{a_i}, v_{i+1}, \ldots, v_{m(\varphi)-1}) \\ \text{for every } \varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta.$$

This finishes the inductive construction. So (1) holds for all $i < \omega$. Now if we apply (1) with $i = m(\varphi)$ to $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Gamma$, we get $(\overline{A}, b, j)_{b\in X, j<m(\varphi)} \models \varphi(k_{a_0}, \ldots, k_{a_{m(\varphi)-1}})$. Hence $\overline{A}_X \models \varphi[a]$, as desired. $\square$

Lemma 6.22. *Suppose that:*
κ is an infinite cardinal,
$\mathscr{L}$ is a language with at most κ non-logical symbols,
$\overline{A}$ is an infinite $\mathscr{L}$-structure of size at most 2^κ.
$X \in [A]^{\le\kappa}$,
$i < \omega$, and Γ is a set of formulas of $\mathscr{L}_X$ with at most v_i free,

for every finite $\Theta \subseteq \Gamma$ there is a $b \in A$ such that $\overline{A} \models \varphi[b]$ for all $\varphi \in \Theta$.

Then $\overline{A}$ has an elementary extension $\overline{B}$ of size at most 2^κ with an element b such that $\overline{B} \models \varphi[b]$ for all $\varphi \in \Gamma$.

Proof. Let the system of individual constants used to form $\mathscr{L}_X$ be denoted by $\langle b_a : a \in X \rangle$. Now form the language $\mathscr{L}_{XA}$ used to define $\mathrm{Eldiag}(\overline{A})$, with new individual constants $\langle c_a : a \in A \rangle$. In addition, add one more new individual constant d, forming an expansion $\mathscr{L}'$ of $\mathscr{L}_{XA}$. In $\mathscr{L}'$ consider the following set Ω of sentences:

$$\mathrm{Eldiag}(\overline{A}) \cup \{\varphi(d) : \varphi \in \Gamma\}.$$

By the elementary diagram theorem followed by the downward Löwenheim-Skolem theorem, it suffices to show that Ω has a model; and for this we take a finite subset Ω' of Ω, consisting of the union of a finite subset Ω'' of $\mathrm{Eldiag}(\overline{A})$ together with a finite subset Ω''' of $\{\varphi(d) : \varphi \in \Gamma\}$. Clearly Ω' has a model consisting of an expansion of $\overline{A}_A$, using the hypotheses of our lemma. □

Theorem 6.23. *Suppose that κ is an infinite cardinal, $\mathscr{L}$ is a language with at most κ non-logical symbols, and $\overline{A}$ is an infinite $\mathscr{L}$-structure of size at most 2^κ. Then $\overline{A}$ has a κ^+-saturated elementary extension $\overline{B}$ of size at most 2^κ.*

Proof. Let $M = \{(X, \Gamma) : X \in [A]^{\leq\kappa}, \Gamma$ is a set of formulas of $\mathscr{L}_X$ each with at most v_0 free and for each finite subset Δ of Γ there is an $a \in A$ such that $\overline{A}_X \models \varphi[a]\}$. Clearly $|M| \leq 2^\kappa$. Hence we can construct an elementary chain of elementary extensions of $\overline{A}$, applying Lemma 6.22 at each stage to a member of M, so that the union $\overline{B}$ of the chain is an elementary extension such that for each member (X, Γ) of M there is a $b \in B$ such that $\overline{B} \models \varphi[b]$ for each $\varphi \in \Gamma$; moreover, $|B| \leq 2^\kappa$. Repeating this κ^+ times we obtain the desired κ^+-saturated structure. □

We conclude our discussion of saturated structures with one of the most important properties of them.

Theorem 6.24. *If κ is an infinite cardinal and $\overline{A}$ is κ-saturated, then every structure $\overline{B}$ which is elementarily equivalent to $\overline{A}$ and of size at most κ can be elementarily embedded in $\overline{A}$.*

Proof. Let $\langle b_\alpha : \alpha < \kappa \rangle$ enumerate B, possibly with repetitions. We now define by recursion a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of A, so that for each α, $(\overline{A}, a_\beta)_{\beta<\alpha} \equiv (\overline{B}, b_\beta)_{\beta<\alpha}$. Suppose that a_β has been defined for all $\beta < \alpha$ so that this condition holds. Note that for $\alpha = 0$ the condition says that merely $\overline{A} \equiv \overline{B}$, which is given. Now we consider the following set Γ of formulas:

$$\{\varphi(x) : \varphi \text{ is a formula in } \mathscr{L}_{\langle b_\beta:\beta,\alpha\rangle} \text{ and } \overline{B} \models \varphi(b_\alpha)\}.$$

If Δ is a finite subset of Γ, then $\overline{B} \models \bigwedge \Delta[b_\alpha]$, so $\overline{B} \models \exists x \bigwedge \Delta$. Hence $\overline{A} \models \exists x \bigwedge \Delta$. So by the definition of κ-saturated, there is an element a_α of A such that $\overline{A} \models \varphi[a_\alpha]$ for all $\varphi \in \Gamma$. Hence $(\overline{A}, a_\beta)_{\beta\leq\alpha} \equiv (\overline{B}, b_\beta)_{\beta\leq\alpha}$, finishing the construction.

Clearly $\{(b_\alpha, a_\alpha) : \alpha < \kappa\}$ is the desired elementary embedding. $\square$

Omitting types

Let T be a theory and $n \in \omega$. An *n-type* of T is a collection t of formulas of the language $\mathscr{L}$ of T with all free variables among $\overline{x}$, where $\overline{x}$ is a sequence of variables of length n, such that $T \models \exists\overline{x}\varphi$ for every finite conjunction φ of members of t.

