Model theory

(Math 6000)

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These notes form an introduction to model theory. The topics are: first-order structures; terms and varieties; first-order languages, satisfaction, and truth; elimination of quantifiers; Löwenheim-Skolem theorems; ultraproducts; the compactness theorem; diagrams; Ehrenfeucht-Fraissé games; interpretations; saturated structures; omitting types; Morley's categoricity theorem; Morley rank; interpolation; countable models; the number of types and models.

CONTENTS

1. Structures

The basic notions of model theory are *structures* and *first-order logic*; model theory is essentially the study of the relationships between these two notions. In this chapter we define the notion of structure, and give several examples which will be discussed later.

Structures consist of a nonempty set A together with certain elements of A, operations on A, and relations on A, with various numbers of arguments. See the precise definition below. Some important examples are: groups, rings, fields, lattices, Boolean algebras, posets, and linearly ordered sets. We will illustrate the general notions in this chapter with some of these special cases.

Some simple set-theoretic background is needed first. We use ω to denote the set of natural numbers $0, 1, 2, \ldots$. Each natural number is considered to actually be the set of all preceding natural numbers. Thus 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}$, etc. Moreover, $\omega\backslash\{0\}$ is the set of all positive integers. The domain of a function f is denoted by $\mathrm{dmn}(f)$. A *finite sequence* is a function whose domain is some positive integer. For any set A and any postive integer m, we denote by ${}^mA$ the set of all finite sequences of members of A, of length m. That is, ${}^mA$ is the set of all functions mapping m into A. An m-ary *relation* on A is a subset of ${}^mA$. An m-ary *operation* on A is a function mapping ${}^mA$ into A. These are general notions which are used in what follows.

A *signature* or *similarity type* or (first-order) *language* is an ordered quadruple $\sigma = (\mathrm{Fcn}, \mathrm{Rel}, \mathrm{Cn}, \mathrm{ar})$ such that Fcn, Rel, Cn are pairwise disjoint sets, and ar is a function mapping $\mathrm{Fcn} \cup \mathrm{Rel}$ into $\omega\backslash 1$. Members of Fcn, Rel, Cn are called function (or operation) symbols, relation symbols, and individual constants respectively. The positive integer $\mathrm{ar}(S)$ is called the *arity* of the function or relation symbol S, and S is said to be $\mathrm{ar}(S)$-ary. We also say that S has *rank* $\mathrm{ar}(S)$. We sometimes say *unary* rather than 1-ary, and *binary* rather than 2-ary. The members of Cn are called *individual constants.* One might think of the function and relation symbols as constants of a different sort.

A *(first-order) structure* with signature σ is a quadruple $\overline{A} = (A, a, b, c)$ such that:

- A is a nonempty set.
- a is a function with $\mathrm{dmn}(a) = \mathrm{Rel}$, and a_R is an $ar(R)$-ary relation on A for every $R \in \mathrm{dmn}(a)$.
- b is a function with $\mathrm{dmn}(b) = \mathrm{Fcn}$, and b_F is an $ar(F)$-ary operation on A for every $F \in \mathrm{dmn}(b)$.
- c is a function mapping Cn into A.

We say then that (A, a, b, c) has signature σ. The set A is called the *universe* of (A, a, b, c). Frequently we just write $\overline{A}$ instead of (A, a, b, c) if the appropriate a, b, c are clear. Frequently we write $R^{\overline{A}}$, $F^{\overline{A}}$, or $k^{\overline{A}}$ in place of $a(R), b(F), c(k)$. The functions, relations, constants $R^{\overline{A}}$, $F^{\overline{A}}$, $k^{\overline{A}}$ are called *fundamental functions, relations, constants* of A, to distinguish them from arbitrary functions, relations, constants on A. In practice, structures are of three sorts as far as motivation is concerned. If there are no relation symbols, a structure is called an *algebra.* This is in a general sense, not the same as the notion of algebra considered in ring and field theory. If there are no function symbols, or individual constants the structure is called a *relational structure*. The general notion is needed in many special cases, though, for example in talking about ordered fields. The associated signatures of these three sorts are called *algebraic*, *relational*, or *mixed*.

EXAMPLES

For the examples it is convenient to use a looser notation. In particular, if one of Fcn, Rel, Cn is empty, we simply omit it.

• Partial orderings. These are structures $(A, <)$ such that $<$ is irreflexive (for no $x \in A$ is $x < x$) and transitive (for any $x, y, z \in A$, if $x < y$ and $y < z$ then $x < z$). Officially we are dealing with a signature (Fcn,Rel,Cn,ar), where $\mathrm{Fcn} = \emptyset$, Rel is a one element set $\{R\}$, $\mathrm{Cn} = \emptyset$, and ar is the function with domain the one-element set Rel with $\mathrm{ar}(R) = 2$. The particular set R which is the only member of Rel is rather irrelevant.

• Groups. We define a group to be a structure $(G, \cdot, ^{-1}, e)$ such that $\cdot$ is a binary operation on G, $^{-1}$ is a one-place operation on G, and $e \in G$, satisfying the familiar laws. Officially our signature is (Fcn,Rel,Cn,ar), where $\mathrm{Fcn} = \{F, G\}$, $\mathrm{Rel} = \emptyset$, $\mathrm{Cn} = \{e\}$, and ar is the function with domain $\{F, G\}$ with $\mathrm{ar}(F) = 2$ and $\mathrm{ar}(G) = 1$.

• Rings. A ring is a structure $(R, +, \cdot, -, 0)$ satisfying the usual properties, where $-$ is the operation of forming the additive inverse of an element. The official definition should be clear.

• Ordered fields. We add to the fundamental operations for rings a binary relation, and thus consider structures of the form $(F, +, \cdot, -, 0, <)$.

• Vector spaces over a field. Since there are two kinds of objects here, it is not immediately clear how to consider vector spaces as structures in our sense. But a standard for this has arisen. We work over a field F which is the same for all structures. The structures have the form $(V, +, -, 0, f_\alpha)_{\alpha \in F}$, where $(V, +, -, 0)$ is an abelian group and for each $\alpha \in F$, f_α is scalar multiplication by α. Officially the signature is $(\{p, m\} \cup F, \emptyset, \{z\}, ar)$ with $ar(p) = 2$, $ar(m) = 1$, and $ar(\alpha) = 1$ for every $\alpha \in F$. Some of the axioms are as follows: $f_\alpha(f_\beta(a)) = f_{\alpha \cdot \beta}(a)$, $f\alpha(a + b) = f_\alpha(a) + f_\alpha(b)$.

SUBSTRUCTURES

We now consider generalizations of several notions of abstract algebra. These involve more than one structure, but all structures will have the same signature. We say that $\overline{A}$ and $\overline{B}$ are *similar* if they have the same signature.

We say that $\overline{A}$ is a *substructure* of $\overline{B}$ iff the following conditions hold:

$A \subseteq B$

For every relation symbol R, $R^{\overline{A}} = R^{\overline{B}} \cap {}^{\mathrm{ar}(R)}A$.

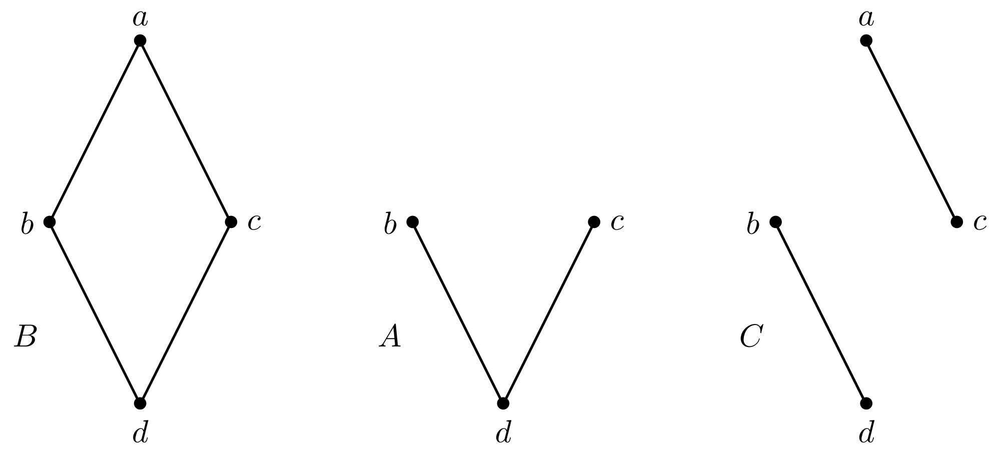
For every function symbol F and all $x \in {}^{\mathrm{ar}(F)}A[F^{\overline{A}}(x) = F^{\overline{B}}(x)]$.

For every individual constant k we have $k^{\overline{A}} = k^{\overline{B}}$.

We write $\overline{A} \leq \overline{B}$ to mean that $\overline{A}$ is a substructure of $\overline{B}$. If $\overline{A} \leq \overline{B}$, then we also call $\overline{B}$ a *superstructure* of $\overline{A}$.

For our main examples we have the following:

Partial orderings. If $(A, <)$ is a substructure of $(B, \prec)$ and $(B, \prec)$ is a partial ordering, then so is $(A, <)$, and $<$ is the intersection of $\prec$ with $A \times A$. This means that $A \subseteq B$ and for any $x, y \in A$, $x < y$ iff $x \prec y$. This is illustrated in the following diagrams, where $(A, <)$ is a substructure of $(B, \prec)$, but $(C, <')$ is not a substructure of $(B, \prec)$.



Groups. If $(H, \circ, f, e')$ is a substructure of a group $(G, \cdot, ^{-1}, e)$, then $(H, \circ, f, e')$ is also a group, H is closed under $\cdot$ and $^{-1}$, $e = e'$, $\circ$ is the restriction of $\cdot$ to $H \times H$, and f is the restriction of $^{-1}$ to H. Note that if we had treated a group as a structure $(G, \cdot)$, then there would be substructures which are not groups. For example, $(\omega, +)$ is a substructure of $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$ is a group but $(\omega, +)$ is not.

Rings. If $(R', +', \cdot', -', 0')$ is a substructure of a ring $(R, +, \cdot, -, 0)$ then $(R', +', \cdot', -', 0')$ is a ring. Note that R might have an identity while R' does not. An example is given by $R = \mathbb{Z}$ and $R' = \{2m : m \in \mathbb{Z}\}$.

Ordered fields. If $(F, +, \cdot, -, 0, <)$ is an ordered field and $(F', +', \cdot', -', 0', <')$ is a substructure of it, then F' is not necessarily an ordered field. An example is given by $F = \mathbb{Q}$ and $F' = \mathbb{Z}$.

Vector spaces. If V is a vector space over a field F, a substructure of V is just a subspace in the usual sense.

If $\overline{A}$ is a structure and $B \subseteq A$ is closed under the fundamental operations of A, then we call B a *subuniverse* of $\overline{A}$. (Saying that B is closed under the fundamental operations of A is supposed to include the assumption that the fundamental constants of A are in B.) Any subuniverse B can be made into a substructure of A in the natural way:

$R^{\overline{B}} = {}^{\mathrm{ar}(R)}B \cap R^{\overline{A}}$;

$F^{\overline{B}}(x) = F^{\overline{A}}(x)$ for all $x \in {}^{\mathrm{ar}(F)}B$.

$k^{\overline{B}} = k^{\overline{A}}$.

Proposition 1.1. *If $\overline{A}$ is a structure and K is a nonempty collection of subuniverses of $\overline{A}$ having at least one common element, then $\bigcap K$ is a subuniverse of $\overline{A}$.* □

This proposition justifies the following definition: if $\emptyset \neq X \subseteq A$, then $\langle X\rangle_A$ is the intersection of all subuniverses of A which contain X. This is called the subuniverse *generated* by X; the associated structure is the substructure *generated* by X. In case the signature has at least one individual constant, we can define $\langle\emptyset\rangle_A$ in the same way.

Note that if there are no function symbols or individual constants, then any nonempty subset of A is a subuniverse.

The following characterization of $\langle X\rangle_A$ is frequently useful.

Proposition 1.2. *Suppose that $\overline{A}$ is a structure and X is a nonempty subset of A. Define $\langle Y_i : i \in \omega\rangle$ by recursion, as follows:*

$$Y_0 = X \cup \{k^{\overline{A}} : k \textit{ an individual constant}\};$$
$$Y_{i+1} = Y_i \cup \{F^{\overline{A}}(x) : F \textit{ a function symbol}, x \in {}^{\mathrm{ar}(F)}Y_i\}.$$

Then $\langle X\rangle_A = \bigcup_{i\in\omega} Y_i$.

Proof. An easy induction shows that $Y_i \subseteq \langle X\rangle_A$ for all $i \in \omega$; so $\supseteq$ holds. Hence it suffices to show that $\bigcup_{i\in\omega} Y_i$ is a subuniverse which contains X. Obviously it contains X and all fundamental constants of $\overline{A}$. Now suppose that F is a function symbol, and $x \in {}^{\mathrm{ar}(F)}\bigcup_{i\in\omega} Y_i$. For each $k < \mathrm{ar}(F)$ let $s(k) < \omega$ be such that $x_k \in Y_{s(k)}$. Let l be the maximum of all $s(k)$ for $k < n_j$. Then $x \in {}^{\mathrm{ar}(F)}Y_l$, and it follows that $F^{\overline{A}}(x) \in Y_{l+1}$, as desired. □

Note that if V is a vector space over F and $X \subseteq V$, then $\langle X\rangle$ is the subspace of V spanned by X.

HOMOMORPHISMS

Again, let $\overline{A}$ and $\overline{B}$ be similar structures. A *homomorphism* from $\overline{A}$ to $\overline{B}$ is a function f mapping the set A into the set B such that the following conditions hold:

- For R an m-ary relation symbol and any sequence $\overline{a}$ of length m of elements of A, $\overline{a} \in R^{\overline{A}}$ iff $f \circ \overline{a} \in R^{\overline{B}}$.

- For F an m-ary operation symbol, if $\overline{a} \in {}^mA$, then $f(F^{\overline{A}}(\overline{a})) = F^{\overline{B}}(f \circ \overline{a})$.

- For any individual constant k we have $f(k^{\overline{A}}) = k^{\overline{B}}$.

A *weak homomorphism* from $\overline{A}$ to $\overline{B}$ is a mapping f satisfying the above conditions for individual constants and function symbols, but with the following weaker condition for relation symbols: For R an m-ary relation symbol and any sequence $\overline{a}$ of length m of elements of A, if $\overline{a} \in R^{\overline{A}}$ then $f \circ \overline{a} \in R^{\overline{B}}$.

In our examples, homomorphisms mean the following.

A homomorphism from a partial ordering $(P, <)$ to a partial ordering $(Q, \prec)$ is a function f mapping P into Q such that for all $a, b \in P$, $a < b$ iff $f(a) \prec f(b)$. Thus in the examples concerning substructures, the identity on A is a homomorphism from A into B, but the identity on C is not a homomorphism from C into B. The identity on C is a weak homomorphism from C into B.

For groups, rings, and ordered fields, homomorphisms have the same meaning as in elementary algebra. For vector spaces, homomorphisms are the same as linear transformations.

An *embedding* or *isomorphism into* or *monomorphism* from $\overline{A}$ to $\overline{B}$ is a one-one homomorphism from $\overline{A}$ into $\overline{B}$. An *isomorphism* is an embedding which is onto. An isomorphism of $\overline{A}$ onto $\overline{A}$ is called an *automorphism* of $\overline{A}$.

We write $\overline{A} \cong \overline{B}$ to indicate that there is an isomorphism from $\overline{A}$ onto $\overline{B}$.

Proposition 1.3. *Suppose that $\overline{A}$ and $\overline{B}$ are similar structures and $f : A \to B$. Then the following conditions are equivalent:*

(i) f is a embedding from $\overline{A}$ into $\overline{B}$.

(ii) f is an isomorphism from $\overline{A}$ onto a substructure of $\overline{B}$.

Proof. (i)⇒(ii): Assume (i). Let $C = \text{rng}(f)$. We check that C is closed under the operations of $\overline{B}$. If F is an m-ary function symbol and $\overline{a} \in {}^mA$, then

$$F^{\overline{B}}(f \circ \overline{a}) = f(F^{\overline{A}}(\overline{a})) \in C.$$

For any individual constant k, $k^{\overline{B}} = f(k^{\overline{A}}) \in C$. Thus C is a subuniverse of $\overline{B}$, and we have a structure $\overline{C}$ which is a substructure of $\overline{B}$. f is an isomorphism from $\overline{A}$ onto $\overline{C}$ since

$$\begin{aligned} f(F^{\overline{A}}(\overline{a})) &= F^{\overline{B}}(f \circ \overline{a}) = F^{\overline{C}}(f \circ \overline{a}); \\ f(k^{\overline{A}}) &= k^{\overline{B}} = k^{\overline{C}}; \\ \overline{a} \in R^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in R^{\overline{B}} \\ &\text{iff} \quad f \circ \overline{a} \in R^{\overline{C}}. \end{aligned}$$

(ii)⇒(i): Assume (ii); say that f is an isomorphism from $\overline{A}$ onto a structure $\overline{C}$ which is a substructure of $\overline{B}$. Then f is an embedding from $\overline{A}$ into $\overline{B}$:

$$\begin{aligned} f(F^{\overline{A}}(\overline{a})) &= F^{\overline{C}}(f \circ \overline{a}) = F^{\overline{B}}(f \circ \overline{a}); \\ f(k^{\overline{A}}) &= k^{\overline{C}} = k^{\overline{B}}; \\ \overline{a} \in R^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in R^{\overline{C}} \\ &\text{iff} \quad f \circ \overline{a} \in R^{\overline{B}}. \end{aligned}$$ □

Proposition 1.4. *Suppose that $\overline{A}$ is a structure, and f is a bijection from A onto a set B. Then there is a structure $\overline{B}$ with universe B such that f is an isomorphism from $\overline{A}$ onto $\overline{B}$.*

Proof. With obvious assumptions, we define

$$\begin{aligned} \mathbf{F}^{\overline{B}}(\overline{b}) &= f(F^{\overline{A}}(f^{-1} \circ \overline{b})); \\ \overline{b} \in \mathbf{R}^{\overline{B}} \quad &\text{iff} \quad f^{-1} \circ \overline{b} \in \mathbf{R}^{\overline{A}}; \\ \mathbf{k}^{\overline{B}} &= f(\mathbf{k}^{\overline{A}}). \end{aligned}$$

The conclusion is easy to check. □

Proposition 1.5. *Suppose that f is an isomorphism from a structure $\overline{A}$ into a structure $\overline{B}$. Then there exist a structure $\overline{C}$ and a function g such that $\overline{A}$ is a substructure of $\overline{C}$, g is an isomorphism from $\overline{C}$ onto $\overline{B}$, and $f \subseteq g$.*

Proof. Let X be a set disjoint from A with the same number of elements as $B \backslash \text{rng}(f)$, and let h be a bijection from X onto $B \backslash \text{rng}(f)$. Let $g = f \cup h$. So g is a bijection from $A \cup X$ onto B. We now apply Proposition 1.4 to $\overline{B}$ and g^{-1} to get a structure $\overline{C}$ with universe $A \cup X$ such that g^{-1} is an isomorphism from $\overline{C}$ onto $\overline{B}$. Hence g is an isomorphism from $\overline{B}$ onto $\overline{C}$. So it remains only to check that $\overline{A}$ is a substructure of $\overline{C}$. With obvious assumptions, we have:

$$\begin{aligned} \mathbf{F}^{\overline{A}}(\overline{a}) &= f^{-1}(f(\mathbf{F}^{\overline{A}}(\overline{a}) \\ &= f^{-1}(\mathbf{F}^{\overline{B}}(f \circ \overline{a})) \\ &= g^{-1}(\mathbf{F}^{\overline{B}}(f \circ \overline{a})) \\ &= \mathbf{F}^{\overline{C}}(g^{-1} \circ f \circ \overline{a}) \\ &= \mathbf{F}^{\overline{C}}(\overline{a}); \\ \overline{a} \in \mathbf{R}^{\overline{A}} \quad &\text{iff} \quad f \circ \overline{a} \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad g \circ \overline{a} \in \mathbf{R}^{\overline{B}} \\ &\text{iff} \quad \overline{a} \in \mathbf{R}^{\overline{C}}; \\ \mathbf{k}^{\overline{A}} &= f^{-1}(f(\mathbf{k}^{\overline{A}})) \\ &= f^{-1}(k^{\overline{B}}) \\ &= g^{-1}(k^{\overline{B}}) \\ &= k^{\overline{C}}. \end{aligned}$$

□

Proposition 1.5 is justification for the common procedure in algebra when one has an isomorphism from a structure $\overline{A}$ into a structure $\overline{B}$, and assumes "without loss of generality" that $\overline{A}$ is actually a substructure of $\overline{B}$.

Let $\overline{A}$ be a structure. A *congruence relation* on $\overline{A}$ is an equivalence relation $\equiv$ on A such that if F is an m-ary function symbol, $\overline{a}$ and $\overline{b}$ are sequences of elements of A of length m, and $a_j \equiv b_j$ for each $j < m_i$, then $F^{\overline{A}}(\overline{a}) \equiv F^{\overline{A}}(\overline{b})$; and if R is an n-ary relation symbol, $\overline{a}$ and $\overline{b}$ are sequences of elements of A of length m, and $a_j \equiv b_j$ for each $j < m_i$, then $\overline{a} \in R^A$ iff $\overline{b} \in R^A$.

The equivalence class of an element a under an equivalence relation R is denoted by $[a]_R$, or simply $[a]$ if R is clear.

Proposition 1.6. *The intersection of a nonempty family of congruence relations on a structure is again a congruence relation on the structure.* □

Proposition 1.7. *If A is a structure and $\equiv$ is a congruence relation on A, then the set $B \stackrel{\text{def}}{=} A/\equiv$ of all equivalence classes under $\equiv$ can be given a structure $\overline{B}$ similar to that of A, such that $k^{\overline{B}} = [k^A]_\equiv$ for any individual constant k, $F^{\overline{B}}([a_0]_\equiv, \ldots, [a_{m-1}]_\equiv) = [F^{\overline{A}}(a_0, \ldots, a_{m-1})]_\equiv$ for any m-ary operation symbol F, and for any n-ary relation symbol R and any $a \in {}^mA$, $\langle [a_0]_\equiv, \ldots, [a_{n-1}]_\equiv \rangle \in R^{\overline{B}}$ iff $\langle a_0, \ldots, a_{m-1} \rangle \in R^A$. Moreover, the function assigning to each $a \in A$ its equivalence class $[a]_\equiv$ is a homomorphism from A onto $\overline{B}$.*

Proof. The definition of congruence relation assures that the definition of $F^{\overline{B}}$ is unambiguous, for any function symbol F. Now for an n-ary relation symbol R, let

$$R^{\overline{B}} = \{\langle [a_0]_\equiv, \ldots, [a_{n-1}]_\equiv \rangle : \langle a_0, \ldots, a_{m-1} \rangle \in R^{\overline{A}}\}.$$

Thus for any $a \in {}^nA$, obviously $a \in R^{\overline{A}}$ implies that $\langle [a_0]_\equiv, \ldots, [a_{n-1}]_\equiv \rangle \in R^{\overline{B}}$. Conversely, suppose that $\langle [a_0]_\equiv, \ldots, [a_{n-1}]_\equiv \rangle \in R^{\overline{B}}$. Then there is a $b \in R^{\overline{A}}$ such that $[a_i]_\equiv = [b_i]_\equiv$ for all $i < n$. So $a_i \equiv b_i$ for all $i < n$, and hence $\langle a_0, \ldots, a_{n-1} \rangle \in R^{\overline{A}}$.

Now the function described at the end of the statement of the proposition is clearly a homomorphism. □

The structure in Proposition 1.7 is denoted by $\overline{A}/\equiv$. This is the *quotient structure* of A under $\equiv$.

If f is a homomorphism from $\overline{A}$ to $\overline{B}$, the *kernel* of f is $\{(a_0, a_1) : a_0, a_1 \in A$ and $f(a_0) = f(a_1)\}$; we denote it by $\ker(f)$.

Proposition 1.8. *If f is a homomorphism from $\overline{A}$ to $\overline{B}$, then $\ker(f)$ is a congruence relation on $\overline{A}$, and $\overline{A}/\ker(f)$ can be isomorphically embedded in $\overline{B}$.*

Proof. Clearly $\ker(f)$ is an equivalence relation on A. Suppose that F is an m-ary function symbol and $\overline{a}$ and $\overline{c}$ are sequences of elements of A of length m, with $(a_i, c_i) \in \ker(f)$ for each $i < m$. Thus $f(a_i) = f(c_i)$ for each $i < m$, and so

$$\begin{aligned} f(F^{\overline{A}}(a_0, \ldots, a_{m-1})) &= F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1})) \\ &= F^{\overline{B}}(f(c_0), \ldots, f(c_{m-1})) \\ &= f(F^{\overline{A}}(c_0, \ldots, c_{m-1})); \end{aligned}$$

thus $(F^{\overline{A}}(a_0, \ldots, a_{m-1}), F^{\overline{A}}(c_0, \ldots, c_{m-1}) \in \ker(f)$. Now suppose that R is an m-ary relation symbol and $(a_i, b_i) \in \ker(f)$ for all $i < m$. Thus $f(a_i) = f(b_i)$ for all $i < m$. Suppose that $\langle a_0, \ldots, a_{m-1}\rangle \in R^{\overline{A}}$. Then $\langle f(a_0), \ldots, f(a_{m-1})\rangle \in R^{\overline{B}}$, hence $\langle f(b_0), \ldots, f(b_{m-1})\rangle \in R^{\overline{B}}$. Hence $\langle b_0, \ldots, b_{m-1}\rangle \in R^{\overline{A}}$. The converse holds by the same argument. So $\ker(f)$ is a congruence relation on $\overline{A}$.

Now let $g = \{([a]_{\ker(f)}, f(a)) : a \in A\}$. We claim that g is an isomorphic embedding from $A/\ker(f)$ into $\overline{B}$. First of all, g is a function and is one-one, since for any $a, c \in A$,

$$[a]_{\ker(f)} = [c]_{\ker(f)} \quad \text{iff} \quad f(a) = f(c).$$

g is a homomorphism, since for an individual constant k we have

$$g(k^{\overline{A}/\ker(f)}) = g([k^{\overline{A}}]_{\ker(f)}) = f(k^{\overline{A}}) = k^{\overline{B}},$$

and for F an m-ary function symbol,

$$\begin{aligned}
g(F^{\overline{A}/\ker(f)}([a_0]_{\ker(f)}, \ldots, [a_{m-1}]_{\ker(f)}) &= g([F^{\overline{A}}(a_0, \ldots, a_{m-1})]_{\ker(f)}) \\
&= f(F^{\overline{A}}(a_0, \ldots, a_{m-1})) \\
&= F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1})) \\
&= F^{\overline{B}}(g([a_0]_{\ker(f)}), \ldots, g([a_{m-1}]_{\ker(f)})),
\end{aligned}$$

and for R an m-ary relation symbol,

$$\begin{aligned}
\langle [a_0]_{\ker(f)}, \ldots, [a_{m-1}]_{\ker(f)}\rangle \in R^{\overline{A}/\ker(f)} \quad &\text{iff} \quad \langle a_0, \ldots, a_{m-1}\rangle \in R^{\overline{A}} \\
&\text{iff} \quad \langle f(a_0), \ldots, f(a_{m-1})\rangle \in R^{\overline{B}} \\
&\text{iff} \quad \langle g([a_0]_{\ker(f)}), \ldots, g([a_{m-1}]_{\ker(f)})\rangle \in R^{\overline{B}}. \square
\end{aligned}$$

PRODUCTS

Suppose that $\langle \overline{A}_i : i \in I\rangle$ is a system of similar structures. We make the set-theoretic product $B \stackrel{\text{def}}{=} \prod_{i \in I} A_i$ into a structure $\prod_{i \in I} \overline{A}_i$ similar to the $\overline{A}_i$'s as follows. For each $i \in I$ let pr_i be the projection of B into A_i, defined by $\mathrm{pr}_i(f) = f_i$ for all $f \in B$. For R an m-ary relation symbol,

$$R^{\overline{B}} = \{f \in {}^mB : \forall i \in I[\mathrm{pr}_i \circ f \in R^{A_i}]\}.$$

For F an m-ary fundamental operation and for any $a \in {}^mB$,

$$F^{\overline{B}}(a) = \langle F^{A_i}(\mathrm{pr}_i \circ a) : i \in I\rangle.$$

for k an individual constant, $k^{\overline{B}} = \langle k^{A_i} : i \in I\rangle$.

This definition works out as follows for our examples.

If $\langle (A_i, <_i) : i \in I \rangle$ is a system of partial orders, then $\prod_{i \in I} \overline{A}_i$ has the relation $\prec$, where $f \prec g$ iff $f_i <_i g_i$ for all $i \in I$.

If $\langle (A_i, \cdot, {}^{-1}, e_i) : i \in I \rangle$ is a system of groups (where we leave off some of the necessary subscripts $_i$), then the operations in $\prod_{i \in I}$ are defined as follows:

$$\begin{aligned} f \cdot g &= \langle f_i \cdot g_i : i \in I \rangle; \\ f^{-1} &= \langle f_i^{-1} : i \in I \rangle; \\ e &= \langle e_i : i \in I \rangle. \end{aligned}$$

If $\langle (A_i, +, \cdot, -, 0_i) : i \in I \rangle$ is a system of rings, then the operations in $\prod_{i \in I} A_i$ are defined as follows:

$$\begin{aligned} f + g &= \langle f_i + g_i : i \in I \rangle; \\ f \cdot g &= \langle f_i \cdot g_i : i \in I \rangle; \\ -f &= \langle -f_i : i \in I \rangle; \\ 0 &= \langle 0_i : i \in I \rangle. \end{aligned}$$

Ordered fields and vector spaces are treated similarly.

Proposition 1.9. *Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures. Let $\overline{B} = \prod_{i \in I} \overline{A}_i$, and let $i \in I$. Then pr_i is a weak homomorphism from $\overline{B}$ onto $\overline{A}_i$.*

Proof. Clearly pr_i maps onto $\overline{A}_i$. If k is an individual constant, then $\mathrm{pr}_i(k^{\overline{B}}) = k^{\overline{A}_i}$. If F is an m-ary function symbol and $f \in {}^m B$. Then

$$\mathrm{pr}_i(F^{\overline{B}}(f)) = (F^{\overline{B}}(f))_i = F^{\overline{A}_i}(\mathrm{pr}_i \circ f).$$

If R is an m-ary relation symbol and $a \in R^{\overline{B}}$, then $\forall j \in I[\mathrm{pr}_j(a) \in R^{\overline{A}_j}]$, and hence $\mathrm{pr}_i(a) \in R^{\overline{A}_i}$. $\square$

An important special case of products is the product of two structures. We define $\overline{A} \times \overline{B}$ to be $\prod_{i \in 2} \overline{C}_i$, where $\overline{C}_0 = \overline{A}$ and $\overline{C}_1 = \overline{B}$.

UNIONS

We can form unions of structures under special circumstances, given by the following theorem. A collection K of subuniverses of a structure A is *directed by* $\subseteq$ iff $\forall A, B \in K \exists C \in K[A \subseteq C$ and $B \subseteq C]$.

Theorem 1.10. *Suppose that $\overline{A}$ is a structure, and K is a nonempty collection of subuniverses of $\overline{A}$ directed by $\subseteq$. Then $\bigcup K$ is a subuniverse of A, such that B is a subuniverse of the corresponding structure $\bigcup K$ for all $B \in K$.*

Proof. Take any n-ary function symbol F and any $x \in {}^n\bigcup K$. For each $j < n$ choose $B_j \in K$ such that $x_j \in B_j$. The condition of the theorem then implies that there is a $C \in K$ such that $B_j \subseteq C$ for all $j < n_i$. Hence $F^{\overline{A}}(x) \in C \subseteq \bigcup K$. $\square$

A subuniverse B of a structure $\overline{A}$ is *finitely generated* iff there is a finite nonempty set F such that $B = \langle F \rangle_A$.

Theorem 1.11. *Any structure is the union of its finitely generated substructures.*

Proof. Let K be the set of all finitely generated substructures of A. The condition of Theorem 1.10 clearly holds, and so $\bigcup K$ is a subuniverse of $\overline{A}$. But clearly $A = \bigcup K$, since for any $a \in A$ we have $\langle \{a\} \rangle_A \in K$. $\square$

ULTRAPRODUCTS

Ultraproducts play an important role in model theory. Here we are not dealing with a generalization of a common notion; ultraproducts are a new thing, although related to products and quotients.

First we need to discuss the purely set-theoretic notion of ultrafilters. Let I be a nonempty set. An *ultrafilter* on I is a collection D of subsets of I satisfying the following conditions:

(1) $I \in D$,

(2) $\emptyset \notin D$.

(3) For all $a, b \in D$, also $a \cap b \in D$.

(4) For all $a \in D$ and all $b \subseteq I$ with $a \subseteq b$, also $b \in D$.

(5) For any $a \subseteq I$, either $a \in D$ or $(I \backslash a) \in D$.

Note that in (5) the "or" is exclusive. Indeed, if $a \in D$ and $(I \backslash a) \in D$, then $\emptyset \in D$ by (3), contradicting (2).

If D satisfies (1)–(4), it is called a *filter* on I. A filter is *proper* iff it is different from $\mathscr{P}(I)$. Note that D is proper iff $\emptyset \notin D$, by (4).

Proposition 1.12. *Let D be a proper filter on a nonempty set I. Then D is an ultrafilter iff it is maximal among proper filters.*

Proof. First suppose that D is an ultrafilter, and $D \subset F$ with F a filter. Say $a \in F \backslash D$. Then $(I \backslash a) \in D$ by (5), and so $\emptyset = a \cap (I \backslash a) \in F$ by (3), so that F is not proper.

Second suppose that D is a filter which is maximal among proper filters. We verify condition (5). Suppose that $a \subseteq I$ and $a \notin D$. Let

$$F = \{x \subseteq I : y \backslash a \subseteq x \text{ for some } y \in D\}.$$

Clearly F is a filter. It is proper, since $\emptyset \in F$ would imply that $y \backslash a = \emptyset$ for some $y \in D$, hence $y \subseteq a$, hence $a \in D$ by (4), contradiction. It follows that $D = F$, and hence $(I \backslash a) \in D$. $\square$

A family $\mathscr{A}$ of subset of a set I has the *finite intersection property* (fip) iff every finite subset of $\mathscr{A}$ has nonempty intersection. A connection of this notion with ultrafilters is as follows.

Theorem 1.13. *Let D be a collection of subsets of a set I. Then D is an ultrafilter on I iff D is maximal with respect to the property of having fip.*

Proof. $\Rightarrow$ is clear. Now assume that D is maximal with respect to the property of having fip. $D \cup \{I\}$ clearly still has fip, so it is equal to D, so that $I \in D$. Obviously $\emptyset \notin D$. Suppose that $a, b \in D$. Then $D \cup \{a \cap b\}$ still has fip, and it follows that $a \cap b \in D$. Similarly, $a \in D$ and $a \subseteq b \subseteq I$ imply that $b \in I$. Finally, suppose that $a \subseteq I$ and $a \notin D$. Then $D \cup \{a\}$ no longer has fip, so there is a finite subset F of D such that $a \cap \bigcap F = \emptyset$. Hence $\bigcap F \subseteq (I \backslash a)$, and so $(I \backslash a) \in D$. $\square$

A basic theorem concerning ultrafilters is as follows.

Theorem 1.14. *Suppose that I is a nonempty set, and $\mathscr{A}$ is a family of subsets of I with the fip. Then there is an ultrafilter D on I such that $\mathscr{A} \subseteq D$.*

Proof. Let $\mathscr{B}$ be the collection of all families $\mathscr{C}$ of subsets of I such that $\mathscr{A} \subseteq \mathscr{C}$ and $\mathscr{C}$ has fip. We partially order $\mathscr{B}$ by inclusion. Then the hypothesis of Zorn's lemma holds for $\mathscr{B}$. In fact, suppose that $\mathscr{D}$ is a collection of members of $\mathscr{B}$ linearly ordered by inclusion. Let $\mathscr{E} = \bigcup \mathscr{D}$. We claim that $\mathscr{E}$ has fip. For, suppose that F is a finite subset of $\mathscr{E}$. For each $X \in F$ choose $\mathscr{F}_X \in \mathscr{E}$ such that $X \in \mathscr{F}_X$. Since $\mathscr{E}$ is linearly ordered by inclusion, there is a $Z \in F$ such that $\mathscr{F}_X \subseteq \mathscr{F}_Z$ for all $X \in F$. Then, since $\mathscr{F}_X$ has fip, we get $\bigcap F \neq \emptyset$. This verifies that the hypothesis of Zorn's lemma holds. Hence we can let D be a maximal member of $\mathscr{B}$. So, D has fip, and is maximal with this property. By Theorem 1.13, D is as desired. $\square$

This ends our set-theoretic interlude concerning ultrafilters.

Theorem 1.15. *Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures and D is a ultrafilter on I. For brevity let $B = \prod_{i \in I} A_i$. Define $\equiv_D$, a relation on B, by*

$$f \equiv_D g \quad \textit{iff} \quad \{i \in I : f_i = g_i\} \in D.$$

Then $\equiv_D$ is an equivalence relation on B, and it satisfies the conditions for a congruence relation on B as far as individual constants and function symbols are concerned. Moreover, if R is an n-ary relation symbol, $a, b \in {}^nB$ and $a_j \equiv_D b_j$ for all $j < n$, then

$$\{i \in I : \mathrm{pr}_i \circ a \in R^{A_i}\} \in D \quad \text{iff} \quad \{i \in I : \mathrm{pr}_i \circ b \in R^{A_i}\} \in D.$$

Proof. Clearly $\equiv_D$ is reflexive and symmetric. For transitivity, suppose that $a \equiv_D b \equiv_D c$. Now

$$\{i \in I : a_i = b_i\} \cap \{i \in I : b_i = c_i\} \subseteq \{i \in I : a_i = c_i\},$$

and the left side is in D; hence also the right side is in D, and it follows that $a \equiv_D c$. So $\equiv_D$ is an equivalence relation on B.

Suppose that F is an m-ary function symbol and $f_j, f'_j \in B$ for $j < m$, with $f_j \equiv_D f'_j$ for all $j < m$. Let $g = F^{\overline{B}}(f_0, \ldots, f_{m-1})$ and $g' = F^{\overline{B}}(f'_0, \ldots, f'_{m-1})$. Then

$$\bigcap_{j<m} \{i \in I : f_j(i) = f'_j(i)\} \subseteq \{i \in I : g(i) = g'(i)\},$$

and the first set is in D, so that the second set is also, proving that $g \equiv_D g'$.

Now assume that R is an n-ary relation symbol, $a, b \in {}^nB$ and $a_j \equiv_D b_j$ for all $i < n$. Then

$$\begin{aligned}
&\bigcap_{j<n} \{i \in I : a_j(i) = b_j(i)\} \cap \{i \in I : \mathrm{pr}_i \circ a \in R^{A_i}\} \\
&\quad = \bigcap_{j<n} \{i \in I : a_j(i) = b_j(i)\} \cap \{i \in I : \mathrm{pr}_i \circ b \in R^{A_i}\},
\end{aligned}$$

and the desired conclusion follows. $\square$

This theorem justifies the following definition. We make the collection $C \stackrel{\text{def}}{=} \prod_{i\in I} A_i/{\equiv_D}$ of equivalence classes under $\equiv_D$ into a structure similar to the A_i's as follows. Let $B = \prod_{i\in I} A_i$.

$$\begin{aligned}
F^{\overline{C}}([a_0], \ldots, [a_{m-1}]) &= [F^{\overline{B}}(a_0, \ldots, a_{m-1})]; \\
([a'_0], \ldots, [a'_{n-1}]) \in R^{\overline{C}} \quad &\text{iff} \quad \{i \in I : (a'_0(i), \ldots, a'_{n-1}) \in R^{A_i}\} \in D; \\
k^{\overline{C}} &= [k^{\overline{B}}],
\end{aligned}$$

where F is an m-ary function symbol, R is an n-ary relation symbol, k is a n individual constant, and each a_i and a'_i is in B. Note that $[f]$ denotes the equivalence class of f under $\equiv_F$.

The structure with universe C so defined is called the *ultraproduct* of $\langle \overline{A}_i : i \in I\rangle$ and is denoted by $\prod_{i\in I} \overline{A}_i/D$.

Theorem 1.16. *Suppose that K is a nonempty collection of subuniverses of a structure A, directed by $\subseteq$. Then there is an ultrafilter D on K such that $\bigcup K$ can be isomorphically embedded in $\prod_{B\in K} B/{\equiv_D}$.*

Proof. For brevity let $M = \prod_{B\in K} B$; $\overline{M}$ is the corresponding structure. For each $B \in K$ let $L_B = \{C \in K : B \subseteq C\}$.

(1) $\{L_B : B \in K\}$ has fip.

For, let $X \in [K]^{<\omega}$. Choose $C \in K$ such that $B \subseteq C$ for all $B \in X$. Then $C \in \bigcap_{B\in X} L_B$.

By (1), let D be an ultrafilter on K containing $\{L_B : B \in K\}$. Let $\overline{N} = \overline{M}/{\equiv_D}$. Fix $a_B \in B$ for every $B \in K$. We define $f : \bigcup K \to M$ by setting, for any $b \in \bigcup K$ and $B \in K$,

$$(f(b))_B = \begin{cases} b & \text{if } b \in B, \\ a_B & \text{otherwise.} \end{cases}$$

Then let $g(b)=[f(b)]$ for every $b\in\bigcup K$. So $g:\bigcup K\to N$.

Now to show that g is a homomorphism, first suppose that $\mathbf{F}$ is an m-ary function symbol and $b_0,\ldots,b_{m-1}\in\bigcup K$.

(2) $[f(\mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1}))]=[\mathbf{F}^{\overline{M}}(f(b_0),\ldots,f(b_{m-1}))]$. ‘

In fact, choose $B\in K$ such that $b_0,\ldots,b_{m-1}\in B$. Then for any $C\in L_B$ we have $\mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1})\in C$, and

$$\begin{aligned}(f((\mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1}))_C &= \mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1})\\ &= \mathbf{F}^{\overline{A}}((f(b_0))_C,\ldots,(f(b_{m-1}))_C)\\ &= (\mathbf{F}^{\overline{M}}(f(b_0),\ldots,f(b_{m-1})))_C,\end{aligned}$$

and (2) follows.

Now we calculate:

$$\begin{aligned}\mathbf{F}^{\overline{N}}(g(b_0),\ldots,g(b_{m-1})) &= \mathbf{F}^{\overline{N}}([f(b_0)],\ldots,[f(b_{m-1})])\\ &= [\mathbf{F}^{\overline{M}}(f(b_0),\ldots,f(b_{m-1}))]\\ &= [f(\mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1}))] \quad \text{by (2)}\\ &= g(\mathbf{F}^{\overline{A}}(b_0,\ldots,b_{m-1}))\\ &= g(\mathbf{F}^{\overline{\bigcup K}}(b_0,\ldots,b_{m-1})).\end{aligned}$$

Next, let $\mathbf{R}$ be an n-ary relation symbol and $b_0,\ldots,b_{n-1}\in\bigcup K$. Choose $B\in K$ such that $b_0,\ldots,b_{n-1}\in B$. Then

$$\begin{aligned}\langle b_0,\ldots,b_{n-1}\rangle\in\mathbf{R}^{\overline{\bigcup K}} \quad &\text{iff}\quad \langle b_0,\ldots,b_{n-1}\rangle\in\mathbf{R}^{\overline{B}}\\ &\text{iff}\quad \langle (f(b_0))_B,\ldots,(f(b_{n-1}))_B\rangle\in\mathbf{R}^{\overline{B}}\\ &\text{iff}\quad L_B\cap\{C\in K:\langle (f(b_0))_C,\ldots,(f(b_{n-1}))_C\rangle\in\mathbf{R}^{\overline{C}}\}\in D\\ &\text{iff}\quad \{C\in K:\langle (f(b_0))_C,\ldots,(f(b_{n-1}))_C\rangle\in\mathbf{R}^{\overline{C}}\}\in D\\ &\text{iff}\quad \langle [f(b_0)],\ldots,[f(b_{n-1})]\rangle\in\mathbf{R}^{N}\\ &\text{iff}\quad \langle g(b_0),\ldots,g(b_{n-1})\rangle\in\mathbf{R}^{N}.\end{aligned}$$

For $\mathbf{k}$ an individual constant,

$$g(\mathbf{k}^{\overline{\bigcup K}})=[f(\mathbf{k}^{\overline{A}})]=[\mathbf{k}^{\overline{M}}]=\mathbf{k}^{\overline{N}}.$$

It remains only to show that g is one-one. Suppose that $b,c\in\bigcup K$ with $b\neq c$. Choose $B\in K$ such that $b,c\in B$. Then for any $C\in L_B$ we have $(f(b))_C=b\neq c=(f(c))_C$. So $L_B\subseteq\{C\in K:(f(b))_C\neq(f(c))_C\}$, so that $\{C\in K:(f(b))_C\neq(f(c))_C\}\in D$. Hence $f(b)\not\equiv_D f(c)$, and so $g(b)=[f(b)]\neq[f(c)]=g(c)$. □

Corollary 1.17. *Any structure can be isomorphically embedded in an ultraproduct of its finitely generated substructures.* □

Exc. 1.1. Let $\mathscr{L}$ be a language with no individual constants. Define an $\mathscr{L}$-structure $\overline{A}$ and subuniverses B, C of $\overline{A}$ such that $B \cap C = \emptyset$.

Exc. 1.2. Carry out the "easy induction" at the beginning of the proof of Proposition 1.2.

Exc. 1.3. If X and Y are nonempty subsets of the universe A of an algebra $\overline{A}$, then $\langle X \cup \langle Y \rangle \rangle = \langle X \cup Y \rangle$.

Exc. 1.4. If K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$, then $\langle \bigcup K \rangle = \langle \bigcup_{X \in K} \langle X \rangle \rangle$.

Exc. 1.5. Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{A}$. Show that $f[C]$ is a subuniverse of $\overline{B}$.

Exc. 1.6. Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{B}$. Show that $f^{-1}[C]$ is a subuniverse of $\overline{A}$.

Exc. 1.7. If X generates $\overline{A}$, f and g are homomorphisms from $\overline{A}$ into $\overline{B}$, and $f \restriction X = g \restriction X$, then $f = g$.

Exc. 1.8. If f is a homomorphism from $\overline{A}$ into $\overline{B}$ and X is a nonempty subset of A, then $f[\langle X \rangle] = \langle f[X] \rangle$.

Exc. 1.9. If $\overline{A}$ is a substructure of $\overline{B}$ and $\equiv$ is a congruence relation on $\overline{B}$, then $\equiv \cap (A \times A)$ is a congruence relation on $\overline{A}$.

Exc. 1.10. Suppose that R is a congruence relation on $\overline{A}$, and S is a congruence relation on $\overline{A}/R$. Define $T = \{(a_0, a_1) \in A \times A : ([a_0]_R, [a_1]_R) \in S\}$. Show that T is a congruence relation on A and $R \subseteq T$.

Exc. 1.11. (Continuing exercise 1.10) Show that the procedure of exercise 1.10 establishes a one-one order-preserving correspondence between congruence relations on A/R and those congruence relations on A with include R.

Exc. 1.12. Suppose that $\langle \overline{A}_i : i \in I \rangle$ is a system of similar structures, and $\overline{B}$ is another structure similar to them. Suppose that f_i is a homomorphism from $\overline{B}$ into $\overline{A}_i$ for each $i \in I$. Show that there is a homomorphism g from $\overline{B}$ into $\prod_{i \in I} \overline{A}_i$ such that $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$.

Exc. 1.13. Show that a product of partial orderings is a partial ordering.

Exc. 1.14. A partial ordering $(A, <)$ is a *linear ordering* iff for any two distinct $x, y \in A$ we have $x < y$ or $y < x$. Give an example of two linear orderings whose product is not a linear ordering.

Exc. 1.15. Give an example of two ordered fields whose product is not even a field.

Exc. 1.16. Let F be a proper filter on a set I. Show that F is an ultrafilter iff for all $a, b \subseteq I$, if $a \cup b \in F$ then $a \in F$ or $b \in F$.

Exc. 1.17. Show that any ultraproduct of linear orderings is a linear ordering.

Exc. 1.18. Suppose that I is a nonempty set, and $\langle J_i : i \in I\rangle$ is a system of nonempty sets. Also suppose that F_i is an ultrafilter on J_i for each $i \in I$, and G is an ultrafilter on I. Let $K = \{(i,j) : i \in I, j \in J_i\}$, and define

$$H = \{X \subseteq K : \{i \in I : \{j \in J_i : (i,j) \in X\} \in F_i\} \in G\}.$$

Show that H is an ultrafilter on K.

Exc. 1.19. Under the notation of exercise 1.18, show that there is an isomorphism f of the structure $\prod_{i\in I}(\prod_{j\in J_i}\overline{A_{ij}}/F_i)/G)$ onto $\prod_{(i,j)\in K}\overline{A_{ij}}/H$ such that:

$$\forall r \in \prod_{i\in I}\left(\prod_{j\in J_i}\overline{A_{ij}}/F_i)/G\right)\left[\left[r = [s]_G \text{ with } s \in \prod_{i\in I}\left(\prod_{j\in J_i}\overline{A_{ij}}/F_i\right)\right]\right.$$
$$\left.\text{and } \forall i \in I\left[s_i = [t_i]_{F_i} \text{ with } t_i \in \prod_{j\in J_i}\overline{A_{ij}}\right] \text{ implies that } f(r) = [\langle t_i(j) : (i,j) \in K\rangle]_H\right].$$

2. Terms and varieties

We now make one important step towards full model theory: terms and equations. Let σ be any signature. We assume given now a simple infinite sequence $v_0, v_1, \ldots$ of distinct objects called *variables*, different from the relation symbols, function symbols, and individual constants of σ. We define the notion of a *term* over σ; these are certain finite sequences of members of the set

$$\mathrm{Fcn} \cup \mathrm{Rel} \cup \mathrm{Cn} \cup \{v_i : i < \omega\}.$$

The collection of terms is the intersection of all sets A of nonzero sequences of members of this set such that:

(1) For any variable v_i, the sequence $\langle v_i \rangle$ is in A.

(2) For any constant k, the sequence $\langle k \rangle$ is in A.

(3) For any m-ary function symbol F and any members $\tau_0, \ldots, \tau_{m-1}$ of A, the sequence

$$\langle F \rangle^\frown \tau_0^\frown \cdots ^\frown \tau_{m-1}$$

is also in A.

Note that we distinguish between an object a and the sequence $\langle a \rangle$ consisting of that object alone. The operation $^\frown$ of *concatenation* is defined by

$$\langle a_0, \ldots, a_{m-1} \rangle^\frown \langle b_0, \ldots, b_{n-1} \rangle = \langle a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1} \rangle.$$

We will usually write $F\sigma_0 \ldots \sigma_{m-1}$ rather than $\langle F \rangle^\frown \tau_0^\frown \cdots ^\frown \tau_{m-1}$.

In our examples, terms look like this. For partial orderings, the only terms are the variables (or rather the sequences $\langle v_i \rangle$). For groups some examples of terms are Fv_0e, $Fv_0Fv_1v_2$, $FFv_0v_1v_2$, GFv_0v_1. In more familiar terminology these are $v_0 \cdot e$, $v_0 \cdot (v_1 \cdot v_2)$, $(v_0 \cdot v_1) \cdot v_2$, and $(v_0 \cdot v_1)^{-1}$. For rings, the terms are the polynomials. Examples, with F, G the binary function symbols, H the unary one, and Z the constant, are Z, HZ, Hv_3, $FFGv_0v_0Fv_0v_0v_1$, which one would normally write as 0, -0, $-v_3$, $(v_0^2 + 2v_0) + v_1$. The terms for ordered fields are the same as those for rings.

Proposition 2.1. *(i) No proper initial segment of a term is a term.*

(ii) If τ is a term, then exactly one of the following conditions holds:

(a) τ is a constant.

(b) τ is a variable.

(c) There exist a function symbol F, say of arity m, and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $F\sigma_0 \ldots \sigma_{m-1}$.

(iii) If F and G are function symbols, say of arities m and n respectively, and if $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1}$ are terms, and if $F\sigma_0 \ldots \sigma_{m-1}$ is equal to $G\tau_0, \ldots \tau_{n-1}$, then $F = G$, $m = n$, and $\sigma_i = \tau_i$ for all $i < m$.

Proof.

(i) holds by "induction on terms": the set of terms for which it is true clearly contains the variables and constants and an easy argument shows that if it holds for $\sigma_0, \ldots, \sigma_{m-1}$, then it also holds for $F\sigma_0 \ldots \sigma_{m-1}$.

(ii) is clear, and (iii) follows from (i). $\square$

The terms form a rudimentary kind of logic. We now make the first connection between logic and structures. Let σ be a term and $\overline{A}$ a structure. For each $a \in {}^{\omega}A$ we define the *value of σ under the assignment a*, denoted by $\sigma^{\overline{A}}(a)$, recursively as follows:

$$\begin{aligned}
k^{\overline{A}}(a) &= k^{\overline{A}} \quad \text{for each constant } k\\
v_i^{\overline{A}}(a) &= a_i;\\
(F\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= F^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))\\
&\qquad \text{with } F \text{ an } m\text{-ary function symbol and } \sigma_0, \ldots, \sigma_{m-1} \text{ terms.}
\end{aligned}$$

Note that Proposition 2.1(iii) is needed to see that this definition is unambigous.

In concrete cases we can just use the intuition that $\sigma^{\overline{A}}(a)$ is the result of replacing each variable v_i in σ by a_i, each individual constant $\mathbf{k}$ by $\mathbf{k}^{\overline{A}}$, each function symbol $\mathbf{F}$ by $\mathbf{F}^{\overline{A}}$, and evaluating the result. The precise definition above is needed when we want to prove something about the evaluation of terms in general.

We give some examples in the case of the group Z of integers under addition, using a looser notation than the official one.

$$\begin{aligned}
(v_0 + -v_1)(0, 1, 2, \ldots) &= 0 - 1 = -1;\\
(v_0 + -v_1)(0, -1, -2, \ldots) &= 0 + 1 = 1;\\
((v_5 + 0) + -v_3)(0, 1, 2, \ldots) &= 5 - 3 = 2;\\
((v_5 + 0) + -v_3)(0, -1, -2, \ldots) &= -5 + 3 = -2.
\end{aligned}$$

Proposition 2.2. *If a and b are assignments, τ is a term, the variables occurring in τ are among $v_0, \ldots, v_{m-1}$, and $a_i = b_i$ for all $i < m$, then $\tau^A(a) = \tau^A(b)$.*

Proof. Induction on terms. $\square$

Because of this proposition, we can write $\tau^A(a)$ for a finite sequence a which covers all the variables of τ.

Proposition 2.3. *If $\overline{A}$ is generated by X, then every element of A has the form $\tau^{\overline{A}}(a)$ for some sequence a of elements of X and some term τ. If $\overline{x}$ is a sequence of elements of A whose range generates $\overline{A}$, then every element of A has the form $\tau^{\overline{A}}(\overline{x})$ for some term τ.* $\square$

Equations and varieties

We introduce now one more part of logic: equality. For a given signature (Fcn,Rel,Cn,ar), we suppose that $\mathbf{e}$ is an object not in the set

$$\mathrm{Fcn} \cup \mathrm{Rel} \cup \mathrm{Cn} \cup \{v_i : i < \omega\}.$$

An *equation* is a sequence of the form $\langle \mathbf{e}\rangle^\frown\sigma^\frown\tau$ with σ and τ terms. This sequence is denoted by $\sigma = \tau$. If $\overline{A}$ is a structure, then we say that $\sigma = \tau$ *holds in* $\overline{A}$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ for every $a \in {}^\omega A$.

For the rest of this chapter, suppose that our signature is algebraic. For any set E of equations, we let $\mathbf{Mod}(E)$ be the class of all structures in which every member of E holds. Note that $\mathbf{Mod}(E)$ is a proper class in the set-theoretic sense; the class of all groups is an example. It is not necessary to completely understand the difference between ordinary sets and proper classes in order to follow the development here. For $\mathbf{K}$ a class of structures, we define

$$\begin{aligned}
\mathbf{SK} &= \{\overline{A} : \overline{A} \text{ is a subalgebra of some member of } \mathbf{K}\};\\
\mathbf{HK} &= \{\overline{A} : \overline{A} \text{ is a homomorphic image of some member of } \mathbf{K}\};\\
\mathbf{PK} &= \{\overline{A} : \overline{A} \text{ is isomorphic to a product of members of } \mathbf{K}\}.
\end{aligned}$$

A class $\mathbf{K}$ of structures is a *variety* iff there is a set E of equations such that $\mathbf{K} = \mathbf{Mod}(E)$.

We are now going to work towards Birkhoff's theorem that $\mathbf{K}$ is a variety iff it is closed under the operations $\mathbf{H}, \mathbf{S}, \mathbf{P}$. This is a typical model-theoretic result, showing the equivalence between a logical notion (a variety) and a notion not involving logic (closure under $\mathbf{H}, \mathbf{S}, \mathbf{P}$).

Proposition 2.4. *Let $\overline{A}$ be a substructure of $\overline{B}$.*

(i) If σ is a term and $a \in {}^\omega A$ then $\sigma^{\overline{A}}(a) = \sigma^{\overline{B}}(a)$.

(ii) If $\sigma = \tau$ is an equation holding in $\overline{B}$, then it also holds in $\overline{A}$. □

Proposition 2.5. *Suppose that h is a homomorphism from $\overline{A}$ into $\overline{B}$.*

(i) If τ is a term and $a \in {}^\omega A$ is an assignment, then $h(\tau^{\overline{A}}(a)) = \tau^{\overline{B}}(h \circ a)$.

(ii) If $\sigma = \tau$ holds in $\overline{A}$ and h is a surjection, then $\sigma = \tau$ holds in $\overline{B}$.

Proof. (i): by induction on τ:

$$\begin{aligned}
h(v_i^{\overline{A}}(a)) &= h(a_i) = (h \circ a)_i = v_i^{\overline{B}}(h \circ a);\\
h(k^{\overline{A}}(a)) &= h(k^{\overline{A}}) = k^{\overline{B}} = k^{\overline{B}}(a) \quad \text{for } k \text{ an individual constant}\\
h((F\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a)) &= h(F^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)))\\
&= F^{\overline{B}}(h(\sigma_0^{\overline{A}}(a)), \ldots, h(\sigma_{m-1}^{\overline{A}}(a)))\\
&= F^{\overline{B}}(\sigma_0^{\overline{B}}(h \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(h \circ a))\\
&= (F\sigma_0 \ldots \sigma_{m-1})^{\overline{B}}(h \circ a)\\
&\qquad \text{for } F \text{ an } m\text{-ary function symbol}\\
&\qquad \text{and } \sigma_0, \ldots, \sigma_{m-1} \text{ terms.}
\end{aligned}$$

(ii): Assume that $\sigma = \tau$ holds in $\overline{A}$ and h is a surjection. Take any $b \in {}^\omega B$. Choose $a \in {}^\omega A$ such that $h \circ a = b$. Then, using (i),

$$\sigma^{\overline{B}}(b) = \sigma^{\overline{B}}(h \circ a) = h(\sigma^{\overline{A}}(a)) = h(\tau^{\overline{A}}(a)) = \tau^{\overline{B}}(h \circ a) = \tau^{\overline{B}}(b).$$ □

Proposition 2.6. *Let $\langle \overline{A_i} : i \in I\rangle$ be a system of similar structures. Let $\overline{B} = \prod_{i\in I} \overline{A_i}$.*

(i) If σ is a term, $a \in {}^\omega B$. and $i \in I$, then $(\sigma^{\overline{B}}(a))_i = \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a)$.

(ii) If $\sigma = \tau$ holds in each $\overline{A_i}$, then it also holds in $\overline{B}$.

Proof. (i) holds by Propositions 1.9 and 2.5. For (ii), suppose that $\sigma = \tau$ holds in each $\overline{A_i}$. Take any $a \in {}^\omega B$ and $i \in I$. Then, using (i),

$$(\sigma^{\overline{B}}(a))_i = \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a) = \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) = (\tau^{\overline{B}}(a))_i;$$

since i is arbitrary, it follows that $\sigma^{\overline{B}}(a) = t^{\overline{B}}(a)$. □

Corollary 2.7. *If* **K** *is a variety, then it is closed under* **H**, **S**, **P**. □

Theorem 2.8. (Birkhoff) *suppose that* **K** *is a class of similar algebraic structures. Then* **K** *is a variety iff it is closed under* **H**, **S**, *and* **P**.

Proof. $\Rightarrow$ holds by Corollary 2.7. Now suppose that **K** is closed under **H**, **S**, and **P**. Let Γ be the set of all equations holding in every member of **K**. We claim that $\mathbf{K} = \mathbf{Mod}(\Gamma)$. $\subseteq$ is clear. Now suppose that $\overline{A}' \in \mathrm{Mod}(\Gamma)$; we want to show that $\overline{A}' \in \mathbf{K}$. By Corollary 1.17 we may assume that $\overline{A}'$ is finitely generated.

An auxiliary role will now be played by the following algebra $\mathfrak{F}$, called the absolutely free algebra. Its elements are the terms in our language, and if F is an m-ary function symbol and $\sigma_0, \ldots, \sigma_{m-1}$ are terms, then $F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}) = F\sigma_0 \ldots \sigma_{m-1}$. For any individual constant k, we define $k^{\mathfrak{F}} = k$. Let $L = \{(\sigma, \tau) : \sigma, \tau$ are terms and $\sigma = \tau$ holds in every member of $\mathbf{K}\}$.

(1) $L = \bigcap\{R : R$ is a congruence on $\mathfrak{F}$ and $\mathfrak{F}/R$ can be isomorphically embedded in some member of $\mathbf{K}\}$.

In fact, for $\supseteq$ suppose that (σ, τ) is a member of the right side; we want to show that $(\sigma, \tau) \in L$, i.e., that $\sigma = \tau$ holds in every member of **K**. So suppose that $\overline{A} \in K$ and $a \in {}^\omega A$. Define $f(\sigma) = \sigma^{\overline{A}}(a)$ for every term σ. Then f is a homomorphism from $\mathfrak{F}$ into $\overline{A}$:

$$\begin{aligned} f(F^{\mathfrak{F}}(\rho_0, \ldots, \rho_{m-1})) &= f(F\rho_0 \ldots \rho_{m-1}) \\ &= (F\rho_0 \ldots \rho_{m-1})^{\overline{A}}(a) \\ &= F^{\overline{A}}(\rho_0^{\overline{A}}(a), \ldots, \rho_{m-1}^{\overline{A}}(a)) \\ &= F^{\overline{A}}(f(\rho_0), \ldots, f(\rho_{m-1})). \end{aligned}$$

For k an individual constant, $f(k^{\mathfrak{F}}) = f(k) = k^{\overline{A}}(a) = k^{\overline{A}}$. Say that $\overline{B}$ is the range of f. Then f is a homomorphism from $\mathfrak{F}$ onto $\overline{B}$, and so by Theorem 1.8, $\mathfrak{F}/\ker(f)$ can be isomorphically embedded in $\overline{A}$. It follows that $(\sigma, \tau) \in \ker(f)$. This means that $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, as desired.

Conversely, suppose that $(\sigma,\tau) \in L$, R is a congruence relation on $\mathfrak{F}$, and f is an isomorphism from $\mathfrak{F}/R$ into $\overline{A} \in K$. We want to show that $(\sigma,\tau) \in R$. Define $a_i = f([v_i]_R)$ for every $i < \omega$.

(2) $\rho^{\overline{A}}(a) = f([\rho]_R)$ for every term ρ.

We prove (2) by induction on ρ. $v_i^{\overline{A}}(a) = a_i = f([v_i]_R)$, as desired. If k is an individual constant, then $k^{\overline{A}}(a) = k^{\overline{A}} = f([k^{\mathfrak{F}}]_R$. Now assume that $\sigma_i^{\overline{A}}(a) = f([\sigma_i]_R)$ for every $i < m$. Then

$$
\begin{aligned}
(F\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= F^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \\
&= F^{\overline{A}}(f([\sigma_0]_R), \ldots, f([\sigma_{m-1}]_R)) \\
&= f(F^{\overline{A}/R}([\sigma_0]_R, \ldots, [\sigma_{m-1}]_R)) \\
&= f([F\sigma_0 \ldots \sigma_{m-1}]_R),
\end{aligned}
$$

finishing the inductive proof of (2).

By (2), since $\sigma = \tau$ holds in $\overline{A}$,

$$f([\sigma]_R) = \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) = f([\tau]_R).$$

Since f is one-one, it follows that $[\sigma]_R = [\tau]_R$, so that $(\sigma,\tau) \in R$, finishing the proof of (1).

(3) $\mathfrak{F}/L \in \mathbf{K}$.

