HW 8

Due Friday April 13

Exercises: 8.1.1, **8.2.2**, **8.3.1**, (8.3.5), (8.3.6), **8.4.1**, **8.4.3**, **8.4.4**, **8.4.5**, **8.5.1**, (8.5.2), **8.5.5**.

To pass this assignment, submit well-written, complete, correct solutions to at least **five** exercises from among the non-parenthesized numbers above.

8.1.1. Lct $\varphi \in L$ dehne a finite set X in the L-structure $\mathcal{M}$. Show that in every $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$, the set defined by φ has the same power as X. Formulate and prove a converse of this.

8.2.2. Given a structure $\mathcal{M}$ and an ultrafilter U on a nonempty set I, prove that the canonical embedding (from Exercise 4.1.2) of $\mathcal{M}$ in its ultrapower $\mathcal{M}^I/U$ is elementary.

8.3.1. Let $\mathcal{M}$ be an L-structure. Prove that the following are equivalent for any formula $\varphi \in L_n(M)$.
(i) $\varphi(\mathcal{M})$ is finite.
(ii) $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ for every $\mathcal{N} \succcurlyeq \mathcal{M}$.
(iii) $\varphi(\mathcal{M}) \subseteq M^n$ for every $\mathcal{N} \succcurlyeq \mathcal{M}$.

(8.3.5) Show by counterexample that the criterion 8.3.4 is not necessary for being an elementary substructure.

(8.3.6) Prove that, given an arbitrary subordering $\mathcal{M} = (M, <) \subseteq \rho$, this inclusion is elementary if and only if $\mathcal{M} \models \mathrm{DLO}_{--}$.

8.4.1. Suppose $\mathcal{M}$ is an L-structure and $\varphi \in L_n (n > 0)$ is such that $\varphi(\mathcal{M})$ is infinite. Then for every $\kappa \in \mathbf{Cn}$ with $\kappa \geq |L|$, there is an L-structure $\mathcal{N} \equiv \mathcal{M}$ of power κ and such that $|\varphi(\mathcal{N})| = \kappa$.

8.4.3. Verify that ultrapowers (along with isomorphic correction) provide us with another tool to construct elementary extensions.

8.4.4 Consider $\mathcal{M}$ and $\mathcal{N}$ as in Example (4) of §6.3 and the disjoint union of $\mathcal{M}$ and $\mathcal{N}$, regarded as an $L_<$-structure $\mathcal{N}'$. Prove that $\mathcal{M}$ and $\mathcal{N}'$ are elementarily equivalent.

8.4.5. Find an example of a theory T and a model of $T_\exists$ that does not contain a model of T (this showing that we cannot improve on Exercise 6.2.6).

8.5.1. Show that DLO_{-+}, DLO_{+-}, and DLO_{++} are complete theories.

(8.5.2) Prove that DLO_{--} is in no uncountable power categorical.

8.5.5. Show that the theory of the structure $\mathcal{M}$ from Exercise 8.4.4 is categorical in all uncountable powers.