Solutions of exercises in Chapter 7

$\boxed{\text{E7.1}}$ *Let $\overline{M}$ be a field, A a subfield, and $a \in M$. Suppose that a is algebraic over A in the usual sense of field theory. Show that a is algebraic over A in the model-theoretic sense.*

Let $f(x)$ be a polynomial with coefficients in A such that $f(a) = 0$. Let $\varphi(x, \overline{b})$ be the formula $f(x) = 0$, where $\overline{b}$ is the system of coefficients of $f(x)$.

$\boxed{\text{E7.2}}$ *Let $\overline{M} = (\omega, <)$. Show that every element of ω is algebraic over $\emptyset$.*

For any $m \in \omega$ let $\varphi(x)$ be the formula "There are exactly m elements less than x."

$\boxed{\text{E7.3}}$ *Let $\overline{A} = ([\omega]^2, R)$, where*

$$R = \{(a, b) : a, b \in [\omega]^2,\ a \neq b \text{ and } a \cap b \neq \emptyset\}.$$

(i) Show that $\{a \in [\omega]^2 : (a, \{0, 1\}) \in R\}$ is neither finite nor cofinite.

(ii) Infer from (i) that $[\omega]^2$ is not minimal.

(iii) If f is a permutation of ω, define $f^+ : [\omega]^2 \to [\omega]^2$ by setting $f^+(a) = f[a]$ for any $a \in [\omega]^2$. Show that f^+ is an automorphism of $\overline{A}$.

(iv) Let $X = \{a \in [\omega]^2 : 0 \in a \text{ and } a \cap \{1, 2\} = \emptyset\}$. Show that X is definable in $\overline{A}$ with parameters.

(v) Show that X is minimal.

(i): For each $m > 1$ we have $(\{0, m\}, \{0, 1\}) \in R$, so the set is not finite. Also, for each $m > 1$ we have $(\{m, m + 1\}, \{0, 1\}) \notin R$, so the set is not cofinite.

(ii): The set is definable with parameters, since it is

$$\{a \in [\omega]^2 : \overline{A} \models \mathbf{R}a\{0, 1\}\}.$$

(iii): f^+ is one-one, since $f[a] = f[b]$ implies that $a = f^{-1}[f[a]] = f^{-1}[f[b]] = b$. f^+ maps onto $[\omega]^2$ since for any $a \in [\omega]^2$ we have $f[f^{-1}[a]] = a$. Now suppose that $a, b \in [\omega]^2$. Suppose that aRb. Then $a \neq b$ and $a \cap b \neq \emptyset$. Hence $f[a] \neq f[b]$ and $f[a] \cap f[b] \neq \emptyset$, so $f^(a)Rf^+(b)$. The converse is similar.

(iv): $X = \{a \in [\omega]^2 : \overline{A} \models aR\{0, 1\} \wedge \neg(aR\{1, 2\}) \wedge \neg(a = \{1, 2\})\}$.

(v): For any $m > 2$ we have $\{0, m\} \in X$, so X is infinite. Now suppose that Z is first-order definable with parameters, and $X \cap Z$ is infinite; we want to show that $X \backslash Z$ is finite. Say $Z = \{b \in [\omega]^2 : \overline{A} \models \varphi(\overline{c}, b)\}$, where $\overline{c}$ is a finite sequence of elements of $[\omega]^2$. We claim that

$$X \backslash Z \subseteq \{\{0, n\} : n \in \bigcup \mathrm{rng}(\overline{c})\}.$$

To prove this, take any $\{0, n\} \in X \backslash Z$. Thus $n \notin \{0, 1, 2\}$. Suppose that $n \notin \bigcup \mathrm{rng}(\overline{c})$. Since $X \cap Z$ is infinite, there is a p such that $p \notin \{0, 1, 2, n\} \cup \bigcup \mathrm{rng}(\overline{c})\}$ and $\{0, p\} \in X \cap Z$. Let f be the transposition (n, p), considered as a permutation of ω. Then f induces an automorphism f^+ of A, defined by $f^+(a) = f[a]$, for any $a \in A$, as above. Note that $f^+(u) = u$ for each $u \in \mathrm{rng}(\overline{c})$. Now $A \models \varphi(\overline{c}, \{0, p\})$, so it follows that $A \models \varphi(\overline{c}, \{0, n\})$. Hence $\{0, n\} \in Z$, contradiction.