We say that such a type t is *isolated* in T iff there is a formula φ with free variables among $\overline{x}$ such that $T \cup \{\exists\overline{x}\varphi\}$ has a model, and $T \models \varphi \to \psi$ for every $\psi \in t$. We say then that φ isolates t in T.

A model $\overline{M}$ of T *omits* t iff there is no sequence $\overline{a}$ of elements of M such that $\overline{M} \models \psi[\overline{a}]$ for all $\psi \in t$.

Proposition 6.25. *If T is complete and has a model which omits a type t, then t is not isolated in T.*

Proof. We prove the contrapositive. Suppose that T is complete and t is isolated in T; say that φ isolates t in T. Let $\overline{M}$ be any model of T. Since $T \cup \{\exists\overline{x}\varphi\}$ has a model and T is complete, it follows that $\overline{M} \models \exists\overline{x}\varphi$. Choose $\overline{a}$ in M such that $\overline{M} \models \varphi[\overline{a}]$. Take any $\psi \in T$. Then $T \models \varphi \to \psi$, and hence $\overline{M} \models \psi[\overline{a}]$. Thus $\overline{M}$ realizes t, i.e., it does not omit t. $\square$

Theorem 6.26. (Omitting types theorem) *Suppose that $\mathscr{L}$ is countable and T is a theory in $\mathscr{L}$ which has a model. Let $\langle n_i : i \in \omega\rangle$ be a sequence of natural numbers, and for each $i < \omega$ let t_i be an n_i-type over T which is not isolated.*

Then T has a countable model which omits each type t_i.

Proof. Expand our given language $\mathscr{L}$ by adjoining a sequence $\langle c_i : i < \omega\rangle$ of new individual constants, forming a new language $\mathscr{L}'$ which is still countable. Let $\langle \varphi_i : i < \omega\rangle$ enumerate all of the sentences of $\mathscr{L}'$. Let $\langle (j_i, \overline{d}_i) : i \in \omega\rangle$ enumerate all pairs $(m, \overline{e})$ such that $m \in \omega$ and $\overline{e}$ is a sequence of length n_m of distinct members of $\{c_0, c_1, \ldots\}$.

Now we are going to construct a sequence

$$T = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_m \subseteq \cdots$$

of theories in $\mathscr{L}'$ satisfying for each $i \in \omega$ the following conditions:

(1) T_i has a model, and $T_i = T \cup \Delta$ for some finite set Δ of sentences.

(2) $\varphi_i \in T_{i+1}$ or $\neg\varphi_i \in T_{i+1}$.

(3) If φ_i has the form $\exists x\psi(x)$ and $\varphi_i \in T_{i+1}$, then there is an $k < \omega$ such that $\psi(c_k) \in T_{i+1}$.

(4) There is a formula $\sigma(\overline{x}) \in t_{j_i}$ such that $\neg\sigma(\overline{d}_i) \in T_{i+1}$.

Having defined T_i, we define T_{i+1} as follows. Let Δ be a finite set of sentences such that $T_i = T \cup \Delta$. Let $\chi = \bigwedge \Delta$. We can write $\chi = \chi(\overline{d}_i, \overline{e})$, where $\overline{d}_i$ and $\overline{e}$ exhaust all the constants c_k that occur in χ. Replace these constants by new distinct variables $\overline{x}$ and $\overline{y}$ respectively, obtaining $\chi(\overline{x}, \overline{y})$. Since T_i has a model, it follows that $T \cup \{\exists\overline{x}\exists\overline{y}\chi(\overline{x}, \overline{y})\}$ has a model. Now t_{j_i} is not isolated, so it follows that there is a formula $\sigma(\overline{x})$ in t_{j_i} such that

$T \not\models \exists \overline{y}\chi(\overline{x}, \overline{y}) \to \sigma(\overline{x})$. Thus there is a model $\overline{M}$ of T and a sequence $\overline{a}$ of elements of M such that $\overline{M} \models \exists \overline{y}\chi(\overline{a}, \overline{y}) \wedge \neg\sigma(\overline{a})$. It follows that $S \overset{\text{def}}{=} T_i \cup \{\neg\sigma(\overline{d}_i)\}$ has a model.

If $S \cup \{\varphi_i\}$ has a model, let $S' = S \cup \{\varphi_i\}$; otherwise let $S' = S \cup \{\neg\varphi_i\}$. In either case, S' has a model.

If φ_i has the form $\exists x\psi(x)$, let p be minimum such that c_p does not occur in any formula of S', and let $T_{i+1} = S' \cup \{\psi(c_p)\}$. If φ_i does not have this form, let $T_{i+1} = S'$.

This finishes the construction. Clearly (1)–(4) hold.

Let $T_\omega = \bigcup_{i\in\omega} T_i$. By (1) and compactness, T_ω has a model $(\overline{M}, a_i)_{i\in\omega}$. Let $\overline{N}$ be the substructure of $\overline{M}$ generated by the individual constants; $\overline{N}$ is an $\mathscr{L}$-structure.

(5) $N = \{a_i : i \in \omega\}$.

In fact, let σ be any variable-free term. Then $(\overline{M}, a_i)_{i\in\omega} \models \exists x[x = \sigma]$, and so by (3) there is a $k < \omega$ such that $(\overline{M}, a_i)_{i\in\omega} \models a_k = \sigma]$. This proves (5).

