$\boxed{1.1}$ *Let $\mathscr{L}$ be a language with no individual constants. Define an $\mathscr{L}$-structure $\overline{A}$ and subuniverses B, C of $\overline{A}$ such that $B \cap C = \emptyset$.*

Let $A = 2$, and let the fundamental operations of $\overline{A}$ be such that $\{0\}$ and $\{1\}$ are closed under them; the fundamental relations can be anything. Then $\{0\}$ and $\{1\}$ are disjoint subuniverses.

$\boxed{1.2}$ *Carry out the "easy induction" at the beginning of the proof of Proposition 1.2.*

Since $\langle X\rangle_A$ is a subuniverse containing X, we have $Y_0 \subseteq \langle X\rangle_A$. Now suppose that $Y_i \subseteq \langle X\rangle_A$. Suppose that F is a function symbol of rank m and $x \in {}^mY_i$. Then $x \in {}^mX$, and so $F(x) \in \langle X\rangle_A$. It follows that $Y_{i+1} \subseteq \langle X\rangle_A$.

$\boxed{1.3}$ *If X and Y are nonempty subsets of the universe A of an algebra $\overline{A}$, then $\langle X \cup \langle Y\rangle\rangle = \langle X \cup Y\rangle$.*

Suppose that B is a subuniverse of A containing $X \cup Y$. Then $Y \subseteq B$, and so $\langle Y\rangle \subseteq B$. Thus $X \cup \langle Y\rangle \subseteq B$. It follows that $\langle X \cup \langle Y\rangle\rangle \subseteq B$. Since B is arbitrary, this shows that $\langle X \cup \langle Y\rangle\rangle \subseteq \langle X \cup Y\rangle$.

Conversely, suppose that C is a subuniverse of A containing $X \cup \langle Y\rangle$. Then $\langle Y\rangle \subseteq C$, and so also $Y \subseteq C$. Thus $X \cup Y \subseteq C$. It follows that $\langle X \cup Y\rangle \subseteq C$. Since C is arbitrary, this proves that $\langle X \cup Y\rangle \subseteq \langle X \cup \langle Y\rangle\rangle$. Together with the preceding paragraph this proves that $\langle X \cup Y\rangle = \langle X \cup \langle Y\rangle\rangle$.

$\boxed{1.4}$ *If K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$, then $\langle \bigcup K\rangle = \langle \bigcup_{X\in K}\langle X\rangle\rangle$.*

Suppose that K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$.

First suppose that B is a subuniverse of A containing $\bigcup K$. Then for each $X \in K$, B contains X, and hence $\langle X\rangle \subseteq B$. thus $\bigcup_{X\in K}\langle X\rangle \subseteq B$, and so $\langle \bigcup_{X\in K}\langle X\rangle\rangle \subseteq B$. Since B is arbitrary, this proves that $\langle \bigcup_{X\in K}\langle X\rangle\rangle \subseteq \langle \bigcup K\rangle$.

Second, suppose that B is a subuniverse of A containing $\bigcup_{X\in K}\langle X\rangle$. Then for each $X \in K$ we have $X \subseteq \langle X\rangle \subseteq B$. So $\bigcup K \subseteq B$, and so also $\langle \bigcup K\rangle \subseteq B$. Since B is arbitrary, this shows that $\langle \bigcup K\rangle \subseteq \langle \bigcup_{X\in K}\langle X\rangle\rangle$. Together with the preceding paragraph this shows that $\langle \bigcup K\rangle = \langle \bigcup_{X\in K}\langle X\rangle\rangle$.

$\boxed{1.5}$ *Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{A}$. Show that $f[C]$ is a subuniverse of $\overline{B}$.*

For k an individual constant, $k^{\overline{B}} = f(k^{\overline{A}}) \in f[C]$. For F an m-ary operation symbol and $b \in {}^mC$,

$$F^{\overline{B}}(f \circ b) = f(F^{\overline{A}}(b)) \in f[C],$$

as desired.

$\boxed{1.6}$ *Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{B}$. Show that $f^{-1}[C]$ is a subuniverse of $\overline{A}$.*

For k an individual constant, $f(k^{\overline{A}}) = k^{\overline{B}} \in C$ and so $k^{\overline{A}} \in f^{-1}[C]$. For F an m-ary operation symbol and $b \in {}^m(f^{-1}[C])$ we have $f(F^{\overline{A}}(b)) = F^{\overline{B}}(f \circ b) \in C$ since $(f \circ b) \in {}^mC$; hence $F^{\overline{A}}(b) \in f^{-1}[C]$.

