Solutions of exercises in Chapter 6

$\boxed{\text{E6.1}}$ *A subset X of a structure $\overline{M}$ is definable iff there is a formula $\varphi(x)$ with only x free such that $X = \{a \in M : \overline{M} \models \varphi[a]\}$. Similarly, for any positive integer m, a subset X of ${}^mM$ is definable iff there is a formula $\varphi(\overline{x})$ with $\overline{x}$ a sequence of m distinct variables including all variables occurring free in φ, such that $X = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$.*

For the language with no nonlogical symbols and for any structure $\overline{M}$ in that language, determine all the definable subsets and m-ary relations over $\overline{M}$. Hint: use Theorem 6.1.

Let $\overline{M}$ be any $\mathscr{L}$-structure. Thus $\overline{M}$ is essentially just a set. Its definable subsets are $\emptyset$ and M, which are clearly defined by $x \neq x$ and $x = x$ respectively. Suppose that $A \subseteq M$ is definable by $\varphi(x)$, with $\emptyset \neq A \neq M$. Let $a \in A$ and $b \in M\backslash A$. Let f be the bijection of M which interchanges a and b. Then $\overline{M} \models \varphi[a]$ but $\overline{M} \not\models \varphi[f(a)]$, contradicting Theorem 6.1.

Now suppose that $m > 1$. For each equivalence relation $\equiv$ on m let

$$R_{\equiv} = \{a \in {}^mM : \forall i, j < m[a_i = a_j \text{ iff } i \equiv j\}.$$

We claim that the definable m-are relations over $\overline{M}$ are just $\emptyset$ and unions of these relations $R_{\equiv}$. To show that these are definable, for $\emptyset$ take $\bigwedge_{i<m} x_i \neq x_j$; and for a nonempty set E of equivalence relations, take the formula

$$\bigvee_{\equiv \in E} \left(\bigwedge_{i \equiv j} (x_i = x_j) \wedge \bigwedge_{i \not\equiv j} (x_i \neq x_j) \right),$$

which we denote by φ_E. Now suppose that K is a nonempty definable m-ary relation on M; say $K = \{a \in {}^mM : \overline{M} \models \psi[a]\}$. For each $a \in K$ let $\equiv_a = \{(i, j) \in m \times m : a_i = a_j\}$, and let $E = \{\equiv_a : a \in K\}$. If $a \in K$, then $\equiv_a \in E$, and hence $\overline{M} \models \varphi_E[a]$. So $K \subseteq \{a \in {}^mM : \overline{M} \models \varphi_E[a]\}$. Now suppose that $\overline{M} \models \varphi_E[a]$ but $a \notin K$. Choose $b \in K$ such that

$$\overline{M} \models \left(\bigwedge_{i \equiv_b j} (x_i = x_j) \wedge \bigwedge_{i \not\equiv_b j} (x_i \neq x_j) \right) [a].$$

Then there is a bijection f of M onto M such that $f(b_i) = a_i$ for all $i < m$. Now $\overline{M} \models \psi[b]$ but $\overline{M} \not\models \psi[a]$, contradiction.

$\boxed{\text{E6.2}}$ *Let Γ be the set of all sentences holding in the structure $(\omega, S, 0)$, where $S(n) = n+1$ for all $n \in \omega$. Prove an elimination of quantifiers theorem for Γ.*

By the general procedure at the beginning of this chapter it suffices to eliminate the quantifier in a formula of the form $\exists x\varphi$, where φ is a conjunction of atomic formulas and their negations, where x actually occurs in each conjunct. Moreover, for any natural numbers m, n we have $\Gamma \models S^m x = S^n x$ if $m = n$, and $\Gamma \models S^m x \neq S^n x$ if $m \neq n$. So we may assume that the atomic and negated atomic formulas have the form $S^m x = S^n u$ and $S^m x \neq S^n u$, where u is 0 or a variable different from x. Now $\Gamma \models S^m x = S^n u \leftrightarrow$

$S^{m+1}x = S^{n+1}u$, so we may assume that m does not depend on any particular conjunct. If a conjunct $S^m x = S^n u$ actually appears, then

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \wedge \psi,$$

where ψ is obtained from φ by replacing $S^m x$ by $S^n u$. Now if $m \leq n$, then $\Gamma \models S^m x = S^n u \leftrightarrow x = S^{n-m}u$, and so

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \leftrightarrow 0 = 0,$$

eliminating the quantifier. If $n < m$, then

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \leftrightarrow u \neq 0 \wedge u \neq S0 \wedge \ldots \wedge u \neq S^{m-n}0,$$

again eliminating the quantifier.