Let $M = \{R : R$ is a congruence relation on $\mathfrak{F}$ such that $\mathfrak{F}/R$ can be isomorphically embedded in a member of $\mathbf{K}\}$. For each $R \in M$, let f_R be an isomorphism from $\mathfrak{F}/R$ into $\overline{A}_R \in \mathbf{K}$. Now if $R \in M$ and $(\sigma,\tau) \in L$, then also $(\sigma,\tau) \in R$, by (1). Hence there is a function g from $\mathfrak{F}/L$ into $\prod_{R\in M} \overline{A}_R$ such that $(g([\sigma]_L))_R = f_R([\sigma]_R)$ for every $\sigma \in \mathfrak{F}$ and $R \in M$. Now g is a homomorphism. To see this, let $\overline{B} = \prod_{R\in M} \overline{A}_R$. Then for an individual constant k we have $(g([k^{\mathfrak{F}}]_L))_R = f_R([k^{\mathfrak{F}}]_R) = k^{\overline{A}_R}$. For an m-ary operation symbol F,

$$
\begin{aligned}
g(F^{\mathfrak{F}/L}([\sigma_0]_L, \ldots, [\sigma_{m-1}]_L) &= g([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1})]_L) \\
&= \langle (g([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))]_L))_R : R \in M\rangle \\
&= \langle f_R([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1})]_R) : R \in M\rangle \\
&= \langle f_R(F^{\mathfrak{F}/R}([\sigma_0]_R, \ldots, [\sigma_{m-1}]_R)) : R \in M\rangle \\
&= \langle F^{\overline{A}_R}(f_R([\sigma_0]_R), \ldots, f_R([\sigma_{m-1}]_R)) : R \in M\rangle \\
&= \langle F^{\overline{A}_R}((g([\sigma_0]_L))_R, \ldots, (g([\sigma_{m-1}]_L)_R) : R \in M\rangle \\
&= F^{\overline{B}}(g([\sigma_0]_L), \ldots, g([\sigma_{m-1}]_L)).
\end{aligned}
$$

Moreover, g is one-one, for if $g([\sigma]_L) = g([\tau]_L)$, then for each $R \in M$ we have $f_R([\sigma]_R) = (g([\sigma]_L))_R = (g([\tau]_L))_R = f_R([\tau]_R)$, hence $[\sigma]_R = [\tau]_R$, hence $(\sigma,\tau) \in R$. This being true for all $R \in M$, it follows that $(\sigma,\tau) \in L$ by (1). So $[\sigma]_L = [\tau]_L$. Now (3) follows.

Now we are ready for the final argument. Choose $x \in {}^{\omega}A$ such that $\mathrm{rng}(x)$ generates $\overline{A}$. Now if $(\sigma, \tau) \in L$, then $\sigma = \tau$ holds in $\overline{A}$, and hence $\sigma^{\overline{A}}(x) = \tau^{\overline{A}}(x)$. It follows that there is a function $f : \mathfrak{F}/L \to \overline{A}$ such that $f([\sigma]_L) = \sigma^{\overline{A}}(x)$ for every $\sigma \in \mathfrak{F}$. Now f is a homomorphism. In fact, if k is an individual constant, then $f(k^{\mathfrak{F}}]_L) = k^{\overline{A}}(x) = k^{\overline{A}}$. If F is an m-ary operation symbol, then

$$\begin{aligned} f(F^{\mathfrak{F}/L}([\sigma_0]_L, \ldots, [\sigma_{m-1}]_L)) &= f([F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))]_L) \\ &= (F^{\mathfrak{F}}(\sigma_0, \ldots, \sigma_{m-1}))^{\overline{A}}(x) \\ &= F^{\overline{A}}(\sigma_0^{\overline{A}}(x), \ldots, \sigma_{m-1}^{\overline{A}}(x)) \\ &= F^{\overline{A}}(f([\sigma_0]_L), \ldots, f([\sigma_{m-1}]_L)). \end{aligned}$$

Furthermore, f maps onto A. For, $f([v_i]_L) = v_i^{\overline{A}}(x) = x_i$, so $\mathrm{rng}(x) \subseteq \mathrm{rng}(f)$. Since $\mathrm{rng}(f)$ is a subalgebra of $\overline{A}$ it follows that $\mathrm{rng}(f) = A$.

So f is a homomorphism from the member $\mathfrak{F}/L$ of $\mathbf{K}$ (by (3)) onto $\overline{A}$, hence $\overline{A} \in \mathbf{K}$. $\square$

We now give an application of Theorem 2.8 of a fairly concrete nature. This is to the notion of relation algebra, which has been intensively studied.

If R is a binary relation, then $R^{-1} = \{(a, b) : (b, a) \in R\}$. If R and S are binary relations, then $R|S = \{(a, c) : \exists b[(a, b) \in R$ and $(b, c) \in S$. A *proper relation algebra* is a structure $\overline{M} = (M, \cup, \cap, {}_R-, \emptyset, R, |, {}^{-1}, id)$ such that R is a binary relation, M is a collection of binary relations contained in R, M is closed under $\cup$ and $\cap$, $\emptyset, R, id \in M$, $\forall S \in M[{}_R-S = R\backslash S \in M]$, M is closed under $|$ and ${}^{-1}$, and id is the set of all ordered pairs $(a, a) \in R$. We define $U(\overline{M}) = \{a : (a, b) \in S$ for some $S \in M\}$. Thus $S \subseteq U(\overline{M}) \times U(\overline{M})$ for all $S \in M$. Note that R is an equivalence relation on $U(\overline{M})$. In fact, $R^{-1} \subseteq R$ and $R|R \subseteq R$, so R is symmetric and transitive. If $a \in U(\overline{M})$, say with $(a, b) \in S \in M$, then $(b, a) \in S^{-1}$, $(a, a) \in S|S^{-1}$ and hence $(a, a) \in R$; so R is reflexive on $U(\overline{M})$.

Let **RRA** be the class of all algebras isomorphic to a proper relation algebra; **RRA** abreviates "representable relation algebra".

Clearly **RRA** is closed under **S**.

Lemma 2.9. *Suppose that $\overline{M}$ is a proper relation algebra, as above, and f is a bijection from $U(\overline{M})$ onto a set V. Then there exist a proper relation algebra $\overline{N}$ and a function g such that $U(\overline{N}) = V$ and g is an isomorphism from $\overline{M}$ onto $\overline{N}$.*

Proof. For any $S \in M$ let

$$g(S) = \{(f(u), f(v)) : (u, v) \in S\},$$

and $\overline{N} = (g[M], \cup, \cap, {}_{g(R)}-, \emptyset, g(R), |, {}^{-1})$. We check the required conditions.

- N is closed under $\cup$, and $g(S \cup T) = g(S) \cup g(T)$ for all $S, T \in M$. In fact,

$$\begin{aligned} (f(u), f(v)) \in g(S \cup T) \quad &\text{iff} \quad (u, v) \in S \cup T \\ &\text{iff} \quad (u, v) \in S \text{ or } (u, v) \in T \\ &\text{iff} \quad (f(u), f(v)) \in g(S) \text{ or } (f(u), f(v)) \in g(T) \\ &\text{iff} \quad (f(u), f(v)) \in g(S) \cup g(T). \end{aligned}$$

- N is closed under $\cap$, and $g(S \cap T) = g(S) \cap g(T)$ for all $S, T \in M$. The proof is similar to that for $\cup$.

- N is closed under ${}_{g(R)}-$, and $g({}_R - S) = {}_{g(R)} - g(S)$. For,

$$\begin{array}{rll} (f(u), f(v)) \in g({}_R - S) & \text{iff} & (f(u), f(v)) \in g(R) \backslash g(S) \\ & \text{iff} & (f(u), f(v)) \notin g(S) \\ & \text{iff} & (u, v) \notin S \\ & \text{iff} & (f(u), f(v)) \in {}_{g(R)} - g(S). \end{array}$$

- g preserves $\emptyset$ and R. Obvious.

- N is closed under $|$, and $g(S|T) = g(S)|g(T)$. For,

$$\begin{array}{rll} (f(u), f(v)) \in g(S|T) & \text{iff} & (u, v) \in S|T \\ & \text{iff} & \exists w[(u, w) \in S \text{ and } (w, v) \in T] \\ & \text{iff} & \exists f(w)[(f(u), f(w)) \in g(S) \text{ and } (f(w), f(v)) \in g(T) \\ & \text{iff} & ((f(u), f(v)) \in g(S)|g(T) \end{array}$$

- N is closed under ${}^{-1}$, $g(S^{-1}) = (g(S))^{-1}$, and $g(id) = id$. This is clear.

Finally, g is clearly one-one. $\square$

Lemma 2.10. RRA *is closed under* **P**.

Proof. Let $\langle \overline{M}_i : i \in I \rangle$ be a system of proper relation algebras. By Lemma 2.9 we may assume that $U(\overline{M}_i) \cap U(\overline{M}_j) = \emptyset$ for $i \neq j$. Write $\overline{M}_i = (M_i, \cup, \cap, {}_{R_i}, \emptyset, R_i, |, {}^{-1}, id\rangle$. For brevity let $\overline{B} = \prod_{i \in I} \overline{M}_i$. The operations of $\overline{B}$ are denoted by $+, \cdot, -, 0, 1, ;, {}^{\cup}, I$. Now for any $x \in B$ we define

$$f(x) = \bigcup_{i \in I} x_i.$$

We claim that f is an isomorphism from B onto a proper relation algebra of the form $(f[B], \cup, \cap, {}_T-, \emptyset, T, |, {}^{-1}, id)$, where $T = \bigcup_{i \in I} R_i$.

Clearly $f(x)$ is a binary relation. f preserves $+$:

$$f(x + y) = \bigcup_{i \in I} (x_i \cup y_i) = \bigcup_{i \in I} x_i \cup \bigcup_{i \in I} y_y = f(x) \cup f(y).$$

f preserves $\cdot$:

$$f(x \cdot y) = \bigcup_{i \in I} (x_i \cap y_i) = \bigcup_{i \in I} x_i \cap \bigcup_{i \in I} y_y = f(x) \cap f(y).$$

f preserves $-$:

$$f(-x) = \bigcup_{i \in I} ({}_{R_i} - x_i) = T \backslash \bigcup_{i \in I} x_i = {}_T - f(x).$$

Obviously $f(0) = \emptyset$ and $f(1) = T$.
f preserves ;:

$$f(x;y) = \bigcup_{i\in I}(x_i|y_i)$$
$$= f(x)|f(y).$$

Here the second equality is seen as follows. If $(a,b) \in \bigcup_{i\in I}(x_i|y_i)$, choose $i \in I$ such that $(a,b) \in (x_i|y_i)$. Then there is a $c \in U(\overline{M}_i)$ such that $(a,c) \in x_i$ and $(c,b) \in y_i$. So $(a,c) \in \bigcup_{i\in I} x_i$ and $(c,b) \in \bigcup_{i\in I} y_i$, so $(a,b) \in f(x)|f(y)$. On the other hand, suppose that $(a,b) \in f(x)|f(y)$. Say $(a,c) \in f(x)$ and $(c,b) \in f(y)$. Choose $i,j \in I$ such that $(a,c) \in x_i$ and $(c,b) \in y_j$. Then $c \in U(\overline{M}_i) \cap U(\overline{M}_j)$, and it follows that $i = j$. Hence $(a,b) \in x_i|y_i$, so that $(a,b) \in f(x;y)$.

f preserves $^\cup$:

$$f(x^\cup) = \bigcup_{i\in I} x_i^{-1} = \left(\bigcup_{i\in I} x_i\right)^{-1} = (f(x))^{-1}.$$

Clearly $f(id) = id$. Finally, it is clear that f is one-one. □

To deal with **H** we first have to consider ultraproducts.

Lemma 2.11. *An ultraproduct of members of* **RRA** *is again a member of* **RRA***.*

Proof. Let $\langle \overline{M}_i : i \in I\rangle$ be a system of proper relation algebras, and let D be an ultrafilter on I. For brevity let $\overline{B} = \prod_{i\in I} \overline{M}_i$, and $\overline{C} = \overline{B}/D$, the ultraproduct. Members of C will be denoted by $[x]$ with $x \in B$. Also, let $V = \prod_{i\in I} U(\overline{M}_i)$, and $W = V/D$, the ultraproduct of these sets. Members of W will be denoted by $[v]'$, with $v \in V$. Now for any $x \in B$ let

$$F([x]) = \{([u]', [v]') : u, v \in V \text{ and } \{i \in I : (u_i, v_i) \in x_i\} \in D\}.$$

First we need to see that F is well-defined. So, suppose that $[x] = [y]$; we want to show that the expressions on the right for $F([x])$ and $F([y])$ are the same. Suppose that $u, v \in V$ and $M \stackrel{\text{def}}{=} \{i \in I : (u_i, v_i) \in x_i\} \in D$. Let $N = \{i \in I : x_i = y_i\}$. Then $N \in D$. For any $i \in M \cap N$ we have $(u_i, v_i) \in y_i$, and hence $([u]', [v]')$ is in the right side for $F([y])$. By symmetry, then, F is well-defined.

Now we claim

$(*)$ If $x \in B$, $u, v \in V$, and $([u]', [v]') \in F([x])$, then $\{i \in I : (u_i, v_i) \in x_i\} \in D$.

In fact, assume the hypothesis of $(*)$. Then there exist $u', v' \in V$ such that $[u]' = [u']'$, $[v]' = [v']'$, and $\{i \in I : (u'_i, v'_i) \in x_i\} \in D$. Now take any $i \in I$ such that $u_i = u'_i$, $v_i = v'_i$, and $(u'_i, v'_i) \in x_i$. Then $(u_i, v_i) \in x_i$. So the conclusion of $(*)$ follows.

F preserves $+$: Suppose that $x, y \in B$ and $u, v \in V$. Then

$$\begin{array}{rll} ([u]', [v]') \in F([x]+[y]) & \text{iff} & ([u]', [v]') \in F([x+y]) \\ & \text{iff} & \{i \in I : (u_i, v_i) \in x_i \cup y_i\} \in D \\ & \text{iff} & \{i \in I : (u_i, v_i) \in x_i\} \in D \text{ or } \{i \in I : (u_i, v_i) \in y_i\} \in D \\ & \text{iff} & ([u]', [v]') \in F([x]) \text{ or } ([u]', [v]') \in F([y]) \\ & \text{iff} & ([u]', [v]') \in F([x]) \cup F([y]). \end{array}$$

The proofs for $\cdot$ and $-$ are similar. Clearly $F(0) = \emptyset$ and $f(1) = 1$.

To show that F preserves ;, suppose that $x, y \in B$, $u, v \in V$. First suppose that $([u]', [v]') \in F(x; y)$. Thus by $(*)$, $M \stackrel{\text{def}}{=} \{i \in I : (u_i, v_i) \in x_i | y_i\} \in D$. Take any $i \in M$. Choose $z_i \in U(\overline{M}_i)$ such that $(u_i, z_i) \in x_i$ and $(z_i, v_i) \in y_i$. Let $z_i \in U(\overline{M}_i)$ be arbitrary for $i \notin M$. Then

$$M \subseteq \{i \in I : (u_i, z_i) \in x_i\} \cap \{i \in I : (z_i, v_i) \in y_i\},$$

so it follows that $\{i \in I : (u_i, z_i) \in x_i\} \in D$ and $\{i \in I : (z_i, v_i) \in y_i\} \in D$. Hence $([u]', [z]') \in F([x])$ and $([z]', [v]') \in F([y])$, so that $([u]', [v]') \in F([x])|F([y])$.

Conversely, suppose that $([u]', [v]') \in F([x])|F([y])$. Choose $z \in V$ with $([u]', [z]') \in F([x])$ and $([z]', [v]') \in F([y])$. Thus

$$\{i \in I : (u_i, z_i) \in x_i\} \in D \quad \text{and} \quad \{i \in I : (z_i, v_i) \in y_i\} \in D,$$

so the intersection N of these two sets is in D. Now

$$N \subseteq \{i \in I : (u_i, v_i) \in x_i | y_i\},$$

so the set on the right is in D, and hence $([u]', [v]') \in F(x; y)$. This finishes the proof that F preserves ;.

Clearly F preserves $^\cup$ and id.

If $[x] \neq [y]$, then $P \stackrel{\text{def}}{=} \{i \in I : x_i \neq y_i\} \in D$. Take any $i \in P$, and choose $(u_i, v_i) \in x_i \triangle y_i$. Let u_i and v_i be any members of $U(\overline{M}_i)$ for $i \notin P$. Then $([u]', [v]') \in F([x \triangle y]) = F([x]) \triangle F([y])$. So $F([x]) \neq F([y])$. □

Lemma 2.12. *If $\overline{M}$ is a proper relation algebra with unit element R, then for any $X \in M$ we have $R|(R\backslash(R|X|R))|R = (R\backslash(R|X|R))$.*

Proof. Suppose that $(a, b) \in R|(R\backslash(R|X|R))|R$ but also $(a, b) \in (R|X|R)$. Say $aRc(R\backslash(R|X|R))dRb$ and $aReXfRb$. Then $cRaReXfRbRd$, hence $cReXfRd$, so that $c(R|X|R)d$, contradiction. Thus $\subseteq$ holds.

If $(a, b) \in (R\backslash(R|X|R))$, then $aRa(R\backslash(R|X|R))bRb$, and so (a, b) is a member of $(R|(R\backslash(R|X|R))|R)$, proving $\supseteq$. □

Lemma 2.13. *A homomorphic image of a member of* **RRA** *is again a member of* **RRA**.

Proof. Let $\overline{M}$ be a proper relation algebra with unit element R, and let f be a homomorphism from $\overline{M}$ onto $\overline{N}$. Then $\overline{M}/\ker(f) \cong \overline{N}$ by Theorem 1.8, so it suffices to show that $\overline{M}/\ker(f)$ is isomorphic to a proper relation algebra.

(1) For any $Z \in M$, the set $N_Z \stackrel{\text{def}}{=} \{y \cap (R|Z|R) : y \in M\}$ is a proper relation algebra, and $\langle y \cap (R|Z|R) : y \in M\rangle$ is a homomorphism from $\overline{M}$ onto the associated structure.

In fact, clearly N_Z is closed under $\cup$, $\cap$, complementation relative to $R|Z|R$, and $\emptyset \in N_Z$. To show that it is closed under $|$, suppose that $x, y \in M$; we want to show that $(x|y)\cap(R|Z|R) = (x\cap(R|Z|R))|(y\cap(R|Z|R))$. First suppose that $(u, v) \in (x|y)\cap(R|Z|R)$. Choose w such that $(u, w) \in x$ and $(w, v) \in y$. Also choose s, t so that $(u, s) \in R$, $(s, t) \in Z$, and $(t, v) \in R$. Then $(u, s) \in R$ and $(t, w) \in R|y^{-1} \subseteq R$, so $(u, w) \in (R|Z|R)$. So $(u, w) \in (x \cap (R|Z|R))$. Similarly, $wRuRs$. so $wRsZtRv$. hence $(w, v) \in (y \cap (R|Z|R))$, so $(u, v) \in (x \cap (R|Z|R))|(y \cap (R|Z|R))$.

Conversely, suppose that $(u, v) \in (x \cap (R|Z|R))|(y \cap (R|Z|R))$. Choose w so that $(u, w) \in (x \cap (R|Z|R))$ and $(w, v) \in (y \cap (R|Z|R))$. Then $(u, v) \in (x|y)$. Also choose p, q, r, s so that $(u, p) \in R$, $(p, q) \in Z$, $(q, w) \in R$, $(w, r) \in R$, $(r, s) \in Z$, and $(Z, v) \in R$. Then $(u, r) \in R|Z|R|R \subseteq R$, so $(u, v) \in (R|Z|R)$. Thus $(u, v) \in ((x|y) \cap (R|Z|R))$, as desired.

To show that N_Z is closed under ${}^{-1}$, suppose that $x \in M$; we want to show that $(x \cap R|Z|R)^{-1} = (x^{-1} \cap R|Z|R)$. First suppose that $(u, v) \in (x \cap R|Z|R)^{-1}$. Then $(v, u) \in (x \cap R|Z|R)$. Choose s, t so that $(v, s) \in R$, $(s, t) \in Z$, and $(t, u) \in R$. Then $(u, s) \in R^{-1}|Z^{-1} \subseteq R$ and $(t, v) \in Z^{-1}|R^{-1} \subseteq R$, so $(u, v) \in (R|Z|R)$. So $(u, v) \in (x^{-1} \cap R|Z|R)$. This shows that $(x \cap R|Z|R)^{-1} \subseteq (x^{-1} \cap R|Z|R)$. Conversely, suppose that $(u, v) \in (x^{-1} \cap R|Z|R)$. Then $(v, u) \in x$. Also, $(u, v) \in R|Z|R$, so choose s, t so that $uRsZtRv$. Then $(v, s) \in x|R \subseteq R$ and $(t, u) \in R|x \subseteq R$, so $(v, u) \in R|Z|R$. Hence $(v, u) \in x \cap (R|Z|R)$, and it follws that $(u, v) \in (x \cap (R|Z|R))^{-1}$.

Clearly $id \in N_Z$. The homomorphism property in (1) is clear.

We may assume that $f(R) \neq f(\emptyset)$, as otherwise clearly $|\text{rng}(f)| = 1$. Let $I = \{Z \in M : f(Z) = 0\}$. Thus $R \notin I$, and if $Z_1, Z_2 \in I$ then also $Z_1 \cup Z_2 \in I$.

Now let D be an ultrafilter on I containing each set $\{V \in I : Z \leq V\}$ for each $Z \in I$. For each $Z \in I$ let $T(Z) = (R\backslash(R|Z|R))$.

Define $h : M \to \prod_{Z\in I} N_{T(Z)}$ by setting $(h(Y))_Z = Y \cap (R|T(Z)|R)$ for each $Y \in M$ and $Z \in I$. Then define $h'(Y) = h(Y)/D$ for any $Y \in M$.

From (1) it follows that h' is a homomorphism. We claim that $\ker(f) \subseteq \ker(h')$. For, suppose that $(Y, Y') \in \ker(f)$. So $f(Y) = f(Y')$, so $f(Y \triangle Y') = 0$. Hence $(Y \triangle Y', 0) \in \ker(f)$, so $Y \triangle Y' \in I$. For any $W \in I$ such that $(Y \triangle Y') \subseteq W$ we have $T(W) \subseteq T(Y \triangle Y')$. Now $Y \triangle Y' \subseteq R|(Y \triangle Y')|R$, so $(Y \triangle Y') \cap (R\backslash(R|(Y \triangle Y')|R) = \emptyset$, which means that $(Y \triangle Y') \cap T(Y \triangle Y') = \emptyset$, so $(Y \triangle Y') \cap T(W) = \emptyset$. It follows that $Y \cap T(W) = Y' \cap T(W)$. This being true for all $W \in I$ such that $(Y \triangle Y') \subseteq W$, it follows that $h'(Y) = h'(Y')$. Thus we have shown that $\ker(f) \subseteq \ker(h')$.

Now suppose that $h'(Y) = h'(Y')$. Then $h(Y)/D = h(Y')/D$, and so $\{Z \in I : h(Y)_Z = h(Y')_Z\} \in D$. Choose any Z such that $h(Y)_Z = h(Y')_Z$. Thus $Y \cap (R|T(Z)|R) = Y' \cap (R|T(Z)|R)$, so $(Y \triangle Y') \cap (R|T(Z)|R) = 0$. By Lemma 2.12 we have $(R|T(Z)|R) = T(Z)$. Hence $(Y \triangle Y') \cap T(Z) = 0$, and it follows that $(Y \triangle Y') \subseteq (R|Z|R)$. Clearly

$f(R|Z|R) = 0$, so also $f(Y \triangle Y') = 0$. So $f(Y) = f(Y')$. This shows that $\ker(f) = \ker(h')$.

Now define $k(Y/\ker(f)) = h'(Y)$, for any $Y \in M$. Then k is well-defined and one-one by what was just shown. Now $\overline{M} \in \mathbf{RRA}$ by Lemma 2.10. $\square$

Theorem 2.14. (Tarski) **RRA** *is a variety.* $\square$

No very nice system of equations characterizing **RRA** is known. It was proved in an article of Monk in 1964 that no finite system of equations works; the proof involves a construction of a system of non-representable relation algebras whose ultraproduct is representable.

Exc. 2.1. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{SHK} \subseteq \mathbf{HSK}$.

Exc. 2.2. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{SHK} \neq \mathbf{HSK}$.

Exc. 2.3. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{PHK} \subseteq \mathbf{HPK}$.

Exc. 2.4. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{PHK} \neq \mathbf{HPK}$. Hint: let $\mathbf{K}$ consist of all fields $\mathbb{Z}_p$ for p a prime, and take an ultraproduct of them with a nonprincipal ultrafilter. Show that the result is an infinite field.

Exc. 2.5. Prove that for any class $\mathbf{K}$ of algebras we have $\mathbf{PSK} \subseteq \mathbf{SPK}$.

Exc. 2.6. Give an example of a class $\mathbf{K}$ of algebras such that $\mathbf{PSK} \neq \mathbf{SPK}$.

Exc. 2.7. Prove that $\mathbf{HSPK}$ is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$. Infer that for any class $\mathbf{K}$ of structures, $\mathbf{HSPK}$ is the smallest variety containing $\mathbf{K}$.

[In 1972, D. Pigozzi determined all possible distinct sequences of $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$; it turns out that there are exactly 18 of them. It would be natural to adjoin an operation $\mathbf{Up}$ such that $\mathbf{UpK}$ is the class of all algebras isomorphic to an ultraproduct of members of $\mathbf{K}$. Then there is an open problem, raised by Henkin, Monk, and Tarski in 1971: does the sequence $\mathbf{P}$, $\mathbf{PUp}$, $\mathbf{PUpP}$, $\mathbf{PUpPUp}$, ... have any repetitions?]

Exc. 2.8. Prove that the following hold in any proper relation algebra with unit R:

(i) $S^{-1}|[-(S|T)] \subseteq -T$.

(ii) $((S|R) \cap id)|R = S|R$.

(iii) $S \subseteq S|S^{-1}|S$.

3. Sentential logic

Here we discuss some more components of our final first-order logic: the logic surrounding words like "not", "and", etc. The language here is simpler than what we have dealt with so far. We have only the following symbols:

n, a symbol for negation.

a, a symbol for conjunction ("and").

Symbols $S_0, S_2, \ldots$, called *sentential variables.*

Symb is the collection of all of these symbols. An *expression* is a finite nonempty sequence of members of Symb. We define operations $\neg$ and $\wedge$ on the set of expressions:

$$\neg\varphi = \langle \mathbf{n}\rangle^\frown\varphi; \quad \varphi\wedge\psi = \langle \mathbf{a}\rangle^\frown\varphi^\frown\psi.$$

The collection of *sentential formulas* is the smallest collection C of expressions such that $\langle S_i\rangle$ is in C for each $i \in \omega$, and C is closed under the operations $\neg$, $\wedge$. Frequently we write S_i instead of $\langle S_i\rangle$.

In analogy to Proposition 2.1 we have:

Proposition 3.1. *(i) No proper initial segment of a sentential formula is a formula.*

(ii) If φ is a sentential formula, then exactly one of the following holds:

(a) φ is a sentential variable.

(b) φ is $\neg\psi$ for some sentential formula ψ.

(c) φ is $\psi\wedge\chi$ for some sentential formulas ψ, χ.

(iii) If φ and ψ are sentential formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.

(iv) If $\varphi, \psi, \varphi', \psi'$ are sentential formulas and $\varphi\wedge\psi = \varphi'\wedge\psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$. □

Now we define satisfaction and truth for this special language. A *sentential assignment* is a function mapping ω into $\{0, 1\}$. We think of 0 as "false" and 1 as "true". Then the value of an arbitrary formula φ under a sentential assignment f is denoted by $\varphi[f]$ and is defined as follows:

$$\begin{aligned} S_i[f] &= f(i); \\ (\neg\varphi)[f] &= 1 - \varphi[f]; \\ (\varphi\wedge\psi)[f] &= \varphi[f]\cdot\psi[f]. \end{aligned}$$

We say that f *satisfies* φ, or that φ is *true* under f iff $\varphi[f] = 1$. A sentential formula is a *tautology* iff it is true under every assignment.

We introduce some further logical notions:

$$\begin{aligned} \varphi\to\psi &= \neg(\varphi\wedge\neg\psi); \\ \varphi\vee\psi &= \neg(\neg\varphi\wedge\neg\psi); \\ \varphi\leftrightarrow\psi &= (\varphi\to\psi)\wedge(\psi\to\varphi). \end{aligned}$$

Here are some common tautologies:

(T1) $S_0 \to S_0$.
(T2) $S_0 \leftrightarrow \neg\neg S_0$.
(T3) $(S_0 \to \neg S_0) \to \neg S_0$.
(T4) $(S_0 \to \neg S_1) \to (S_1 \to \neg S_0)$.
(T5) $S_0 \to (\neg S_0 \to S_1)$.
(T6) $(S_0 \to S_1) \to [(S_1 \to S_2) \to (S_0 \to S_2)]$.
(T7) $[S_0 \to (S_1 \to S_2)] \to [(S_0 \to S_1) \to (S_0 \to S_2)]$.
(T8) $(S_0 \wedge S_1) \to (S_1 \wedge S_0)$.
(T9) $(S_0 \wedge S_1) \to S_0$.
(T10) $(S_0 \wedge S_1) \to S_1$.
(T11) $S_0 \to [S_1 \to (S_0 \wedge S_1)]$.
(T12) $S_0 \to (S_0 \vee S_1)$.
(T13) $S_1 \to (S_0 \vee S_1)$.
(T14) $(S_0 \to S_2) \to [(S_1 \to S_2) \to ((S_0 \vee S_1) \to S_2)]$.
(T15) $\neg(S_0 \wedge S_1) \leftrightarrow (\neg S_0 \vee \neg S_1)$.
(T16) $S_0 \wedge S_1 \leftrightarrow \neg(\neg S_0 \vee \neg S_1)$.
(T17) $\neg(S_0 \vee S_1) \leftrightarrow (\neg S_0 \wedge \neg S_1)$.
(T18) $[S_0 \vee (S_1 \vee S_2)] \leftrightarrow [(S_0 \vee S_1) \vee S_2]$.
(T19) $[S_0 \wedge (S_1 \wedge S_2)] \leftrightarrow [(S_0 \wedge S_1) \wedge S_2]$.
(T20) $[S_0 \wedge (S_1 \vee S_2)] \leftrightarrow [(S_0 \wedge S_1) \vee (S_0 \wedge S_2)]$.
(T21) $[S_0 \vee (S_1 \wedge S_2)] \leftrightarrow [(S_0 \vee S_1) \wedge (S_0 \vee S_2)]$.
(T22) $S_0 \wedge S_1 \leftrightarrow \neg(S_0 \to \neg S_1)$l.

Some of these tautologies show that we could have selected different primitive notions for the sentential part of first-order logic. Thus:

$\neg$ and $\vee$ suffice, by (T16).

$\neg$ and $\to$ suffice, by (T22).

We now define general conjunctions and disjunctions:

$$\bigwedge_{i\le 0} \varphi_i = \varphi_0;$$

$$\bigwedge_{i\le m+1} \varphi_i = \left(\bigwedge_{i\le m} \varphi_i\right) \wedge \varphi_{m+1};$$

$$\bigvee_{i\le 0} \varphi_i = \varphi_0;$$

$$\bigvee_{i\le m+1} \varphi_i = \left(\bigvee_{i\le m} \varphi_i\right) \vee \varphi_{m+1}.$$

We also might write $\varphi_0 \wedge \ldots \wedge \varphi_m$ in place of $\bigwedge_{i \leq m} \varphi_i$; similarly for $\bigvee$. Sometimes we will not explicitly give an order; for example we might write $\bigvee_{i \in I} \varphi_i$. In such a case, any order should be ok.

For any sentential formula φ, let $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$.

Lemma 3.2. *If f and g are sentential assignments which agree on every i such that S_i occurs in φ, then $\varphi[f] = \varphi[g]$.*

Proof. By induction on φ. $\square$

Theorem 3.3. (Disjunctive normal form) *If φ is a sentential formula which is true under some sentential assignment, and if every sentential variable S_i occurring in φ has $i < m$, then there is a nonempty set $M \subseteq {}^m2$ such that the following formula is a tautology:*

$$\varphi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} S_i^{\varepsilon(i)}.$$

Proof. Let

$$M = \{\varepsilon \in {}^m2 : \varphi[f] = 1 \text{ for some } f \supseteq \varepsilon\}.$$

Note that M is nonempty, since φ is true under some assignment. Now take any sentential assignment f. Note that

$$\left(\bigwedge_{i<m} S_i^{f(i)}\right)[f] = 1.$$

Hence the right side of the formula in the Theorem is true under f iff $f \restriction m \in M$, and this is true by Lemma 3.2 iff $\varphi[f] = 1$. $\square$

EXERCISES

Exc. 3.1. Define $\varphi|\psi = \neg\varphi \wedge \neg\psi$. (The Sheffer stroke.). Show that $\neg$ and $\wedge$ can be defined in terms of $|$.

Exc. 3.2. A formula φ involving only $S_0, \ldots, S_m$ determines a function $t_\varphi : {}^{m+1}2 \to 2$ defined by $t_\varphi(x) = \varphi[x]$ for any $x \in {}^{m+1}2$. Show that any member of $\bigcup_{0<m<\omega} {}^{({}^m2)}2$ can be obtained in this way.

Exc. 3.3. Show that the following formula is a tautology:

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)]$$

(This formula can be used as a single axiom in an axiomatic development of sentential logic.)

4. First-order logic

In this chapter we finish introducing the notion of first-order logic, and connect this notion to satisfaction and truth in structures.

Let a signature $\sigma = (\mathrm{Fcn}, \mathrm{Rel}, \mathrm{Cn}, \mathrm{ar})$ be given. Now in addition to the variables and the logical symbol $\mathbf{e}$ for equality introduced in the chapter 2, we add some more symbols, assumed to be distinct and different from all the previous symbols:

$\mathbf{n}$, a symbol for negation.

$\mathbf{a}$, a symbol for conjunction ("and").

$\mathbf{x}$, a symbol for existential quantification ("there exists").

So altogether the set Symb of *symbols* for the given signature σ is the set

$$\mathrm{Fcn} \cup \mathrm{Rel} \cup \mathrm{Cn} \cup \{v_i : i < \omega\} \cup \{\mathbf{e}, \mathbf{n}, \mathbf{a}, \mathbf{x}\}.$$

Thus a given first-order language is completely determined by its signature; so we may define a first-order language as just a signature. An *expression* is a finite nonempty sequence of members of Symb. Thus terms and equations are expressions. We now introduce some operations on expressions, so that we will rarely see the actual symbols $\mathbf{e}, \mathbf{n}, \mathbf{a}, \mathbf{x}$:

$\neg\varphi = \langle \mathbf{n} \rangle^\frown \varphi.$

$\varphi \wedge \psi = \langle \mathbf{a} \rangle^\frown \varphi^\frown \psi.$

$\exists v_i \varphi = \langle \mathbf{x}, v_i \rangle^\frown \varphi.$

An *atomic equality formula* is an equation. An *atomic non-equality formula* is an expression of the form $R\sigma_0 \ldots \sigma_{m-1}$ for some relation symbol R of rank m and some terms $\sigma_0, \ldots \sigma_{m-1}$. Note that we are being a little sloppy here, we really mean

$$\langle R \rangle^\frown \sigma_0^\frown \cdots^\frown \sigma_{m-1}.$$

The collection of *formulas* is the intersection of all sets A of expressions such that:

Each atomic formula (equality or non-equality) is in A.

If φ is in A, then so is $\neg\varphi$.

If φ and ψ are in A, then so is $\varphi \wedge \psi$,

If φ is in A and $i \in \omega$, then $\exists v_i \varphi$ is also in A.

In analogy to Proposition 2.1 we have:

Proposition 4.1. *(i) No proper initial segment of a formula is a formula.*

(ii) If φ is a formula, then exactly one of the following holds:

(a) φ is an atomic equality formula.

(b) φ is an atomic non-equality formula.

(c) φ is $\neg\psi$ for some formula ψ.

(d) φ is $\psi \wedge \chi$ for some formulas ψ, χ.

(e) φ is $\exists v_i \psi$ for some $i \in \omega$ and some formula ψ.

(iii) If φ and ψ are formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.

(iv) If $\varphi, \psi, \varphi', \psi'$ are formulas and $\varphi \wedge \psi = \varphi' \wedge \psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$.

(v) If $i, j \in \omega$, φ, ψ are formulas, and $\exists v_i \psi = \exists v_j \psi$, then $i = j$ and $\varphi = \psi$. □

Now we can define the relationship between first-order logic and structures. First a useful abbreviation: if $a \in {}^\omega A$, $i \in \omega$, and $c \in A$, then a^i_c is the member of ${}^\omega A$ such that for any $j \in \omega$,

$$(a^i_c)_j = \begin{cases} c & \text{if } j = i, \\ a_j & \text{if } j \neq i. \end{cases}$$

Now suppose that $\overline{A}$ is a structure and $a \in {}^\omega A$. We define what it means for a to *satisfy* a formula φ in $\overline{A}$, abbreviated by $\overline{A} \models \varphi[a]$. The definition goes by recursion on φ. Proposition 4.1 is needed to assure the unambiguity of the definition. A rigorous justification of definitions by various sorts of recursion can be found in books on set theory; the intent is clear in our case, though.

$$\begin{aligned}
\overline{A} \models \sigma = \tau[a] \quad &\text{iff} \quad \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a); \\
\overline{A} \models R\sigma_0 \ldots \sigma_{m-1}[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a) \rangle \in R^{\overline{A}}; \\
\overline{A} \models \neg\varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \varphi[a]); \\
\overline{A} \models (\varphi \wedge \psi)[a] \quad &\text{iff} \quad \overline{A} \models \varphi[a] \text{ and } \overline{A} \models \psi[a]); \\
\overline{A} \models \exists v_i \varphi[a] \quad &\text{iff} \quad \text{there is a } c \in A \text{ such that } \overline{A} \models \varphi[a^i_c].
\end{aligned}$$

We define some more logical notions:

$$\begin{aligned}
\varphi \to \psi &= \neg(\varphi \wedge \neg\psi); \\
\varphi \vee \psi &= \neg(\neg\varphi \wedge \neg\psi); \\
\varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi); \\
\forall v_i \varphi &= \neg\exists v_i \neg\varphi.
\end{aligned}$$

Proposition 4.2. *Let $\overline{A}$ be a structure and $a \in {}^\omega A$. Then:*

$$\begin{aligned}
\overline{A} \models (\varphi \to \psi)[a] \quad &\textit{iff} \quad \overline{A} \models \varphi[a] \textit{ implies that } \overline{A} \models \psi[a]; \\
\overline{A} \models (\varphi \vee \psi)[a] \quad &\textit{iff} \quad \overline{A} \models \varphi[a] \textit{ or } \overline{A} \models \psi[a] \textit{ (or both)}; \\
\overline{A} \models (\varphi \leftrightarrow \psi)[a] \quad &\textit{iff} \quad (\overline{A} \models \varphi[a] \textit{ iff } \overline{A} \models \psi[a]); \\
\overline{A} \models \forall v_i \varphi[a] \quad &\textit{iff} \quad \overline{A} \models \varphi[a^i_c] \textit{ for every } c \in A.
\end{aligned}$$

□

We give some examples of these notions for the signatures introduced at the beginning.

Partial orderings. Consider the partial ordering $\overline{A} \stackrel{\text{def}}{=} (\mathscr{P}(\omega), \subseteq)$ of all subsets of ω.

$\overline{A} \models v_0 < v_1[\emptyset, \{0\}, \{0\}, \ldots]$ since the empty set is a proper subset of any nonempty set.

$\overline{A} \models \forall v_0 \forall v_1 \exists v_2[(v_0 < v_2 \vee v_0 = v_2) \wedge (v_1 < v_2 \vee v_1 = v_2)[a]$ for any $a \in {}^\omega A$ since for any $a_0, a_1 \subseteq \omega$ we can take $a_2 = a_0 \cup a_1$ to satisfy this formula.

Groups. For any group $\overline{A}$ we have:

$\overline{A} \models \forall v_0[v_0 \cdot v_0 = e][a]$ iff every non-identity element of $\overline{A}$ has order 2.

$\overline{A} \models (\neg(v_2 = e) \wedge v_2 \cdot (v_2 \cdot v_2) = e)[a]$ iff a_2 has order 3.

Rings. For any structure $\overline{A}$ for this language we have

$\overline{A} \models \forall v_0(v_0 + (-v_0) = 0)[a]$ iff $a + -a = 0$ for all $a \in A$; this expresses one of the usual axioms for rings.

Ordered fields. For $\overline{A}$ any ordered field, $\forall v_0[\neg(v_0 = 0) \rightarrow 0 < v_0 \cdot v_0][a]$ is the little theorem that the square of any nonzero element is positive.

Having introduced the main relationship between structures and logic, we can now define the most important concepts derived from this relationship. Let $\overline{A}$ be a stucture, φ a formula, $\mathbf{K}$ a class of similar structures, and Γ a set of formulas.

- φ *holds in* $\overline{A}$ iff $\overline{A} \models \varphi[a]$ for every $a \in {}^\omega A$. In this case we also say that φ is *true in* $\overline{A}$, or that $\overline{A}$ is a *model of* φ.
- $\overline{A}$ is a *model of* Γ iff it is a model of each member of Γ.
- $\Gamma \models \varphi$ iff every model of Γ is a model of φ. We read this as "Γ models φ".
- $\mathbf{K} \models \varphi$ iff every member of $\mathbf{K}$ models φ.

Now in this chapter we go through some of the simplest notions concerning models. We begin by applying the material of Chapter 3 concerning sentential logic. Now a tautology in a first-order language is a formula which can be obtained from a sentential tautology by simultaneously replacing each subformula S_i by φ_i, for some formulas φ_i.

Theorem 4.3. *If φ is a tautology of signature σ, then φ holds in every structure of signature σ.*

Proof. Let φ be obtained from a sentential tautology ρ by simultaneously replacing each subformula S_i by χ_i, for some formulas χ_i, for each $i \in \omega$. (Of course only finitely many S_i's actually occur in ψ.) Let $\overline{A}$ be any structure of signature σ, and let $b \in {}^\omega A$; we want to show that $\overline{A} \models \varphi[b]$. To this end we produce a sentential assignment f. Namely, for each $i \in \omega$ let

$$f(i) = \begin{cases} 1 & \text{if } \overline{A} \models \chi_i[b], \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim:

(*) For any subformula ψ of ρ, if ψ' is obtained from ψ by simultaneously replacing each subformula S_i of ψ by χ_i, for each $i \in \omega$, then $\overline{A} \models \psi'[b]$ iff $\psi[f] = 1$.

We prove this by induction on ψ:

If ψ is S_i, then ψ' is χ_i, and our condition holds by definition. If inductively ψ is $\neg\tau$, then ψ' is $\neg\tau'$, and

$$\begin{aligned}
\overline{A} \models \psi'[b] \quad & \text{iff} \quad \text{not}(\overline{A} \models \tau'[b]) \\
& \text{iff} \quad \text{not}(\tau[f] = 1) \\
& \text{iff} \quad \tau[f] = 0 \\
& \text{iff} \quad \psi[f] = 1.
\end{aligned}$$

Finally if inductively ψ is $\tau \wedge \xi$, then ψ' is $\tau' \wedge \xi'$, and

$$\begin{aligned}
\overline{A} \models \psi'[b] \quad & \text{iff} \quad \overline{A} \models \tau'[b] \text{ and } \overline{A} \models \xi'[b] \\
& \text{iff} \quad \tau[f] = 1 \text{ and } \xi[f] = 1 \\
& \text{iff} \quad \psi[f] = 1.
\end{aligned}$$

This finishes the proof of (*), and with $\rho = \varphi$ in (*) the theorem is proved. $\square$

Theorem 4.4. (Disjunctive normal form) *Suppose that Γ is a set of formulas. Let Δ be the smallest set of formulas containing Γ and closed under $\neg$ and $\wedge$. Suppose that $\varphi \in \Delta$ and φ has a model. Then there exist a positive integer m, a sequence $\psi \in {}^m\Gamma$, and a set $M \subseteq {}^m 2$, such that*

$$\emptyset \models \varphi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} \psi_i^{\varepsilon(i)},$$

where χ^1 is χ and χ^0 is $\neg\chi$, for any formula χ.

Proof. For each finite subset F of Γ let Θ_F be the smallest set of formulas containing F and closed under $\neg$ and $\wedge$. Note that

$$\Delta = \bigcup \{\Theta_F : F \text{ is a finite subset of } \Gamma\}.$$

Hence there is a finite subset F of Γ such that $\varphi \in \Theta_F$. Clearly F is nonempty. Let $m = |F|$, and let $\langle \psi_i : i < m \rangle$ enumerate F. Let Ω be the set of all sentential formulas involving only S_i for $i < m$. For each $\theta \in \Omega$ let θ' be obtained from θ by simultaneously replacing each subformula S_i of θ by ψ_i. Then $\{\theta' : \theta \in \Omega\}$ contains F and is closed under $\neg$ and $\wedge$. Hence $\Theta_F \subseteq \{\theta' : \theta \in \Omega\}$. So choose $\theta \in \Omega$ so that $\theta' = \varphi$.

(1) θ is true under some sentential assignment.

In fact, we define $f : \omega \to \{0, 1\}$ as follows. Let $\overline{A}$ be a model of φ, and choose any $a \in {}^\omega A$. Define

$$f(i) = \begin{cases} 1 & \text{if } i < m \text{ and } \overline{A} \models \psi_i[a], \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim

(2) For every sentential formula χ involving only S_i with $i < m$, we have $\chi[f] = 1$ iff $\overline{A} \models \chi'[a]$.

This is clear by induction. It follows that $\theta[f] = 1$, as desired in (1).

Now by Theorem 3.3 choose a nonempty $M \subseteq {}^m2$ such that

$$\theta \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} S_i^{\varepsilon(i)}$$

is a tautology. Then by Theorem 4.3 we have

$$\emptyset \models \varphi \leftrightarrow \bigvee_{\varepsilon \in M} \bigwedge_{i<m} \psi_i^{\varepsilon(i)}. \qquad \square$$

Now we turn to elementary results involving quantifiers. We first need to talk about free and bound occurrences of variables.

Proposition 4.5. *Suppose that σ is a signature, φ is a σ-formula, and $\exists$ occurs in φ at a certain place: say $\varphi_i = \mathbf{x}$ (remember that formulas are just certain sequences, so we can look at the i-th entry of one). Then there is a unique formula ψ of the form $\exists v_j \chi$, χ a formula, which is a segment of φ beginning at the i-th place.* $\square$

In the situation of this proposition, all occurrences of v_j in the indicated ψ-part of φ are said to be *in the scope of the quantifier $\exists$ appearing at the i-th place in φ*, and each such occurrence is said to be a *bound occurrence* of v_j. Thus the notion of "bound" refers to a particular occurrence, and it is always with respect to a particular formula. An occurrence of a variable in a formula φ is a *free occurrence* if it is not a bound occurrence.

Proposition 4.6. *Suppose that σ is a first-order language, $\overline{A}$ is an σ-structure, φ is a σ-formula, $a, b \in {}^\omega A$, and $a_i = b_i$ for every $i \in \omega$ such that v_i occurs free somewhere in φ. Then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$.*

Proof. We proceed by induction on φ. If φ is $\sigma = \tau$ for some terms σ and τ, then to occur free in φ means the same thing as just occurring in σ or τ. So by Proposition 2.2 we have $\sigma^{\overline{A}}(a) = \sigma^{\overline{A}}(b)$, $\tau^{\overline{A}}(a) = \tau^{\overline{A}}(b)$, and so

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \\ &\text{iff} \quad \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \\ &\text{iff} \quad \overline{A} \models \varphi[b], \end{aligned}$$

as desired. The case of an atomic formula starting with a relation symbol is treated similarly. The induction hypotheses with respect to $\neg$ and $\rightarrow$ are easy:

$$\begin{aligned} \overline{A} \models \neg\psi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\ &\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \\ &\text{iff} \quad \overline{A} \models \neg\psi[a]; \\ \overline{A} \models (\psi \wedge \chi)[a] \quad &\text{iff} \quad \overline{A} \models \psi[a]) \text{ and } \overline{A} \models \chi[a] \\ &\text{iff} \quad \overline{A} \models \psi[b]) \text{ or } \overline{A} \models \chi[b] \\ &\text{iff} \quad \overline{A} \models (\psi \wedge \chi)[b]. \end{aligned}$$

The induction step involving $\exists$ is more delicate. By symmetry it suffices to go one direction only. So, suppose that $\overline{A} \models \exists v_i \psi[a]$. Then there is a $c \in A$ such that $\overline{A} \models \psi[a^i_c]$. Note that $(a^i_c)_j = (b^i_c)_j$ for every j such that v_j occurs free in ψ. Hence by the inductive hypothesis, $\overline{A} \models \psi[b^i_c]$. Hence $\overline{A} \models \exists v_i \psi[b]$. □

This proposition enables us to use more liberal notation for satisfaction of formulas, just as above for terms. Thus we might write $\varphi(x, y)$ to indicate that we are dealing with a formula whose free variables are among x and y (i.e., all free occurrences of variables in φ are free occurrences of x or y), and then write something like $A \models \varphi[a, b]$ for particular elements $a, b \in A$.

A *sentence* is a formula φ such that no variable occurs free in it. In this case we can write simply $\overline{A} \models \varphi$ or $\overline{A} \models \neg\varphi$. A *theory* is a collection of sentences. A theory Γ is *complete* iff for any sentence φ, either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$. This is an important notion which will play a role in much of what follows.

Now we turn to the important but somewhat complicated matter of substituting terms in terms, terms in formulas, etc. The first fact is easily shown by induction (exercise!).

Proposition 4.7. *(i) Suppose that τ is a term and we replace zero or more occurrences of a variable v_i in τ by a term σ, obtaining thereby a sequence ρ. Then ρ is a term.*

(ii) Suppose that φ is a formula, τ is a term, and we replace zero or more occurrences of a variable v_i in φ by τ (except not if the occurrence of v_i is just after the symbol $\exists$), obtaining thereby a sequence ψ. Then ψ is a formula. □

The following basic lemma explains how satisfaction is related to substitutions in a formula.

Lemma 4.8. *Let φ be a formula, v_i a variable, τ a term, and suppose*

(*) no free occurrence of v_i in φ is within the scope of a quantifier on a variable occurring in τ.

Let φ' be obtained from φ by replacing every free occurrence of v_i in φ by τ. Then for any structure $\overline{A}$ and any $a \in {}^\omega A$ we have $\overline{A} \models \varphi'[a]$ iff $\overline{A} \models \varphi\left[a^i_{\tau^{\overline{A}}(a)}\right]$.

Proof. We give this proof in full. The harder cases here are the atomic case and the quantifier case.

First we prove the following related fact involving only terms:

(1) If ρ is a term and ρ' is obtained from ρ by replacing all occurrences of v_i in ρ by τ, then for any $a \in {}^\omega A$ we have $\rho'^{\overline{A}}(a) = \rho^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right)$.

We prove this, of course, by induction on ρ. If $\rho = v_j$ with $j \neq i$, then also $\rho' = v_j$, and a and $a^i_{\tau^{\overline{A}}(a)}$ agree at j, so the conclusion is clear. If $\rho = v_i$, then $\rho' = \tau$, and $\rho'^{\overline{A}}(a) = \tau^{\overline{A}}(a) = \rho^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right)$. If k is an individual constant, then $\rho' = \rho$ and the conclusion is obvious.

For the inductive step, suppose that ρ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$ and we know (1) for each of the terms $\tau_0, \ldots, \tau_{m-1}$. Note that ρ' is $\mathbf{F}\tau'_0 \ldots \tau'_{m-1}$. Then

$$\begin{aligned}
\rho'^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\tau_0'^{\overline{A}}(a), \ldots, \tau_{m-1}'^{\overline{A}}(a)) \\
&= \mathbf{F}^{\overline{A}}\left(\tau_0^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right), \ldots, \tau_{m-1}^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right)\right) \\
&= \rho^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right),
\end{aligned}$$

as desired. This completes the inductive proof of (1).

Now for formulas, we also proceed by induction on φ. In case φ is $\sigma = \rho$, the formula φ' is $\sigma' = \rho'$ with notation as for (1), and so by (1),

$$\begin{aligned}
\overline{A} \models \varphi'[a] \quad &\text{iff} \quad \sigma'^{\overline{A}}(a) = \rho'^{\overline{A}}(a) \\
&\text{iff} \quad \sigma^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right) = \rho^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models (\sigma = \rho)\left[a^i_{\tau^{\overline{A}}(a)}\right],
\end{aligned}$$

as desired.

Next, suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Then φ' is $\mathbf{R}\sigma'_0 \ldots \sigma'_{m-1}$. So

$$\begin{aligned}
\overline{A} \models \varphi'[a] \quad &\text{iff} \quad \langle \sigma_0'^{\overline{A}}(a), \ldots, \sigma_{m-1}'^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \left\langle \sigma_0^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right), \ldots, \sigma_{m-1}^{\overline{A}}\left(a^i_{\tau^{\overline{A}}(a)}\right)\right\rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\tau^{\overline{A}}(a)}\right],
\end{aligned}$$

as desired.

Next, suppose that φ is $\neg\psi$, where we know our result for ψ. Then φ' is $\neg\psi'$, and

$$\begin{aligned}
\overline{A} \models \varphi'[a] \quad &\text{iff} \quad \text{not } (\overline{A} \models \psi'[a]) \\
&\text{iff} \quad \text{not } \left(\overline{A} \models \psi\left[a^i_{\tau^{\overline{A}}(a)}\right]\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\tau^{\overline{A}}(a)}\right],
\end{aligned}$$

as desired.

Next, suppose that φ is $\psi \wedge \chi$, where we know our result for ψ and χ. Then φ' is $\psi' \wedge \chi'$, and

$$\begin{aligned}
\overline{A} \models \varphi'[a] \quad &\text{iff} \quad \overline{A} \models \psi'[a] \text{ and } \overline{A} \models \chi'[a] \\
&\text{iff} \quad \overline{A} \models \psi\left[a^i_{\tau^{\overline{A}}(a)}\right] \text{ and } \overline{A} \models \chi\left[a^i_{\tau^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\tau^{\overline{A}}(a)}\right],
\end{aligned}$$

as desired.

Finally, suppose that φ is $\exists v_j\chi$.

Case 1. $j = i$. Then $\varphi' = \varphi$ in this case, since v_i does not occur free anywhere in φ. Hence the desired conclusion follows by 4.6.

Case 2. $j \neq i$.

Subcase 2.1. v_i does not occur free in φ. Then $\varphi' = \varphi$, and the desired equivalence is true by 4.6 again.

Subcase 2.2. v_i occurs free in φ. This is the main case. Note that φ' is $\exists v_j\chi'$, where χ' is obtained from χ by replacing every free occurrence of v_i in χ by τ; moreover, (*) holds for χ as well. And by (*) and the assumption of this case, v_j does not occur in τ. Hence:

(**) For any $x \in A$ we have $\tau^{\overline{A}}(a) = \tau^{\overline{A}}(a^j_x)$, and $(a^i_{\tau^{\overline{A}}(a)})^j_x = (a^j_x)^i_{\tau^{\overline{A}}(a)}$.

Now suppose that $\overline{A} \models \varphi'[a]$. Thus $\overline{A} \models \exists v_j\chi'[a]$. Choose $x \in A$ such that $\overline{A} \models \chi'[a^j_x]$. Then by the inductive hypothesis, $\overline{A} \models \chi[(a^j_x)^i_{\tau^{\overline{A}}(a^j_x)}]$. By the above we then get $\overline{A} \models \chi[(a^j_x)^i_{\tau^{\overline{A}}(a)}]$ and further $\overline{A} \models \chi[(a^i_{\tau^{\overline{A}}(a)})^j_x]$. Hence $\overline{A} \models \exists v_j\chi[a^i_{\tau^{\overline{A}}(a)}]$, as desired.

The other direction is similar: suppose that $\overline{A} \models \exists v_j\chi[a^i_{\tau^{\overline{A}}(a)}]$. Choose $x \in A$ such that $\overline{A} \models \chi[(a^i_{\tau^{\overline{A}}(a)})^j_x]$. Then $\overline{A} \models \chi[(a^j_x)^i_{\tau^{\overline{A}}(a)}]$, hence $\overline{A} \models \chi[(a^j_x)^i_{\tau^{\overline{A}}(a^j_x)}]$. So by the inductive hypothesis $\overline{A} \models \chi'[a^j_x]$, hence $\overline{A} \models \exists v_j\chi'[a]$. □

From this lemma we can get some important elementary facts, given in the following theorems. First we need another simple lemma

Lemma 4.9. *Let φ be a formula, and let the sequence ψ be obtained from φ by replacing zero or more bound occurrences of v_i by v_j. Then ψ is a formula.*

Proof. By induction on φ. □

Theorem 4.10. (Change of bound variables) *Suppose that $\mathscr{L}$ is a first-order language and φ is a formula of $\mathscr{L}$. Let v_i and v_j be two variables, and suppose that v_j does not occur in φ. Let ψ be obtained by replacing every bound occurrence of v_i in φ by v_j. Then for any $\mathscr{L}$-structure and any $a \in {}^{\omega}A$ we have $\overline{A} \models (\varphi \leftrightarrow \psi)[a]$.*

Proof. The proof is by induction on φ, and all steps except the quantifier case are easy. So, suppose that φ is $\exists v_k\varphi'$, and we know the theorem for φ'. Note that $k \neq j$, since v_j does not occur in φ. We now consider two cases.

Case 1. $i \neq k$. Then ψ is $\exists v_k\psi'$, where ψ' is obtained from φ' by replacing all bound occurrences of v_i by v_j. Clearly v_j does not occur in φ', so the inductive assumption gives $\overline{A} \models (\varphi' \leftrightarrow \psi')[a]$. Clearly, then, $\overline{A} \models (\varphi \leftrightarrow \psi)[a]$.

Case 2. $i = k$. Then ψ is $\forall v_j\psi'$, where ψ' is obtained from φ' by replacing all occurrences of v_i, free or bound, by v_j. Let ψ'' be obtained from φ' by replacing all bound occurrences of v_i by v_j. Thus ψ' is obtained from ψ'' by replacing all free occurrences of v_i by v_j.

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{there is a } u \in A \text{ such that } \overline{A} \models \varphi'[a^i_u] \\ &\text{iff} \quad \text{there is a } u \in A \text{ such that } \overline{A} \models \varphi'[a^i_u{}^j_u]. \end{aligned}$$

Here the second equivalence holds because v_j does not occur in φ, hence not in φ', so that 4.6 applies. Now by the inductive hypothesis, for any $b \in {}^{\omega}A$ we have $\overline{A} \models (\varphi' \leftrightarrow \psi'')[b]$. Hence by the above equivalences,

$$(*) \qquad \overline{A} \models \varphi[a] \quad \text{iff} \quad \text{there is a } u \in A \text{ such that } \overline{A} \models \psi''[a^{i\ j}_{u\ u}].$$

Now no free occurrence of v_i in ψ'' is within the scope of a quantifier on v_j. So, we can apply 4.8, with $\varphi, \varphi', v_i, \tau, a$ replaced by $\psi'', \psi', v_i, v_j, a^{i\ j}_{u\ u}$ respectively. Note that

$$(a^{i\ j}_{u\ u})^{i}_{v_j^{\overline{A}}(a^{i\ j}_{u\ u})} = a^{i\ j}_{u\ u};$$

hence we get, from (*),

$$\begin{aligned} \overline{A} \models \varphi[a] \quad & \text{iff} \quad \text{there is a } u \in A \text{ such that } \overline{A} \models \psi'[a^{i\ j}_{u\ u}] \\ & \text{iff} \quad \text{there is a } u \in A \text{ such that } \overline{A} \models \psi'[a^{j}_{u}] \quad \text{by 4.6} \\ & \text{iff} \quad \overline{A} \models \psi[a]. \end{aligned}$$

□

Theorem 4.11. (Universal specification) *For any formula φ, any variable v_i, and any term τ such that*

(*) no free occurrence of v_i in φ is within the scope of a quantifier on a variable occurring in τ,

and for any $\mathscr{L}$-structure $\overline{A}$ and any $a \in {}^{\omega}A$ we have

$$\overline{A} \models (\forall v_i \varphi \to \varphi')[a],$$

where φ' is obtained from φ by replacing each free occurrence of v_i in φ by τ.

Proof. Suppose that $\overline{A} \models \forall v_i \varphi[a]$. Then $\overline{A} \models \varphi[a^{i}_{\tau^{\overline{A}}(a)}]$, so by Lemma 4.8, $\overline{A} \models \varphi'[a]$. □

The funny condition (*) in this theorem is really necessary, and understanding it helps in seeing the real difference between free and bound occurrences. For example, consider the formula $\varphi \stackrel{\text{def}}{=} \forall y \exists x (y < x)$. Then $(\omega, <) \models \varphi[a]$ for any $a \in {}^{\omega}\omega$, while if we blindly substitute x for y in $\exists x(y < x)$ we obtain the formula $\exists x(x < x)$ which does not hold under any assignment in $(\omega, <)$. We would like our substitution to lead from "true" formulas to "true" formulas.

We can use these theorems to describe exactly what we mean in going from a formula $\varphi(x, y)$ to another formula $\varphi(\sigma, \tau)$, where σ and τ are terms. Here $\varphi(x, y)$ may have free occurrences of variables other than x and y. Let $u_0, \ldots, u_{m-1}$ list all of the variables that appear in σ or τ. We replace any bound occurrences of any of these in φ by new variables, obtaining ψ. (For definiteness, one should take the first m new variables and do the replacement one-by-one. "new" means not occurring in φ, σ, or τ, and different from x and y.) Then, by definition, $\varphi(\sigma, \tau)$ is the result of replacing x and y in ψ simultaneously by

σ and τ. We illustrate this notation in the statement and proof of the following corollary, which in itself is not very important.

Corollary 4.12. *Suppose that x and y are distinct variables. Then*

$$\models \exists x(x = y \wedge \varphi(x, y)) \leftrightarrow \varphi(y, y).$$

Proof. Here we are using the informal notation just introduced. Thus when we write $\varphi(x, y)$ we just mean φ, but we are indicating variables x and y which may appear in φ. Then $\varphi(y, y)$ means that we first replace all bound occurrences of y in φ by some new variable, obtaining thereby a formula ψ, and then in ψ we replace all free occurrences of x by y, obtaining $\varphi(y, y)$. The idea here is that the change of bound variables does no harm, and fixes things up so that the substitution of y for x does not lead to any unwanted "clashes of bound variables".

Now, down to the proof of the corollary. Say x is v_i and y is v_j. First suppose that $\overline{A} \models \exists x(x = y \wedge \varphi(x, y))[a]$. Accordingly, choose $u \in A$ so that $\overline{A} \models (v_i = v_j \wedge \varphi)[a^i_u]$. Thus $a_j = u$ and $\overline{A} \models \varphi[a^i_u]$. Hence by the change of bound variable theorem, $\overline{A} \models \psi[a^i_u]$. Now since $a_j = u$, we have $v_j^{\overline{A}}(a) = u$ and hence $a^i_u = a^i_{v_j^{\overline{A}}(a)}$. Hence by Lemma 4.8 it follows that $\overline{A} \models \varphi(y, y)[a]$.

Conversely, suppose that $\overline{A} \models \varphi(y, y)[a]$. Then by Lemma 4.8 we get $\overline{A} \models \psi[a^i_{v_j^{\overline{A}}(a)}]$. Clearly also $\overline{A} \models (v_i = v_j)\left[a^i_{v_j^{\overline{A}}(a)}\right]$. So $\overline{A} \models \exists x(x = y \wedge \varphi(x, y))[a]$, as desired. □

Lemma 4.13. *(i) $\models \neg\forall x\varphi \leftrightarrow \exists x(\neg\varphi)$.*

(ii) $\models \neg\exists x\varphi \leftrightarrow \forall x(\neg\varphi)$.

(iii) If x does not occur free in ψ, then $\models (\forall x\phi \wedge \psi) \leftrightarrow \forall x(\phi \wedge \psi)$.

(iv) If x does not occur free in ψ, then $\models (\exists x\varphi \wedge \psi) \leftrightarrow \exists x(\varphi \wedge \psi)$.

(v) $\models \exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \exists x\psi$.

Proof. (i) and (ii) are clear. For (iii), suppose that $\overline{A}$ is a structure and $a \in {}^\omega A$. First suppose that $\overline{A} \models (\forall x\varphi \wedge \psi)[a]$. Say x is v_i. To show that $\overline{A} \models \forall x(\varphi \wedge \psi)$, let $u \in A$. Since $\overline{A} \models \forall x\varphi[a]$, it follows that $\overline{A} \models \varphi[a^i_u]$, and hence $\overline{A} \models (\varphi \wedge \psi)[a^i_u]$ by 4.6, as desired. If $\overline{A} \models \forall x(\varphi \wedge \psi)[a]$, then for any $u \in A$ we have $\overline{A} \models (\varphi \wedge \psi)[a^i_u]$, hence $\overline{A} \models \varphi[a^i_u]$ and $\overline{A} \models \psi[a]$ by 4.6. So $\overline{A} \models \forall x\varphi \wedge \psi)[a]$, as desired.

And (iv) is proved similarly. Finally, (v) is easy. □

Of course (iii) and (iv) in 4.13 hold also if the quantifiers are on ψ, x not occurring free in φ; we implicitly assume this version of (iii) and (iv) sometimes.

A formula is said to be in *prenex normal form* if it has the form

$$Q_0 x_0 \ldots Q_{m-1} x_{m-1} \varphi,$$

where each Q_i is $\forall$ or $\exists$, and φ is quantifier-free.

Theorem 4.14. *For any formula φ there is a formula ψ in prenex normal form such that $\models \varphi \leftrightarrow \psi$ and the same variables occur free in φ and ψ.*

Proof. We go by induction on φ. The atomic case is trivial, and passage to negation is clear by 4.13(i),(ii). Now take φ of the form $\chi \wedge \theta$. So it suffices to show that the conjunction of two formulas in prenex normal form is equivalent to one formula in prenex normal form. By changing bound variables, we may assume that all quantifiers in either of the two disjuncts are on variables not appearing in the other disjunct. Then 4.13(iii),(iv) give the desired result.

This time the induction step from φ to $\exists v_i\varphi$ is trivial. $\square$

EXERCISES

Exc. 4.1. Prove Proposition 4.7.