$\boxed{1.7}$ *If X generates $\overline{A}$, f and g are homomorphisms from $\overline{A}$ into $\overline{B}$, and $f \upharpoonright X = g \upharpoonright X$, then $f = g$.*

It suffices to show that $Y \stackrel{\text{def}}{=} \{a \in A : f(a) = g(a)\}$ is a subuniverse of $\overline{A}$ containing X. We are given that $X \subseteq Y$. For any individual constant k, $f(k^{\overline{A}}) = k^{\overline{B}} = f(k^{\overline{A}})$, so $k^{\overline{A}} \in Y$. Now suppose that F is an n-ary operation symbol and $a \in {}^nY$. Then

$$f(F^{\overline{A}}(a)) = F^{\overline{B}}(f \circ a) = F^{\overline{B}}(g \circ a) = g(F^{\overline{A}}(a)),$$

and so $F^{\overline{A}}(a) \in Y$. Thus Y is a subuniverse of $\overline{A}$.

$\boxed{1.8}$ *If f is a homomorphism from $\overline{A}$ into $\overline{B}$ and X is a nonempty subset of A, then $f[\langle X\rangle] = \langle f[X]\rangle$.*

By exercise 1.5, $f[\langle X\rangle]$ is a subuniverse of $\overline{B}$ containing $f[X]$. Hence $\langle f[X]\rangle \subseteq f[\langle X\rangle]$.

Now by exercise 1.6, $f^{-1}[\langle f[X]\rangle]$ is a subuniverse of $\overline{A}$, and it obviously contains X. So $\langle X\rangle \subseteq f^{-1}[\langle f[X]\rangle]$, and so $f[\langle X\rangle] \subseteq \langle f[X]\rangle$.

$\boxed{1.9}$ *If $\overline{A}$ is a substructure of $\overline{B}$ and $\equiv$ is a congruence relation on $\overline{B}$, then $\equiv \cap (A \times A)$ is a congruence relation on $\overline{A}$.*

Clearly $\equiv \cap (A \times A)$ is an equivalence relation on $\overline{A}$. Now suppose that F is an m-ary operation symbol, $x, y \in {}^mA$, and $x_i \equiv y_i$ for all $i < m$. Then

$$F^{\overline{A}}x = F^{\overline{B}}x \equiv F^{\overline{B}}y = F^{\overline{A}}y.$$

Finally, if R is an m-ary relation symbol, $x, y \in {}^mA$, and $x_i \equiv y_i$ for all $i < m$, then

$$a \in R^{\overline{A}} \text{ iff } a \in R^{\overline{B}} \text{ iff } b \in R^{\overline{B}} \text{ iff } b \in R^{\overline{A}}.$$

$\boxed{1.10}$ *Suppose that R is a congruence relation on $\overline{A}$, and S is a congruence relation on $\overline{A}/R$. Define $T = \{(a_0, a_1) \in A \times A : ([a_0]_R, [a_1]_R) \in S\}$. Show that T is a congruence relation on A and $R \subseteq T$.*

T is reflexive on A: given $a \in A$, we have $[a]_R S [a]_R$, so aTa.

T is symmetric: given xTy, we have $[x]_R S [y]_R$, hence $[y]_R S [x]_R$, hence yTx,

T is transitive: given $xTyTz$, we have $[x]_R S [y]_R S [z]_R$, hence $[x]_R S [z]_R$, hence xTz.

Now suppose that F is an m-ary operation symbol, $x, y \in {}^mA$, and $x_i T y_i$ for all $i < m$. Then $[x_i]_R S [y_i]_R$ for all $i < m$, so

$$F^{\overline{A}/R}([x_0]_R, \ldots, [x_{m-1}]_R) S F^{\overline{A}/R}([y_0]_R, \ldots, [y_{m-1}]_R).$$

Now $F^{\overline{A}/R}([x_0]_R, \ldots, [x_{m-1}]_R) = [F^{\overline{A}}(x_0, \ldots, x_{m-1})]_R$ and $F^{\overline{A}/R}([y_0]_R, \ldots, [y_{m-1}]_R) = [F^{\overline{A}}(y_0, \ldots, y_{m-1})]_R$, so

$$[F^{\overline{A}}(x_0, \ldots, x_{m-1})]_R S [F^{\overline{A}}(y_0, \ldots, y_{m-1})]_R;$$

hence $F^{\overline{A}}(x_0, \ldots, x_{m-1}) T F^{\overline{A}}(y_0, \ldots, y_{m-1})$.