Hence we may assume that no conjunct $S^m x = S^n u$ actually appears in φ. Thus φ has the form

$$S^m x \neq S^{n(0)}u_0 \wedge \ldots \wedge S^m x \neq S^{n(k)}u_k,$$

where each u_i is a variable not equal to x, or is the individual constant 0. Then we claim that $\Gamma \models \exists x\varphi \leftrightarrow 0 = 0$, i.e, $\Gamma \models \exists x\varphi$. For, suppose that $a \in {}^\omega\omega$. Say x is v_j. Choose $v \in \omega$ such that $S^m v \neq S^{n(0)}u_0(a), \ldots, S^m v \neq S^{n(k)}u_k(a)$. Thus $(\omega, S, 0) \models \varphi[a^i_v]$, as desired.

$\boxed{\text{E6.3}}$ *Let T be the theory of an infinite equivalence relation each of whose equivalence classes has exactly two elements. Use an Ehrenfeucht game to show that T is complete.*

To be precise, let T consist of the following sentences:

$\forall x[xEx]$

E is symmetric and transitive

there are infinitely many elements in the model

$\forall v_0$ there are exactly two elements equivalent to v_0.

Now assume that $\overline{A}$ and $\overline{B}$ are models of Γ and m is a positive integer. The strategy of ISO is as follows. Suppose that we are at the i-th turn and NON-ISO chooses 0 and an element $a \in A$. The move of ISO depends on the following possibilities. If the turns so far have not produced a partial isomorphism, then ISO selects any element of B. Suppose that the turns so far have produced a partial isomorphism f.

Case 1. No element of A equivalent to a has been selected yet. Then ISO picks an element of B not equivalent to any element selected so far.

Case 2. There is an element $a' \in A$ which has already been selected which is equivalent to a, while a itself has not been previously selected. Then ISO picks an element of B equivalent to $f(a')$ which has not yet been selected.

Case 3. a has already been selected. Then ISO picks $f(a)$.

If NON-ISO chooses 1 and an element of B, ISO does a similar thing, interchanging the roles of A and B.

Clearly this produces a partial isomorphism. $\square$

$\boxed{\text{E6.4}}$ *Let T be any theory. Show that the class of all substructures of models of T is the class of all models of a set of universal sentences, i.e., sentences of the form $\forall\overline{x}\varphi$ with φ quantifier free and $\overline{x}$ a finite string of variables containing all variables free in φ.*

Let $\Gamma = \{\forall\overline{x}\varphi : \overline{x}$ is a finite string of variables containing all variables free in φ, and $T \models \forall\overline{x}\varphi\}$.

Suppose that $\overline{A} \leq \overline{B} \models T$. Clearly then $\overline{A} \models \Gamma$.

Conversely, suppose that $\overline{A} \models \Gamma$. Then in order to show that $\overline{A}$ can be embedded in a model of T it suffices to show that $T \cup \mathrm{Diag}(\overline{A})$ has a model. Suppose not. Then $T \cup \mathrm{Diag}(\overline{A}) \models \exists x(x \neq x)$. Hence by Lemma 6.29 there is an existential sentence ψ such that $T \models \psi \to \exists x(x \neq x)$; so $T \models \neg\psi$. Hence $\neg\psi \in \Gamma$. But also by Lemma 6.29, $\overline{A} \models \psi$, contradicting $\overline{A} \models \Gamma$.