Exc. 4.2. Suppose that $\varphi, \psi, \chi, \theta$ are formulas, $\models \chi \leftrightarrow \theta$, and ψ is obtained from φ by replacing one or more occurrences of χ in φ by θ. Show that $\models \varphi \leftrightarrow \psi$. Hint: use induction on φ.

Exc. 4.3. Show that $\models \exists x\forall y\varphi \rightarrow \forall y\exists x\varphi$.

Exc. 4.4. Show that

$$\models \exists x[\varphi \wedge \psi \wedge \exists y(\varphi \wedge \neg\psi)] \rightarrow \exists y(\exists x\varphi \wedge \neg x = y).$$

Exc. 4.5. Show that

$$\models \exists x\varphi \wedge \exists y\psi \wedge \exists z\chi \rightarrow \exists x\exists y\exists z[\exists x(\exists y\chi \wedge \exists z\psi) \wedge \exists y(\exists z\varphi \wedge \exists x\chi) \wedge \exists z(\exists x\psi \wedge \exists y\varphi)].$$

Exc. 4.6. In $(\omega, 0, \mathbb{S})$, show that every singleton $\{m\}$ for $m \in \omega$ is definable, i.e., there is a formula $\varphi_m(x)$ with only x free such that $\{m\} = \{a \in \omega : (\omega, 0, \mathbb{S}) \models \varphi[a]\}$. Here $\mathbb{S}$ is the successor function, which assigns $m+1$ to each natural number m.

Exc. 4.7. Let $\mathscr{L}$ be a relational language. A formula φ is *standard* if every nonequality atomic part of φ has the form $\mathbf{R}v_0 \ldots v_{m-1}$, where $\mathbf{R}$ is m-ary. (Normally any sequence of m variables is allowed.) Show that for every formula φ there is a standard formula ψ such that $\models \varphi \leftrightarrow \psi$.

Exc. 4.8. In the language of rings, write down a single sentence whose models are exactly all rings.

Exc. 4.9. We describe an extension of first-order logic that can be used to make the set theoretical notation $\{a \in A : \varphi\}$ formal (rather than being treated as an abbreviation). Let $\mathscr{L}$ be a first order language, with an individual constant $\mathbf{Z}$ which will play a special role (in set theory, this can be the empty set as introduced in a definition). We define *description terms* and *description formulas* simultaneously:

(a) Any variable or individual constant is a description term.

(b) If $\mathbf{O}$ is an operation symbol of positive rank m and $\tau_0, \ldots, \tau_{m-1}$ are description terms, then $\mathbf{O}\tau_0 \ldots \tau_{m-1}$ is a description term.

(c) If $i < \omega$ and φ is a formula, then $Tv_i\varphi$ is a description term. (This is the *description operator*. $Tv_i\varphi$ should be read "the v_i such that φ, or $\mathbf{Z}$ if there is not a unique v_i such that φ".

(d) If σ and τ are description terms, then $\sigma = \tau$ is an atomic description formula.

(e) If $\mathbf{R}$ is an m-ary relation symbol and $\tau_0, \ldots, \tau_{m-1}$ are description terms, then $\mathbf{R}\tau_0 \ldots \tau_{m-1}$ is an atomic description formula.

(f) If φ and ψ are description formulas and $i < \omega$, then the following are description formulas: $\neg\varphi$, $(\varphi \wedge \psi)$, and $\exists v_i \varphi$.

Next we define the value of description terms, and satisfaction of description formulas in an $\mathscr{L}$-structure simultaneously. Let $\overline{A}$ be an $\mathscr{L}$-structure, and let $a \in {}^{\omega}A$.

(a) $v_i^{\overline{A}} = a_i$.

(b) If τ is the term $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$, then

$$\tau^{\overline{A}}(a) = \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)).$$

(c) If τ is the term $Tv_i\varphi$, then

$$\tau^{\overline{A}} = \begin{cases} \text{the } x \in A \text{ such that } \overline{A} \models \varphi[a] \text{ and } a_i = x & \text{if there is a unique such } x, \\ \mathbf{Z}^{\overline{A}} & \text{otherwise.} \end{cases}$$

(d) If φ is $\sigma = \tau$, then $\overline{A} \models \varphi[a]$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

(e) If φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, then $\overline{A} \models \varphi[a]$ iff $(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \in r_{\mathbf{R}}$.

(f) $\overline{A} \models \neg\varphi[a]$ iff it is not the case that $\overline{A} \models \varphi[a]$.

(g) $\overline{A} \models (\varphi \wedge \psi)[a]$ iff $\overline{A} \models \varphi[a]$ and $\overline{A} \models \psi[a]$.

(h) $\overline{A} \models \exists v_i \varphi[a]$ iff there is an $x \in A$ such that $\overline{A} \models \varphi[a_x^i]$.

Show that for any description formula φ there is an ordinary formula ψ with the same free variables such that $\models \varphi \leftrightarrow \psi$.

Exc. 4.10. We modify the definition of first-order language by using parentheses. Thus we add two symbols (and) to our logical symbols.

We retain in this context the same definition of terms as before. But we change the definition of formula as follows:

An atomic formula is a sequence of one of the following two sorts: $(\sigma = \tau)$, with σ and τ terms; or $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, where $\mathbf{R}$ is a relation symbol of rank m and $\sigma_0, \ldots, \sigma_{m-1}$ are terms. Then we define the collection of formulas to be the intersection of all sets of sequences of symbols such that every atomic formula is in A, and if φ and ψ are in A and $i < \omega$, then each of the following is in A: $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\exists v_i \varphi)$.

Prove the analog of Proposition 4.1(i), adding an additional condition, that in a formula the number of left parentheses is equal to the number of right parentheses, while in any proper initial segment of a formula, either there are no parentheses, or there are more left parentheses than right ones.

Exc. 4.11. Show that 0 and $\mathbb{S}$ are definable in $(\omega, <)$. That is, there are formulas $\varphi(x)$ and $\psi(x, y)$ with only the indicated free variables such that for all $a \in \omega$, $a = 0$ iff $(\omega, <) \models \varphi[a]\}$, and for all $a, b \in \omega$, $\mathbb{S}a = b$ iff $(\omega, <) \models \psi[a, b]$. Here we are working in the language of orderings.

5. The compactness theorem

Here we prove the *compactness theorem*: If a set of sentences is such that every finite subset of it has a model, then the whole set has a model. The theorem will be an easy consequence of Łoś's theorem on ultraproducts.

Theorem 5.1. (Łoś) *Suppose that $\mathscr{L}$ is a first-order language, $\overline{A} = \langle \overline{A_i} : i \in I\rangle$ is a system of $\mathscr{L}$-structures, F is an ultrafilter on I, and $a \in {}^{\omega}\prod_{i\in I} a_i$. The values of a will be denoted by $a^0, a^1, \ldots$. Let $\pi : \prod_{i\in I} A_i \to \prod_{i\in I} A_i/F$ be the natural mapping, taking each element of $\prod_{i\in I} A_i$ to its equivalence class under $\equiv_F^A$. For each $i \in I$ let $pr_i : \prod_{j\in I} A_j \to A_i$ be defined by setting $pr_i(x) = x_i$ for all $x \in \prod_{i\in I} A_i$. Suppose that φ is any formula of $\mathscr{L}$. Then*

$$\prod_{i\in I} \overline{A_i}/F \models \varphi[\pi \circ a] \text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F.$$

Proof. Because the situation and the notation are complicated, we are going to give the proof in full. For brevity let $\overline{B} = \prod_{i\in I} \overline{A_i}/F$. First we show

(1) For any term τ, $\tau^{\overline{B}}(\pi \circ a) = [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F$.

We prove (1) by induction on τ. For τ a variable v_k,

$$\tau^{\overline{B}}(\pi \circ a) = (\pi \circ a)(k) = [a^k]_F = [\langle a_i^k : i \in I\rangle]_F = [\langle v_k^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F,$$

as desired. For τ an individual constant $\mathbf{k}$,

$$\mathbf{k}^{\overline{B}}(\pi \circ a) = \mathbf{k}^{\overline{B}} = [\langle \mathbf{k}^{\overline{A_i}} : i \in I\rangle]_F = [\langle \mathbf{k}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F.$$

The inductive step:

$$\begin{aligned}
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{B}}(\pi \circ a) &= \mathbf{F}^{\overline{B}}(\sigma_0^{\overline{B}}(\pi \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(\pi \circ a)) \\
&= \mathbf{F}^{\overline{B}}([\langle \sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F, \ldots, [\langle \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F) \\
&= [\langle \mathbf{F}^{\overline{A_i}}(\sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a), \ldots, \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a)) : i \in I\rangle]_F \\
&= [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F,
\end{aligned}$$

as desired.

Now we begin the real proof of the theorem, proceding, of course, by induction on φ. Suppose that φ is $\sigma = \tau$. Then

$$\begin{aligned}
\overline{B} \models (\sigma = \tau)[\pi \circ a] &\text{ iff } \sigma^{\overline{B}}(\pi \circ a) = \tau^{\overline{B}}(\pi \circ a) \\
&\text{ iff } [\langle \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F = [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F \\
&\text{ iff } \{i \in I : \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a) = \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a)\} \in F \\
&\text{ iff } \{i \in I : \overline{A_i} \models (\sigma = \tau)[\mathrm{pr}_i \circ a]\} \in F,
\end{aligned}$$

as desired.

Now suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Then

$$\begin{aligned}\overline{B} \models \varphi[\pi \circ a] &\text{ iff } (\sigma_0^{\overline{B}}(\pi \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(\pi \circ a)) \in \mathbf{R}^{\overline{B}}\\
&\text{ iff } ([\langle \sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F, \ldots, [\langle \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F) \in \mathbf{R}^{\overline{B}}\\
&\text{ iff } \{i \in I : (\sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a), \ldots, \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a)) \in \mathbf{R}^{\overline{A_i}}\} \in F\\
&\text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F,\end{aligned}$$

as desired.

The inductive step when φ is $\neg\psi$:

$$\begin{aligned}\overline{B} \models \varphi[\pi \circ a] &\text{ iff not}(\overline{B} \models \varphi[\pi \circ a])\\
&\text{ iff not}(\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a]\} \in F)\\
&\text{ iff } I\backslash\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a]\} \in F\\
&\text{ iff } \{i \in I : \text{not}(\overline{A_i} \models \psi[\mathrm{pr}_i \circ a])\} \in F\\
&\text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F,\end{aligned}$$

as desired.

The induction step for $\wedge$:

$$\begin{aligned}\overline{B} \models (\psi \wedge \chi)[\pi \circ a] &\overline{B} \models \psi[\pi \circ a]) \text{ and } \overline{B} \models \chi[\pi \circ a]\\
&\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a]\} \in F) \text{ and } \{i \in I : \overline{A_i} \models \chi[\mathrm{pr}_i \circ a]\} \in F\\
&\text{ iff } \{i \in I : \overline{A_i} \models \psi[\mathrm{pr} \circ a] \text{ and } \overline{A_i} \models \chi[\mathrm{pr} \circ a]\} \in F\\
&\text{ iff } \{i \in I : \overline{A_i} \models (\psi \wedge \chi)[\mathrm{pr}_i \circ a]\} \in F,\end{aligned}$$

as desired.

It remains only to consider φ of the form $\exists v_k\psi$, which is the main case. We do each direction in the desired equivalence separately. First suppose that $\overline{B} \models \varphi[\pi \circ a]$. Choose $u \in \prod_{i\in I} A_i$ such that $\overline{B} \models \psi[(\pi \circ a)^k_{[u]_F}]$. Now $(\pi \circ a)^k_{[u]_F} = \pi \circ a^k_u$, so we can apply the induction hypothesis and get $\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \in F$. But for each $i \in I$ we have $(\mathrm{pr}_i \circ a^k_u) = (\mathrm{pr}_i \circ a)^k_{u(i)}$, so

$$\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \subseteq \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\},$$

and hence $\{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F$. This finishes half of what we want.

Conversely, suppose that $\{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F$. For each i in this set pick $u(i) \in A_i$ such that $\overline{A_i} \models \psi[(\mathrm{pr}_i \circ a)^k_{u(i)}]$ (using the axiom of choice). For other i's in I let $u(i)$ be any old element of A_i, just to fill out u to make it a member of $\prod_{i\in I} A_i$. Now $(\mathrm{pr}_i \circ a)^k_{u(i)} = \mathrm{pr}_i \circ a^k_u$. Thus $\{i \in I : \overline{A_1} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \in F$. By the inductive hypothesis it follows that $\overline{B} \models \psi[\pi \circ a^k_u]$. But $\pi \circ a^k_u = (\pi \circ a)^k_{[u]_F}$, so we finally get $\overline{B} \models \varphi[\pi \circ a]$. $\square$

Corollary 5.2. *Suppose that $\mathscr{L}$ is a first-order language, $\overline{A} = \langle \overline{A_i} : i \in I \rangle$ is a system of $\mathscr{L}$-structures, and F is an ultrafilter on I. Suppose that φ is any sentence of $\mathscr{L}$. Then*

$$\prod_{i \in I} \overline{A_i}/F \models \varphi \text{ iff } \{i \in I : \overline{A_i} \models \varphi\} \in F. \qquad \square$$

The compactness theorem is now easy to prove:

Theorem 5.3. (The Compactness Theorem) *Suppose that Γ is a set of sentences in a first-order language $\mathscr{L}$, and every finite subset of Γ has a model. Then Γ has a model.*

Proof. For each finite subset Δ of Γ, let $\overline{A}_\Delta$ be a model of Δ. Let $I = \{\Delta \subseteq \Gamma : \Delta$ is finite$\}$. For each $\Delta \in I$, let

$$x_\Delta = \{\Theta \in I : \Delta \subseteq \Theta\}.$$

Then the family $\{x_\Delta : \Delta \in I\}$ of subsets of I has fip. In fact, let $\Delta_0, \ldots, \Delta_{m-1} \in I$. Then clearly

$$\Delta_0 \cup \ldots \cup \Delta_{m-1} \in x_{\Delta_0} \cap \ldots \cap x_{\Delta_{m-1}},$$

as desired. Hence by Theorem 1.14, let F be an ultrafilter on I such that $x_\Delta \in F$ for all $\Delta \in I$. We claim now that $\prod_{\Delta \in I} \overline{A}_\Delta/F$ is a model of Γ. To see this, take any $\varphi \in \Gamma$. Then for any $\Delta \in x_{\{\varphi\}}$ we have $\varphi \in \Delta$, and $\overline{A}_\Delta$ is a model of Δ, hence of φ. Thus $\{\Delta \in I : \overline{A}_\Delta \models \varphi\} \supseteq x_\Delta$, so $\{\Delta \in I : \overline{A}_\Delta \models \varphi\} \in F$. By Corollary 5.2 it follows that $\prod_{\Delta \in I} \overline{A}_\Delta/F \models \varphi$, as desired. $\square$

We now indicate a few applications of the compactness theorem. There are more in the exercises.

Corollary 5.4. *Let $\mathscr{L}$ be a first order language. Then there does not exist a set Γ of sentences of $\mathscr{L}$ such that the models of Γ are exactly the finite $\mathscr{L}$-structures.*

Proof. Suppose that such a set Γ exists. Then let

$$\Theta = \Gamma \cup \left\{ \exists v_0 \ldots \exists v_{m-1} \left(\bigwedge_{i<j<m} \neg(v_i = v_j) \right) : m \in \omega \right\}.$$

Note that for a given m, the sentence $\exists v_0 \ldots \exists v_{m-1}(\bigwedge_{i<j<m} \neg(v_i = v_j)$ holds in a structure $\overline{A}$ iff A has at least m elements. Since there are $\mathscr{L}$-structures of any finite size, it follows that every finite subset of Θ has a model. So Θ itself has a model. But such a model must satisfy the indicated sentences for all $m < \omega$, and so must be infinite, contradiction. $\square$

The argument just given really shows the following stronger result:

Corollary 5.5. *Let $\mathscr{L}$ be a first order language and Γ a set of sentences of $\mathscr{L}$. If Γ has models of arbitrarily large finite size (i.e., if for every $m \in \omega$ Γ has a finite model with at least m elements), then Γ has an infinite model.* $\square$

Corollary 5.6. *If a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.*

Proof. Suppose that the conclusion fails. Note that the hypothesis implies that every model of $\Gamma \cup \{\neg\varphi\}$ is finite. The conclusion failing means that for every $m \in \omega$ there is a model of Γ with at least m elements in which φ fails. So $\Gamma \cup \{\neg\varphi\}$ has models of arbitrarily large finite size, hence by 5.5 it has an infinite model, contradiction. $\square$

Note that Corollary 5.6 can be applied in the theory of fields: let Γ be a collection of sentences saying that the $\mathscr{L}$-structure is a field of characteristic p (p fixed). Thus a sentence holds in every infinite field of characteristic p (p a prime) iff there is an m such that it holds in every field of characteristic p with at least m elements.

Corollary 5.7. *Let $\mathscr{L}$ be the language of ordering. Then there is no set Γ of sentences whose models are exactly the well-ordering structures.*

Proof. Suppose there is such a set. Let us expand the language $\mathscr{L}$ to a new one $\mathscr{L}'$ by adding an infinite sequence $\mathbf{c}_m$, $m \in \omega$, of individual constants. Then consider the following set Θ of sentences: all members of Γ, plus all sentences $\mathbf{c}_{m+1} < \mathbf{c}_m$ for $m \in \omega$. Clearly every finite subset of Θ has a model, so let $\overline{A} = (A, <, a_i)_{i<\omega}$ be a model of Θ itself. (Here a_i is the 0-ary function, i.e., element of A, corresponding to $\mathbf{c}_i$.) Then $a_0 > a_1 > \cdots$; so $\{a_i : i \in \omega\}$ is a nonempty subset of A with no least element, contradiction. $\square$

The applications of the compactness theorem so far are negative: certain things cannot be done in first-order logic. Positive applications are more interesting, but generally more lengthy. We give one example.

Proposition 5.8. *If $(A, <)$ is a partial ordering structure, then there is a relation $\prec$ such that $(A, \prec)$ is a simple ordering structure and $<$ is a subset of $\prec$.*

Proof. First we prove this for A finite. This goes by induction on $|A|$. It is obvious if $|A| = 1$. If we have proved it for all partial ordering structures $(A, <)$ with $|A| = m$, suppose that $(A, <)$ is a partial ordering structure with $|A| = m + 1$. Fix $a_0 \in A$, and let $A' = A\backslash\{a_0\}$ and $<' = < \cap (A' \times A')$. Let $\prec'$ be a simple ordering on A' such that $<'$ is a subset of $\prec'$. Let

$$X = \{u \in A' : \exists v(v < a_0 \text{ and } u \preceq' v)\},$$

and then we define a relation $\prec$ on A by setting, for any $x, y \in A$,

$$x \prec y \text{ iff } \begin{cases} x, y \in A' \text{ and } x \prec' y \\ \text{or } x \in X \text{ and } y = a_0 \\ \text{or } x = a_0 \text{ and } y \in A\backslash(X \cup \{a_0\}). \end{cases}$$

It is straightforward but tedious to check that this gives a simple order extending $<$; we go through the details. Clearly $\prec$ is irreflexive. Suppose now that $x \prec y \prec z$; we want to show that $x \prec z$. If $x, y, z \in A'$ this is obvious. So three cases remain. *Case 1.* $x = a_0$. Then $y \in A\backslash(X \cup \{a_0\})$. Then $y \notin X$, and this implies that $z \neq a_0$. Suppose that $z \in X$. Choose accordingly v such that $v < a_0$ and $z \preceq' v$. Now also $y \prec' z$, so $y \prec' v$, and hence $y \in X$, contradiction. Thus $z \notin X$, so $x \prec z$. *Case 2.* $y = a_0$. Thus $x \in X$ and $z \in A \backslash (X \cup \{a_0\})$. Since $x \in X$, choose v so that $v < a_0$ and $x \preceq' v$. Obviously $z \neq x$. Suppose that $z \prec' x$. Then $z \prec' v$ and so $z \in X$, contradiction. The only remaining possibility is that $x \prec' z$, hence $x \prec z$. *Case 3.* $z = a_0$. Thus $y \in X$ and $x \prec' y$. It is clear then from the definition of X that also $x \in X$, and so $x \prec' z$.

Next we show that $(A, \prec)$ is a simple ordering structure. Suppose that x and y are distinct elements of A. If both of them are in A', then $x \prec' y$ or $y \prec' x$, hence $x \prec y$ or $y \prec x$. Suppose that one of them, say x, is a_0. Then either $y \in X$, in which case $y \prec x$, or $y \notin X$, in which case $x \prec y$.

It remains just to show that $<$ is a subset of $\prec$. Suppose that $x < y$. If $x, y \in A'$, then $x \prec' y$, hence $x \prec y$. Suppose that $x = a_0$. If $y \in X$, choose v so that $v < a_0$ and $y \preceq' v$. Then $v < a_0 < y$, so $v < y$, hence $v \prec' y$, contradiction. So $y \notin X$, and hence $x \prec y$. Finally, suppose that $y = a_0$. Then $x \in X$ by definition of X, so $x \prec y$.

This finishes the case when A is finite. Note that all of this is just standard set theory, no logic involved.

Now we take a partial ordering structure $(A, <)$ with A infinite. Let $\mathscr{L}$ be a first-order language which has a binary relation symbol $<$ and, for each $a \in A$, an individual constant $\mathbf{c}_a$. In this language we consider the following set Γ of sentences:

$$\begin{array}{l} \text{Sentences saying that } < \text{ gives a simple order} \\ \mathbf{c}_a < \mathbf{c}_b \quad \text{whenever } a < b, \text{ for all } a, b \in A. \end{array}$$

By our previous argument, every finite subset of Γ has a model. So Γ itself has a model $(B, <, b_a)_{a \in A}$, where b_a is the denotation of $\mathbf{c}_a$ in the model. Now we define $a \prec a'$ iff $a, a' \in A$ and $b_a < b_{a'}$. Clearly $(A, \prec)$ is a simple ordering structure. If $a, a' \in A$ and $a < a'$, then the sentence $\mathbf{c}_a < \mathbf{c}_{a'}$ is in Γ, and so this sentence holds in $(B, <, b_a)_{a \in A}$, which means that $b_a < b_{a'}$ and hence $a \prec a'$. $\square$

The following concept will play an important role in what follows. Two structures $\overline{M}, \overline{N}$ are *elementarily equivalent* iff for every sentence φ, $\overline{M} \models \varphi$ iff $\overline{N} \models \varphi$.

EXERCISES

Exc. 5.1. Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Let F be a nonprincipal ultrafilter on a set I. For each $a \in A$ let $f(a) = [\langle a : i \in I \rangle]_F$. Show that f is an embedding of $\overline{A}$ into ${}^I\overline{A}/F$, and $\overline{A}$ is elementarily equivalent to ${}^I\overline{A}/F$.

Exc. 5.2. We work in the language for ordered fields; see Chapter 1. In general, an element $a \in M$ is *definable* iff there is a formula $\varphi(x)$ with one free variable x such that $\{b \in M : \overline{M} \models \varphi[b]\} = \{a\}$.

(i) Show that 1 is definable in $\mathbb{R}$.

(ii) Show that every positive integer is definable in $\mathbb{R}$.

(iii) Show that every positive rational is definable in $\mathbb{R}$.

(iv) If $\overline{M}$ is an extension of $\mathbb{R}$, an element ε of M is *infinitesimal* iff $0 < \varepsilon < r$ for every positive rational r. Let $\overline{M}$ be an ultrapower of $\mathbb{R}$ using a nonprincipal ultrafilter on ω. Thus $\overline{M}$ is isomorphic to an extension of $\mathbb{R}$ by exercise 5.1. Show that $\overline{M}$ has an infinitesimal.

(v) Use the compactness theorem to show the existence of an ordered field $\overline{M}$ which has an infinitesimal and is elementarily equivalent to $\mathbb{R}$.

Exc. 5.3. Consider the structure $\overline{N} = (\omega, +, \cdot, 0, 1, <)$. We look at models of $\Gamma = \{\varphi : \varphi$ is a sentence and $\overline{N} \models \varphi\}$.

(i) For every $m \in \omega$ there is a formula φ_m with one free variable x such that $\overline{N} \models \varphi_m[m]$ and $\overline{N} \models \exists! x \varphi_m(x)$.

(ii) $\overline{N}$ can be embedded in any model of Γ.

(iii) Show that Γ has a model with an infinite element in it, i.e., an element greater than each $m \in \omega$.

Exc. 5.4. (Continuing exercise 5.3.) An element p of a model $\overline{M}$ of Γ is a *prime* iff $p > 1$ and for all $a, b \in M$, if $p = a \cdot b$ then $a = 1$ or $a = p$.

(i) Prove that if $\overline{M}$ is a model of Γ with an infinite element, then it has an infinite prime element.

(ii) Show that the following conditions are equivalent:

(a) There are infinitely many (ordinary) primes p such that $p + 2$ is also prime. (The famous twin prime conjecture, unresolved at present.)

(b) There is a model $\overline{M}$ of Γ having at least one infinite prime p such that $p + 2$ is also a prime.

(c) For every model $\overline{M}$ of Γ having an infinite element, there is an infinite prime p such that $p + 2$ is also a prime.

Exc. 5.5. Let G be a group which has elements of arbitrarily large finite order. Show that there is a group H elementarily equivalent to G which has an element of infinite order.

Exc. 5.6. Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.

6. Basic model theory

We survey important notions and results in model theory.

Isomorphisms

Theorem 6.1. *Suppose that h is an isomorphism from $\overline{A}$ onto $\overline{B}$, where these are $\mathscr{L}$-structures. Suppose that $a \in {}^{\omega}A$, σ is a term, and φ is any formula. Then $h(\sigma^{\overline{A}}(a)) = \sigma^{\overline{B}}(h \circ a)$. And $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[h \circ a]$. Finally, $\overline{A}$ and $\overline{B}$ are elementarily equivalent.*

Proof. We prove the first statement by induction on σ. For σ a variable v_i we have $h(v_i^{\overline{A}}(a)) = h(a_i) = (h \circ a)_i = v_i^{\overline{B}}(h \circ a)$. For k an individual constant, $h(k^{\overline{A}}(a)) = h(k^{\overline{A}}) = k^{\overline{B}} = k^{\overline{B}}(h \circ a)$. The inductive step, with $\tau = \mathbf{F}\sigma_0 \ldots \sigma_{m-1}$:

$$\begin{aligned} h(\tau^{\overline{A}}(a)) &= h(\mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))) \\ &= \mathbf{F}^{\overline{B}}(h(\sigma_0^{\overline{A}}(a)), \ldots, h(\sigma^{\overline{A}}(a))) \\ &= \mathbf{F}^{\overline{B}}(\sigma_0^{\overline{B}}(h \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(h \circ a)) \\ &= \tau^{\overline{B}}(h \circ a), \end{aligned}$$

as desired.

We prove the second statement by induction on φ. The atomic cases:

$$\begin{aligned} \overline{A} \models (\sigma = \tau)[a] &\text{ iff } \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \\ &\text{ iff } h(\sigma^{\overline{A}}(a)) = h(\tau^{\overline{A}}(a)) \\ &\text{ iff } \sigma^{\overline{B}}(h \circ a) = \tau^{\overline{B}}(h \circ a) \\ &\text{ iff } \overline{B} \models (\sigma = \tau)[h \circ a]; \end{aligned}$$

$$\begin{aligned} \overline{A} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}[a] &\text{ iff } (\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)) \in \mathbf{R}^{\overline{A}} \\ &\text{ iff } (h(\sigma_0^{\overline{A}}(a)), \ldots, h(\sigma_{m-1}^{\overline{A}}(a))) \in \mathbf{R}^{\overline{B}} \\ &\text{ iff } (\sigma_0^{\overline{B}}(h \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(h \circ a)) \in \mathbf{R}^{\overline{B}} \\ &\text{ iff } \overline{B} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}[h \circ a]. \end{aligned}$$

The induction steps for $\neg$ and $\wedge$:

$$\begin{aligned} \overline{A} \models \neg\psi[a] &\text{ iff not}(\overline{A} \models \psi[a]) \\ &\text{ iff not}(\overline{B} \models \psi[h \circ a]) \\ &\text{ iff } \overline{B} \models \neg\psi[h \circ a]; \end{aligned}$$

$$\begin{aligned} \overline{A} \models (\psi \wedge \chi)[a] &\text{ iff } \overline{A} \models \psi[a]) \text{ and } \overline{A} \models \chi[a] \\ &\text{ iff } (\overline{B} \models \psi[h \circ a]) \text{ and } \overline{B} \models \chi[h \circ a] \\ &\text{ iff } \overline{B} \models (\psi \wedge \chi)[h \circ a]. \end{aligned}$$

Finally, for the $\exists$ induction step, by symmetry we go one direction only. Suppose that $\overline{A} \models \exists v_i \psi[a]$. Choose $u \in A$ such that $\overline{A} \models \psi[a^i_u]$. Then by the inductive hypothesis, $\overline{B} \models \psi[h \circ a^i_u]$. Now $h \circ a^i_u = (h \circ a)^i_{h(u)}$, so $\overline{B} \models \varphi[h \circ a]$.

The final statement of the theorem now follows in a clear fashion. $\square$

Although this theorem is simple, it is important and useful. An example of a non-obvious use of it is to check this little fact: in the language with equality alone, with any nonempty set A as a structure to look at, the only definable subsets of A are $\emptyset$ and A. They are defined by the following formulas $\varphi(x)$: $x \neq x$ and $x = x$. Suppose that $\emptyset \subset X \subset A$ and $X = \{a \in A : A \models \psi[a]\}$, where ψ is a formula having only x free. Choose $a \in X$ and $b \in A \backslash X$. Let h be the transposition (a, b), as a permutation of A. (Here (a, b) is not the ordered pair (a, b).) Thus $A \models \psi[a]$, so by the theorem, $A \models \psi[b]$, so $b \in X$, contradiction.

Note that a theory is complete iff any two of its models are elementarily equivalent.

Elimination of quantifiers

The elimination of quantifier method is best understood by going through a simple example. Roughly speaking, a theory admits elimination of quantifiers if every formula in the theory is equivalent in the theory to a formula built up from simple sentences and quantifier-free formulas using only sentential connectives. In the example we give, only a very simple sentence is involved in these building blocks. The proof is rather direct, and for more complicated theories indirect methods are usually easier. We return to this later in this chapter.

Theorem 6.2. *In the theory $(\mathscr{L}, \Gamma)$ of dense linear order without first or last elements, for every formula φ there is a formula ψ built up from atomic formulas and the sentence $\exists v_0(v_0 = v_0)$ using $\neg$ and $\wedge$, with the same free variables as φ, such that $\Gamma \models \varphi \leftrightarrow \psi$.*

Proof. To clarify what this theory is: the language $\mathscr{L}$ has only one non-logical constant, a binary relation symbol $<$. Γ consists of the following sentences:

$$\neg\exists v_0(v_0 < v_0);$$
$$\forall v_0\forall v_1\forall v_2(v_0 < v_1 \wedge v_1 < v_2 \rightarrow v_0 < v_2);$$
$$\forall v_0\forall v_1(v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0);$$
$$\forall v_0\forall v_1[v_0 < v_1 \rightarrow \exists v_2(v_0 < v_2 \wedge v_2 < v_1)];$$
$$\forall v_0\exists v_1(v_1 < v_0);$$
$$\forall v_0\exists v_1(v_0 < v_1).$$

Now to prove the theorem, we proceed by induction on φ. The atomic case, and the induction steps involving $\neg$ and $\wedge$ are obvious. Now assume that we know the result for φ', and φ is $\exists x\varphi'$. Thus there is a quantifier-free formula ψ' with the same free variables as φ' such that $\Gamma \models \varphi' \leftrightarrow \psi'$. Hence $\Gamma \models \exists x\varphi' \leftrightarrow \exists x\psi'$. So we have arrived at this conclusion:

(*) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is quantifier-free.

Now let θ be a formula in disjunctive normal form with the same free variables as χ such that $\models \chi \leftrightarrow \theta$. So $\models \exists x\chi \leftrightarrow \exists x\theta$. Using the universally valid formula $\exists x(\mu \vee \rho) \leftrightarrow \exists x\mu \vee \exists x\rho$, we thus see:

(**) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of atomic formulas and their negations.

Now we can make several additional simplifications. By the universally valid formula $\models \exists x(\mu \wedge \rho) \leftrightarrow \exists x\mu \wedge \rho$ if x does not occur in ρ, we may assume that each conjunct of χ actually involves x. We may assume that in χ no formula is repeated, and no formula is found along with its negation (if the latter happens, φ is actually equivalent to $\neg\exists v_0(v_0 = v_0)$). The formulas $x = x$ and $\neg(x = x)$ may be eliminated. The formula $y = x$ can be replaced by $x = y$. If $x = y$ is one of the conjuncts of χ, then $\models \exists x\chi \leftrightarrow \theta$, where θ is obtained from χ by replacing each occurrence of x by y. And we may rearrange the conjuncts. So we arrive at:

(***) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of the following form:

$$(****)\qquad \begin{aligned}&\neg(x = y_1) \wedge \ldots \wedge \neg(x = y_k) \wedge\\ &x < z_1 \wedge \ldots \wedge x < z_l \wedge\\ &w_1 < x \wedge \ldots \wedge w_m < x \wedge\\ &\neg(x < u_1) \wedge \ldots \wedge \neg(x < u_n) \wedge\\ &\neg(s_1 < x) \wedge \ldots \wedge \neg(s_p < x).\end{aligned}$$

Here some of the integers k, l, m, n, p can be zero; the variables $y_1, \ldots, y_k$ are all distinct among themselves and different from x. Similarly for the z_i's and w_i's. The variables u_i are distinct among themselves, but one of them might equal x. Similarly for the variables s_i.

What we have described so far is a procedure common to most applications of the elimination of quantifiers method. In particular, we have not used the order axioms yet.

Now we can use those axioms: since $\Gamma \models \forall v_0 \neg(v_0 < v_0)$, we may assume that none of the u_i's or s_i's are equal to x. Also, $\Gamma \models (\neg(v_0 < v_1) \leftrightarrow v_0 = v_1 \vee v_1 < v_0)$; so using this, distributive laws, and the way equality was eliminated above, we may assume that $n = 0 = p$. Furthermore, $\Gamma \models (\neg(v_0 = v_1) \leftrightarrow v_0 < v_1 \vee v_1 < v_0$. So again using distributive laws we may assume that $k = 0$. Thus we have arrived at

(*****) It suffices to prove the theorem under the assumption that φ has the form $\exists x\chi$, where χ is a conjunction of the following form:

$$x < z_1 \wedge \ldots \wedge x < z_l \wedge w_1 < x \wedge \ldots \wedge w_m < x.$$

If $l = 0$, this is equivalent under Γ to $\exists v_0(v_0 = v_0)$ by the "no last element" sentence. If $m = 0$, it is equivalent under Γ to $\exists v_0(v_0 = v_0)$ by the "no first element" sentence. If $l \neq 0 \neq m$, it is equivalent under Γ to

$$z_1 < w_1 \wedge \ldots \wedge z_1 < w_m \wedge$$

$$z_2 < w_1 \wedge \ldots \wedge z_2 < w_m \wedge$$
$$\ldots$$
$$z_l < w_1 \wedge \ldots \wedge z_l < w_m,$$

by the "denseness" sentence. □

Corollary 6.3. *The only definable subsets of* $\mathbb{Q}$ *in the structure* $(\mathbb{Q}, <)$ *are* $\emptyset$ *and* $\mathbb{Q}$.

Proof. Suppose that A is a definable subset of $\mathbb{Q}$ in $(\mathbb{Q}, <)$. Thus there is a formula φ with only v_0 as a possible free variable such that $A = \{a \in \mathbb{Q} : (\mathbb{Q}, <) \models \varphi[a]\}$. By Theorem 6.2, let ψ be a formula built up from $\exists v_0(v_0 = v_0)$ and atomic formulas using only $\neg$ and $\wedge$, and with only v_0 as a possible free variable such that $\Gamma \models \varphi \leftrightarrow \psi$. This means that up to logical equivalence ψ is either $\exists v_0(v_0 = v_0)$ or $\neg\exists v_0(v_0 = v_0)$. So $A = \emptyset$ or $A = \mathbb{Q}$. □

Ehrenfeucht-Fraissé games

We explain now a game-theoretic characterization of elementary equivalence, due independently to Ehrenfeucht and Fraissé. Actually Fraissé's method does not explicitly involve games, while Ehrenfeucht's does; but they are really easily seen to be equivalent. We restrict ourselves to the game formulation.

The formulation of the game depends on the notion of a partial isomorphism. Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures. A *partial isomorphism* between $\overline{A}$ and $\overline{B}$ is a function f mapping a subset of A into B such that for any $m \in \omega$, any atomic formula φ with free variables among $v_0, \ldots, v_{m-1}$, and any $a \in {}^m\mathrm{dmn}(f)$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$.

Now we describe the $(m, \overline{A}, \overline{B})$-elementary game, where m is a positive integer and $\overline{A}$ and $\overline{B}$ are $\mathscr{L}$-structures. The game takes place between two players, ISO and NON-ISO, and each player gets m moves. For ease of reference we think of ISO as feminine, NON-ISO as maxculine. NON-ISO moves first, and the two players take turns through m plays, numbered $0, \ldots, m-1$. At his i-th turn, NON-ISO picks $\varepsilon \in \{0, 1\}$, and if $\varepsilon = 0$ he picks an element $a_i \in A$, while if $\varepsilon = 1$ he picks an element $b_i \in B$. Then ISO uses the ε that NON-ISO chose, and she picks $b_i \in B$ if $\varepsilon = 0$, and $a_i \in A$ if $\varepsilon = 1$ (the opposite thing from NON-ISO). Thus each play in the game produces a pair (a_i, b_i). At the end of the game, ISO wins if $\{(a_i, b_i) : i < m\}$ is a partial isomorphism, otherwise NON-ISO wins. A *winning strategy* for ISO is a rule that unambiguously tells her how to play at each step, given what has happened up to that point. More precisely, let us define a *state* to be a quadruple (a, b, ε, c) such that, for some $i < m-1$ we have $a \in {}^iA$, $b \in {}^iB$, $\varepsilon \in \{0, 1\}$, and $c \in A$ if $\varepsilon = 0$, $c \in B$ if $\varepsilon = 1$. Then a *strategy* for ISO is a function F defined on all states such that $F(a, b, \varepsilon, c) \in B$ if $\varepsilon = 0$ and $F(a, b, \varepsilon, c) \in A$ if $\varepsilon = 1$. A *complete play* of the game is a triple $(\varepsilon, a, b) \in {}^m2 \times {}^mA \times {}^mB$. Such a complete play is *played according to the strategy F for ISO* provided that, for each $i < m-1$, either

$\varepsilon_i = 0$ and $b_i = F(a \restriction i, b \restriction i, 0, a_i)$ or
$\varepsilon_i = 1$ and $a_i = F(a \restriction i, b \restriction i, 1, b_i)$.

Finally, we say that F is a *winning strategy* for ISO provided that ISO wins whenever she plays according to F.

Theorem 6.4. (Ehrenfeucht) *Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures, and let m be a positive integer. Suppose that* ISO *has a winning strategy for the $(m, \overline{A}, \overline{B})$-elementary game. Then $\overline{A} \models \varphi$ iff $\overline{B} \models \varphi$, for every sentence φ in prenex normal form with at most m initial quantifiers.*

Proof. Let F be a winning strategy for ISO for the $(m, \overline{A}, \overline{B})$-elementary game. Let φ be a sentence $Q_0 v_0 \ldots Q_{m-1} v_{m-1} \psi$ where ψ is quantifier-free and each Q_i is $\exists$ or $\forall$. If a sentence in prenex normal form has fewer than m initial quantifiers or some are repeated, then the outer most of the repeated ones can be deleted; then we can put at the beginning extra quantifiers on variables not appearing in the sentence to make the total equal to m. And if the variables are not $v_0, \ldots, v_{m-1}$ we can rename bound variables to make them $v_0, \ldots, v_{m-1}$ in that order. Thus it suffices to treat the case indicated.

We prove the following statement by downward induction, from $k = m$ to $k = 0$:

(1) For every $k \leq m$ and every complete play (ε, a, b) of the game according to F,

$$\overline{A} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[a \restriction k] \quad \text{iff} \quad \overline{B} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[b \restriction k].$$

The case $k = m$ is given by the assumption that ISO wins the game. Now suppose that (1) holds for $k + 1$; we prove it for k. So, assume the hypothesis of (1) for k. Because of the symmetry involved, we now take only the case $Q_k = \forall$, and prove only $\Rightarrow$. Thus assume that $\overline{A} \models Q_k v_k \ldots Q_{m-1} v_{m-1} \psi[a \restriction k]$. Let y be any element of B; we want to show that $\overline{B} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[b_0, \ldots, b_{k-1}, y]$. We now consider the following complete play of the game according to F. The players play just as in (ε, a, b) through stage k, thus producing $a_0, \ldots, a_{k-1}$ and $b_0, \ldots b_{k-1}$. Now NON-ISO chooses 1 and y. So ISO chooses (according to F) $x \in A$. Then let the play continue according to F, with NON-ISO choosing what he pleases, producing a complete play (ε', a', b'). . Since we are assuming

$$\overline{A} \models \forall v_k Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[a \restriction k],$$

we get

$$\overline{A} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[a_0, \ldots, a_{k-1}, x].$$

Then the inductive hypothesis yields

$$\overline{B} \models Q_{k+1} v_{k+1} \ldots Q_{m-1} v_{m-1} \psi[b_0, \ldots, b_{k-1}, y],$$

as desired. $\square$

Corollary 6.5. (Ehrenfeucht) *If* ISO *has a winning strategy for the $(m, \overline{A}, \overline{B})$-elementary game for every positive integer m, then $\overline{A}$ and $\overline{B}$ are elementarily equivalent.*

We give one application of elementary games.

Proposition 6.6. *Let Γ be the theory of an infinite equivalence relation with infinitely many equivalence classes, each of which is infinite. Then Γ is complete. More precisely,* *the language has only one non-logical constant, a binary relation symbol E, and Γ consists of the following sentences:*

$\forall x[xEx]$

E is symmetric and transitive

$$\forall v_0 \ldots \forall v_{n-1}\left[\bigwedge_{i<j<n} \neg(v_iEv_j) \rightarrow \exists v_n\left[\bigwedge_{i<n} \neg(v_iEv_n)\right]\right]$$

for each positive integer n

$$\forall v_0 \ldots \forall v_{n-1}\left[\bigwedge_{i<j<n} \neg(v_i = v_j) \wedge \bigwedge_{i<j<n} v_iEv_j \rightarrow \exists v_n\left[v_0Ev_n \wedge \bigwedge_{i<n} \neg(v_i = v_n)\right]\right]$$

for each positive integer n.

Proof. Assume that $\overline{A}$ and $\overline{B}$ are models of Γ and m is a positive integer. The strategy of ISO is as follows. Suppose that we are at the i-th turn and NON-ISO chooses 0 and an element $a \in A$. The move of ISO depends on the following possibilities. If the turns so far have not produced a partial isomorphism, then ISO selects any element of B. Suppose that the turn so far have produced a partial isomorphism f.

Case 1. No element of A equivalent to a has been selected yet. Then ISO picks an element of B not equivalent to any element selected so far.

Case 2. There is an element $a' \in A$ which has already been selected which is equivalent to a, while a itself has not been previously selected. Then ISO picks an element of B equivalent to $f(a')$ which has not yet been selected.

Case 3. a has already been selected. Then ISO picks $f(a)$.

If NON-ISO choose 1 and an element of B, ISO does a similar thing, interchanging the roles of A and B.

Clearly this produces a partial isomorphism. □

Diagrams

A language $\mathscr{L}'$ is an *expansion* of a language $\mathscr{L}$ iff $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding new non-logical symbols. Given an $\mathscr{L}$-structure $\overline{A}$, an *expansion* of $\overline{A}$ to $\mathscr{L}'$ is a structure obtained from $\mathscr{A}$ by adjoining interpretations of the new symbols.

Going the other way, $\mathscr{L}$ is called a *reduct* of $\mathscr{L}'$ and $\overline{A}$ is called a reduct of $\overline{B}$.

Right now we are interested in the case of adding new individual constants. Let $\mathscr{L}$ be any language, and let A be any set. Then $\mathscr{L}_A$ is the expansion of $\mathscr{L}$ by adding new individual constants c_a for $a \in A$. Given any $\mathscr{L}$-structure $\overline{B}$ and a system $\langle b_a : a \in A\rangle$ of elements of B, by $\overline{B}_b = (\overline{B}, b_a)_{a\in A}$ we mean the expansion of $\overline{B}$ to an $\mathscr{L}_A$-structure such that $c_a^{\overline{B}_b} = b_a$ for all $a \in A$. As a special case we have the expansion $(\overline{A}, a)_{a\in A}$ of $\overline{A}$ itself, which we denote by $\overline{A}_A$ instead of $\overline{A}_{\mathrm{Id}\restriction A}$.

Let $\mathscr{L}$ be a language and $\overline{A}$ an $\mathscr{L}$-structure. The *diagram* of $\overline{A}$, denoted by $\mathrm{Diag}(\overline{A})$, is the set of all atomic sentences and negations of atomic sentences which hold in the structure $\overline{A}_A$. This takes place in the language $\mathscr{L}_A$.

Lemma 6.7. *Assume that σ is a term in $\mathscr{L}$, $e \in {}^{\omega}A$, σ_e is obtained from σ by replacing each variable v_i by $c_{e(i)}$ (the individual constant of $\mathscr{L}_A$ associated with $e(i)$), $\overline{D}$ is an $\mathscr{L}$-structure, and $d \in {}^{A}D$.*

Then $\sigma^{\overline{D}}(d \circ e) = \sigma_e^{\overline{D}_d}$.

Proof. This is immediate from the proof of Lemma 4.8. □

Theorem 6.8. (Diagram lemma) *For any $\mathscr{L}$-structures $\overline{A}$ and $\overline{B}$, and any $f : A \to B$ the following conditions are equivalent:*

(i) f is an isomorphism from $\overline{A}$ into $\overline{B}$.

(ii) $(\overline{B}, f(a))_{a\in A}$ is a model of $\mathrm{Diag}(\overline{A})$.

Proof. (i)⇒(ii): Let f be an isomorphism from $\overline{A}$ into $\overline{B}$.

Suppose that σ and τ are variable-free terms of $\mathscr{L}_A$ and $\overline{A}_A \models \sigma = \tau$. Thus $\sigma^{\overline{A}_A} = \tau^{\overline{A}_A}$. Now there is a sequence $e \in {}^{\omega}A$ and terms ρ, ξ such that $\sigma = \rho_e$ and $\tau = \xi_e$. Thus $\rho_e^{\overline{A}_A} = \xi_e^{\overline{A}_A}$. By Lemma 6.7 we then get $\rho^{\overline{A}}(e) = \xi^{\overline{A}}(e)$. (The "$d$" in Lemma 6.7 is the identity.) By Proposition 2.4(i) we then get $\rho^{\overline{B}}(f \circ e) = \xi^{\overline{B}}(f \circ e)$. Then by Lemma 6.7 again we have $\sigma^{\overline{B}_f} = \rho_e^{\overline{B}_f} = \rho^{\overline{B}}(f \circ e) = \xi^{\overline{B}}(f \circ e) = \xi_e^{\overline{B}_f} = \tau^{\overline{B}_f}$. Hence $\overline{B}_f \models \sigma = \tau$.

Almost exactly the same proof shows that $\overline{A}_A \models \neg(\sigma = \tau)$ implies that $\overline{B}_f \models \neg(\sigma = \tau)$.

Next, suppose that R is an m-ary relation symbol, $\sigma_0, \ldots, \sigma_{m-1}$ are variable-free terms of $\mathscr{L}_A$, and $\overline{A}_A \models R\sigma_0 \ldots \sigma_{m-1}$. Thus $\langle \sigma_0^{\overline{A}_A}, \ldots, \sigma_{m-1}^{\overline{A}_A} \rangle \in R^{\overline{A}_A}$. Now there is a sequence $e \in {}^{\omega}A$ and terms ρ_i for $i < m$ such that $\rho_{ie} = \sigma_i$ for each $i < m$. Thus $\langle \rho_{0e}^{\overline{A}_A}, \ldots, \rho_{(m-1)e}^{\overline{A}_A} \rangle \in R^{\overline{A}_A}$. By Lemma 6.7 we then get $\langle \rho_0^{\overline{A}}(e), \ldots, \rho_{m-1}^{\overline{A}}(e) \rangle \in R^{\overline{A}_A}$. Hence $\langle f(\rho_0^{\overline{A}}(e)), \ldots, f(\rho_{m-1}^{\overline{A}}(e)) \rangle \in R^{\overline{B}}$. By Proposition 2.4(i), $f(\rho_i^{\overline{A}}(e)) = \rho_i^{\overline{B}}(f \circ e)$ for each $i < m$. By Lemma 6.7 again, $\rho_i^{\overline{B}}(f \circ e) = \rho_{ie}^{\overline{B}_f}$ for each $i < m$. Thus $\langle \rho_{0e}^{\overline{B}_f}, \ldots, \rho_{(m-1)e}^{\overline{B}_f} \rangle \in R^{\overline{B}_f}$. Since $\rho_{ie} = \sigma_i$ for each $i < m$, we infer that $\overline{B}_f \models R\sigma_0 \ldots \sigma_{m-1}$.

The same steps work for $\neg R\sigma_0 \ldots \sigma_{m-1}$.

(ii)⇒(i): Assume (ii). Assume that $\overline{B}_f$ is a model of the diagram of $\overline{A}$. We claim that f is the desired isomorphism. To show that f is one-to-one, suppose that $x, y \in A$ and $f(x) = f(y)$. Then $\overline{B}_f \models c_x = c_y$. Hence $\overline{A}_A \models c_x = c_y$, as otherwise $\neg(c_x = c_y) \in \mathrm{Diag}(\overline{A})$, contradicting the fact that $\overline{B}_f$ is a model of the diagram. Hence $x = y$. So f is one-one.

For k an individual constant, the sentence $c_{k^{\overline{A}}} = k$ is in $\mathrm{Diag}(\overline{A})$, hence $\overline{B}_f \models c_{k^{\overline{A}}} = k$, so that $f(k^{\overline{A}}) = c_{k^{\overline{A}}}^{\overline{B}_f} = k^{\overline{B}_f} = k^{\overline{B}}$.

Next, suppose that F is an m-ary operation symbol and $e \in {}^{m}A$; we want to show that $f(F^{\overline{A}}(e_0, \ldots, e_{m-1})) = F^{\overline{B}}(f(e_0), \ldots, f(e_{m-1}))$. Let $u = F^{\overline{A}}(e_0, \ldots, e_{m-1})$. Then

$Fc_{e_0}\ldots c_{e_{m-1}} = c_u$ is in the diagram, and so $\overline{B}_f$ is a model of it. Hence $(Fc_{e_0}\ldots c_{m-1})^{\overline{B}_f} = c_u^{\overline{B}_f}$, so

$$F^{\overline{B}_f}(f(e_0),\ldots,f(e_{m-1})) = f(u) = f(F^{\overline{A}}(e_0,\ldots,e_{m-1})),$$

as desired.

Finally, suppose that R is an m-ary relation symbol, $e \in {}^mA$, and $e \in R^{\overline{A}}$. Then $Rc_{e_0}\ldots c_{e_{m-1}}$ is in the diagram, and so it holds in $\overline{B}_f$. Hence $\langle f(e_0),\ldots,f(e_{m-1})\rangle \in R^{\overline{B}_f}$. Similarly starting with $e \notin R^{\overline{A}}$. □

For the next theorem, recall the notion of finite generation from just before Theorem 1.11.

Theorem 6.9. (Henkin) *Let Γ be a set of sentences in a language $\mathscr{L}$, and let $\overline{A}$ be an $\mathscr{L}$-structure. Then $\overline{A}$ can be isomorphically embedded in some model of Γ iff every finitely generated substructure of $\overline{A}$ can be so embedded.*

Proof. $\Rightarrow$ is trivial, so assume that every finitely generated substructure of $\overline{A}$ can be isomorphically embedded in a model of Γ; we want to show that $\overline{A}$ itself can be so embedded. By the diagram lemma it suffices to show that the set $\Gamma \cup \mathrm{Diag}(A)$ has a model. By the compactness theorem it suffices to take finite subsets Δ of Γ and Θ of $\mathrm{Diag}(\overline{A})$ and show that $\Delta \cup \Theta$ has a model. Let $B = \{a \in A : c_a$ occurs in some sentence of Θ; so B is a finite subset of A. Hence there is a finitely generated substructure $\overline{C}$ of $\overline{A}$ such that $B \subseteq C$. Let $f : \overline{C} \to \overline{D}$ be an isomorphism from $\overline{C}$ into a model $\overline{D}$ of Γ. Choose $g : A \to D$ extending f. We claim that $\overline{D}_g$ is a model of Θ. To prove this, take any $\varphi \in \Theta$.

First suppose that φ is $\sigma = \tau$ with σ and τ variable-free terms of $\mathscr{L}_A$. Then there are terms ρ, ξ of $\mathscr{L}$ and $e \in {}^\omega B$ such that $\rho_e = \sigma$ and $\xi_e = \tau$. Note that $f \circ e = g \circ e$. Then

$$\begin{aligned}
\sigma^{\overline{D}_g} &= \rho_e^{\overline{D}_g} = \rho^{\overline{D}}(g \circ e) = \rho^{\overline{D}}(f \circ e) = f(\rho^{\overline{C}}(e)) = f(\rho^{\overline{A}}(e)) = f(\rho_e^{\overline{A}_A}) = f(\sigma^{\overline{A}_A}) \\
&= f(\tau^{\overline{A}_A}) = f(\xi_e^{\overline{A}_A}) = f(\xi^{\overline{A}}(e) = f(\xi^{\overline{C}}(e)) = \xi^{\overline{D}}(f \circ e) = \xi^{\overline{D}}(g \circ e) = \xi_e^{\overline{D}_g} = \tau^{\overline{D}_g}.
\end{aligned}$$

Hence $\overline{D}_g \models \sigma = \tau$.

The case when φ is $\neg(\sigma = \tau)$ is treated similarly.

Second, suppose that φ is $R\sigma_0\ldots\sigma_{m-1}$ with each σ_i a variable-free term of $\mathscr{L}_A$. Then there are terms ρ_i of $\mathscr{L}$ for $i < m$ and $e \in {}^\omega B$ such that $\rho_{ie} = \sigma_i$ for all $i < m$. Again, $f \circ e = g \circ e$.

We have $\overline{A}_A \models R\sigma_0\ldots\sigma_{m-1}$. Thus

$$\langle \sigma_0^{\overline{A}_A},\ldots,\sigma_{m-1}^{\overline{A}_A}\rangle \in R^{\overline{A}}.$$

So

$$\langle \rho_{0e}^{\overline{A}_A},\ldots,\rho_{(m-1)e}^{\overline{A}_A}\rangle \in R^{\overline{A}}.$$

Hence Lemma 6.7 yields

$$\langle \rho_0^{\overline{A}}(e),\ldots,\rho_{m-1}^{\overline{A}}(e)\rangle \in R^{\overline{A}}.$$

Now an easy induction shows that $\eta^{\overline{A}}(e) = \eta^{\overline{C}}(e)$ for any term η. Hence

$$\langle \rho_0^{\overline{C}}(e),\ldots,\rho_{m-1}^{\overline{C}}(e)\rangle \in R^{\overline{C}}.$$

Now the fact that f is an isomorphism gives

$$\langle f(\rho_0^{\overline{C}}(e)), \ldots, f(\rho_{m-1}^{\overline{C}}(e))\rangle \in R^{\overline{D}}.$$

By the argument in the equality case, $f(\rho_i^{\overline{C}}(e)) = \sigma_i^{\overline{D}_g}$ for each $i < m$. Hence

$$\langle \sigma_0^{\overline{D}_g}, \ldots \sigma_{m-1}^{\overline{D}_g}\rangle \in R^{\overline{D}}.$$

Hence $\overline{D}_g \models R\sigma_0 \ldots \sigma_{m-1}$.

The case $\neg R\sigma_0 \ldots \sigma_{m-1}$ is treated similarly. $\square$

Elementary extensions

Let $\overline{A}$ and $\overline{B}$ be $\mathscr{L}$-structures. An *elementary embedding* of $\overline{A}$ into $\overline{B}$ is a function $f : A \to B$ satisfying the following condition:

$(*)$ For every formula φ and every $a \in {}^{\omega}A$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$.

In case f is the identity on A we say that $\overline{A}$ is an *elementary substructure* of $\overline{B}$ and $\overline{B}$ is an *elementary extension* of $\overline{A}$, and write $\overline{A} \preceq \overline{B}$. Then the condition $(*)$ takes the form

$(**)$ $A \subseteq B$, and for every formula α and every $a \in {}^{\omega}A$, $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$.

Proposition 6.10. *If f is an elementary embedding of $\overline{A}$ into $\overline{B}$, then f is an isomorphism from $\overline{A}$ onto a substructure $\overline{C}$ of $\overline{B}$, and $\overline{C}$ is an elementary substructure of $\overline{B}$.*

In particular, if $\overline{A} \preceq \overline{B}$, then $\overline{A}$ is a substructure of $\overline{B}$.

Proof. First we show that f is one-one. Suppose that x and y are distinct members of A. Let φ be the formula $\neg(v_0 = v_1)$. Then $\overline{A} \models \varphi[x, y]$, hence $\overline{B} \models \varphi[f(x), f(y)]$, so $f(x) \neq f(y)$. Thus f is one-one.

Let $C = \text{rng}(f)$. Suppose that k is an individual constant. Let φ be the formula $v_0 = k$. Then $\overline{A} \models \varphi[k^{\overline{A}}, k]$, and so $\overline{B} \models \varphi[f(k^{\overline{A}}), k]$, i.e., $f(k^{\overline{A}}) = k^{\overline{B}}$. Thus $k^{\overline{B}} \in C$, and f preserves k. Suppose that F is an m-ary operation symbol and $a \in {}^{m}A$. Let $c = F^{\overline{A}}(a)$. Let φ be the formula $Fv_0 \ldots v_{m-1} = v_m$. Then $\overline{A} \models \varphi[a_0, \ldots, a_{m-1}, c]$, and so $\overline{B} \models \varphi[f(a_0), \ldots, f(a_{m-1}), f(c)]$, i.e. $F^{\overline{B}}(f(a_0), \ldots, f(a_{m-1}) = f(c) = f(F^{\overline{A}}(a_0, \ldots, a_{m-1})$. Hence C is closed under $F^{\overline{B}}$, and f preserves F.

Now we have shown that C is a subuniverse of $\overline{B}$; and we have also checked the isomorphism properties of f for individual constants and for operation symbols. Now suppose that R is an m-ary relation symbol and $a \in {}^{m}A$. Let φ be the formula $Rv_0 \ldots v_{m-1}$. Suppose that $a \in R^{\overline{A}}$. Then $\overline{A} \models \varphi[a_0, \ldots, a_{m-1}]$. Hence $\overline{B} \models \varphi[f(a_0), \ldots, f(a_{m-1})]$, i.e., $f \circ a \in R^{\overline{B}}$. The converse is similar. So f is an isomorphism from $\overline{A}$ onto $\overline{C}$.

To show that $\overline{C}$ is an elementary substructure of $\overline{B}$, suppose that $c \in {}^{\omega}C$ and φ is any formula. We can write $c = f \circ a$ for some $a \in {}^{\omega}A$. Hence $\overline{C} \models \varphi[c]$ iff $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[f \circ a]$, as desired. $\square$

Since the condition for elementary extension can be applied to sentences, it follows that $\overline{A} \preceq \overline{B}$ implies that $\overline{A}$ and $\overline{B}$ are elementarily equivalent. But $\overline{A} \preceq \overline{B}$ means more than just $\overline{A} \leq \overline{B}$ and $\overline{A}$ elementarily equivalent to $\overline{B}$. For example, let $\overline{A}$ be $\omega\backslash 1$ under its usual order, and $\overline{B}$ be ω under its usual order. Then $\overline{A} \leq \overline{B}$ and they are elementarily equivalent since they are isomorphic; see Theorem 6.1. But $\overline{A}$ is not an elementary substructure of $\overline{B}$. In fact, let φ be the formula $\forall v_1[v_0 < v_1 \vee v_0 = v_1]$. Then $\overline{A} \models \varphi[1]$, but $\overline{B} \not\models \varphi[1]$ since $0 < 1$.

Lemma 6.11. (Tarski) *Suppose that $\overline{A}$ is a substructure of $\overline{B}$. Then the following conditions are equivalent:*

(i) $\overline{A} \preceq \overline{B}$.

(ii) For every formula φ, every $i < \omega$, and every $x \in {}^\omega A$, if there is a $b \in B$ such that $\overline{B} \models \varphi[x^i_b]$, then there is an $a \in A$ such that $\overline{B} \models \varphi[x^i_a]$.

Proof. (i)⇒(ii): Assume that $\overline{A} \preceq \overline{B}$, φ is a formula, $i < \omega$, $x \in {}^\omega A$, $b \in B$, and $\overline{B} \models \varphi[x^i_b]$. Then $\overline{B} \models \exists v_i\varphi[a]$, so $\overline{A} \models \exists v_i\varphi[a]$. Choose $a \in A$ such that $\overline{A} \models \varphi[x^i_a]$. Then also $\overline{B} \models \varphi[x^i_a]$.

(ii)⇒(i): Assume (ii). Now we prove by induction on ψ that for any $a \in {}^\omega A$ we have $\overline{A} \models \psi[a]$ iff $\overline{B} \models \psi[a]$. This is true for ψ atomic, using Proposition 2.3. The induction steps involving $\neg$ and $\wedge$ are clear. Now suppose that ψ is $\exists v_i\varphi$.

Suppose that $\overline{A} \models \psi[a]$. Choose $u \in A$ such that $\overline{A} \models \varphi[a^i_u]$. Then $\overline{B} \models \varphi[a^i_u]$ by the inductive hypothesis. Hence $\overline{B} \models \psi[a]$.

Suppose that $\overline{B} \models \psi[a]$. By (ii), choose $u \in A$ such that $\overline{B} \models \varphi[a^i_u]$. Then $\overline{A} \models \varphi[a^i_u]$ by the inductive hypothesis. Hence $\overline{A} \models \psi[a]$. □

Proposition 6.12. *If $\overline{A}$ is an elementary substructure of $\overline{B}$, and $\overline{C}$ is an elementary substructure of $\overline{B}$ containing A, then $\overline{A} \preceq \overline{C}$.*

Proof. For any formula φ and any $a \in {}^\omega A$ we have $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$ iff $\overline{C} \models \varphi[a]$. □

For the next fact, see the definition of unions of structures given in Chapter 1.

Proposition 6.13. *Suppose that $\langle A_i : i \in I\rangle$ is a system of structures, $(I, \leq)$ is a directed set, and $A_i \preceq A_j$ if $i, j \in I$ with $i \leq j$. Then $A_i \preceq \bigcup_{j\in I} A_j$.*

Proof. Let $\overline{B} = \bigcup_{i\in I} \overline{A}_i$. We prove the following by induction on φ;

($*$) For every $i \in I$, every $m \in \omega$ such that $j < m$ for every variable v_j occurring in φ, and every $a \in {}^m A_i$, $\overline{A}_i \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$.

The atomic case, and the inductive steps for $\neg$ and $\wedge$, are clear. Now suppose that φ is $\exists v_k\psi$. If $\overline{A}_i \models \varphi[a]$, the inductive hypothesis easily gives $\overline{B} \models \varphi[a]$. Now suppose that $\overline{B} \models \varphi[a]$. Choose $b \in B$ such that $\overline{B} \models \psi[a^k_b]$. Then there is a $j \in I$ such that $i < j$ and $a^k_b \in A_j$. Hence $\overline{A}_j \models \psi[a^k_b]$ by the inductive hypothesis. So $\overline{A}_j \models \varphi[a]$, and $\overline{A}_i \models \varphi[a]$ since $\overline{A}_i \preceq \overline{A}_j$. □

By the *elementary diagram* of $\overline{A}$, denoted by Eldiag($\overline{A}$), we mean the set of all sentences in $\mathscr{A}_A$ which hold in $\overline{A}_A$.

Lemma 6.14. *Let $\overline{A}$ be an $\mathscr{L}$-structure, $e \in {}^{\omega}A$, φ an $\mathscr{L}$-formula, and let φ_e be obtained from φ by replacing every free occurrence of v_i by $c_{e(i)}$, the constant corresponding to $e(i)$ in the language $\mathscr{L}_A$. Then $\overline{A}_A \models \varphi_e$ iff $\overline{A} \models \varphi[e]$.*

Proof. Immediate from Lemma 4.8. □

Theorem 6.15. (Elementary diagram theorem) *Let $\overline{A}$ and $\overline{B}$ be similar structures and $f : A \to B$. Then the following conditions are equivalent:*

(i) f is an elementary embedding of $\overline{A}$ into $\overline{B}$.

(ii) $(\overline{B}, f(a))_{a\in A}$ is a model of $\mathrm{Eldiag}(\overline{A})$.

Proof. (i)⇒(ii): Let f be an elementary embedding of $\overline{A}$ into $\overline{B}$. We claim that $\overline{B}_f$ is a model of $\mathrm{Eldiag}(\overline{A})$. For, let φ be a sentence of $\mathscr{L}_A$ which holds in $\overline{A}_A$. Then there exist a formula ψ in $\mathscr{L}$ and an $a \in {}^{\omega}A$ such that φ is obtained from ψ by replacing each free occurrence of v_i in ψ by c_{a_i}, the constant of $\mathscr{L}_A$ corresponding to a_i. Then by Lemma 6.14 we have $\overline{A} \models \psi[a]$. Hence $\overline{B} \models \psi[f \circ a]$. Then by Lemma 6.14 again we have $\overline{B}_f \models \varphi$, as desired.

