2. Existence of exact upper bounds

Last Change: Sept 4, 2009

We introduce several notions leading up to an existence theorem for exact upper bounds: projections, strongly increasing sequences, a partition property, and the bounding projection property.

We start with the important notion of **projections**. By a *projection framework* we mean a triple (A, I, S) consisting of a nonempty set A, an ideal I on A, and a sequence $\langle S_a : a \in A\rangle$ of nonempty sets of ordinals. Suppose that we are given such a framework. We define $\sup(S)$ in the natural way: it is a function with domain A, and $(\sup(S))(a) = \sup(S_a)$ for every $a \in A$. Thus $\sup(S) \in {}^A\mathrm{Ord}$. Now suppose also that we have a function $f \in {}^A\mathrm{Ord}$ such that $f <_I \sup(S)$. Then we define the *projection* of f onto $\prod_{a\in A} S_a$, denoted by $f^+ = \mathrm{proj}(f, S)$ by setting, for any $a \in A$,

$$f^+(a) = \begin{cases} \min(S_a\backslash f(a)) & \text{if } f(a) < \sup(S_a), \\ f(a) & \text{otherwise.} \end{cases}$$

Note that $f \le f^+$. Actually $f^+ \notin \prod_{a\in A} S_a$ in general.

[This differs from Abraham, Magidor in some small details. We assume that each S_a is nonempty, while they don't. We define $f^+(a) = f(a)$ if $f(a) \ge \sup(S_a)$, while they define it to be 0 then.]

Proposition 2.1. *Let a projection framework be given, with the notation above.*

(i) If $f \in {}^A\mathrm{Ord}$ and $f <_I \sup(S)$, then there is a $g \in \prod_{a\in A} S_a$ such that $f^+ =_I g$, $f \le_I g$, and if $h \in \prod_{a\in A} S_a$ and $f \le_I h$, then $g \le_I h$.

(ii) If $f_1, f_2 \in {}^A\mathrm{Ord}$, $f_1 <_I \sup(S)$, $f_2 <_I \sup(S)$, and $f_1 \le f_2$, then $f_1^+ \le f_2^+$.

(iii) If $f_1, f_2 \in {}^A\mathrm{Ord}$, $f_1 <_I \sup(S)$, $f_2 <_I \sup(S)$, and $f_1 \le_I f_2$, then $f_1^+ \le_I f_2^+$.

Proof. For (i), define

$$g(a) = \begin{cases} f^+(a) & \text{if } f(a) < \sup(S_a), \\ \min(S_a) & \text{otherwise.} \end{cases}$$

Then $\{a \in A : f^+(a) \ne g(a)\} \subseteq \{a \in A : f(a) \ge \sup(S_a)\} \in I$. So $f^+ =_I g$. Since $f \le f^+$, it follows that $f \le_I g$. Now suppose that $h \in \prod_{a\in A} S_a$ and $f \le_I h$. If $f^+(a) = g(a)$ and $f(a) \le h(a)$, then either $f(a) < \sup(S_a)$ and $g(a) = f^+(a) = \min(S_a\backslash f(a)) \le h(a)$, or $\sup(S_a) \le f(a)$ and $g(a) = \min(S_a) \le f(a) \le h(a)$; in any case, $g(a) \le h(a)$. Hence $g \le_I h$. So (i) holds.

(ii) and (iii) are clear. □

Another important notion in discussing exact upper bounds is as follows. Let I be an ideal over A, L a set of ordinals, and $f = \langle f_\xi : \xi \in L\rangle$ a sequence of members of ${}^A\mathrm{Ord}$. Then we say that f is *strongly increasing under* I iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash(Z_\xi \cup Z_\eta)[f_\xi(a) < f_\eta(a)]].$$

Under the same assumptions we say that f is *very strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash Z_\eta[f_\xi(a) < f_\eta(a)].$$

Proposition 2.2. *Under the above assumptions, f is very strongly increasing iff for every $\xi \in L$ we have*

$$(*) \qquad \sup\{f_\alpha + 1 : \alpha \in L \cap \xi\} \leq_I f_\xi.$$

Proof. $\Rightarrow$: suppose that f is very strongly increasing, with sets Z_ξ as indicated. Let $\xi \in L$. Suppose that $a \in A\backslash Z_\xi$. Then for any $\alpha \in L \cap \xi$ we have $f_\alpha(a) < f_\xi(a)$, and so $\sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} \leq f_\xi(a)$; it follows that $(*)$ holds.

$\Leftarrow$: suppose that $(*)$ holds for each $\xi \in L$. For each $\xi \in L$ let

$$Z_\xi = \{a \in A : \sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} > f_\xi(a)\};$$

it follows that $Z_\xi \in I$. Now suppose that $\alpha \in L$ and $\alpha < \xi$. Suppose that $a \in A\backslash Z_\xi$. Then $f_\alpha(a) < f_\alpha(a) + 1 \leq \sup\{f_\beta(a) + 1 : \beta \in L \cap \xi\} \leq f_\xi(a)$, as desired. □

Lemma 2.3. (The sandwich argument) *Suppose that $h = \langle h_\xi : \xi \in L\rangle$ is strongly increasing, L has no largest element, and ξ' is the successor in L of ξ for every $\xi \in L$. Also suppose that $f_\xi \in {}^A Ord$ is such that*

$$h_\xi <_I f_\xi \leq_I h_{\xi'} \text{ for every } \xi \in L.$$

Then $\langle f_\xi : \xi \in L\rangle$ is also strongly increasing.

Proof. Let $\langle Z_\xi : \xi \in L\rangle$ testify that h is strongly increasing. For every $\xi \in L$ let

$$W_\xi = \{a \in A : h_\xi(a) \geq f_\xi(a) \text{ or } f_\xi(a) > h_{\xi'}(a)\}.$$

Thus by hypothesis we have $W_\xi \in I$. Let $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi'}$ for every $\xi \in L$. Then if $\xi_1 < \xi_2$, both in L, and if $a \in A\backslash(Z^{\xi_1} \cup Z^{\xi_2})$, then

$$f_{\xi_1}(a) \leq h_{\xi_1'}(a) \leq h_{\xi_2}(a) < f_{\xi_2}(a).$$ □

Proposition 2.4. *Let I be a proper ideal over A, let $\lambda > |A|$ be a regular cardinal, and let $f = \langle f_\xi : \xi < \lambda\rangle$ be a $<_I$ increasing sequence of functions in ${}^A Ord$.*

Then f contains a strongly increasing subsequence of length λ iff f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ *for all $a \in A$.*