$\boxed{\text{E6.5}}$ *Suppose that $\Gamma \cup \{\varphi\}$ is a set of sentences in a language $\mathscr{L}$. Suppose that Γ and φ have the same models. Prove that there is a finite subset Δ of Γ with the same models as Γ.*

Applying Lemma 6.28 with Γ, Δ, φ replaced by $\emptyset, \Gamma, \varphi$ respectively, we get a finite conjunction ψ of members of Γ such that $\models \psi \to \varphi$. On the other hand, obviously $\models \varphi \to \psi$. Thus the collection of conjuncts of ψ has the same models as Γ.

$\boxed{\text{E6.6}}$ *Suppose that T and T' are theories in a language $\mathscr{L}$. Show that the following conditions are equivalent:*

(i) Every model of T' can be embedded in a model of T.

(ii) Every universal sentence which holds in all models of T also holds in all models of T'.

(i)⇒(ii): Assume (i), suppose that φ is a universal sentence holding in all models of T, and suppose that $\overline{A}$ is a model of T'. By (i), choose $\overline{B} \models T$ such that $\overline{A} \leq \overline{B}$. Clearly $\overline{A} \models \varphi$, as desired.

(ii)⇒(i): Assume (ii), and suppose that $\overline{C}$ is a model of T'. Let Γ be a set of universal sentences as given in exercise E6.4:

$$\{\overline{A} : \overline{A} \models \Gamma\} = \{\overline{A} : \exists\overline{B} \models T(\overline{A} \leq \overline{B})\}.$$

Clearly then $T \models \Gamma$, so $\overline{C} \models \Gamma$ by (ii), and hence (i) holds.

$\boxed{\text{E6.7}}$ *Let T be a theory in a language $\mathscr{L}$. Let $\mathbf{K}$ be the class of all models of T. Show that the following conditions are equivalent:*

(i) $\mathbf{SK} = \mathbf{K}$.

(ii) There is a collection Γ of universal sentences such that $\mathbf{K}$ is the class of all models of Γ.

This is immediate from exercise E6.4.

$\boxed{\text{E6.8}}$ *Suppose that $\overline{A} \leq \overline{B}$. Prove that $\overline{A} \preceq \overline{B}$ iff $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)_{a\in A}$.*

$\Rightarrow$: Suppose that $\overline{A} \preceq \overline{B}$ and $\overline{A}, a)_{a\in A} \models \varphi$. Thus $\varphi \in \text{Eldiag}(\overline{A})$, so by Theorem 6.15, $(\overline{B}, a)_{a\in A} \models \varphi$. The converse follows by applying this argument to $\neg\varphi$.

$\Leftarrow$. Assume that $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)_{a\in A}$. In particular, $(\overline{B}, a)a \in A$ is a model of $\text{Eldiag}(\overline{A})$, so by Theorem 6.15, $\overline{A} \preceq \overline{B}$.

$\boxed{\text{E6.9}}$ *Suppose that m is a positive integer, $\varphi(\overline{x})$ is a formula with free variables $\overline{x}$ of length m, and $\overline{M}$ is a structure. Define $\varphi(\overline{M}) = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$. Show that the following conditions are equivalent:*

(i) $\varphi(\overline{M})$ is finite.

(ii) $\varphi(\overline{M}) = \varphi(\overline{N})$ whenever $\overline{M} \preceq \overline{N}$.

Assume (i), and suppose that $\overline{M} \preceq \overline{N}$. Say $|\varphi(\overline{M})| = n$. Since $\overline{M} \preceq \overline{N}$, the m-tuples from M that satisfy φ in $\overline{M}$ also satisfy φ in $\overline{N}$. The statement (i) can be expressed by a sentence (with fixed m), and it holds in $\overline{M}$, hence in $\overline{N}$. so (ii) follows.

Now assume that (i) fails; we show that (ii) fails. To the language $\mathscr{L}_A$ adjoin new constants $\overline{d}$ for $i < m$, of length m. We consider the following set of sentences:

$$\text{Eldiag}(\overline{M}) \cup \left\{ \bigvee_{i<m} c_{b(i)} \neq d_i : \overline{M} \models \varphi[b(0), \ldots, b(m-1)] \right\}.$$

By (i) failing and the compactness theorem this set has a model, and this gives an elementary extension $\overline{N}$ of $\overline{M}$ in which $\varphi(\overline{M}) \subset \varphi(\overline{N})$.