(ii)⇒(i): Assume that $\overline{B}_f$ is a model of $\mathrm{Eldiag}(\overline{A})$. We claim that f is an elementary embedding of $\overline{A}$ into $\overline{B}$. By the proof of Theorem 6.8, f is an isomorphism from $\overline{A}$ into $\overline{B}$. Now suppose that φ is a formula of $\mathscr{L}$ and $a \in {}^{\omega}A$. Suppose that $\overline{A} \models \varphi[a]$. Then by Lemma 6.14, $\overline{A}_A \models \psi$, where ψ is obtained from φ by replacing each free occurrence of v_i by $c_{a(i)}$, for all $i < \omega$. Hence ψ is in $\mathrm{Eldiag}(\overline{A})$, so $\overline{B}_f$ is a model of it. By Lemma 6.14, $\overline{B} \models \varphi[f \circ a]$. The converse holds by considering $\neg\varphi$. □

Löwenheim-Skolem theorems

The theorems are existence statements concerning starting with one structure and finding smaller elementary substructures, or larger elementary extensions. To start with we need a result from elementary set theory describing the closure of a set under finitary partial operations.

Given a set A, a *partial operation* on A is a function f which, for some positive integer m, maps a subset of ${}^{m}A$ into A. The integer m is uniquely determined by f if $f \neq \emptyset$, and we denoter it by $\rho(f)$. We say that a subset X of A is *closed* under a nonempty partial function f iff for every a which is in the domain of f and is a member of ${}^{\rho(f)}X$ we have $f(a) \in X$. If $\mathscr{F}$ is a collection of partial functions on A and $X \subseteq A$, then $\mathrm{cl}(X, \mathscr{F})$ is the intersection of all subsets of A which contain X are are closed under each member of $\mathscr{F}$.

Theorem 6.16. *Let A be any set, and $\mathscr{F}$ a collection of partial finitary operations on A. Suppose that $X \subseteq A$.*

(i) Define $Y_0 = X$ and for any $i \in \omega$ let

$$Y_{i+1} = Y_i \cup \{f(a) : \emptyset \neq f \in \mathscr{F}, a \in \mathrm{dmn}(f) \cap {}^{\rho(f)}Y_i\}.$$

Then $\mathrm{cl}(X, \mathscr{F}) = \bigcup_{i\in\omega} Y_i$.

(ii) $\mathrm{cl}(X, \mathscr{F}) \leq \max(|\mathscr{F}|, |X|, \omega)$.

Proof. By induction, $Y_i \subseteq \mathrm{cl}(X, \mathscr{F})$ for every $i < \omega$. Hence $\bigcup_{i\in\omega} Y_i \subseteq \mathrm{cl}(X, \mathscr{F})$. For the other inclusion it suffices to show that $X \subseteq \bigcup_{i\in\omega} Y_i$ and $\bigcup_{i\in\omega} Y_i$ is closed under each

$f \in \mathscr{F}$. Since $Y_0 = X$, it is obvious that $X \subseteq \bigcup_{i\in\omega} Y_i$. Now suppose that $f \in \mathscr{F}$ and a is in the domain of f and $\text{rng}(a) \subseteq \bigcup_{i\in\omega} Y_i$. Clearly there is an $i < \omega$ such that $\text{rng}(a) \subseteq Y_i$. Then $f(a) \in Y_{i+1}$, as desired. This proves (i).

For (ii), it suffices to show that $|Y_i| \le \max(|\mathscr{F}|, |X|, \omega)$ for each $i \in \omega$. This is clear, by induction on i. □

Theorem 6.17. (Downward Löwenheim-Skolem theorem) *Suppose that $\overline{B}$ is an infinite $\mathscr{L}$-structure, $X \subseteq B$, κ is a cardinal, $|X| \le \kappa \le |B|$, and $\mathscr{L}$ has at most κ non-logical symbols.*

Then $\overline{B}$ has an elementary substructure $\overline{A}$ such that $X \subseteq A$ and $|A| = \kappa$.

Proof. Let $<$ well-order B. Let F consist of all triples (φ, i, m) such that φ is a formula with free variables among $v_0, \ldots, v_{m-1}$ and $i < m$. With each $(\varphi, i, m) \in F$ we associate a partial m-ary function $f_{(\varphi,i,m)}$ on B. The domain of $f_{(\varphi,i,m)}$ is $\{a \in {}^mB : \exists u \in B[\overline{B} \models \varphi[a^i_u]]\}$, and its value at any such is the $<$-least u such that $\overline{B} \models \varphi[a^i_u]$. Let $\mathscr{F} = \{f_{(\varphi,i,m)} : (\varphi, i, m) \in F$. Let Y be a subset of B containing X and of size κ. Finally, set $A = \text{cl}(Y, \mathscr{F})$. Clearly $X \subseteq A$ and $|A| = \kappa$.

First we prove that A is a subuniverse of $\overline{B}$. Let k be an individual constant of $\mathscr{L}$. Let φ be the formula $v_0 = k$. Note that $\text{dmn}(f_{(\varphi,0,1)}) = \{a \in {}^1B : \exists u \in B[\overline{B} \models \varphi[a^0_u]]\} = {}^1B$, and for any $a \in B[f_{(\varphi,0,1)}(a) = k^{\overline{B}}$. Thus $k^{\overline{B}} \in A$.

Next let F be an m-ary operation symbol, and suppose that $a \in {}^mA$. Let φ be the formula $Fv_0, \ldots, v_{m-1} = v_m$. Then $\text{dmn}(f_{(\varphi,m,m+1)}) = \{b \in {}^{m+1}B : \exists u \in B[\overline{B} \models \varphi[b^m_u]]\} = {}^{m+1}B$. It follows that

$$f_{(\varphi,m,m+1)}(a_0, \ldots, a_{m-1}, a_0) = F^{\overline{B}}(a_0, \ldots, a_{m-1}) \in A.$$

So A is a subuniverse of $\overline{B}$ and we let $\overline{A}$ be the associated substructure.

To show that $\overline{A} \preceq \overline{B}$ we will apply Tarski's lemma 6.11. To this end, let φ be a formula, $i < \omega$, $x \in {}^\omega A$, and suppose that there is a $b \in B$ such that $\overline{B} \models \varphi[x^i_b]$. Choose n so that $j < n$ for every variable v_j that occurs free in φ, and also so that $i < n$. Let $u = f_{(\varphi,i,n)}(x \restriction n)$. Then by definition, $\overline{B} \models \varphi[(x \restriction n)^i_u]$. Also, $u \in A$. Hence $\overline{B} \models \varphi[x^i_u]$. This verifies (ii) in Tarski's lemma. Hence $\overline{A} \preceq \overline{B}$. □

A somewhat philosophical application of the downward Löwenheim-Skolem theorem is *Skolem's paradox*: if there is a model of the axioms for set theory, then there is a countable model, even though from the axioms one can prove the existence of uncountable sets. The "solution" of the paradox is that the model only "thinks" that the sets are uncountable; they are really countable. This is connected to the notion of absoluteness for models of set theory.

Theorem 6.18. (Upward Löwenheim-Skolem theorem) *If $\overline{A}$ is an $\mathscr{L}$-structure, $|A| \ge \omega$, and κ is a cardinal such that $|A| \le \kappa$ and $\mathscr{L}$ has at most κ non-logical constants, then $\overline{A}$ has an elementary extension $\overline{B}$ of size κ. In case $|A| = \kappa$, we can insist that $A \neq B$.*

Proof. Expand $\mathscr{L}_A$ to a language $\mathscr{L}'$ by adding κ many new individual constants $\langle k_\alpha : \alpha < \kappa\rangle$. By the elementary diagram theorem 6.15 it suffices to show that the following set has a model:

$$\text{Eldiag}(\overline{A}) \cup \{\neg(k_\alpha = k_\beta) : \alpha < \beta < \kappa\} \cup \{\neg(c_a = k_\alpha) : a \in A, \alpha < \kappa\}.$$

To apply the compactness theorem, take a finite subset Γ of this set. We can write $\Gamma = \Delta_0 \cup \Delta_1 \cup \Delta_2$, where $\Delta_0, \Delta_1, \Delta_2$ are finite subsets of Eldiag$(\overline{A})$, $\{\neg(k_\alpha = k_\beta) : \alpha < \beta < \kappa\}$ and $\{\neg(c_a = k_\alpha) : a \in A, \alpha < \kappa\}$ respectively. We consider the following $\mathscr{L}'$-structure: $(\overline{A}, a, d_\alpha)_{a\in A,\alpha<\kappa}$, where $d \in {}^\kappa A$ is such that, if Θ is the set of all $\alpha < \kappa$ such that k_α occurs in some formula of $\Delta_1 \cup \Delta_2$, then $d \restriction \Theta$ is one-one and for all $\alpha \in \Theta$, d_α is different from all $a \in A$ such that c_a occurs in some formula of Δ_2. Clearly this gives a model of Γ.

So by the compactness theorem we get a model $(\overline{B}, b_a, d_\alpha)_{a\in A,\alpha<\kappa}$ of Γ. By the elementary diagram theorem 6.15, there is an elementary embedding f of $\overline{A}$ into $\overline{B}$. Clearly $|B| \geq \kappa$, and $f[A] \neq B$. We may assume that $\overline{A} \preceq \overline{B}$. Take any $b \in B\backslash A$. By the downward Löwenheim-Skolem theorem, $\overline{B}$ has an elementary substructure $\overline{C}$ of size κ with $A \cup \{b\} \subseteq C$. Clearly $\overline{A} \preceq \overline{C}$ and $\overline{C}$ is as desired. □

The Löwenheim-Skolem theorems can be used to prove our first major theorem concerning complete theories–Vaught's criterion. For any cardinal κ, we call a theory (weakly) κ-*categorical* if any two models of Γ of size κ are (elementarily equivalent) isomorphic. For κ finite, this notion is trivial:

Proposition 6.19. *If κ is a finite cardinal and Γ is a theory with a model of size κ, then the following conditions are equivalent.*

(i) Γ is κ-categorical.

(ii) Γ is weakly κ-categorical.

If Γ is complete, then all models of Γ have size κ, and both of these conditions hold.

Proof. By Theorem 6.1, (i) implies (ii). Now suppose that (ii) holds, and $\overline{A}$ and $\overline{B}$ are models of Γ; we want to show that they are isomorphic. First suppose that our language $\mathscr{L}$ has only finitely many non-logical symbols. Then Diag$(\overline{A})$ is finite. Let f be a bijection from A onto some positive integer m, and form a formula φ by replacing each constant c_a in $\bigwedge$ Diag$(\overline{A})$ corresponding to an element $a \in A$ by $v_{f(a)}$. Then the sentence

$$\exists v_0 \ldots \exists v_{m-1}\left[\varphi \wedge \forall v_m \bigvee_{i<m} (v_i = v_m)\right]$$

holds in $\overline{A}$, and hence also in $\overline{B}$. Hence there is a $g \in {}^A B$ such that $\overline{B}_g$ is a model of Diag$(\overline{A})$, and with g a surjection. By the proof of the diagram lemma 6.8, g is an isomorphism from $\overline{A}$ onto $\overline{B}$.

Second, suppose that $\mathscr{L}$ has infinitely many non-logical symbols. A *finite reduct* of $\mathscr{L}$ is a reduct of $\mathscr{L}$ with only finitely many non-logical symbols. Similarly for a finite reduct of $\overline{A}$ or of $\overline{B}$. Now for each finite reduct $\overline{A}_F$ of $\overline{A}$, given by a finite set F of non-logical symbols of $\mathscr{L}$, let I_F be the set of all isomorphisms from $\overline{A}_F$ onto $\overline{B}_F$. Note that if $F \subseteq G$, both being finite sets of non-logical symbols of $\mathscr{L}$, then $I_G \subseteq I_F$. Let J be a set I_F of smallest size. Then each member of J is an isomorphism from $\overline{A}$ onto $\overline{B}$. For, if S is any non-logical constant of $\mathscr{L}$, let $G = F \cup \{S\}$. Then $I_G \subseteq I_F$, hence $I_G = I_F = L$, and so each member of L preserves S.

This finishes the proof of equivalence of (i) and (ii).

Now suppose that Γ is complete. Since Γ has a model $\overline{A}$ of size κ, the sentence

$$\exists v_0 \ldots \exists v_{\kappa-1} \left[\bigwedge_{i<j<\kappa} \neg(v_i = v_j) \wedge \forall v_\kappa \left[\bigvee_{i<\kappa} (v_i = v_\kappa) \right] \right]$$

holds in $\overline{A}$, hence in all models of Γ since it is complete. So all models of Γ have size κ. Clearly (i) and (ii) hold. $\square$

Theorem 6.20. (Vaught's Criterion) *Let Γ be a theory in a language $\mathscr{L}$ with only infinite models, and let κ be a cardinal $\geq$ the number of non-logical symbols of $\mathscr{L}$. Suppose that Γ is weakly κ-categorical. Then Γ is complete.*

Let $\overline{A}$ and $\overline{B}$ be models of Γ; we want to show that they are elementarily equivalent. We can apply the uppward or downward Löwenheim-Skolem theorem to get a model $\overline{C}$ elementarily equivalent to $\overline{A}$ and of size κ. Similarly with $\overline{B}$, so the conclusion follows. $\square$

Some typical applications of Vaught's criterion are:

- The theory of dense linear orders with no end points. ($\aleph_0$-categorical)
- The theory of atomless Boolean algebras. ($\aleph_0$-categorical)
- The theory of algebraically closed fields of a given characteristic. ($\aleph_1$-categorical)

κ-saturated structures

We generalize a procedure and notation described in the subsection on diagrams above. Given an $\mathscr{L}$-structure $\overline{A}$ and a subset X of A, let $\mathscr{L}_X$ be the expansion of $\mathscr{L}$ obtained by adding new individual constants c_a for each $a \in X$. If $\overline{B}$ is an $\mathscr{L}$-structure and $b \in {}^X B$, then $\overline{B}_b$ is the expansion of $\overline{B}$ to an $\mathscr{L}_X$ structure in which the denotation of c_a is b_a for each $a \in X$.

Now let κ be an infinite cardinal. An $\mathscr{L}$-structure $\overline{A}$ is *κ-saturated* iff for every $X \in [A]^{<\kappa}$, every $i < \omega$, and every set Γ of formulas of $\mathscr{L}_X$ each with only v_i possibly free, the following condition (c) implies condition (x):

(c) For every finite $\Theta \subseteq \Gamma$ there is a $b \in A$ such that $\overline{A}_X \models \varphi[b]$ for all $\varphi \in \Theta$.

(x) There is a $b \in A$ such that $\overline{A}_X \models \varphi[b]$ for all $\varphi \in \Gamma$.

Here "(c)" stands for "consistent" and "(x)" stands for "exists". By the compactness theorem, a structure with a consistent set of formulas has an elementary extension with an element satisfying all of the formulas. A κ-saturated structure already itself has such an element. κ-saturated structures are very useful in model theory. The main result we present here is an existence theorem for them.

κ-saturation is defined in terms of a single element satisfying a set of formulas. What about two or three or more elements? According to the next proposition, this generalization is not necessary.

Proposition 6.21. *Suppose that κ is an infinite cardinal and $\overline{A}$ is an $\mathscr{L}$-structure. Then the following conditions are equivalent:*

(i) $\overline{A}$ is κ-saturated.

(ii) For each $X \in [A]^{<\kappa}$ and for every set Γ of formulas of $\mathscr{L}_X$, if for each finite $\Theta \subseteq \Gamma$ there is an $a \in {}^{\omega}A$ such that $\forall \varphi \in \Theta[\overline{A}_X \models \varphi[a]]$, then there is an $a \in {}^{\omega}A$ such that $\forall \varphi \in \Gamma[\overline{A}_X \models \varphi[]]$.

Proof. Obviously (ii) implies (i). Now suppose that (i) holds. Let Δ be the closure of Γ under $\wedge$. For each $\varphi \in \Delta$ choose a positive integer $m(\varphi)$ such that for every $i \in \omega$, if v_i occurs free in φ, then $i < m(\varphi)$. Now in addition to the individual constants c_a, $a \in X$ used to form $\mathscr{L}_X$, we introduce more individual constants k_i for $i < \omega$.

Now we define $a \in {}^{\omega}A$ by recursion. Suppose that $i < \omega$ and $a \restriction i$ has been defined so that

$$(1) \qquad (\overline{A}, b, a_j)_{b \in X, j<i} \models \exists v_i \ldots \exists v_{m(\varphi)-1} \varphi(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, \ldots, v_{m(\varphi)-1}) \text{ for every } \varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta.$$

Clearly (1) holds for $i = 0$. Let Ω be the set of all formulas

$$(2) \qquad \exists v_{i+1} \ldots \exists v_{m(\varphi)-1} \varphi(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, v_{i+1}, \ldots, v_{m(\varphi)-1})$$

with $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta$. These are formulas in $\mathscr{L}_Y$, where $Y = X \cup \{a_j : j < i\}$, and each of them has at most v_i free. Note that $|Y| < \kappa$. Now if Ω' is a finite subset of Ω, then there is a finite subset Δ' of Δ such that Ω' consists of all formulas (2) such that $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta'$. Now $\bigwedge \Delta' \in \Delta$, so by the inductive assumption the formula

$$\exists v_i \ldots \exists v_{m(\varphi)-1} \bigwedge \Delta'(k_{a_0}, \ldots, k_{a_{i-1}}, v_i, \ldots v_{m(\varphi)-1})$$

holds in $(\overline{A}, b, a_j)_{b \in X, j<i}$. Hence there is a $u \in A$ such that

$$(\overline{A}, b, a_j)_{b \in X, j<i} \models \exists v_{i+1} \ldots \exists v_{m(\varphi)-1} \bigwedge \Delta'(k_{a_0}, \ldots, k_{a_i}, v_i, \ldots v_{m(\varphi)-1})[u].$$

Now it follows from (i) that there is an $a_i \in A$ such that

$$(\overline{A}, b, a_j)_{b \in X, j \leq i} \models \exists v_{i+1} \ldots \exists v_{m(\varphi)-1} \varphi(k_{a_0}, \ldots, k_{a_i}, v_{i+1}, \ldots, v_{m(\varphi)-1}) \text{ for every } \varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Delta.$$

This finishes the inductive construction. So (1) holds for all $i < \omega$. Now if we apply (1) with $i = m(\varphi)$ to $\varphi(v_0, \ldots, v_{m(\varphi)-1}) \in \Gamma$, we get $(\overline{A}, b, j)_{b \in X, j<m(\varphi)} \models \varphi(k_{a_0}, \ldots, k_{a_{m(\varphi)-1}})$. Hence $\overline{A}_X \models \varphi[a]$, as desired. $\square$

Lemma 6.22. *Suppose that:*

κ is an infinite cardinal,

$\mathscr{L}$ is a language with at most κ non-logical symbols,

$\overline{A}$ is an infinite $\mathscr{L}$-structure of size at most 2^{κ}.

$X \in [A]^{\leq\kappa}$,
$i < \omega$, and Γ is a set of formulas of $\mathscr{L}_X$ with at most v_i free,
for every finite $\Theta \subseteq \Gamma$ there is a $b \in A$ such that $\overline{A} \models \varphi[b]$ for all $\varphi \in \Theta$.

Then $\overline{A}$ has an elementary extension $\overline{B}$ of size at most 2^κ with an element b such that $\overline{B} \models \varphi[b]$ for all $\varphi \in \Gamma$.

Proof. Let the system of individual constants used to form $\mathscr{L}_X$ be denoted by $\langle b_a : a \in X \rangle$. Now form the language $\mathscr{L}_{XA}$ used to define $\mathrm{Eldiag}(\overline{A})$, with new individual constants $\langle c_a : a \in A \rangle$. In addition, add one more new individual constant d, forming an expansion $\mathscr{L}'$ of $\mathscr{L}_{XA}$. In $\mathscr{L}'$ consider the following set Ω of sentences:

$$\mathrm{Eldiag}(\overline{A}) \cup \{\varphi(d) : \varphi \in \Gamma\}.$$

By the elementary diagram theorem followed by the downward Löwenheim-Skolem theorem, it suffices to show that Ω has a model; and for this we take a finite subset Ω' of Ω, consisting of the union of a finite subset Ω'' of $\mathrm{Eldiag}(\overline{A})$ together with a finite subset Ω''' of $\{\varphi(d) : \varphi \in \Gamma\}$. Clearly Ω' has a model consisting of an expansion of $\overline{A}_A$, using the hypotheses of our lemma. □

Theorem 6.23. *Suppose that κ is an infinite cardinal, $\mathscr{L}$ is a language with at most κ non-logical symbols, and $\overline{A}$ is an infinite $\mathscr{L}$-structure of size at most 2^κ. Then $\overline{A}$ has a κ^+-saturated elementary extension $\overline{B}$ of size at most 2^κ.*

Proof. Let $M = \{(X, \Gamma) : X \in [A]^{\leq\kappa}, \Gamma$ is a set of formulas of $\mathscr{L}_X$ each with at most v_0 free and for each finite subset Δ of Γ there is an $a \in A$ such that $\overline{A}_X \models \varphi[a]\}$. Clearly $|M| \leq 2^\kappa$. Hence we can construct an elementary chain of elementary extensions of $\overline{A}$, applying Lemma 6.22 at each stage to a member of M, so that the union $\overline{B}$ of the chain is an elementary extension such that for each member (X, Γ) of M there is a $b \in B$ such that $\overline{B} \models \varphi[b]$ for each $\varphi \in \Gamma$; moreover, $|B| \leq 2^\kappa$. Repeating this κ^+ times we obtain the desired κ^+-saturated structure. □

We conclude our discussion of saturated structures with one of the most important properties of them.

Theorem 6.24. *If κ is an infinite cardinal and $\overline{A}$ is κ-saturated, then every structure $\overline{B}$ which is elementarily equivalent to $\overline{A}$ and of size at most κ can be elementarily embedded in $\overline{A}$.*

Proof. Let $\langle b_\alpha : \alpha < \kappa \rangle$ enumerate B, possibly with repetitions. We now define by recursion a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of A, so that for each α, $(\overline{A}, a_\beta)_{\beta<\alpha} \equiv (\overline{B}, b_\beta)_{\beta<\alpha}$. Suppose that a_β has been defined for all $\beta < \alpha$ so that this condition holds. Note that for $\alpha = 0$ the condition says that merely $\overline{A} \equiv \overline{B}$, which is given. Now we consider the following set Γ of formulas:

$$\{\varphi(x) : \varphi \text{ is a formula in } \mathscr{L}_{\langle b_\beta : \beta, \alpha \rangle} \text{ and } \overline{B} \models \varphi(b_\alpha)\}.$$

If Δ is a finite subset of Γ, then $\overline{B} \models \bigwedge \Delta[b_\alpha]$, so $\overline{B} \models \exists x \bigwedge \Delta$. Hence $\overline{A} \models \exists x \bigwedge \Delta$. So by the definition of κ-saturated, there is an element a_α of A such that $\overline{A} \models \varphi[a_\alpha]$ for all $\varphi \in \Gamma$. Hence $(\overline{A}, a_\beta)_{\beta \leq \alpha} \equiv (\overline{B}, b_\beta)_{\beta \leq \alpha}$, finishing the construction.

Clearly $\{(b_\alpha, a_\alpha) : \alpha < \kappa\}$ is the desired elementary embedding. □

Omitting types

Let T be a theory and $n \in \omega$. An *n-type* of T is a collection t of formulas of the language $\mathscr{L}$ of T with all free variables among $\overline{x}$, where $\overline{x}$ is a sequence of variables of length n, such that $T \models \exists \overline{x} \varphi$ for every finite conjunction φ of members of t.

We say that such a type t is *isolated* in T iff there is a formula φ with free variables among $\overline{x}$ such that $T \cup \{\exists \overline{x} \varphi\}$ has a model, and $T \models \varphi \rightarrow \psi$ for every $\psi \in t$. We say then that φ isolates t in T.

A model $\overline{M}$ of T *omits* t iff there is no sequence $\overline{a}$ of elements of M such that $\overline{M} \models \psi[\overline{a}]$ for all $\psi \in t$.

Proposition 6.25. *If T is complete and has a model which omits a type t, then t is not isolated in T.*

Proof. We prove the contrapositive. Suppose that T is complete and t is isolated in T; say that φ isolates t in T. Let $\overline{M}$ be any model of T. Since $T \cup \{\exists \overline{x} \varphi\}$ has a model and T is complete, it follows that $\overline{M} \models \exists \overline{x} \varphi$. Choose $\overline{a}$ in M such that $\overline{M} \models \varphi[\overline{a}]$. Take any $\psi \in T$. Then $T \models \varphi \rightarrow \psi$, and hence $\overline{M} \models \psi[\overline{a}]$. Thus $\overline{M}$ realizes t, i.e., it does not omit t. □

Theorem 6.26. (Omitting types theorem) *Suppose that $\mathscr{L}$ is countable and T is a theory in $\mathscr{L}$ which has a model. Let $\langle n_i : i \in \omega \rangle$ be a sequence of natural numbers, and for each $i < \omega$ let t_i be an n_i-type over T which is not isolated.*

Then T has a countable model which omits each type t_i.

Proof. Expand our given language $\mathscr{L}$ by adjoining a sequence $\langle c_i : i < \omega \rangle$ of new individual constants, forming a new language $\mathscr{L}'$ which is still countable. Let $\langle \varphi_i : i < \omega \rangle$ enumerate all of the sentences of $\mathscr{L}'$. Let $\langle (j_i, \overline{d}_i) : i \in \omega \rangle$ enumerate all pairs $(m, \overline{e})$ such that $m \in \omega$ and $\overline{e}$ is a sequence of length n_m of distinct members of $\{c_0, c_1, \ldots\}$.

Now we are going to construct a sequence

$$T = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_m \subseteq \cdots$$

of theories in $\mathscr{L}'$ satisfying for each $i \in \omega$ the following conditions:

(1) T_i has a model, and $T_i = T \cup \Delta$ for some finite set Δ of sentences.

(2) $\varphi_i \in T_{i+1}$ or $\neg \varphi_i \in T_{i+1}$.

(3) If φ_i has the form $\exists x \psi(x)$ and $\varphi_i \in T_{i+1}$, then there is an $k < \omega$ such that $\psi(c_k) \in T_{i+1}$.

(4) There is a formula $\sigma(\overline{x}) \in t_{j_i}$ such that $\neg \sigma(\overline{d}_i) \in T_{i+1}$.

Having defined T_i, we define T_{i+1} as follows. Let Δ be a finite set of sentences such that $T_i = T \cup \Delta$. Let $\chi = \bigwedge \Delta$. We can write $\chi = \chi(\overline{d}_i, \overline{e})$, where $\overline{d}_i$ and $\overline{e}$ exhaust all the constants c_k that occur in χ. Replace these constants by new distinct variables $\overline{x}$ and $\overline{y}$ respectively, obtaining $\chi(\overline{x}, \overline{y})$. Since T_i has a model, it follows that $T \cup \{\exists\overline{x}\exists\overline{y}\chi(\overline{x}, \overline{y})\}$ has a model. Now t_{j_i} is not isolated, so it follows that there is a formula $\sigma(\overline{x})$ in t_{j_i} such that $T \not\models \exists\overline{y}\chi(\overline{x}, \overline{y}) \to \sigma(\overline{x})$. Thus there is a model $\overline{M}$ of T and a sequence $\overline{a}$ of elements of M such that $\overline{M} \models \exists\overline{y}\chi(\overline{a}, \overline{y}) \wedge \neg\sigma(\overline{a})$. It follows that $S \stackrel{\text{def}}{=} T_i \cup \{\neg\sigma(\overline{d}_i)\}$ has a model.

If $S \cup \{\varphi_i\}$ has a model, let $S' = S \cup \{\varphi_i\}$; otherwise let $S' = S \cup \{\neg\varphi_i\}$. In either case, S' has a model.

If φ_i has the form $\exists x\psi(x)$, let p be minimum such that c_p does not occur in any formula of S', and let $T_{i+1} = S' \cup \{\psi(c_p)\}$. If φ_i does not have this form, let $T_{i+1} = S'$.

This finishes the construction. Clearly (1)–(4) hold.

Let $T_\omega = \bigcup_{i\in\omega} T_i$. By (1) and compactness, T_ω has a model $(\overline{M}, a_i)_{i\in\omega}$. Let $\overline{N}$ be the substructure of $\overline{M}$ generated by the individual constants; $\overline{N}$ is an $\mathscr{L}$-structure.

(5) $N = \{a_i : i \in \omega\}$.

In fact, let σ be any variable-free term. Then $(\overline{M}, a_i)_{i\in\omega} \models \exists x[x = \sigma]$, and so by (3) there is a $k < \omega$ such that $(\overline{M}, a_i)_{i\in\omega} \models a_k = \sigma]$. This proves (5).

(6) For any sentence φ of $\mathscr{L}'$ the following conditions are equivalent:

(a) $(\overline{M}, a_i)_{i\in I} \models \varphi$;

(b) $(\overline{N}, a_i)_{i\in I} \models \varphi$;

(c) $T_\omega \models \varphi$.

In fact, (a) and (c) are equivalent since T_ω is complete by (2). We prove the equivalence of (a) and (b) by induction on φ. For φ atomic, the equivalence is clear. The induction steps using $\neg$ and $\wedge$ are straightfoward. Finally, take a sentence $\exists x\psi(x)$. If $(\overline{M}, a_i)_{i\in\omega} \models \exists x\psi(x)$, then by (3) there is a $k < \omega$ such that $(\overline{M}, a_i)_{i\in\omega} \models \psi(c_k)$. Then $(\overline{N}, a_i)_{i\in\omega} \models \psi(c_k)$ by the inductive hypothesis, so $(\overline{N}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Now suppose that $(\overline{N}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Then there is a $k < \omega$ such that $(\overline{N}, a_i)_{i\in\omega} \models \psi(c_k)$, so $(\overline{M}, a_i)_{i\in\omega} \models \psi(c_k)$ by the inductive hypothesis, hence $(\overline{M}, a_i)_{i\in\omega} \models \exists x\psi(x)$. Thus (6) holds.

It follows that $\overline{N}$ is a model of T. Moreover, by (4) it omits each type t_i. □

Model completeness

A theory T is *model complete* iff for any two models $\overline{A}$ and $\overline{B}$ of T, $\overline{A} \leq \overline{B}$ iff $\overline{A} \preceq \overline{B}$. We prove here some general results about this notion.

Given a theory T, a *weakly prime* model of T is a model of T which can be embedded in any model of T. The qualifier "weakly" is there since *prime* model requires an elementary embedding.

Proposition 6.27. *If T is model complete and has a weakly prime model, then T is complete.*

Proof. Let $\overline{A}$ and $\overline{B}$ be arbitrary models of T, and let $\overline{C}$ be a weakly prime model of T. Then $\overline{C}$ can be elementarily embedded in $\overline{A}$ and in $\overline{B}$, so $\overline{A}$ and $\overline{B}$ are elementarily equivalent. □

This simple proposition can be used to show the completeness of various theories.

A formula is *existential* iff it has the form $\exists\overline{x}\varphi$ with φ quantifier-free; here $\overline{x}$ is a finite sequence of variables. Similarly one defines *universal* formula.

Lemma 6.28. *Suppose that $\Gamma\cup\Delta\cup\{\varphi\}$ is a collection of sentences in $\mathscr{L}$, and $\Gamma\cup\Delta\models\varphi$. Then there is a finite conjunction ψ of members of Δ such that $\Gamma\models\psi\to\varphi$.*

Proof. Assume the hypotheses. Then $\Gamma\cup\Delta\cup\{\neg\varphi\}$ does not have a model, so some finite subset fails to have a model. We may assume that the finite subset has the form $\Gamma'\cup\Delta'\cup\{\neg\varphi\}$ with Γ' and Δ' finite subsets of Γ,Δ respectively. Clearly then $\psi\stackrel{\text{def}}{=}\bigwedge\Delta'$ is as desired. $\square$

Lemma 6.29. *If $\overline{A}$ is an $\mathscr{L}$-structure, Γ is a set of $\mathscr{L}$-sentences, φ is an $\mathscr{L}$-sentence, Δ is the diagram of $\overline{A}$, and $\Gamma\cup\Delta\models\varphi$, then there is an existential sentence ψ of $\mathscr{L}$ such that $\Gamma\models\psi\to\varphi$ and $\overline{A}\models\psi$.*

Proof. Assume the hypotheses. By Lemma 6.28, there is a conjunction χ of finitely many members of Δ such that $\Gamma\models\chi\to\varphi$. Replace the constants c_a in χ associated with the diagram by variables, obtaining a formula $\theta(\overline{x})$, with $\overline{x}$ a sequence of variables. Then $\Gamma\models\exists\overline{x}\theta\to\varphi$. Clearly $\overline{A}\models\exists\overline{x}\theta\to\varphi$. $\square$

Theorem 6.30. *For any theory T the following conditions are equivalent:*

(i) T is model complete.

(ii) For all models $\overline{A},\overline{B}$ of T such that $\overline{A}\leq\overline{B}$, and for every existential formula φ and every sequence $a\in{}^{\omega}A$, if $\overline{B}\models\varphi[a]$, then $\overline{A}\models\varphi[a]$.

(iii) For every formula φ there is an existential formula ψ with the same free variables as φ such that $T\models\varphi\leftrightarrow\psi$.

Proof. (i)$\Rightarrow$(ii) is obvious. Now assume that (ii) holds. First we claim

(1) For any universal formula φ there is an existential formula ψ with the same free variables such that $T\models\varphi\leftrightarrow\psi$.

To prove (1), let the free variables of φ be $v_{i(0)},\ldots,v_{i(m-1)}$. Expand the language by adjoining new individual constants $c_0,\ldots,c_{m-1}$, obtaining a language $\mathscr{L}'$, and let φ' be the sentence $\mathscr{L}'$ obtained from φ by replacing $v_{i(j)}$ by c_j for each $j<m$. Let $\Delta=\{\psi:\psi$ is a universal sentence in $\mathscr{L}'$ and $T\models\neg\varphi'\to\psi\}$. Now we claim

(2) $$T\cup\Delta\cup\{\varphi'\}\text{ does not have a model.}$$

In fact, suppose that $(\overline{A},a_i)_{i<m}$ is a model of this set. Let Θ be the diagram of $(\overline{A},a_i)_{i<m}$ in $\mathscr{L}'_A$. Then

(3) $$T\cup\Theta\models\varphi'.$$

In fact, let $(\overline{B},b_i,t_a)_{i<m,a\in A}$ be a model of $T\cup\Theta$. Then by Theorem 6.8, t is an isomorphism from $(\overline{A},a_i)_{i<m}$ into $(\overline{B},b_i)_{i<m}$. Now $(\overline{A},a_i)_{i<m}\models\varphi'$, so by (ii), $(\overline{B},b_i)_{i<m}\models\varphi'$. This proves (3).

Now from (3) and Lemma 6.29 we see that there is an existential sentence ψ of $\mathscr{L}'$ such that $T \models \psi \to \varphi'$ and $(\overline{A}, a_i)_{i<m} \models \psi$. Thus $\neg\psi \in \Delta$. But $(\overline{A}, a_i)_{i<m} \models \psi$ and also $(\overline{A}, a_i)_{i<m}$ is a model of Δ, contradiction. Hence (2) holds.

Thus $T \cup \Delta \models \neg\varphi'$, and so by Lemma 6.28 there is a conjunction ξ of members of Δ such that $T \models \xi \to \neg\varphi'$. Clearly ξ is logically equivalent to a universal sentence. The definition of Δ shows that $T \models \neg\varphi' \to \xi$. Hence (1) holds.

Now the proof of the (iii) itself follows by an easy induction. It is obviously true for atomic φ. Assuming it true for φ, let ψ be an existential formula with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$. Then by (1) there is an existential formula χ with the same free variables as $\neg\psi$ such that $T \models \neg\psi \leftrightarrow \chi$; so χ works for $\neg\varphi$. The cases for $\wedge$ and $\exists$ are clear.

(iii)⇒(i): Assume (iii), and suppose that $\overline{A}$ and $\overline{B}$ are models of T with $\overline{A} \leq \overline{B}$. Suppose that φ is a formula and $a \in {}^{\omega}A$; we want to show that $\overline{A} \models \varphi[a]$ iff $\overline{B} \models \varphi[a]$. First assume that $\overline{A} \models \varphi[a]$. By (iii), let ψ be an existential formula with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$. Hence $\overline{A} \models \psi[a]$, so $\overline{B} \models \psi[a]$, hence $\overline{B} \models \varphi[a]$. Second assume that $\overline{A} \not\models \varphi[a]$. Then $\overline{A} \models \neg\varphi[a]$, so $\overline{B} \models \neg\varphi[a]$ by what was just shown, hence $\overline{B} \not\models \varphi[a]$. □

Elimination of quantifiers revisited; Real-closed fields

Throughout this section we assume that our language has at least one individual constant. We say that a theory T has *full elimination of quantifiers* iff for every formula φ there is a quantifier-free formula ψ with the same free variables as φ such that $T \models \varphi \leftrightarrow \psi$.

The earlier argument for a special case of elimination of quantifiers essentially proves along the way the following result.

Theorem 6.31. *If for every quantifier-free formula $\varphi(x, \overline{y})$ there is a quantifier-free formula $\psi(\overline{y})$ such that $T \models \exists x\varphi(x, \overline{y}) \leftrightarrow \psi(\overline{y})$, then T has full elimination of quantifiers.*

Lemma 6.32. *Let $\varphi(\overline{x})$ be a formula with the indicated free variables. Then the following conditions are equivalent:*

(i) There is a quantifier free formula $\psi(\overline{x})$ such that $T \models \varphi \leftrightarrow \psi$. ($\psi(\overline{x})$ has no free variables except $\overline{x}$.)

(ii) For any structures $\overline{A}, \overline{B}, \overline{C}$, if $\overline{B}$ and $\overline{C}$ are models of T, $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, and if $\overline{a}$ is a sequence of elements of A, then $\overline{B} \models \varphi[\overline{a}]$ iff $\overline{C} \models \varphi[\overline{a}]$.

Proof. (i)⇒(ii): assume (i) and the hypotheses of (ii). Then

$$\begin{aligned} \overline{B} \models \varphi[\overline{a}] \quad &\text{iff} \quad \overline{B} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{A} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{C} \models \psi[\overline{a}] \\ &\text{iff} \quad \overline{C} \models \varphi[\overline{a}] \end{aligned}$$

□

(ii)⇒(i): Assume (ii). Let $\overline{d} = \langle d_i : i < m \rangle$ be a system of new individual constants whose length is that of $\overline{x}$, thus expanding $\mathscr{L}$ to $\mathscr{L}'$. Let

$$\Gamma \stackrel{\text{def}}{=} \{\psi(\overline{d}) : \psi(\overline{x}) \text{ is a quantifier-free formula in } \mathscr{L}' \text{ and } T \models \varphi(\overline{d}) \to \psi(\overline{d})\}.$$

Here $\psi(\overline{x})$ has no free variables except for $\overline{x}$. We claim

$(*)$ $T \cup \Gamma \models \varphi(\overline{d})$.

In fact, suppose that this fails. Let $(\overline{B}, b_i)_{i<m}$ be a model of $T \cup \Gamma \cup \{\neg\varphi(\overline{d})\}$, and let A be the subuniverse of $\overline{B}$ generated by $\{b_i : i < m\}$, and let $\Sigma = T \cup \text{diag}((\overline{A}, b_i)_{i<m}) \cup \{\varphi(\overline{d})\}$. We claim that Σ has a model. Otherwise, by Proposition 6.28 there is a finite conjunction ψ of members of $\text{diag}(\overline{A}, b_i)_{i<m})$ such that $T \models \psi \to \neg\varphi(\overline{d})$. Now for each constant c_u from the diagram of $(\overline{A}, b_i)_{i<m})$ which appears in ψ, there is a term $\tau_u(d_i : i < m)$ of $\mathscr{L}'$ such that $u = \tau_u^{\overline{A}}(b_i : i < m)$; we replace c_u in ψ by $\tau_u(d_i : i < m)$; this gives a formula ψ'. Note that

$$(\overline{A}, b_i)_{i<m} \models c_u = \tau_u[u, b_i : i < m] \quad \text{and} \quad \models \bigwedge\{c_u = \tau_u : c_u \text{ appears in } \psi\} \to (\psi \leftrightarrow \psi').$$

It follows that $(\overline{A}, b_i)_{i<m} \models \psi'$, and hence also $(\overline{B}, b_i)_{i<m} \models \psi'$. Also, $T \models \varphi(\overline{d}) \to \neg\psi'$, so $\neg\psi' \in \Gamma$. This is a contradiction.

It follows that Σ has a model. By the diagram lemma, this gives a structure $(\overline{C}, b_i)_{i<m}$ which is a model of $T \cup \{\varphi(\overline{d})$ and is an extension of $(\overline{A}, b_i)_{i<m}$. This contradicts (ii).

Hence $(*)$ holds. Hence by Lemma 6.28 there is a finite conjunction ψ of members of Γ such that $T \models \psi(\overline{d}) \to \varphi(\overline{d})$. Clearly ψ is as desired. □

Corollary 6.33. *Let T be a theory in a language $\mathscr{L}$. Suppose that for all quantifier-free formulas $\varphi(x, \overline{y})$ and all $\mathscr{L}$-structures $\overline{A}, \overline{B}, \overline{C}$, if $\overline{B}$ and $\overline{C}$ are models of T and $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, and if $\overline{a}$ is a system of elements of A of the length of $\overline{y}$, then $\overline{B} \models \exists x\varphi(x, \overline{a})$ implies that $\overline{C} \models \exists x\varphi(x, \overline{a})$.*

Under these conditions, T has full quantifier elimination.

Proof. Assume the hypothesis. By Theorem 6.31 it suffices to take any quantifier-free formula $\varphi(x, \overline{y})$ and show that there is a quantifier-free formula $\psi(\overline{y})$ such that $T \models \exists x\varphi(x, \overline{y}) \leftrightarrow \psi(\overline{y})$. The existence of ψ is assured by Lemma 6.32. □

Let T be a theory in a language $\mathscr{L}$. A triple $(\overline{A}, f, \overline{M})$ is *algebraically prime for* T iff $\overline{M}$ is a model of T, f is an embedding of $\overline{A}$ into $\overline{M}$, and for every model $\overline{N}$ of T and embedding g of $\overline{A}$ into $\overline{N}$ there is an embedding h of $\overline{M}$ into $\overline{N}$ such that $h \circ f = g$. This is illustrated in the following diagram:

$$
\begin{tikzcd}
\overline{M} \arrow[rr, dotted, "h"] & & \overline{N} \\
& \overline{A} \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Theorem 6.34. *Suppose that T is a theory in a language $\mathscr{L}$, and the following two conditions hold.*

(i) For any structure $\overline{A}$ which is embeddable in a model of T, there is an algebraically prime triple of the form $(\overline{A}, f, \overline{D})$ for T.

(ii) If $\overline{B}$ and $\overline{C}$ are models of T, $\overline{B} \leq \overline{C}$, $\varphi(x, \overline{y})$ is a quantifier-free formula, $\overline{b}$ is a sequence of elements of B, and $\overline{C} \models \exists x\varphi(x, \overline{b})$, then $\overline{B} \models \exists x\varphi(x, \overline{b})$.

Then T has full quantifier elimination.

Proof. We verify the condition of Corollary 6.33. Thus suppose that $\varphi(x, \overline{y})$ is a quantifier-free formula, $\mathscr{L}$-structures $\overline{A}, \overline{B}, \overline{C}$ are given, $\overline{B}$ and $\overline{C}$ are models of T and $\overline{A}$ is a substructure of both $\overline{B}$ and $\overline{C}$, $\overline{a}$ is a system of elements of A of the length of $\overline{y}$, and $\overline{B} \models \exists x\varphi(x, \overline{a})$. We want to show that $\overline{C} \models \exists x\varphi(x, \overline{a})$. Let $(\overline{A}, f, \overline{D})$ be an algebraically prime triple for T. By the definition of algebraically prime, there are embeddings g, h of $\overline{D}$ into $\overline{B}, \overline{C}$ respectively such that $g \circ f$ is the identity on A and $h \circ f$ is the identity on A. See the diagram:

$$
\begin{tikzcd}
\overline{B} & \overline{D} \arrow[l, "g"'] \arrow[r, "h"] & \overline{C} \\
& \overline{A} \arrow[ul, "\leq"] \arrow[u, "f"] \arrow[ur, "\leq"'] &
\end{tikzcd}
$$

Now $\overline{B} \models \exists x\varphi(x, g \circ f \circ \overline{a})$, hence $\overline{D} \models \exists x\varphi(x, f \circ \overline{a})$, hence $\overline{C} \models \exists x\varphi(x, h \circ g \circ \overline{a})$, hence $\overline{C} \models \exists x\varphi(x, \overline{a})$. $\square$

Theorem 6.34 gives a very useful criterion for eliminating quantifiers. We use it to prove the following classical result of Tarski. Tarski originally proved this result by the straightforward method described at the beginning of this chapter; but the details are formidable.

Theorem 6.35. *The theory of real-closed ordered fields has full elimination of quantifiers.*

Proof. We appeal to algebraic facts found in the appendix *real* to these notes. By Corollary 12 of *real*, we are dealing with the class of models of a first-order theory T. Theorem 20 of *real* gives (i) of Theorem 6.34.

For (ii), suppose that $\overline{B}$ and $\overline{C}$ are real-closed ordered fields, $\overline{B} \leq \overline{C}$, $\varphi(x, \overline{y})$ is a quantifier-free formula, $\overline{b}$ is a sequence of elements of B, and $\overline{C} \models \exists x\varphi(x, \overline{b})$.

We may assume that $\varphi(x, \overline{b})$ is a conjunction of atomic formulas and their negations, and we may assume that x actually occurs in each conjunct. Thus each atomic formula has the form $p(x, \overline{b}) = q(x, \overline{b})$ or $p(x, \overline{b}) < q(x, \overline{b})$ for certain polynomials $p(x, \overline{b})$ and $q(x, \overline{b})$. Since

$$T \models p(x, \overline{b}) = q(x, \overline{b}) \leftrightarrow p(x, \overline{b}) - q(x, \overline{b}) = 0,$$

and similarly for $<$, we may assume that each atomic formula has the form $p(x, \overline{b}) = 0$ or $0 < q(x, \overline{b})$ for certain polynomials $p(x, \overline{b})$ and $q(x, \overline{b})$. Moreover,

$$T \models \neg(p(x, \overline{b}) = 0) \leftrightarrow 0 < p(x, \overline{b}) \vee 0 < -p(x, \overline{b}) \quad \text{and}$$
$$T \models \neg(0 < p(x, \overline{b})) \leftrightarrow p(x, \overline{b}) = 0 \vee 0 < -p(x, \overline{b}).$$

Hence we may assume that $\varphi(x, \overline{y})$ is a conjunction of formulas of the form $p(x, \overline{b}) = 0$ and $0 < p(x, \overline{b})$. If any of the polynomials in conjuncts $p(x, \overline{b}) = 0$ is nonzero, then any root in B must already be in A since $\overline{A}$ does not have any proper formally real extensions, by the definition of real-closed. In this case $\exists x\varphi(x\overline{b})$ is true in $\overline{A}$, as desired. Thus we may assume that $\varphi(x, \overline{y})$ is a conjunction of formulas of the form $0 < p(x, \overline{b})$.

Now there are only finitely many roots of the polynomials $p(x, \overline{b})$ in B; let them be $c_1, \ldots, c_k$. As before, these are all in A. Choose $d \in B$ such that $\overline{B} \models \varphi[d, \overline{b}]$. Without loss of generality, $c_1, \ldots, c_i < d < c_{i+1}, \ldots, c_k$. Let $e \in A$ be such that $c_1, \ldots, c_i < e < c_{i+1}, \ldots, c_k$. Then by Proposition 15 of *real*, $f(d, \overline{b}) > 0$ iff $f(e, \overline{b}) > 0$, for every conjunct $f(x, \overline{b}) > 0$ of $\varphi(x, \overline{b})$. It follows that $\overline{A} \models \varphi[e, \overline{b}]$. $\square$

Infinitary languages

Let κ and λ be infinite cardinals. The logic $L_{\kappa\lambda}$ allows conjunctions and disjunctions of sets of formulas of size $< \kappa$, and simultaneous quantification over fewer than λ variables. We do not go into the formal definition of such languages, which is straightforward. Note that $L_{\omega\omega}$ is essentially the first-order logic we have been dealing with.

Note that the compactness theorem fails in general for these languages. For example, the following set of sentences in $L_{\omega_1\omega}$ does not have a model, although every finite subset does:

$$\{\forall x[x = c_0 \vee x = c_1 \vee \ldots \vee x = c_n \vee \ldots]\} \quad \text{(length } \omega)$$
$$\cup \{d \neq c_0 \wedge d \neq c_1 \wedge \ldots \wedge d \neq c_n : n \in \omega\},$$

where $d, c_0, c_1, \ldots$ are individual constants.

Attempts to generalize the compactness theorem have led to two important large cardinals:

• κ is *strongly compact* iff $\kappa > \omega$ and for every signature and every set Γ of sentences in $L_{\kappa\kappa}$ in that signature, if every subset of Γ of size less that κ has a model, then so does Γ.

• κ is *weakly compact* iff $\kappa > \omega$ and for every signature and every set Γ of sentences in $L_{\kappa\kappa}$ in that signature, subject to $|\Gamma| \le \kappa$, if every subset of Γ of size less that κ has a model, then so does Γ.

Both these cardinals are strongly inaccessible, with many strong inaccessibles below them.

$L_{\omega_1\omega}$ seems to be the most tractable of these infinitary logics. We prove one important theorem about it.

Theorem 6.36. (D. Scott) *Let $\mathscr{L}$ be a countable language, and $\overline{M}$ a countable $\mathscr{L}$-structure. Then there is an $L_{\omega_1\omega}$-sentence ψ in $\mathscr{L}$ such that ψ has countable models, and all such models are isomorphic to $\overline{M}$.*

Proof. For each $n \in \omega$, each $a \in {}^nM$, and each $\alpha < \omega_1$ we define a formula φ^β_a. The definition goes by induction on α. Let

$$\begin{aligned}\varphi^0_a = \bigwedge\{\chi : \chi &\text{ is a formula with free variables among } v_0, \ldots, v_{n-1},\\ &\chi \text{ is atomic or the negation of an atomic formula, and } \overline{M} \models \chi[a]\}.\end{aligned}$$

For α limit we let

$$\varphi^\alpha_a = \bigwedge_{\beta<\alpha} \varphi^\beta_a.$$

Finally,

$$\varphi^{\alpha+1}_a = \varphi^\alpha_a \wedge \bigwedge_{b\in M} \exists v_n \varphi^\alpha_{a^\frown\langle b\rangle} \wedge \forall v_n \bigvee_{b\in M} \varphi^\alpha_{a^\frown\langle b\rangle}.$$

Then the following conditions hold; proof by induction on α:

(1) $\overline{M} \models \varphi^\alpha_a[a]$.

(2) $\models \varphi^\alpha_a \to \varphi^\beta_a$ for $\beta < \alpha < \omega_1$.

By (2), $\langle\{b \in {}^nM : \overline{M} \models \varphi^\alpha_a[b]\} : \alpha < \omega_1\rangle$ is a $\subseteq$-increasing sequence of subsets of ${}^nM$. Since ${}^nM$ is countable, it follows that there is an $\alpha_a < \omega_1$ such that $\overline{M} \models \varphi^{\alpha_a}_a \leftrightarrow \varphi^\beta_a$ whenevery $\alpha_a \le \beta < \omega_1$. Let β be the supremum of all α_a for $a \in \bigcup_{n\in\omega} {}^nM$. Then $\beta < \omega_1$ and

(3) For all finite sequences a of elements of M and all $\alpha \ge \beta$ we have $\overline{M} \models \varphi^\alpha_a \leftrightarrow \varphi^\beta_a$.

Now we define the desired sentence ψ:

$$\psi = \varphi^\beta_\emptyset \wedge \bigwedge\{\forall v_0 \ldots \forall v_{n-1}(\varphi^\beta_a \to \varphi^{\beta+1}_a) : n \in \omega,\ a \in {}^nM\}.$$

By (1) and (3), $\overline{M}$ is a model of ψ. Now let $\overline{N}$ be any countable model of ψ. Write $M = \{a_i : i < \omega\}$ and $N = \{b_i : i < \omega\}$ We construct sequences $\langle e_i : i < \omega\rangle$ and $\langle (c_i, d_i) : i < \omega\rangle$ by recursion, so that for each i we have

$$\overline{M} \models \varphi^{\beta}_{e\restriction i}[c \restriction i] \quad \text{and } \overline{N} \models \varphi^{\beta}_{e\restriction i}[d \restriction i]. \tag{4}$$

This holds for $i = 0$ since $\models \psi \to \varphi^{\beta}_{\emptyset}$. Now assume that it holds for all $j \leq 2i$. Let c_{2i} be the element a_k with smallest index k such that $a_k \notin \{c_j : j < 2i\}$. By (4), $\overline{M} \models \varphi^{\beta}_{e\restriction 2i}[c \restriction 2i]$ and $\overline{M} \models \psi$, so $\overline{M} \models \varphi^{\beta+1}_{e\restriction 2i}[c \restriction 2i]$ and hence $\overline{M} \models \forall v_{2i} \bigvee_{u\in M} \varphi^{\beta}_{(e\restriction 2i)^\frown\langle u\rangle}[c \restriction 2i]$. So we can choose $e_{2i} \in M$ such that $\overline{M} \models \varphi^{\beta}_{(e\restriction 2i)^\frown\langle e_{2i}\rangle}[(c \restriction 2i)^\frown\langle c_{2i}\rangle]$. Now also $\overline{N} \models \varphi^{\beta}_{e\restriction 2i}[d \restriction 2i]$ and, since $\overline{N} \models \psi$, also $\overline{N} \models \varphi^{\beta}_{e\restriction 2i}[d \restriction 2i] \to \psi^{\beta+1}_{e\restriction 2i}[d \restriction 2i]$. Hence $\overline{N} \models \psi^{\beta+1}_{e\restriction 2i}[d \restriction 2i]$, so $\overline{N} \models \exists v_{2i}\varphi^{\beta}_{(e\restriction 2i)^\frown\langle e_{2i}\rangle}[d \restriction 2i]$. So we get an element d_{2i} of N such that $\overline{N} \models \varphi^{\beta}_{(e\restriction 2i)^\frown\langle e_{2i}\rangle}[(d \restriction 2i)^\frown\langle d_{2i}\rangle]$. Thus (4) holds for $2i + 1$. Now we repeat this argument with the roles of $\overline{M}$ and $\overline{N}$ interchanged, starting with an element $d_{2i+1} = b_l$ with l the smallest index such that $b_l \notin \{d_j : j \leq 2i\}$, to obtain c_{2i+1} such that (4) holds for $2i + 2$. This finishes the construction.

By (2) we have $\overline{M} \models \varphi^{0}_{c\restriction i}[c \restriction i]$ and $\overline{N} \models \varphi^{0}_{c\restriction i}[d \restriction i]$ for all i, and by construction $\{a_i : i < \omega\} = \{c_i : i < \omega\}$ and $\{b_i : i < \omega\} = \{d_i : i < \omega\}$, so $\{(c_i, d_i) : i < \omega\}$ is the desired isomorphism from $\overline{M}$ onto $\overline{N}$. $\square$

EXERCISES

Exc. 6.1. A subset X of a structure $\overline{M}$ is *definable* iff there is a formula $\varphi(x)$ with only x free such that $X = \{a \in M : \overline{M} \models \varphi[a]\}$. Similarly, for any positive integer m, a subset X of ${}^mM$ is *definable* iff there is a formula $\varphi(\overline{x})$ with $\overline{x}$ a sequence of m distinct variables including all variables occurring free in φ, such that $X = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$.

For the language with no nonlogical symbols and for any structure $\overline{M}$ in that language, determine all the definable subsets and m-ary relations over $\overline{M}$. Hint: use Theorem 6.1.

Exc. 6.2. Let Γ be the set of all sentences holding in the structure $(\omega, S, 0)$, where $S(n) = n + 1$ for all $n \in \omega$. Prove an elimination of quantifiers theorem for Γ.

Exc. 6.3. Let T be the theory of an infinite equivalence relation each of whose equivalence classes has exactly two elements. Use an Ehrenfeucht game to show that T is complete.

Exc. 6.4. Let T be any theory. Show that the class of all substructures of models of T is the class of all models of a set of universal sentences, i.e., sentences of the form $\forall\overline{x}\varphi$ with φ quantifier free and $\overline{x}$ a finite string of variables containing all variables free in φ.

Exc. 6.5. Suppose that $\Gamma \cup \{\varphi\}$ is a set of sentences in a language $\mathscr{L}$. Suppose that Γ and φ have the same models. Prove that there is a finite subset Δ of Γ with the same models as Γ.

Exc. 6.6. Suppose that T and T' are theories in a language $\mathscr{L}$. Show that the following conditions are equivalent:

(i) Every model of T' can be embedded in a model of T.

(ii) Every universal sentence which holds in all models of T also holds in all models of T'.

Exc. 6.7. Let T be a theory in a language $\mathscr{L}$. Let $\mathbf{K}$ be the class of all models of T. Show that the following conditions are equivalent:

(i) $\mathbf{SK} = \mathbf{K}$.

(ii) There is a collection Γ of universal sentences such that $\mathbf{K}$ is the class of all models of Γ.

Exc. 6.8. Suppose that $\overline{A} \leq \overline{B}$. Prove that $\overline{A} \preceq \overline{B}$ iff $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)a \in A$.

Exc. 6.9. Suppose that m is a positive integer, $\varphi(\overline{x})$ is a formula with free variables $\overline{x}$ of length m, and $\overline{M}$ is a structure. Define $\varphi(\overline{M}) = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$. Show that the following conditions are equivalent:

(i) $\varphi(\overline{M})$ is finite.

(ii) $\varphi(\overline{M}) = \varphi(\overline{N})$ whenever $\overline{M} \preceq \overline{N}$.

Exc. 6.10. Prove that if K is a set of models of a complete theory T then there is a structure $\overline{M}$ such that every member of K can be elementarily embedded in $\overline{M}$.

Exc. 6.11. Suppose that $\overline{A}$ and $\overline{B}$ are elementarily equivalent, κ-saturated, and both of size κ. Show that they are isomorphic.

Exc. 6.12. For any natural number n and any structure $\overline{M}$, an *n-type* of $\overline{M}$ is a collection Γ of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, a sequence of distinct variables of length n, such that $\overline{M}_M \models \exists\overline{x}\varphi$ for every conjunction of finitely many members of Γ. Prove that if Γ is a collection of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, then Γ is an n-type over $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $\overline{N} \models \varphi[\overline{a}]$ for every $\varphi \in \Gamma$.

Exc. 6.13. If $\overline{M}$ is a structure, $A \subseteq M$, and $n \in \omega$, then an *n-type over A* of $\overline{M}$ is an n-type of $\overline{M}$ all of whose additional constants come from A. Given an n-tupe $\overline{a}$ of elements of M, the n-type over A of $\overline{a}$ in $\overline{M}$, denoted by $\mathrm{tp}^{\overline{M}}(\overline{a})/A)$, is the set $\{\varphi(\overline{x}) : \varphi$ is a formula with free variables among $\overline{x}$, $\overline{x}$ has length n, and $\overline{M}_A \models \varphi[\overline{a}]\}$. An n-type S over A is *complete* iff $\varphi \in S$ or $\neg\varphi \in S$ for every formula in the language $\mathscr{L}_A$ with free variables among $\overline{x}$.

Prove that S is a complete n-type over A in $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S = \mathrm{tp}^{\overline{N}}(\overline{a}/A)$.

Exc. 6.14. Let t be an n-type over A of $\overline{M}$. We say that t is *isolated* iff there is a formula $\varphi(\overline{x})$ in $\mathscr{L}_A$ such that $\overline{M}_A \models \exists\overline{x}\varphi$ and $\overline{M}_A \models \forall\overline{x}(\varphi \rightarrow \psi)$ for every $\psi \in t$. We then say that φ *isolates* t.

Prove that if φ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

Exc. 6.15. Show that $\mathrm{tp}^{\overline{M}}(\overline{a}{}^\frown\overline{b}/A)$ is isolated iff both $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated.

Exc. 6.16. Let T be a complete theory with a model. A formula $\varphi(\overline{x})$ is *complete* in T iff $T \cup \{\exists\overline{x}\varphi\}$ has a model, and for every formula $\psi(\overline{x})$, either $T \models \varphi \rightarrow \psi$ or $T \models \varphi \rightarrow \neg\psi$. Here $\overline{x}$ is a sequence of variables containing all variables free in φ or ψ.

A formula $\theta(\overline{x})$ is *completable* in T iff there is a complete formula $\varphi(\overline{x})$ such that $T \models \varphi \to \theta$.

A structure $\overline{M}$ is *atomic* iff every tuple $\overline{a}$ of elements of M satisfies a complete formula in the theory of $\overline{M}$.

A theory T is *atomic* iff for every formula $\theta(\overline{x})$ such that $T \cup \{\exists \overline{x} \theta(\overline{x})$ has a model, θ is completable in T.

Show that if T is a complete theory in a countable language, then T has a countable atomic model iff T is atomic. Hint: in the direction $\Leftarrow$, for each $n \in \omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0, \ldots, v_{n-1}$, and apply the omitting types theorem.

7. Morley's theorem

This chapter is devoted to the proof of Morley's theorem, which says that in a countable language, if Γ is a theory with only infinite models and Γ is κ-categorical for some uncountable cardinal κ, then it is κ-categorical for every uncountable cardinal κ. In the course of developing the proof we will introduce several new model-theoretic concepts. We follow Marker, **Model theory, an introduction.**

Unless otherwise mentioned, T is a complete theory in a countable language having only infinite models.

Some useful notation is as follows. If $\overline{M}$ is a structure and $\varphi(\overline{v})$ is a formula with parameters in M, with $\overline{v}$ of length m, then by $\varphi(\overline{M})$ we mean the set $\{\overline{a} \in {}^mM : \overline{M} \models \varphi(\overline{a})\}$. Here we make a slight abuse of notation, in that we write $\overline{M} \models \varphi(\overline{a})$ when we should write something like $(\overline{M}, \overline{b}) \models \varphi(\overline{v}, \overline{w})[\overline{a}, \overline{b}]$, where $\overline{b}$ is the sequence of parameters in φ. Similar abuses will take place later without comment.

Let $\overline{M}$ be an $\mathscr{L}$-structure, and suppose that $A \cup \{b\} \subseteq M$. We say that *b is algebraic over A* iff there is a formula $\varphi(x, \overline{a})$ with $\overline{a} \in A$ such that $\varphi(\overline{M}, \overline{a})$ is finite and $\overline{M} \models \varphi(b, \overline{a})$. Now if $A \subseteq D \subseteq M$ we define

$$\mathrm{acl}_D(A) = \{b \in D : b \text{ is algebraic over } A\}.$$

Lemma 7.1. *(i)* $A \subseteq \mathrm{acl}_D(A)$.

(ii) $\mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$.

(iii) If $A \subseteq B$ *then* $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(B)$.

(iv) If $a \in \mathrm{acl}_D(A)$, *then* $a \in \mathrm{acl}_D(A_0)$ *for some finite* $A_0 \subseteq A$.

Proof. (i): For any $a \in A$, let $\varphi(x, a)$ be the formula $x = a$.

(ii): Suppose that $b \in \mathrm{acl}_D(\mathrm{acl}_D(A))$. Accordingly, choose $\varphi(v, \overline{w})$ and $\overline{a} \in \mathrm{acl}_D(A)$ such that

$$\overline{M} \models \varphi(b, \overline{a}) \text{ and } \{y \in M : \overline{M} \models \varphi(y, \overline{a})\} \text{ is finite.}$$

Say $\overline{a}$ has length n. Then for all $i < n$ we get $\psi_i(v, \overline{u}^i)$ and $\overline{c}^i \in A$ such that

$$\overline{M} \models \psi_i(a_i, \overline{c}^i) \text{ and } \{y \in M : \overline{M} \models \psi_i(y, \overline{c}^i)\} \text{ is finite.}$$

Let $k = |\{y \in M : \overline{M} \models \varphi(y, \overline{a})\}|$. Now let $\chi(v, \overline{u}^0, \ldots, \overline{u}^{n-1})$ be the formula

$$\exists v_0 \ldots v_{n-1} \left[\bigwedge_{i<n} \psi_i(v_i, \overline{u}^i) \wedge \varphi(v, v_0, \ldots, v_{n-1}) \wedge (\exists! k) v_j \varphi(v_j, v_0, \ldots, v_{n-1}) \right].$$

Here $(\exists! k)v_j \ldots$ abbreviates "there are exactly k v_j such that $\ldots$", which is easy to express in our language.

Now we want to show that $\overline{M} \models \chi(b, \overline{c}^0, \ldots, \overline{c}^{n-1})$. For any $i < n$ we have $\overline{M} \models \psi_i(a_i, \overline{c}^i)$, and so

$$\overline{M} \models \bigwedge_{i<n} \psi_i(a_i, \overline{c}^i) \wedge \varphi(b, \overline{a}) \wedge (\exists! k) v_j \varphi(v_j, \overline{a}).$$

Hence $\overline{M} \models \chi(b, \overline{c}^0, \ldots, \overline{c}^{n-1})$.

Next we want to show that $\{y \in M : \overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$ is finite. Let

$$K = \prod_{i<m} \{y \in M : \overline{M} \models \psi_i(y, \overline{c}^i)\}.$$

Thus K is finite. Suppose that $\overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$. Choose $\overline{e}$ such that

$$\overline{M} \models \bigwedge_{i<n} \psi_i(e_i, \overline{c}^i) \wedge \varphi(y, \overline{e}) \wedge (\exists! k) v_j \varphi(v_j, \overline{e}).$$

Then $\overline{e} \in K$. Hence there are at most $|K| \cdot k$ elements y such that $\overline{M} \models \chi(y, \overline{c}^0, \ldots, \overline{c}^{n-1})\}$.

