$\boxed{4.1}$ *Prove Proposition 4.7.*

For (i), we proceed by induction on τ. If τ is v_j with $j \neq i$, or is an individual constant, then $\rho = \tau$ and so ρ is a term. Suppose that τ is v_i. Then $\rho = \tau$ if 0 occurrences of v_i are replaced, or is σ is v_i is replaced by σ. At any rate, ρ is a term. Finally, suppose that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Then ρ is $\mathbf{F}\sigma'_0 \ldots \sigma'_{m-1}$, where σ'_i is obtained from σ_i by replacing 0 or more occurrences of v_i by σ. By the inductive hypothesis, each σ'_i is a term. Hence ρ is a term.

For (ii) we proceed by induction on φ. Suppose that φ is $\sigma = \xi$. Then ψ is $\sigma' = \xi'$, where σ' is obtained from σ by replacing 0 or more occurrences of v_i by τ, and ξ' is obtained from ξ by replacing 0 or more occurrences of v_i by τ. By (i), σ' and ξ' are terms. So ψ is a formula. Next, suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Then ψ is $\mathbf{R}\sigma'_0 \ldots \sigma'_{m-1}$, where each σ'_i is obtained from σ_i by replacing 0 or more occurrences of v_i by τ. By (i), each σ'_i is a term. So ψ is a formula.

Next suppose inductively that φ is $\neg\chi$. Then ψ is $\neg\chi'$, where χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$). By the inductive hypothesis, χ' is a formula. Hence ψ is a formula.

Suppose inductively that φ is $\chi \wedge \rho$. Then ψ is $\chi' \wedge \rho'$, and χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$); ρ' is similarly obtained from ρ. By the inductive hypothesis, χ' and ρ' are formulas. So ψ is a formula.

Finally, suppose inductively that φ is $\exists v_j\chi$. Then ψ is $\exists v_j\chi'$, and χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$). By the inductive hypothesis, χ' is a formula. So ψ is a formula.

$\boxed{4.2}$ *Suppose that $\varphi, \psi, \chi, \theta$ are formulas, $\models \chi \leftrightarrow \theta$, and ψ is obtained from φ by replacing one or more occurrences of χ in φ by θ. Show that $\models \varphi \leftrightarrow \psi$. Hint: use induction on φ.*

We proceed by induction on φ. Note that if $\chi = \varphi$ the conclusion is clear. So we assume that $\chi \neq \varphi$. Then the atomic case vacuously holds, since an atomic formula has no proper subformula. Suppose that the statement is true for φ_0, and φ is $\neg\varphi_0$. Then ψ is $\neg\psi_0$, where ψ_0 is obtained from φ_0 by replacing one or more occurrences of χ by θ. So $\models \varphi_0 \leftrightarrow \psi_0$ by the inductive assumption. Clearly then $\models \varphi \leftrightarrow \psi$.

The other cases are treated similarly. For the quantifier case, one needs to observe that $\models \varphi \leftrightarrow \psi$ implies that $\models \exists v_i\varphi \leftrightarrow \exists v_i\psi$.

$\boxed{4.3}$ *Show that $\models \exists x\forall y\varphi \rightarrow \forall y\exists x\varphi$.*

Let $\overline{M}$ be any $\mathscr{L}$-structure, with $\mathscr{L}$ the implicit language we are working in. We use v_i for x and v_j for y to apply the rigorous definition of satisfaction; here $i \neq j$. Suppose that $a \in {}^{\omega}M$ and $\overline{M} \models \exists v_i\forall v_i\varphi[a]$. Then there is a $u \in M$ such that $\overline{M} \models \forall v_j\varphi[a^i_u]$. To prove that $\overline{M} \models \forall v_j\exists v_i\varphi[a]$, take any $w \in M$. Then $\overline{M} \models \varphi[a^{ij}_{uw}]$, hence $\overline{M} \models \exists v_i\varphi[a^j_w]$. Hence $\overline{M} \models \forall v_j\exists v_i\varphi[a]$.

