HW 4

Due Friday, February 16

Exercises: 3.5.1, (3.5.3), 3.6.1, (3.6.2), (4.1.1), (4.1.2), (4.1.3), 4.1.4, 4.2.1, (4.2.2), (4.2.3), 4.2.4, (4.2.5), 4.3.1, 4.3.3.

(To pass this assignment, submit well-written, complete, correct solutions to at least **four** exercises from among the non-parenthesized numbers above.)

Sec 3.5.

3.5.1. An L-theory T is complete iff $\varphi \vee \psi \in T$ implies $\varphi \in T$ or $\psi \in T$.

(3.5.3) Find a fommla defining the set of prime numbers in the standard model of Peano arithmetic.

Sec 3.6.

3.6.1. Prove that thc set of L-sentences true in $\emptyset_L$ is neither consistent nor deductively closed.

(3.6.2) Determine the set of L-sentences true in $\emptyset_L$ precisely.

Sec 4.1.

(4.1.1) Verify that the definition of reduced product does not depend on the representatives chosen.

(4.1.2) Show that every structure is embeddable in each of its reduced powers.

(4.1.3) Let F be the Frechet filter on $\mathbb{N}$ and $\mathcal{N}$ the standard model of peano arithmetic. Prove that $\mathcal{N}^{\mathbb{N}}/F$ contains no prime divisors, hence the embedding from the previous exercise does not preserve prime divisors.

4.1.4. Prove that every reduced power of a structure $\mathcal{N}$ modulo a principal filter is isomorphic to a (direct) power of $\mathcal{N}$.

Sec 4.2.

(4.2.2) Prove the statement left to the reader in the proof of Theorem 4.2.1.

(4.2.3) Let U be an ultrafilter containing the Fréchet filter on $\mathbb{N}$, and let $\mathcal{N}$ be the standard model of Peano arithmetic. Prove that $\mathcal{N}^{\mathbb{N}}/U$ contains elements with infinitely many prime divisors (cf. Exercise 4.1.3).

4.2.4. A **primitive positive formula** (or *pp-formula*) is a formula of the form $\exists \mathbf{x}\, \varphi$ where φ is a finite conjunction of atomic formulas. Prove that if we restrict to pp-formulas, then Los' Theorem holds in arbitrary reduced products.

(4.2.5) Show that the ultrapowers of a structure $\mathcal{N}$ modulo a principal ultralilter are isomorphic to $\mathcal{N}$. [cf. 4.1.4 and 4.2.1.]

Sec 4.3.

4.3.1. Let I be a nonempty set. A subset F of (I) is said to have the **finite intersection property** if $F \neq \emptyset$ and no intersection of finitely many members of F is empty. Show that a nonempty set of subsets of a nonempty set I is contained in a filter on I if and only if that subset has the finite intersection property.

(4.3.2) Prove that if a set Σ of sentences axiomatizes a finitely axiomatizable class of structures, then this class is axiomatized aleady by a finite subset of Σ.

4.3.3. Show that a class $\mathbf{K}$ of structures is finitely axiomatizable if and only if $\mathbf{K}$ and its complement (within the class of all structures) are both axiomatizable.