Now suppose that U is an m-ary relation symbol, $a, b \in {}^mA$, and $a_i T b_i$ for all $i < m$. Then $[a_i]_R S [b_i]_R$ for all $i < m$, and

$$\begin{aligned} a \in U^{\overline{A}} \quad &\text{iff} \quad \langle [a_i]_R : i < m \rangle \in U^{\overline{A}/R} \\ &\text{iff} \quad \langle [b_i]_R : i < m \rangle \in U^{\overline{A}/R} \quad \text{since } S \text{ is a congruence relation on } \overline{A}/R \\ &\text{iff} \quad b \in U^{\overline{A}}. \end{aligned}$$

Finally, if xRy, then $[x]_R = [y]_R$, hence $[x]_R S [y]_R$, hence xTy.

$\boxed{1.11}$ *(Continuing exercise 1.10) Show that the procedure of exercise 1.10 establishes a one-one order-preserving correspondence between congruence relations on A/R and those congruence relations on A with include R.*

For each congruence relation S on A/R let F_S be the congruence relation T defined in exercise 1.10. Suppose that S_0 and S_1 are distinct congruence relations on A/R. Say $[a]_R S_0 [b]_R$ and not$([a]_R S_1 [b]_R$. Then $aF_{S_0}b$. Suppose that $aF_{S_1}b$. Then $[a]_R S_1 [b]_R$, contradiction. Thus F is a one-one function.

Now suppose that U is a congruence relation on A which includes R. Define

$$S = \{(x, y) \in (A/R) \times (A/R) : \exists a, b \in A[x = [a]_R \wedge y = [b]_R \wedge aUb\}.$$

We claim that S is a congruence relation on A/R and $F_S = U$.

S is reflexive, since if $a \in A$ then aUa, hence $[a]_R S [a]_R$.

S is symmetric: suppose xSy. Choose $a, b \in A$ such that $x = [a]_R$, $y = [b]_R$, and aUb. Then bUa, so ySx.

S is transitive: suppose that $xSySz$. Choose $a, b, c, d \in A$ such that $x = [a]_R$, $y = [b]_R$, aUb, $y = [c]_R$, $z = [d]_R$, and cUd. Then $[b]_R = [c]_R$, so bRc, hence bUc. So $aUbUcUd$, hence aUd and so xSz.

Now let F be an m-ary operation symbol and $x, y \in {}^mS$ with $x_i S y_i$ for all $i < m$. Choose $a, b \in {}^mA$ so that $\forall i < m[x_i = [a_i]_R \wedge y_i = [b_i]_R \wedge a_i U b_i]$. Then $F^{\overline{A}}(a) U f^{\overline{A}}(b)$. Moreover, $F^{A/R}(x) = [F^A(a)]_R$ and $F^{A/R}(y) = [F^A(b)]_R$. Hence $F^{A/R}(x) S F^{A/R}(y)$.

Let T be an m-ary relation symbol, and let $x, y \in {}^m(A/R)$ with $x_i S y_i$ for all $i < m$. Then there exist $x'_i, y'_i \in A$ so that $x_i = [x'_i]_R$ and $y_i = [y'_i]_R$ and $x'_i U y'_i$ for all $i < m$. Then

$$\begin{aligned} x \in T^{\overline{A}/R} \quad &\text{iff} \quad x' \in T^{\overline{A}} \\ &\text{iff} \quad y' \in T^{\overline{A}} \quad \text{since } U \text{ is a congruence relation on } \overline{A} \\ &\text{iff} \quad y \in T^{\overline{A}/R}. \end{aligned}$$

Thus S is a congruence relation on A/R. Now take any $a, b \in A$. If aF_Sb, then $[a]_RS[b]_R$. Hence there are $c, d \in A$ such that $[a]_R = [c]_R$, $[b]_R = [d]_R$, and cUd. Since $R \subseteq U$, we have $aUcUdUb$, and so aUb.

Conversely, suppose that aUb. Then $[a]_RS[b]_R$ and so aF_Sb.