$\boxed{\text{E7.4}}$ *Let V be an infinite vector space over a finite field F. We consider V as a structure $(V,+,f_a)_{a\in F}$, where $f_a(v)=av$ for any $v\in V$ and $a\in F$. Show that V is minimal.*

Suppose that X is definable with parameters; say

$$X=\{a\in A : A\models\varphi(\overline{c},a)\}.$$

Let Y be the span of $\mathrm{rng}(\overline{c})$. Thus Y is finite. We claim that $X\subseteq Y$ or $A\backslash X\subseteq Y$. Suppose that $X\not\subseteq Y$; choose $x\in X\backslash Y$. Take any $y\in A\backslash Y$. Let B be a basis for Y. Both x and y are not in Y, and thus there exist bases B',B'' of A such that $B\subseteq B',B''$, $x\in B'$, and $y\in B''$. Hence there is a one-one function f from B' onto B'' such that f is the identity on B and $f(x)=y$. Now f extends to an automorphism of A and it follows that $y\in X$. So we have shown that $A\backslash Y\subseteq X$. Hence $A\backslash X\subseteq Y$, as desired.

$\boxed{\text{E7.5}}$ *(continuing exc. 7.4) Prove that for any subset A of V, $\mathrm{acl}(A)=\mathrm{span}(A)$.*

First suppose that $b\in\mathrm{span}(A)$. Then there exist $a_0,\ldots,a_{n-1}\in A$ and $\varepsilon\in{}^nF$ such that $b=\varepsilon_0a_0+\cdots+\varepsilon_{n-1}a_{n-1}$. The formula $x=\varepsilon_0a_0+\cdots+\varepsilon_{n-1}a_{n-1}$ shows that $b\in\mathrm{acl}(A)$.

Now suppose that $b\in\mathrm{acl}(A)\backslash\mathrm{span}(A)$; we want to get a contradiction. Say $\overline{V}\models\varphi(b,\overline{a})$ and $\varphi(\overline{V},\overline{a})$ is finite, where $\overline{a}\in A$. Thus $b\notin\mathrm{span}(\overline{a})$. Choose c not in $\mathrm{span}(a)$ and also $c\notin\varphi(\overline{V},\overline{a})$. Then there is an automorphism f of $\overline{V}$ such that $f\circ\overline{a}$ is the identity and $f(b)=c$. Then $\overline{V}\models\varphi(c,\overline{a})$, contradiction.

$\boxed{\text{E7.6}}$ *(continuing excs. 7.4, 7.5) By exercise 7.4 and Lemma 7.2, the following holds in $\overline{V}$: if $a\in\mathrm{span}(A\cup\{b\})\backslash\mathrm{span}(A)$, then $b\in\mathrm{span}(A\cup\{a\})$. Prove this statement using ordinary linear algebra.*

Assume the hypothesis. Then we can write $a=\varepsilon_0c_0+\cdots+\varepsilon_{n-1}c_{n-1}+\delta b$ with each $c_i\in A$. Since $a\notin\mathrm{span}(A)$, we must have $\delta\neq 0$. Then

$$b=a+-\varepsilon_0c_0+\cdots+-\varepsilon_{n-1}c_{n-1},$$

which shows that $b\in\mathrm{span}(A\cup\{a\})$.