Proof. $\Rightarrow$: Let $\langle \eta(\xi) : \xi < \lambda\rangle$ be a strictly increasing sequence of ordinals less than λ, thus with supremum λ since λ is regular, and assume that $\langle f_{\eta(\xi)} : \xi < \lambda\rangle$ is strongly increasing. Hence for each $\xi < \lambda$ let $Z_\xi \in I$ be chosen correspondingly. We define for each $a \in A$

$$h(a) = \sup_{a \notin Z_\xi} f_{\eta(\xi)}(a)$$

for each $\xi < \lambda$. To see that h is an exact upper bound for f, we are going to apply 1.19. If $f_{\eta(\xi)}(a) > h(a)$, then $a \in Z_\xi \in I$. Hence $f_{\eta(\xi)} \leq_I h$ for each $\xi < \lambda$. Then for any $\xi < \lambda$ we have $f_\xi \leq_I f_{\eta(\xi)} \leq_I h$, so h bounds every f_ξ. Now suppose that $d <_I h$. Let $M = \{a \in A : d(a) \geq h(a)\}$; so $M \in I$. For each $a \in A \backslash M$ we have $d(a) < h(a)$, and so there is a $\xi_a < \lambda$ such that $d(a) < f_{\eta(\xi_a)}(a)$ and $a \notin Z_{\xi_a}$. Since $|A| < \lambda$ and λ is regular, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in A \backslash M} \xi_a$ is less than λ. We claim that $d <_I f_{\eta(\rho)}$ (as desired). In fact, suppose that $a \in A \backslash (M \cup Z_\rho)$. Then $a \in A \backslash (Z_{\xi_a} \cup Z_\rho)$, and so $d(a) < f_{\eta(\xi_a)}(a) \leq f_{\eta(\rho)}(a)$. Thus $d <_I f_{\eta(\rho)}$.

It remains to show that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$. Actually this does not hold in general for h as we have defined it. So we define a new h' in terms of h. First we need:

(1) There is a $W \in I$ such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A \backslash W$.

In fact, let
$$W = \{a \in A : \exists \xi_a < \lambda \forall \xi' \in [\xi_a, \lambda)[a \in Z_{\xi'}]\}.$$

Since $|A| < \lambda$, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in W} \xi_a$ is less than λ. Clearly $W \subseteq Z_\rho$, so $W \in I$. For $a \in A \backslash W$ we have $\forall \xi < \lambda \exists \xi' \in [\xi, \lambda)[a \notin Z_{\xi'}]$. This gives an increasing sequence $\langle \sigma_\nu : \nu < \lambda \rangle$ of ordinals less than λ such that $a \notin Z_{\sigma_\nu}$ for all $\nu < \lambda$. By the strong increasing property it follows that $f_{\eta(\sigma_0)}(a) < f_{\eta(\sigma_1)}(a) < \cdots$. Now $|\{f_{\eta(\xi)} : a \notin Z_\xi\}| \leq \lambda$, so $\mathrm{cf}(h(a)) \leq \lambda$. Hence $h(a)$ has cofinality λ. This proves (1).

Now we take W as in (1). Since I is a proper ideal, choose $a_0 \in A \backslash W$, and define

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A \backslash W, \\ h(a_0) & \text{if } a \in W. \end{cases}$$

Then $h =_I h'$, and it follows that h' satisfies the properties needed.

$\Leftarrow$: Assume that f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$. Now we define by recursion two sequences $\langle g_\xi : \xi < \lambda \rangle$ and $\langle \eta(\xi) : \xi < \lambda \rangle$. Suppose defined for all $\nu < \xi$, in such a way that $g_\nu < h$ and $\eta(\nu) < \lambda$ for each $\nu < \xi$. Then by the cofinality assumption, $\sup_{\nu < \xi} g_\nu < h$. Hence by the exact upper bound condition, there is a $\rho(\xi) < \lambda$ such that $\sup_{\nu < \xi} g_\nu <_I f_{\rho(\xi)}$. We may assume that also $\sup_{\nu < \xi} \eta(\nu) < \rho(\xi)$. Let
$$W = \{a \in A : (\sup_{\nu < \xi} g_\nu)(a) \geq f_{\rho(\xi)}(a)\};$$
$$V = \{a \in A : f_{\rho(\xi)}(a) \geq h(a)\}.$$

Now we define g_ξ:

$$g_\xi(a) = \begin{cases} f_{\rho(\xi)}(a) & \text{if } a \in A \backslash (W \cup V), \\ (\sup_{\nu < \xi} g_\nu)(a) + 1 & \text{if } a \in W \cup V. \end{cases}$$

Clearly then we have $g_\nu < g_\xi$ for all $\nu < \xi$. Now choose $\eta(\xi) < \lambda$ and greater than $\rho(\xi)$ and each $\eta(\nu)$ for $\nu < \xi$. This finishes the construction. Clearly $g_\xi =_I f_{\rho(\xi)} <_I f_{\eta(\xi)} <_I f_{\rho(\xi+1)} \leq_I g_{\xi+1}$ for all $\xi < \lambda$. Hence by Lemma 2.3 we get that $\langle f_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing. $\square$

Now we define a partition property. Suppose that I is an ideal over a set A, λ is an uncountable regular cardinal $> |A|$, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of members of ${}^A\mathrm{Ord}$, and κ is a regular cardinal such that $|A| < \kappa \leq \lambda$. The following property of these things is denoted by $(*)_\kappa$:

$(*)_\kappa$ For all unbounded $X \subseteq \lambda$ there is an $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing.

Proposition 2.5. *Assume the above notation, with $\kappa < \lambda$. Then $(*)_\kappa$ holds iff the set*

$$\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa \textit{ and } \langle f_\xi : \xi \in X_0\rangle \textit{ is strongly increasing for some unbounded } X_0 \subseteq \delta\}$$

is stationary in λ.

Proof. Let S be the indicated set of ordinals δ.

$\Rightarrow$: Assume $(*)_\kappa$ and suppose that $C \subseteq \lambda$ is a club. Choose $C_0 \subseteq C$ of order type κ such that $\langle f_\xi : \xi \in C_0\rangle$ is strongly increasing. Let $\delta = \sup(C_0)$. Clearly $\delta \in C \cap S$.

$\Leftarrow$: Assume that S is stationary in λ, and suppose that $X \subseteq \lambda$ is unbounded. Define

$$C = \{\alpha \in \lambda : \alpha \text{ is a limit ordinal and } X \cap \alpha \text{ is unbounded in } \alpha\}.$$

We check that C is club in λ. For closure, suppose that $\alpha < \lambda$ is a limit ordinal and $C \cap \alpha$ is unbounded in α; we want to show that $\alpha \in C$. So, we need to show that $X \cap \alpha$ is unbounded in α. To this end, take any $\beta < \alpha$; we want to find $\gamma \in X \cap \alpha$ such that $\beta < \gamma$. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\beta < \delta$. By the definition of C we have that $X \cap \delta$ is unbounded in δ. So we can choose $\gamma \in X \cap \delta$ such that $\beta < \gamma$. Since $\gamma < \delta < \alpha$, γ is as desired. So, indeed, C is closed.