(6) For any sentence φ of $\mathscr{L}'$ the following conditions are equivalent:
- (a) $(\overline{M}, a_i)_{i\in I} \models \varphi$;
- (b) $(\overline{N}, a_i)_{i\in I} \models \varphi$;
- (c) $T_\omega \models \varphi$.

In fact, (a) and (c) are equivalent since T_ω is complete by (2). We prove the equivalence of (a) and (b) by induction on φ. For φ atomic, the equivalence is clear. The induction steps using $\neg$ and $\wedge$ are straightfoward. Finally, take a sentence $\exists x\psi(x)$. If $(\overline{M}, a_i)_{i\in\omega} \models \exists x\psi(x)$, then by (3) there is a $k < \omega$ such that $(\overline{M}, a_i)_{i\in\omega} \models \psi(c_k)$. Then $(\overline{N}, a_i)_{i\in\omega} \models \psi(c_k)$ by the inductive hypothesis, so $(\overline{N}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Now suppose that $(\overline{N}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Then there is a $k < \omega$ such that $(\overline{N}, a_i)_{i\in\omega} \models \psi(c_k)$, so $(\overline{M}, a_i)_{i\in\omega} \models \psi(c_k)$ by the inductive hypothesis, hence $(\overline{M}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Thus (6) holds.

It follows that $\overline{N}$ is a model of T. Moreover, by (4) it omits each type t_i. □

Model completeness

A theory T is *model complete* iff for any two models $\overline{A}$ and $\overline{B}$ of T, $\overline{A} \le \overline{B}$ iff $\overline{A} \preceq \overline{B}$. We prove here some general results about this notion.

Given a theory T, a *weakly prime* model of T is a model of T which can be embedded in any model of T. The qualifier "weakly" is there since *prime* model requires an elementary embedding.

Proposition 6.27. *If T is model complete and has a weakly prime model, then T is complete.*

Proof. Let $\overline{A}$ and $\overline{B}$ be arbitrary models of T, and let $\overline{C}$ be a weakly prime model of T. Then $\overline{C}$ can be elementarily embedded in $\overline{A}$ and in $\overline{B}$, so $\overline{A}$ and $\overline{B}$ are elementarily equivalent. □

This simple proposition can be used to show the completeness of various theories.

A formula is *existential* iff it has the form $\exists\overline{x}\varphi$ with φ quantifier-free; here $\overline{x}$ is a finite sequence of variables. Similarly one defines *universal* formula.

Lemma 6.28. *Suppose that $\Gamma \cup \Delta \cup \{\varphi\}$ is a collection of sentences in $\mathscr{L}$, and $\Gamma \cup \Delta \models \varphi$. Then there is a finite conjunction ψ of members of Δ such that $\Gamma \models \psi \to \varphi$.*

Proof. Assume the hypotheses. Then $\Gamma \cup \Delta \cup \{\neg\varphi\}$ does not have a model, so some finite subset fails to have a model. We may assume that the finite subset has the form $\Gamma' \cup \Delta' \cup \{\neg\varphi\}$ with Γ' and Δ' finite subsets of Γ, Δ respectively. Clearly then $\psi \stackrel{\text{def}}{=} \bigwedge \Delta'$ is as desired. □

Lemma 6.29. *If $\overline{A}$ is an $\mathscr{L}$-structure, Γ is a set of $\mathscr{L}$-sentences, φ is an $\mathscr{L}$-sentence, Δ is the diagram of $\overline{A}$, and $\Gamma \cup \Delta \models \varphi$, then there is an existential sentence ψ of $\mathscr{L}$ such that $\Gamma \models \psi \to \varphi$ and $\overline{A} \models \psi$.*

Proof. Assume the hypotheses. By Lemma 6.28, there is a conjunction χ of finitely many members of Δ such that $\Gamma \models \chi \to \varphi$. Replace the constants c_a in χ associated with the diagram by variables, obtaining a formula $\theta(\overline{x})$, with $\overline{x}$ a sequence of variables. Then $\Gamma \models \exists\overline{x}\theta \to \varphi$. Clearly $\overline{A} \models \exists\overline{x}\theta \to \varphi$. □

Theorem 6.30. *For any theory T the following conditions are equivalent:*

(i) T is model complete.

(ii) For all models $\overline{A}, \overline{B}$ of T such that $\overline{A} \leq \overline{B}$, and for every existential formula φ and every sequence $a \in {}^{\omega}A$, if $\overline{B} \models \varphi[a]$, then $\overline{A} \models \varphi[a]$.

(iii) For every formula φ there is an existential formula ψ with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$.

Proof. (i)⇒(ii) is obvious. Now assume that (ii) holds. First we claim

(1) For any universal formula φ there is an existential formula ψ with the same free variables such that $T \models \varphi \leftrightarrow \psi$.

To prove (1), let the free variables of φ be $v_{i(0)}, \ldots, v_{i(m-1)}$. Expand the language by adjoining new individual constants $c_0, \ldots, c_{m-1}$, obtaining a language $\mathscr{L}'$, and let φ' be the sentence $\mathscr{L}'$ obtained from φ by replacing $v_{i(j)}$ by c_j for each $j < m$. Let $\Delta = \{\psi : \psi$ is a universal sentence in $\mathscr{L}'$ and $T \models \neg\varphi' \to \psi\}$. Now we claim

(2) $$T \cup \Delta \cup \{\varphi'\} \text{ does not have a model.}$$

In fact, suppose that $(\overline{A}, a_i)_{i<m}$ is a model of this set. Let Θ be the diagram of $(\overline{A}, a_i)_{i<m}$ in $\mathscr{L}'_A$. Then

(3) $$T \cup \Theta \models \varphi'.$$

In fact, let $(\overline{B}, b_i, t_a)_{i<m, a\in A}$ be a model of $T \cup \Theta$. Then by Theorem 6.8, t is an isomorphism from $(\overline{A}, a_i)_{i<m}$ into $(\overline{B}, b_i)_{i<m}$. Now $(\overline{A}, a_i)_{i<m} \models \varphi'$, so by (ii), $(\overline{B}, b_i)_{i<m} \models \varphi'$. This proves (3).