(iii) and (iv) are clear. $\square$

If $\overline{M}$ is a structure, m is a positive integer, and $D \subseteq {}^m M$, then we say that D is *definable with parameters* iff there is a formula $\varphi(\overline{v})$ in $\mathscr{L}_M$ with $\overline{v}$ of length m such that $D = \{a \in {}^m M : \overline{M}_M \models \varphi(a)\}$.

A subset D of M^n is *minimal* in $\overline{M}$ iff D is infinite, and for any set $Y \subseteq D$ definable with parameters, either Y is finite or $D \backslash Y$ is finite. In case $\varphi(\overline{v}, \overline{a})$ defines D, we also say that φ is minimal.

Lemma 7.2. *Suppose that $D \subseteq M$ is definable and minimal in $\overline{M}$ and $A \cup \{a, b\} \subseteq D$. Suppose that $a \in \mathrm{acl}_D(A \cup \{b\}) \backslash \mathrm{acl}_D(A)$. Then $b \in \mathrm{acl}_D(A \cup \{a\})$.*

Proof. Assume the hypotheses. Thus there is a formula $\varphi(a, b)$ with additional parameters from A, and a positive integer n, such that $\overline{M} \models \varphi(a, b)$ and $|\{x \in D : \overline{M} \models \varphi(x, b)\}| = n$. Let $\psi(w)$ be the formula with parameters from A asserting that $|\{x \in D : \varphi(x, w)\}| = n$. If $\psi(w)$ defines a finite subset of D, then $b \in \mathrm{acl}_D(A)$. Hence $A \cup \{b\} \subseteq \mathrm{acl}_D(A)$, hence by Lemma 7.1 $a \in \mathrm{acl}_D(A \cup \{b\}) \subseteq \mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$, contradiction. It follows that $\psi(w)$ defines a cofinite subset of D.

If $\{y \in D : \overline{M} \models \varphi(a, y) \wedge \psi(y)\}$ is finite then since b is in this set we get $b \in \mathrm{acl}_D(A \cup \{a\})$, as desired. Thus we may assume that $\{y \in D : \overline{M} \models \varphi(a, y) \wedge \psi(y)\}$ is cofinite in D; say that its complement has size l. Let $\chi(x)$ be the formula expressing that

$$|D \backslash \{y \in D : \varphi(x, y) \wedge \psi(y)\}| = l.$$

since $\overline{M} \models \chi(a)$, our assumption that $a \notin \mathrm{acl}_D(A)$ implies that $\chi(\overline{M})$ is cofinite. Let $a_0, \ldots, a_n$ be distinct members of $\chi(\overline{M})$. Then for each $i \leq n$ the set $B_i \stackrel{\text{def}}{=} \{y \in D : \overline{M} \models \varphi(a_i, y) \wedge \psi(y)\}$ is cofinite. Let $c \in \bigcap_{i \leq n} B_i$. Thus $\varphi(a_i, c)$ for each $i \leq n$, so $|\{x \in D : \overline{M} \models \varphi(x, c)\}| \geq n + 1$, contradicting the choice of $\psi(c)$. $\square$

Suppose that $D \subseteq M^n$. We say that D is *strongly minimal* in $\overline{M}$ iff D is minimal in any elementary extension of $\overline{M}$. Similarly for a formula φ.

Given $A \subseteq D$, we call A *independent* iff $\forall a \in A[a \notin \mathrm{acl}_D(A \backslash \{a\})]$. For $C \subseteq D$ we say that A is *independent over* C iff $\forall a \in A[a \notin \mathrm{acl}_D(C \cup (A \backslash \{a\}))$. Note then that $A \cap C = \emptyset$.

For $\overline{a}$ a sequence of elements of M and $A \subseteq M$ we define

$$\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \{\varphi(\overline{v}) : \varphi \text{ is a formula with parameters from } A \text{ and } \overline{M} \models \varphi(\overline{a})\}.$$

Note that if $\overline{a}$ is the empty sequence, then $\mathrm{tp}(\overline{a}/A)$ is simply the set of all sentences with parameters from A that hold in $\overline{M}$. If A is empty, we just omit it.

Lemma 7.3. *Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:*

(i) $A = \emptyset$.

(ii) $A \subseteq \overline{M}_0 \prec \overline{M}, \overline{N}$.

Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from A, $n \in \omega$, $a \in {}^n\varphi(\overline{M})$, $\mathrm{rng}(\overline{a})$ is independent over A, and $b \in {}^n\varphi(\overline{N})$, $\mathrm{rng}(\overline{b})$ is independent over A.

Then $\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \mathrm{tp}^{\overline{N}}(\overline{b}/A)$.

Proof. Induction on n. For $n = 0$ the conclusion is clear if (i) holds, since $\overline{M} \equiv \overline{N}$. The conditions in (ii) also clearly give the conclusion.

Now assume the result for n, and suppose that $a \in {}^{n+1}\varphi(\overline{M})$, $\mathrm{rng}(\overline{a})$ is independent over A, $b \in {}^{n+1}\varphi(\overline{N})$, and $\mathrm{rng}(\overline{b})$ is independent over A. So by the inductive hypothesis,

$$\mathrm{tp}^{\overline{M}}((\overline{a} \upharpoonright n)/A) = \mathrm{tp}^{\overline{N}}((\overline{b} \upharpoonright n)/A). \tag{1}$$

Let $\psi(\overline{v})$ be a formula with parameters from A such that $\overline{M} \models \psi(\overline{a})$. Now $a_n \in \varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ and $a_n \notin \mathrm{acl}_D(A \cup \{a_0, \ldots, a_{n-1}\})$, so $\varphi(\overline{M}) \cap \psi(a_0, \ldots, a_{n-1}, \overline{M})$ is infinite. Since φ is strongly minimal, this set is actually cofinite in $\varphi(\overline{M})$. So there is an integer m such that

$$\overline{M} \models |\{v : \varphi(v) \wedge \neg\psi(a_0, \ldots, a_{n-1}, v\}| = m.$$

Thus the formula $\chi(\overline{w})$ expressing that

$$|\{v : \varphi(v) \wedge \neg\psi(w_0, \ldots, w_{n-1}, v\}| = m$$

is in $\mathrm{tp}^{\overline{M}}((\overline{a} \upharpoonright n)/A)$, and hence by (1) we get

$$\overline{N} \models |\{v : \varphi(v) \wedge \neg\psi(b_0, \ldots, b_{n-1}, v\}| = m.$$

Since $b_n \notin \mathrm{acl}_D(A \cup \{b_0, \ldots, b_{n-1}\})$, it follows that $\overline{N} \models \psi(\overline{b})$, as desired. □

If X is an infinite subset of M, then X *is an indiscernible set* over $\overline{M}$ iff for any formula $\varphi(\overline{v})$ and any two sequences $\overline{x}, \overline{y}$ of distinct elements of X we have $\overline{M} \models \varphi(\overline{x}) \leftrightarrow \varphi(\overline{y})$.

Corollary 7.4. *Suppose that $\overline{M}, \overline{N} \models T$ and one of the following conditions holds:*

(i) $A = \emptyset$.

(ii) $A \subseteq \overline{M}_0 \prec \overline{M}, \overline{N}$.

Assume that $\varphi(v)$ is strongly minimal over $\overline{M}$ and has parameters from A, and B and C are infinite subsets of $\varphi(\overline{M})$ each independent over A. Then B and C are sets of *indiscernibles over $\overline{M}$, and for any $n \in \omega$ and one-one sequences $\overline{b} \in {}^nB$ and $\overline{c} \in {}^nC$ we have* $\mathrm{tp}(\overline{b}/A) = \mathrm{tp}(\overline{c}/A)$. $\square$

If $Y \subseteq D$, we say that $A \subseteq Y$ is a *basis* for Y iff A is independent and $\mathrm{acl}_D(A) = \mathrm{acl}_D(Y)$.

Lemma 7.5. *Assume that $D \subseteq M$ is minimal in $\overline{M}$.*

(i) A union of a chain of independent sets over a set $C \subseteq D$ is again independent over C. (Hence we can apply Zorn's lemma in this context.)

(ii) For any $Y \subseteq D$, any maximal independent subset subset of Y is a basis for Y.

Proof. (i): Let $\mathscr{A}$ be a chain of independent sets over C. Suppose that $a \in \mathrm{acl}_D(C \cup (\bigcup \mathscr{A} \backslash \{a\}))$. Thus a is algebraic over $C \cup (\bigcup \mathscr{A} \backslash \{a\})$, and so there is a formula $\varphi(x, \overline{c}, \overline{b})$ with $\overline{c} \in C$ and $\overline{b} \in \bigcup \mathscr{A} \backslash \{a\}$ such that $\varphi(\overline{M}, \overline{c}, \overline{b})$ is finite and $\overline{M} \models \varphi(a, \overline{c}, \overline{b})$. Then there is an $X \in \mathscr{A}$ such that $\overline{b} \in X$, so that $a \in \mathrm{acl}_D(C \cup (X \backslash \{a\}))$, contradiction.

(ii): Suppose that A is a maximal independent subset of Y. Obviously $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(Y)$. Suppose that $a \in \mathrm{acl}_D(Y) \backslash \mathrm{acl}_D(A)$. Let $\varphi(x, \overline{y})$ be a formula with $\overline{y} \in Y$ such that $\varphi(\overline{M}, \overline{y})$ is finite and $\overline{M} \models \varphi(a, \overline{y})$. If each $y_i \in \mathrm{acl}_D(A)$, then $a \in \mathrm{acl}_D(\mathrm{rng}(\overline{y})) \subseteq \mathrm{acl}_D(\mathrm{acl}_D(A)) = \mathrm{acl}_D(A)$, contradiction. So there is an i such that $y_i \notin \mathrm{acl}_D(A)$. If $b \in A$ and $b \in \mathrm{acl}_D(\{y_i\} \cup (A \backslash \{b\}))$, then $b \notin \mathrm{acl}_D(A) \backslash \{b\})$ by independence, and so $y_i \in \mathrm{acl}_D(A)$ by Lemma 7.2, contradiction. Hence $A \cup \{y_i\}$ is independent, contradiction. $\square$

Lemma 7.6. *Let D be strongly minimal over $\overline{M}$. Then:*

(i) Let $A, B \subseteq D$ be independent with $A \subseteq \mathrm{acl}_D(B)$. Then:

(a) Suppose that $A_0 \subseteq A$, $B_0 \subseteq B$, $A_0 \cup B_0$ is a basis for $\mathrm{acl}_D(B)$, and $a \in A \backslash A_0$. Then there is a $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is a basis for $\mathrm{acl}_D(B)$.

(b) $|A| \leq |B|$.

(ii) If A and B are bases for $Y \subseteq D$, then $|A| = |B|$.

Proof. (i): Assume the hypotheses. (a): Assume the hypotheses. Then $a \in A \subseteq \mathrm{acl}_D(B) = \mathrm{acl}_D(A_0 \cup B_0)$, so by Lemma 7.1(iv) there is a finite $X \subseteq A_0 \cup B_0$ such that $a \in \mathrm{acl}_D(X)$. Let $C \subseteq B_0$ be of smallest size such that $a \in \mathrm{acl}_D(A_0 \cup C)$. Thus C is finite, and $A_0 \cap C = \emptyset$ by the minimality of C. Since A is independent and $a \notin A_0$, we have $C \neq \emptyset$. Fix $b \in C$. Now $a \in \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}) \cup \{b\}) \backslash \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}))$, so by Lemma 7.2, $b \in \mathrm{acl}_D(A_0 \cup (C \backslash \{b\}) \cup \{a\})$. Hence $A_0 \cup B_0 \subseteq \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}))$, and hence

$$\begin{aligned}
\mathrm{acl}_D(B) &= \mathrm{acl}_D(A_0 \cup B_0) \\
&\subseteq \mathrm{acl}_D(\mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}))) \\
&= \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\})) \\
&\subseteq \mathrm{acl}_D(B).
\end{aligned}$$

Thus $\mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\})) = \mathrm{acl}_D(B)$. We claim that $X \stackrel{\text{def}}{=} A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is independent. For, suppose that $x \in X$ and $x \in \mathrm{acl}_D(X \backslash \{x\})$.

Case 1. $x = a$. Thus $a \in \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, hence $A_0 \cup \{a\} \cup (B_0 \backslash \{b\}) \subseteq \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, hence $b \in \mathrm{acl}_D(B) = \mathrm{acl}_D(A_0 \cup \{a\} \cup (B_0 \backslash \{b\}) \subseteq \mathrm{acl}_D(A_0 \cup (B_0 \backslash \{b\})$, contradicting the fact that $A_0 \cup B_0$ is independent. (Recall that $A_0 \cap C = \emptyset$, hence $b \notin A_0$.)

Case 2. $x \neq a$. Now $X\backslash\{x\} = \{a\} \cup (A_0 \cup (B_0\backslash\{b\}))\backslash\{x\})$ and $x \notin \mathrm{acl}_D((A_0 \cup (B_0\backslash\{b\}))\backslash\{x\})$ by the independence of $A_0 \cup B_0$. So by Lemma 2 we get $a \in \mathrm{acl}_D(A_0 \cup (B_0\backslash\{b\}))$, i.e., Case 1, contradiction.

(b): *Case 1.* B is finite; say $|B| = n$. Suppose that $a_0, \ldots, a_n$ are distinct elements of A. We now define distinct elements b_i of B for $i < n$ by recursion. Suppose they have been defined for all $j < i$, where $0 \leq i < n-1$, so that $\{a_j : j < i\} \cup (B\backslash\{b_j : j < i\}$ is a basis for $\mathrm{acl}_D(B)$. Since $a_i \in A\backslash\{a_j : j < i\}$, we can apply (a) to obtain b_i such that $\{a_j : j \leq i\} \cup (B\backslash\{b_j : j \leq i\}$ is a basis for $\mathrm{acl}_D(B)$.

It follows that $\{a_j : j < n\}$ is a basis for $\mathrm{acl}_D(B)$. Hence $a_n \in \mathrm{acl}_D(\{a_j : j < n\})$, contradicting A independent.

Thus we must have $|A| \leq |B|$.

Case 2. B is infinite. By Case 1, $|A \cap \mathrm{acl}(B_0)| \leq |B_0|$ for each finite subset B_0 of B. Now

$$A \subseteq \bigcup_{\substack{B_0 \subseteq B,\\ B_0 \text{ finite}}} (A \cap \mathrm{acl}(B_0),]$$

so clearly $|A| \leq |B|$.

(ii) follows from (i)(b). □

For D strongly minimal, the *dimension* of D, $\dim(D)$, is the cardinality of a basis for D.

Lemma 7.7. *If D is strongly minimal and uncountable, then* $\dim(D) = |D|$.

Proof. Since the language is countable, also $\mathrm{acl}(B)$ is countable for every finite subset B of D. If $X \subseteq D$ and $|X| < |D|$, then

$$|\mathrm{acl}(X)| \leq \left| \bigcup_{\substack{B \subseteq X,\\ B \text{ finite}}} \mathrm{acl}(B) \right| \leq |X| \cdot \omega < |D|. \qquad \square$$

Let $\overline{a}$ be a sequence of elements of M and $A \subseteq M$. We say that $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is *isolated* if there is a formula $\varphi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ such that for every formula $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ we have $\overline{M} \models \forall\overline{v}[\varphi(\overline{v}) \rightarrow \chi(\overline{v})]$.

Lemma 7.8. *If $A \cup \{b\} \subseteq M$ and b is algebraic over A, then $\mathrm{tp}^{\overline{M}}(b/A)$ is isolated.*

Proof. Let $\overline{a} \in A$ and $\varphi(v, \overline{a})$ be such that $\overline{M} \models \varphi(b, \overline{a})$ and $\{y \in M : \overline{M} \models \varphi(y, \overline{a})\}$ is finite. Let

$$\begin{aligned} B = \{d \in M : \overline{M} \models \varphi(d, \overline{a}) &\text{ and there exist a formula } \psi(v, \overline{c}) \text{ with } \overline{c} \in A \\ &\text{such that } \overline{M} \models \psi(b, \overline{c}) \text{ and } \overline{M} \models \neg\psi(d, \overline{c})\}. \end{aligned}$$

Note that B is finite. For each $d \in B$, choose ψ_d and $\overline{c}_d$ as indicated. Let $\varphi'(v, \overline{e})$ be the formula

$$\varphi(v, \overline{a}) \wedge \bigwedge_{d \in B} \psi_d(v, \overline{c}_d).$$

Thus $\overline{M} \models \varphi'(b, \overline{e})$, and so $\varphi'(v, \overline{e}) \in \mathrm{tp}^{\overline{M}}(b/A)$. Now suppose that $\chi(v, \overline{u}) \in \mathrm{tp}^{\overline{M}}(b/A)$, but there is a $d \in M$ such that $\overline{M} \models \varphi'(d, \overline{e}) \wedge \neg\chi(d, \overline{u})$; we want to get a contradiction. We have $\overline{M} \models \varphi(d, \overline{a})$, so it follows that $d \in B$, hence $\overline{M} \models \neg\psi_d(d, \overline{c})$; but this contradicts $\overline{M} \models \varphi'(d, \overline{e})$. □

Lemma 7.9. *Suppose that $\overline{M}, \overline{N} \models T$, $\varphi(v)$ is strongly minimal, and* $\dim(\varphi(\overline{M})) = \dim(\varphi(\overline{N}))$. *Then there is a bijection $f : \varphi(\overline{M}) \to \varphi(\overline{N})$ such that for every formula $\psi(\overline{w})$ and every $\overline{a} \in \varphi(\overline{M})$, $\overline{M} \models \psi(\overline{a})$ iff $\overline{N} \models \psi(f \circ \overline{a})$.*

Proof. Assume the hypotheses. Let B be a base for $\varphi(\overline{M})$, and let C be a base for $\varphi(\overline{N})$. Thus $|B| = |C|$, and we let $h : B \to C$ be a bijection. Let

$$\begin{aligned} I = \{g : \; & g : B' \to C' \text{ is a surjection, } B \subseteq B' \subseteq \varphi(\overline{M}), C \subseteq C' \subseteq \varphi(\overline{N}) \text{ and} \\ & \forall\chi\forall\overline{a} \in B'[\overline{M} \models \chi(\overline{a}) \leftrightarrow \overline{N} \models \chi(g \circ \overline{a})]\}. \end{aligned}$$

Note that every $g \in I$ is injective; consider the formula $x \neq y$. Now $h \in I$, since for any $\chi(\overline{w})$ and any $\overline{a} \in B$,

$$\begin{aligned} \overline{M} \models \chi(\overline{a}) \quad & \text{iff} \quad \chi(\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a}) \\ & \text{iff} \quad \chi(\overline{w}) \in \mathrm{tp}^{\overline{N}}(h \circ \overline{a}) \quad \text{by Corollary 7.4} \\ & \text{iff} \quad \overline{N} \models \chi(h \circ \overline{a}). \end{aligned}$$

Clearly we can apply Zorn's lemma to I and obtain a maximal member g of it, with associated sets B', C'. We claim that $\mathrm{dmn}(g) = \varphi(\overline{M})$ and $\mathrm{rng}(g) = \varphi(\overline{N})$. By symmetry we prove only that $\mathrm{dmn}(g) = \varphi(\overline{M})$. In fact, suppose that this is not true. Let $b \in \varphi(\overline{M}) \backslash B'$. Since $\mathrm{acl}(B) = \varphi(\overline{M})$, we also have $\mathrm{acl}(B') = \varphi(\overline{M})$, and so $b \in \mathrm{acl}(B')$. Hence by Lemma 7.8 let $\psi(v, \overline{c}) \in \mathrm{tp}^{\overline{M}}(b/B')$ isolate $\mathrm{tp}^{\overline{M}}(b/B')$, where $\overline{c} \in B'$. Now $\overline{M} \models \exists x\psi(x, \overline{c})$, so from $g \in I$ we get $\overline{N} \models \exists x\psi(x, g \circ \overline{c})$. Say $\overline{N} \models \psi(d, g \circ \overline{c})$. Extend g to $g' : B' \cup \{b\} \to C' \cup \{d\}$ by setting $g'(b) = d$. So g' is a surjection from $B' \cup \{b\}$ to $C' \cup \{d\}$. Now take any formula $\chi(v, \overline{w})$ and any $\overline{e} \in B'$. Then

$$\begin{aligned} \overline{M} \models \chi(b, \overline{e}) \quad & \Rightarrow \quad \chi(v, \overline{e}) \in \mathrm{tp}^{\overline{M}}(b) \\ & \Rightarrow \quad \overline{M} \models \forall v[\psi(v, \overline{c}) \to \chi(v, \overline{e}) \\ & \Rightarrow \quad \overline{N} \models \forall v[\psi(v, g \circ \overline{c}) \to \chi(v, g \circ \overline{e}) \\ & \Rightarrow \quad \overline{N} \models \chi(d, g \circ \overline{e}); \end{aligned}$$

this shows that $g' \in I$, contradiction. □

A theory T is *strongly minimal* iff the formula $v = v$ is strongly minimal for each model $\overline{M}$ of T.

For each infinite cardinal κ, $I(T, \kappa)$ is the number of nonisomorphic models of T of size κ.

Theorem 7.10. *Suppose that T is strongly minimal.*

(i) If $\overline{M}, \overline{N} \models T$, then $\overline{M} \cong \overline{N}$ iff $\dim(\overline{M}) = \dim(\overline{N})$.
(ii) T is κ-categorical for each uncountable cardinal κ.
(iii) $I(T, \omega) \leq \omega$.

Proof. (i) is immediate from Lemma 7.9. (ii) follows from (i) by Lemma 7.7. (iii) follows from (i) since $\dim(\overline{M}) \leq \omega$ for any countable model $\overline{M}$ of T. □

A set Γ of formulas is *finitely satisfiable* in $\overline{M}$ iff for every finite subset Δ of Γ there is an $a \in {}^{\omega}M$ such that $\overline{M} \models \varphi[a]$ for all $\varphi \in \Delta$. For any model $\overline{M}$ of T, any subset A of M, and any positive integer n, an *n-type* over $\overline{M}$ is a set of formulas with free variables among $v_0, \ldots, v_{n-1}$ and with parameters from A which is finitely satisfiable over $\overline{M}$. It is a *complete* n-type iff for any formula φ with free variables among $v_0, \ldots, v_{n-1}$ and parameters from A, either φ or $\neg\varphi$ is a member of it. We let $S_n^{\overline{M}}(A)$ be the set of all complete n-types over A with respect to $\overline{M}$. Note that $|S_n^{\overline{M}}(A)| \leq 2^{\max(\omega, \mathrm{A})}$. T is *κ-stable* iff for every $\overline{M} \models T$, every $A \subseteq M$ of size κ, and every positive integer n we have $|S_n^{\overline{M}}(A)| = \kappa$.

Lemma 7.11. *If T is ω-stable and $\overline{M} \models T$, then there is a minimal formula for $\overline{M}$.*

Proof. Suppose not. We define formulas φ_f for each $f \in {}^{<\omega}2$ by induction on $\mathrm{dmn}(f)$. Let $\varphi_\emptyset$ be the formula $v = v$. Now suppose that φ_f has been defined so that $\varphi_v(\overline{M})$ is infinite. Since φ_f is not minimal, there is a formula ψ with parameters such that $\varphi_f(\overline{M}) \cap \psi(\overline{M})$ and $\varphi_f(\overline{M}) \wedge \neg\psi(\overline{M})$ are infinite. We let $\varphi_{f\frown\langle 0\rangle}$ be $\varphi_f \wedge \psi$ and $\varphi_{f\frown\langle 1\rangle}$ be $\varphi_f \wedge \neg\psi$.

Let A be the set of all parameters appearing in any formula φ_f for $f \in {}^{<\omega}2$. So A is countable. For each $f \in {}^{\omega}2$ the set

$$\{\varphi_{f\restriction n} : n \in \omega\}$$

is finitely satisfiable in $\overline{M}$ and hence is contained in a complete type t_f over $\overline{M}$. This gives 2^{ω} complete types over A, contradicting ω-stability. □

Lemma 7.12. *If $\overline{M}$ is ω-saturated and $\varphi(\overline{v}, \overline{a})$ is a minimal formula in $\overline{M}$, then $\varphi(\overline{v}, \overline{a})$ is a strongly minimal.*

Proof. Suppose not. Let $\overline{M} \prec \overline{N}$ with $\psi({}^{n}\overline{N}, \overline{b})$ an infinite and coinfinite subset of $\varphi({}^{n}\overline{N}, \overline{a})$, where $\overline{b} \in N$. Then $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a})$ is a complete type in $\overline{N}$, hence it is finitely satisfiable in $\overline{N}$, so it is finitely satisfiable in $\overline{M}$. Thus it is a complete type in $\overline{M}$ over $\overline{a}$. So by the ω-saturation of $\overline{M}$, it is satisfiable in $\overline{M}$, say by $\overline{b}'$. Thus $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b}'/\overline{a})$. Now for any positive integer p,

$$\overline{N} \models \exists_{\geq p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{b})],$$

hence

$$\exists_{\geq p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{w})] \in \mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}),$$

hence

$$\exists_{\geq p}\overline{v}[\varphi(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{w})] \in \mathrm{tp}^{\overline{M}}(\overline{b}'/\overline{a}),$$

$$\overline{M} \models \exists_{\geq p}\overline{v}[\varphi(\overline{v},\overline{a}) \wedge \psi(\overline{v},\overline{b}')].$$

It follows that $\varphi({}^n\overline{M}) \cap \psi({}^n\overline{M})$ is infinite. Similarly, $\varphi({}^n\overline{M}) \cap \neg\psi({}^n\overline{M})$ is infinite, contradiction. $\square$

A *Vaughtian pair* for T is a pair $(\overline{M}, \overline{N})$ of models of T such that there is a formula $\varphi(\overline{v})$ such that $\overline{M} \prec \overline{N}$, $M \neq N$, $\varphi(\overline{M})$ is infinite, and $\varphi(\overline{M}) = \varphi(\overline{N})$.

Lemma 7.13. *Suppose that T does not have any Vaughtian pairs, $\overline{M} \models T$, and $\varphi(\overline{v},\overline{w})$ is a formula with parameters from M, with $\overline{v}$ of length m and $\overline{w}$ of length k. Then there is a natural number n such that for all $\overline{a} \in M$, if $|\varphi(\overline{M},\overline{a})| > n$, then $\varphi(\overline{M},\overline{a})$ is infinite.*

Proof. Suppose not. For each $n \in \omega$ let $\overline{a}_n \in M$ be such that $\varphi(\overline{M},\overline{a}_n)$ is finite, but of size $> n$.

Adjoin to the language a new one-place relation symbol U. Let Γ be the set of formulas of the following four types:

(1) $\forall\overline{x}\left[\bigwedge_{i<p} Ux_i \rightarrow [\psi \leftrightarrow \psi^U]\right]$, for each formula ψ with free variables among $\overline{x}$, where $\overline{x} = \langle x_i : i < p\rangle$, and ψ^U indicates relativization of quantifiers to U.

(2) $\exists x \neg Ux$.

(3) $\exists_{\geq s}\overline{v}\varphi(\overline{v},\overline{w})$ for each $s \in \omega$.

(4) $\varphi(\overline{v},\overline{w}) \rightarrow \bigwedge_{i<k} Uw_i$.

Now let $\overline{N}$ be a proper elementary extension of $\overline{M}$. For each $n \in \omega$ we have $\varphi(\overline{M},\overline{a}_n) = \varphi(\overline{N},\overline{a}_n)$, since $\varphi(\overline{M},\overline{a}_n)$ is finite. Each finite subset of Γ is satisfiable in the structure $(\overline{N}, M)$. Hence by the compactness theorem we get an elementary extension $(\overline{N}', M')$ of $(\overline{N}, M)$ such that Γ is realizable in $(\overline{N}', M')$, say by $\overline{a}$. Let $\overline{M}'$ be the structure with universe M'. Then by (1), $\overline{N}'$ is an elementary extension of $\overline{M}'$, and it is a proper extension by (2). By (3), $\varphi(\overline{N}',\overline{a})$ is infinite, and by (4) we have $\varphi(\overline{N}',\overline{a}) \subseteq M'$, hence $\varphi(\overline{N}',\overline{a}) = \varphi(\overline{M}',\overline{a})$ by elementarity. Thus $(\overline{M}', \overline{N}')$ is a Vaughtian pair, contradiction. $\square$

Lemma 7.14. *If T has no Vaughtian pairs, then for every $\overline{M} \models T$ and every formula φ with parameters from $\overline{M}$, if φ is minimal for $\overline{M}$ then it is strongly minimal for $\overline{M}$.*

Proof. Suppose not. Let φ be $\varphi(\overline{v})$, with parameters from M. Then there is an elementary extension $\overline{N}$ of $\overline{M}$ and a formula $\psi(\overline{v},\overline{b})$ with $\overline{b} \in N$ such that $\varphi(\overline{N}) \cap \psi(\overline{N},\overline{b})$ and $\varphi(\overline{N}) \cap \neg\psi(\overline{N},\overline{b})$ are infinite. By Lemma 7.13 applied twice, let $n \in \omega$ be such that for all $\overline{a} \in M$,

$$|\varphi(\overline{M}) \cap \psi(\overline{M},\overline{a})| > n \rightarrow \varphi(\overline{M}) \cap \psi(\overline{M},\overline{a}) \text{ is infinite, and}$$
$$|\varphi(\overline{M}) \cap \neg\psi(\overline{M},\overline{a})| > n \rightarrow \varphi(\overline{M}) \cap \neg\psi(\overline{M},\overline{a}) \text{ is infinite.}$$

Thus by the minimality of φ,

$$\overline{M} \models \forall\overline{w}[|\varphi(\overline{M}) \cap \psi(\overline{M},\overline{w})| \leq n \vee |\varphi(\overline{M}) \cap \neg\psi(\overline{M},\overline{w})| \leq n].$$

So this also holds in $\overline{N}$, and it follows that $\varphi(\overline{N}) \cap \psi(\overline{N}, \bar{b})$ is finite or $\psi(\overline{N}) \cap \neg\psi(\overline{N}, \bar{b})$ is finite, contradiction. $\square$

Corollary 7.15. *If T is ω-stable and has no Vaughtian pairs, then for every $\overline{M} \models T$ there is a strongly minimal formula over $\overline{M}$.* $\square$

Corollary 7.16. *If T has no Vaughtian pairs, $\overline{M} \models T$, and $\varphi(\overline{v})$ is a formula with parameters from M, and if $\varphi(\overline{M})$ is infinite, then no proper elementary submodel of $\overline{M}$ contains both $\varphi(\overline{M})$ and the parameters of $\varphi(\overline{v})$.*

Proof. Suppose that $\overline{N}$ is a proper elementary submodel of $\overline{M}$ which contains both $\varphi(\overline{M})$ and the parameters of $\varphi(\overline{v})$. Then for any $\overline{a} \in N$, $\overline{N} \models \varphi(\overline{a})$ implies that $\overline{M} \models \varphi(\overline{a})$ by elementarity. Conversely, if $\overline{M} \models \varphi(\overline{a})$ with $\overline{a} \in M$, then $\overline{a} \in N$ by assumption, so $\overline{N} \models \varphi(\overline{a})$ by elementarity. Thus $\varphi(\overline{M}) = \varphi(\overline{N})$. So $(\overline{M}, \overline{N})$ is a Vaughtian pair, contradiction. $\square$

Lemma 7.17. *Suppose that T is ω-stable, $\overline{M} \models T$, $A \subseteq M$, $\varphi(\overline{v})$ is a formula with parameters from A, and $\overline{M} \models \exists\overline{v}\varphi(\overline{v})$. Then there is an $\overline{a} \in M$ such that $\varphi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$ and $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Suppose that this does not hold. We construct formulas ψ_f for each $f \in {}^{<\omega}2$. Let $\psi_\emptyset = \varphi$. Suppose that we have constructed $\psi_f(\overline{v})$, a formula with parameters from A, so that

(*) $\overline{M} \models \exists\overline{v}\psi_f(\overline{v})$, and for all $\overline{a} \in M$, if $\varphi_f(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is not isolated.

This is true for $f = \emptyset$ by assumption. We claim

(**) There is a formula $\chi(\overline{v})$ with parameters from A such that $\overline{M} \models \exists\overline{v}[\psi_f(\overline{v}) \wedge \chi(\overline{v})]$ and $\overline{M} \models \exists\overline{v}[\psi_f(\overline{v}) \wedge \neg\chi(\overline{v})]$.

Suppose not. Take any $\overline{a}$ such that $\overline{M} \models \psi_f(\overline{a})$. Suppose that $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$. Now by (**) failing we have

$\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \chi(\overline{v})]$ or $\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \neg\chi(\overline{v})]$.

But $\overline{M} \models \chi(\overline{a})$ and $\overline{M} \models \psi_f(\overline{a})$, so it follows that $\overline{M} \models \forall\overline{v}[\psi_f(\overline{v}) \rightarrow \chi(\overline{v})]$. This proves that $\psi_f(\overline{v})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, contradiction. Hence (**) holds. We take such a formula $\chi(\overline{v})$ and define $\psi_{f\frown\langle 0\rangle}$ to be $\psi_f(\overline{v}) \wedge \chi(\overline{v})$ and $\psi_{f\frown\langle 1\rangle}$ to be $\psi_f(\overline{v}) \wedge \neg\chi(\overline{v})$. This finishes the construction.

But this clearly gives 2^ω types over A, contradicting ω-stability. $\square$

If $\overline{M}, \overline{N}$ are structures, $A \subseteq M$, and $f : A \rightarrow N$, we say that f is *partial elementary* iff for every formula $\varphi(\overline{v})$ without parameters and every $\overline{a} \in A$, $\overline{M} \models \varphi(\overline{a})$ iff $\overline{N} \models \varphi(f \circ \overline{a})$.

$\overline{M}$ is a *prime* model of T iff $\overline{M}$ can be elementarily embedded in every model of T. If $\overline{M} \models T$ and $A \subseteq M$, we say that $\overline{M}$ is *prime over A for T* iff for every model $\overline{N}$ of T, every partial elementary $f : A \rightarrow N$ can be extended to an elementary $f^+ : \overline{M} \rightarrow \overline{N}$.

Lemma 7.18. *If $\overline{a} \in {}^mM$, $\overline{b} \in {}^nM$, $A \subseteq M$, and $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$ is isolated, then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Let $\varphi(\overline{v},\overline{w})$, a formula with parameters in A, isolate $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. We claim that $\exists\overline{w}\varphi(\overline{v},\overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$. First, $\overline{M} \models \varphi(\overline{a},\overline{b})$, so $\overline{M} \models \exists\overline{w}\varphi(\overline{a},\overline{w})$. Second, suppose that $\overline{M} \models \chi(\overline{a})$, where χ has parameters in A. Then $\chi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$, so $\overline{M} \models \forall\overline{v}\forall\overline{w}[\varphi(\overline{v},\overline{w}) \rightarrow \chi(\overline{v})]$. Hence $\overline{M} \models \forall\overline{v}[\exists\overline{w}\varphi(\overline{v},\overline{w}) \rightarrow \chi(\overline{v})]$ by elementary logic. $\square$

Lemma 7.19. *Suppose that $A \subseteq B \subseteq M$, and $\overline{M} \models T$. Suppose that every $\overline{b} \in B$ realizes an isolated type over A, and suppose that $\mathrm{tp}^{\overline{M}}(\overline{a}/B)$ is isolated. Then $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated.*

Proof. Suppose that $\varphi(\overline{v},\overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/B)$, where $\overline{b} \in B$ are the parameters of φ. By hypothesis, let $\theta(\overline{w})$ isolate $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$. We claim that $\varphi(\overline{v},\overline{w}) \wedge \theta(\overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. For, $\overline{M} \models \varphi(\overline{a},\overline{b})$ and $\overline{M} \models \theta(\overline{b})$, so $\overline{M} \models \varphi(\overline{a},\overline{b}) \wedge \theta(\overline{b})$. Now suppose that $\overline{M} \models \chi(\overline{a},\overline{b})$. Hence $\overline{M} \models \forall\overline{v}[\varphi(\overline{v},\overline{b}) \rightarrow \chi(\overline{v},\overline{b})$. Hence the formula

$$\forall\overline{v}[\varphi(\overline{v},\overline{b}) \rightarrow \chi(\overline{v},\overline{b})$$

is in $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$, and it follows that

$$\overline{M} \models \forall\overline{w}[\theta(\overline{w}) \rightarrow \forall\overline{v}[\varphi(\overline{v},\overline{b}) \rightarrow \chi(\overline{v},\overline{b})].$$

Hence by elementary logic,

$$\overline{M} \models \forall\overline{w}\forall\overline{v}[\theta(\overline{w}) \wedge \varphi(\overline{v},\overline{b}) \rightarrow \chi(\overline{v},\overline{b})].$$

So we have shown that $\varphi(\overline{v},\overline{w}) \wedge \theta(\overline{w})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$. Now by Lemma 7.18 it follows that $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$ is isolated. $\square$

Theorem 7.20. *Let T be ω-stable. Suppose that $\overline{M} \models T$ and $A \subseteq M$. Then there is an $\overline{M}_0 \preceq \overline{M}$ which is prime over A for T, and is such that every element of M_0 realizes an isolated type over A with respect to $\overline{M}_0$.*

Proof. We define a sequence $\langle A_\alpha : \alpha \leq \delta\rangle$ by recursion, where δ is also defined in the construction. Let $A_0 = A$. If α is a limit ordinal and A_β has been defined for all $\beta < \alpha$, then we let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now suppose that A_α has been defined. If no element of $M\backslash A_\alpha$ realizes an isolated type over A_α (in particular, if $M = A_\alpha$), we stop and let $\delta = \alpha$. Otherwise we pick an element $a_\alpha \in M\backslash A_\alpha$ realizing an isolated type over A_α and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

(1) A_δ is closed under the fundamental functions of $\overline{M}$.

In fact, suppose that $\mathbf{F}$ is an m-ary function symbol and $\overline{a} \in {}^mA_\delta$. Now $\mathrm{tp}^{\overline{M}}(\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta)$ is isolated over A_δ. For, suppose that $\varphi(v) \in \mathrm{tp}^{\overline{M}}(\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta)$. Thus $\overline{M} \models \varphi(\mathbf{F}^{\overline{M}}(\overline{a}))$, and so $\overline{M} \models \forall v[\mathbf{F}^{\overline{M}}(\overline{a}) = v \to \psi(v)]$, so that $\mathbf{F}^{\overline{M}}(\overline{a}) = v$ isolates $\mathbf{F}^{\overline{M}}(\overline{a})/A_\delta$. It follows that $\mathbf{F}^{\overline{M}}(\overline{a}) \in A_\delta$.

Let $\overline{M}_0$ be the substructure of $\overline{M}$ with universe A_δ.

(2) $\overline{M}_0 \preceq \overline{M}$.

We apply Tarski's lemma. Suppose that $\varphi(v, \overline{a})$ is a formula with parameters $\overline{a} \in A_\delta$, and $\overline{M} \models \exists v\varphi(v, \overline{a})$. By Lemma 7.17, choose $b \in M$ such that $\varphi(v, \overline{a}) \in \mathrm{tp}^{\overline{M}}(b/\overline{a})$ and $\mathrm{tp}^{\overline{M}}(b/\overline{a})$ is isolated. By construction we have $b \in A_\delta$, as desired.

Now suppose that $\overline{N} \models T$ and $f : A \to \overline{N}$ is partial elementary. We now define $f_0 \subseteq \cdots \subseteq f_\delta$ by recursion so that $f_\alpha : A_\alpha \to \overline{N}$ is partial elementary. Let $f_0 = f$. If $\alpha \leq \delta$ is a limit ordinal and f_β has been defined for all $\beta < \alpha$, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Clearly f_α is partial elementary. Now suppose that f_α has been defined, where $\alpha < \delta$, with $f_\alpha : A_\alpha \to \overline{N}$ partial elementary. Then by construction, $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, where $a_\alpha \in M\backslash A_\alpha$ and $\mathrm{tp}^{\overline{M}}(a_\alpha/A_\alpha)$ is isolated. Let $\varphi(v, \overline{b})$ be a formula with parameters $\overline{b} \in A_\alpha$ which isolates $\mathrm{tp}^{\overline{M}}(a_\alpha/A_\alpha)$. Thus the following conditions hold:

(3) $\overline{M} \models \varphi(a_\alpha, \overline{b})$.

(4) For every formula $\chi(v, \overline{c})$ with parameters $\overline{c} \in A_\alpha$, if $\overline{M} \models \chi(a_\alpha, \overline{c})$ then $\overline{M} \models \forall v[\varphi(v, \overline{b}) \to \chi(v, \overline{c})]$.

Now by (3) we have $\overline{M} \models \exists v\varphi(v, \overline{b})$, so by the assumption that f_α is partial elementary we have $\mathbb{N} \models \exists v\varphi(v, f_\alpha \circ \overline{b})$. Choose $d \in N$ so that $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$. Let $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, d)\}$. To show that $f_{\alpha+1}$ is partial elementary, suppose that $\chi(v, \overline{c})$ is a formula with parameters $\overline{c} \in A_\alpha$, and $\overline{M} \models \chi(a_\alpha, \overline{c})$. So by (4) we have $\overline{M} \models \forall v[\varphi(v, \overline{b}) \to \chi(v, \overline{c})]$, hence $\overline{N} \models \forall v[\varphi(v, f \circ \overline{b}) \to \chi(v, f_\alpha \circ \overline{c})]$. Now $\overline{N} \models \varphi(d, f_\alpha \circ \overline{b})$, so $\overline{N} \models \chi(d, f_\alpha \circ \overline{c})$. Hence $f_{\alpha+1}$ is partial elementary.

This finishes the construction of the f_α's. In particular, f_δ is an elementary mapping of $\overline{M}_0$ into $\overline{N}$, as desired.

It remains to show that every element of M_0 realizes an isolated type over A with respect to $\overline{M}_0$. We prove by induction on α that every element of A_α realizes an isolated type over A with respect to $\overline{M}$, for each $\alpha \leq \delta$. This is true for $\alpha = 0$, since any element $a \in A$ is isolated over A by the formula $v = a$. The inductive step to a limit ordinal α is obvious. Now suppose that $b \in A_{\alpha+1}$. Then b is isolated over A_α by construction, so b is isolated over A by the inductive hypothesis and Lemma 7.19.

Clearly being isolated over A with respect to $\overline{M}$ implies isolated over A with respect to $\overline{M}_0$. □

Corollary 7.21. *If T is ω-stable, then it has a prime model.*

Proof. Take $A = \emptyset$ in Theorem 7.20. □

Corollary 7.22. *If T is ω-stable and has no Vaughtian pairs, and if $\varphi(\overline{v})$ is a formula with parameters in M such that $\varphi(\overline{M})$ is infinite, then $\overline{M}$ is prime over $\varphi(\overline{M})$.*

Proof. By Theorem 7.20 there is an $\overline{N} \preceq \overline{M}$ which is prime over $\varphi(\overline{M})$. Since $\varphi(\overline{M}) \subseteq N$, we have $\varphi(\overline{N}) = \varphi(\overline{M})$. Since T has no Vaughtian pairs, it follows that $\overline{N} = \overline{M}$. □

Theorem 7.23. *If T is ω-stable and has no Vaughtian pairs, then T is κ-categorical for every uncountable cardinal κ.*

Proof. Assume the hypotheses, with κ uncountable. Suppose that $\overline{M}, \overline{N} \models T$ with $|M| = |N| = \kappa$. Let $\overline{M}_0$ be a prime model of T by Corollary 7.21. Wlog $\overline{M}_0 \preceq \overline{M}, \overline{N}$. By Corollary 7.15 let $\varphi(\overline{v})$ be strongly minimal over $\overline{M}_0$.

(1) $|\varphi(\overline{M})| = |\varphi(\overline{N})| = \kappa$.

For, suppose that $|\varphi(\overline{M})| < \kappa$. By the downward Löwenheim-Skolem theorem, let $\overline{P}$ be an elementary substructure of $\overline{M}$ containing both $\varphi(\overline{M})$ and the parameters of φ, with $|P| < \kappa$. This contradicts Corollary 7.16. Hence $|\varphi(\overline{M})| = \kappa$. By symmetry, $|\varphi(\overline{N})| = \kappa$.

By Lemma 7.7, $\dim(\varphi(\overline{M})) = \dim(\varphi(\overline{N}))$, an hence there is a bijection $f : \varphi(\overline{M}) \to \varphi(\overline{N})$ which is a partial elementary embedding of $\varphi(\overline{M})$ into $\overline{N}$, by Lemma 7.9. By Corollary 7.22, $\overline{M}$ is prime over $\varphi(\overline{M})$, and hence f can be extended to an elementary embedding of $\overline{M}$ into $\overline{N}$. By Corollary 7.16, f maps onto $\overline{N}$. □

We now give some results of a general nature before turning to the converse of Theorem 7.23. We will use Ramsey's theorem from set theory, and we begin with a proof of it.

Ramsey's Theorem. *Suppose that M is an infinite set, n and r are positive integers, and $f : [M]^n \to r$. (r is considered as equal to $\{0, \ldots, r-1\}$.) Then there exist an $i < r$ and an infinite $N \subseteq M$ such that $f(a) = i$ for all $a \in [N]^n$.*

Proof. We may assume that $M = \omega$. We proceed by induction on n. First suppose that $n = 1$. Thus $f : [\omega]^1 \to r$, so $\omega = \bigcup_{i \in r}\{j \in \omega : f(\{j\}) = i\}$. It follows that there is an $i \in r$ such that $N \stackrel{\text{def}}{=} \{j \in \omega : f(\{j\}) = i\}$ is infinite, as desired.

Now assume that the theorem holds for $n \geq 1$, and suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \backslash \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H_m^S \subseteq S \backslash \{m\}$ such that g_m is constant on $[H_m^S]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each m_i will be in ω, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H_{m_i}^{S_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r} \{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \stackrel{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qquad \square$$

Now we return to model theory. Let $(I, <)$ be a linear order, $\overline{M}$ a structure, and $\langle a_i : i \in I\rangle$ a system of distinct elements of M. We say that $\langle a_i : i \in I\rangle$ is a *system of order indiscernibles for* $\overline{M}$ iff for every formula $\varphi(w_1, \ldots, w_m)$ with free variables among the distinct variables $w_1, \ldots, w_m$ and all sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ of elements of I we have

$$\overline{M} \models \varphi(a_{i_1}, \ldots, a_{i_m}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_m}).$$

Theorem 7.24. *Let T be a theory with infinite models, and let $(I, <_I)$ be an infinite linear order. Then T has a model with a system $\langle a_i : i \in I\rangle$ of order indiscernibles.*

Proof. We will work with the standard sequence $v_1, v_2, \ldots$ of variables; all variables are assumed to be among these. Adjoin to the language a system $\langle c_i : i \in I\rangle$ of distinct new individual constants. Let Γ be the union of the following set of sentences:

(1) T;

(2) $c_i \neq c_j$ for $i \neq j$.

(3) $\varphi(c_{i_1}, \ldots, c_{i_p}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_p})$ for every formula $\varphi(v_1, \ldots, v_p)$ with free variables exactly the variables $v_1, \ldots, v_p$ and all sequences $i_1 <_I \cdots <_I i_p$ and $j_1 <_I \cdots <_I j_p$ of elements of I.

We claim that every finite subset of Γ has a model. So, suppose that $\Delta \subseteq \Gamma$ is finite. Let I_0 be the set of all $i \in I$ such that c_i occurs in one of the formulas in Δ. Let $\varphi_1, \ldots, \varphi_m$ be all of the formulas occuring in the third part of Δ as above, and for each $k \in [1, m]$ let p_k be the "p" involved. Let $n = \max\{p_k : 1 \leq k \leq n\}$. Let $\overline{M}$ be an infinite model of T, and fix any linear order $<_M$ of M. We now define $F : [M]^n \to \mathscr{P}(m)$ as follows. Given $A \in [M]^n$ with $A = \{a_1, \ldots, a_n\}$, $a_1 <_M \cdots <_M a_n$, let

$$F(A) = \{k : \overline{M} \models \varphi_k[a_1, \ldots, a_n]\}.$$

By Ramsey's theorem let $X \in [M]^\omega$ and $\eta \in \mathscr{P}(m)$ be such that $F(A) = \eta$ for all $A \in [X]^n$. Let $I_0 = \{s_0, \ldots, s_{m-1}\}$ with $s_0 <_I \cdots <_I s_{m-1}$. Let $x_0 <_M \cdots <_M x_{m+n-1}$ be elements of X. Define $a_{s_k} = x_k$ for all $k < m$. Thus for any $i, j \in I_0$ we have $i <_I j$ iff $a_i <_M a_j$.

Now $(\overline{M}, a_i)_{i \in I_0}$ is a model of Δ. In fact, this is clear for the first two kinds of sentences above. Now take one of the third sort:

$\varphi_k(c_{i_1}, \ldots, c_{i_{p_k}}) \leftrightarrow \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}})$ where $\varphi_k(v_1, \ldots, v_{p_k})$ is a formula with free variables exactly the variables $v_1, \ldots, v_{p_k}$ and with sequences $i_1 <_I \cdots <_I i_{p_k}$ and $j_1 <_I \cdots <_I j_{p_k}$ of elements of I_0. Using the additional n elements of X mentioned above, extend $a_{i_1}, \ldots, a_{i_{p_k}}$ to a sequence $\overline{b} \in {}^nX$ strictly increasing in the sense of $<_M$, and extend $a_{j_1}, \ldots, a_{j_{p_k}}$ to a sequence $\overline{c} \in {}^nX$ strictly increasing in the sense of $<_M$. Then

$$\begin{aligned}
(\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{i_1}, \ldots, c_{i_{p_k}}) \quad &\text{iff} \quad \overline{M} \models \varphi_k[\overline{b}] \\
&\text{iff} \quad k \in F(\operatorname{rng}(\overline{b})) \\
&\text{iff} \quad k \in \eta \\
&\text{iff} \quad k \in F(\operatorname{rng}(\overline{c})) \\
&\text{iff} \quad \overline{M} \models \varphi_k[\overline{c}] \\
&\text{iff} \quad (\overline{M}, a_i)_{i \in I_0} \models \varphi_k(c_{j_1}, \ldots, c_{j_{p_k}}).
\end{aligned}$$

This finishes the proof that $(\overline{M}, a_i)_{i \in I_0}$ is a model of Δ.

Hence by the compactness theorem, let $(\overline{N}, d_i)_{i \in I}$ be a model of Γ. We claim that $\overline{N}$ is as desired. For, suppose that $\varphi(\overline{w})$ is a formula with every free variable occurring in the sequence $\overline{w}$ of distinct variables, $\overline{w} = \langle w_1, \ldots, w_q \rangle$, and $i_1 <_I \cdots <_I i_q$, $j_1 <_I \cdots <_I j_q$. Let the variables actually occurring free in φ be $w_{s(1)}, \ldots, w_{s(r)}$, with $1 \leq s(1) < \cdots < s(r) \leq q$. Let φ' be obtained from φ by replacing $w_{s(1)}, \ldots, w_{s(r)}$ by $v_1, \ldots, v_r$ respectively, after changing bound variables to avoid clashes. Then φ' is a formula with exactly the free variables $v_1, \ldots, v_r$. Moreover, $i_{s(1)} <_I \cdots <_I i_{s(r)}$ and $j_{s(1)} <_I \cdots <_I j_{s(r)}$. Hence

$$(\overline{N}, d_i)_{i \in I} \models \varphi'(c_{i_{s(1)}}, \ldots, c_{i_{s(r)}}) \leftrightarrow \varphi'(c_{j_{s(1)}}, \ldots, c_{j_{s(r)}}).$$

It follows that

$$\begin{aligned}
&\overline{N} \models \varphi'(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \leftrightarrow \varphi'(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}}); \\
&\overline{N} \models \varphi(d_{i_{s(1)}}, \ldots, d_{i_{s(r)}}) \leftrightarrow \varphi(d_{j_{s(1)}}, \ldots, d_{j_{s(r)}}); \\
&\overline{N} \models \varphi(d_{i_1}, \ldots, d_{i_q}) \leftrightarrow \varphi(d_{j_1}, \ldots, d_{j_q}).
\end{aligned} \qquad \square$$

A theory T in a language $\mathscr{L}$ *has built-in Skolem functions* iff for every positive integer n, every system $v, w_1, \ldots, w_n$ of distinct variables, and every formula $\varphi(v, w_1, \ldots, w_n)$ without parameters whose free variables are among $v, w_1, \ldots, w_n$, there is an m-ary function symbol f such that

$$T \models \forall \overline{w}[\exists v \varphi(v, \overline{w}) \to \varphi(f(\overline{w}), \overline{w})].$$

Theorem 7.25. *Let T be a theory in a language $\mathscr{L}$. Then there exist a language $\mathscr{L}^* \supseteq \mathscr{L}$ and a theory $T^* \supseteq T$ in $\mathscr{L}^*$ such that:*

(i) T^* has built-in Skolem functions.

(ii) Each model of T can be expanded to a model of T^*.

(iii) $|\mathscr{L}^*| = |\mathscr{L}| + \omega$.

Proof. Fix $c \in M$ We define $\mathscr{L}_0, \mathscr{L}_1, \ldots$ and $T_0, T_1, \ldots$ by recursion. Let $\mathscr{L}_0 = \mathscr{L}$ and $T_0 = T$. Having defined $\mathscr{L}_m$ and T_m, for each formula $\varphi(v, w_1, \ldots, w_n)$ as in the above definition, introduce an n-ary function symbol f_φ, and add the following sentence to T_m:

$$\forall \overline{w}[\exists v \varphi(v, \overline{w}) \to \varphi(f_\varphi(\overline{w}), \overline{w})].$$

This finishes the construction. Let $\mathscr{L}^* = \bigcup_{m\in\omega} \mathscr{L}_m$ and $T^* = \bigcup_{m\in\omega} T_m$. The desired conditions are easy to check. $\square$

Theorem 7.26. *Let $\mathscr{L}$ be countable and let T be an $\mathscr{L}$-theory with an infinite model. Suppose that κ is an infinite cardinal. Then there is a model $\overline{M}$ of T of size κ such that for every $A \subseteq M$ and every positive integer n, $\overline{M}$ realizes at most $|A| + \omega$ n-types over A.*

Proof. By Theorem 7.24, let $\overline{N}$ be a model of T with a system $\langle a_\alpha : \alpha < \kappa\rangle$ of order indiscernibles with respect to $(\kappa, <)$. Let $I = \{a_\alpha : \alpha < \kappa\}$. Let $\mathscr{L}^*$ and T^* be as in Theorem 7.25. Let M be the closure under all of the functions of $\overline{N}^*$ of I. Then M is the universe of some substructure $\overline{M}^*$ of $\overline{N}^*$. Let $\overline{M}$ be the reduct of $\overline{M}^*$ to the language $\mathscr{L}$. So $\overline{M} \models T$, and $|M| = \kappa$. Suppose that $A \subseteq M$. For each $b \in M$ we can write $b = t_b(x_b)$, where t_b is a term and x_b is a strictly increasing sequence $\langle x_b(0), \ldots, x_b(m_b - 1)\rangle$ of elements of I. Let $X = \{y \in I : y = x_b(u)$ for some $b \in A$ and $u < m_b\}$. Now for any $c \in {}^nM$ we define (with $c = \langle c(i) : i < n\rangle$)

$$L_c = \langle \tau_{c(i)} : i < n\rangle;$$
$$N_c = \{(i, j, u, v) : i < j < n, u < m_{c(i)}, v < m_{c(j)}\};$$
$$\text{for } (i, j, u, v) \in N_c,\ F_c(i, j, u, v) = \begin{cases} 0 & \text{if } x_{c(i)}(u) < x_{c(j)}(v), \\ 1 & \text{if } x_{c(i)}(u) = x_{c(j)}(v), \\ 2 & \text{if } x_{c(i)}(u) > x_{c(j)}(v); \end{cases}$$
$$P_c = \{(i, u, y) : i < n, u < m_i, y \in X\};$$
$$\text{for } (i, u, y) \in P_c,\ G_c(i, u, y) = \begin{cases} 0 & \text{if } x_{c(i)}(u) = y, \\ 1 & \text{if } x_{c(i)}(u) < y, \\ 2 & \text{if } x_{c(i)}(u) > y; \end{cases}$$
$$T(c) = \langle L_c, F_c, G_c\rangle.$$

Now we claim that if $c, d \in {}^nM$ and $T(c) = T(d)$, then $\mathrm{tp}^{\overline{M}}(c/A) = \mathrm{tp}^{\overline{M}}(d/A)$. For, assume that $T(c) = T(d)$, and let $\varphi(\overline{v}, a)$ be given, with $a \in {}^lA$. Let

$$Y_c = \{x_{c(i)}(u) : i < n, u < m_i\} \cup \{x_{a(i)}(u) : i < l, u < m_i\};$$
$$Y_d = \{x_{d(i)}(u) : i < n, u < m_i\} \cup \{x_{a(i)}(u) : i < l, u < m_i\}.$$

Clearly $|Y_c| = |Y_d|$. Let $\langle z_i^c : i < e\rangle$ and $\langle z_i^d : i < e\rangle$ enumerate Y_c and Y_d respectively, in the order $<_I$. Let $\langle w_i : i < e\rangle$ be a sequence of new variables. Say $x_{c(i)}(u) = z_{k(i,u)}^c$ and

$x_{a(i)}(u) = z^c_{l(i,u)}$. Then by $T(c) = T(d)$ we have $x_{d(i)}(u) = z^d_{k(i,u)}$ and $x_{a(i)}(u) = z^d_{l(i,u)}$. Let φ' be the formula

$$\varphi(\langle t_{c(i)}(w_{k(i,0)}, \ldots, w_{k(i,m_i-1)}) : i < n\rangle, \langle t_{a(i)}(w_{l(i,0)}, \ldots, w_{l(i,m_i-1)}) : i < l\rangle).$$

Then

$$\begin{aligned} \overline{M} \models \varphi(c,a) \quad &\text{iff} \quad \overline{M} \models \varphi'(z^c) \\ &\text{iff} \quad \overline{M} \models \varphi'(z^d) \\ &\text{iff} \quad \overline{M} \models \varphi(d,a). \end{aligned}$$

This proves our claim. Now clearly there are at most $|A| + \omega$ choices for $T(c)$, so the conclusion of the theorem follows. □

Now we again make the standing assumption that T is a complete theory in a countable language with only infinite models.

Theorem 7.27. *If T is κ-categorical for some uncountable κ, then T is ω-stable.*

Proof. Suppose that T is not ω-stable. Then there is a model $\overline{M}$ of T, a countable subset A of M, and a positive integer n, such that $|S_n^{\overline{M}}(A)| > \omega$. Let $\overline{M}'$ be a countable elementary submodel of $\overline{M}$ containing A. Then $\overline{M}' \models T$ and $|S_n^{\overline{M}'}(A)| > \omega$. Hence $\overline{M}'$ has an elementary extension $\overline{N}_0$ of size κ which realises uncountably many n-types over A. By Theorem 7.26 there is a model $\overline{N}_1$ of T such that for every countable $B \subseteq N_1$, $\overline{N}_1$ realizes only countably many n types over B. Hence $\overline{N}_0$ and $\overline{N}_1$ are not isomorphic. □

If $\overline{M}$ is an infinite structure and κ is an infinite cardinal, we say that $\overline{M}$ is *κ-homogeneous* iff for every $A \in [M]^{<\kappa}$, every partial elementary map $f : A \to \overline{M}$, and every $a \in M$, there is a partial elementary map $f^+ : A \cup \{a\} \to \overline{M}$ which extends f. We say that $\overline{M}$ is homogeneous iff it is $|M|$-homogeneous.

Lemma 7.28. *Suppose that $\overline{M}$ and $\overline{N}$ are $\mathscr{L}$ structures, n is a positive integer, $a \in {}^nM$, and $b \in {}^nN$. Then the following conditions are equivalent:*

(i) $\mathrm{tp}^{\overline{M}}(a) = \mathrm{tp}^{\overline{N}}(b)$.

(ii) There is a partial elementary map $f : \mathrm{rng}(a) \to N$ such that $b = f \circ a$.

Proof. (i)⇒(ii): Assume (i). Define $f(a_i) = b_i$ for all $i < n$. f is well defined, since $a_i = a_j$ implies that $v_i = v_j \in \mathrm{tp}^{\overline{M}}(a) = \mathrm{tp}^{\overline{N}}(b)$, hence $b_i = b_j$. Clearly f is partial elementary.

(ii)⇒(i): clear. □

Lemma 7.29. *Suppose that κ is an infinite cardinal, $\overline{M}$ is κ-homogeneous, n is a positive integer, $\overline{a}, b \in {}^nM$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(b)$, and $c \in M$. Then there is a $d \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.*

Proof. This is immediate from Lemma 7.28. □

Lemma 7.30. *The following are equivalent:*

(i) $\overline{M}$ *is* ω*-homogeneous.*

(ii) For every positive integer n*, all* $\overline{a}, b \in {}^nM$*, and all* $c \in M$*, if* $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(b)$*, then there is a* $d \in M$ *such that* $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). So by Lemma 7.28 there is an elementary map $f : \mathrm{rng}(\overline{a}) \to M$ such that $b = f \circ \overline{a}$. By (i), extend f to an elementary map $f^+ : \mathrm{rng}(\overline{a}) \cup \{c\} \to M$. Let $d = f(c)$. Then by Lemma 7.28 again, $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}(b^\frown\langle d\rangle)$.

(ii)⇒(i): Assume (ii) and suppose that $f : A \to M$ is partial elementary, where A is a finite subset of M, and suppose that $c \in M$. Say $\mathrm{rng}(\overline{a}) = A$. By Lemma 7.28 we have $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(f \circ \overline{a})$. Hence by (ii) choose $d \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}}((f\circ\overline{a})^\frown\langle d\rangle)$. By Lemma 28 we get a partial elementary map g such that $(f\circ\overline{a})^\frown\langle d\rangle = g \circ (\overline{a}^\frown\langle c\rangle)$. Thus g extends f and $g(c) = d$, as desired. □

Theorem 7.31. *If* $\overline{M}$ *and* $\overline{N}$ *are countable homogeneous models of* T *and for each positive integer* n *they realize the same* n*-types, then they are isomorphic.*

Proof. Let $a_0, a_1, \ldots$ enumerate M and $b_0, b_1, \ldots$ enumerate N. We now define by recursion partial elementary maps $f_0, f_1, \ldots$ from subsets of M into $\overline{N}$. Let $f = \emptyset$; so it is partial elementary into $\overline{N}$ because T is complete. Now suppose that a partial elementary map f_s has been defined from a finite subset of M into $\overline{N}$. Let $\overline{c}$ be a sequence enumerating the domain of f.

Case 1. s is even, say $s = 2i$. By hypothesis, let $d, e \in N$ such that $\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle a_i\rangle) = \mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle)$. Hence $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{N}}(d)$. Also, by Lemma 7.28, $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{N}}(f_s \circ \overline{c})$. So $\mathrm{tp}^{\overline{N}}(d) = \mathrm{tp}^{\overline{N}}(f_s \circ \overline{c})$. Since $\overline{N}$ is homogeneous, by Lemma 7.29 there is a $u \in N$ such that $\mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f_s \circ \overline{c}) \circ \langle u\rangle)$. Let $f_{s+1} = f_s \cup \{(a_i, u)\}$. Then

$$\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle a_i\rangle) = \mathrm{tp}^{\overline{N}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f_s \circ \overline{c})\circ\rangle u\rangle) = \mathrm{tp}^{\overline{N}}(f_{s+1} \circ (\overline{c}^\frown\langle a_i\rangle)),$$

so by Lemma 7.28 f_{s+1} is partial elementary.

Case 2. s is odd, say $s = 2i + 1$. This is treated similarly. Choose $d, e \in M$ such that $\mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f \circ \overline{c})^\frown\langle b_i\rangle)$. Hence $\mathrm{tp}^{\overline{M}}(d) = \mathrm{tp}^{\overline{N}}(f \circ \overline{c})$. Also, by Lemma 7.28 $\mathrm{tp}^{\overline{M}})(\overline{c}) = \mathrm{tp}^{\overline{N}}(f \circ \overline{c})$. So $\mathrm{tp}^{\overline{M}}(\overline{c}) = \mathrm{tp}^{\overline{M}}(d)$. Since $\overline{M}$ is homogeneous, by Lemma 7.27 there is a $u \in M$ such that $\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle u\rangle) = \mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle)$. Now if there is an i such that $c_i = u$, then $d_i = e$, hence $f(c_i) = b_i$. Hence $f_{\sigma+1} \stackrel{\text{def}}{=} f_s \cup \{(u, b_i)\}$ is a function. Also,

$$\mathrm{tp}^{\overline{M}}(\overline{c}^\frown\langle u\rangle) = \mathrm{tp}^{\overline{M}}(d^\frown\langle e\rangle) = \mathrm{tp}^{\overline{N}}((f \circ \overline{c})^\frown\langle b_i\rangle) = \mathrm{tp}^{\overline{N}}(f_{s+1} \circ (\overline{c}^\frown\langle u\rangle),$$

so by Lemma 7.28 f_{s+1} is partial elementary.

Clearly $\bigcup_{s\in\omega} f_s$ is as desired. □

We consider an expansion $\overline{L}_U$ of our language $\overline{L}$ obtained by adjoining a one-place relation symbol U. For each formula $\varphi(v_0, \ldots, v_{n-1})$ of $\mathscr{L}$ we associate a formula $\varphi^U(v_0, \ldots, v_{n-1})$ of $\mathscr{L}_U$, as follows:

If φ is atomic, then φ^U is $Uv_0 \wedge \ldots \wedge Uv_{n-1} \wedge \varphi$.
$(\neg\psi)^U = \neg\psi^U$.
$(\psi \wedge \chi)^U = \psi^U \wedge \chi^U$.
$(\exists w\psi)^U = \exists w[Uw \wedge \psi^U]$.