$\boxed{4.4}$ *Show that*

$$\models \exists x[\varphi \wedge \psi \wedge \exists y(\varphi \wedge \neg\psi)] \rightarrow \exists y(\exists x\varphi \wedge \neg x = y).$$

Again we suppose that x is v_i and y is v_j, with $i \neq j$. Suppose that $\overline{A}$ is an $\mathscr{L}$-structure, $a \in {}^{\omega}A$, and $\overline{A} \models \exists v_i[\varphi \wedge \psi \wedge \exists v_j(\varphi \wedge \neg\psi)][a]$. Choose $u \in A$ such that $\overline{A} \models [\varphi \wedge \psi \wedge \exists v_j(\varphi \wedge \neg\psi)][a^i_u]$. Then choose $w \in A$ such that $\overline{A} \models (\varphi \wedge \neg\psi)[a^{i\,j}_{u\,w}]$. Thus $\overline{A} \models \psi[a^i_u]$ while $\overline{A} \models \neg\psi[a^{i\,j}_{u\,w}]$, hence it is not the case that $\overline{A} \models \psi[a^{i\,j}_{u\,w}]$. It follows that $a^i_u \neq a^{i\,j}_{u\,w}$, and the only way this can happen is that $a_j \neq w$. We now consider two cases.

Case 1. $a_i \neq w$. Then $\overline{A} \models \neg(v_i = v_j)[a^j_w]$, $\overline{A} \models \varphi[a^{i\,j}_{u\,w}]$, hence $\overline{A} \models \exists v_i\varphi[a^j_w]$, so $\overline{A} \models \exists v_j[\exists v_i\varphi \wedge \neg(v_i = v_j)][a]$, as desired.

Case 2. $a_i = w$. Then $\overline{A} \models \neg(v_i = v_j)[a]$, $\overline{A} \models \varphi[a^i_u]$, hence $\overline{A} \models \exists v_i\varphi[a]$, so $\overline{A} \models \exists v_j[\exists v_i\varphi \wedge \neg(v_i = v_j)][a]$, as desired.

$\boxed{4.5}$ *Show that*

$$\models \exists x\varphi \wedge \exists y\psi \wedge \exists z\chi \rightarrow \exists x\exists y\exists z[\exists x(\exists y\chi \wedge \exists z\psi) \wedge \exists y(\exists z\varphi \wedge \exists x\chi) \wedge \exists z(\exists x\psi \wedge \exists y\varphi)].$$

Again, let x, y, z be v_i, v_j, v_k respectively, with i, j, k different. Suppose that $\overline{A}$ is an $\mathscr{L}$-structure and $a \in {}^{\omega}A$. Suppose that $\overline{A} \models (\exists v_i\varphi \wedge \exists v_j\psi \wedge \exists v_k\chi)[a]$. Accordingly, choose $r, s, t \in A$ such that

(1) $$\overline{A} \models \varphi[a^i_r]$$

(2) $$\overline{A} \models \psi[a^j_s]$$

(3) $$\overline{A} \models \chi[a^k_t]$$

We claim that

(4) $$\overline{A} \models [\exists v_i(\exists v_j\chi \wedge \exists v_k\psi) \wedge \exists v_j(\exists v_k\varphi \wedge \exists v_i\chi) \wedge \exists v_k(\exists v_i\psi \wedge \exists v_j\varphi)][a^{i\,j\,k}_{r\,s\,t}].$$

Clearly this will prove the desired conclusion. By (3) we have $\overline{A} \models \exists v_j\chi[a^{j\,k}_{s\,t}]$, and by (2) we have $\overline{A} \models \exists v_k\psi[a^{j\,k}_{s\,t}]$. Hence

$$\overline{A} \models (\exists v_j\chi \wedge \exists v_k\psi)[a^{j\,k}_{s\,t}];$$

hence

(5) $$\overline{A} \models \exists v_i(\exists v_j\chi \wedge \exists v_k\psi)[a^{i\,j\,k}_{r\,s\,t}].$$

By (1) we have $\overline{A} \models \exists v_k\varphi[a^{i\,k}_{r\,t}]$, and by (3) we have $\overline{A} \models \exists v_i\chi[a^{i\,k}_{r\,t}]$. Hence

$$\overline{A} \models (\exists v_k\varphi \wedge \exists v_i\chi)[a^{i\,k}_{r\,t}];$$

hence

(6) $$\overline{A} \models \exists v_j((\exists v_k\varphi \wedge \exists v_i\chi)[a^{i\,j\,k}_{r\,s\,t}].$$