To show that C is unbounded in λ, take any $\beta < \lambda$; we want to find an $\alpha \in C$ such that $\beta < \alpha$. Since X is unbounded in λ, we can choose a sequence $\gamma_0 < \gamma_1 < \cdots$ of elements of X with $\beta < \gamma_0$. Now λ is uncountable and regular, so $\sup_{n\in\omega} \gamma_n < \lambda$, and it is the member of C we need.

Now choose $\delta \in C \cap S$. This gives us an unbounded set X_0 in δ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing. Now also $X \cap \delta$ is unbounded, since $\delta \in C$. Hence we can define by induction two increasing sequences $\langle \eta(\xi) : \xi < \kappa\rangle$ and $\langle \nu(\xi) : \xi < \kappa\rangle$ such that each $\eta(\xi)$ is in X_0, each $\nu(\xi)$ is in X, and $\eta(\xi) < \nu(\xi) \leq \eta(\xi + 1)$ for all $\xi < \kappa$. It follows by 2.3 that $X_1 \overset{\text{def}}{=} \{\nu(\xi) : \xi < \kappa\}$ is a subset of X as desired in $(*)_\kappa$. □

Finally, we introduce the bounding projection property.

Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in Ord^A, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$.

We say that f has the *bounding projection property for* κ iff whenever $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S)$, then for some $\xi < \lambda$, the function $\mathrm{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f. (Note that (A, I, S) is a projection framework.)

Lemma 2.6. *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \le \lambda$. Then f has the bounding projection property for κ.*

Proof. Assume the hypothesis of the lemma and of the bounding projection property for κ. For every $\xi < \lambda$ let

$$f^+_\xi = \mathrm{proj}(f_\xi, S).$$

Suppose that the conclusion of the bounding projection property fails. Then for every $\xi < \lambda$, the function f^+_ξ is not a bound for f, and so there is a $\xi' < \lambda$ such that $f_{\xi'} \not\le_I f^+_\xi$. Since $f_\xi \le f^+_\xi$, we must have $\xi < \xi'$. Clearly for any $\xi'' \ge \xi'$ we have $f_{\xi''} \not\le_I f^+_\xi$. Thus for every $\xi'' \ge \xi'$ we have $< (f^+_\xi, f_{\xi''}) \notin I$. Now we define a sequence $\langle \xi(\mu) : \mu < \lambda\rangle$ of elements of λ by recursion. Let $\xi(0) = 0$. Suppose that $\xi(\mu)$ has been defined. Choose $\xi(\mu+1) > \xi(\mu)$ so that $< (f^+_{\xi(\mu)}, f_{\xi''}) \notin I$ for every $\xi'' \ge \xi(\mu+1)$. If ν is limit and $\xi(\mu)$ has been defined for all $\mu < \nu$, let $\xi(\nu) = \sup_{\mu<\nu} \xi(\mu)$. Then let X be the range of this sequence. Thus

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi', \text{ then } < (f^+_\xi, f_{\xi'}) \notin I.$$

Since $(*)_\kappa$ holds, there is a subset $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing. Let $\langle Z_\xi : \xi \in X_0\rangle$ be as in the definition of strongly increasing.

For every $\xi \in X_0$, let ξ' be the successor of ξ in X_0. Note that

$$< (f^+_\xi, f_{\xi'}) \backslash (Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \ge \sup(S(a))\}) \notin I,$$

and hence it is nonempty. So, choose

$$a_\xi \in < (f^+_\xi, f_{\xi'}) \backslash (Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \ge \sup(S(a))\}).$$

Note that this implies that $f^+_\xi(a_\xi) \in S(a_\xi)$. Since $\kappa > |A|$, we can find a single $a \in A$ such that $a = a_\xi$ for all ξ in a subset X_1 of X_0 of size κ. Now for $\xi_1 < \xi_2$ with both in X_1, we have

$$f^+_{\xi_1}(a) < f_{\xi'_1}(a) \le f_{\xi_2}(a) \le f^+_{\xi_2}(a).$$

[The first inequality is a consequence of $a = a_{\xi_1} \in < (f^+_{\xi_1}, f_{\xi'_1})$, the second follows from $\xi'_1 \le \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A \backslash (Z_{\xi'_1} \cup Z_{\xi_2}),$$

and the third is true by the definition of $f^+_{\xi_2}$.]

Thus $\langle f^+_\xi(a) : \xi \in X_1\rangle$ is a strictly increasing sequence of members of $S(a)$. This contradicts our assumption that $|S(a)| < \kappa$. $\square$

Lemma 2.7. *Suppose that I is a proper ideal over A, $\lambda \ge |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathrm{Ord}$ satisfying the bounding projection property for $|A|^+$. Suppose that h is a least upper bound for f. Then h is an exact upper bound.*

Proof. Assume the hypotheses, and suppose that $g <_I h$; we want to find $\xi < \lambda$ such that $g <_I f_\xi$. By increasing h on a subset of A in the ideal, we may assume that $g < h$ everywhere. (See Proposition 1.24.) Define $S_a = \{g(a), h(a)\}$ for every $a \in A$. By the bounding projection property we get a $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=} \text{proj}(f_\xi, \langle S_a : a \in A\rangle)$ is an upper bound for f. We shall prove that $g <_I f_\xi$, as required.

Since h is a least upper bound, it follows that $h \leq_I f^+_\xi$. Thus $M \stackrel{\text{def}}{=} \{a \in A : h(a) > f^+_\xi(a)\} \in I$. Also, the set $N \stackrel{\text{def}}{=} \{a \in A : f_\xi(a) \geq \sup(S_a)\}$ is in I. Suppose that $a \in A \backslash (M \cup N)$. Then $g(a) < h(a) \leq f^+_\xi(a) = \min(S_a \backslash f_\xi(a))$, and since $g(a) \in S_a$, this implies that $g(a) < f_\xi(a)$. So $g <_I f_\xi$, as desired. $\square$

Theorem 2.8. (Existence of exact upper bounds) *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathrm{Ord}$ that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the hypotheses. By 2.7 it suffices to show that f has a least upper bound, and to do this we will apply 1.18. Suppose that f does not have a least upper bound. Since it obviously has an upper bound, this means, by 1.18:

(1) For every upper bound $h \in {}^A\mathrm{Ord}$ for f there is another upper bound h' for f such that $h' \leq_I h$ and $\{a \in A : h'(a) < h(a)\} \notin I$.