Proposition 7.32. *If $\overline{M}$ is a substructure of $\overline{N}$, $\varphi(v_0, \ldots, v_{n-1})$ is a formula of $\mathscr{L}$, and $a \in {}^nM$, then $\overline{M} \models \varphi(\overline{a})$ iff $(\overline{N}, U) \models \varphi^U(\overline{a})$.*

Proof. An easy induction on φ. $\square$

Theorem 7.33. *If there is a Vaughtian pair $(\overline{M}, \overline{N})$, then there is one in which N is countable.*

Proof. Let φ be a formula such that $\varphi(\overline{M})$ is infinite and $\varphi(\overline{M}) = \varphi(\overline{N})$. Let $\overline{a}$ be the parameters from M occurring in φ. We consider the structure $(\overline{N}, M)$ in the language $\mathscr{L}_U$. Let $(\overline{N}_0, M_0)$ be a countable elementary substructure of $(\overline{N}, M)$ such that $\overline{a} \in M_0$. Among the sentences holding in $(\overline{N}, M)$ are those asserting that M is closed under the fundamental function of $\overline{N}$. Hence M_0 is closed under the fundamental functions of $\overline{N}_0$, and hence M_0 is the universe of a substructure $\overline{M}_0$ of $\overline{N}_0$. For any formula $\psi(b)$ with $b \in M_0$ we have, using Proposition 7.32,

$$\begin{aligned}
\overline{M}_0 \models \psi(b) \quad &\text{iff} \quad (\overline{N}_0, M_0) \models \psi^U(b) \\
&\text{iff} \quad (\overline{N}, M) \models \psi^U(b) \\
&\text{iff} \quad \overline{M} \models \psi(b) \\
&\text{iff} \quad \overline{N} \models \psi(b) \\
&\text{iff} \quad \overline{N}_0 \models \psi(b).
\end{aligned}$$

Thus $\overline{M}_0 \preceq \overline{N}_0$. Moreover, the sentence $\exists x\neg Ux$ holds in $(\overline{N}, M)$, hence also in $(\overline{N}_0, M_0)$, so that $\overline{M}_0 \neq \overline{N}_0$.

Clearly $\varphi(\overline{M}_0)$ is infinite and $\varphi(\overline{M}_0) = \varphi(\overline{N}_0)$. $\square$

Lemma 7.34. *Suppose that $\overline{M} \preceq \overline{N}$ and in the language $\mathscr{L}_U$ we have $(\overline{N}, M) \preceq (\overline{N}', M')$. Then M' is the universe of a structure $\overline{M}'$, and $\overline{M} \preceq \overline{M}' \preceq \overline{N}'$.*

Proof. Clearly $M \subseteq M'$, and M' is closed under the fundamental functions of $\overline{N}'$, and hence is the universe of a structure $\overline{M}'$. If φ is a formula and $\overline{a} \in M$, then by Proposition 7.32,

$$\overline{M} \models \varphi(\overline{a}) \quad \text{iff} \quad (\overline{N}, M) \models \varphi^U(\overline{a}) \quad \text{iff} \quad (\overline{N}', M') \models \varphi^U(\overline{a}) \quad \text{iff} \quad \overline{M}' \models \varphi(\overline{a}).$$

Thus $\overline{M} \preceq \overline{M}'$.

Next we claim that

$$(1) \qquad (\overline{N}, M) \models \forall\overline{v}[Uv_0 \wedge \ldots \wedge Uv_{n-1} \rightarrow (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

In fact, suppose that $\overline{a} \in M$ is given. Then

$$(\overline{N}, M) \models \varphi(\overline{a}) \quad \text{iff} \quad \overline{N} \models \varphi(\overline{a}) \quad \text{iff} \quad \overline{M} \models \varphi(\overline{a}) \quad \text{iff} \quad (\overline{N}, M) \models \varphi^U(\overline{a}).$$

This proves (1). Hence we also get

$$(2) \qquad (\overline{N}', M') \models \forall \overline{v}[Uv_0 \wedge \ldots \wedge Uv_{n-1} \to (\varphi(\overline{v}) \leftrightarrow \varphi^U(\overline{v}))].$$

Now let $b \in M'$. Then using (2),

$$\overline{M}' \models \varphi(b) \quad \text{iff} \quad (\overline{N}', M') \models \varphi^U(b) \quad \text{iff} \quad (\overline{N}', M') \models \varphi(b) \quad \text{iff} \quad \overline{N}' \models \varphi(b). \qquad \square$$

Lemma 7.35. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ countable, $\overline{a} \in M$, $\overline{b} \in N$.*

Then there exist countable $M', \overline{N}'$ and $\overline{c}$ such that $(\overline{N}, M) \prec (\overline{N}', M')$ and $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}'}(\overline{c}/\overline{a})$.

Proof. Say $\overline{b}$ is of length n. In $\mathscr{L}_U$ let $\Gamma(\overline{v})$ be the following set of formulas:

Eldiag$(\overline{N}, M)$
$\{\bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v}, \overline{a}) : \overline{N} \models \varphi(\overline{b}, \overline{a})\}$.

If $\varphi_0, \ldots, \varphi_{m-1}$ are such that $\overline{N} \models \varphi_i(\overline{b}, \overline{a})$ for all $i < m$, then $\overline{N} \models \exists \overline{v} \bigwedge_{i<m} \varphi_i(\overline{v}, \overline{a})$, hence $\overline{M} \models \exists \overline{v} \bigwedge_{i<m} \varphi_i(\overline{v}, \overline{a})$, hence by Proposition 7.32,

$$(\overline{N}, M) \models \exists \overline{v} \left(\bigwedge_{i<n} Uv_i \wedge \bigwedge_{i<m} \varphi_i^U(\overline{v}, \overline{a}) \right).$$

This shows that every finite subset of $\Gamma(\overline{v})$ is satisfiable. Hence there exist a countable $(\overline{N}', M')$ and $\overline{c} \in M'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$ and $(\overline{N}', M') \models \varphi^U(\overline{c}, \overline{a})$ whenever $\overline{N} \models \varphi(\overline{b}, \overline{a})$. If $\overline{N} \models \varphi(\overline{b}, \overline{a})$, then $\overline{M}' \models \varphi(\overline{c}, \overline{a})$. $\square$

Corollary 7.36. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^*, \overline{N}^*$ such that $(\overline{N}, M) \preceq (\overline{N}^*, M^*)$, and for every $\overline{a} \in M$ and every $\overline{b} \in N$ there is a $\overline{c} \in M^*$ such that $\mathrm{tp}^{\overline{N}}(\overline{b}/\overline{a}) = \mathrm{tp}^{\overline{M}^*}(\overline{c}/\overline{a})$.*

Proof. Iterate Lemma 7.35. $\square$

Lemma 7.37. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $\overline{a}, \overline{b}, c \in N$, $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, $d \in N^\star$ and $\mathrm{tp}^{\overline{N}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{N}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Apply the compactness theorem to the set

Eldiag$(\overline{N}, M)$
$\{\varphi(\overline{b}, u) : \overline{N} \models \varphi(\overline{a}, c)\}$ (u a new constant) $\square$

Corollary 7.38. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, and for all $\overline{a}, \overline{b}, c \in N$, if $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$, then there is a $d \in N^\star$ such that $\mathrm{tp}^{\overline{N}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{N}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Iterate Lemma 7.37. □

Lemma 7.37a. *Suppose that $\overline{M} \preceq \overline{N}$, $\overline{N}$ is countable, $\overline{a}, \overline{b}, c \in M$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b})$. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, $d \in M^\star$ and $\mathrm{tp}^{\overline{M}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Apply the compactness theorem to the set

$\mathrm{Eldiag}(\overline{N}, M)$
$\{Uu \wedge \varphi^U(\overline{b}, u) : \overline{M} \models \varphi(\overline{a}, c)\}$ (u a new constant) □

Corollary 7.38a. *Suppose that $\overline{M} \preceq \overline{N}$ and $\overline{N}$ is countable. Then there exist countable $\overline{M}^\star, \overline{N}^\star$ and d such that $(\overline{N}, M) \preceq (\overline{N}^\star, M^\star)$, and for all $\overline{a}, \overline{b}, c \in M$, if $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{N}}(\overline{b})$, then there is a $d \in M^\star$ such that $\mathrm{tp}^{\overline{M}^\star}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{M}^\star}(\overline{b}^\frown\langle d\rangle)$.*

Proof. Iterate Lemma 7.37a. □

Lemma 7.39. *Suppose that $\overline{M} \prec \overline{N}$ (so $M \neq N$), and $\overline{N}$ is countable. Then there exist countable $\overline{M}', \overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, $\overline{N}'$ and $\overline{M}'$ are homogeneous and they realize the same n-types for all positive integers n. Moreover, they are isomorphic.*

Proof. We define an elementary chain $\langle(\overline{P}_i, Q_i) : i \in \omega\rangle$ by recursion. Let $\overline{P}_0 = \overline{N}$ and $Q_0 = M$. Suppose that $(\overline{P}_{3i}, Q_{3i})$ has been defined. Apply Corollary 7.36 to get an elementary extension $(\overline{P}_{3i+1}, Q_{3i+1})$ of $(\overline{P}_{3i}, Q_{3i})$ such that every type realized in $\overline{P}_{3i}$ is realized in $\overline{Q}_{3i+1}$. Note that these types are realized in $\overline{P}_{3i+1}$. Next, apply Corollary 7.38a to obtain an elementary extension $(\overline{P}_{3i+2}, Q_{3i+2})$ of $(\overline{P}_{3i+1}, Q_{3i+1})$ such that for all $\overline{a}, b, c \in Q_{3i+1}$, if $\mathrm{tp}^{\overline{Q}_{3i+1}}(\overline{a}) = \mathrm{tp}^{\overline{Q}_{3i+1}}(b)$, then there is a $d \in Q_{3i+2}$ such that $\mathrm{tp}^{\overline{Q}_{3i+1}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{Q}_{3i+2}}(b^\frown\langle d\rangle)$. Finally, apply Corollary 7.38 to obtain an elementary extension $(\overline{P}_{3i+3}, Q_{3i+3})$ of $(\overline{P}_{3i+2}, Q_{3i+2})$ such that for all $\overline{a}, b, c \in P_{3i+2}$, $\mathrm{tp}^{\overline{P}_{3i+2}}(\overline{a}) = \mathrm{tp}^{\overline{P}_{3i+2}}(b)$ implies that $\mathrm{tp}^{\overline{P}_{3i+3}}(\overline{a}^\frown\langle c\rangle) = \mathrm{tp}^{\overline{P}_{3i+3}}(b^\frown\langle d\rangle)$ for some $d \in P_{3i+3}$.

This finishes the construction. Let $\overline{N}' = \bigcup_{i\in\omega} \overline{P}_i$ and $M' = \bigcup_{i\in\omega} Q_i$. The desired conclusion is clear, using Theorem 7.31 for the last statement. □

Suppose that $\omega \le \lambda < \kappa$. We say that T *has a* (κ, λ)*-model* iff there exist an $\overline{M} \models T$ and a formula $\varphi(\overline{v})$ such that $|M| = \kappa$ and $|\varphi(\overline{M})| = \lambda$.

Lemma 7.40. *If $\omega \le \lambda < \kappa$ and T has a (κ, λ)-model, then T has a Vaughtian pair.*

Proof. Let $\overline{N}$ be a (κ, λ)-model, with associated formula $\varphi(\overline{v})$. By the downward Löwenheim-Skolem theorem, let $\overline{M}$ be an elementary substructure of $\overline{N}$ of size λ such that $\varphi(\overline{N}) \subseteq M$. Clearly then $\varphi(\overline{N}) = \varphi(\overline{M})$, so that $(\overline{M}, \overline{N})$ is a Vaughtian pair. □

Theorem 7.41. *If T has a Vaughtian pair, then T has an $(\aleph_1, \aleph_0)$-model.*

Proof. Assume that T has a Vaughtian pair. By Lemma 7.33 we may assume that $(\overline{M}, \overline{N})$ is a Vaughtian pair with N countable. Say $\varphi(\overline{M}) = \varphi(\overline{N})$ is infinite. Also, $M \neq N$. By Lemma 7.39 there are countable $\overline{M}', \overline{N}'$ such that $(\overline{N}, M) \preceq (\overline{N}', M')$, $\overline{N}'$ and $\overline{M}'$ are homogeneous, the realize the same n-types for every positive integer n, and they are isomorphic. Still $M' \neq N'$. Now $(\overline{N}, M) \models \forall \overline{v}[\varphi(\overline{v}) \leftrightarrow \bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v})]$, so also $(\overline{N}', M') \models \forall \overline{v}[\varphi(\overline{v}) \leftrightarrow \bigwedge_{i<n} Uv_i \wedge \varphi^U(\overline{v})]$, and this implies that $\varphi(\overline{M}') = \varphi(\overline{N}')$.

We now define by recursion a sequence $\langle \overline{P}_\alpha : \alpha < \omega_1 \rangle$ of models. Let $\overline{P}_0 = \overline{N}'$. Now suppose that $\overline{P}_\alpha$ has been defined so that $\overline{P}_\alpha \cong \overline{N}'$. Then also $\overline{P}_\alpha \cong \overline{M}'$, so P_α has an elementary extension $\overline{P}_{\alpha+1}$ such that $(\overline{N}', M') \cong (\overline{P}_{\alpha+1}, P_\alpha)$. To see this, let g be an isomorphism from $\overline{P}_\alpha$ onto $\overline{M}'$, and let Q be a set such that $Q \cap (N' \backslash M') = Q \cap P_\alpha = \emptyset$ and $|Q| = |N' \backslash M'$. Let $P_{\alpha+1} = P_\alpha \cup Q$, and let $f : P_{\alpha+1} \to N'$ be a bijection such that $f \restriction P_\alpha = g$ while $f \restriction Q$ is a bijection from Q onto $N' \backslash M'$. We can make $P_{\alpha+1}$ into a structure so that f is an isomorphism from $\overline{P}_{\alpha+1}$ onto $\overline{N}'$. Then $\overline{P}_\alpha$ is an elementary substructure of $\overline{P}_{\alpha+1}$, since for $a \in {}^\omega P_\alpha$ we have

$$\overline{P}_\alpha \models \varphi[a] \quad \text{iff} \quad \overline{M}' \models \varphi[g \circ a] \quad \text{iff} \quad \overline{N}' \models \varphi[g \circ a] \quad \text{iff} \quad \overline{P}_{a+1} \models \varphi[a].$$

For α limit, let $\overline{P}_\alpha = \bigcup_{\beta<\alpha} \overline{P}_\beta$. Since then $\overline{P}_\alpha$ is the union of models isomorphic to $\overline{N}'$, it is clearly homogeneous and realizes the same types as $\overline{N}'$. Hence it is isomorphic to $\overline{N}'$. This finishes the construction.

Let $\overline{P}_{\omega_1} = \bigcup_{\alpha<\omega_1} \overline{P}_\alpha$. Then $|P_{\omega_1}| = \omega_1$. Now by induction we have $\varphi(\overline{P}_\alpha) = \varphi(\overline{M}')$ for all $\alpha \leq \omega_1$. Hence $|\varphi(\overline{P}_{\omega_1})| = \omega$. □

Lemma 7.42. *Suppose that T is ω-stable, $\overline{M} \models T$, and $|M| \geq \aleph_1$. Then $\overline{M}$ has a proper elementary extension $\overline{N}$ such that for every finite sequence $\overline{w}$ of variables and every $\Gamma(\overline{w})$ of formulas with free variables among $\overline{w}$ and with parameters from M and with $\Gamma(\overline{w})$ countable, if $\Gamma(\overline{w})$ is realized in $\overline{N}$, then it is also realized in $\overline{M}$.*

Proof. First we claim

(1) There is a formula $\varphi(v)$ with parameters from M such that $|\varphi(\overline{M})| \geq \aleph_1$, and for every formula $\psi(v)$ with parameters from M, either $|\varphi(\overline{M}) \cap \psi(\overline{M})| < \aleph_1$ or $|\varphi(\overline{M}) \cap \neg\psi(\overline{M})| < \aleph_1$.

Suppose not. Then it is easy to define formulas φ_f for $f \in {}^{<\omega}2$ such that the following conditions hold for each f:

(2) $\varphi_\emptyset$ is the formula $v = v$.

(3) $|\varphi_f(\overline{M})| \geq \aleph_1$.

(4) $\varphi_{f^\frown\langle 0\rangle}(\overline{M}) \cap \varphi_{f^\frown\langle 1\rangle}(\overline{M}) = \emptyset$.

This gives 2^ω types over M, contradicting the ω-stability of T. So (1) holds.

Choose $\varphi(v)$ as in (1), and let

$$p = \{\psi(v) : \psi(v) \text{ is a formula with parameters from } M, \text{ and } |\varphi(\overline{M}) \cap \psi(\overline{M})| \geq \aleph_1\}.$$

Note that $\varphi(\overline{M}) \cap \psi(\overline{M})$ is a co-countable subset of $\varphi(\overline{M})$, and an intersection of countably many co-countable subset of a set is still co-countable. Hence

(5) p is finitely satisfiable.

From (1) it also follows that p is a complete type.

Let $\overline{M}'$ be a proper elementary extension of $\overline{M}$ containing an element c which realizes p, and choose $d \in M' \backslash M$. Now we apply Theorem 20 to get an elementary substructure $\overline{N}$ of $\overline{M}'$ which is prime over $M \cup \{c, d\}$ for T and is such that every finite sequence of elements of N realizes an isolated type over $M \cup \{c, d\}$. Thus $M \cup \{c, d\} \subseteq N$, so clearly $\overline{M} \prec \overline{N}$. Now suppose that $\Gamma(\overline{w})$ is a set of formulas with free variables among $\overline{w}$, with parameters from M and $\Gamma(\overline{w})$ is countable, and such that it is realized in $\overline{N}$, say by b. Let $\theta(\overline{w}, v)$ be a formula which isolates $\mathrm{tp}^{\overline{N}}(b/M \cup \{c\})$.

(6) $\exists \overline{w} \theta(\overline{w}, v) \in p$.

In fact, otherwise $\neg \exists \overline{w} \theta(\overline{w}, v) \in p$, hence $\overline{M}' \models \neg \exists \overline{w} \theta(\overline{w}, c)$, hence $\overline{N} \models \neg \exists \overline{w} \theta(\overline{w}, c)$. This contradicts $\overline{N} \models \theta(b, c)$.

(7) $\forall \overline{w}[\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})] \in p$ for every $\gamma(\overline{w}) \in \Gamma(\overline{w})$.

For, otherwise $\exists \overline{w}[\theta(\overline{w}, v) \wedge \neg \gamma(\overline{w}] \in p$, hence $\overline{M}' \models \exists \overline{w}[\theta(\overline{w}, c) \wedge \neg \gamma(\overline{w}]$, hence $\overline{N} \models \exists \overline{w}[\theta(\overline{w}, c) \wedge \neg \gamma(\overline{w}]$, contradicting $\overline{N} \models \varphi(b, c) \wedge \gamma(b)$.

Now let

$$\Delta = \{\exists \overline{w} \theta(\overline{w}, v)\} \cup \{\forall \overline{w}[\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})] : \gamma(\overline{w}) \in \Gamma(\overline{w})\}.$$

If $\delta(v) \in \Delta$, then $\delta(v) \in p$, and so $|\varphi(\overline{M}) \backslash \delta(\overline{M})| < \aleph_1$. It follows that $\bigcap_{\delta(v) \in \Delta} \delta(\overline{M}) \neq \emptyset$, i.e. there is a $c' \in M$ such that $\overline{M} \models \delta(c)$ for every $\delta(v) \in \Gamma(v)$. In particular, $\overline{M} \models \exists \overline{w} \theta(\overline{w}, c')$, so we can choose $b' \in M$ such that $\overline{M} \models \theta(b', c)$. Now for each $\gamma(\overline{w}) \in \Gamma(\overline{w})$ the formula $\forall \overline{w}[\theta(\overline{w}, v) \rightarrow \gamma(\overline{w})]$ is in Δ, so it follows that $\overline{M} \models \gamma(b')$. $\square$

Theorem 7.43. *Suppose that T is ω-stable and has an $(\aleph_1, \aleph_0)$-model. Then for any $\kappa > \aleph_1$ it has a $(\kappa, \aleph_0)$-model.*

Proof. Let $\overline{M} \models T$ with $|M| = \aleph_1$, and let $\varphi(\overline{v})$ be a formula with $|\varphi(\overline{M})| = A_0$. We now construct an elementary chain $\langle \overline{N}_\alpha : \alpha < \kappa \rangle$ by recursion. Let $\overline{N}_0 = \overline{M}$. Now suppose that $\overline{N}_\alpha$ has been defined so that $\varphi(\overline{M}) = \varphi(\overline{N}_\alpha)$. We apply Lemma 7.42 to obtain a proper elementary extension $\overline{N}_{\alpha+1}$ of $\overline{N}_\alpha$ such that if $G(\overline{w})$ is a countable type over M realized in $N_{\alpha+1}$, then it is realized in $\overline{N}_\alpha$. Let

$$\Gamma_\alpha(\overline{v}) = \{\varphi(\overline{v})\} \cup \{\overline{v} \neq \overline{a} : \overline{a} \in M \text{ and } \overline{M} \models \varphi(\overline{a})\}$$

Thus Γ_α is a countable type over $\overline{M}$, but it is not realized in $\overline{N}_\alpha$. Hence it is not realized in $\overline{N}_{\alpha+1}$. It follows that $\varphi(\overline{N}_{\alpha+1}) = \varphi(\overline{M})$.

For α limit we let $\overline{N}_\alpha = \bigcup_{\beta < \alpha} \overline{N}_\beta$. Clearly still $\varphi(\overline{N}_\alpha) = \varphi(\overline{M})$.

Finally, $\bigcup_{\alpha < \kappa} \overline{N}_\alpha$ is as desired. $\square$

Theorem 7.44. *If $\overline{M}$ is an infinite structure and κ is a cardinal $\geq |M|$, then $\overline{M}$ has an elementary extension $\overline{N}$ of cardinality κ such that for every formula $\varphi(\overline{v})$ with parameters from N, if $\varphi(\overline{N})$ is infinite then $|\varphi(\overline{N})| = \kappa$.*

Proof. For each formula $\varphi(\overline{v})$ adjoin κ many tuples of new constants of the length of $\overline{v}$, and apply the compactness theorem to the set consisting of Eldiag($\overline{M}$) together with sentences saying, for each $\varphi(\overline{v})$ such that $\varphi(\overline{M})$ is infinite, that the κ many tuples for this formula are all distinct and statisfy φ. □

Theorem 7.45. *Suppose that κ is uncountable and T is κ-categorical. Then T has no Vaughtian pairs.*

Proof. Assume the hypothesis. By Theorem 7.27, T is ω-stable. Suppose that there is a Vaughtian pair. Then by Theorem 7.41 T has an $(\aleph_1, \aleph_0)$-model, and then by Theorem 7.43 it has a $(\kappa, \aleph_0)$-model $\overline{M}$. So $|M| = \kappa$ and $|\varphi(\overline{M})| = \aleph_0$ for some formula $\varphi(\overline{v})$. By Theorem 7.44, there is a model $\overline{N}$ of T in which $|\varphi(\overline{N})| = |N| = \kappa$. This contradicts κ-categoricity.

Theorem 7.46. (Baldwin, Lachlan) *Let κ be uncountable. Then the following conditions are equivalent:*

(i) T is κ-categorical

(ii) T is ω-stable and has no Vaughtian pairs.

Proof. (i)⇒(ii): Theorems 7.27 and 7.45.

(ii)⇒(i): Theorem 7.23. □

Theorem 7.47. (Morley) *T is κ categorical for some uncountable κ iff it is κ-categorical for every uncountable κ.* □

EXERCISES

Exc. 7.1. Let $\overline{M}$ be a field, A a subfield, and $a \in M$. Suppose that a is algebraic over A in the usual sense of field theory. Show that a is algebraic over A in the model-theoretic sense.

Exc. 7.2. Let $\overline{M} = (\omega, <)$. Show that every element of ω is algebraic over $\emptyset$.

Exc. 7.3. Let $\overline{A} = ([\omega]^2, R)$, where

$$R = \{(a, b) : a, b \in [\omega]^2,\ a \neq b \text{ and } a \cap b \neq \emptyset\}.$$

(i) Show that $\{a \in [\omega]^2 : (a, \{0, 1\}) \in R\}$ is neither finite nor cofinite.

(ii) Infer from (i) that $[\omega]^2$ is not minimal.

(iii) If f is a permutation of ω, define $f^+ : [\omega]^2 \to [\omega]^2$ by setting $f^+(a) = f[a]$ for any $a \in [\omega]^2$. Show that f^+ is an automorphism of $\overline{A}$.

(iv) Let $X = \{a \in [\omega]^2 : 0 \in a \text{ and } a \cap \{1, 2\} = \emptyset\}$. Show that X is definable in $\overline{A}$ with parameters.

(v) Show that X is minimal.

Exc. 7.4. Let V be an infinite vector space over a finite field F. We consider V as a structure $(V, +, f_a)_{a \in F}$, where $f_a(v) = av$ for any $v \in V$ and $a \in F$. Show that V is minimal.

Exc. 7.5. (continuing exc. 7.4) Prove that for any subset A of V, $\mathrm{acl}(A) = \mathrm{span}(A)$.

Exc. 7.6. (continuing excs. 7.4, 7.5) By exercise 7.4 and Lemma 7.2, the following holds in $\overline{V}$: if $a \in \mathrm{span}(A \cup \{b\}) \backslash \mathrm{span}(A)$, then $b \in \mathrm{span}(A \cup \{a\})$. Prove this statement using ordinary linear algebra.

Exc. 7.7. Give an example of a set Γ of sentences and two sentences φ and ψ, such that $\Gamma \models \varphi$ iff $\Gamma \models \psi$, but $\Gamma \not\models (\varphi \leftrightarrow \psi)$.

Exc. 7.8. Show that for Γ a set of sentences and for sentences φ, ψ, if $\Gamma \models \varphi \leftrightarrow \psi$ then $\Gamma \models \varphi$ iff $\Gamma \models \psi$.

Exc. 7.9. Prove that the following two conditions are equivalent:

(i) $\overline{M} \models \varphi[a]$ iff $\overline{M} \models \psi[a]$.

(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)[a]$.

Exc. 7.10. Prove that the following two conditions are equivalent, for any sentences φ, ψ:

(i) $\overline{M} \models \varphi$ iff $\overline{M} \models \psi$.

(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)$.

Exc. 7.11. In the language with no non-logical symbols, show that ω is an indiscernible set in ω.

Exc. 7.12. (Continuing exercises 7.4, 7.5, 7.6) Let $A = \{w_1, w_2\}$, two members of V, and let $b = w_1$. Thus $b \in \mathrm{span}(A)$. According to Lemma 7.8, $\mathrm{tp}^{\overline{V}}(b/A)$ is isolated. Give a formula $\varphi(v_0, \overline{a})$ with $\overline{a} \in A$ which isolates $\mathrm{tp}^{\overline{V}}(b/A)$.

Exc. 7.13. Suppose that $\overline{M}$ is an infinite structure, $\varphi(v_0)$ is a formula with at most v_0 free, and $\varphi(\overline{M})$ is infinite. Show that $\overline{M}$ has a proper elementary extension $\overline{N}$ such that $(\overline{M}, \overline{N})$ is not a Vaughtian pair.

8. Morley rank

Let $\overline{M}$ be an $\mathscr{L}$-structure and $\varphi(\overline{v})$ a formula of $\mathscr{L}_M$. We define $\mathrm{RM}(\overline{M},\varphi,\alpha)\in\{0,1\}$ for every ordinal α by recursion on α.

- $\mathrm{RM}(\overline{M},\varphi,0)=1$ iff $\varphi(\overline{M})$ is nonempty.

- For α limit, $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ iff $\mathrm{RM}(\overline{M},\varphi,\beta)=1$ for all $\beta<\alpha$.

- $\mathrm{RM}(\overline{M},\varphi,\alpha+1)=1$ iff there exist formulas $\psi_0(\overline{v}),\psi_1(\overline{v}),\ldots$ such that $\psi_0(\overline{M}),\psi_1(\overline{M}),\ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}(\overline{M},\psi_i,\alpha)=1$ for every $i\in\omega$.

Proposition 8.1. *If* $\overline{M}\models\varphi\to\psi$ *and* $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$, *then* $\mathrm{RM}(\overline{M},\psi,\alpha)=1$.

Proof. Induction on α. □

Proposition 8.2. *If* $\mathrm{RM}(\overline{M},\varphi,\alpha)=0$, *then* $\mathrm{RM}(\overline{M},\varphi,\beta)=0$ *for all* $\beta\geq\alpha$.

Proof. Induction on β. □

Now we define the *Morley rank* $\mathrm{RM}^{\overline{M}}(\varphi)$ of φ in $\overline{M}$ as follows. If $\varphi(\overline{M})=\emptyset$, then $\mathrm{RM}^{\overline{M}}(\varphi)=-1$. If α is minimum such that $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ and $\mathrm{RM}(\overline{M},\varphi,\alpha+1)=0$, then $\mathrm{RM}^{\overline{M}}(\varphi)=\alpha$. If $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ for all α, then $\mathrm{RM}^{\overline{M}}(\varphi)=\infty$.

We will also define the Morley rank of structures below.

Proposition 8.3. $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$ *iff* $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha$.

Proof. Suppose that $\mathrm{RM}^{\overline{M}}(\varphi)=\beta<\alpha$. Then $\mathrm{RM}(\overline{M},\varphi,\beta+1)=0$, and so by Proposition 8.2, $\mathrm{RM}(\overline{M},\varphi,\alpha)=0$.

Suppose that $\mathrm{RM}^{\overline{M}}(\varphi)=\beta\geq\alpha$. Then $\mathrm{RM}(\overline{M},\varphi,\beta)=1$, and so by Proposition 8.2, also $\mathrm{RM}(\overline{M},\varphi,\alpha)=1$. □

By Proposition 8.3, the definition of Morley rank can be reformulated as follows.

- $\mathrm{RM}^{\overline{M}}(\varphi)=-1$ if $\varphi(\overline{M})=\emptyset$.

- $\mathrm{RM}^{\overline{M}}(\varphi)\geq 0$ if $\varphi(\overline{M})\neq\emptyset$.

- For α limit, $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha$ iff $\mathrm{RM}^{\overline{M}}(\varphi)\geq\beta$ for every $\beta<\alpha$.

- $\mathrm{RM}^{\overline{M}}(\varphi)\geq\alpha+1$ iff there exist formulas $\psi_0(\overline{v}),\psi_1(\overline{v}),\ldots$ such that $\psi_0(\overline{M}),\psi_1(\overline{M}),\ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}^{\overline{M}}(\psi_i)\geq\alpha$ for every $i\in\omega$.

Proposition 8.4. *Suppose that* $\overline{M}$ *is* ω*-saturated,* $\varphi(\overline{v},\overline{w})$ *is a formula,* $\overline{a},\overline{b}\in M$, *and* $\mathrm{tp}^{\overline{M}}(\overline{a})=\mathrm{tp}^{\overline{M}}(\overline{b})$. *Then* $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a}))=\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b}))$.

Proof. We prove by induction on α that if $\varphi(\overline{v},\overline{w})$ is any formula, $\overline{a},\overline{b} \in M$, and $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b})$, then $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \alpha$ iff $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \alpha$. To begin with,

$$\begin{aligned}
\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq 0 \quad &\text{iff} \quad \varphi(\overline{M},\overline{a}) \neq \emptyset \\
&\text{iff} \quad \text{there is a } \overline{c} \text{ such that } \overline{M} \models \varphi[\overline{c},\overline{a}] \\
&\text{iff} \quad \exists \overline{v}\varphi(\overline{v},\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a}) \\
&\text{iff} \quad \exists \overline{v}\varphi(\overline{v},\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{b}) \\
&\text{iff} \quad \varphi(\overline{M},\overline{b}) \neq \emptyset \\
&\text{iff} \quad \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq 0.
\end{aligned}$$

If α is limit and we know the result for all $\beta < \alpha$, then

$$\begin{aligned}
\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \alpha \quad &\text{iff} \quad \text{for all } \beta < \alpha,\ \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \beta \\
&\text{iff} \quad \text{for all } \beta < \alpha,\ \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \beta \\
&\text{iff} \quad \mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \alpha.
\end{aligned}$$

Now suppose that the equivalence is true for α, and $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{a})) \geq \alpha + 1$; we prove that $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v},\overline{b})) \geq \alpha + 1$. By symmetry this is all that is required. Now there are $\mathscr{L}_M$-formulas $\psi_0, \psi_1, \ldots$ such that $\langle \psi_i(\overline{M}) : i < \omega \rangle$ is a system of pairwise disjoint subsets of $\varphi(\overline{M},\overline{a})$ and $\mathrm{RM}(\psi_i) \geq \alpha$ for all $i < \omega$. For each $i < \omega$ there is a sequence $\overline{c}_i$ of elements of M such that ψ_i is $\psi_i(\overline{v},\overline{c}_i)$. We now define $\overline{d}_0, \overline{d}_1, \ldots$. Suppose that $\overline{d}_0, \ldots, \overline{d}_m$ have been defined so that

$$(*) \qquad \mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_m) = \mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_m).$$

Let

$$\Delta = \{\chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) : \overline{M} \models \chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1})\}.$$

Suppose that Δ' is a finite subset of Δ. Then

$$\overline{M} \models \bigwedge\{\chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\},$$

so

$$\overline{M} \models \exists\overline{w} \bigwedge\{\chi(\overline{a},\overline{c}_0,\ldots,\overline{c}_m,\overline{w}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\}.$$

It follows that the formula

$$\exists\overline{w} \bigwedge\{\chi(\overline{v},\overline{u}_0,\ldots,\overline{u}_m,\overline{w}) : \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w}) \in \Delta'\}$$

is in $\mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_m)$, and hence by $(*)$ it is in $\mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_m)$. Thus $\overline{M} \models \exists\overline{w} \bigwedge \Delta'$. Now since $\overline{M}$ is ω-saturated, there is a $\overline{d}_{m+1}$ in M such that $\overline{M} \models \chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_{m+1})$ for each formula $\chi(\overline{b},\overline{d}_0,\ldots,\overline{d}_m,\overline{w})$ in Δ. It follows that

$$\mathrm{tp}^{\overline{M}}(\overline{a},\overline{c}_0,\ldots,\overline{c}_{m+1}) = \mathrm{tp}^{\overline{M}}(\overline{b},\overline{d}_0,\ldots,\overline{d}_{m+1}).$$

This finishes the definition of $\overline{d}_0, \overline{d}_1, \ldots$. So we have

$$\mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \overline{c}_1, \ldots) = \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{d}_0, \overline{d}_1, \ldots).$$

Now for $i \neq j$ we have $\psi_i(\overline{M}, \overline{c}_i) \cap \psi_i(\overline{M}, \overline{c}_j) = \emptyset$. Hence

$$\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \wedge \psi_j(\overline{v}, \overline{w}_j)] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \overline{c}_1, \ldots),$$

and hence

$$\neg\exists\overline{v}[\psi_i(\overline{v}, \overline{w}_i) \wedge \psi_j(\overline{v}, \overline{w}_j)] \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{d}_0, \overline{d}_1, \ldots),$$

which means that $\psi_i(\overline{M}, \overline{d}_i) \cap \psi_j(\overline{M}, \overline{d}_j) = \emptyset$. Also, $\psi_i(\overline{M}, \overline{c_i}) \subseteq \varphi(\overline{M}, \overline{a})$, so

$$\forall\overline{v}[\psi_i(\overline{v}, \overline{w}) \rightarrow \varphi(\overline{M}, \overline{x})] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \overline{c}_1, \ldots),$$

and so

$$\forall\overline{v}[\psi_i(\overline{v}, \overline{w}) \rightarrow \varphi(\overline{M}, \overline{x})] \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{d}_0, \overline{d}_1, \ldots),$$

from which it follows that $\psi_i(\overline{M}, \overline{d}_i) \subseteq \varphi(\overline{M}, \overline{b})$. From $(**)$ the inductive hypothesis gives $\mathrm{RM}(\psi_i(\overline{v}, \overline{d}_i)) \geq \alpha$. So $\mathrm{RM}^{\overline{M}}(\varphi(\overline{v}, \overline{b})) \geq \alpha + 1$.

Proposition 8.5. *Suppose that $\overline{M}$ and $\overline{N}$ are ω-saturated models of T and $\overline{M} \preceq \overline{N}$. Then $\mathrm{RM}^{\overline{M}}(\varphi) = \mathrm{RM}^{\overline{N}}(\varphi)$ for any $\mathscr{L}_M$-formula φ.*

Proof. We prove by induction on α that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha$ iff $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha$. For $\alpha = 0$,

$$\begin{aligned} \mathrm{RM}^{\overline{M}}(\varphi) \geq 0 \quad & \text{iff} \quad \varphi(\overline{M}) \neq \emptyset \\ & \text{iff} \quad \overline{M} \models \exists\overline{v}\varphi(\overline{v}) \\ & \text{iff} \quad \overline{N} \models \exists\overline{v}\varphi(\overline{v}) \\ & \text{iff} \quad \varphi(\overline{N}) \neq \emptyset \\ & \text{iff} \quad \mathrm{RM}^{\overline{N}}(\varphi) \geq 0. \end{aligned}$$

For α limit, assuming the equivalence for all $\beta < \alpha$,

$$\begin{aligned} \mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha \quad & \text{iff} \quad \text{for all } \beta < \alpha[\mathrm{RM}^{\overline{M}}(\varphi) \geq \beta] \\ & \text{iff} \quad \text{for all } \beta < \alpha[\mathrm{RM}^{\overline{N}}(\varphi) \geq \beta] \\ & \text{iff} \quad \mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha. \end{aligned}$$

Now assume the equivalence for α. Suppose that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha+1$. Then there are formulas $\psi_0(\overline{v}), \psi_1(\overline{v}), \ldots$ of $\mathscr{L}_M$ such that $\psi_0(\overline{M}), \psi_1(\overline{M}), \ldots$ are pairwise disjoint subsets of $\varphi(\overline{M})$ and $\mathrm{RM}^{\overline{M}}(\psi_i) \geq \alpha$ for all $i < m$. By the inductive hypothesis, $\mathrm{RM}^{\overline{N}}(\psi_i) \geq \alpha$ for all $i < m$. For distinct $i, j < \omega$ we have $\overline{M} \models \neg\exists\overline{v}(\psi_i(\overline{v}) \wedge \psi_j(\overline{v}))$, and hence $\overline{N} \models \neg\exists\overline{v}(\psi_i(\overline{v}) \wedge \psi_j(\overline{v}))$. So $\psi_0(\overline{N}), \psi_1(\overline{N}), \ldots$ are pairwise disjoint. Also, $\overline{M} \models \forall\overline{v}[\psi_i(\overline{v}) \rightarrow \varphi(\overline{v})]$ for each $i < \omega$, so

$\overline{N} \models \forall \overline{v}[\psi_i(\overline{v}) \to \varphi(\overline{v})]$. Hence each $\psi_i(\overline{N})$ is a subset of $\varphi(\overline{N})$. It follows that $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha + 1$.

Suppose now that $\mathrm{RM}^{\overline{N}}(\varphi) \geq \alpha + 1$. Then there are formulas $\psi_0(\overline{v}), \psi_1(\overline{v}), \ldots$ of $\mathscr{L}_N$ such that $\psi_0(\overline{N}), \psi_1(\overline{N}), \ldots$ are pairwise disjoint subsets of $\varphi(\overline{N})$ and $\mathrm{RM}^{\overline{N}}(\psi_i) \geq \alpha$ for all $i < m$. Write $\varphi(\overline{v}) = \varphi(\overline{v}, \overline{a})$ with $\overline{a} \in M$ and $\psi_i(\overline{v}) = \psi_i(\overline{v}, \overline{b}_i)$ with $\overline{b}_i \in N$. We now define $\overline{c}_i$ in M for $i < \omega$ so that

$$(*) \qquad \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots, \overline{b}_{m-1})$$

for every $m \in \omega$. Note that $(*)$ holds for $m = 0$ since $\overline{M} \preceq \overline{N}$. Suppose now that $(*)$ holds for m. Let

$$\Delta = \{\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) : \overline{N} \models \chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m)\}.$$

Suppose that Δ' is a finite subset of Δ. Then

$$\overline{N} \models \bigwedge\{\chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\},$$

so

$$\overline{N} \models \exists \overline{w} \bigwedge\{\chi(\overline{a}, \overline{b}_0, \ldots, \overline{b}_{m-1}, \overline{w}) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\}.$$

Hence by $(*)$ for m we get

$$\overline{M} \models \exists \overline{w} \bigwedge\{\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) : \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}) \in \Delta'\}.$$

Since $\overline{M}$ is ω-saturated, there is a $\overline{c}_m$ in M such that $\overline{M} \models \chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{c}_m)$ for each formula $\chi(\overline{a}, \overline{c}_0, \ldots, \overline{c}_{m-1}, \overline{w}$ in Δ. It follows that $(*)$ holds for $m + 1$.

Now suppose that $i, j \in \omega$ with $i \neq j$. Then $\overline{N} \models \neg\exists \overline{v}[\psi_i(\overline{v}, \overline{b_i}) \wedge \psi_j(\overline{v}, \overline{b_j})$, so that $\neg\exists \overline{v}[\psi_i(\overline{v}, \overline{w_i}) \wedge \psi_j(\overline{v}, \overline{w_j}) \in \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots)$. Hence by $(*)$, $\neg\exists \overline{v}[\psi_i(\overline{v}, \overline{w_i}) \wedge \psi_j(\overline{v}, \overline{w_j}) \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots)$. Hence $\psi_i(\overline{M}, \overline{c_i}) \cap \psi_i(\overline{M}, \overline{c}_j) = \emptyset$. Also, $\overline{N} \models \forall \overline{v}[\psi_i(\overline{v}, \overline{b}_i) \to \varphi(\overline{v}, \overline{a})]$, so $\forall \overline{v}[\psi_i(\overline{v}, \overline{w}_i) \to \varphi(\overline{v}, \overline{a})] \in \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots)$. Hence by $(*)$, $\forall \overline{v}[\psi_i(\overline{v}, \overline{w}_i) \to \varphi(\overline{v}, \overline{a})] \in \mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots)$. Hence $\psi_i(\overline{M}, \overline{c}_i) \subseteq \varphi(\overline{M}, \overline{a})$. Now since $\overline{M} \preceq \overline{N}$, we have

$$\mathrm{tp}^{\overline{M}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m)$$

for each $m \in \omega$, and hence by $(*)$, also $\mathrm{tp}^{\overline{N}}(\overline{a}, \overline{c}_0, \ldots, \overline{c}_m) = \mathrm{tp}^{\overline{N}}(\overline{a}, \overline{b}_0, \ldots, \overline{b}_m)$ for each $m \in \omega$. Hence by Proposition 8.4 we get $\mathrm{RM}^{\overline{N}}(\psi_i(\overline{v}, \overline{c}_i) \geq \alpha$, and so by the inductive hypothesis $\mathrm{RM}^{\overline{N}}(\psi_i(\overline{v}, \overline{c}_i) \geq \alpha$ for each $i < \omega$. It follows that $\mathrm{RM}^{\overline{M}}(\varphi) \geq \alpha + 1$. $\square$

Proposition 8.6. (amalgamation) *Suppose that $\overline{M}$, $\overline{N}_0$, and $\overline{N}_1$ are structures, and $f : M \to N_0$ and $g : M \to N_1$ are elementary embeddings. Then there exist a structure $\overline{P}$ and elementary embeddings $h : \overline{N}_0 \to \overline{P}$ and $k : \overline{N}_1 \to \overline{P}$ such that $h \circ f = k \circ g$; so the following diagram commutes:*

$$
\begin{tikzcd}
 & \overline{N}_0 \arrow[dr, "h"] & \\
\overline{M} \arrow[ur, "f"] \arrow[dr, "g"'] & & \overline{P} \\
 & \overline{N}_1 \arrow[ur, "k"'] &
\end{tikzcd}
$$

Proof. Our first goal is to obtain isomorphic copies $\overline{N}_0'$ and $\overline{N}_1'$ of $\overline{N}_0$ and $\overline{N}_1$ such that $\overline{M} \preceq \overline{N}_0'$, $\overline{M} \preceq \overline{N}_1'$, and $N_0' \cap N_1' = M$.

Let Q_0 and Q_1 be sets such that $Q_0 \cap M = \emptyset = Q_1 \cap M = Q_0 \cap Q_1$, $|Q_0| = |N_0 \backslash f[M]|$, and $|Q_1| = |N_1 \backslash g[M]|$. Let $f' : N_0 \backslash f[M] \to Q_0$ and $g' : N_1 \backslash g[M] \to Q_1$ be bijections. Let $N_0' = M \cup Q_0$ and $N_1' = M \cup Q_1$. Note that $N_0' \cap N_1' = M$. Define $f'' : N_0 \to N_0'$ by setting, for any $a \in N_0$,

$$f''(a) = \begin{cases} f^{-1}(a) & \text{if } a \in f[M], \\ f'(a) & \text{if } a \in N_0 \backslash f[M], \end{cases}$$

and define $g'' : N_1 \to N_1'$ by setting, for any $a \in N_1$,

$$g''(a) = \begin{cases} g^{-1}(a) & \text{if } a \in g[M], \\ g'(a) & \text{if } a \in N_1 \backslash g[M]. \end{cases}$$

Clearly f'' and g'' are bijections.

We now define structures on N_0' and N_1'. If R is an m-ary relation symbol, then

$$\begin{aligned} R^{\overline{N}_0'} &= \{a \in {}^m N_0' : (f'')^{-1} \circ a \in R^{\overline{N}_0}\}; \\ R^{\overline{N}_1'} &= \{a \in {}^m N_1' : (g'')^{-1} \circ a \in R^{\overline{N}_1}\}. \end{aligned}$$

If F is an m-ary function symbol, then

$$\begin{aligned} F^{\overline{N}_0'}(a) &= f''(F^{\overline{N}_0}((f'')^{-1} \circ a)) \quad \text{for any } a \in {}^m N_0'; \\ F^{\overline{N}_1'}(a) &= g''(F^{\overline{N}_1}((g'')^{-1} \circ a)) \quad \text{for any } a \in {}^m N_1'. \end{aligned}$$

Then it is easy to check that f'' is an isomorphism from $\overline{N}_0$ onto $\overline{N}_0'$ and g'' is an isomorphism from $\overline{N}_1$ onto $\overline{N}_1'$. Now take any formula φ and any $a \in {}^\omega M$. Then

$$\begin{aligned} \overline{M} \models \varphi[a] \quad &\text{iff} \quad \overline{N}_0 \models \varphi[f \circ a] \\ &\text{iff} \quad \overline{N}_0' \models \varphi[f'' \circ f \circ a] \\ &\text{iff} \quad \overline{N}_0' \models \varphi[f^{-1} \circ f \circ a] \\ &\text{iff} \quad \overline{N}_0' \models \varphi[a]. \end{aligned}$$

Thus $\overline{M} \preceq \overline{N}'_0$. Similarly $\overline{M} \preceq \overline{N}'_1$.

Now we claim that $\mathrm{Eldiag}(\overline{N}'_0) \cup \mathrm{Eldiag}(\overline{N}'_1)$ has a model. If not, by the compactness theorem some finite subset fails to have a model. Say Δ_0 is a finite subset of $\mathrm{Eldiag}(\overline{N}'_0)$, Δ_1 is a finite subset of $\mathrm{Eldiag}(\overline{N}'_1)$, and $\Delta_0 \cup \Delta_1$ does not have a model. Then $\bigwedge \Delta_1$ has the form $\psi(c_{a(0)}, \ldots, c_{a(m-1)}, c_{b(0)}, \ldots, c_{b(n-1)})$ with each $a(i) \in M$ and each $d(i) \in N'_1 \backslash M$. Thus

$$\Delta_0 \models \neg\psi(c_{a(0)}, \ldots, c_{a(m-1)}, c_{b(0)}, \ldots, c_{b(n-1)}).$$

Replacing $c_{b(i)}$ by a variable w_i, we get

$$\Delta_0 \models \forall \overline{u} \neg\psi(c_{a(0)}, \ldots, c_{a(m-1)}, \overline{u}).$$

Now $(\overline{N}'_0)_{N'_0}$ is a model of Δ_0, hence $\overline{N}'_0 \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$, hence $\overline{M} \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$, hence $\overline{N}'_1 \models \forall \overline{u} \neg\psi(a(0), \ldots, a(m-1), \overline{u})$. But this is impossible. Hence we have shown that $\mathrm{Eldiag}(\overline{N}'_0) \cup \mathrm{Eldiag}(\overline{N}'_1)$ has a model. Such a model has the form $(\overline{P}, h(s), k(t))_{s \in N'_0, t \in N'_1}$, where $h(a) = k(a)$ for all $a \in M$. By the elementary diagram lemma 6.15, h is an elementary embedding of $\overline{N}'_0$ into $\overline{P}$, and k is an elementary embedding of $\overline{N}'_1$ into $\overline{P}$.

Hence $h \circ f''$ is an elementary embedding of $\overline{N}_0$ into $\overline{P}$, $k \circ g''$ is an elementary embedding of $\overline{N}_1$ into $\overline{P}$, and for any $a \in M$,

$$h(f''(f(a)) = h(f^{-1}(f(a))) = h(a) = k(a) = k(g^{-1}(g(a))) = k(g''(g(a))). \qquad \square$$

Corollary 8.7. *Suppose that $\overline{M}$ is an $\mathscr{L}$-structure and $\overline{N}_0$ and $\overline{N}_1$ are ω-saturated elementary extensions of $\overline{M}$. Then for any formula φ of $\mathscr{L}_M$,* $\mathrm{RM}^{\overline{N}_0}(\varphi) = \mathrm{RM}^{\overline{N}_1}(\varphi)$.

Proof. By Proposition 8.6 let $\overline{N}_2$ be f and g be elementary embeddings of $\overline{N}_0$ and $\overline{N}_1$ into a structure $\overline{N}_2$. Let $\overline{N}_3$ be an ω-saturated elementary extension of $\overline{N}_2$. Then by Proposition 8.5, $\mathrm{RM}^{\overline{N}_0}(\varphi) = \mathrm{RM}^{\overline{N}_3}(\varphi) = \mathrm{RM}^{\overline{N}_1}(\varphi)$. $\square$

Proposition 8.8. *If $\varphi(\overline{v})$ and $\psi(\overline{v})$ are formulas of $\mathscr{L}_M$ and $\overline{M} \models \forall \overline{v}[\varphi \to \psi]$, then* $\mathrm{RM}(\varphi) \leq \mathrm{RM}(\psi)$.

Proof. We prove by induction on α that $\mathrm{RM}(\varphi) \geq \alpha$ implies that $\mathrm{RM}(\psi) \geq \alpha$. For $\alpha = 0$, if $\mathrm{RM}(\varphi) \geq 0$, then $\varphi(\overline{M}) \neq \emptyset$; hence $\mathrm{RM}(\psi) \neq \emptyset$ and so $\mathrm{RM}(\psi) \geq 0$. Suppose that α is a limit ordinal and we know the implication for all $\beta < \alpha$. Suppose that $\mathrm{RM}(\varphi) \geq \alpha$. Then $\forall \beta < \alpha[\mathrm{RM}(\varphi) \geq \beta]$, hence $\forall \beta < \alpha[\mathrm{RM}(\psi) \geq \beta]$, hence $\mathrm{RM}(\psi) \geq \alpha$. Now suppose that we know the implication for α, and $\mathrm{RM}(\varphi) \geq \alpha + 1$. Then there are pairwise disjoint subsets $\psi_i(\overline{M})$ of $\varphi(\overline{M})$ such that $\mathrm{RM}(\psi_i) \geq \alpha$. Since also $\psi_i(\overline{M}) \subseteq \psi(\overline{M})$, it follows that $\mathrm{RM}(\psi) \geq \alpha + 1$. $\square$

Proposition 8.9. $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) = \max(\mathrm{RM}(\varphi(\overline{v})), \mathrm{RM}(\psi(\overline{v}))$.

Proof. It suffices to show that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$ iff $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha$ or $\mathrm{RM}(\psi(\overline{v})) \geq \alpha$ for every ordinal α, by induction. For $\alpha = 0$,

$$\begin{aligned} \mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq 0 \quad &\text{iff} \quad \varphi(\overline{M}) \neq \emptyset \text{ or } \psi(\overline{M}) \neq \emptyset \\ &\text{iff} \quad \mathrm{RM}(\varphi(\overline{v})) \geq 0 \text{ or } \mathrm{RM}(\psi(\overline{v})) \geq 0. \end{aligned}$$

For α limit, assume the result for any $\beta < \alpha$. Suppose that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$. Then for any $\beta < \alpha$ we have $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \beta$, and hence by the inductive hypothesis, $\mathrm{RM}(\varphi(\overline{v})) \geq \beta$ or $\mathrm{RM}(\psi(\overline{v})) \geq \beta$. If $\mathrm{RM}(\varphi(\overline{v})) \geq \beta$ for all $\beta < \alpha$, then $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha$. If $\mathrm{RM}(\varphi(\overline{v})) < \beta$ for some $\beta < \alpha$, then $\mathrm{RM}(\psi(\overline{v})) \geq \beta$ for all $\beta < \alpha$ and so $\mathrm{RM}(\psi(\overline{v})) \geq \alpha$. Thus $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha$ implies that $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha$ or $\mathrm{RM}(\psi(\overline{v})) \geq \alpha$. The converse holds by Proposition 8.8.

Now assume the result for α and suppose that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha + 1$. Choose formulas $\chi_i(\overline{v})$ for $i \in \omega$ such that the sets $\chi_i(\overline{M})$ are pairwise disjoint, contained in $\varphi(\overline{M}) \cup \psi(\overline{M})$, and each of Morley rank $\geq \alpha$. Suppose that $\mathrm{RM}(\varphi(\overline{v})) < \alpha + 1$ and $\mathrm{RM}(\psi(\overline{v})) < \alpha + 1$. Now the sets $\chi_i(\overline{M}) \wedge \varphi(\overline{M})$ are pairwise disjoint and contained in $\varphi(\overline{M})$. It follows that there is an $m \in \omega$ such that $\mathrm{RM}(\chi_i(\overline{v}) \wedge \varphi(\overline{v})) < \alpha$ for all $i \geq m$. Similarly, there is an $n \in \omega$ such that $\mathrm{RM}(\chi_i(\overline{v}) \wedge \psi(\overline{v})) < \alpha$ for all $i \geq n$. Let $p = \max(m, n)$. Then by the inductive hypothesis, $\mathrm{RM}(\chi_p(\overline{v}) < \alpha$, contradiction. This shows that $\mathrm{RM}(\varphi(\overline{v}) \vee \psi(\overline{v})) \geq \alpha + 1$ implies that $\mathrm{RM}(\varphi(\overline{v})) \geq \alpha + 1$ or $\mathrm{RM}(\psi(\overline{v})) \geq \alpha + 1$. The converse holds by Proposition 8.8. □

Proposition 8.10. *If $\varphi(\overline{M}) \neq \emptyset$, then $\mathrm{RM}(\varphi(\overline{v}) = 0$ iff $\varphi(\overline{M})$ is finite.*

Proof. $\Rightarrow$: suppose that $\varphi(\overline{M})$ is infinite, and choose distinct $\overline{a}_i \in M$ such that $\overline{M} \models \varphi[\overline{a}_i)$ for all $i \in \omega$. Then let ψ_i be the formula $\overline{v} = \overline{a}_i$ for each $i \in \omega$. Then the sets $\psi_i(\overline{M})$ are pairwise disjoint, nonempty, and are subsets of $\varphi(\overline{M})$. Since $\mathrm{RM}(\psi_i(\overline{v})) \geq 0$ for each $i \in \omega$, it follows that $\mathrm{RM}(\varphi(\overline{v}) \geq 1$.

$\Leftarrow$: If $\varphi(\overline{M})$ is finite, then there do not exist infinitely many pairwise disjoint nonempty subsets of it. So $\mathrm{RM}(\varphi(\overline{v})) \leq 0$. Since $\varphi(\overline{M}) \neq \emptyset$, actually $\mathrm{RM}(\varphi(\overline{v})) = 0$. □

Proposition 8.11. *Suppose that $\overline{M}$ is a structure, φ is an $\mathscr{L}_M$-formula, and $\mathrm{RM}^{\overline{M}}(\varphi) = \alpha$ for some ordinal α. Then there is a positive integer d such that if $\psi_1, \ldots, \psi_m$ are $\mathscr{L}_M$-formulas such that $\psi_1(\overline{M}), \ldots, \psi_m(\overline{M})$ are pairwise disjoint subsets of $\varphi(\overline{M})$ each of Morley rank α, then $m \leq d$.*

Proof. By recursion we construct subsets T_i of ${}^{<\omega}2$ and formulas φ_σ for $\sigma \in T_i$. Let $T_0 = \{\emptyset\}$ and $\varphi_\emptyset = \varphi$. Suppose T_i has been defined along with the formulas φ_σ for $\sigma \in T_i$. Take any $\sigma \in T_i$. If there is a formula ψ in $\mathscr{L}_M$ such that $\mathrm{RM}^{\overline{M}}(\varphi_\sigma \wedge \psi) = \mathrm{RM}^{\overline{M}}(\varphi_\sigma \wedge \neg\psi) = \alpha$, we take such a formula ψ, put $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ in T_{i+1}, and define $\varphi_{\sigma^\frown\langle 0\rangle} = \varphi_\sigma \wedge \psi$ and $\varphi_{\sigma^\frown\langle 1\rangle} = \varphi_\sigma \wedge \neg\psi$. If such a formula ψ does not exist, we put σ in T_{i+1}.

By induction we have:

(1) for every $i \in \omega$, $\bigcup_{\sigma \in T_i} \varphi_\sigma(\overline{M}) = \varphi(\overline{M})$.

Now let $T = \bigcup_{i \in \omega} T_i$. We claim that T is finite. Suppose not. Then by König's tree lemma, there is an increasing sequence $\langle \sigma_i : i \in \omega \rangle$ of members of T. Let $\chi_i = \varphi_{\sigma_i} \wedge \neg \varphi_{\sigma_{i+1}}$ for all $i \in \omega$. Then $\langle \chi_i(\overline{M}) : i \in \omega \rangle$ is a system of pairwise disjoint subsets of $\varphi(\overline{M})$ and each χ_i has Morley rank α, contradicting $\mathrm{RM}(\varphi) = \alpha$.

Since T is finite, there is an $i \in \omega$ such that $T_i = T_j$ for all $j \geq i$. Let $\langle \psi_1, \ldots \psi_d \rangle$ enumerate T_i. By (1), $\langle \psi_i(\overline{M}) : i = 1, \ldots d \rangle$ is a partition of $\varphi(\overline{M})$.

Now suppose that $\theta_1, \ldots, \theta_m$ is a sequence of $\mathscr{L}_M$-formulas each of rank α such that $\langle \theta_1(\overline{M}), \ldots \theta_m(\overline{M}) : 1 \leq i \leq m \rangle$ is a sequence of pairwise disjoint subsets of $\varphi(\overline{M})$. We claim that $m \leq d$ (as desired). Suppose that $m > d$. Now for each $i \leq d$ there is at most one $j \leq m$ such that $\mathrm{RM}(\psi_i \wedge \theta_j) = \alpha$. Hence there is a $j \leq m$ such that $\mathrm{RM}(\psi_i \wedge \theta_j) < \alpha$ for all $i \leq d$. But $\theta_j(\overline{M})$ is a subset of $\varphi(\overline{M})$, and so by (1), $\overline{M} \models \theta_j \leftrightarrow \bigvee_{i \leq d}(\psi_i \wedge \theta_j)$. This contradicts Proposition 8.9. $\square$

The smallest d satisfying the conditions of Proposition 8.11 is called the *Morley degree* of φ in $\overline{M}$, and is denoted by $\deg_{\overline{M}}(\varphi)$.

Corollary 8.12. *A formula φ is minimal over $\overline{M}$ iff* $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$.

Proof. $\Rightarrow$: Let φ be minimal. Then $\varphi(\overline{M})$ is infinite, hence nonempty, so $\mathrm{RM}^{\overline{M}}(\varphi) \geq 0$; by Proposition 8.10, $\mathrm{RM}^{\overline{M}}(\varphi) \geq 1$. Now $\varphi(\overline{M})$ cannot be partitioned into two infinite definable subsets, so $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$.

$\Leftarrow$: Assume that $\mathrm{RM}^{\overline{M}}(\varphi) = \deg_{\overline{M}}(\varphi) = 1$. Then by Proposition 8.10, $\varphi(\overline{M})$ is infinite. Since $\deg_{\overline{M}}(\varphi) = 1$, it cannot be partitioned into two infinite definable subsets. So φ is minimal. $\square$

The *Morley rank* of a theory T is the rank of the formula $v = v$ in any ω-saturated model of T.

The theory T is *totally transcendental* iff its rank is less than ∞.

If $\overline{M}$ is ω-saturated, a formula $\varphi(\overline{v}, \overline{w})$ has the *order property* over $\overline{M}$ iff there are sequences $\langle \overline{a}_i : i < \omega \rangle$ and $\langle \overline{b}_i : i < \omega \rangle$ such that for all $i, j \in \omega$, $\overline{M} \models \varphi(\overline{a}_i, \overline{b}_j)$ iff $i < j$.

Proposition 8.13. *If T is totally transcendental and $\overline{M}$ is an ω-saturated model of T, then no formula has the order property over $\overline{M}$.*

Proof. Suppose to the contrary that T is totally transcendental and $\overline{M}$ is an ω-saturated model of T with a formula $\varphi(\overline{v}, \overline{w})$ having the order property over $\overline{M}$, say with sequences $\langle \overline{a}_i : i < \omega \rangle$ and $\langle \overline{b}_i : i < \omega \rangle$ such that for all $i, j \in \omega$, $\overline{M} \models \varphi(\overline{a}_i, \overline{b}_j)$ iff $i < j$. Adjoin new individual constants $\overline{c}_q$ and $\overline{d}_q$ for $q \in \mathbb{Q}$ and consider the following set of sentences in the expanded language:

$$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{c}_q, \overline{d}_r) : q < r\} \cup \{\neg \varphi(\overline{c}_q, \overline{d}_r) : r \leq q\}.$$

Clearly every finite subset of this set has a model, so the whole set has a model. This gives an elementary extension $\overline{N}$ of $\overline{M}$ such that there are systems $\langle \overline{s}_q : q \in \mathbb{Q} \rangle$ and $\langle \overline{t}_q : q \in \mathbb{Q} \rangle$ of elements of N such that for all $q, r \in \mathbb{Q}$, $\overline{N} \models \varphi(\overline{s}_q, \overline{t}_r)$ iff $q < r$. Let $\overline{P}$ be and ω-saturated elementary extension of $\overline{N}$. Now for any $r \in \mathbb{Q}$, the set $\{q \in \mathbb{Q} : \overline{P} \models \varphi(\overline{s}_q, \overline{t}_r)\}$

is an infinite convex set. Let $\psi(\overline{v})$ be a formula of smallest Morley rank and degree such that $\{q \in \mathbb{Q} : \overline{P} \models \psi(\overline{s}_q)\}$ is infinite and convex. Choose r in the interior of this set. Let $\psi_0(\overline{v})$ be $\psi(\overline{v}) \wedge \varphi(\overline{v}\overline{t}_r)$ and let $\psi_1(\overline{v})$ be $\psi(\overline{v}) \wedge \neg\varphi(\overline{v}\overline{t}_r)$. Each set $\{q \in \mathbb{Q} : \overline{P} \models \psi_i(\overline{s}_q)$ is infinite and closed downwards. By Proposition 8.8, each ψ_i has Morley rank $\leq$ that of ψ; so $\mathrm{RM}^{\overline{P}}(\psi_i) = \mathrm{RM}^{\overline{P}}(\psi)$. But clearly both ψ_0 and ψ_1 have degree less than that of ψ, contradiction.

If p is an n-type over $A \subseteq M$ then we define $\mathrm{RM}(p)$ to be $\min\{\mathrm{RM}(\varphi) : \varphi \in p\}$. We let φ_p be a formula such that $\mathrm{RM}(p) = \mathrm{RM}(\varphi_p)$ and also with $\deg(\varphi_p)$ minimum among all formulas ψ such that $\mathrm{RM}(p) = \mathrm{RM}(\psi)$.

Lemma 8.14. *If $p, q \in S_n(A)$, $\mathrm{RM}(p), \mathrm{RM}(q) < \infty$, and $p \neq q$, then $\varphi_p \neq \varphi_q$.*

Proof. Let $\psi \in p \backslash q$. Then $\varphi_p \wedge \psi \in p$. So $\mathrm{RM}(\varphi_p \wedge \psi) \leq \mathrm{RM}(\varphi_p)$, so by the minimality of $\mathrm{RM}(\varphi_p)$ we get $\mathrm{RM}(\varphi_p \wedge \psi = \mathrm{RM}(\varphi_p)$. Similarly, $\mathrm{RM}(\varphi_q \wedge \neg\psi = \mathrm{RM}(\varphi_q)$. If $\varphi_p = \varphi_q$, then $\mathrm{RM}(\varphi_p \wedge \psi) = \mathrm{RM}(\varphi_p \wedge \neg\psi) = \mathrm{RM}(\varphi_p)$, and so $\deg(\varphi_p \wedge \psi) < \deg(\varphi_p)$, contradiction. □

Theorem 8.15. *If T is a theory in a countable language, then T is totally transcendental iff T is ω-stable.*

Proof. $\Rightarrow$: Assume that T is totally transcendental. Let $\overline{M}$ be a model of T, and let $A \subseteq M$ be countable. For each $p \in S_n(A)$ we have $\mathrm{RM}^{\overline{M}}(p) < \infty$, so φ_p exists. By Lemma 8.14 there are only countably many possible formulas φ_p, so $|S_n(A)| \leq \omega$.