Now from (3) and Lemma 6.29 we see that there is an existential sentence ψ of $\mathscr{L}'$ such that $T \models \psi \to \varphi'$ and $(\overline{A}, a_i)_{i<m} \models \psi$. Thus $\neg\psi \in \Delta$. But $(\overline{A}, a_i)_{i<m} \models \psi$ and also $(\overline{A}, a_i)_{i<m}$ is a model of Δ, contradiction. Hence (2) holds.

Thus $T \cup \Delta \models \neg\varphi'$, and so by Lemma 6.28 there is a conjunction ξ of members of Δ such that $T \models \xi \to \neg\varphi'$. Clearly ξ is logically equivalent to a universal sentence. The definition of Δ shows that $T \models \neg\varphi' \to \xi$. Hence (1) holds.

Now the proof of the (iii) itself follows by an easy induction. It is obviously true for atomic φ. Assuming it true for φ, let ψ be an existential formula with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$. Then by (1) there is an existential formula χ with the same free variables as $\neg\psi$ such that $T \models \neg\psi \leftrightarrow \chi$; so χ works for $\neg\varphi$. The cases for $\wedge$ and $\exists$ are clear.

(iii)$\Rightarrow$(i): Assume (iii), and suppose that $\overline{A}$ and $\overline{B}$ are models of T with $\overline{A} \leq \overline{B}$. Suppose that φ is a formula and $a \in {}^{\omega}A$; we want to show that $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$. First assume that $\overline{A} \models \varphi[a]$. By (iii), let ψ be an existential formula with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$. Hence $\overline{A} \models \psi[a]$, so $\overline{B} \models \psi[a]$, hence $\overline{B} \models \varphi[a]$. Second assume that $\overline{A} \not\models \varphi[a]$. Then $\overline{A} \models \neg\varphi[a]$, so $\overline{B} \models \neg\varphi[a]$ by what was just shown, hence $\overline{B} \not\models \varphi[a]$. $\square$

Elimination of quantifiers revisited; Real-closed fields

Throughout this section we assume that our language has at least one individual constant. We say that a theory T has *full elimination of quantifiers* iff for every formula φ there is a quantifier-free formula ψ with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$.

The earlier argument for a special case of elimination of quantifiers essentially proves along the way the following result.

Theorem 6.31. *If for every quantifier-free formula $\varphi(x, \overline{y})$ there is a quantifier-free formula $\psi(\overline{y})$ such that $T \models \exists x\varphi(x, \overline{y}) \leftrightarrow \psi(\overline{y})$, then T has full elimination of quantifiers.*

Lemma 6.32. *Let $\varphi(\overline{x})$ be a formula with the indicated free variables. Then the following conditions are equivalent:*

(i) There is a quantifier free formula $\psi(\overline{x})$ such that $T \models \varphi \leftrightarrow \psi$. ($\psi(\overline{x})$ has no free variables except $\overline{x}$.)

(ii) For any structures $\overline{A}, \overline{B}, \overline{C}$, if $\overline{B}$ and $\overline{C}$ are models of T, $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, and if $\overline{a}$ is a sequence of elements of A, then $\overline{B} \models \varphi[\overline{a}]$ iff $\overline{C} \models \varphi[\overline{a}]$.

Proof. (i)$\Rightarrow$(ii): assume (i) and the hypotheses of (ii). Then

$$\begin{aligned} \overline{B} \models \varphi[\overline{a}] \quad &\text{iff} \quad \overline{B} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{A} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{C} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{C} \models \varphi[\overline{a}] \end{aligned}$$ $\square$

(ii)$\Rightarrow$(i): Assume (ii). Let $\overline{d} = \langle d_i : i < m \rangle$ be a system of new individual constants whose length is that of $\overline{x}$, thus expanding $\mathscr{L}$ to $\mathscr{L}'$. Let

$$\Gamma \overset{\text{def}}{=} \{\psi(\overline{d}) : \psi(\overline{x}) \text{ is a quantifier-free formula in } \mathscr{L}' \text{ and } T \models \varphi(\overline{d}) \rightarrow \psi(\overline{d})\}.$$

Here $\psi(\overline{x})$ has no free variables except for $\overline{x}$. We claim

$(*)$ $T \cup \Gamma \models \varphi(\overline{d})$.