$\boxed{\text{E6.10}}$ *Prove that if K is a set of models of a complete theory T then there is a structure $\overline{M}$ such that every member of K can be elementarily embedded in $\overline{M}$.*

Let κ be an infinite cardinal greater than the size of all members of K, and let $\overline{A}$ be a κ^+-saturated model of T. Then $\overline{A}$ is as desired, by Theorem 6.24.

$\boxed{\text{E6.11}}$ *Suppose that $\overline{A}$ and $\overline{B}$ are elementarily equivalent, κ-saturated, and both of size κ. Show that they are isomorphic.*

Write $A = \{a_\alpha : a < \kappa\}$ and $B = \{b_\alpha : \alpha < \kappa\}$. We now define $\langle c_\alpha : \alpha < \kappa\rangle$ and $\langle d_\alpha : \alpha < \kappa\rangle$ by recursion. Suppose they have been defined for all $\beta < \alpha$ so that $(\overline{A}, c_\beta)_{\beta<\alpha} \equiv (\overline{B}, d_\beta)_{\beta<\alpha}$. We now define $c_\alpha, d_\alpha, d_{\alpha+1}, c_{\alpha+1}$. Let $c_\alpha = a_\gamma$, with γ minimum such that $a_\gamma \notin \{c_\beta < \beta < \alpha\}$. Let Γ be the set of all formulas $\varphi(x)$ in $\mathscr{L}_{\langle c_\beta:\beta<\alpha\rangle}$ such that $(\overline{A}, c_\beta)_{\beta<\alpha} \models \varphi[c_\alpha]$. Then $(\overline{A}, c_\beta)_{\beta<\alpha} \models \exists x\Delta$ for every conjunction D of finitely many elements of Γ, so also $(\overline{B}, d_\beta)_{\beta<\alpha} \models \exists x\Delta$. Hence since $\overline{B}$ is κ-saturated we get an element d_α of B such that $(\overline{B}, d_\beta)_{\beta<\alpha} \models \exists\varphi[d_\alpha]$ for every $\varphi \in \Gamma$. It follows that $(\overline{A}, c_\beta)_{\beta\leq\alpha} \equiv (\overline{B}, d_\beta)_{\beta\leq\alpha}$. We define $d_{\alpha+1}$ and $c_{\alpha+1}$ analogously.

After this construction we have $A = \{c_\alpha : \alpha < \kappa\}$, $B = \{d_\alpha : \alpha < \kappa\}$, and $(\overline{A}, c_\alpha)_{\alpha<\kappa} \equiv (\overline{B}, d_\alpha)_{\alpha<\kappa}$. Hence $\{(c_\alpha, d_\alpha) : \alpha < \kappa\}$ is the desired isomorphism.

$\boxed{\text{E6.12}}$ *For any natural number n and any structure $\overline{M}$, an n-type of $\overline{M}$ is a collection Γ of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, a sequence of distinct variables of length n, such that $\overline{M}_M \models \exists\overline{x}\varphi$ for every conjunction of finitely many members of Γ. Prove that if Γ is a collection of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, then Γ is an n-type* *over $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $\overline{N} \models \varphi[\overline{a}]$ for every $\varphi \in \Gamma$.*

$\Rightarrow$: Expand $\mathscr{L}_M$ with new individual constants $\overline{d}$ of length n. Every finite subset of

$$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{d}) : \varphi \in \Gamma\}$$

has a model by hypothesis, so the compactness theorem yields the required $\overline{N}$.

$\Leftarrow$: $\overline{N} \models \exists\overline{x}\varphi$ for every finite subset of Γ, so $\overline{M}$ models this too, as desired.