Hence $F_S = U$.

$\boxed{1.12}$ *Suppose that $\langle \overline{A}_i : i \in I\rangle$ is a system of similar structures, and $\overline{B}$ is another structure similar to them. Suppose that f_i is a homomorphism from $\overline{B}$ into $\overline{A}_i$ for each $i \in I$. Show that there is a homomorphism g from $\overline{B}$ into $\prod_{i\in I} \overline{A}_i$ such that $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$.*

Define $(g(b))_i = f_i(b)$ for all $b \in B$ and $i \in I$. Thus $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$. If k is an individual constant, then

$$g(k^{\overline{B}}) = \langle k^{\overline{A}_i} : i \in I\rangle = k^{\overline{C}}.$$

For R an m-ary relation symbol and $b \in {}^mB$,

$$\begin{aligned} g \circ b \in R^{\overline{C}} \quad &\text{iff} \quad \forall i \in I[\mathrm{pr}_i \circ g \circ b \in R^{\overline{A}_i}] \\ &\text{iff} \quad \forall i \in I[f_i \circ b \in R^{\overline{A}_i}] \\ &\text{iff} \quad b \in R^{\overline{B}}. \end{aligned}$$

For F an m-ary operation symbol and $b \in {}^mB$,

$$\begin{aligned} g(F^{\overline{B}}(b)) &= \langle f_i(F^{\overline{B}}(b)) : i \in I\rangle \\ &= \langle F^{\overline{A}_i}(f_i \circ b) : i \in I\rangle \\ &= \langle F^{\overline{A}_i}(\mathrm{pr}_i \circ g \circ b) : i \in I\rangle \\ &= F^{\overline{C}}(g \circ b). \end{aligned}$$

$\boxed{1.13}$ *Show that a product of partial orderings is a partial ordering.*

See page 2 for the official definition of a partial ordering. Suppose that $\langle (A_i, <_i) : i \in I\rangle$ is a system of partial orderings. Let $B = \prod_{i\in I} A_i$.

Irreflexive: if $b \in B$ and $b <_B b$, then $\forall i[b_i < b_i]$, contradicting irreflexivity on each factor.

Transitive: suppose that $b <_B c <_B d$. Thus $\forall i \in I[b_i <_I c_i <_i d_i]$, so $\forall i \in I[b_i <_i d_i]$, hence $b <_B d$.

$\boxed{1.14}$ *A partial ordering $(A, <)$ is a* **linear ordering** *iff for any two distinct $x, y \in A$ we have $x < y$ or $y < x$. Give an example of two linear orderings whose product is not a linear ordering.*

Take $\mathbb{Z} \times \mathbb{Z}$. Then $(1, 2)$ and $(3, 0)$ are incomparable.

$\boxed{1.15}$ *Give an example of two ordered fields whose product is not even a field.*

We consider $\mathbb{R} \times \mathbb{R}$. Then $(1, 0)$ is nonzero, but does not have an inverse.

$\boxed{1.16}$ *Let F be a proper filter on a set I. Show that F is an ultrafilter iff for all $a, b \subseteq I$, if $a \cup b \in F$ then $a \in F$ or $b \in F$.*

$\Rightarrow$: Suppose $a \notin F$ and $b \notin F$. Then $(I \backslash a) \in F$ and $(I \backslash b) \in F$, so $(I \backslash a) \cap (I \backslash b) \in F$. It follows that $a \cup b \notin F$, as otherwise $\emptyset = (I \backslash a) \cap (I \backslash b) \cap (a \cup b) \in F$.

$\Leftarrow$: If $a \subseteq I$, then $I = a \cup (I \backslash a) \in F$, hence $a \in F$ or $(I \backslash a) \in F$.

$\boxed{1.17}$ *Show that any ultraproduct of linear orderings is a linear ordering.*

We consider an ultraproduct $B \stackrel{\text{def}}{=} \prod_{i \in I} \overline{A}_i / D$. By exercise E1.13 it suffices to show that any two elements of B are comparable. Let $a, b \in \prod_{i \in I} A_i$. Then

$$I = \{i \in I : a_i < b_i\} \cup \{i \in I : a_i = b_i\} \cup \{i \in I : b_i < a_i\}.$$

By exercisee 1.16, since $I \in D$ one of these three sets is in D, giving $[a] < [b]$, $[a] = [b]$, or $[b] < [a]$ respectively.