$\Leftarrow$: Suppose that T is ω-stable but T is not totally transcendental. Thus there is an ω-saturated model $\overline{M}$ of T such that $\mathrm{RM}^{\overline{M}}(v = v) = \infty$. Let $\beta = \sup\{\mathrm{RM}(\psi) : \psi$ is an $\mathscr{L}_M$-formula and $\mathrm{RM}(\psi) < \infty\}$. We now define formulas φ_f for $f \in {}^{<\omega}2$. Let $\varphi_\emptyset$ be $v = v$. Suppose that φ_f has been defined so that $\mathrm{RM}(\varphi_f) = \infty$. Then there is a formula ψ such that $\mathrm{RM}(\varphi_f \wedge \psi) \geq \beta + 1$ and $\mathrm{RM}(\varphi_f \wedge \neg\psi) \geq \beta + 1$. Hence $\mathrm{RM}(\varphi_f \wedge \psi) = \infty$ and $\mathrm{RM}(\varphi_f \wedge \psi) = \infty$. let $\varphi_{f \frown \langle 0 \rangle}$ be $\varphi_f \wedge \psi$ and $\varphi_{f \frown \langle 1 \rangle}$ be $\varphi_f \wedge \neg\psi$. This completes the construction, and shows that T is not ω-stable, contradiction. □

9. Interpolation

Lemma 9.1. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are languages and $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\overline{B}$ is an $\mathscr{L}_1$-structure and $\overline{C}$ is an $\mathscr{L}_2$-structure. Suppose that $\overline{a}$ is a sequence of elements of B, $\overline{c}$ is a sequence of elements of C, and $(\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \equiv (\overline{C} \upharpoonright \mathscr{L}, \overline{c})$.*

Then there exist an elementary extension $\overline{D}$ of $\overline{B}$ and an elementary embedding g of $\overline{C} \upharpoonright \mathscr{L}$ into $\overline{D} \upharpoonright \mathscr{L}$ such that $g \circ \overline{c} = \overline{a}$.

Note that the sequences $\overline{a}$ and $\overline{c}$ can both be empty, or both infinite; but they are of the same length.

Proof. This proof is patterned after that of Proposition 8.6. Our first goal is to obtain an isomorphic copy $\overline{C}'$ of $\overline{C}$ so that $B \cap C' = \mathrm{rng}(\overline{a})$. Let Q be a set such that $Q \cap B = \emptyset$ and $|Q| = |C \backslash \mathrm{rng}(\overline{c})|$. Let f be a bijection from $C \backslash \mathrm{rng}(\overline{c})$ onto Q. Define $C' = \mathrm{rng}(\overline{a}) \cup Q$. Note that $B \cap C' = \mathrm{rng}(\overline{a})$. Define $f' : C \to C'$ by setting, for any $d \in C$,

$$f'(d) = \begin{cases} a_i & \text{if } d = c_i, \\ f(d) & \text{if } d \in C \backslash \mathrm{rng}(\overline{c}). \end{cases}$$

We now define a structure on C'. If R is an m-ary relation symbol of $\mathscr{L}_2$, let

$$R^{\overline{C}'} = \{d \in {}^mC' : (f')^{-1} \circ c \in R^{\overline{C}}\},$$

while if F is an m-ary function symbol of $\mathscr{L}_2$ and $d \in {}^mC'$ define

$$F^{\overline{C}'}(d) = f'(F^{\overline{C}}((f')^{-1} \circ d)).$$

Clearly f' is an isomorphism from $\overline{C}$ onto $\overline{C}'$. Moreover, $(\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \equiv (\overline{C}' \upharpoonright \mathscr{L}, \overline{a})$. In fact, for any sentence $\varphi(\overline{a})$ of $\mathscr{L}$ we have

$$\begin{aligned} (\overline{B} \upharpoonright \mathscr{L}, \overline{a}) \models \varphi(\overline{a}) \quad &\text{iff} \quad (\overline{C} \upharpoonright \mathscr{L}, \overline{c}) \models \varphi(\overline{c}) \\ &\text{iff} \quad (\overline{C}' \upharpoonright \mathscr{L}, \overline{a}) \models \varphi(\overline{a}). \end{aligned}$$

Now we claim that $\mathrm{Eldiag}(\overline{B}) \cup \mathrm{Eldiag}(\overline{C'} \upharpoonright \mathscr{L})$ has a model. Here the same constants are used in $\mathrm{Eldiag}(\overline{B})$ and $\mathrm{Eldiag}(\overline{C'} \upharpoonright \mathscr{L})$ for the members of $\mathrm{rng}(\overline{a})$. If not, by the compactness theorem some finite subset fails to have a model. Say Δ_0 is a finite subset of $\mathrm{Eldiag}(\overline{B})$ and Δ_1 is a finite subset of $\mathrm{Eldiag}(\overline{C}' \upharpoonright \mathscr{L})$ such that $\Delta_0 \cup \Delta_1$ does not have a model. Then $\bigwedge \Delta_1$ has the form $\psi(c_{a(i_0)}, \ldots c_{a(i_{n-1})}, c_{d(0)}, \ldots, c_{d(m-1)})$ with each $d(i)$ in $C' \backslash \mathrm{rng}(a)$. Thus

$$\Delta_0 \models \neg\psi(c_{a(i_0)}, \ldots c_{a(i_{n-1})}, c_{d(0)}, \ldots, c_{d(m-1)}).$$

Now the constants $c_{d(i)}$ do not occur in the formulas of Δ_0. Hence, replacing each $c_{b(i)}$ by a new variable w_i we get

$$\Delta_0 \models \forall \overline{w} \neg\psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

Since $\overline{B}_B$ is a model of Δ_0, we get

$$\overline{B}_B \models \forall\overline{w}\neg\psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}),$$

hence

$$\overline{B}_{\mathrm{rng}(a)} \models \forall\overline{w}\neg\psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

Hence by the above,

$$\overline{C}'_{\mathrm{rng}(a)} \models \forall\overline{w}\neg\psi(c_{a(i_0)}, \ldots, c_{a(i_{n-1})}, \overline{w}).$$

But this is impossible.

Hence $\mathrm{Eldiag}(\overline{B}) \cup \mathrm{Eldiag}(\overline{C'} \restriction \mathscr{L})$ has a model, say $(\overline{D}, h(b), k(c))_{b\in B, c\in C'}$, where $h(a_i) = k(a_i)$ for all i. By the elementary diagram lemma, h is an elementary embedding of $\overline{B}$ into $\overline{D}$ and k is an elementary embedding of $\overline{C}' \restriction \mathscr{L}$ into $\overline{D} \restriction \mathscr{L}$. Now let $\overline{D}'$ be an elementary extension of $\overline{B}$ and l an isomorphism of $\overline{D}$ with $\overline{D}'$ such that $l \circ h$ is the identity on B. Now $l \circ k \circ f'$ is an elementary embedding of $\overline{C} \restriction \mathscr{L}$ into $\overline{D}' \restriction \mathscr{L}$, and

$$l \circ k \circ f' \circ \overline{c} = l \circ k \circ \overline{a} = l \circ h \circ \overline{a} = \overline{a}. \qquad \square$$

Theorem 9.2. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are languages and $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\overline{B}$ is an $\mathscr{L}_1$-structure and $\overline{C}$ is an $\mathscr{L}_2$-structure. Suppose that $\overline{a}$ is a sequence of elements of $B \cap C$, and $(\overline{B} \restriction \mathscr{L}, \overline{a}) \equiv (\overline{C} \restriction \mathscr{L}, \overline{a})$.*

Then there exist an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure $\overline{D}$ and a function g such that $\overline{B} \preceq (\overline{D} \restriction \mathscr{L}_1)$ and g is an elementary embedding of $\overline{C}$ into $\overline{D} \restriction \mathscr{L}_2$ such that $g \circ \overline{a} = \overline{a}$.

Proof. Define $B_0 = B$ and $C_0 = C$. We apply Lemma 9.1 to get $B_0 \preceq B_1$ and an elementary embedding $g_0 : C \restriction \mathscr{L} \to B_1 \restriction \mathscr{L}$ such that $g_0 \circ \overline{a} = \overline{a}$. Let $\overline{c}$ enumerate C_0. Then we have $(C_0 \restriction \mathscr{L}, \overline{c}) \equiv (B_1 \restriction \mathscr{L}, g_0 \circ \overline{c})$, so we apply Lemma 9.1 to get $C_0 \preceq C_1$ and $h_1 : B_1 \restriction \mathscr{L} \to C_1 \restriction \mathscr{L}$ such that $h_1 \circ g_0 \circ \overline{c} = \overline{c}$. This means that $h_1 \circ g$ is the identity on C_0. Thus so far we have the following diagram:

$$\begin{tikzcd}
 & B_0 \arrow[rr, "\preceq"] & & B_1 \arrow[dr, "h_1"] & \\
\overline{a} \arrow[ur, "\subseteq"] \arrow[rr, "\subseteq"'] & & C_0 \arrow[ur, "g_0"] \arrow[rr, "\preceq"'] & & C_1
\end{tikzcd}$$

Now suppose that C_i, B_{i+1}, C_{i+1}, g_i, and h_{i+1} have been defined so that B_{i+1} is an $\mathscr{L}_1$-structure, $C_i \preceq C_{i+1}$ are $\mathscr{L}_2$-structures, $g_i : C_i \restriction \mathscr{L} \to B_{i+1} \restriction \mathscr{L}$ is an elementary embedding, $h_{i+1} : B_{i+1} \restriction \mathscr{L} \to C_{i+1} \restriction \mathscr{L}$ is an elementary embedding, and $h_{i+1} \circ g_i$ is the identity on C_i. Let $\overline{b}$ enumerate B_{i+1}. Then $(B_{i+1} \restriction \mathscr{L}, \overline{b}) \equiv (C_{i+1} \restriction \mathscr{L}, h_{i+1} \circ \overline{b})$. So we can apply Lemma 9.1 and get $B_{i+1} \preceq B_{i+2}$ and an elementary embedding g_{i+1} of $C_{i+1} \restriction \mathscr{L}$ into $B_{i+2} \restriction \mathscr{L}$ such that $g_{i+1} \circ h_{i+1} \circ \overline{b} = \overline{b}$. Thus $g_{i+1} \circ h_{i+1}$ is the identity on B_{i+1}. Then let $\overline{d}$ enumerate C_{i+1}. So we have $(C_{i+1} \restriction \mathscr{L}, \overline{d}) \equiv (B_{i+2} \restriction \mathscr{L}, g_{i+1} \circ \overline{d})$, so by Lemma 9.1 we get $C_{i+1} \preceq C_{i+2}$ and an elementary embedding h_{i+2} of $B_{i+2} \restriction \mathscr{L}$ into $C_{i+2} \restriction \mathscr{L}$ such that $h_{i+2} \circ g_{i+1}$ is the identity on C_{i+1}.

This completes the inductive definition; we have

$$\begin{tikzcd}
 & B_0 \arrow[rr, "\preceq"] & & B_1 \arrow[rr, "\preceq"] \arrow[dr, "h_1"] & & B_2 \arrow[rr, "\preceq"] \arrow[dr, "h_2"] & & B_3 & \cdots \\
\overline{a} \arrow[ur, "\subseteq"] \arrow[rr, "\subseteq"'] & & C_0 \arrow[ur, "g_0"] \arrow[rr, "\preceq"'] & & C_1 \arrow[ur, "g_1"] \arrow[rr, "\preceq"'] & & C_2 \arrow[ur, "g_2"] & \cdots
\end{tikzcd}$$

We claim that $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots$. For, if $c \in C_i$, then $g_{i+1}(c) = g_{i+1}(h_{i+1}(g_i(c))) = g_i(c)$. Also, $h_1 \subseteq h_2 \subseteq h_3 \subseteq \cdots$. For, if $b \in B_i$, then $h_{i+1}(b) = h_{i+1}(g_i(h_i(b))) = h_i(b)$.

Let $B_\omega = \bigcup_{i<\omega} B_i$ (an $\mathscr{L}_1$-structure) and $C_\omega = \bigcup_{i<\omega} C_i$ (an $\mathscr{L}_2$-structure). Let $k = \bigcup_{i<\omega} g_i$. Then k is an embedding of $C_\omega \restriction \mathscr{L}$ into $B_\omega \restriction \mathscr{L}$. Actually k is onto; for, given $b \in B_\omega$, say $b \in B_i$. Then $k(h_i(b)) = g_i(h_i(b) = b$. Thus k is an isomorphism of $C_\omega \restriction \mathscr{L}$ onto $B_\omega \restriction \mathscr{L}$. Now we expand B_ω to an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure D by defining, for any symbol S in $\mathscr{L}_2 \backslash \mathscr{L}_1$ $S^D = k(S^{C_\omega})$ (in the natural sense). Thus $B \preceq B_\omega = D \restriction \mathscr{L}_1$. We claim that $g_0 : C \to D \restriction \mathscr{L}_2$ is an elementary embedding. Since $C \preceq C_\omega$, it suffices to show that k is an isomorphism of C_ω onto $D \restriction \mathscr{L}_2$. This is clear by the definition above. $\square$

If $\mathscr{L}$ and $\mathscr{L}'$ are languages with $\mathscr{L} \subseteq \mathscr{L}'$, and T is a theory in $\mathscr{L}'$, then we denote by $T_\mathscr{L}$ the set of all sentences φ of $\mathscr{L}$ such that $T \models \varphi$.

Lemma 9.3. *Let $\mathscr{L}$ and $\mathscr{L}'$ be languages with $\mathscr{L} \subseteq \mathscr{L}'$, and let T be a theory in $\mathscr{L}'$. Let $\overline{M}$ be an $\mathscr{L}$-structure. Then $\overline{M} \models T_\mathscr{L}$ iff there is a model $\overline{N}$ of T such that $\overline{M} \preceq \overline{N} \restriction \mathscr{L}$.*

Proof. $\Rightarrow$: Assume that $\overline{M} \models T_\mathscr{L}$. It suffices to show that $S \stackrel{\text{def}}{=} T \cup \text{Eldiag}(\overline{M})$ has a model. To apply the compactness theorem, suppose that there is a finite subset of it with no model. This finite subset has the form $\Delta_0 \cup \Delta_1$ with Δ_0 a finite subset of T and Δ_1 a finite subset of $\text{Eldiag}(\overline{M})$. This yields $T \models \neg \bigwedge \Delta_1$. Replacing the diagram constants in Δ_1 by variables, we obtain a formula $\varphi(\overline{w})$ such that $T \models \forall \overline{w} \neg \varphi(\overline{w})$, with $\overline{M} \models \exists \overline{w} \varphi(\overline{w})$. Then $\forall \overline{w} \neg \varphi(\overline{w}) \in T_\mathscr{L}$, hence $\overline{M} \models \forall \overline{w} \neg \varphi(\overline{w})$, contradiction.

$\Leftarrow$: obvious. $\square$

Theorem 9.4. *Let $\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$ be languages with $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that T_1 and T_2 are theories in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, and $T_1 \cup T_2$ does not have a model. Then there is a sentence φ of $\mathscr{L}$ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$.*

Proof. By the compactness theorem it suffices to show that $(T_1)_\mathscr{L} \cup T_2$ does not have a model. Suppose that $\overline{M}$ is a model of $(T_1)_\mathscr{L} \cup T_2$. By Lemma 9.3 let $\overline{N}$ be a model of T_1 such that $\overline{M} \restriction \mathscr{L} \preceq \overline{N} \restriction \mathscr{L}$. Then $\overline{M} \restriction \mathscr{L} \equiv \overline{N} \restriction \mathscr{L}$, so by Theorem 9.2 there exist an $(\mathscr{L}_1 \cup \mathscr{L}_2)$-structure $\overline{D}$ and a function g such that $\overline{N} \preceq \overline{D} \restriction \mathscr{L}_1$ and g is an elementary embedding of $\overline{M}$ into $\overline{D} \restriction \mathscr{L}_2$. Since $\overline{N} \models T_1$ and $\overline{N} \preceq \overline{D} \restriction \mathscr{L}_1$, it follows that $\overline{D} \models T_1$. Since $\overline{M} \models T_2$ and g is an elementary embedding of $\overline{M}$ into $\overline{D} \restriction \mathscr{L}_2$, it follows that $\overline{D} \models T_2$. So $\overline{D}$ is a model of $T_1 \cup T_2$, contradiction. $\square$

Corollary 9.5. (Craig's interpolation theorem) *If φ and ψ are sentences and $\models \varphi \to \psi$, then there is a sentence χ such that $\models \varphi \to \chi$, $\models \chi \to \psi$, and the non-logical symbols that occur in χ occur in both φ and ψ.*

Proof. Assume that φ and ψ are sentences and $\models \varphi \to \psi$. Let $\mathscr{L}_1$ consist of all of the non-logical symbols occurring in φ, and let $\mathscr{L}_2$ consist of all of the non-logical symbols occurring in ψ. Let $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Let $T_1 = \{\theta : \theta \text{ is a sentence of } \mathscr{L}_1 \text{ and } \models \varphi \to \theta\}$ and let $T_2 = \{\theta : \theta \text{ is a sentence of } \mathscr{L}_2 \text{ and } \models \neg\psi \to \theta\}$. Then $T_1 \cup T_2$ does not have a model. Hence by Theorem 9.4 there is a sentence θ of $\mathscr{L}$ such that $T_1 \models \theta$ and $T_2 \models \neg\theta$. Hence $\varphi \to \theta$ and $\theta \to \psi$. □

Corollary 9.6. (Robinson's consistency theorem) *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be languages, and let T_1 and T_2 be theories in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, both of which have models. Let $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Suppose that $\{\varphi : \varphi$ is a sentence of $\mathscr{L}$ and $T_1 \models \varphi$ and $T_2 \models \varphi\}$ is a complete theory in $\mathscr{L}$. Then $T_1 \cup T_2$ has a model.*

Proof. Suppose not. Then by Theorem 9.4 there is a sentence φ of $\mathscr{L}$ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$. This contradicts the completeness of the above theory. □

Proposition 9.7. (Padoa's method) *Let $\mathscr{L}$ be a language and let S be a non-logical symbol of $\mathscr{L}$, and let T be a theory in $\mathscr{L}$. Suppose that $\overline{M}$ and $\overline{N}$ are models of T such that $\overline{M} \upharpoonright (\mathscr{L} - S) = \overline{N} \upharpoonright (\mathscr{L} - S)$ while $S^{\overline{M}} \neq S^{\overline{N}}$.*

Then S is not definable in $\mathscr{L}$ under T. That is, if S is an m-ary relation symbol then there does not exist a formula $\varphi(v_0, \ldots, v_{m-1})$ of $\mathscr{L} - S$ such that $T \models \forall\overline{v}[\varphi \leftrightarrow S\overline{v}]$; and similarly for function symbols and individual constants.

Proof. Assume the hypotheses, but suppose that such a formula φ exists. Then for $\overline{b}$ in M we have

$$\begin{aligned}
\overline{M} \models S\overline{b} \quad &\text{iff} \quad \overline{M} \models \varphi[\overline{b}] \\
&\text{iff} \quad \overline{M} \upharpoonright (\mathscr{L} - S) \models \varphi[\overline{b}] \\
&\text{iff} \quad \overline{N} \upharpoonright (\mathscr{L} - S) \models \varphi[\overline{b}] \\
&\text{iff} \quad \overline{N} \models \varphi[\overline{b}].
\end{aligned}$$

Thus $S^{\overline{M}} = S^{\overline{N}}$, contradiction. □

Theorem 9.8. *Let $\mathscr{L}$ and $\mathscr{L}^+$ be languages with $\mathscr{L} \subseteq \mathscr{L}^+$. Let T be a theory in $\mathscr{L}^+$ and $\varphi(\overline{x})$ a formula of $\mathscr{L}^+$. Then the following are equivalent:*

(i) If $\overline{A}$ and $\overline{B}$ are models of T and $\overline{A} \upharpoonright \mathscr{L} = \overline{B} \upharpoonright \mathscr{L}$, then for all tuples $\overline{a}$ in A, $\overline{A} \models \varphi[\overline{a}]$ iff $\overline{B} \models \varphi[\overline{a}]$.

(ii) There is a formula $\psi(\overline{x})$ of $\mathscr{L}$ such that $T \models \forall\overline{x}[\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})]$.

Proof. Clearly (ii)⇒(i). Now assume (i). Let

$$\Phi = T \cup \{\psi(\overline{c}) : \psi(\overline{x}) \text{ is a formula of } \mathscr{L} \text{ and } T \cup \varphi(\overline{c}) \models \psi(\overline{c})\}.$$

Clearly it suffices to show that any model $(\overline{A}, \overline{a})$ of Φ is a model of $\varphi(\overline{c})$.

(*) There is a model $(\overline{B}, \overline{a})$ of $T \cup \{\varphi(\overline{c})\}$ such that $(\overline{A} \restriction \mathscr{L}, \overline{a}) \preceq (\overline{B} \restriction \mathscr{L}, \overline{a})$.

To prove this, it suffices to show that the set

$$T \cup \mathrm{eldiag}(\overline{A} \restriction \mathscr{L}) \cup \{\varphi(\overline{c})\}$$

has a model, where in eldiag$(\overline{A} \restriction \mathscr{L})$ the tuple $\overline{c}$ corresponds to $\overline{a}$. Suppose not. Then we can write

$$T \vdash \varphi(\overline{c}) \to \forall \overline{y} \neg \psi(\overline{c}, \overline{y}),$$

where $(\overline{A} \restriction \mathscr{L}, \overline{a}) \models \exists \overline{y} \psi(\overline{c}, \overline{y})$ and ψ is an $\mathscr{L}$-formula. But this means that $\forall \overline{y} \neg \psi(\overline{c}, \overline{y})$ is in Φ, contradiction. Thus (*) holds.

Let $\mathscr{L}'$ be obtained from $\mathscr{L}^+$ by replacing each symbol S in $\mathscr{L}^+ \backslash \mathscr{L}$ by a new symbol S' of the same kind. Thus $\mathscr{L}' \cap \mathscr{L}^+ = \mathscr{L}$. With each $\mathscr{L}^+$-structure $\overline{C}$ let $\overline{C}'$ be the $\mathscr{L}'$-structure such that $S^{\overline{C}'} = S^{\overline{C}}$ if S is in $\mathscr{L}$, and $(S')^{\overline{C}'} = S^{\overline{C}}$ for S a symbol of $\mathscr{L}^+ \backslash \mathscr{L}$. With each formula φ of $\mathscr{L}^+$, let φ' be obtained from φ by a similar replacement. Clearly $\overline{C} \models \varphi(\overline{b})$ iff $\overline{C}' \models \varphi'(\overline{b})$ for any tuple $\overline{b}$.

Now we apply Theorem 9.2 to $\overline{A}'$ and $\overline{B}$. We obtain an $(\mathscr{L}' \cup \mathscr{L}^+)$-structure $\overline{D}$ such that $\overline{A}' \preceq \overline{D} \restriction \mathscr{L}'$ and an elementary embedding $g : \overline{B} \to \overline{D} \restriction \mathscr{L}^+$. We can write $\overline{D} \restriction \mathscr{L}' = \overline{E}'$ for some $\mathscr{L}^+$-structure $\overline{E}$. Then $\overline{D} \restriction \mathscr{L}^+$ and $\overline{E}$ are models of T and $\overline{D} \restriction \mathscr{L} = \overline{E} \restriction \mathscr{L}$. Since $\overline{B} \models \varphi(\overline{a})$, it follows from (i) that $\overline{E} \models \varphi(\overline{a})$. Clearly $\overline{A} \preceq \overline{E}$, so $\overline{A} \models \varphi(\overline{a})$, as desired. $\square$

Corollary 9.9. (Beth's definability theorem) *Let $\mathscr{L}$ and $\mathscr{L}^+$ be languages with $\mathscr{L} \subseteq \mathscr{L}^+$. Let T be a theory in $\mathscr{L}^+$ and S a nonlogical symbol of $\mathscr{L}^+$. Then the following are equivalent:*

(i) If $\overline{A}$ and $\overline{B}$ are models of T and $\overline{A} \restriction \mathscr{L} = \overline{B} \restriction \mathscr{L}$, then $S^{\overline{A}} = S^{\overline{B}}$.

(ii) There is a formula $\psi(\overline{x})$ of $\mathscr{L}$ such that $T \models \forall \overline{x}[S(\overline{x}) \leftrightarrow \psi(\overline{x})]$. $\square$

10. Countable models

A structure $\overline{M}$ is *atomic* iff $\mathrm{tp}^{\overline{M}}(\overline{a})$ is isolated, for all positive integers m and all $\overline{a} \in {}^nM$.

Theorem 10.1. *Let $\mathscr{L}$ be a countable language, and T a complete theory in $\mathscr{L}$ with infinite models. Let $\overline{M}$ be a model of T. Then $\overline{M}$ is prime iff it is countable and atomic.*

Proof. $\Rightarrow$: Assume that $\overline{M}$ is prime. If t is a type of T which is not isolated, then by the omitting types theorem 6.26, T has a model $\overline{N}$ which omits t. Since $\overline{M}$ can be elementarily embedded in $\overline{N}$, it follows that $\overline{M}$ also omits t. Thus for any type t of T, t isolated $\Rightarrow M$ omits t. Hence any type which $\overline{M}$ realizes is isolated; this means that $\overline{M}$ is atomic. since T has countable models by the downward Löwenheim-Skolem theorem, and $\overline{M}$ can be elementarily embedded in any model of T, $\overline{M}$ is countable.

$\Leftarrow$: Suppose that $\overline{M}$ is countable and atomic, and $\overline{N}$ is any model of T; we want to construct an elementary embedding of $\overline{M}$ into $\overline{N}$. Let $\langle a_i : i \in \omega\rangle$ enumerate the elements of M, and for each $i \in \omega$ let $\theta_i(\overline{v})$ isolate $\mathrm{tp}^{\overline{M}}(a_0, \ldots, a_i)$. We will now construct elementary maps $f_0 \subseteq f_1 \subseteq \ldots$ from subsets of M into N, where the domain of f_i is $\{a_0, \ldots, a_{i-1}\}$. Let $f_0 = \emptyset$. It is elementary since $\overline{M} \equiv \overline{N}$ (because T is complete). Suppose that f_i has been constructed, an elementary map. Then $\overline{M} \models \theta_i(a_0, \ldots, a_i)$, hence $\overline{M} \models \exists v\theta_i(a_0, \ldots, a_{i-1}, v)$. Since f_i is an elementary map, it follows that $\overline{N} \models \exists v\theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), v)$. Choose $b \in N$ such that $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), b)$, and let $f_{i+1} = f_i \cup \{(i, b)\}$. To see that f_{i+1} is elementary, suppose that $\overline{M} \models \psi(a_0, \ldots, a_i)$, Thus $\psi(\overline{v}) \in \mathrm{tp}^{\overline{M}}(a_0, \ldots, a_i)$, so $T \models \theta_i \to \psi$. Since $\overline{N} \models \theta_i(f_i(a_0), \ldots, f_i(a_{i-1}), b)$, it follows that $\overline{N} \models \psi(f_i(a_0), \ldots, f_i(a_{i-1}), b)$. Thus f_{i+1} is elementary.

Now $\bigcup_{i\in\omega} f_i$ is an elementary, as desired. □

Corollary 10.2. *If $\mathscr{L}$ is a countable language and T is a complete theory with infinite models, then T has a prime model iff T has an atomic model.*

Proof. $\Rightarrow$: by Theorem 10.1.

$\Leftarrow$: Suppose that $\overline{M}$ is an atomic model of T. Let $\overline{N}$ be a countable elementary substructure of $\overline{M}$. Then for any $n \in \omega$ and any $\overline{a} \in {}^nN$ we have $\mathrm{tp}^{\overline{N}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{a})$, and hence $\mathrm{tp}^{\overline{N}}(\overline{a})$ is isolated. So $\overline{N}$ is prime by Theorem 10.1. □

Proposition 10.3. *If $\overline{M}$ is an atomic model of T, then it is ω-homogeneous.*

Proof. See just before Lemma 7.28 for the definition of ω-homogeneous. Suppose that A is a finite subset of M and $f : A \to M$ is a partial elementary map. Let $\overline{a}$ enumerate A. Let $b \in M$. Let $\varphi(\overline{v}, w)$ isolate the type $\mathrm{tp}^{\overline{M}}(\overline{a}, b)$. Then $\overline{M} \models \exists w\varphi(\overline{a}, w)$. So, since f is partial elementary, we get $\overline{M} \models \exists w\varphi(f \circ \overline{a}, w)$ Choose $d \in M$ such that $\overline{M} \models \varphi(f \circ \overline{a}, d)$. Let $g = f \cup \{(b, d)\}$. To see that g is partial elementary, suppose that $\overline{M} \models \psi(\overline{a}, b)$. Then $T \models \varphi \to \psi$ and $\overline{M} \models \varphi(f \circ \overline{a}, d)$, so $\overline{M} \models \psi(f \circ \overline{a}, d)$, as desired. □

Corollary 10.4. *If T is a complete theory in a countable language, then any two prime models of T are isomorphic.*

Proof. Let $\overline{M}$ and $\overline{N}$ be prime models of T. Then by Theorem 10.1 they are both countable and atomic. By Proposition 10.3 they are also both ω-homogeneous. Now every type realized in $\overline{M}$ is isolated. Also, if t is an isolated type, say isolated by $\varphi(\overline{v})$, then $T \models \exists\overline{v}\varphi(\overline{v})$, hence $\overline{M}$ realizes t. So a type is isolated iff it is realized in $\overline{M}$. The same is true of $\overline{N}$, so $\overline{M}$ and $\overline{N}$ realize the same types. Hence they are isomorphic by Theorem 7.31. □

Theorem 10.5. *If $\overline{M}$ is κ-saturated, then $\overline{M}$ is κ-homogeneous.*

Proof. Suppose that $A \in [M]^{<\kappa}$, $f : A \to M$ is partial elementary, and $b \in M \backslash A$. Let

$$\Gamma = \{\varphi(v, f \circ \overline{a}) : \exists m \in \omega[\overline{a} \in {}^mA \text{ and } \overline{M} \models \varphi(b, \overline{a})]\}.$$

Let Δ be a finite subset of Γ. For each member χ of Δ, choose $m_\chi \in \omega$, φ_χ, and $\overline{a}_\chi \in {}^mA$ such that $\overline{M} \models \varphi_\chi(b, \overline{a}_\chi)$ and $\chi = \varphi_\chi(v, f \circ \overline{a}_\chi)$. Then there is an $n \in \omega$, $\overline{a}' \in {}^nA$, and a formula ψ such that the following conditions hold:

(1) $\overline{M} \models \psi(b, \overline{a}')$.

(2) $\models \bigwedge \Delta \leftrightarrow \psi(v, f \circ \overline{a}')$.

Now from (1) we get $\overline{M} \models \exists v\psi(v, \overline{a}')$. Hence, since f is elementary, $\overline{M} \models \exists v\psi(v, f \circ \overline{a}')$. Hence by (2), $\overline{M} \models \exists v \bigwedge \Delta$. Thus Γ is finitely satisfiable. So since $\overline{M}$ is ω-saturated we get $c \in M$ such that $\overline{M} \models \varphi(c, f \circ \overline{a})$ for each $\varphi(v, f \circ \overline{a}) \in \Gamma$. So $f \cup \{(b, c)\}$ is elementary. □

Theorem 10.6. *Suppose that $\overline{M}$ is a model of T. Then $\overline{M}$ is ω-saturated iff $\overline{M}$ is ω-homogeneous and for every $m \in \omega$, $\overline{M}$ realizes all types in $S_m(T)$.*

Proof. $\Rightarrow$: by Theorem 10.5.

$\Leftarrow$: Let $m, n \in \omega$, $\overline{a} \in {}^mM$, and $p \in S_n^{\overline{M}}(\overline{a})$. Let $q \in S_{n+m}(T)$ be the type $\{\varphi(\overline{v}, \overline{w}) : \varphi(\overline{v}, \overline{a}) \in p\}$. Since $\overline{M}$ realizes all types in $S_{m+n}(T)$, choose $(\overline{b}, \overline{c})$ realizing q. Now

$(*)$ $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{c})$.

In fact, let $\psi(\overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{a})$. Let ψ' be $\overline{v} = \overline{v} \wedge \psi$. Then $\psi'(\overline{v}, \overline{a}) \in p$, since otherwise $\neg\psi'(\overline{v}, \overline{a}) \in p$ and hence, since every finite subset of p is satisfiable in $\overline{M}$, we get $\overline{e}$ such that $\overline{M} \models \neg\psi'(\overline{e}, \overline{a})$, i.e., $\overline{M} \models \neg\psi(\overline{a})$, contradiction. So $\psi'(\overline{v}, \overline{a}) \in p$, hence $\psi'(\overline{v}, \overline{w}) \in q$, hence $\overline{M} \models \psi'(\overline{b}, \overline{c})$, so $\overline{M} \models \psi(\overline{c})$. This proves $(*)$.

Now by ω-homogeneity we get $\overline{d}$ such that $\mathrm{tp}^{\overline{M}}(\overline{b}, \overline{c}) = \mathrm{tp}^{\overline{M}}(\overline{d}, \overline{a})$. Thus for any $\varphi(\overline{v}, \overline{a}) \in p$ we have $\varphi(\overline{v}, \overline{w}) \in q$, hence $\overline{M} \models \psi(\overline{b}, \overline{c})$, so $\varphi(\overline{v}, \overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{b}, \overline{c})$, hence $\varphi(\overline{v}, \overline{w}) \in \mathrm{tp}^{\overline{M}}(\overline{d}, \overline{a})$, so $\overline{M} \models \varphi(\overline{d}, \overline{a})$. This shows that $\overline{d}$ realizes p. □

Corollary 10.7. *If $\overline{M}$ and $\overline{N}$ are countable saturated models of T, then $\overline{M} \cong \overline{N}$.*

Proof. By Theorem 10.6, both $\overline{M}$ and $\overline{N}$ are ω-homogeneous and realize all types in any $S_m(T)$. By Theorem 7.31 they are isomorphic. □

Now we need some variants of 7.38–7.39.

Lemma 10.8. *Suppose that $\overline{M}$ is a model of T and $\overline{a}, \overline{b}, c \in M$, $\mathrm{tp}^{\overline{M}}(\overline{a}) = \mathrm{tp}^{\overline{M}}(\overline{b})$. Then there exist an elementary extension $\overline{N}$ of $\overline{M}$ such that $|M| = |N|$ and an element $d \in N$ such that $\mathrm{tp}^{\overline{N}}(\overline{a}, c) = \mathrm{tp}^{\overline{N}}(\overline{b}, d)$.*

Proof. Apply the compactness theorem to the set

$$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{b}, u) : \overline{M} \models \varphi(\overline{a}, c)\} \text{ (u a new constant)} \qquad \square$$

Theorem 10.9. *Suppose that $\overline{M}$ is a model of T. Then there exists an elementary extension $\overline{N}$ of $\overline{M}$ such that $|M| = |N|$ and for all $\overline{a}, \overline{b}, c \in M$ there is a $d \in N$ such that $\mathrm{tp}^{\overline{N}}(\overline{a}, c) = \mathrm{tp}^{\overline{N}}(\overline{b}, d)$.*

Proof. Iterate Lemma 10.8. $\square$

Theorem 10.10. *Suppose that $\overline{M}$ is a model of T. Then there is an elementary extension $\overline{N}$ of $\overline{M}$ such that $\overline{N}$ is ω-homogeneous and $|M| = |N|$.*

Proof. Iterate Lemma 10.9 ω times. $\square$

Theorem 10.11. *T has a countable saturated model iff $|S_n(T)| \leq \aleph_0$ for all n.*

Proof. $\Rightarrow$: If $\overline{M}$ is a countable saturated model, then it realizes only countably many types; but it realizes all types, so $|S_n(T)| \leq \aleph_0$ for all n.

$\Leftarrow$: Let $t_0, t_1, \ldots$ list all members of $\bigcup_{n\in\omega} S_n(T)$. By the compactness theorem, for any countable model $\overline{N}$ of T and any $i \in \omega$ there is a countable elementary extension $\overline{P}$ of $\overline{N}$ with an element which realizes t_i. So if we start with $\overline{M}$ and iterate this process ω times we obtain an elementary chain $\overline{M} = \overline{N}_0 \preceq \overline{N}_1 \preceq \cdots$ such that each $\overline{N}_i$ is countable and $\overline{N}_{i+1}$ has an element realizing t_i. Let $\overline{P} = \bigcup_{i\in\omega} \overline{N}_i$. Then $\overline{P}$ is an elementary extension of $\overline{M}$ which realizes every type over T. By Theorem 10.10 let $\overline{Q}$ be a countable elementary extension of $\overline{P}$ which is ω-homogeneous. By Theorem 10.6, $\overline{Q}$ is ω-saturated. $\square$

Theorem 10.12. *If $\mathscr{L}$ is countable, T is a theory in $\mathscr{L}$, and $|S_n(T)| \leq \omega$ for all $n \in \omega$, then T has a countable atomic model.*

Proof. By Theorem 6.26 (the omitting types theorem), let $\overline{M}$ be a countable model of T which omits all non-isolated types. Thus $\overline{M}$ is atomic. $\square$

Corollary 10.13. *If $\mathscr{L}$ is countable and T is a theory in $\mathscr{L}$ which has an ω-saturated model, then T has a countable atomic model.*

Proof. By Theorems 10.11 and 10.12. $\square$

Theorem 10.14. *For T a theory in a countable language the following are equivalent:*

(i) T is $\aleph_0$-categorical.

(ii) For every $n < \omega$, every type in $S_n(T)$ is isolated.

(iii) $|S_n(T)| < \aleph_0$ for every $n \in \omega$.

(iv) For every $n \in \omega$ there is a finite set Γ of formulas with free variables among $v_0, \ldots, v_{n-1}$ such that for every formula φ with free variables among $v_0, \ldots, v_{n-1}$ there is a $\psi \in \Gamma$ such that $T \models \varphi \leftrightarrow \psi$.

Proof. (i)⇒(ii): Suppose that (ii) fails: there exist $n < \omega$ and a type $p \in S_n(T)$ which is not isolated. By the omitting types theorem 6.26, there is a countable model $\overline{M}$ of T which omits p. But clearly there is also a countable model $\overline{N}$ which admits p. Thus $\overline{M} \not\cong \overline{N}$, so (i) fails.

(ii)⇒(iii): Assume (ii), but suppose that (iii) fails: there is an $n \in \omega$ such that $S_n(T)$ is infinite. For each $p \in S_n(T)$ let φ_p isolate p. Let $\overline{c}$ be a sequence of new constants of length n, and consider the set

$$T' \stackrel{\text{def}}{=} T \cup \{\neg\varphi_p(\overline{c}) : p \in S_n(T)\}.$$

We claim that T' has a model. For, take any finite subset T'' of T. Let P be the set of all types p such that $\neg\varphi_p(\overline{c})$ is in T''. Let $q \in S_n(T)$ be different from each member of P. Now $T \models \varphi_q \to \neg\varphi_p$ for each $p \in P$; otherwise $T \models \varphi_q \to \varphi_p$ for some $p \in P$ and then $T \models \varphi_q \to \psi$ for each $\psi \in p$, so that $p \subseteq q$, hence $p = q$, contradiction. Now take a model $\overline{M}$ of T which realizes q, say $\overline{M} \models \varphi_q(\overline{a})$. Interpreting $\overline{c}$ by $\overline{a}$ in this model gives a model of T'', as desired.

Thus T' has a model, which gives a model $\overline{N}$ of T with a sequence $\overline{b}$ satisfying in $\overline{N}$ each formula $\neg\varphi_p$. Thus $\mathrm{tp}^{\overline{N}}(\overline{b})$ cannot be in $S_n(T)$, contradiction.

(iii)⇒(iv): Assume (iii), and suppose that $n \in \omega$. For distinct $p, q \in S_n(T)$ choose $\varphi_{pq} \in p \backslash q$. For each $p \in S_n(T)$ let $\psi_p = \bigwedge\{\varphi_{pq} : q \neq p\}$. Thus $\psi_p \in p$ while $\psi_p \notin q$ for all $q \neq p$. Now given χ with variables among $v_0, \ldots, v_{n-1}$ we have $T \models \chi \leftrightarrow \bigvee\{\psi_p : \chi \in p\}$.

iv)⇒(i): Assume (iv). Let $\overline{M}$ be a countable model of T; we show that $\overline{M}$ is atomic; so T is $\aleph_0$-categorical by Theorems 10.1 and 10.4. If $\overline{a} \in {}^nM$, then $\mathrm{tp}^{\overline{M}}(\overline{a})$ is isolated by

$$\bigwedge\{\varphi_i(\overline{v}) : \overline{M} \models \varphi_i(\overline{a})\} \wedge \bigwedge\{\neg\varphi_i(\overline{v}) : \overline{M} \models \neg\varphi_i(\overline{a})\} \qquad \square$$

11. The number of types and models

Let $\overline{M}$ be a structure and $A \subseteq M$. For $\varphi(\overline{v})$ a formula of $\mathscr{L}_A$ with free variables $\overline{v}$ of length n we define

$$[\varphi]_A^{\overline{M}} = \{p \in S^{\overline{M}}(A) : \varphi \in p\}.$$

Theorem 11.1. *Let $\overline{M}$ be a structure and $A \subseteq M$. Suppose that Γ is a set of formulas of $\mathscr{L}_A$ with free variables $\overline{v}$ of length n, and $S^{\overline{M}}(A) = \bigcup_{\varphi \in \Gamma}[\varphi]_A^{\overline{M}}$. Then there is a finite subset Γ' of Γ such that $S^{\overline{M}}(A) = \bigcup_{\varphi \in \Gamma'}[\varphi]_A^{\overline{M}}$.*

Proof. Suppose not. Let $\overline{c}$ be a system of new constants, and let Δ be the following set of sentences of $\mathscr{L}_A$:

$$\{\psi : \overline{M}_A \models \psi\} \cup \{\neg\varphi(\overline{c}) : \varphi(\overline{v}) \in \Gamma\}.$$

We claim that every finite subset of Δ has a model. For, let Δ' be a finite subset of Δ. Then we can write $\Delta' = \Delta'' \cup \Delta'''$, where Δ'' is a finite subset of $\{\psi : \overline{M}_A \models \psi\}$ and $\Delta''' = \{\neg\varphi(\overline{c}) : \varphi(\overline{c}) \in \Gamma'\}$ for some finite subset Γ' of Γ. By the "suppose not" assumption, there is a type $p \in S^{\overline{M}}(A) \backslash \bigcup_{\varphi \in \Gamma'}[\varphi]_A^{\overline{M}}$. Let $\overline{N}$ be an elementary extension of $\overline{M}$ having a sequence $\overline{a}$ of elements which realizes p. Then $\overline{N}_A$ together with $\overline{a}$ corresponding to $\overline{c}$ provides a model of Δ'.

Thus by the compactness theorem, Δ has a model $(\overline{P}, \overline{b})$. Let $p = \mathrm{tp}^{\overline{P}}(\overline{b})$, and choose $\varphi \in \Gamma$ such that $p \in [\varphi]_A^{\overline{M}}$. Thus $\varphi(\overline{v}) \in p$, so $\overline{P} \models \varphi(\overline{b})$, which contradicts $\overline{P}$ being a model of Γ. $\square$

Theorem 11.2. *Suppose that $\mathscr{L}$ is countable, $\overline{M}$ is an $\mathscr{L}$-structure, and $A \subseteq M$ is countable. Assume that $|S_n^{\overline{M}}(A)| > \aleph_0$. Then $|S_n^{\overline{M}}(A)| = 2^{\aleph_0}$.*

Proof. Assume the hypotheses. Since $\mathscr{L}$ and A are countable, $|S^{\overline{M}}(A)| \leq 2^{\aleph_0}$. So it suffices to find $2^{\aleph_0}$ many types over A.

(*) If $\varphi(\overline{v})$ is a formula of $\mathscr{L}_A$ with $\overline{v}$ a string of variables of length n, and $|[\varphi]_A^{\overline{M}}| > \aleph_0$, then there is an $\mathscr{L}_A$-formula $\psi(\overline{v})$ such that $|[\varphi \wedge \psi]_A^{\overline{M}}| > \aleph_0$ and $|[\varphi \wedge \neg\psi]_A^{\overline{M}}| > \aleph_0$.

In fact, suppose not. Let $p = \{\psi(\overline{v}) : |[\varphi \wedge \psi]_A^{\overline{M}}| > \aleph_0$. Clearly for every formula $\psi(\overline{v}$, either $\psi(\overline{v}) \in p$ or $\neg\psi(\overline{v}) \in p$, but not both. We claim that p is finitely satisfiable. For, let $\psi_0, \ldots, \psi_{m-1} \in p$. If $\neg(\psi_0 \wedge \ldots \wedge \psi_{m-1}) \in p$, then

$$|[\neg\psi_0]_A^{\overline{M}} \cup \ldots \cup [\neg\psi_{m-1}]_A^{\overline{M}}| > \aleph_0,$$

and hence $|\neg\psi_i]_A^{\overline{M}}| > \aleph_0$ for some i, so that $\neg\psi_i \in p$, contradiction. It follows that $(\psi_0 \wedge \ldots \wedge \psi_{m-1}) \in p$, so that

$$|[\varphi \wedge \psi_0 \wedge \ldots \wedge \psi_{m-1}]_A^{\overline{M}}| > \aleph_0,$$

in particular $[\varphi\wedge\psi_0\wedge\ldots\wedge\psi_{m-1}]^{\overline{M}}_A\neq\emptyset$, so there is a type $q\in[\varphi\wedge\psi_0\wedge\ldots\wedge\psi_{m-1}]^{\overline{M}}_A$, so $\varphi\wedge\psi_0\wedge\ldots\wedge\psi_{m-1}\in q$, so that $\varphi\wedge\psi_0\wedge\ldots\wedge\psi_{m-1}$ is satisfiable. So, we have proved that p is finitely satisfiable. So $p\in S^{\overline{M}}_n(A)$. If $\psi\notin p$, then $|[\varphi\wedge\psi]^{\overline{M}}_A|\le\aleph_0$. Now

$$[\varphi]^{\overline{M}}_A=\bigcup_{\psi\notin p}[\varphi\wedge\psi]^{\overline{M}}_A\cup\{p\},$$

so $[\varphi]^{\overline{M}}_A$ is countable, contradiction. Thus $(*)$ holds.

Now $S^{\overline{M}}(A)=\bigcup\{[\varphi]^{\overline{M}}_A:\varphi(\overline{v})$ a formula with the string $\overline{v}$ of n variables$\}$, and there are only countably many formulas, so there is a formula $\varphi(\overline{v})$ such that $|[\varphi]^{\overline{M}}_A|>\aleph_0$. Let $\varphi_\emptyset$ be such a formula. Now suppose that φ_σ has been defined, where $\sigma\in{}^{<\omega}2$, so that $|[\varphi_\sigma]^{\overline{M}}_A|>\aleph_0$. By $(*)$ there is a formula ψ such that $|[\varphi_\sigma\wedge\psi]^{\overline{M}}_A|>\aleph_0$ and $|[\varphi_\sigma\wedge\neg\psi]^{\overline{M}}_A|>\aleph_0$. Let $\varphi_{\sigma\frown\langle 1\rangle}$ be $\varphi_\sigma\wedge\psi$ and $\varphi_{\sigma\frown\langle 0\rangle}$ be $\varphi_\sigma\wedge\neg\psi$. This gives $2^{\aleph_0}$ types. $\square$

Recall from Chapter 7 that $I(T,\kappa)$ is the number of non-isomorphic models of T of size κ. We want to describe completely what is known about $I(T,\aleph_0)$:

(1) There is a complete theory T with $I(T,\aleph_0)=1$. This means that T is ω-categorical. A simple example of such a theory is the theory of dense linear orders with no endpoints.

(2) There is no theory T with $I(T,\aleph_0)=2$. This theorem, due to Vaught, is proved below.

(3) For $3\le\kappa\le\omega$ there is a complete theory T with $I(T,\aleph_0)=\kappa$. We give examples below.

(4) There is a theory T with $I(T,\aleph_0)=2^{\aleph_0}$. We give an example below.

(5) Vaught's conjecture is that (in ZFC) there is no theory T such that $\aleph_0<I(T,\aleph_0)<2^{\aleph_0}$.

(6) (A theorem of Morley) If $I(T,\aleph_0)>\aleph_1$, then $I(T,\aleph_0)=2^{\aleph_0}$. We will not give a proof of this; see Marker's book.

Lemma 11.3. *If $\overline{M}$ is ω-saturated and A is a finite subset of M, then $\overline{M}_A$ is ω-saturated.*

Proof. Assume the hypotheses, and suppose that B is a finite subset of M. Suppose that Γ is a set of formulas of $\mathscr{L}_{A\cup B}$ with only v free, and for every finite subset Γ' of Γ there is a $b\in M$ such that $\overline{M}_{A\cup B}\models\bigwedge\Gamma'(b)$. Since $A\cup B$ is finite, it follows that there is a $b\in M$ such that $\overline{M}\models\varphi(b)$ for every $\varphi\in\Gamma$. $\square$

Theorem 11.4. *If T has fewer than $2^{\aleph_0}$ countable models up to isomorphism, then T has a countable saturated and a countable prime model.*

Proof. By 10.1, 10.11, and 10.13 it suffices to show that $S_n(T)$ is countable for all $n\in\omega$. Suppose that $|S_n(T)|>\aleph_0$ for some n. By Theorem 11.2, $|S_n(T)|=2^{\aleph_0}$. Each type is realized in a countable model, and a countable model realizes at most countably many types. Hence there must be $2^{\aleph_0}$ countable models up to isomorphism. $\square$

Theorem 11.5. (Vaught) $I(T,\aleph_0)\neq 2$.

Proof. Suppose that $I(T, \aleph_0) = 2$. By Theorem 11.4, T has a countable saturated model $\overline{M}$ and a countable atomic model $\overline{N}$. Since T is not $\aleph_0$-categorical, by Theorem 10.14 there exist an $n \in \omega$ and a type $p \in S_n(T)$ such that p is not isolated. Since $\overline{M}$ is saturated, p is realized in $\overline{M}$. By Theorem 10.1 and the definition of atomic, p is not realized in $\overline{N}$. Hence $\overline{M}$ is not isomorphic to $\overline{N}$.

Let $\overline{a}$ in M realize p. Let $T^* = \{\varphi : \varphi$ is a sentence of $\mathscr{L}_{\overline{a}}$ and $\overline{M}_{\overline{a}} \models \varphi\}$. Now since T is not $\aleph_0$-categorical, by Theorem 10.14 we have

$$\begin{aligned}\exists m \in \omega[&\text{for every finite set } \Gamma \text{ of formulas with free variables among } v_0, \ldots, v_{m-1}\\ &\text{there is a formula } \varphi \text{ with free variables among } v_0, \ldots, v_{m-1}\\ &\text{such that for every } \psi \in \Gamma,\ T \not\models \varphi \leftrightarrow \psi\end{aligned}$$

Thus there is an $m \in \omega$ such that there is an infinite set Γ of formulas with free variables among $v_0, \ldots, v_{m-1}$ such that for any two distinct $\varphi, \psi \in \Gamma$ we have $T \not\models \varphi \leftrightarrow \psi$. Hence also for any two distinct $\varphi, \psi \in \Gamma$ we have $T^* \not\models \varphi \leftrightarrow \psi$. It follows that T^* is not $\aleph_0$-categorical. Now by Lemma 11.3, $\overline{M}_{\overline{a}}$ is a saturated $\overline{a}$-structure. Hence it has a countable atomic model $(\overline{P}, \overline{b})$. Now $\overline{b}$ realizes p in $\overline{P}$. Hence $\overline{P}$ is not isomorphic to $\overline{N}$. Also, since T^* is not $\aleph_0$-categorical, there is a non-isolated $\mathscr{L}_{\overline{a}}$ type q. This type is not realized in $(\overline{P}, \overline{b})$, so $(\overline{P}, \overline{b})$ is not saturated. Hence by Lemma 11.3, also $\overline{P}$ is not saturated. So $\overline{M}$ is not isomorphic to $\overline{P}$. Thus $\overline{M}, \overline{N}, \overline{P}$ are pairwise non-isomorphic, so $I(T, \aleph_0) \geq 3$, contradiction. □

Now suppose that $3 \leq n \in \omega$. We describe a theory T_n which has exactly n non-isomorphic models of size $\aleph_0$. For $n = 3$ this example is due to Ehrenfeucht; for $n > 3$ to Vaught. The language has a binary relation symbol $<$, one-place relation symbols U_i for $i < n - 2$, and individual constants $c_0, c_1, \ldots$. T_n consists of all consequences of the following sentences:

(1) A sentence saying that $<$ is a dense linear order with no endpoints.

(2) Sentences $c_i < c_j$ for all $i < j$.

(3) Sentences $U_0 c_i$ for all $i < \omega$.

(4) A sentence saying that $\langle U_i : i < n - 2 \rangle$ is a partition.

(5) Sentences saying that each set U_i is dense; i.e. the sentences $\forall x \forall y[x < y \rightarrow \exists z[U_i z \wedge x < z \wedge z < y]$.

Note that for $n = 3$ there is just one set U_i, namely U_0, and then (4) says that U_0 is the whole universe, and (5) is redundant.

To see that T_n has a model we need the following set-theoretical result.

Theorem 11.6. *If L is a dense linear order without endpoints, then L is the disjoint union of two dense subsets.*

Proof. Let $\langle a_\alpha : \alpha < \kappa \rangle$ be a well-order of L, with $\kappa = |L|$. We put each a_α in A or B by recursion, as follows. Suppose that we have already done this for all $\beta < \alpha$. Let $C = \{a_\beta : \beta < \alpha$ and $a_\beta < a_\alpha\}$, and let $D = \{a_\beta : \beta < \alpha$ and $a_\beta > a_\alpha\}$. We take two possibilities.

Case 1. C has a largest element a_β, D has a smallest element a_γ, and $a_\beta, a_\gamma \in A$. Then we put a_α in B.

Case 2. Otherwise, we put a_α in A.

Now we want to see that this works. So, suppose that elements $a_\xi < a_\eta$ of L are given. Let $a_\beta < a_\gamma$ be the elements of L with smallest indices which are in the interval (a_ξ, a_η). If one of these is in A and the other in B, this gives elements of A and B in (a_ξ, a_η). So, suppose that they are both in A, or both in B. Let a_ν be the member of L with smallest index that is in (a_β, a_γ). Thus $a_\xi < a_\beta < a_\nu < a_\gamma < a_\eta$, so by the minimality of β and γ we have $\beta, \gamma < \nu$. Thus $\beta < \nu$ and $a_\beta < a_\nu$.

(1) a_β is the largest element of $\{a_\rho : \rho < \nu,\, a_\rho < a_\nu\}$.

In fact, a_β is in this set, as just observed. If $a_\beta < a_\rho$, $\rho < \nu$, and $a_\rho < a_\nu$, then also $a_\rho < a_\gamma$ since $a_\nu < a_\gamma$, so the definition of ν is contradicted. Hence (1) holds.

(2) a_γ is the smallest element of $\{a_\rho : \rho < \nu,\, a_\rho > a_\nu\}$.

In fact, $\gamma < \nu$ as observed just before (1), and $a_\gamma > a_\nu$ by the definition of a_ν. If $a_\rho < a_\gamma$, $\rho < \nu$, and $a_\rho > a_\nu$, then also $a_\rho > a_\beta$ since $a_\nu > a_\beta$, so the definition of ν is contradicted. Hence (2) holds.

So by construction, if $a_\beta, a_\gamma \in A$ then $a_\nu \in B$, while if $a_\beta, a_\gamma \in B$, then $a_\nu \in A$. So again we have found elements of both A and B which are in (a_ξ, a_η). $\square$

By this theorem it is clear how to construct models of the theories T_n.

Now suppose that $3 \le n < \omega$. We describe n models of T_n of size $\aleph_0$. For each model we take the underlying set to be $\mathbb{Q}$, and form U'_i for $i < n-2$ using Theorem 11.6.

First model: $\langle c'_i : i < \omega\rangle$ is a strictly increasing sequence of elements of U'_0 which is cofinal in $\mathbb{Q}$.

Second model: $\langle c'_i : i < \omega\rangle$ is a strictly increasing sequence of elements of U'_0 which is not cofinal in $\mathbb{Q}$, and has no limit in $\mathbb{Q}$.

j-th model, $3 \le j \le n$: $\langle c'_i : i < \omega\rangle$ is a strictly increasing sequence of elements of U'_0 which has a limit in U'_{j-3}.

Clearly these are models of T_n, and they are pairwise non-isomorphic. Now let $\overline{M}$ be any denumerable model of T_n. For each j with $3 \le j \le n$ let f_j be an order isomorphism of $U^{\overline{M}}_{j-3}$ onto U'_{j-3}. Let $g = \bigcup_{3\le j\le n} f_j$. Then g is an isomorphism of $\overline{M}$ onto the above model of T_n, taking $c^{\overline{M}}_i$ to c'_i. The exact model is given by whether $\langle c^{\overline{M}}_i : i < \omega\rangle$ is cofinal, has no limit, or has a limit in some $U^{\overline{M}}_j$.

To show that T_n is complete one can use elimination of quantifiers. In fact, starting from a formula $\exists x\varphi$ with φ quantifier free, one can use the following members of T_n to simplify φ:

$\forall x\forall y[\neg(x = y) \leftrightarrow x < y \vee y < x]$.
$\forall x\forall y[\neg(x < y) \leftrightarrow x = y \vee y < x]$.
$\forall x[\neg U_i x \leftrightarrow \bigvee_{j\ne i} U_j x]$.

Using standard procedures with these formulas, one can assume that φ is a conjunction of atomic formulas, with no conjunct of the form $x = \tau$; so one can assume that φ has the following form:

$$\begin{aligned}&y_1 < x \wedge \ldots \wedge y_m < x \wedge x < z_1 \wedge \ldots \wedge x < z_n\\ &\wedge c_{i(0)} < x \wedge \ldots \wedge c_{i(p)} < x \wedge x < c_{j(0)} \wedge \ldots \wedge x < c_{j(q)} \wedge U_k x\end{aligned}$$

Then $\exists x\varphi$ is equivalent under T_n to the following quantifier free formula: Each y_s and each $c_{i(s)}$ are less than each z_t and each $c_{j(t)}$.

Then the quantifier free form of a sentence is equivalent to $c_0 = c_0$ or $\neg c_0 = c_0$, giving completeness.

A complete theory with exactly $\aleph_0$ denumerable models is the theory of algebraically closed fields of characteristic 0. The denumerable models are given by the transcendence degree; a natural number, or ω. The theory is complete since it is $\aleph_1$-categorical.

For a theory with exactly $2^{\aleph_0}$ denumerable models we take the language of orderings. First we describe $2^{\aleph_0}$ non-isomorphic denumerable linear orderings, and then we modify them to be models of a certain complete theory of linear orderings. The construction depends on infinite sums of orderings, and we go into that first. Let α be an ordinal, and for each $\xi < \alpha$ let $(L_\xi, <_\xi)$ be a linear order. We define a new structure

$$\left(\sum_{\xi<\alpha} L_\xi, <\right).$$

Namely,

$$\sum_{\xi<\alpha} L_\xi = \{(x,\xi) : \xi < \alpha,\ x \in L_\xi\},$$

and $(x,\xi) < (y,\eta)$ iff $\xi < \eta$, or $\xi = \eta$ and $x <_\xi y$. We show that this gives a linear order. Clearly $<$ is irreflexive. Now suppose that $(x,\xi) < (y,\eta) < (z,\rho)$. Clearly then $\xi \le \eta \le \rho$. If $\xi < \eta$ or $\eta < \rho$, then $\xi < \rho$ and hence $(x,\xi) < (z,\rho)$. Assume that $\xi = \eta = \rho$. Then $x <_\xi y <_\xi z$, so $x <_\xi z$ and hence $(x,\xi) < (\zeta,\rho)$. Thus $<$ is transitive.

Next, let $(x,\xi),(y,\eta) \in \sum_{\eta<\alpha} L_\eta$. If $\xi < \eta$, then $(x,\xi) < (y,\eta)$, and if $\eta < \xi$ then $(y,\eta) < (x,\xi)$. Suppose that $\xi = \eta$. Then $(x,\xi) < (y,\eta)$ if $x <_\xi y$, $(y,\eta) < (x,\xi)$ if $y <_\xi x$, and $(x,\xi) = (y,\eta)$ if $\xi = \eta$.

Now we give the construction of models of T. Let L_0 be a linear order similar to $\omega^* + \omega + 1$; specifically, let it consist of a copy of $\mathbb{Z}$ followed by one element a greater than every integer, and let L_1 be a linear order similar to $\omega^* + \omega + 2$; say it consists of a copy of $\mathbb{Z}$ followed by two elements $a < b$ greater than every integer. For any $f \in {}^\omega 2$ let

$$M_f = \sum_{\alpha<\omega} L_{f(\alpha)}.$$

We show that if $f, g \in {}^\omega 2$ then M_f and M_g are not isomorphic. Assume that M_f is isomorphic to M_g and show that $f = g$. Let F be an isomorphism of M_f onto M_g. Clearly

(1) For all $x \in M_f$ and all $\xi < \omega$ the following conditions are equivalent:

(a) x does not have an immediate predecessor, and there are exactly ξ elements less than x which do not have an immediate predecessor.

(b) $x = (a, \xi)$.

A similar statement holds for M_g.

Now take any $\xi < \omega$. Then by (1),

$$M_f \models (v_0 \text{ does not have an immediate predecessor, and there are exactly } \sigma \text{ elements less than } v_0 \text{ which do not have an immediate predecessor})[(a, \xi)].$$

It follows that

$$M_g \models (v_0 \text{ does not have an immediate predecessor, and there are exactly } \sigma \text{ elements less than } v_0 \text{ which do not have an immediate predecessor})[F(a, \xi)].$$

Hence by (1) for M_g we have $F(a, \xi) = (a, \xi)$ for all $\xi < \omega$.

Next we claim

(2) For any $x \in M_f$ the following conditions are equivalent:

(a) x does not have an immediate predecessor, but it has an immediate successor y which in turn does not have an immediate successor.

(b) $x = (a, \xi)$ for some ξ such that $f(\xi) = 1$.

This is obvious, and a similar condition for M_g holds.

Now the property given in (2)(a) is preserved under isomorphisms, so by the above, for any $\xi < \omega$,

$$\begin{aligned} f(\xi) = 1 \quad &\text{iff} \quad (a, \xi) \text{ satisfies (2)(a)} \\ &\text{iff} \quad F(a, \xi) \text{ satisfies (2)(a)} \\ &\text{iff} \quad g(\xi) = 1. \end{aligned}$$

Thus $f = g$, as desired.

As we said, we want to modify these models so that they will be models of a complete theory. The complete theory we want is the theory T consisting of all consequences of the following sentences:

A sentence saying that $<$ is a linear order.
A sentence saying that every element has an immediate successor and also an immediate predecessor.

To show that T is complete we will use an Ehrenfeucht game, specifically Corollary 6.5. Suppose that $\overline{A}$ and $\overline{B}$ are models of T, and m is a positive integer. We want to describe a winning strategy for ISO.

If $a < b$ in $\overline{A}$, we write (a, b) for the interval $\{x : a < x < b\}$. Similarly for $\overline{B}$. The strategy of ISO is this: at the first move, she plays any member of B if NON-ISO chose a member of A, and she plays any member of A if NON-ISO chose a member of B. After the n-th play, $n < m$, elements $a_0, \ldots, a_{n-1}$ in A and $b_0, \ldots, b_{n-1}$ in B have been chosen. By symmetry we describe how ISO plays if NON-ISO chooses a member a_n of A. We now consider two cases.

Case 1. $\{(a_i, b_i) : i < n\}$ is a partial isomorphism, and if $a_{i(0)}, \ldots, a_{i(p)}$ are the distinct elements of $\{a_i : i < n\}$, with $a_{i(0)} < \cdots < a_{i(p)}$, and if $j < p$, then $|(a_{i(j)}, a_{i(j+1)})| < 3^{m-n}$ implies that $|(b_{i(j)}, b_{i(j+1)})| = |(a_{i(j)}, a_{i(j+1)})|$, while $|(a_{i(j)}, a_{i(j+1)})| \geq 3^{m-n}$ implies that $|(b_{i(j)}, b_{i(j+1)})| \geq 3^{m-n}$. Now if $a_n = a_k$ for some $k < n$, ISO chooses $b_n = b_k$. Suppose that $a_n \neq a_k$ for all $k < n$. We consider several subcases.

Subcase 1.1. $a_n < a_{i(0)}$. If $|(a_n, a_{i(0)})| < 3^{m-n-1}$, ISO picks $b_n < b_{i(0)}$ such that $|(b_n, b_{i(0)})| = |(a_n, a_{i(0)})|$. If $|(a_n, a_{i(0)})| \geq 3^{m-n-1}$, ISO picks b_n such that $|(b_n, b_{i(0)})| = 3^{m-n-1}$.

Subcase 1.2. $a_{i(p)} < a_n$. If $|(a_{i(p)}, a_n)| < 3^{m-n-1}$, ISO picks $b_n > b_{i(p)}$ such that $|(b_{i(p)}, b_n)| = |(a_{i(p)}, a_n)|$. If $|(a_{i(p)}, a_n)| \geq 3^{m-n-1}$, ISO picks $b_n > b_{i(p)}$ such that $|(b_{i(p)}, b_n)| = 3^{m-n-1}$.