By (2) we have $\overline{A} \models \exists v_i\psi[a^{i\,j}_{r\,s}]$, and by (1) we have $\overline{A} \models \exists v_j\varphi[a^{i\,j}_{r\,s}]$. Hence

$$\overline{A} \models (\exists v_i\psi \wedge \exists v_j\varphi)[a^{i\,j}_{r\,s}];$$

hence

$$\overline{A} \models \exists v_k(\exists v_i\psi \wedge \exists v_j\varphi)[a^{i\ j\ k}_{r\ s\ t}]. \tag{7}$$

Clearly (5)–(7) give the desired result (4).

$\boxed{4.6}$ *In $(\omega, 0, \mathbb{S})$, show that every singleton $\{m\}$ for $m \in \omega$ is definable, i.e., there is a formula $\varphi_m(x)$ with only x free such that $\{m\} = \{a \in \omega : (\omega, 0, \mathbb{S}) \models \varphi[a]\}$. Here $\mathbb{S}$ is the successor function, which assigns $m+1$ to each natural number m.*

By recursion, let $\overline{0} = 0$ and $\overline{m+1} = S\overline{m}$. Then a formula φ_m as required is $v_0 = \overline{m}$.

$\boxed{4.7}$ *Let $\mathscr{L}$ be a relational language. A formula φ is standard if every nonequality atomic part of φ has the form $\mathbf{R}v_0 \ldots v_{m-1}$, where $\mathbf{R}$ is m-ary. (Normally any sequence of m variables is allowed.) Show that for every formula φ there is a standard formula ψ such that $\models \varphi \leftrightarrow \psi$.*

Clearly it suffices to do this only for atomic formulas $\mathbf{R}v_{i_0} \ldots v_{i_{m-1}}$. Let j be greater than both m and each i_k. Let φ be the formula

$$\exists v_j \ldots \exists v_{j+m-1}\left[\bigwedge_{k<m}(v_{j+k} = v_{i_k}) \wedge \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right]\right].$$

To show that this works, let $\overline{M}$ be any $\mathscr{L}$-structure and $a \in {}^{\omega}M$. Then

$\overline{M} \models \varphi[a]$ iff there are $b(0), \ldots, b(m-1) \in M$ such that

$$\overline{M} \models \bigwedge_{k<m}(v_{j+k} = v_{i_k}) \wedge \exists v_0 \ldots \exists v_{m-1}$$

$$\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right][a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and

$$\overline{M} \models \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right][a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that

$$\overline{M} \models \bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}[(a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)})^{0}_{c(0)} \cdots {}^{m-1}_{c(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that $c(k) = b(k)$ for all $k < m$ and $\overline{M} \models \mathbf{R}v_0 \ldots v_{m-1}[(a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)})^{0}_{c(0)} \cdots {}^{m-1}_{c(m-1)}]$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that $c(k) = b(k)$ for all $k < m$ and

$$\begin{aligned}
&\langle c(0), \ldots, c(m-1)\rangle \in \mathbf{R}^{\overline{M}} \\
\text{iff}\quad &\langle a(i_0), \ldots, a_{i_{m-1}}\rangle \in \mathbf{R}^{\overline{M}} \\
\text{iff}\quad &\overline{M} \models \mathbf{R}v_{i_0}\ldots v_{i_{m-1}}[a].
\end{aligned}$$

$\boxed{4.8}$ *In the language of rings, write down a single sentence whose models are exactly all rings.*

$$\begin{aligned}
&\forall x\forall y[x+y=y+x] \wedge \forall x\forall y\forall z[x+(y+z)=(x+y)+z] \wedge \forall x[x+0=x] \\
&\wedge \forall x[x+(-x)=0] \wedge \forall x\forall y\forall z[x\cdot(y\cdot z)=(x\cdot y)\cdot z] \\
&\wedge \forall x\forall y\forall z[x\cdot(y+z)=(x\cdot y)+(x\cdot z)] \wedge \forall x\forall y\forall z[(y+z)\cdot x=(y\cdot x)+(z\cdot x)]
\end{aligned}$$