In fact, suppose that this fails. Let $(\overline{B}, b_i)_{i<m}$ be a model of $T\cup\Gamma\cup\{\neg\varphi(\overline{d})\}$, and let A be the subuniverse of $\overline{B}$ generated by $\{b_i : i < m\}$, and let $\Sigma = T\cup\mathrm{diag}((\overline{A}, b_i)_{i<m})\cup\{\varphi(\overline{d})\}$. We claim that Σ has a model. Otherwise, by Proposition 6.28 there is a finite conjunction ψ of members of $\mathrm{diag}(\overline{A}, b_i)_{i<m})$ such that $T \models \psi \to \neg\varphi(\overline{d})$. Now for each constant c_u from the diagram of $(\overline{A}, b_i)_{i<m})$ which appears in ψ, there is a term $\tau_u(d_i : i < m)$ of $\mathscr{L}'$ such that $u = \tau_u^{\overline{A}}(b_i : i < m)$; we replace c_u in ψ by $\tau_u(d_i : i < m)$; this gives a formula ψ'. Note that

$$(\overline{A}, b_i)_{i<m} \models c_u = \tau_u[u, b_i : i < m] \quad \text{and} \quad \models \bigwedge\{c_u = \tau_u : c_u \text{ appears in } \psi\} \to (\psi \leftrightarrow \psi').$$

It follows that $(\overline{A}, b_i)_{i<m} \models \psi'$, and hence also $(\overline{B}, b_i)_{i<m} \models \psi'$. Also, $T \models \varphi(\overline{d}) \to \neg\psi'$, so $\neg\psi' \in \Gamma$. This is a contradiction.

It follows that Σ has a model. By the diagram lemma, this gives a structure $(\overline{C}, b_i)_{i<m}$ which is a model of $T \cup \{\varphi(\overline{d})$ and is an extension of $(\overline{A}, b_i)_{i<m}$. This contradicts (ii).

Hence $(*)$ holds. Hence by Lemma 6.28 there is a finite conjunction ψ of members of Γ such that $T \models \psi(\overline{d}) \to \varphi(\overline{d})$. Clearly ψ is as desired. □

Corollary 6.33. *Let T be a theory in a language $\mathscr{L}$. Suppose that for all quantifier-free formulas $\varphi(x, \overline{y})$ and all $\mathscr{L}$-structures $\overline{A}, \overline{B}, \overline{C}$, if $\overline{B}$ and $\overline{C}$ are models of T and $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, and if $\overline{a}$ is a system of elements of A of the length of $\overline{y}$, then $\overline{B} \models \exists x\varphi(x, \overline{a})$ implies that $\overline{C} \models \exists x\varphi(x, \overline{a})$.*

Under these conditions, T has full quantifier elimination.

Proof. Assume the hypothesis. By Theorem 6.31 it suffices to take any quantifier-free formula $\varphi(x, \overline{y})$ and show that there is a quantifier-free formula $\psi(\overline{y})$ such that $T \models \exists x\varphi(x, \overline{y}) \leftrightarrow \psi(\overline{y})$. The existence of ψ is assured by Lemma 6.32. □

Let T be a theory in a language $\mathscr{L}$. A triple $(\overline{A}, f, \overline{M})$ is *algebraically prime for* T iff $\overline{M}$ is a model of T, f is an embedding of $\overline{A}$ into $\overline{M}$, and for every model $\overline{N}$ of T and embedding g of $\overline{A}$ into $\overline{N}$ there is an embedding h of $\overline{M}$ into $\overline{N}$ such that $h \circ f = g$. This is illustrated in the following diagram:

$$\begin{tikzcd}
\overline{M} \arrow[rr, dotted, "h"] & & \overline{N} \\
& \overline{A} \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Theorem 6.34. *Suppose that T is a theory in a language $\mathscr{L}$, and the following two conditions hold.*

(i) For any structure $\overline{A}$ which is embeddable in a model of T, there is an algebraically prime triple of the form $(\overline{A}, f, \overline{D})$ for T.

(ii) If $\overline{B}$ and $\overline{C}$ are models of T, $\overline{B} \leq \overline{C}$, $\varphi(x, \overline{y})$ is a quantifier-free formula, $\overline{b}$ is a sequence of elements of B, and $\overline{C} \models \exists x\varphi(x, \overline{b})$, then $\overline{B} \models \exists x\varphi(x, \overline{b})$.

Then T has full quantifier elimination.

Proof. We verify the condition of Corollary 6.33. Thus suppose that $\varphi(x, \overline{y})$ is a quantifier-free formula, $\mathscr{L}$-structures $\overline{A}, \overline{B}, \overline{C}$ are given, $\overline{B}$ and $\overline{C}$ are models of T and $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, $\overline{a}$ is a system of elements of A of the length of $\overline{y}$, and $\overline{B} \models \exists x\varphi(x, \overline{a})$. We want to show that $\overline{C} \models \exists x\varphi(x, \overline{a})$. Let $(\overline{A}, f, \overline{D})$ be an algebraically prime triple for T. By the definition of algebraically prime, there are embeddings g, h of $\overline{D}$ into $\overline{B}, \overline{C}$ respectively such that $g \circ f$ is the identity on A and $h \circ f$ is the identity on A. See the diagram:

$$\begin{tikzcd}
\overline{B} & \overline{D} \arrow[l, "g"'] \arrow[r, "h"] & \overline{C} \\
& \overline{A} \arrow[ul, "\leq"] \arrow[u, "f"] \arrow[ur, "\leq"'] &
\end{tikzcd}$$

Now $\overline{B} \models \exists x\varphi(x, g \circ f \circ \overline{a})$, hence $\overline{D} \models \exists x\varphi(x, f \circ \overline{a})$, hence $\overline{C} \models \exists x\varphi(x, h \circ g \circ \overline{a})$, hence $\overline{C} \models \exists x\varphi(x, \overline{a})$. $\square$

Theorem 6.34 gives a very useful criterion for eliminating quantifiers. We use it to prove the following classical result of Tarski. Tarski originally proved this result by the straightforward method described at the beginning of this chapter; but the details are formidable.

Theorem 6.35. *The theory of real-closed ordered fields has full elimination of quantifiers.*

Proof. We appeal to algebraic facts found in the appendix *real* to these notes. By Corollary 12 of *real*, we are dealing with the class of models of a first-order theory T. Theorem 20 of *real* gives (i) of Theorem 6.34.