In fact, 1.18 says that there is another upper bound h' for f such that $h' \leq_I h$ and it is not true that $h =_I h'$. Hence $\{a \in A : h(a) < h'(a)\} \in I$ and $\{a \in A : h(a) \neq h'(a)\} \notin I$. So

$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} \notin I \quad \text{and}$$
$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} = \{a \in A : h'(a) < h(a)\},$$

so (1) follows.

Now we shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) : a \in A\rangle$ of sets of ordinals satisfying the following conditions:

(2) $|S^\alpha(a)| \leq |A|$ for each $a \in A$;

(3) $f_\xi(a) < \sup S^\alpha(a)$ for all $\xi \in \lambda$ and $a \in A$;

(4) If $\alpha < \beta$ and $a \in A$, then $S^\alpha(a) \subseteq S^\beta(a)$, and if δ is a limit ordinal, then $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

We also define sequences $\langle h_\alpha : \alpha < |A|^+\rangle$ and $\langle h'_\alpha : \alpha < |A|^+\rangle$ of functions and $\langle \xi(\alpha) : \alpha < |A|^+\rangle$ of ordinals. In fact, we will define h_α, h'_α, and $\xi(\alpha)$ after defining $S^{\alpha+1}$.

The definition of S^α for α limit is fixed by (4), and the conditions (2)–(4) continue to hold. To define S^0, pick any function k that bounds f (everywhere) and define $S^0(a) = \{k(a)\}$ for all $a \in A$; so (2)–(4) hold.

Suppose that $S^\alpha = \langle S^\alpha(a) : a \in A\rangle$ has been defined, satisfying (2)–(4); we define $S^{\alpha+1}$. By the bounding projection property for $|A|^+$, there is a $\xi(\alpha) < \lambda$ such that $h_\alpha \stackrel{\text{def}}{=} \text{proj}(f_{\xi(\alpha)}, S^\alpha)$ is an upper bound for f under $<_I$. Then

(5) if $\xi(\alpha) \leq \xi' < \lambda$, then $h_\alpha =_I \operatorname{proj}(f_{\xi'}, S^\alpha)$.

In fact, recall that $h_\alpha(a) = \min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a))$ for every $a \in A$. Now suppose that $\xi(\alpha) < \xi' < \lambda$. Let $M = \{a \in A : f_{\xi(\alpha)}(a) \geq f_{\xi'}(a)\}$. So $M \in I$. For any $a \in A \backslash M$ we have $f_{\xi(\alpha)}(a) < f_{\xi'}(a)$, and hence

$$\min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a)) \leq \min(S^\alpha(a) \backslash f_{\xi'}(a));$$

it follows that $h_\alpha \leq_I \operatorname{proj}(f_{\xi'}, S^\alpha)$. For the other direction, recall that h_α is an upper bound for f under $<_I$. So $f_{\xi'} \leq_I h_\alpha$. If a is any element of A such that $f_{\xi'}(a) \leq h_\alpha(a)$ then, since $h_\alpha(a) \in S^\alpha(a)$, we get $\min(S^\alpha(a) \backslash f_{\xi'}(a)) \leq h_\alpha(a)$. Thus $\operatorname{proj}(f_{\xi'}, S^\alpha) \leq_I h_\alpha$.

This checks (5).

Now we apply (1) to get an upper bound h'_α for f such that $h'_\alpha \leq_I h_\alpha$ and $< (h'_\alpha, h_\alpha) \notin I$. We now define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{h'_\alpha(a)\}$ for any $a \in A$.

(6) If $\xi(\alpha) \leq \xi < \lambda$, then $\operatorname{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$.

For, we have $f_\xi \leq_I h'_\alpha$ and, by (5), $h_\alpha =_I \operatorname{proj}(f_\xi, S^\alpha)$. If $a \in A$ is such that $f_\xi(a) \leq h'_\alpha(a)$, $h'_\alpha(a) \leq h_\alpha(a)$, and $h_\alpha(a) = \operatorname{proj}(f_\xi, S^\alpha)(a)$, then $\min(S^\alpha(a) \backslash f_\xi(a)) = h_\alpha(a) \geq h'_\alpha(a) \geq f_\xi(a)$, and hence

$$\operatorname{proj}(f_\xi, S^{\alpha+1})(a) = \min(S^{\alpha+1}(a) \backslash f_\xi(a)) = h'_\alpha(a).$$

It follows that $\operatorname{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$, as desired in (6).

Now since $|A|^+ < \lambda$, let $\xi < \lambda$ be greater than each $\xi(\alpha)$ for $\alpha < |A|^+$. Define $H_\alpha = \operatorname{proj}(f_\xi, S^\alpha)$ for each $\alpha < |A|^+$. Since $\xi > \xi(\alpha)$, we have $H_\alpha =_I h_\alpha$ by (5). Note that $H_{\alpha+1} = \operatorname{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$; so $< (H_{\alpha+1}, H_\alpha) \notin I$. Now clearly by the construction we have $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$ when $\alpha_1 < \alpha_2 < |A|^+$. Hence we get

(7) if $\alpha_1 < \alpha_2 < |A|^+$, then $H_{\alpha_2} \leq H_{\alpha_1}$, and $< (H_{\alpha_2}, H_{\alpha_1}) \notin I$.

Now for every $\alpha < |A|^+$ pick $a_\alpha \in A$ such that $H_{\alpha+1}(a_\alpha) < H_\alpha(a_\alpha)$. We have $a_\alpha = a_\beta$ for all α, β in some subset of $|A|^+$ of size $|A|^+$, and this gives an infinite decreasing sequence of ordinals, contradiction. $\square$

Lemma 2.9. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A Ord$, $|A|^+ \leq \kappa \leq \lambda$, f satisfies the bounding projection property for κ, and g is an exact upper bound for f. Then*

$$\{a \in A : g(a) \text{ is non-limit, or } \operatorname{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let $P = \{a \in A : g(a)$ is non-limit, or $\operatorname{cf}(g(a)) < \kappa\}$. If $a \in P$ and $g(a)$ is a limit ordinal, choose $S(a) \subseteq g(a)$ cofinal in $g(a)$ and of order type $< \kappa$. If $g(a) = 0$ let $S(a) = \{0\}$, and if $g(a) = \beta + 1$ for some β let $S(a) = \{\beta\}$. Finally, if $g(a)$ is limit but is not in P, let $S(a) = \{g(a)\}$.

Now for any $\xi < \lambda$ let

$$\begin{aligned} N_\xi &= \{a \in A : f_\xi(a) \geq f_{\xi+1}(a)\} \quad \text{and} \\ Q_\xi &= \{a \in A : f_{\xi+1}(a) \geq g(a)\}. \end{aligned}$$

Then clearly

$(*)$ If $a \in A \backslash (N_\xi \cup Q_\xi)$, then $f_\xi(a) < \sup(S(a))$.