$\boxed{\text{E6.13}}$ *If $\overline{M}$ is a structure, $A \subseteq M$, and $n \in \omega$, then an n-type over A of $\overline{M}$ is an n-type of $\overline{M}$ all of whose additional constants come from A. Given an n-tupe $\overline{a}$ of elements of M, the n-type over A of $\overline{a}$ in $\overline{M}$, denoted by $\mathrm{tp}^{\overline{M}}(\overline{a})/A)$, is the set $\{\varphi(\overline{x}) : \varphi$ is a formula with free variables among $\overline{x}$, $\overline{x}$ has length n, and $\overline{M}_A \models \varphi[\overline{a}]\}$. An n-type S over A is complete iff $\varphi \in S$ or $\neg\varphi \in S$ for every formula in the language $\mathscr{L}_A$ with free variables among $\overline{x}$.*

Prove that S is a complete n-type over A in $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S = \mathrm{tp}^{\overline{N}}(\overline{a}/A)$.

$\Rightarrow$: Assume that S is complete. By Exercise E6.12, there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S \subseteq \mathrm{tp}^{\overline{N}}(\overline{a}/A)$. Clearly equality holds since S is complete.

$\Leftarrow$: clear.

$\boxed{\text{E6.14}}$ *Let t be an n-type over A of $\overline{M}$. We say that t is isolated iff there is a formula $\varphi(\overline{x})$ in $\mathscr{L}_A$ such that $\overline{M}_A \models \exists\overline{x}\varphi$ and $\overline{M}_A \models \forall\overline{x}(\varphi \to \psi)$ for every $\psi \in t$. We then say that φ isolates t.*

Prove that if φ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

Assume the hypothesis. Choose $\overline{b}$ such that $\overline{M}_A \models \varphi[\overline{b}]$. If $\psi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\overline{M}_A \models \varphi(\overline{b}) \to \psi(\overline{b})$. Hence $\mathrm{tp}^{\overline{M}}(\overline{a}/A) \subseteq \mathrm{tp}^{\overline{M}}(\overline{b}/A)$, so $\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \mathrm{tp}^{\overline{M}}(\overline{b}/A)$. Hence $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

$\boxed{\text{E6.15}}$ *Show that $\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$ is isolated iff both $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated.*

$\Rightarrow$: Suppose that $\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$ is isolated. Let $\varphi(\overline{x}, \overline{y})$ be such that $\overline{M}_A \models \exists\overline{x}\exists\overline{y}\varphi$ and $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$ for every $\psi \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. We claim that $\varphi(\overline{x}, \overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\exists\overline{x}\varphi$ isolates $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$.

By Exercise E6.14 we have $\overline{M}_A \models \varphi[\overline{a}, \overline{b}]$. Hence $\overline{M}_{A\cup\mathrm{rng}(\overline{b})} \models \exists\overline{x}\varphi(\overline{x}, \overline{b})$. Now suppose that $\psi(\overline{x}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$. Then we can write $\psi(\overline{x}) = \psi(\overline{x}, \overline{b})$, and $\psi(\overline{x}, \overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. Hence $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$. Hence $\overline{M}_{A\cup\mathrm{rng}(\overline{b})} \models \forall\overline{x}[\varphi(\overline{x}, \overline{b}) \to \psi(\overline{x})]$. This proves that $\varphi(\overline{x}, \overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$.

For the second type, clearly $\overline{M}_A \models \exists\overline{y}\exists\overline{x}\varphi$. Now suppose that $\psi(\overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{b}/A)$. Then also $\psi(\overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. Hence $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$. So $\overline{M}_A \models \forall\overline{y}[\exists\overline{x}\varphi \to \psi]$. This proves that $\exists\overline{x}\varphi$ isolates $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$.

$\Leftarrow$: Assume that $\mathrm{tp}^{\overline{M}}(\overline{a}/A\cup\mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated. Let $\psi(\overline{x},\overline{b})$ isolate the type $\mathrm{tp}^{\overline{M}}(\overline{a}/A\cup\mathrm{rng}(\overline{b}))$ and $\varphi(\overline{y})$ isolate $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$. We claim that $\theta \stackrel{\text{def}}{=} \psi(\overline{x},\overline{y})\wedge\varphi(\overline{y})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$.