For (ii), suppose that $\overline{B}$ and $\overline{C}$ are real-closed ordered fields, $\overline{B} \leq \overline{C}$, $\varphi(x, \overline{y})$ is a quantifier-free formula, $\overline{b}$ is a sequence of elements of B, and $\overline{C} \models \exists x\varphi(x, \overline{b})$.

We may assume that $\varphi(x, \overline{b})$ is a conjunction of atomic formulas and their negations, and we may assume that x actually occurs in each conjunct. Thus each atomic formula has the form $p(x, \overline{b}) = q(x, \overline{b})$ or $p(x, \overline{b}) < q(x, \overline{b})$ for certain polynomials $p(x, \overline{b})$ and $q(x, \overline{b})$. Since

$$T \models p(x, \overline{b}) = q(x, \overline{b}) \leftrightarrow p(x, \overline{b}) - q(x, \overline{b}) = 0,$$

and similarly for $<$, we may assume that each atomic formula has the form $p(x, \overline{b}) = 0$ or $0 < q(x, \overline{b})$ for certain polynomials $p(x, \overline{b})$ and $q(x, \overline{b})$. Moreover,

$$T \models \neg(p(x, \overline{b}) = 0) \leftrightarrow 0 < p(x, \overline{b}) \vee 0 < -p(x, \overline{b}) \quad \text{and}$$
$$T \models \neg(0 < p(x, \overline{b})) \leftrightarrow p(x, \overline{b}) = 0 \vee 0 < -p(x, \overline{b}).$$

Hence we may assume that $\varphi(x, \overline{y})$ is a conjunction of formulas of the form $p(x, \overline{b}) = 0$ and $0 < p(x, \overline{b})$. If any of the polynomials in conjuncts $p(x, \overline{b}) = 0$ is nonzero, then any root in B must already be in A since $\overline{A}$ does not have any proper formally real extensions, by the definition of real-closed. In this case $\exists x \varphi(x\overline{b})$ is true in $\overline{A}$, as desired. Thus we may assume that $\varphi(x, \overline{y})$ is a conjunction of formulas of the form $0 < p(x, \overline{b})$.

Now there are only finitely many roots of the polynomials $p(x, \overline{b})$ in B; let them be $c_1, \ldots, c_k$. As before, these are all in A. Choose $d \in B$ such that $\overline{B} \models \varphi[d, \overline{b}]$. Without loss of generality, $c_1, \ldots, c_i < d < c_{i+1}, \ldots, c_k$. Let $e \in A$ be such that $c_1, \ldots, c_i < e < c_{i+1}, \ldots, c_k$. Then by Proposition 15 of *real*, $f(d, \overline{b}) > 0$ iff $f(e, \overline{b}) > 0$, for every conjunct $f(x, \overline{b}) > 0$ of $\varphi(x, \overline{b})$. It follows that $\overline{A} \models \varphi[e, \overline{b}]$. $\square$

Infinitary languages

Let κ and λ be infinite cardinals. The logic $L_{\kappa\lambda}$ allows conjunctions and disjunctions of sets of formulas of size $< \kappa$, and simultaneous quantification over fewer than λ variables. We do not go into the formal definition of such languages, which is straightforward. Note that $L_{\omega\omega}$ is essentially the first-order logic we have been dealing with.

Note that the compactness theorem fails in general for these languages. For example, the following set of sentences in $L_{\omega_1\omega}$ does not have a model, although every finite subset does:

$$\{\forall x[x = c_0 \vee x = c_1 \vee \ldots \vee x = c_n \vee \ldots]\} \quad (\text{length } \omega)$$
$$\cup \{d \neq c_0 \wedge d \neq c_1 \wedge \ldots \wedge d \neq c_n : n \in \omega\},$$

where $d, c_0, c_1, \ldots$ are individual constants.

Attempts to generalize the compactness theorem have led to two important large cardinals:

- κ is *strongly compact* iff $\kappa > \omega$ and for every signature and every set Γ of sentences in $L_{\kappa\kappa}$ in that signature, if every subset of Γ of size less that κ has a model, then so does Γ.

- κ is *weakly compact* iff $\kappa > \omega$ and for every signature and every set Γ of sentences in $L_{\kappa\kappa}$ in that signature, subject to $|\Gamma| \leq \kappa$, if every subset of Γ of size less that κ has a model, then so does Γ.

Both these cardinals are strongly inaccessible, with many strong inaccessibles below them.

$L_{\omega_1\omega}$ seems to be the most tractable of these infinitary logics. We prove one important theorem about it.

Theorem 6.36. (D. Scott) *Let $\mathscr{L}$ be a countable language, and $\overline{M}$ a countable $\mathscr{L}$-structure. Then there is an $L_{\omega_1\omega}$-sentence ψ in $\mathscr{L}$ such that ψ has countable models, and all such models are isomorphic to $\overline{M}$.*

Proof. For each $n \in \omega$, each $a \in {}^nM$, and each $\alpha < \omega_1$ we define a formula φ_a^β. The definition goes by induction on α. Let

$$\varphi_a^0 = \bigwedge\{\chi : \chi \text{ is a formula with free variables among } v_0, \ldots, v_{n-1},$$
$$\chi \text{ is atomic or the negation of an atomic formula, and } \overline{M} \models \chi[a]\}.$$

For α limit we let

$$\varphi_a^\alpha = \bigwedge_{\beta<\alpha} \varphi_a^\beta.$$

Finally,

$$\varphi_a^{\alpha+1} = \varphi_a^\alpha \wedge \bigwedge_{b\in M} \exists v_n \varphi^\alpha_{a\frown\langle b\rangle} \wedge \forall v_n \bigvee_{b\in M} \varphi^\alpha_{a\frown\langle b\rangle}.$$

Then the following conditions hold; proof by induction on α:

(1) $\overline{M} \models \varphi_a^\alpha[a]$.