It follows that $\{a \in A : f_\xi(a) \geq \sup(S)(a)\} \subseteq N_\xi \cup Q_\xi \in I$. Hence the hypothesis of the bounding projection property holds. Applying it, we get $\xi < \lambda$ such that $f_\xi^+ \stackrel{\text{def}}{=} \text{proj}(f_\xi, \langle S(a) : a \in A\rangle) <_I$-bounds f. Since g is a least upper bound for f, we get $g \leq_I f_\xi^+$, and hence $M \stackrel{\text{def}}{=} \{a \in A : f_\xi^+(a) < g(a)\} \in I$. By $(*)$, for any $a \in P \backslash (N_\xi \cup Q_\xi)$ we have $f_\xi^+(a) = \min(S(a) \backslash f_\xi(a)) < g(a)$. This shows that $P \backslash (N_\xi \cup Q_\xi) \subseteq M$, hence $P \subseteq N_\xi \cup Q_\xi \cup M \in I$, so $P \in I$, as desired. $\square$

Theorem 2.10. *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathrm{Ord}$, and $|A|^+ \leq \kappa \leq \lambda$, with κ regular. Then the following are equivalent:*

(i) $(*)_\kappa$ holds for f.

(ii) f satisfies the bounding projection property for κ.

(iii) f has an exact upper bound g such that

$$\{a \in A : g(a) \textit{ is non-limit, or } \mathrm{cf}(g(a)) < \kappa\} \in I.$$

Proof. (i)⇒(ii): Lemma 2.6.

(ii)⇒(iii): Since $(*)_\kappa$ clearly implies $(*)_{|A|^+}$, this implication is true by Theorem 2.8 and Lemma 2.9.

(iii)⇒(i): Assume (iii). By modifying g on a set in the ideal we may assume that $g(a)$ is a limit ordinal and $\mathrm{cf}(g(a)) \geq \kappa$ for all $a \in A$. For each $a \in A$ choose a club $S(a) \subseteq g(a)$ of order type $\mathrm{cf}(g(a))$. Thus the order type of $S(a)$ is $\geq \kappa$. We prove that $(*)_\kappa$ holds. So, assume that $X \subseteq \lambda$ is unbounded; we want to find $X_0 \subseteq X$ of order type κ over which f is strongly increasing. To do this, we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \prod S$ and an index $\xi(\alpha) \in X$ such that

(1) $h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}$.

(2) The sequence $\langle h_\alpha : \alpha < \kappa \rangle$ is $<$-increasing (increasing everywhere; and hence it certainly is strongly increasing).

After we have done this, the sandwich argument (Lemma 2.3) shows that $\langle f_{\xi(\alpha)} : \alpha < \kappa \rangle$ is strongly increasing and of order type κ, giving the desired result.

The functions h_α are defined as follows.

(3) $h_0 \in \prod S$ is arbitrary.

(4) For a limit ordinal $\delta < \kappa$ let $h_\delta = \sup_{\alpha < \delta} h_\alpha$.

(5) Having defined h_α, we define $h_{\alpha+1}$ as follows. Since g is an exact upper bound and $h_\alpha < g$, choose $\xi(\alpha)$ such that $h_\alpha <_I f_{\xi(\alpha)}$. Also, since $f_\xi <_I g$ for all $\xi < \lambda$, the projections $f_\xi^+ = \text{proj}(f, S)$ are defined. We define

$$h_{\alpha+1}(a) = \begin{cases} \max(h_\alpha(a), f_{\xi(\alpha)}^+(a)) + 1 & \text{if } f_{\xi(\alpha)}(a) < g(a), \\ h_\alpha(a) + 1 & \text{if } f_{\xi(\alpha)}(a) \geq g(a). \end{cases}$$

Thus we have

$$h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}, \text{ for every } \alpha. \tag{I.6}$$

So conditions (1) and (2) hold. $\square$

To proceed further we need the following *club guessing theorem.*

Theorem 2.11. (Club guessing) *Suppose that κ is a regular cardinal, λ is a cardinal such that $\mathrm{cf}(\lambda) \geq \kappa^{++}$, and $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. Then there is a sequence $\langle C_\delta : \delta \in S^\lambda_\kappa \rangle$ such that:*

(i) For every $\delta \in S^\lambda_\kappa$ the set $C_\delta \subseteq \delta$ is club, of order type κ.

(ii) For every club $D \subseteq \lambda$ there is a $\delta \in D \cap S^\lambda_\kappa$ such that $C_\delta \subseteq D$.

The sequence $\langle C_\delta : \delta \in S^\lambda_\kappa \rangle$ is called a *club guessing sequence* for S^λ_κ.

Proof. First we take the case of uncountable κ. Fix a sequence $C' = \langle C'_\delta : \delta \in S^\lambda_\kappa \rangle$ such that $C'_\delta \subseteq \delta$ is club in δ of order type κ, for every $\delta \in S^\lambda_\kappa$. For any club E of λ, let

$$C' \restriction E = \langle C'_\delta \cap E : \delta \in S^\lambda_\kappa \cap E' \rangle,$$

where $E' = \{\delta \in E : E \cap \delta \text{ is unbounded in } \delta\}$. Clearly E' is also club in λ. Also note that $C'_\delta \cap E$ is club in δ for each $\delta \in S^\lambda_\kappa \cap E'$. We claim:

(1) There is a club E of λ such that for every club D of λ there is a $\delta \in D \cap E' \cap S^\lambda_\kappa$ such that $C'_\delta \cap E \subseteq D$.

Note that if we prove (1), then the theorem follows by defining $C_\delta = C'_\delta \cap E$ for all $\delta \in E' \cap S^\lambda_\kappa$, and $C_\delta = C'_\delta$ for $\delta \in S^\kappa_\lambda \backslash E'$.

Assume that (1) is false. Hence for every club $E \subseteq \lambda$ there is a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap E' \cap S^\lambda_\kappa$ we have

$$C'_\delta \cap E \nsubseteq D_E.$$

We now define a sequence $\langle E^\alpha : \alpha < \kappa^+ \rangle$ of clubs of λ decreasing under inclusion, by induction on α:

(2) $E^0 = \lambda$.

(3) If $\gamma < \kappa^+$ is a limit ordinal and E^α has been defined for all $\alpha < \gamma$, we set $E^\gamma = \bigcap_{\alpha<\gamma} E^\alpha$. Since $\gamma < \kappa^+ < \mathrm{cf}(\lambda)$, E^γ is club in λ.

(4) If E^α has been defined, let $E^{\alpha+1}$ be the set of all limit points of $E^\alpha \cap D_{E^\alpha}$, i.e., the set of all $\varepsilon < \lambda$ such that $E^\alpha \cap D_{E^\alpha} \cap \varepsilon$ is unbounded in ε.