$\boxed{4.9}$ *We describe an extension of first-order logic that can be used to make the set theoretical notation $\{a \in A : \varphi\}$ formal (rather than being treated as an abbreviation). Let $\mathscr{L}$ be a first order language, with an individual constant* $\mathbf{Z}$ *which will play a special role (in set theory, this can be the empty set as introduced in a definition). We define description terms and description formulas simultaneously:*

(a) Any variable or individual constant is a description term.

(b) If $\mathbf{O}$ *is an operation symbol of positive rank m and $\tau_0, \ldots, \tau_{m-1}$ are description terms, then $\mathbf{O}\tau_0\ldots\tau_{m-1}$ is a description term.*

(c) If $i < \omega$ and φ is a formula, then $Tv_i\varphi$ is a description term. (This is the description operator. $Tv_i\varphi$ should be read "the v_i such that φ, or $\mathbf{Z}$ *if there is not a unique v_i such that φ".*

(d) If σ and τ are description terms, then $\sigma = \tau$ is an atomic description formula.

(e) If $\mathbf{R}$ *is an m-ary relation symbol and $\tau_0, \ldots, \tau_{m-1}$ are description terms, then $\mathbf{R}\tau_0\ldots\tau_{m-1}$ is an atomic description formula.*

(f) If φ and ψ are description formulas and $i < \omega$, then the following are description formulas: $\neg\varphi$, $(\varphi \wedge \psi)$, and $\exists v_i\varphi$.

Next we define the value of description terms, and satisfaction of description formulas in an $\mathscr{L}$-structure simultaneously. Let $\overline{A}$ be an $\mathscr{L}$-structure, and let $a \in {}^{\omega}A$.

(a) $v_i^{\overline{A}} = a_i$.

(b) If τ is the term $\mathbf{F}\sigma_0\ldots\sigma_{m-1}$, then

$$\tau^{\overline{A}}(a) = \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)).$$

(c) If τ is the term $Tv_i\varphi$, then

$$\tau^{\overline{A}} = \begin{cases} \textit{the } x \in A \textit{ such that } \overline{A} \models \varphi[a] \textit{ and } a_i = x & \textit{if there is a unique such } x, \\ \mathbf{Z}^{\overline{A}} & \textit{otherwise.} \end{cases}$$

(d) If φ is $\sigma = \tau$, then $\overline{A} \models \varphi[a]$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

(e) If φ is $\mathbf{R}\sigma_0\ldots\sigma_{m-1}$, then $\overline{A} \models \varphi[a]$ iff $(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \in r_{\mathbf{R}}$.

(f) $\overline{A} \models \neg\varphi[a]$ iff it is not the case that $\overline{A} \models \varphi[a]$.

(g) $\overline{A} \models (\varphi \wedge \psi)[a]$ iff $\overline{A} \models \varphi[a]$ and $\overline{A} \models \psi[a]$.

(h) $\overline{A} \models \exists v_i\varphi[a]$ *iff there is an* $x \in A$ *such that* $\overline{A} \models \varphi[a^i_x]$.

Show that for any description formula φ *there is an ordinary formula* ψ *with the same free variables such that* $\models \varphi \leftrightarrow \psi$.

We prove by simultaneous induction on terms and formulas that if φ is a description formula and τ is a description term, and if v_i does not occur in τ, then there are ordinary formulas ψ and χ such that $\models \varphi \leftrightarrow \psi$ and $\models (v_i = \tau) \leftrightarrow \chi$.

(1) Suppose that τ is v_j. Take χ to be $v_i = v_j$.

(2) Suppose that τ is an individual constant $\mathbf{c}$. Take χ to be $v_i = \mathbf{c}$.