$\boxed{1.18}$ *Suppose that I is a nonempty set, and $\langle J_i : i \in I \rangle$ is a system of nonempty sets. Also suppose that F_i is an ultrafilter on J_i for each $i \in I$, and G is an ultrafilter on I. Let $K = \{(i, j) : i \in I, j \in J_i\}$, and define*

$$H = \{X \subseteq K : \{i \in I : \{j \in J_i : (i, j) \in X\} \in F_i\} \in G\}.$$

Show that H is an ultrafilter on K.

• $K \in H$: For any $i \in I$ we have $\{j \in J_i : (i, j) \in K\} = J_i \in F_i$, so that $\{i \in I : \{j \in J_i : (i, j) \in K\} \in F_i\} = I \in G\}$. Hence $K \in H$.

• $\emptyset \notin H$: Suppose $\emptyset \in H$. Thus $\{i \in I : \{j \in J_i : (i, j) \in \emptyset\} \in F_i\} \in G$, in particular, there is an $i \in I$ such that $\{j \in J_i : (i, j) \in \emptyset\} \in F_i$. In particular there is a $j \in J$ such that $(i, j) \in \emptyset$, contradiction.

• Suppose that $X, Y \in H$; we show that $X \cap Y \in H$. Let

$$\begin{aligned} X' &= \{i \in I : \{j \in J_i : (i, j) \in X\} \in F_i\}; \\ Y' &= \{i \in I : \{j \in J_i : (i, j) \in Y\} \in F_i\}; \\ Z' &= \{i \in I : \{j \in J_i : (i, j) \in X \cap Y\} \in F_i\}. \end{aligned}$$

Since $X, Y \in H$, we have $X', Y' \in G$; hence $X' \cap Y' \in G$. We claim that $X' \cap Y' \subseteq Z'$; hence $Z' \in G$ and so $X \cap Y \in H$. To prove this claim, suppose that $i \in X' \cap Y'$. Then $\{j \in J_i : (i, j) \in X\}$ and $\{j \in J_i : (i, j) \in Y\}$ are both members of F_i, and hence so is their intersection. Now

$$\{j \in J_i : (i, j) \in X\} \cap \{j \in J_i : (i, j) \in Y\} \subseteq \{j \in J_i : (i, j) \in X \cap Y\},$$

so it follows that $\{j \in J_i : (i, j) \in X \cap Y\} \in F_i$; hence $i \in Z'$. This proves out claim.

• Suppose that $X \subseteq K$ and $X \notin H$; we prove that $(K \backslash X) \in H$. Since $X \notin H$, with X' as above we have $X' \notin G$; so $(I \backslash X') \in G$. We claim that

$$(I \backslash X') \subseteq \{i \in I : \{j \in J_i : (i, j) \in (K \backslash X)\} \in F_i\};$$

from this claim it follows that $\{i \in I : \{j \in J_i : (i,j) \in (K\backslash X)\} \in F_i\} \in G$, and hence $(K\backslash X) \in H$.

To prove ths claim, suppose that $i \in (I\backslash X')$. Thus $\{j \in J_i : (i,j) \in X\} \notin F_i$, and hence $\{j \in J_i : (i,j) \in (K\backslash X)\} \in F_i$, as desired.

$\boxed{1.19}$ *Under the notation of exercise 1.18, show that there is an isomorphism f of the structure $\prod_{i\in I}(\prod_{j\in J_i} \overline{A_{ij}}/F_i)/G)$ onto $\prod_{(i,j)\in K} \overline{A_{ij}}/H$ such that:*

$$\forall r \in \prod_{i\in I}\left(\prod_{j\in J_i} \overline{A_{ij}}/F_i)/G\right)\left[\left[r = [s]_G \text{ with } s \in \prod_{i\in I}\left(\prod_{j\in J_i} \overline{A_{ij}}/F_i\right)\right]\right.$$
$$\left.\text{and } \forall i \in I\left[s_i = [t_i]_{F_i} \text{ with } t_i \in \prod_{j\in J_i} \overline{A_{ij}}\right] \text{ implies that } f(r) = [\langle t_i(j) : (i,j) \in K\rangle]_H\right].$$

First we show that f is well-defined. Thus suppose that r, s, t are as above, and suppose that also $r = [s']_G$ with $s' \in \prod_{i\in I}\left(\prod_{j\in J_i} \overline{A_{ij}}/F_i\right)$ and $s'_i = [t'_i]_{F_i}$ for each $i \in I$, with $t' \in \prod_{j\in J_i} \overline{A_{ij}}$. Then $\{i \in I : s_i = s'_i\} \in G$, hence $\{i \in I : \{j \in J_i : t_{ij} = t'_{ij}\} \in F_i\} \in G$. So $\{(i,j) \in K : t_{ij} = t'_{ij}\} \in H$, as desired.