This defines the sequence. Let $E = \bigcap_{\alpha<\kappa^+} E^\alpha$. Then E is club in λ. Take any $\delta \in S^\lambda_\kappa \cap E$. Since $|C'_\delta| = \kappa$ and the sequence $\langle E^\alpha : \alpha < \kappa^+ \rangle$ is decreasing, there is an $\alpha < \kappa^+$ such that $C'_\delta \cap E = C_\delta \cap E^\alpha$. So $C'_\delta \cap E^\alpha = C'_\delta \cap E^{\alpha+1}$. Hence $C'_\delta \cap E^\alpha \subseteq D_{E_\alpha}$, contradiction.

Thus the case κ uncountable has been finished.

Now we take the case $\kappa = \omega$. For $S = S^\lambda_{\aleph_0}$ fix $C = \langle C_\delta : \delta \in S \rangle$ so that C_δ is club in δ with order type ω. We denote the n-th element of C_δ by $C_\delta(n)$. For any club $E \subseteq \lambda$ and any $\delta \in S \cap E'$ we define

$$C^E_\delta = \{\max(E \cap (C_\delta(n) + 1)) : n \in \omega\}.$$

This set is cofinal in δ. In fact, given $\alpha < \delta$, there is a $\beta \in E \cap \delta$ such that $\alpha < \beta$ since $\delta \in E'$, and there is an $n \in \omega$ such that $\beta < C_\delta(n)$. Then $\alpha < \max(E \cap (C_\delta(n) + 1))$, as desired. There may be repetitions in the description of C_δ^E, but $\max(E \cap (C_\delta(n) + 1)) \leq \max(E \cap (C_\delta(m) + 1))$ if $n < m$, so C_δ^E has order type ω. We claim

(5) There is a closed unbounded $E \subseteq \lambda$ such that for every club $D \subseteq \lambda$ there is a $\delta \in D \cap S \cap E'$ such that $C_\delta^E \subseteq D$. [This proves the club guessing property.]

Suppose that (5) fails. Thus for every closed unbounded $E \subseteq \lambda$ there exist a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap S \cap E'$ we have $C_\delta^E \not\subseteq D$. Then we construct a descending sequence E^α of clubs in λ as in the case $\kappa > \omega$, for $\alpha < \omega_1$. Thus for each $\alpha < \omega_1$ and each $\delta \in D_{E^\alpha} \cap S \cap (E^\alpha)'$ we have $C_\delta^{E^\alpha} \not\subseteq D_{E^\alpha}$. Let $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Take any $\delta \in S \cap E$. For $n \in \omega$ and $\alpha < \beta$ we have

$$E^\alpha \cap (C_\delta(n) + 1) \supseteq E^\beta \cap (C_\delta(n) + 1),$$

and so $\max(E^\alpha \cap (C_\delta(n)+1)) \geq \max(E^\beta \cap (C_\delta(n)+1))$; it follows that there is an $\alpha_n < \omega_1$ such that $\max(E^\beta \cap (C_\delta(n) + 1)) = \max(E^{\alpha_n} \cap (C_\delta(n) + 1))$ for all $\beta > \alpha_n$. Choose γ greater than all α_n. Thus

(6) For all $\varepsilon > \gamma$ and all $n \in \omega$ we have $\max(E^\varepsilon \cap (C_\delta(n) + 1)) = \max(E^\gamma \cap (C_\delta(n) + 1))$.

But there is a $\rho \in C_\delta^{E^\gamma} \backslash D_{E^\gamma}$; say that $\rho = \max(E^\gamma \cap (C_\delta(n) + 1))$. Then $\rho = \max(E^{\gamma+1} \cap (C_\delta(n) + 1)) \in E^{\gamma+1} = (E^\gamma \cap D_{E^\gamma})' \in D_{E^\gamma}$, contradiction. □

Lemma 2.12. *Suppose that:*
(i) I is an ideal over A.
(ii) κ and λ are regular cardinals such that $|A| < \kappa$ and $\kappa^{++} < \lambda$.
(iii) $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of length λ of functions in ${}^A\mathrm{Ord}$ that is $<_I$-increasing and satisfies the following condition:

For every $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa^{++}$ there is a club $E_\delta \subseteq \delta$ such that for some $\delta' \geq \delta$ with $\delta' < \lambda$,

$$(\star) \qquad \sup\{f_\alpha : \alpha \in E_\delta\} \leq_I f_{\delta'}.$$

Under these assumptions, $(*)_\kappa$ holds for f.

Proof. Assume the hypotheses. Let $S = S_\kappa^{\kappa^{++}}$; so S is stationary in κ^{++}. By 2.11, let $\langle C_\delta : \delta \in S \rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

Now let $U \subseteq \lambda$ be unbounded; we want to find $X_0 \subseteq U$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing. To do this we first define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++} \rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0) = 0$. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i+1)$. For each $\alpha \in S$ we define

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \quad \text{and}$$
$$\sigma_\alpha = \begin{cases} \text{least } \sigma \in (\xi(i), \lambda) \text{ such that } h_\alpha \leq_I f_\sigma & \text{if there is such a } \sigma, \\ \xi(i)+1 & \text{otherwise.} \end{cases}$$

Now we let $\xi(i+1)$ be the least member of U which is greater than $\sup\{\sigma_\alpha : \alpha \in S\}$. It follows that

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $h_\alpha <_I f_{\xi(i+1)}$.

Now the set $F \stackrel{\text{def}}{=} \{\xi(k) : k \in \kappa^{++}\}$ is closed, and has order type κ^{++}. Let $\delta = \sup(F)$. Then F is a club of δ, and $\mathrm{cf}(\delta) = \kappa^{++}$. Hence by the hypothesis (iii) of the lemma, there is a club $E_\delta \subseteq \delta$ and a $\delta' \in [\delta, \lambda)$ such that $(\star)$ in the lemma holds. Note that $F \cap E_\delta$ is club in δ.

Let $D = \xi^{-1}[F \cap E_\delta]$. Since ξ is strictly increasing and continuous, it follows that D is club in κ^{++}. Hence by (2) there is an $\alpha \in D \cap S$ such that $C_\alpha \subseteq D$. Hence

$$\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq F \cap E_\delta$$

is club in $\xi(\alpha)$ of order type κ. Then by $(\star)$ we have

$$\sup\{f_\rho : \rho \in \overline{C}_\alpha\} \leq_I f_{\delta'}.$$

Now

(5) For every $\rho < \rho'$ both in $\overline{C}_\alpha$, we have $\sup\{f_\zeta : \zeta \in \overline{C}_\alpha \cap (\rho+1)\} <_I f_{\rho'}$.

