HW 6

Due Friday, March 9

Exercises:
(6.1.1), **6.1.2**, (6.1.3), (6.1.4), **6.1.5**, **6.1.6**, **6.1.7**, (6.1.8),
6.2.1, (6.2.2), **6.2.3**, (6.2.4), **6.2.5**, **6.2.6**,
(6.3.1), (6.3.2), (6.3.3), **6.3.4**, **6.3.5**

(To pass this assignment, submit well-written, complete, correct solutions to at least **five** exercises from among the non-parenthesized numbers above.)

Section 6.1

(6.1.1) Write out a detailed proof of Lemma 6.1.1.

6.1.2. Verify the following remarks (from Section 6.1 of the text):

1. $f\colon M \xrightarrow{\Delta} \mathcal{N}$ iff $(\mathcal{M}, M) \Rrightarrow_{\Delta(M)} (N, f[M])$.
2. Let $\Delta \subseteq L_0$. Then $f\colon \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ if and only if $\mathcal{M} \Rrightarrow_{\Delta} \mathcal{N}$ and $f\colon M \to N$.
3. $f\colon \mathcal{M} \to \mathcal{N}$ if and only if $f\colon \mathcal{M} \xrightarrow{\mathbf{at}} \mathcal{N}$.
4. Suppose $\Delta \subseteq L$ contains **at**, the set of all atomic formulas, as well as all negations of unnested relational atomic formulas, i.e., all formulas $\neg R(\overline{x})$ where $R \in \mathbf{R}$. Then $f\colon \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ implies that f is a strong homomorphism (while the converse is not true).
5. $f\colon M \to N$ is injective if and only if $f\colon \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ for the set $\Delta = \{x \neq y\}$.
6. If $\Delta \subseteq L_0$ contains, along with every sentence, also its negation, then $\mathcal{M} \Rrightarrow_{\Delta} \mathcal{N}$ implies $\mathcal{M} \equiv_{\Delta} \mathcal{N}$.
7. If $\Delta \subseteq L$ contains, along with every formula, also its negation, then $f\colon \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ implies that for all $\varphi \in \Delta$ and tuples $\overline{a}$ from M we have $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(f[\overline{a}])$.
8. This can be used to find, for any structure $\mathcal{N} \models \mathrm{Th}\mathcal{M}$, a disjoint structure $\mathcal{N} \models \mathrm{Th}\mathcal{M}$: consider a bijection of f of M onto an arbitrary disjoint set N and define thereon a structure $\mathcal{N}$ according to the preimages. Then $f\colon \mathcal{M} \cong \mathcal{N}$ and hence, by the preceding result, $\mathcal{N} \models \mathrm{Th}\mathcal{M}$.
9. Another important consequence is that sets definable without parameters are invariant under automorphisms, i.e., given an L-structure $\mathcal{M}$ and an automorphism $f \in \mathrm{Aut}\mathcal{M}$, we have $f[\psi(\mathcal{M})] = \psi(\mathcal{M})$ for all $\psi \in L$.

(6.1.3) Prove $\mathrm{Th}_{\mathbf{qf}}(\mathcal{M}, M) \subseteq D(\mathcal{M})^{\vdash}$.

(6.1.4) Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathrm{Th}_{\mathbf{qf}}(\mathcal{M}, M) = \mathrm{Th}_{\mathbf{qf}}(\mathcal{N}, M)$, hence also $\mathrm{Th}_{\mathbf{qf}}(\mathcal{N}, M) \subseteq D(\mathcal{M})^{\vdash}$.

6.1.5. Verify that the diagram of a structure is logically equivalent to the subset of all its unnested atomic formulas and all its negations of unnested atomic formulas.

6.1.6. Show that the diagram of a finite structure in a finite signature is finitely axiomatizable.

6.1.7. Write down the diagrm of a field with three elements (up to TF-equivalence).

(6.1.8) Generalize Remark 9 (Section 6.1) appropriately to parametrically defined sets.

Section 6.2

(6.2.1) Prove $T_\forall = \mathrm{Th}\{\mathcal{M} : \mathcal{M} \hookrightarrow \mathcal{N} \models T\}$.

(6.2.2) Derive Corollary 6.2.5 directly from (the remark after) Lemma 6.2.2.

6.2.3 Prove Corollary 6.2.6.

(6.2.4) Fomulate and prove the dual of Proposition 6.2.3 for $\exists$-formulas (that is, the generalization of Corollary 6.2.6 to arbitrary sets of formulas $\Phi(\mathbf{x})$).

In the next exercises we deal with the dual of the remark after Lemma 6.2.2, i.e., with the models of $T_\exists\cdot$

6.2.5 Given theories S and T, show that $S_\forall \cup T$ is consistent if and only if there are $\mathcal M \models T arid N F S such that M w JV (if and only if thure is a sentence in 5 whose ncgation is in Ta) ·

(6.2.6) Prove the following two equations.

1. $\mathrm{Mod}\, T_\exists = \{\mathcal{N} : \text{ there are } \mathcal{M} \hookrightarrow \mathcal{N}' \equiv \mathcal{N} \text{ such that } \mathcal{M} \models T\}$
2. $T_\exists = \mathrm{Th}\{\mathcal{N} : \text{ there are } \mathcal{M} \hookrightarrow \mathcal{N}' \equiv \mathcal{N} \text{ such that } \mathcal{M} \models T\}$

Section 6.3

(6.3.1) Show that the substructure of a L-structure generated by the empty set is empty if and only if L has no constant symbols.

(6.3.2) Let $\mathcal{M}$ be an L-structure and $X \subseteq M$. Verify that the universe, M_X, of $\mathcal{M}_X$ is the set $\{t^{\mathcal{M}}(\mathbf{a}) : t \text{ is an } L\text{-term}, \mathbf{a} \text{ is a tuple from } X\}$.

(6.3.3) Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures and $X \subseteq M$. Suppose $f\colon X \to N$ satisfies $(\mathcal{M}, X) \equiv_{\mathbf{at}} (\mathcal{N}, f[X])$ (and is thus injective). Extend f to an isomorphism $F\colon \mathcal{M}_X \cong \mathcal{N}_{f[X]}$ and prove $(\mathcal{M}, M_X) \equiv_{\mathbf{at}} (\mathcal{N}, F[M_X])$.

6.3.4 Prove every linear ordering can be embedded in a dense linear ordering.

6.3.5 Check the last two examples: (8) The property of being locally finite clearly is hereditary, but not elementary; (9) The property of being divisible is elementary, but not hereditary.