(3) Suppose that $\mathbf{O}$ is an operation symbol of rank $m > 0$, and $\tau_0, \ldots, \tau_{m-1}$ are description terms about which we know our result. Choose n greater than i and all v_j occurring in any term τ_k. Choose ordinary formulas $\theta_0, \ldots, \theta_{m-1}$ such that $\models (v_{n+j} = \tau_j) \leftrightarrow \theta_j$. Then the following formula works for $\mathbf{O}\tau_0 \ldots \tau_{m-1}$:

$$\exists v_n \ldots v_{n+m-1}\left(\bigwedge_{j<m} \theta_j \wedge v_i = \mathbf{O}v_n \ldots v_{n+m-1}\right).$$

(4) Suppose that τ is $Tv_j\psi$. Let σ be an ordinary formula such that $\models \psi \leftrightarrow \sigma$. Then the following formula works for τ, where v_k is a new variable:

$$[\sigma(v_i) \wedge \forall v_k(\sigma(v_k) \to v_i = v_k)] \vee [\neg\exists v_i[\sigma(v_i) \wedge \forall v_k(\sigma(v_k) \to v_i = v_k)] \wedge v_i = \mathbf{Z}].$$

(5) If φ is $\sigma = \rho$ for description terms σ, ρ, choose ordinary formulas η, ξ such that $\models (v_m = \sigma) \leftrightarrow \eta$ and $\models (v_n = \rho) \leftrightarrow \xi$, where v_m and v_n are new variables. Then we can take the following formula for ψ:

$$\exists v_m \exists v_n[\eta \wedge \xi \wedge v_m = v_n]$$

The other cases are straightforward.

$\boxed{4.10}$ *We modify the definition of first-order language by using parentheses. Thus we add two symbols (and) to our logical symbols.*

We retain in this context the same definition of terms as before. But we change the definition of formula as follows:

An atomic formula is a sequence of one of the following two sorts: $(\sigma = \tau)$, *with* σ *and* τ *terms; or* $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, *where* $\mathbf{R}$ *is a relation symbol of rank* m *and* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms. Then we define the collection of formulas to be the intersection of all sets of sequences of symbols such that every atomic formula is in* A, *and if* φ *and* ψ *are in* A *and* $i < \omega$, *then each of the following is in* A: $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\exists v_i\varphi)$.

Prove the analog of Proposition 4.1(i), adding an additional condition, that in a formula the number of left parentheses is equal to the number of right parentheses, while in any proper initial segment of a formula, either there are no parentheses, or there are more left parentheses than right ones.

We go by induction on the formula φ. It is clear for atomic formulas. Suppose that it is true for φ and ψ.

Then $(\neg\varphi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\neg\varphi)$ are of these kinds: $\emptyset$; (; $(\neg$; and $(\neg\sigma$ with σ an initial segment of φ, possibly equal to φ. Our condition is clear in each case.

Similarly, $(\varphi\wedge\psi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\varphi\wedge\psi)$ are of these kinds: $\emptyset$; (; $(\sigma$ with σ an initial segment of φ, possibly equal to φ; $(\varphi\wedge$; and $(\varphi\wedge\sigma$ with σ an initial segment of ψ, possibly equal to ψ. Our condition is clear in each case.

Similarly, $(\exists v_i\varphi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\exists v_i\varphi)$ are of these kinds: $\emptyset$; (; $(\exists$; $(\exists v_i$; and $(\exists v_i\sigma$ with σ an initial segment of φ, possibly equal to φ. Our condition is clear in each case.

$\boxed{4.11}$ *Show that* 0 *and* $\mathbb{S}$ *are definable in* $(\omega,<)$. *That is, there are formulas* $\varphi(x)$ *and* $\psi(x,y)$ *with only the indicated free variables such that for all* $a\in\omega$, $a=0$ *iff* $(\omega,<)\models\varphi[a]\}$, *and for all* $a,b\in\omega$, $\mathbb{S}a=b$ *iff* $(\omega,<)\models\psi[a,b]$. *Here we are working in the language of orderings.*

Let φ be the formula $\neg\exists y[y<x]$. Clearly $a=0$ iff $(\omega,<)\models\varphi[a]\}$. Let ψ be the formula

$$x<y\wedge\neg\exists z[x<z\wedge z<y].$$

Clearly $\mathbb{S}a=b$ iff $(\omega,<)\models\psi[a,b]$.