To prove this, note that there is an $i < \kappa^{++}$ such that $\rho = \xi(i)$. Now follow the definition of $\xi(i+1)$. There C_α was considered (among all other closed unbounded sets in the guessing sequence), and h_α was formed at that stage. Now

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \leq \sup\{f_\eta : \eta \in \xi[C_\alpha]\} = \sup\{f_\eta : \eta \in \overline{C}_\alpha\} \leq_I f_{\delta'},$$

so the first case in the definition of σ_α holds. Thus by (3), $h_\alpha <_I f_{\xi(i+1)}$. Clearly $\xi(i+1) \leq \rho'$, so (5) follows.

Now let $\langle \eta(\nu) : \nu < \kappa \rangle$ be the strictly increasing enumeration of $\overline{C}_\alpha$, and set

$$X_0 = \{\eta(\omega \cdot \rho + 2m) : \rho < \kappa, 0 < m \in \omega\}.$$

Suppose that $\zeta \in X_0$. Say $\zeta = \eta(\omega \cdot \rho + 2m)$ with $\rho < \kappa$ and $0 < m \in \omega$. If $\sigma \in X_0 \cap \zeta$, then $\sigma < \eta(\omega \cdot \rho + 2m - 1) < \zeta$, all in $\overline{C}_\alpha$, so

$$\begin{aligned} \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \zeta\} &\leq_I f_{\eta(\omega\cdot\rho+2m-1)} \\ &= \sup\{f_\sigma : \sigma \in \overline{C}_\alpha \cap (\eta(\omega \cdot \rho + 2m - 1) + 1)\} \\ &<_I f_\zeta \quad \text{by (5)} \end{aligned}$$

Hence by 2.2, $\langle f_\zeta : \zeta \in X\rangle$ is strongly increasing. $\square$

Lemma 2.13. *Suppose that I is a proper ideal over a set A of regular cardinals such that $|A| < \min(A)$. Assume that $\lambda > |A|$ is a regular cardinal such that $(\prod A, <_I)$ is λ-directed, and $\langle g_\xi : \xi < \lambda\rangle$ is a sequence of members of $\prod A$.*

Then there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of length λ in $\prod A$ such that:

(i) $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$.

(ii) $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in I$.

Proof. Let f_0 be any member of $\prod A$. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in $\prod A$ that $<$-extends f_ξ and g_ξ.

At limit stages δ, there are three cases. In the first case, $\mathrm{cf}(\delta) \leq |A|$. Fix some $E_\delta \subseteq \delta$ club of order type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i : i \in E_\delta\}.$$

For any $a \in A$ we have $\mathrm{cf}(\delta) \leq |A| < \min(A) \leq a$, and so $f_\delta(a) < a$. Thus $f_\delta \in \prod A$.

In the second case, $\mathrm{cf}(\delta) = \kappa^{++}$, where κ is regular, $|A| < \kappa$, and $\{a \in A : a \leq \kappa^{++}\} \in I$. Then we define f'_δ as in the first case. Then for any $a \in A$ with $a > \kappa^{++}$ we have $f'_\delta(a) < a$, and so $\{a \in A : a \leq f'_\delta(a)\} \in I$, and we can modify f'_δ on this set which is in I to obtain our desired f_δ.

In the third case, neither of the first two cases holds. Then we let f_δ be any $\leq_I$-upper bound of $\{f_\xi : \xi < \delta\}$; it exists by the λ-directedness assumption.

This completes the construction. Obviously (i) holds. For (ii), suppose that κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in I$. If $|A| < \kappa$, the desired conclusion follows by 2.12. In case $\kappa \leq |A|$, note that $\langle f_\xi : \xi < \kappa\rangle$ is $<$-increasing, and so is certainly strongly increasing. $\square$

Notation. *For any set X of cardinals, let*

$$X^{(+)} = \{\alpha^+ : \alpha \in X\}.$$

Theorem 2.14. (Representation of μ^+ as a true cofinality) *Suppose that μ is a singular cardinal with uncountable cofinality. Then there is a club C in μ such that*

$$\mu^+ = \mathrm{tcf}\left(\prod C^{(+)}, <_{J^{\mathrm{bd}}}\right),$$

where J^{bd} is the ideal of all bounded subsets of $C^{(+)}$.

Proof. Let C_0 be any closed unbounded set of limit cardinals less than μ such that $|C_0| = \mathrm{cf}(\mu)$ and all cardinals in C_0 are above $\mathrm{cf}(\mu)$. Then

(1) all members of C_0 which are limit points of C_0 are singular.

In fact, suppose on the contrary that $\kappa \in C_0$, κ is a limit point of C_0, and κ is regular. Thus $C_0 \cap \kappa$ is unbounded in κ, so $|C_0 \cap \kappa| = \kappa$. But $\mathrm{cf}(\mu) < \kappa$ and $|C_0| = \mathrm{cf}\mu$, contradiction. So (1) holds. Hence wlog every member of C_0 is singular.

Now we claim

(2) $(\prod C_0^{(+)}, <_{J^{\rm bd}})$ is μ-directed.

In fact, suppose that $F \subseteq \prod C_0^{(+)}$ and $|F| < \mu$. For $a \in C_0^{(+)}$ with $|F| < a$ let $h(a) = \sup_{f\in F} f(a)$; so $h(a) \in a$. For $a \in C_0^{(+)}$ with $a \le |F|$ let $h(a) = 0$. Clearly $f \le_{J^{\rm bd}} h$ for all $f \in F$. So (2) holds.

(3) $(\prod C_0^{(+)}, <_{J^{\rm bd}})$ is μ^+-directed.

In fact, by (2) it suffices to find a bound for a subset F of $\prod C_0^{(+)}$ such that $|F| = \mu$. Write $F = \bigcup_{\alpha<{\rm cf}(\mu)} G_\alpha$, with $|G_\alpha| < \mu$ for each $\alpha < {\rm cf}(\mu)$. By (2), each G_α has an upper bound k_α under $<_{J^{\rm bd}}$. Then $\{k_\alpha : \alpha < {\rm cf}(\mu)\}$ has an upper bound h under $<_{J^{\rm bd}}$. Clearly h is an upper bound for F.

Now we are going to apply 2.13 to $J^{\rm bd}$, $C_0^{(+)}$, and μ^+ in place of I, A, and λ; and with anything for g. Clearly the hypotheses hold, so we get a $<_{J^{\rm bd}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+\rangle$ in $\prod C_0^{(+)}$ such that $(*)_\kappa$ holds for f, for every regular cardinal $\kappa < \mu$. By 2.10 and 2.9, f has an exact upper bound h such that for every regular $\kappa < \mu$,

$$(\star) \qquad \{a \in C_0^{(+)} : h(a) \text{ is non-limit, or } {\rm cf}(h(a)) < \kappa\} \in J^{\rm bd}.$$

Now the identity function k on $C_0^{(+)}$ is obviously is an upper bound for f, so $h \le_{J^{\rm bd}} k$. By modifying h on a set in $J^{\rm bd}$ we may assume that $h(a) \le a$ for all $a \in C_0^{(+)}$. Now we claim

($\star\star$) The set $C_1 \stackrel{\rm def}{=} \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\}$ contains a club of μ.