(2) $\models \varphi_a^\alpha \rightarrow \varphi_a^\beta$ for $\beta < \alpha < \omega_1$.

By (2), $\langle\{b \in {}^nM : \overline{M} \models \varphi_a^\alpha[b]\} : \alpha < \omega_1\rangle$ is a $\subseteq$-increasing sequence of subsets of ${}^nM$. Since ${}^nM$ is countable, it follows that there is an $\alpha_a < \omega_1$ such that $\overline{M} \models \varphi_a^{\alpha_a} \leftrightarrow \varphi_a^\beta$ whenevery $\alpha_a \leq \beta < \omega_1$. Let β be the supremum of all α_a for $a \in \bigcup_{n\in\omega} {}^nM$. Then $\beta < \omega_1$ and

(3) For all finite sequences a of elements of M and all $\alpha \geq \beta$ we have $\overline{M} \models \varphi_a^\alpha \leftrightarrow \varphi_a^\beta$.

Now we define the desired sentence ψ:

$$\psi = \varphi_\emptyset^\beta \wedge \bigwedge\{\forall v_0 \ldots \forall v_{n-1}(\varphi_a^\beta \rightarrow \varphi_a^{\beta+1}) : n \in \omega,\ a \in {}^nM\}.$$

By (1) and (3), $\overline{M}$ is a model of ψ. Now let $\overline{N}$ be any countable model of ψ. Write $M = \{a_i : i < \omega\}$ and $N = \{b_i : i < \omega\}$ We construct sequences $\langle e_i : i < \omega\rangle$ and $\langle(c_i, d_i) : i < \omega\rangle$ by recursion, so that for each i we have

$$\text{(4)} \qquad \overline{M} \models \varphi^\beta_{e\restriction i}[c \restriction i] \quad \text{and } \overline{N} \models \varphi^\beta_{e\restriction i}[d \restriction i].$$

This holds for $i = 0$ since $\models \psi \rightarrow \varphi_\emptyset^\beta$. Now assume that it holds for all $j \leq 2i$. Let c_{2i} be the element a_k with smallest index k such that $a_k \notin \{c_j : j < 2i\}$. By (4), $\overline{M} \models \varphi^\beta_{e\restriction 2i}[c \restriction 2i]$ and $\overline{M} \models \psi$, so $\overline{M} \models \varphi^{\beta+1}_{e\restriction 2i}[c \restriction 2i]$ and hence $\overline{M} \models \forall v_{2i} \bigvee_{u\in M} \varphi^\beta_{(e\restriction 2i)\frown\langle u\rangle}[c \restriction 2i]$. So we can choose $e_{2i} \in M$ such that $\overline{M} \models \varphi^\beta_{(e\restriction 2i)\frown\langle e_{2i}\rangle}[(c \restriction 2i)\frown\langle c_{2i}\rangle]$. Now also $\overline{N} \models \varphi^\beta_{e\restriction 2i}[d \restriction 2i]$ and, since $\overline{N} \models \psi$, also $\overline{N} \models \varphi^\beta_{e\restriction 2i}[d \restriction 2i] \rightarrow \psi^{\beta+1}_{e\restriction 2i}[d \restriction 2i]$. Hence $\overline{N} \models \psi^{\beta+1}_{e\restriction 2i}[d \restriction 2i]$, so $\overline{N} \models \exists v_{2i}\varphi^\beta_{(e\restriction 2i)\frown\langle e_{2i}\rangle}[d \restriction 2i]$. So we get an element d_{2i} of N such that $\overline{N} \models \varphi^\beta_{(e\restriction 2i)\frown\langle e_{2i}\rangle}[(d \restriction 2i)\frown\langle d_{2i}\rangle]$. Thus (4) holds for $2i + 1$. Now we repeat this argument with the roles of $\overline{M}$ and $\overline{N}$ interchanged, starting with an element $d_{2i+1} = b_l$ with l the smallest index such that $b_l \notin \{d_j : j \leq 2i\}$, to obtain c_{2i+1} such that (4) holds for $2i+2$. This finishes the construction.

By (2) we have $\overline{M} \models \varphi^0_{c\restriction i}[c \restriction i]$ and $\overline{N} \models \varphi^0_{c\restriction i}[d \restriction i]$ for all i, and by construction $\{a_i : i < \omega\} = \{c_i : i < \omega\}$ and $\{b_i : i < \omega\} = \{d_i : i < \omega\}$, so $\{(c_i, d_i) : i < \omega\}$ is the desired isomorphism from $\overline{M}$ onto $\overline{N}$. □

EXERCISES

Exc. 6.1. A subset X of a structure $\overline{M}$ is *definable* iff there is a formula $\varphi(x)$ with only x free such that $X = \{a \in M : \overline{M} \models \varphi[a]\}$. Similarly, for any positive integer m, a subset X of ${}^mM$ is *definable* iff there is a formula $\varphi(\overline{x})$ with $\overline{x}$ a sequence of m distinct variables including all variables occurring free in φ, such that $X = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$.

For the language with no nonlogical symbols and for any structure $\overline{M}$ in that language, determine all the definable subsets and m-ary relations over $\overline{M}$. Hint: use Theorem 6.1.