Reversing these steps, we see that f is injective.

Given $u \in \prod_{(i,j)\in K} \overline{A_{ij}}/H$, write $u = [t]_H$ with $t \in \prod_{(i,j)\in K} \overline{A_{ij}}$. For each $i \in I$ let $s_i = [\langle t_{ij} : j \in J_i\rangle]_{F_i}$. Then $f([s]_G) = u$. Thus f is surjective.

For the remainder of the proof we introduce the following abbreviations:

$$\begin{aligned}
\overline{B}_i &= \prod_{j\in J_i} \overline{A}_{ij};\\
\overline{C}_i &= \overline{B}_i/F_i;\\
\overline{D} &= \prod_{i\in I} \overline{C}_i;\\
\overline{E} &= \overline{D}/G;\\
\overline{L} &= \prod_{(i,j)\in K} \overline{A}_{ij};\\
\overline{M} &= \overline{L}/H.
\end{aligned}$$

Now let k be an individual constant. Then $k^{\overline{E}} = [k^{\overline{D}}]_G$, $k_i^{\overline{D}} = [k^{\overline{B}_i}]_{F_i}$ for each $i \in I$, and $k_j^{\overline{B}_i} = k^{\overline{A}_{ij}}$ for each $i \in I$ and $j \in J_i$. Hence $f(k^{\overline{E}}) = f([k^{\overline{D}}]_G) = [\langle k^{\overline{A}_{ij}} : (i,j) \in K\rangle]_H = k^{\overline{M}}$.

Next, let m be a positive integer and $r^0, \ldots, r^{m-1} \in E$. For each $k < m$ choose $s^k \in E$ with $r^k = [s^k]_G$. Then for each $i \in I$ write $s_i^k = [t_i^k]_{F_i}$.

Suspose that R is an m-ary relation symbol. Then

$$\begin{aligned}
\langle r^0, \ldots, r^{m-1}\rangle \in R^{\overline{E}} \quad & \text{iff} \quad \{i \in I : \langle s_i^0, \ldots, s_i^{m-1}\rangle \in R^{\overline{C}_i}\} \in G \\
& \text{iff} \quad \{i \in I : \{j \in J_i : \langle t_{ij}^0, \ldots, t_{ij}^{m-1}\rangle \in R^{\overline{A_i}}\} \in F_i\} \in G \\
& \text{iff} \quad \{(i,j) \in K : \langle t_{ij}^0, \ldots, t_{ij}^{m-1}\rangle \in R^{\overline{A}_{ij}}\} \in H \\
& \text{iff} \quad \langle [t^0]_H, \ldots, [t^{m-1}]_H\rangle \in R^{\overline{M}} \\
& \text{iff} \quad \langle f(r^0), \ldots, f(r^{m-1})\rangle \in R^{\overline{M}}.
\end{aligned}$$

Now suppose that Q is an m-ary operation symbol. Then

$$\begin{aligned}
f(Q^{\overline{E}}(r^0, \ldots, r^{m-1})) &= f([Q^{\overline{D}}(s^0, \ldots, s^{m-1})]_G) \\
&= f([\langle (Q^{\overline{C}_i}(s_i^0, \ldots, s_i^{m-1}) : i \in I\rangle)]_G) \\
&= f([\langle [\langle Q^{\overline{A}_{ij}}(t_{ij}^0, \ldots, t_{ij}^{m-1}) : j \in J_i\rangle]_{F_i} : i \in I\rangle]_G) \\
&= \langle [Q^{\overline{A}_{ij}}(t_{ij}^0, \ldots, t_{ij}^{m-1})]_H : (i,j) \in K\rangle \\
&= Q^{\overline{M}}([t^0]_H, \ldots, [t^{m-1}]_H) \\
&= Q^{\overline{M}}(f(r^0), \ldots, f(r^{m-1})).
\end{aligned}$$