Assume otherwise. Then for every club K, $K \cap (\mu\backslash C_1) \ne 0$. This means that $\mu\backslash C_1$ is stationary, and hence $S \stackrel{\rm def}{=} C_0\backslash C_1$ is stationary. For each $\alpha \in S$ we have $h(\alpha^+) < \alpha^+$. Hence ${\rm cf}(h(\alpha^+)) < \alpha$ since α is singular. Hence by Fodor's theorem ${\rm cf}(h(\alpha^+))$ is bounded by some $\kappa < \mu$ on a stationary subset of S. This contradicts $(\star)$.

Thus ($\star\star$) holds, and so there is a club $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for all $\alpha \in C$. Now $\langle f_\xi \restriction C^{(+)} : \xi < \mu^+\rangle$ is $<_{J^{\rm bd}}$-increasing. We claim that it is cofinal in $(\prod C^{(+)}, <_{J^{\rm bd}})$. For, suppose that $g \in \prod C^{(+)}$. Let g' be the extension of g to $\prod C_0^{(+)}$ such that $g'(a) = 0$ for any $a \in C_0\backslash C$. Then $g' <_{J^{\rm bd}} h$, and so there is a $\xi < \mu^+$ such that $g' <_{J^{\rm bd}} f_\xi$. So $g <_{J^{\rm bd}} f_\xi \restriction C^{(+)}$, as desired. This shows that $\mu^+ = {\rm tcf}(\prod C^{(+)}, <_{J^{\rm bd}})$. □

Theorem 2.15. *If μ is a singular cardinal of countable cofinality, then there is an unbounded set $D \subseteq \mu$ of regular cardinals such that*

$$\mu^+ = {\rm tcf}\left(\prod D, <_{J^{\rm bd}}\right).$$

Proof. Let C_0 be a set of regular cardinals with supremum μ, of order type ω.

(1) $\prod C_0/J^{\rm bd}$ is μ-directed.

For, let $X \subseteq \prod C_0$ with $|X| < \mu$. For each $a \in C_0$ such that $|X| < a$, let $h(a) = \sup\{f(a) : f \in X\}$, and extend h to all of C_0 in any way. Clearly $h \in \prod C_0$ and it is an upper bound in the $<_{J^{\rm bd}}$ sense for X.

From (1) it is clear that $\prod C_0/J^{\mathrm{bd}}$ is also μ^+-directed. By 2.13 we then get a $<_{J^{\mathrm{bd}}}$-increasing sequence $\langle f_\xi : \xi < \mu^+\rangle$ which satisfies $(*)_\kappa$ for every regular $\kappa < \mu^+$. By 2.9 and 2.10, f has an exact upper bound h such that $\{a \in C_0 : h(a)$ is non-limit or $\mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}$ for every regular $\kappa < \mu^+$. We may assume that $h(a) \le a$ for all $a \in C_0$, since the identity function is clearly an upper bound for f; and we may assume that each $h(a)$ is a limit ordinal of uncountable cofinality since $\{a \in C_0 : \mathrm{cf}(h(a)) < \omega_1\} \in J^{\mathrm{bd}}$.

(2) $\mathrm{tcf}\left(\prod_{a\in C_0}\mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right) = \mu^+$.

To prove this, for each $a \in C_0$ let D_a be club in $h(a)$ of order type $\mathrm{cf}(h(a))$, and let $\langle \eta_{a\xi} : \xi < \mathrm{cf}(h(a))\rangle$ be the strictly increasing enumeration of D_a. For each $\xi < \mu^+$ we define $f'_\xi \in \prod_{a\in C_0}\mathrm{cf}(h(a))$ as follows. Since $f_\xi <_{J^{\mathrm{bd}}} h$, the set $\{a \in C_0 : f_\xi(a) \ge h(a)\}$ is bounded, so choose $a_0 \in C_0$ such that for all $b \in C_0$ with $a_0 \le b$ we have $f_\xi(b) < h(b)$. For such a b we define $f'_\xi(b)$ to be the least ν such that $f_\xi(b) < \eta_{b\nu}$. Then we extend f'_α in any way to a member of $\prod_{a\in C_0}\mathrm{cf}(h(a)))$.

(3) $\xi < \sigma < \mu^+$ implies that $f'_\xi \le_{J^{\mathrm{bd}}} f'_\sigma$.

This is clear by the definitions.

Now for each $l \in \prod_{a\in C_0}\mathrm{cf}(h(a)))$ define $k_l \in \prod C_0$ by setting $k_l(a) = \eta_{al(a)}$ for all a. So $k_l < h$. Since h is an exact upper bound for f, choose $\xi < \mu^+$ such that $k_l <_{J^{\mathrm{bd}}} f_\xi$. Choose a such that $k_l(b) < f_\xi(b)$ for all $b \ge a$. Then for all $b \ge a$, $\eta_{bl(b)} < \eta_{bf'_\xi(b)}$, and hence $l(b) < f'_\xi(b)$. This proves that $l <_{J^{\mathrm{bd}}} f'_\xi$. This proves the following two statements.

(4) $\{f'_\xi : \xi < \mu^+\}$ is cofinal in $\left(\prod_{a\in C_0}\mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right)$.

(5) $\{f'_\xi : \xi < \mu^+\}$ is μ^+-directed with respect to $<_{J^{\mathrm{bd}}}$.

These facts yield (2).

Now let $B = \{\mathrm{cf}(h(a)) : a \in C_0\}$. Define

$$X \in J \text{ iff } X \subseteq B \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in J^{\mathrm{bd}}.$$

By 1.28 we get $\mathrm{tcf}(\prod B/J) = \mu^+$. It suffices now to show that J is the ideal of bounded subsets of B. Suppose that $X \in J$, and choose $a \in C_0$ such that $h^{-1}[\mathrm{cf}^{-1}[X]] \subseteq \{b \in C_0 : b < a\}$. By the choice of h, $X \subseteq \{b \in A : \mathrm{cf}(h(b)) < a\} \in J^{\mathrm{bd}}$, so X is bounded. Conversely, if X is bounded, choose $a \in B$ such that $X \subseteq \{b \in B : b \le a\}$. Now

$$\begin{aligned} h^{-1}[\mathrm{cf}^{-1}[X]] &= \{b \in C_0 : \mathrm{cf}(h(b)) \in X\} \\ &= \{b \in C_0 : \mathrm{cf}(h(b)) \le a\}, \end{aligned}$$

and this is bounded by the choice of h.