By exercise E6.14 we have $\varphi\in\mathrm{tp}^{\overline{M}}(\overline{b}/A)$, so $\overline{M}_A\models\varphi[\overline{b}]$. Now $\overline{M}_A\models\exists\overline{x}\psi(\overline{x},\overline{b})$, so $\overline{M}\models\exists\overline{x}\exists\overline{y}(\psi\wedge\varphi)$.

Now suppose that $\chi\in\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$; we want to show that $\overline{M}_A\models\forall\overline{x}\forall\overline{y}[\theta\to\chi]$. Now

$$\overline{M}_A\models\forall\overline{x}[\psi(\overline{x},\overline{b})\to\chi(\overline{x},\overline{b})].$$

Hence

$$\overline{M}_A\models\varphi(\overline{d})\to\forall\overline{x}[\psi(\overline{x},\overline{d})\to\chi(\overline{x},\overline{d})],$$

so

$$\overline{M}_A\models\varphi(\overline{d})\to[\psi(\overline{c},\overline{d})\to\chi(\overline{c},\overline{d})],$$

hence

$$\overline{M}_A\models\theta(\overline{c},\overline{d})\to\chi(\overline{c},\overline{d})],$$

as desired.

$\boxed{\text{E6.16}}$ *Let T be a complete theory with a model. A formula $\varphi(\overline{x})$ is complete in T iff $T\cup\{\exists\overline{x}\varphi\}$ has a model, and for every formula $\psi(\overline{x})$, either $T\models\varphi\to\psi$ or $T\models\varphi\to\neg\psi$. Here $\overline{x}$ is a sequence of variables containing all variables free in φ or ψ.*

A formula $\theta(\overline{x})$ is completable in T iff there is a complete formula $\varphi(\overline{x})$ such that $T\models\varphi\to\theta$.

A structure $\overline{M}$ is atomic iff every tuple $\overline{a}$ of elements of M satisfies a complete formula in the theory of $\overline{M}$.

A theory T is atomic iff for every formula $\theta(\overline{x})$ such that $T\cup\{\exists\overline{x}\theta(\overline{x})\}$ has a model, θ is completable in T.

Show that if T is a complete theory in a countable language, then T has a countable atomic model iff T is atomic. Hint: in the direction $\Leftarrow$, for each $n\in\omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0,\ldots,v_{n-1}$, and apply the omitting types theorem.

$\Rightarrow$: Assume that T has an atomic model $\overline{M}$, and suppose that $\theta(\overline{x})$ is a formula such that $T\cup\{\exists\overline{x}\theta(\overline{x})\}$ has a model. Since T is complete, we have $T\models\exists\overline{x}\theta(\overline{x})$, and so $\overline{M}\models\exists\overline{x}\theta(\overline{x})$. Choose $\overline{a}$ in M such that $\overline{M}\models\theta[\overline{a}]$. Since $\overline{M}$ is atomic, there is a complete formula $\varphi(\overline{x})$ such that $\overline{M}\models\varphi[\overline{a}]$. Since $\varphi(\overline{x})$ is complete, it follows that $T\models\varphi\to\theta$. Thus θ is competable. This proves that T is atomic.

$\Leftarrow$: Assume that T is atomic. For each $n\in\omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0,\ldots,v_{n-1}$. Then t_n is not isolated. For suppose it is, and let $\varphi(\overline{v})$ be a formula such that $T\cup\{\exists\overline{v}\varphi(\overline{v})$ has a model, and $T\models\varphi\to\psi$ for every $\psi\in t_n$. Now T is atomic, so φ is completable. Let χ be a complete formula such that $T\models\chi\to\varphi$. But $\neg\chi\in t_n$, so $T\models\varphi\to\neg\chi$. Hence $T\models\chi\to\neg\chi$, contradicting the fact that $T\cup\{\exists\overline{v}\chi\}$ has a model.

Now by the omitting types theorem, let $\overline{M}$ be a countable model of T which omits each type t_n. Thus for each $\overline{a}$ in M, say of length m, there is a φ in t_m such that $\overline{M}\models\neg\varphi[\overline{a}]$. Since $\neg\varphi$ is complete, this shows that $\overline{M}$ is atomic.