Exc. 6.2. Let Γ be the set of all sentences holding in the structure $(\omega, S, 0)$, where $S(n) = n+1$ for all $n \in \omega$. Prove an elimination of quantifiers theorem for Γ.

Exc. 6.3. Let T be the theory of an infinite equivalence relation each of whose equivalence classes has exactly two elements. Use an Ehrenfeucht game to show that T is complete.

Exc. 6.4. Let T be any theory. Show that the class of all substructures of models of T is the class of all models of a set of universal sentences, i.e., sentences of the form $\forall\overline{x}\varphi$ with φ quantifier free and $\overline{x}$ a finite string of variables containing all variables free in φ.

Exc. 6.5. Suppose that $\Gamma \cup \{\varphi\}$ is a set of sentences in a language $\mathscr{L}$. Suppose that Γ and φ have the same models. Prove that there is a finite subset Δ of Γ with the same models as Γ.

Exc. 6.6. Suppose that T and T' are theories in a language $\mathscr{L}$. Show that the following conditions are equivalent:

(i) Every model of T' can be embedded in a model of T.

(ii) Every universal sentence which holds in all models of T also holds in all models of T'.

Exc. 6.7. Let T be a theory in a language $\mathscr{L}$. Let $\mathbf{K}$ be the class of all models of T. Show that the following conditions are equivalent:

(i) $\mathbf{SK} = \mathbf{K}$.

(ii) There is a collection Γ of universal sentences such that $\mathbf{K}$ is the class of all models of Γ.

Exc. 6.8. Suppose that $\overline{A} \leq \overline{B}$. Prove that $\overline{A} \preceq \overline{B}$ iff $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)a \in A$.

Exc. 6.9. Suppose that m is a positive integer, $\varphi(\overline{x})$ is a formula with free variables $\overline{x}$ of length m, and $\overline{M}$ is a structure. Define $\varphi(\overline{M}) = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$. Show that the following conditions are equivalent:

(i) $\varphi(\overline{M})$ is finite.

(ii) $\varphi(\overline{M}) = \varphi(\overline{N})$ whenever $\overline{M} \preceq \overline{N}$.

Exc. 6.10. Prove that if K is a set of models of a complete theory T then there is a structure $\overline{M}$ such that every member of K can be elementarily embedded in $\overline{M}$.

Exc. 6.11. Suppose that $\overline{A}$ and $\overline{B}$ are elementarily equivalent, κ-saturated, and both of size κ. Show that they are isomorphic.

Exc. 6.12. For any natural number n and any structure $\overline{M}$, an *n-type* of $\overline{M}$ is a collection Γ of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, a sequence of distinct variables of length n, such that $\overline{M}_M \models \exists \overline{x}\varphi$ for every conjunction of finitely many members of Γ. Prove that if Γ is a collection of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, then Γ is an n-type over $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $\overline{N} \models \varphi[\overline{a}]$ for every $\varphi \in \Gamma$.

Exc. 6.13. If $\overline{M}$ is a structure, $A \subseteq M$, and $n \in \omega$, then an *n-type over* A of $\overline{M}$ is an n-type of $\overline{M}$ all of whose additional constants come from A. Given an n-tupe $\overline{a}$ of elements of M, the n-type over A of $\overline{a}$ in $\overline{M}$, denoted by $\mathrm{tp}^{\overline{M}}(\overline{a})/A)$, is the set $\{\varphi(\overline{x}) : \varphi$ is a formula with free variables among $\overline{x}$, $\overline{x}$ has length n, and $\overline{M}_A \models \varphi[\overline{a}]\}$. An n-type S over A is *complete* iff $\varphi \in S$ or $\neg\varphi \in S$ for every formula in the language $\mathscr{L}_A$ with free variables among $\overline{x}$.

Prove that S is a complete n-type over A in $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S = \mathrm{tp}^{\overline{N}}(\overline{a}/A)$.

Exc. 6.14. Let t be an n-type over A of $\overline{M}$. We say that t is *isolated* iff there is a formula $\varphi(\overline{x})$ in $\mathscr{L}_A$ such that $\overline{M}_A \models \exists\overline{x}\varphi$ and $\overline{M}_A \models \forall\overline{x}(\varphi \to \psi)$ for every $\psi \in t$. We then say that φ *isolates* t.

Prove that if φ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

Exc. 6.15. Show that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$ is isolated iff both $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated.

Exc. 6.16. Let T be a complete theory with a model. A formula $\varphi(\overline{x})$ is *complete* in T iff $T \cup \{\exists\overline{x}\varphi\}$ has a model, and for every formula $\psi(\overline{x})$, either $T \models \varphi \to \psi$ or $T \models \varphi \to \neg\psi$. Here $\overline{x}$ is a sequence of variables containing all variables free in φ or ψ.

A formula $\theta(\overline{x})$ is *completable* in T iff there is a complete formula $\varphi(\overline{x})$ such that $T \models \varphi \to \theta$.

A structure $\overline{M}$ is *atomic* iff every tuple $\overline{a}$ of elements of M satisfies a complete formula in the theory of $\overline{M}$.

A theory T is *atomic* iff for every formula $\theta(\overline{x})$ such that $T \cup \{\exists\overline{x}\theta(\overline{x})$ has a model, θ is completable in T.

Show that if T is a complete theory in a countable language, then T has a countable atomic model iff T is atomic. Hint: in the direction $\Leftarrow$, for each $n \in \omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0, \ldots, v_{n-1}$, and apply the omitting types theorem.