$\boxed{\text{E7.7}}$ *Give an example of a set Γ of sentences and two sentences φ and ψ, such that, $\Gamma\models\varphi$ iff $\Gamma\models\psi$ but $\Gamma\not\models(\varphi\leftrightarrow\psi)$.*

Let $\mathscr{L}$ have just one non-logical symbol, a unary relation symbol $\mathbf{P}$. Let $\Gamma=\emptyset$, $\varphi=\exists x\mathbf{P}x$, $\psi=\exists x\neg\mathbf{P}x$. Then $\not\models\varphi$, using the structure $(\omega,\emptyset)$. Also $\not\models\psi$, using the structure $(\{0\},\{0\})$. Hence $\Gamma\models\varphi$ iff $\Gamma\models\psi$. Finally $\not\models(\varphi\leftrightarrow\psi)$, using the structure $(\omega,\emptyset)$.

$\boxed{\text{E7.8}}$ *Show that for Γ a set of sentences and for sentences φ,ψ, if $\Gamma\models\varphi\leftrightarrow\psi$ then $\Gamma\models\varphi$ iff $\Gamma\models\psi$.*

Assume the hypothesis, and suppose that $\Gamma\models\varphi$. Suppose that $\overline{M}$ is a model of Γ. Since $\Gamma\models\varphi\leftrightarrow\psi$ and $\Gamma\models\varphi$, it follows that $\Gamma\models\psi$. Similarly in the other direction.

$\boxed{\text{E7.9}}$ *Prove that the following two conditions are equivalent:*

(i) $\overline{M}\models\varphi[a]$ *iff* $\overline{M}\models\psi[a]$.

(ii) $\overline{M}\models(\varphi\leftrightarrow\psi)[a]$.

Obvious.

$\boxed{\text{E7.10}}$ *Prove that the following two conditions are equivalent, for any sentences φ, ψ:*
(i) $\overline{M} \models \varphi$ iff $\overline{M} \models \psi$.
(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)$.

Obvious.

$\boxed{\text{E7.11}}$ *In the language with no non-logical symbols, show that ω is an indiscernible set in ω.*

Given $\varphi(\overline{x})$ and two sequences $\overline{a}$ and $\overline{b}$ of distinct elements of ω, there is a permutation f of ω such that $f \circ \overline{a} = \overline{b}$. Thus $\omega \models \varphi(\overline{a})$ iff $\omega \models \varphi(\overline{b})$. Now use exercise E7.9.

$\boxed{\text{E7.12}}$ *(Continuing exercises 7.4, 7.5, 7.6) Let $A = \{w_1, w_2\}$, two members of V, and let $b = w_1$. Thus $b \in \operatorname{span}(A)$. According to Lemma 7.8, $\operatorname{tp}^{\overline{V}}(b/A)$ is isolated. Give a formula $\varphi(v_0, \overline{a})$ with $\overline{a} \in A$ which isolates $\operatorname{tp}^{\overline{V}}(b/A)$.*

Let φ be $v_0 = w_1$. Thus $\varphi \in \operatorname{tp}^{\overline{V}}(b/A)$. Suppose that $\psi \in \operatorname{tp}^{\overline{V}}(b/A)$. Then $\overline{V} \models \psi(w_1)$, and hence $\overline{V} \models \forall v_0[\varphi \to \psi]$.

$\boxed{\text{E7.13}}$ *Suppose that $\overline{M}$ is an infinite structure, $\varphi(v_0)$ is a formula with at most v_0 free, and $\varphi(\overline{M})$ is infinite. Show that $\overline{M}$ has a proper elementary extension $\overline{N}$ such that $(\overline{M}, \overline{N})$ is not a Vaughtian pair.*

Let L be the language of $\overline{M}$, and expand the language L_M by adding another individual constant d. Then every finite subset of

$$\operatorname{Eldiag}(\overline{M}) \cup \{a \neq d : a \in \varphi(\overline{M})\} \cup \{\varphi(d)\}$$

has a model, so the result follows by the compactness theorem and Theorem 6.15.