Subcase 1.3. There is a $j < p$ such that $a_{i(j)} < a_n < a_{i(j+1)}$, and $|(a_{i(j)}, a_{i(j+1)})| < 3^{m-n}$. Then by the assumption of this case, $|(b_{i(j)}, b_{i(j+1)})| = |(a_{i(j)}, a_{i(j+1)})|$. ISO chooses $b_n > b_{i(j)}$ so that $|(b_{i(j)}, b_n)| = |(a_{i(j)}, a_n)|$.

Subcase 1.4. There is a $j < p$ such that $a_{i(j)} < a_n < a_{i(j+1)}$, $|(a_{i(j)}, a_{i(j+1)})| \geq 3^{m-n}$, and $|(a_{i(j)}, a_n)| < 3^{m-n-1}$. Then ISO chooses $b_n > b_{i(j)}$ so that $|(b_{i(j)}, b_n)| = |(a_{i(j)}, a_n)|$. Then if $n + 1 < m$ we have

$$|(a_n, a_{i(j+1)})| \geq 3^{m-n} - 1 - 3^{m-n-1} + 1 = 2 \cdot 3^{m-n-1} \geq 3^{m-n-1},$$

and similarly $|(b_n, b_{i(j+1)})| \geq 3^{m-n-1}$.

Subcase 1.5. There is a $j < p$ such that $a_{i(j)} < a_n < a_{i(j+1)}$, $|(a_{i(j)}, a_{i(j+1)})| \geq 3^{m-n}$, $|(a_{i(j)}, a_n)| \geq 3^{m-n-1}$, and $|(a_n, a_{i(j+1)})| < 3^{m-n-1}$. Then ISO chooses $b_n < b_{i(j+1)}$ such that $|(a_n, a_{i(j+1)})| = |(b_n, b_{i(j+1)})|$. Then if $n + 1 < m$ we have

$$|(b_{i(j)}, b_n)| \geq 3^{m-n} - 1 - 3^{m-n-1} + 1 = 2 \cdot 3^{m-n-1} \geq 3^{m-n-1}.$$

Subcase 1.6. There is a $j < p$ such that $a_{i(j)} < a_n < a_{i(j+1)}$, $|(a_{i(j)}, a_{i(j+1)})| \geq 3^{m-n}$, $|(a_{i(j)}, a_n)| \geq 3^{m-n-1}$, and $|(a_n, a_{i(j+1)})| \geq 3^{m-n-1}$. Then ISO chooses $b_n < b_{i(j+1)}$ such that $|(b_n, b_{i(j+1)})| = 3^{m-n-1}$. Then if $n + 1 < m$ we have

$$|(b_{i(j)}, b_n)| \geq 3^{m-n} - 1 - 3^{m-n-1} = 2 \cdot 3^{m-n-1} - 1 \geq 3^{m-n-1}.$$

Case 2. The conditions of Case 1 fail. Then ISO chooses $b_n = b_0$.

By induction, Case 1 always holds, and ISO wins. This finishes the proof that T is complete.

Now we modify each M_f by letting $N_f = M_f \times \mathbb{Z}$, ordered lexicographically. Clearly N_f is a model of T. We claim that an isomorphism F from N_f to N_g induces an isomorphism from M_f to M_g. In fact, for each $x \in M_f$ let $F'(x)$ be the first coordinate of $F(x, 0)$. suppose that $x, y \in M_f$ and $x < y$. Then $F(x, 0) < F(y, 0)$ so $F'(x) \leq F'(y)$. If $F'(x) = F'(y)$, then $(F(x, 0), F(y, 0))$ is finite. But $((x, 0)(y, 0))$ is infinite, contradiction. Hence $x < y$ implies that $F'(x) < F'(y)$. To show that F' is onto, let $z \in M_g$. Choose $(x, m) \in N_f$ such that $F(x, m) = (z, 0)$. If $m < 0$, then $F(x, 0) = (z, -m)$, and if $0 \leq m$, then $F(x, 0) = (z, m)$. So z is in the range of F'.

It follows that $\{N_f : f \in {}^{\omega}2\}$ is a set of $2^{\aleph_0}$ non-isomorphic models of T.

Notes on real-closed fields

These notes develop the algebraic background needed to understand the model theory of real-closed fields. To understand these notes, a standard graduate course in algebra is sufficient. We make references to Hungerford below for some of the facts that are used.

A field F is *formally real* iff -1 is not a sum of squares in F. A field F is *real-closed* iff F is formally real, but no proper algebraic extension is formally real. Note also that in a formally real field a sum of nonzero squares is never 0.

Theorem 1. *Every formally real field can be embedded in a real-closed field.*

Proof. Let F be formally real, and let K be the algebraic closure of F. By Zorn's lemma let H be an extension of F which is a subfield of K, maximal in the collection of all formally real subfields of K. Then H is real-closed. In fact, let L be a proper algebraic extension of H. We may assume that L is obtained from H by adjoining a root of an irreducible polynomial $f(x) \in H[x]$. Then $f(x)$ has a root in K also, and by standard field theory there is an isomorphism g from L into K pointwise fixing H. (See Hungerford, p. 236, Corollary 1.9.) Thus $g[L]$ is not formally real, and so also L is not formally real. □

Proposition 2. *If F is a real-closed field, then every sum of squares in F is a square.*

Proof. Suppose that a is a sum of squares in F, but a is not a square. Then $x^2 - a$ is irreducible over F. Let $F(\alpha)$ be obtained from F by adjoining a root of this polynomial. Then $F(\alpha)$ is no longer formally real, so we can write

$$-1 = (a_0 + b_0\alpha)^2 + \cdots + (a_{m-1} + b_{m-1}\alpha)^2$$

for certain elements $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}$ of F. Multiplying out, we obtain

$$-1 = a_0^2 + b_0^2 a + \cdots + a_{m-1}^2 + b_{m-1}^2 a + u\alpha$$

for some $u \in F$. Hence $-1 = a_0^2 + b_0^2 a + \cdots + a_{m-1}^2 + b_{m-1}^2 a$; but clearly the right side here is a sum of squares of elements of F, contradiction. □

Proposition 3. *If F is a real-closed field, then for every $a \in F$, either a or $-a$ is a square.*

Proof. Suppose that a is not a square. Then $x^2 - a$ is irreducible in $F[x]$, and so if $F(\alpha)$ is the result of adjoining a root α of it, then $F(\alpha)$ is no longer formally real. Hence by the computation in the preceding proof,

$$-1 = a_0^2 + b_0^2 a + \cdots + a_{m-1}^2 + b_{m-1}^2 a.$$

By Proposition 2 write $1 + a_0^2 + \cdots + a_{m-1}^2 = c^2$ and $b_0^2 + \cdots + b_{m-1}^2 = d^2$. Thus this equation becomes $-c^2 = d^2 a$, hence clearly $-a$ is a square. □

An *ordered field* is a field F together with a linear ordering $<$ on F such that for any $a, b, c \in F$, if $a < b$ then $a + c < b + c$, and if $a < b$ and $c > 0$ then $a \cdot c < b \cdot c$.

Proposition 4. *Suppose that F is formally real, and for every element a of F, either a or $-a$ is a square. Then every sum of squares is a square.*

Proof. Suppose that $-(a_0^2+\cdots+a_{m-1}^2)=b^2$ with all the a_i's nonzero, $m>0$. Then $a_0^2+\cdots+a_{m-1}^2+b^2=0$, contradiction. $\square$

Proposition 5. *Suppose that F is formally real and for any $a \in F$, either a is a square or $-a$ is a square. Then there is a linear order $<$ on F which makes F into an ordered field.*

Proof. Define $a<b$ iff $a \neq b$ and $b-a$ is a square. This gives an ordered field:

(1) Clearly $a \not< a$.

(2) If $a<b<c$, then $b-a$ and $c-b$ are squares, hence $c-a$ is a sum of squares, and so is a square by Proposition 4. We must have $c \neq a$, as otherwise 0 would be the sum of the nonzero squares $b-a$ and $c-b$.

(3) Given $a, b \in F$, either $a-b$ or $b-a$ is a square, and so $a<b$, $a=b$, or $b<a$.

(4) If $a<b$, obviously also $a+c<b+c$.

(5) Suppose that $a<b$ and $c>0$. Thus $b-a$ and c are squares, and hence so is $b \cdot c-a \cdot c$, so $a \cdot c<b \cdot c$. $\square$

So we have shown that any real-closed field can be extended to an ordered field. Clearly any ordered field is of characteristic 0, so any real-closed field has characteristic 0. Also, any formally real field can be extended to an ordered field, and hence also has characteristic 0.

Proposition 6. *If F is formally real, $f(x)$ is a monic irreducible polynomial of odd degree >1, and $F(\alpha)$ is obtained from F by adjoining a root α of $f(x)$, then $F(\alpha)$ is formally real.*

Proof. Suppose not, and let $f(x)$ be irreducible of minimum odd degree such that $F(\alpha)$ is not formally real, with α a root of $f(x)$ in some extension. Say that $f(x)$ has degree $2n+1$, with $n>0$. We use the specific construction of $F(\alpha)$ as $F[x]/(f(x))$. Now -1 is a sum of squares in $F(\alpha)$, and this implies that in $F[x]$ we have polynomials such that

$$(*) \qquad 1+(g_0(x))^2+\cdots+(g_{m-1}(x))^2=f(x)h(x),$$

Here each $g_i(x)$ has degree at most $2n$. Let u be the greatest degree of any of the polynomials $g_i(x)$. For each i such that u is the degree of $g_i(x)$ write $g_i(x)=a_i x^u+h_i(x)$ with $a_i \neq 0$ and $h_i(x)$ of degree less than u. Then the leading term of the left side of $(*)$ is $(\sum\{a_i^2\})x^{2u}$, the sum being taken over all i with $g_i(x)$ of degree u. The coefficient $\sum\{a_i^2\}$ is nonzero since F is formally real. It follows that the left side of $(*)$ has degree $2u \leq 4n$. Hence $h(x)$ has odd degree at most $2n-1$. By the minimality of n, if β is a root of some irreducible factor of $h(x)$ of odd degree, then $F(\beta)$ is formally real. Substituting β into $(*)$, we see that 0 is a sum of nonzero squares in $F(\beta)$, contradiction. $\square$

Corollary 7. *In a real-closed field, every polynomial of odd degree has a root.* $\square$

Now we go through some material which is a reorganization of the proof of the fundamental theorem of algebra found on pages 265–267 of Hungerford. Let us call a field F *special* iff it is formally real, every polynomial of odd degree has a root, and for every $a \in F$, either a is a square or $-a$ is a square. Thus every real-closed field is special. This is a temporary notation, since we are going to show that conversely every special field is real-closed. Note that every special field can be expanded to an ordered field, by Proposition 5.

Lemma 7. *Let K be special field, and i a root of x^2+1 in an extension of K. Then every element of $K(i)$ has a square root.*

Proof. Let $a+bi$ be any element of $K(i)$, where $a, b \in K$. Then $a^2 \leq a^2+b^2$, so $|a| \leq \sqrt{a^2+b^2}$. It follows that $a+\sqrt{a^2+b^2} \geq 0$, so we can choose $c \geq 0$ so that $a+\sqrt{a^2+b^2} = 2c^2$. Similarly we get $d \geq 0$ such that $-a+\sqrt{a^2+b^2} = 2d^2$. Then

$$c^2 - d^2 = \frac{a+\sqrt{a^2+b^2}}{2} - \frac{-a+\sqrt{a^2+b^2}}{2} = a$$

and

$$2cd = \left(\sqrt{a+\sqrt{a^2+b^2}}\right)\left(\sqrt{-a+\sqrt{a^2+b^2}}\right) = \sqrt{-a^2+a^2+b^2} = |b|.$$

Thus if $b \geq 0$ we have $(c+di)^2 = a+bi$, and if $b < 0$ we have $(c-di)^2 = a+bi$. □

Lemma 8. *Let K be a special field, and i a root of x^2+1 in an extension of K. Then every quadratic polynomial in $K(i)[x]$ has a root.*

Proof. Let x^2+sx+t be any monic quadratic in $K(i)[x]$. Thus $x^2+sx+t = (x+\frac{s}{2})^2 + t - \frac{s^2}{4}$. Let $u \in K(i)$ be such that $u^2 = \frac{s^2}{4} - t$. Clearly $u - \frac{s}{2}$ is a root of x^2+sx+t. □

Lemma 9. *Let K be a special field, and i a root of x^2+1 in an extension of K. Then $K(i)$ is algebraically closed.*

Proof. Suppose that L is a finite dimensional extension of $K(i)$. We may assume that L is the splitting field of a set of polynomials in $K[x]$, and hence is Galois over K. It suffices to show that $L = K(i)$. Let $[L : K] = m$. Thus $m \geq 2$. Write $m = 2^n r$ with r odd. Let G be a Sylow 2-subgroup of $\mathrm{Aut}_K L$. Then the fixed field E of G over K is such that $[E : K] = [\mathrm{Aut}_K L : G]$ by the fundamental theorem of Galois theory (Theorem 2.5, page 245, in Hungerford). That degree is thus odd. By Lemma 3.17, page 266, in Hungerford, there is thus an irreducible polynomial of odd degree over K. Since K is special, it follows that this degree is 1.

Thus $r = 1$ and $m = 2^n$. Hence $\mathrm{Aut}_{K(i)} L = 2^{n-1}$. Suppose that $n > 1$. By one of the Sylow theorems, $\mathrm{Aut}_{K(i)} L$ has a subgroup H of index 2. Let F be the fixed field of H. Then $[F : K(i)] = [\mathrm{Aut}_{K(i)} L : H] = 2$. This contradicts Lemma 8. □

Lemma 10. *If K is special, then K is real-closed.*

Proof. Suppose that L is a proper algebraic extension of K. By Lemma 9 we may assume that it is a subfield of $K(i)$; hence it equals $K(i)$, and is not formally real. $\square$

Theorem 11. *K is real-closed iff it is special.*

Corollary 12. *Let $(K, <)$ be an ordered field. Then it is real-closed iff every positive element has a square root and every polynomial over K of odd degree has a root in K.*

Proof. $\Rightarrow$: By Theorem 11 and Proposition 5. $\Leftarrow$: by Theorem 11. $\square$

Proposition 13. *If K is a real-closed field, then every polynomial over K splits into linear and quadratic factors.*

Proof. If $f(x)$ is an irreducible polynomial in $K[x]$, then if α is a root of it in $K(i)$, we have $2 = [K(i) : K] = [K(i) : K(\alpha)] \cdot [K(\alpha) : K]$, and so $[K(\alpha) : K]$ is 1 or 2. $\square$

Proposition 14. *Suppose that K is a real-closed field and $f(x)$ is an irreducible quadratic polynomial in $K[x]$. Then either $f(a) > 0$ for all $a \in K$, or $f(a) < 0$ for all $a \in K$.*

Proof. Write $f(x) = bx^2 + cx + d$. Actually for this proposition clearly we may assume that $b = 1$. Now $f(x) = (x + \frac{c}{2})^2 + d - \frac{c^2}{4}$. Since $f(x)$ is irreducible and every positive element of K has a square root, we must have $d - \frac{c^2}{4} > 0$. Hence the desired conclusion is clear. $\square$

Proposition 15. *Suppose that K is a real-closed field, and $f(x) \in K[x]$. Suppose that $a < b$ in K and $f(a) \cdot f(b) < 0$. Then there is a c with $a < c < b$ such that $f(c) = 0$.*

Proof. Write $f(x) = g_0(x) \cdot \ldots \cdot g_{m-1}(x)$ with each $g_i(x)$ irreducible, and hence by Proposition 13, either linear or quadratic. From Proposition 14 it follows that some linear factor $ux + v$ must have different signs at a and at b. By symmetry say that $ua+v < 0 < ub+v$. Thus $ua < -v < ub$, hence $u > 0$ and $a < \frac{-v}{u} < b$ and $f(\frac{-v}{u}) = 0$. $\square$

Proposition 16. *Suppose that K is a real-closed field and F is a subfield of K. Then there is a smallest real-closed field H such that $F \subseteq H \subseteq K$. Moreover, if K' is any real-closed field such that $F \subseteq K'$, then H is embeddable in K' by an embedding whcih is the identity on F.*

Proof. Let $\langle f_\alpha(x) : \alpha < \kappa \rangle$ list out all irreducible polynomials in $F[x]$ of odd degree plus all irreducible polynomials $x^2 - a$ with a a sum of squares of F. Now form a sequence of extensions G_α of F, each a subfield of K, adjoining a root in K of each of these polynomials if they have remained irreducible. Call the union of all fields obtained in this way L_1. Then repeat this process with L_1, forming L_2, L_3, etc. The union of all these is the desired real-closed field H.

Given K' as in the second part of the Proposition, the identity on F can be extended to embeddings of the G_α's and the L_m's one after the other by standard field theory. $\square$

Finally, we need to go into the question of what structures can be isomorphically embedded into real-closed ordered fields. The answer is: ordered integral domains. We prove a strong version of this. Generalizing the notion of ordered field as defined on page one, we say that an ordered integral domain is a pair $(D, <)$ such that D is an integral domain and $<$ is a linear ordering of D such that for any $a, b, c \in D$, if $a < b$ then $a + c < b + c$, and if $a < b$ and $c > 0$ then $a \cdot c < b \cdot c$.

Proposition 17. *If $(D, <)$ is a substructure of an ordered field, then $(D, <)$ is an ordered integral domain.* $\square$

Proposition 18. *Suppose that D is an integral domain, and P is a set of nonzero elements of D closed under $+$ and $\cdot$, and such that for any nonzero $a \in D$, $a \in P$ or $-a \in P$. Then there is a relation $<$ such that $(D, <)$ is an ordered integral domain and $P = \{a \in D : 0 < a\}$.*

Proof. Define $a < b$ iff $b - a \in P$. Since $0 \notin P$, it follows that $<$ is irreflexive. Suppose that $a < b < c$. Then $b - a \in P$ and $c - b \in P$, so $c - a = c - b + b - a \in P$; so $a < c$. Finally, given distinct elements $a, b \in D$, we have $a - b \in P$ or $b - a \in P$, hence $a < b$ or $b < a$. So $<$ is a linear ordering of D.

Suppose that $a < b$ and $c \in D$. Then $b + c - (a + c) = b - a \in P$, so $a + c < b + c$.

Suppose that $a < b$ and $0 < c$. Then $b - a, c \in P$, so $b \cdot c - a \cdot c = (b - a) \cdot c \in P$, so $a \cdot c < b \cdot c$.

Thus $(D, <)$ is an ordered integral domain. Clearly $P = \{a \in D : 0 < a\}$. $\square$

Lemma 19. *Suppose that $(F, <)$ is an ordered field, G is an extension field of F, $m \in \omega$, $a_0, \ldots, a_{m-1}$ are positive elements of F, and*

$$F \subset F(\sqrt{a_0}) \subset F(\sqrt{a_0}, \sqrt{a_1}) \subset \ldots \subset F(\sqrt{a_0}, \sqrt{a_1}, \ldots, \sqrt{a_{m-1}}) = G.$$

Note that all inclusions here are proper.

Under these conditions, G is formally real.

Proof. As is well-known, any element of G can be written in the form

$$\sum_{M \subseteq m} b_M \cdot \sqrt{\prod_{i \in M} a_i}$$

with each b_M in F. The square of such an element has the form

$$\sum_{M \subseteq m} b_M^2 \cdot \prod_{i \in M} a_i + \sum_{\emptyset \neq M \subseteq M} c_M \cdot \sqrt{\prod_{i \in M} a_i}$$

where each $c_M \in F$. It follows that if -1 is a sum of squares in G, then -1 is a sum of elements of the form

$$\sum_{M \subseteq m} b_M^2 \cdot \prod_{i \in M} a_i.$$

Note, however, that each such element is positive, contradiction. $\square$

Lemma 20. *Let $(F,<)$ be an ordered field, and let P be the set of all positive elements of F. Let $\prec$ be any well-order of P. We now define an increasing sequence of extensions G_ξ of F by recursion for $\xi < \alpha$, where α will also be determined in the construction. Let $G_0 = F$. Suppose that G_η has been constructed for all $\eta < \xi$, with $\xi \geq 1$. Let $H_\xi = \bigcup_{\eta<\xi} G_\eta$. If each member of P has a square root in H_ξ, we let $\alpha = \xi$ and the construction stops. Otherwise we let a_ξ be the $\prec$-first element of P not having a square root in H_ξ, and define $G_\xi = H(\sqrt{a_\xi})$. This finishes the construction.*

The conclusion is that H_α is formally real and every positive element of F has a square root in H_α.

Proof. If -1 is a sum of squares in H_α, then there is a finite subset N of α such that -1 is a sum of squares in $F(\langle a_\xi : \xi \in N\rangle)$. This contradicts Lemma 19.

The other part of the conclusion is assured by the construction. $\square$

Theorem 21. *Suppose that $(D,<)$ is an ordered integral domain. Then there is a real-closed ordered field $(F,<)$ and a function f such that f is an isomorphism from $(D,<)$ into $(F,<)$, and for any real-closed ordered field $(G,<)$ and isomorphism g from $(D,<)$ into $(G,<)$, there is an isomorphism h of $(F,<)$ into $(G,<)$ such that $h \circ f = g$.*

This is illustrated by the following diagram:

$$\begin{tikzcd}
(F,<) \arrow[rr, dotted, "h"] & & (G,<) \\
& (D,<) \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. We recall the standard embedding of an integral domain D into a field H. A relation $\sim$ is defined on $D \times (D\backslash\{0\})$ by setting $(a,b) \sim (c,d)$ iff $ad = bc$. This is an equivalence relation on $D \times (D\backslash\{0\})$, and H is the collection of equivalence classes. There are operations $+$ and $\cdot$ on H such that $[(a,b)] + [(c,d)] = [ad+bc, bd]$ and $[(a,b)] \cdot [(c,d)] = [(ac,bd)]$. This makes H into a field. The function f taking a to $[(a,1)]$ for each $a \in D$ is an isomorphism from D into H.

Now we define $P = \{[(a,b)] \in H : ab > 0\}$.

(1) If $[(c,d)] = [(a,b)]$ and $ab > 0$, then $cd > 0$.

In fact, $cb = da$; multiplying by bd, we get $cdb^2 = abd^2$. Since $ab > 0$ and $d^2 > 0$, it follows that $cdb^2 > 0$. So $cd > 0$ as otherwise, multiplying by b^2 would give $cdb^2 \leq 0$. Thus (1) holds.

If $ab > 0$ and $cd > 0$, then $abcd > 0$. Hence P is closed under $\cdot$.

If $ab > 0$ and $cd > 0$, then $(ad + bc)(bd) = abd^2 + cdb^2 > 0$. Hence P is closed under $+$.

Given $[(a, b)] \neq [(0, 1)]$, we have $a \neq 0$. Hence $ab > 0$ or $-ab > 0$. So for any non-zero element x of H we have $x \in P$ or $-x \in P$.

Let $<$ on H be defined from P as in Proposition 17.

From all of this it follows that f is an isomorphism from $(D, <)$ into $(H, <)$. Next we apply the construction in Lemma 19 to get a formally real extension K of H such that every positive element of H has a square root in K. By Theorem 1 let L be a real-closed extension of K; by Theorem 15 let M be the smallest real-closed field such that $K \subseteq M \subseteq L$. By Propositions 3 and 5 there is a relation $<$ making M an ordered field. The construction in Proposition 5 assures that every nonzero element of M which is a square is positive in the order $<$. Our construction shows that every positive element of H is a square in K and hence in M. It follows that $(H, <)$ is a substructure of $(M, <)$.

Now suppose that N is a real-closed ordered field and g is an isomorphism from $(D, <)$ into N. We would like to define $h : H \to N$ by setting $h([(a, b)]) = g(a) \cdot g(b)^{-1}$ for any $[a, b] \in H$. To show that h is well-defined, suppose that $[(a, b)] = [(c, d)]$. Thus $ad = bc$, so $g(a)g(d) = g(b)g(c)$ and hence

$$g(a) \cdot g(b)^{-1} = g(a)g(d)g(d)^{-1}g(b)^{-1} = g(b)g(c)g(d)^{-1}g(b)^{-1} = g(c)g(d)^{-1}.$$

Hence h is well-defined. Similarly h is one-one. It is straighforward to check that h really is an isomorphism from H into N.

By standard uniqueness facts in field theory, h extends to an isomorphism k from M into N. Now $a \in M$ is nonnegative in M iff it is a square in M iff $k(a)$ is a square in N iff $k(a)$ is nonnegative in N. So k is in fact an isomorphism from M into N. Clearly $k \circ f = g$. $\square$

Solutions to exercises

Solutions of exercises in Chapter 1

$\boxed{1.1}$ *Let $\mathscr{L}$ be a language with no individual constants. Define an $\mathscr{L}$-structure $\overline{A}$ and subuniverses B, C of $\overline{A}$ such that $B \cap C = \emptyset$.*

Let $A = 2$, and let the fundamental operations of $\overline{A}$ be such that $\{0\}$ and $\{1\}$ are closed under them; the fundamental relations can be anything. Then $\{0\}$ and $\{1\}$ are disjoint subuniverses.

$\boxed{1.2}$ *Carry out the "easy induction" at the beginning of the proof of Proposition 1.2.*

Since $\langle X\rangle_A$ is a subuniverse containing X, we have $Y_0 \subseteq \langle X\rangle_A$. Now suppose that $Y_i \subseteq \langle X\rangle_A$. Suppose that F is a function symbol of rank m and $x \in {}^mY_i$. Then $x \in {}^mX$, and so $F(x) \in \langle X\rangle_A$. It follows that $Y_{i+1} \subseteq \langle X\rangle_A$.

$\boxed{1.3}$ *If X and Y are nonempty subsets of the universe A of an algebra $\overline{A}$, then $\langle X \cup \langle Y\rangle\rangle = \langle X \cup Y\rangle$.*

Suppose that B is a subuniverse of A containing $X \cup Y$. Then $Y \subseteq B$, and so $\langle Y\rangle \subseteq B$. Thus $X \cup \langle Y\rangle \subseteq B$. It follows that $\langle X \cup \langle Y\rangle\rangle \subseteq B$. Since B is arbitrary, this shows that $\langle X \cup \langle Y\rangle\rangle \subseteq \langle X \cup Y\rangle$.

Conversely, suppose that C is a subuniverse of A containing $X \cup \langle Y\rangle$. Then $\langle Y\rangle \subseteq C$, and so also $Y \subseteq C$. Thus $X \cup Y \subseteq C$. It follows that $\langle X \cup Y\rangle \subseteq C$. Since C is arbitrary, this proves that $\langle X \cup Y\rangle \subseteq \langle X \cup \langle Y\rangle\rangle$. Together with the preceding paragraph this proves that $\langle X \cup Y\rangle = \langle X \cup \langle Y\rangle\rangle$.

$\boxed{1.4}$ *If K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$, then $\langle\bigcup K\rangle = \langle\bigcup_{X\in K}\langle X\rangle\rangle$.*

Suppose that K is a nonempty set of nonempty subsets of the universe A of a structure $\overline{A}$.

First suppose that B is a subuniverse of A containing $\bigcup K$. Then for each $X \in K$, B contains X, and hence $\langle X\rangle \subseteq B$. thus $\bigcup_{X\in K}\langle X\rangle \subseteq B$, and so $\langle\bigcup_{X\in K}\langle X\rangle\rangle \subseteq B$. Since B is arbitrary, this proves that $\langle\bigcup_{X\in K}\langle X\rangle\rangle \subseteq \langle\bigcup K\rangle$.

Second, suppose that B is a subuniverse of A containing $\bigcup_{X\in K}\langle X\rangle$. Then for each $X \in K$ we have $X \subseteq \langle X\rangle \subseteq B$. So $\bigcup K \subseteq B$, and so also $\langle\bigcup K\rangle \subseteq B$. Since B is arbitrary, this shows that $\langle\bigcup K\rangle \subseteq \langle\bigcup_{X\in K}\langle X\rangle\rangle$. Together with the preceding paragraph this shows that $\langle\bigcup K\rangle = \langle\bigcup_{X\in K}\langle X\rangle\rangle$.

$\boxed{1.5}$ *Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{A}$. Show that $f[C]$ is a subuniverse of $\overline{B}$.*

For k an individual constant, $k^{\overline{B}} = f(k^{\overline{A}}) \in f[C]$. For F an m-ary operation symbol and $b \in {}^mC$,

$$F^{\overline{B}}(f \circ b) = f(F^{\overline{A}}(b)) \in f[C],$$

as desired.

$\boxed{1.6}$ *Suppose that f is a homomorphism from $\overline{A}$ into $\overline{B}$, and C is a nonempty subuniverse of $\overline{B}$. Show that $f^{-1}[C]$ is a subuniverse of $\overline{A}$.*

For k an individual constant, $f(k^{\overline{A}}) = k^{\overline{B}} \in C$ and so $k^{\overline{A}} \in f^{-1}[C]$. For F an m-ary operation symbol and $b \in {}^m(f^{-1}[C])$ we have $f(F^{\overline{A}}(b)) = F^{\overline{B}}(f \circ b) \in C$ since $(f \circ b) \in {}^mC$; hence $F^{\overline{A}}(b) \in f^{-1}[C]$.

$\boxed{1.7}$ *If X generates $\overline{A}$, f and g are homomorphisms from $\overline{A}$ into $\overline{B}$, and $f \restriction X = g \restriction X$, then $f = g$.*

It suffices to show that $Y \stackrel{\text{def}}{=} \{a \in A : f(a) = g(a)\}$ is a subuniverse of $\overline{A}$ containing X. We are given that $X \subseteq Y$. For any individual constant k, $f(k^{\overline{A}}) = k^{\overline{B}} = f(k^{\overline{A}})$, so $k^{\overline{A}} \in Y$. Now suppose that F is an n-ary operation symbol and $a \in {}^nY$. Then

$$f(F^{\overline{A}}(a)) = F^{\overline{B}}(f \circ a) = F^{\overline{B}}(g \circ a) = g(F^{\overline{A}}(a)),$$

and so $F^{\overline{A}}(a) \in Y$. Thus Y is a subuniverse of $\overline{A}$.

$\boxed{1.8}$ *If f is a homomorphism from $\overline{A}$ into $\overline{B}$ and X is a nonempty subset of A, then $f[\langle X\rangle] = \langle f[X]\rangle$.*

By exercise 1.5, $f[\langle X\rangle]$ is a subuniverse of $\overline{B}$ containing $f[X]$. Hence $\langle f[X]\rangle \subseteq f[\langle X\rangle]$.

Now by exercise 1.6, $f^{-1}[\langle f[X]\rangle]$ is a subuniverse of $\overline{A}$, and it obviously contains X. So $\langle X\rangle \subseteq f^{-1}[\langle f[X]\rangle]$, and so $f[\langle X\rangle] \subseteq \langle f[X]\rangle$.

$\boxed{1.9}$ *If $\overline{A}$ is a substructure of $\overline{B}$ and $\equiv$ is a congruence relation on $\overline{B}$, then $\equiv \cap (A \times A)$ is a congruence relation on $\overline{A}$.*

Clearly $\equiv \cap (A \times A)$ is an equivalence relation on $\overline{A}$. Now suppose that F is an m-ary operation symbol, $x, y \in {}^mA$, and $x_i \equiv y_i$ for all $i < m$. Then

$$F^{\overline{A}}x = F^{\overline{B}}x \equiv F^{\overline{B}}y = F^{\overline{A}}y.$$

Finally, if R is an m-ary relation symbol, $x, y \in {}^mA$, and $x_i \equiv y_i$ for all $i < m$, then

$$a \in R^{\overline{A}} \text{ iff } a \in R^{\overline{B}} \text{ iff } b \in R^{\overline{B}} \text{ iff } b \in R^{\overline{A}}.$$

$\boxed{1.10}$ *Suppose that R is a congruence relation on $\overline{A}$, and S is a congruence relation on $\overline{A}/R$. Define $T = \{(a_0, a_1) \in A \times A : ([a_0]_R, [a_1]_R) \in S\}$. Show that T is a congruence relation on A and $R \subseteq T$.*

T is reflexive on A: given $a \in A$, we have $[a]_R S [a]_R$, so aTa.

T is symmetric: given xTy, we have $[x]_R S [y]_R$, hence $[y]_R S [x]_R$, hence yTx,

T is transitive: given $xTyTz$, we have $[x]_R S [y]_R S [z]_R$, hence $[x]_R S [z]_R$, hence xTz.

Now suppose that F is an m-ary operation symbol, $x, y \in {}^mA$, and $x_i T y_i$ for all $i < m$. Then $[x_i]_R S [y_i]_R$ for all $i < m$, so

$$F^{\overline{A}/R}([x_0]_R, \ldots, [x_{m-1}]_R) S F^{\overline{A}/R}([y_0]_R, \ldots, [y_{m-1}]_R).$$

Now $F^{\overline{A}/R}([x_0]_R,\ldots,[x_{m-1}]_R) = [F^{\overline{A}}(x_0,\ldots,x_{m-1})]_R$ and $F^{\overline{A}/R}([y_0]_R,\ldots,[y_{m-1}]_R) = [F^{\overline{A}}(y_0,\ldots,y_{m-1})]_R$, so

$$[F^{\overline{A}}(x_0,\ldots,x_{m-1})]_R S[F^{\overline{A}}(y_0,\ldots,y_{m-1})]_R;$$

hence $F^{\overline{A}}(x_0,\ldots,x_{m-1})TF^{\overline{A}}(y_0,\ldots,y_{m-1})$.

Now suppose that U is an m-ary relation symbol, $a,b\in {}^mA$, and a_iTb_i for all $i<m$. Then $[a_i]_RS[b_i]_R$ for all $i<m$, and

$$\begin{aligned} a\in U^{\overline{A}} \quad &\text{iff}\quad \langle [a_i]_R : i<m\rangle \in U^{\overline{A}/R}\\ &\text{iff}\quad \langle [b_i]_R : i<m\rangle \in U^{\overline{A}/R} \quad \text{since } S \text{ is a congruence relation on } \overline{A}/R\\ &\text{iff}\quad b\in U^{\overline{A}}. \end{aligned}$$

Finally, if xRy, then $[x]_R=[y]_R$, hence $[x]_RS[y]_R$, hence xTy.

$\boxed{1.11}$ *(Continuing exercise 1.10) Show that the procedure of exercise 1.10 establishes a one-one order-preserving correspondence between congruence relations on A/R and those congruence relations on A with include R.*

For each congruence relation S on A/R let F_S be the congruence relation T defined in exercise 1.10. Suppose that S_0 and S_1 are distinct congruence relations on A/R. Say $[a]_RS_0[b]_R$ and not($[a]_RS_1[b]_R$. Then $aF_{S_0}b$. Suppose that $aF_{S_1}b$. Then $[a]_RS_1[b]_R$, contradiction. Thus F is a one-one function.

Now suppose that U is a congruence relation on A which includes R. Define

$$S=\{(x,y)\in (A/R)\times(A/R) : \exists a,b\in A[x=[a]_R\wedge y=[b]_R\wedge aUb\}.$$

We claim that S is a congruence relation on A/R and $F_S=U$.

S is reflexive, since if $a\in A$ then aUa, hence $[a]_RS[a]_R$.

S is symmetric: suppose xSy. Choose $a,b\in A$ such that $x=[a]_R$, $y=[b]_R$, and aUb. Then bUa, so ySx.

S is transitive: suppose that $xSySz$. Choose $a,b,c,d\in A$ such that $x=[a]_R$, $y=[b]_R$, aUb, $y=[c]_R$, $z=[d]_R$, and cUd. Then $[b]_R=[c]_R$, so bRc, hence bUc. So $aUbUcUd$, hence aUd and so xSz.

Now let F be an m-ary operation symbol and $x,y\in {}^mS$ with x_iSy_i for all $i<m$. Choose $a,b\in {}^mA$ so that $\forall i<m[x_i=[a_i]_R\wedge y_i=[b_i]_R\wedge a_iUb_i]$. Then $F^{\overline{A}}(a)Uf^{\overline{A}}(b)$. Moreover, $F^{A/R}(x)=[F^A(a)]_R$ and $F^{A/R}(y)=[F^A(b)]_R$. Hence $F^{A/R}(x)SF^{A/R}(y)$.

Let T be an m-ary relation symbol, and let $x,y\in {}^m(A/R)$ with x_iSy_i for all $i<m$. Then there exist $x'_i,y'_i\in A$ so that $x_i=[x'_i]_R$ and $y_i=[y'_i]_R$ and $x'_iUy'_i$ for all $i<m$. Then

$$\begin{aligned} x\in T^{\overline{A}/R} \quad &\text{iff}\quad x'\in T^{\overline{A}}\\ &\text{iff}\quad y'\in T^{\overline{A}} \quad \text{since } U \text{ is a congruence relation on } \overline{A}\\ &\text{iff}\quad y\in T^{\overline{A}/R}. \end{aligned}$$

Thus S is a congruence relation on A/R. Now take any $a, b \in A$. If aF_Sb, then $[a]_R S [b]_R$. Hence there are $c, d \in A$ such that $[a]_R = [c]_R$, $[b]_R = [d]_R$, and cUd. Since $R \subseteq U$, we have $aUcUdUb$, and so aUb.

Conversely, suppose that aUb. Then $[a]_R S[b]_R$ and so aF_Sb.

Hence $F_S = U$.

$\boxed{1.12}$ *Suppose that $\langle \overline{A}_i : i \in I\rangle$ is a system of similar structures, and $\overline{B}$ is another structure similar to them. Suppose that f_i is a homomorphism from $\overline{B}$ into $\overline{A}_i$ for each $i \in I$. Show that there is a homomorphism g from $\overline{B}$ into $\prod_{i\in I} \overline{A}_i$ such that $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$.*

Define $(g(b))_i = f_i(b)$ for all $b \in B$ and $i \in I$. Thus $\mathrm{pr}_i \circ g = f_i$ for all $i \in I$. If k is an individual constant, then

$$g(k^{\overline{B}}) = \langle k^{\overline{A}_i} : i \in I\rangle = k^{\overline{C}}.$$

For R an m-ary relation symbol and $b \in {}^mB$,

$$\begin{aligned} g \circ b \in R^{\overline{C}} \quad &\text{iff} \quad \forall i \in I[\mathrm{pr}_i \circ g \circ b \in R^{\overline{A}_i}] \\ &\text{iff} \quad \forall i \in I[f_i \circ b \in R^{\overline{A}_i}] \\ &\text{iff} \quad b \in R^{\overline{B}}. \end{aligned}$$

For F an m-ary operation symbol and $b \in {}^mB$,

$$\begin{aligned} g(F^{\overline{B}}(b)) &= \langle f_i(F^{\overline{B}}(b)) : i \in I\rangle \\ &= \langle F^{\overline{A}_i}(f_i \circ b) : i \in I\rangle \\ &= \langle F^{\overline{A}_i}(\mathrm{pr}_i \circ g \circ b) : i \in I\rangle \\ &= F^{\overline{C}}(g \circ b). \end{aligned}$$

$\boxed{1.13}$ *Show that a product of partial orderings is a partial ordering.*

See page 2 for the official definition of a partial ordering. Suppose that $\langle (A_i, <_i) : i \in I\rangle$ is a system of partial orderings. Let $B = \prod_{i\in I} A_i$.

Irreflexive: if $b \in B$ and $b <_B b$, then $\forall i[b_i < b_i]$, contradicting irreflexivity on each factor.

Transitive: suppose that $b <_B c <_B d$. Thus $\forall i \in I[b_i <_I c_i <_i d_i]$, so $\forall i \in I[b_i <_i d_i]$, hence $b <_B d$.

$\boxed{1.14}$ *A partial ordering $(A, <)$ is a* **linear ordering** *iff for any two distinct $x, y \in A$ we have $x < y$ or $y < x$. Give an example of two linear orderings whose product is not a linear ordering.*

Take $\mathbb{Z} \times \mathbb{Z}$. Then $(1, 2)$ and $(3, 0)$ are incomparable.

$\boxed{1.15}$ *Give an example of two ordered fields whose product is not even a field.*

We consider $\mathbb{R} \times \mathbb{R}$. Then $(1, 0)$ is nonzero, but does not have an inverse.

$\boxed{1.16}$ *Let F be a proper filter on a set I. Show that F is an ultrafilter iff for all $a, b \subseteq I$, if $a \cup b \in F$ then $a \in F$ or $b \in F$.*

$\Rightarrow$: Suppose $a \notin F$ and $b \notin F$. Then $(I \backslash a) \in F$ and $(I \backslash b) \in F$, so $(I \backslash a) \cap (I \backslash b) \in F$. It follows that $a \cup b \notin F$, as otherwise $\emptyset = (I \backslash a) \cap (I \backslash b) \cap (a \cup b) \in F$.

$\Leftarrow$: If $a \subseteq I$, then $I = a \cup (I \backslash a) \in F$, hence $a \in F$ or $(I \backslash a) \in F$.

$\boxed{1.17}$ *Show that any ultraproduct of linear orderings is a linear ordering.*

We consider an ultraproduct $B \stackrel{\text{def}}{=} \prod_{i \in I} \overline{A}_i / D$. By exercise E1.13 it suffices to show that any two elements of B are comparable. Let $a, b \in \prod_{i \in I} A_i$. Then

$$I = \{i \in I : a_i < b_i\} \cup \{i \in I : a_i = b_i\} \cup \{i \in I : b_i < a_i\}.$$

By exercisee 1.16, since $I \in D$ one of these three sets is in D, giving $[a] < [b]$, $[a] = [b]$, or $[b] < [a]$ respectively.

$\boxed{1.18}$ *Suppose that I is a nonempty set, and $\langle J_i : i \in I \rangle$ is a system of nonempty sets. Also suppose that F_i is an ultrafilter on J_i for each $i \in I$, and G is an ultrafilter on I. Let $K = \{(i, j) : i \in I, j \in J_i\}$, and define*

$$H = \{X \subseteq K : \{i \in I : \{j \in J_i : (i, j) \in X\} \in F_i\} \in G\}.$$

Show that H is an ultrafilter on K.

• $K \in H$: For any $i \in I$ we have $\{j \in J_i : (i, j) \in K\} = J_i \in F_i$, so that $\{i \in I : \{j \in J_i : (i, j) \in K\} \in F_i\} = I \in G\}$. Hence $K \in H$.

• $\emptyset \notin H$: Suppose $\emptyset \in H$. Thus $\{i \in I : \{j \in J_i : (i, j) \in \emptyset\} \in F_i\} \in G$, in particular, there is an $i \in I$ such that $\{j \in J_i : (i, j) \in \emptyset\} \in F_i$. In particular there is a $j \in J$ such that $(i, j) \in \emptyset$, contradiction.

• Suppose that $X, Y \in H$; we show that $X \cap Y \in H$. Let

$$\begin{aligned} X' &= \{i \in I : \{j \in J_i : (i, j) \in X\} \in F_i\}; \\ Y' &= \{i \in I : \{j \in J_i : (i, j) \in Y\} \in F_i\}; \\ Z' &= \{i \in I : \{j \in J_i : (i, j) \in X \cap Y\} \in F_i\}. \end{aligned}$$

Since $X, Y \in H$, we have $X', Y' \in G$; hence $X' \cap Y' \in G$. We claim that $X' \cap Y' \subseteq Z'$; hence $Z' \in G$ and so $X \cap Y \in H$. To prove this claim, suppose that $i \in X' \cap Y'$. Then $\{j \in J_i : (i, j) \in X\}$ and $\{j \in J_i : (i, j) \in Y\}$ are both members of F_i, and hence so is their intersection. Now

$$\{j \in J_i : (i, j) \in X\} \cap \{j \in J_i : (i, j) \in Y\} \subseteq \{j \in J_i : (i, j) \in X \cap Y\},$$

so it follows that $\{j \in J_i : (i, j) \in X \cap Y\} \in F_i$; hence $i \in Z'$. This proves out claim.

• Suppose that $X \subseteq K$ and $X \notin H$; we prove that $(K \backslash X) \in H$. Since $X \notin H$, with X' as above we have $X' \notin G$; so $(I \backslash X') \in G$. We claim that

$$(I \backslash X') \subseteq \{i \in I : \{j \in J_i : (i, j) \in (K \backslash X)\} \in F_i\};$$

from this claim it follows that $\{i \in I : \{j \in J_i : (i,j) \in (K\backslash X)\} \in F_i\} \in G$, and hence $(K\backslash X) \in H$.

To prove ths claim, suppose that $i \in (I\backslash X')$. Thus $\{j \in J_i : (i,j) \in X\} \notin F_i$, and hence $\{j \in J_i : (i,j) \in (K\backslash X)\} \in F_i$, as desired.

$\boxed{1.19}$ *Under the notation of exercise 1.18, show that there is an isomorphism f of the structure $\prod_{i\in I}(\prod_{j\in J_i}\overline{A_{ij}}/F_i)/G)$ onto $\prod_{(i,j)\in K}\overline{A_{ij}}/H$ such that:*

$$\forall r \in \prod_{i\in I}\left(\prod_{j\in J_i}\overline{A_{ij}}/F_i)/G\right)\left[\left[r = [s]_G \text{ with } s \in \prod_{i\in I}\left(\prod_{j\in J_i}\overline{A_{ij}}/F_i\right)\right]\right.$$

$$\left.\text{and } \forall i \in I\left[s_i = [t_i]_{F_i} \text{ with } t_i \in \prod_{j\in J_i}\overline{A_{ij}}\right] \text{ implies that } f(r) = [\langle t_i(j) : (i,j) \in K\rangle]_H\right].$$

First we show that f is well-defined. Thus suppose that r, s, t are as above, and suppose that also $r = [s']_G$ with $s' \in \prod_{i\in I}\left(\prod_{j\in J_i}\overline{A_{ij}}/F_i\right)$ and $s'_i = [t'_i]_{F_i}$ for each $i \in I$, with $t' \in \prod_{j\in J_i}\overline{A_{ij}}$. Then $\{i \in I : s_i = s'_i\} \in G$, hence $\{i \in I : \{j \in J_i : t_{ij} = t'_{ij}\} \in F_i\} \in G$. So $\{(i,j) \in K : t_{ij} = t'_{ij}\} \in H$, as desired.

Reversing these steps, we see that f is injective.

Given $u \in \prod_{(i,j)\in K}\overline{A_{ij}}/H$, write $u = [t]_H$ with $t \in \prod_{(i,j)\in K}\overline{A_{ij}}$. For each $i \in I$ let $s_i = [\langle t_{ij} : j \in J_i\rangle]_{F_i}$. Then $f([s]_G) = u$. Thus f is surjective.

For the remainder of the proof we introduce the following abbreviations:

$$\begin{aligned}
\overline{B}_i &= \prod_{j\in J_i}\overline{A}_{ij};\\
\overline{C}_i &= \overline{B}_i/F_i;\\
\overline{D} &= \prod_{i\in I}\overline{C}_i;\\
\overline{E} &= \overline{D}/G;\\
\overline{L} &= \prod_{(i,j)\in K}\overline{A}_{ij};\\
\overline{M} &= \overline{L}/H.
\end{aligned}$$

Now let k be an individual constant. Then $k^{\overline{E}} = [k^{\overline{D}}]_G$, $k_i^{\overline{D}} = [k^{\overline{B}_i}]_{F_i}$ for each $i \in I$, and $k_j^{\overline{B}_i} = k^{\overline{A}_{ij}}$ for each $i \in I$ and $j \in J_i$. Hence $f(k^{\overline{E}}) = f([k^{\overline{D}}]_G) = [\langle k^{\overline{A}_{ij}} : (i,j) \in K\rangle]_H = k^{\overline{M}}$.

Next, let m be a positive integer and $r^0, \ldots, r^{m-1} \in E$. For each $k < m$ choose $s^k \in E$ with $r^k = [s^k]_G$. Then for each $i \in I$ write $s_i^k = [t_i^k]_{F_i}$.

Suppose that R is an m-ary relation symbol. Then

$$\begin{aligned}
\langle r^0, \ldots, r^{m-1} \rangle \in R^{\overline{E}} \quad &\text{iff} \quad \{i \in I : \langle s_i^0, \ldots, s_i^{m-1} \rangle \in R^{\overline{C}_i}\} \in G \\
&\text{iff} \quad \{i \in I : \{j \in J_i : \langle t_{ij}^0, \ldots, t_{ij}^{m-1} \rangle \in R^{\overline{A_i}}\} \in F_i\} \in G \\
&\text{iff} \quad \{(i,j) \in K : \langle t_{ij}^0, \ldots, t_{ij}^{m-1} \rangle \in R^{\overline{A}_{ij}}\} \in H \\
&\text{iff} \quad \langle [t^0]_H, \ldots, [t^{m-1}]_H \rangle \in R^{\overline{M}} \\
&\text{iff} \quad \langle f(r^0), \ldots, f(r^{m-1}) \rangle \in R^{\overline{M}}.
\end{aligned}$$

Now suppose that Q is an m-ary operation symbol. Then

$$\begin{aligned}
f(Q^{\overline{E}}(r^0, \ldots, r^{m-1})) &= f([Q^{\overline{D}}(s^0, \ldots, s^{m-1})]_G) \\
&= f([\langle (Q^{\overline{C}_i}(s_i^0, \ldots, s_i^{m-1}) : i \in I \rangle)]_G) \\
&= f([\langle [\langle Q^{\overline{A}_{ij}}(t_{ij}^0, \ldots, t_{ij}^{m-1}) : j \in J_i \rangle]_{F_i} : i \in I \rangle]_G) \\
&= \langle [Q^{\overline{A}_{ij}}(t_{ij}^0, \ldots, t_{ij}^{m-1})]_H : (i,j) \in K \rangle \\
&= Q^{\overline{M}}([t^0]_H, \ldots, [t^{m-1}]_H) \\
&= Q^{\overline{M}}(f(r^0), \ldots, f(r^{m-1})).
\end{aligned}$$

Solutions of exercises in Chapter 2

$\boxed{2.1}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{SHK} \subseteq \mathbf{HSK}$.

Suppose that $\overline{M} \in \mathbf{SHK}$. Then there are $\overline{N} \in \mathbf{K}$ and $\overline{P}$ such that there is a homomorphism f from $\overline{N}$ onto $\overline{P}$ and $\overline{M}$ is a subalgebra of $\overline{P}$. Let $Q = f^{-1}[M]$. Clearly Q is a subuniverse of $\overline{N}$. Let $\overline{Q}$ be the associated subalgebra of $\overline{N}$. Then $f \restriction Q$ is a homomorphism from $\overline{Q}$ onto $\overline{M}$. Hence $\overline{M} \in \mathbf{HSK}$.

$\boxed{2.2}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{SHK} \neq \mathbf{HSK}$.

Let $\mathbf{K} = \{\mathbb{Q}\}$ in the signature of rings. Then $\mathbb{Z}$ is a subalgebra of $\mathbb{Q}$, and $\mathbb{Z}_4$ is a homomorphic image of $\mathbb{Z}$. Thus $\mathbb{Z}_4 \in \mathbf{HSK}$. But $\mathbf{HK}$ has only two elements, $\mathbb{Q}$ and the one-element ring, and $\mathbf{SHK}$ does not have any finite member with more than one element.

$\boxed{2.3}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{PHK} \subseteq \mathbf{HPK}$.

Let $\overline{M} \in \mathbf{PHK}$. Then there is a system $\langle \overline{N}_i : i \in I\rangle$ of members of $\mathbf{K}$, another system $\langle \overline{P}_i : i \in I\rangle$ of algebra, a system $\langle f_i : i \in I\rangle$ of functions, and a function g, such that $\forall i \in I[f_i$ is a homomorphism from $\overline{N}_i$ onto $\overline{P}_i$, and g is an isomorphism from $\prod_{i\in I} \overline{P}_i$ onto $\overline{M}$. Define $h : \prod_{i\in I} N_i \to \overline{M}$ by setting

$$h(x) = g(\langle f_i(x_i) : i \in I\rangle)$$

for any $x \in \prod_{i\in I} N_i$. Then h is a homomorphism from $\prod_{i\in I} \overline{N}_i$ onto $\overline{M}$, hence $\overline{M} \in \mathbf{HPK}$.

$\boxed{2.4}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{PHK} \neq \mathbf{HPK}$. *Hint: let* $\mathbf{K}$ *consist of all fields* $\mathbb{Z}_p$ *for p a prime, and take an ultraproduct of them with a nonprincipal ultrafilter. Show that the result is an infinite field.*

Let $I = \{p : p$ is a prime$\}$, and let D be a nonprincipal ultrafilter on I. We use the signature $+, \cdot, -, 0, 1$ for the fields $\mathbb{Z}_p$. Now $A \stackrel{\text{def}}{=} \prod_{i\in I} \mathbb{Z}_p/D$ is a commutative ring with identity, since equations are used to define such. Now take a nonzero element $[x]$ of A. Then $M \stackrel{\text{def}}{=} \{p \in I : x(p) \neq 0\}$ is in D. There is a y such that $y(p) = 1/x(p)$ for all $p \in M$. It follows that $\{p \in I : x(p) \cdot y(p) = 1\} \in D$; hence $[x] \cdot [y] = [1]$. So A is a field. Now suppose that A is finite; say $A = \{[x_i] : i < n\}$. Let $N = \{p \in I : n < p\}$. If $p \in N$, then there is a $y(p) \in \mathbb{Z}_p$ such that $y(p) \notin \{x_i(p) : i < n\}$. Define $y(p) = 0$ for $p \in I\backslash N$. Then for any $i < n$,

$$N \subseteq \{p \in I : y(p) \neq x_i(p)\},$$

hence the set on the right is in D, and so $[y] \neq [x_i]$. This contradiction shows that A is infinite after all.

Note that $A \in \mathbf{HPK}$. Now a product of two or more fields in not a field, as there are zero divisors in such a product. Hence the only fields in $\mathbf{PHK}$ are isomorphic to members of $\mathbf{K}$, and so are finite. So $A \notin \mathbf{PHK}$.

$\boxed{2.5}$ *Prove that for any class* $\mathbf{K}$ *of algebras we have* $\mathbf{PSK} \subseteq \mathbf{SPK}$.

Let $\overline{M} \in \mathbf{PSK}$. Then there exist a system $\langle \overline{N}_i : i \in I\rangle$ of members of $\mathbf{K}$, a system $\langle \overline{P}_i : i \in I\rangle$ of algebras such that $\overline{P}_i$ is a subalgebra of $\overline{N}_i$ for each $i \in I$, and an isomorphism f from $\overline{M}$ onto $\prod_{i\in I}\overline{P}_i$. Then $\prod_{i\in I}\overline{P}_i$ is a subalgebra of $\prod_{i\in I}\overline{N}_i$. By Proposition 1.5 there exist an algebra $\overline{C}$ and an isomorphism g from $\overline{C}$ onto $\prod_{i\in I}\overline{N}_i$ such that $\overline{M}$ is a subalgebra of $\overline{C}$ and $f \subseteq g$. Thus $\overline{C} \in \mathbf{PK}$, so $\overline{M} \in \mathbf{SPK}$.

$\boxed{2.6}$ *Give an example of a class* $\mathbf{K}$ *of algebras such that* $\mathbf{PSK} \neq \mathbf{SPK}$.

Let $\mathbf{K} = \{\mathbb{Z}_2\}$, the 2-element field. Then every member of $\mathbf{PSK}$ is finite or nondenumerable, but $\mathbf{SPK}$ has a countably infinite member.

$\boxed{2.7}$ *Prove that* $\mathbf{HSPK}$ *is closed under* $\mathbf{H}$, $\mathbf{S}$, *and* $\mathbf{P}$. *Infer that for any class* $\mathbf{K}$ *of structures,* $\mathbf{HSPK}$ *is the smallest variety containing* $\mathbf{K}$.

$\mathbf{HHSPK} = \mathbf{HSPK}$. Next, using exercise 2.1, $\mathbf{HSPK} \subseteq \mathbf{SHSPK} \subseteq \mathbf{HSSPK} = \mathbf{HSPK}$. Finally, using exercises 2.3 and 2.5,

$$\mathbf{HSPK} \subseteq \mathbf{PHSPK} \subseteq \mathbf{HPSPK} \subseteq \mathbf{HSPPK} = \mathbf{HSPK}.$$

For the second part of the exercise, suppose that $\mathbf{K}$ is a class of algebras, $\mathbf{L}$ is a variety, and $\mathbf{K} \subseteq \mathbf{L}$. Then $\mathbf{HSPK} \subseteq \mathbf{HSPL} = \mathbf{L}$.

$\boxed{2.8}$ *Prove that the following hold in any proper relation algebra with unit R:*
(i) $S^{-1}|[-(S|T)] \subseteq -T$.
(ii) $((S|R) \cap id)|R = S|R$.
(iii) $S \subseteq S|S^{-1}|S$.

(i): suppose that $(a,b) \in S^{-1}|[-(S|T)]$ and also $(a,b) \in T$; we want to get a contradiction. Choose c such that $(a,c) \in S^{-1}$ and $(c,b) \in -(S|T)$. Then $(c,a) \in S$ and $(a,b) \in T$, so $(c,b) \in (S|T)$, contradiction.

(ii): We have

$$\begin{aligned}
(a,b) \in ((S|R) \cap id)|R \quad &\text{iff} \quad \exists c[(a,c) \in ((S|R) \cap id) \text{ and } (c,b) \in R \\
&\text{iff} \quad (a,a) \in (S|R) \text{ and } (a,b) \in R \\
&\text{iff} \quad (a,a) \in (S|R) \\
&\text{iff} \quad \exists d[(a,d) \in S \text{ and } (d,a) \in R] \\
&\text{iff} \quad \exists d[(a,d) \in S \text{ and } (d,b) \in R] \\
&\text{iff} \quad (a,b) \in (S|R).
\end{aligned}$$

(iii): suppose that $(a,b) \in S$. Then $(b,a) \in S^{-1}$, hence $(a,a) \in (S|S^{-1})$, hence $(a,b) \in [S|S^{-1}|S]$.

Solutions of exercises in Chapter 3

$\boxed{3.1}$ *Define $\varphi|\psi = \neg\varphi \wedge \neg\psi$. (The Sheffer stroke.). Show that $\neg$ and $\wedge$ can be defined in terms of $|$.*

$\neg\varphi$ is equivalent to $\varphi|\varphi$. Then $\varphi \wedge \psi$ is equivalent to $(\neg\varphi)|(\neg\psi)$.

$\boxed{3.2}$ *A formula φ involving only $S_0, \ldots, S_m$ determines a function $t_\varphi : {}^{m+1}2 \to 2$ defined by $t_\varphi(x) = \varphi[x]$ for any $x \in {}^{m+1}2$. Show that any member of $\bigcup_{0<m<\omega} {}^{({}^m 2)}2$ can be obtained in this way.*

Let $0 < m < \omega$ and let $f \in {}^{{}^m 2}2$. If f takes on only the value 0, then $f = t_\varphi$ with φ the formula $S_0 \wedge \neg S_0$. Suppose that f has at least one value 1. Let $M = \{x \in {}^m 2 : f(x) = 1\}$. Consider the following formula φ:

$$\bigvee_{x \in M} \bigwedge_{i<m} S_i^{x(i)}.$$

Note that for any $x, y \in {}^m 2$ we have $\left(\bigwedge_{i<m} S_i^{x(i)}\right)[y] = 1$ iff $x = y$. It follows that $\varphi[y] = 1$ iff $y \in M$. Hence $t_\varphi = f$.

$\boxed{3.3}$ *Show that the following formula is a tautology:*

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)]$$

(This formula can be used as a single axiom in an axiomatic development of sentential logic.)

A truth table for this formula would involve 32 rows; we want to avoid that. We argue by contradiction. Suppose that f is an assignment which gives our formula the value 0; we want to get a contradiction. It follows that

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau)[f] = 1 \tag{1}$$

and

$$[(\tau \to \varphi) \to (\theta \to \varphi)][f] = 0;$$

from this last condition we get

$$(\tau \to \varphi)[f] = 1 \tag{2}$$

and

$$(\theta \to \varphi)[f] = 0,$$

and the last condition here yields

$$\theta[f] = 1 \quad \text{and} \quad \varphi[f] = 0. \tag{3}$$

Hence from (2) we get

$$\tau[f] = 0.$$

Then (1) yields

$$\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\}[f] = 0,$$

from which we obtain

$$[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)][f] = 1 \tag{4}$$

and

$$\chi[f] = 0,$$

which yields

$$(\neg\chi)[f] = 1. \tag{5}$$

But from (3) we get $(\neg\theta)[f] = 0$, and hence by (5), $(\neg\chi \to \neg\theta)[f] = 0$. So by (4) we have $(\varphi \to \psi)[f] = 0$, so that $\varphi[f] = 1$ and $\psi[f] = 0$. This contradicts (3).

Solutions of exercises in Chapter 4

$\boxed{4.1}$ *Prove Proposition 4.7.*

For (i), we proceed by induction on τ. If τ is v_j with $j \neq i$, or is an individual constant, then $\rho = \tau$ and so ρ is a term. Suppose that τ is v_i. Then $\rho = \tau$ if 0 occurrences of v_i are replaced, or is σ is v_i is replaced by σ. At any rate, ρ is a term. Finally, suppose that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Then ρ is $\mathbf{F}\sigma'_0 \ldots \sigma'_{m-1}$, where σ'_i is obtained from σ_i by replacing 0 or more occurrences of v_i by σ. By the inductive hypothesis, each σ'_i is a term. Hence ρ is a term.

For (ii) we proceed by induction on φ. Suppose that φ is $\sigma = \xi$. Then ψ is $\sigma' = \xi'$, where σ' is obtained from σ by replacing 0 or more occurrences of v_i by τ, and ξ' is obtained from ξ by replacing 0 or more occurrences of v_i by τ. By (i), σ' and ξ' are terms. So ψ is a formula. Next, suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Then ψ is $\mathbf{R}\sigma'_0 \ldots \sigma'_{m-1}$, where each σ'_i is obtained from σ_i by replacing 0 or more occurrences of v_i by τ. By (i), each σ'_i is a term. So ψ is a formula.

Next suppose inductively that φ is $\neg\chi$. Then ψ is $\neg\chi'$, where χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$). By the inductive hypothesis, χ' is a formula. Hence ψ is a formula.

Suppose inductively that φ is $\chi \wedge \rho$. Then ψ is $\chi' \wedge \rho'$, and χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$); ρ' is similarly obtained from ρ. By the inductive hypothesis, χ' and ρ' are formulas. So ψ is a formula.

Finally, suppose inductively that φ is $\exists v_j\chi$. Then ψ is $\exists v_j\chi'$, and χ' is obtained from χ by replacing 0 or more occurrences of v_i by τ (except not if v_i occurs just after the symbol $\exists$). By the inductive hypothesis, χ' is a formula. So ψ is a formula.

$\boxed{4.2}$ *Suppose that $\varphi, \psi, \chi, \theta$ are formulas, $\models \chi \leftrightarrow \theta$, and ψ is obtained from φ by replacing one or more occurrences of χ in φ by θ. Show that $\models \varphi \leftrightarrow \psi$. Hint: use induction on φ.*

We proceed by induction on φ. Note that if $\chi = \varphi$ the conclusion is clear. So we assume that $\chi \neq \varphi$. Then the atomic case vacuously holds, since an atomic formula has no proper subformula. Suppose that the statement is true for φ_0, and φ is $\neg\varphi_0$. Then ψ is $\neg\psi_0$, where ψ_0 is obtained from φ_0 by replacing one or more occurrences of χ by θ. So $\models \varphi_0 \leftrightarrow \psi_0$ by the inductive assumption. Clearly then $\models \varphi \leftrightarrow \psi$.

The other cases are treated similarly. For the quantifier case, one needs to observe that $\models \varphi \leftrightarrow \psi$ implies that $\models \exists v_i\varphi \leftrightarrow \exists v_i\psi$.

$\boxed{4.3}$ *Show that $\models \exists x\forall y\varphi \rightarrow \forall y\exists x\varphi$.*

Let $\overline{M}$ be any $\mathscr{L}$-structure, with $\mathscr{L}$ the implicit language we are working in. We use v_i for x and v_j for y to apply the rigorous definition of satisfaction; here $i \neq j$. Suppose that $a \in {}^{\omega}M$ and $\overline{M} \models \exists v_i\forall v_i\varphi[a]$. Then there is a $u \in M$ such that $\overline{M} \models \forall v_j\varphi[a^i_u]$. To prove that $\overline{M} \models \forall v_j\exists v_i\varphi[a]$, take any $w \in M$. Then $\overline{M} \models \varphi[a^{ij}_{uw}]$, hence $\overline{M} \models \exists v_i\varphi[a^j_w]$. Hence $\overline{M} \models \forall v_j\exists v_i\varphi[a]$.

$\boxed{4.4}$ *Show that*

$$\models \exists x[\varphi \wedge \psi \wedge \exists y(\varphi \wedge \neg\psi)] \rightarrow \exists y(\exists x\varphi \wedge \neg x = y).$$

Again we suppose that x is v_i and y is v_j, with $i \neq j$. Suppose that $\overline{A}$ is an $\mathscr{L}$-structure, $a \in {}^\omega A$, and $\overline{A} \models \exists v_i[\varphi \wedge \psi \wedge \exists v_j(\varphi \wedge \neg\psi)][a]$. Choose $u \in A$ such that $\overline{A} \models [\varphi \wedge \psi \wedge \exists v_j(\varphi \wedge \neg\psi)][a^i_u]$. Then choose $w \in A$ such that $\overline{A} \models (\varphi \wedge \neg\psi)[a^{i\ j}_{u\ w}]$. Thus $\overline{A} \models \psi[a^i_u]$ while $\overline{A} \models \neg\psi[a^{i\ j}_{u\ w}]$, hence it is not the case that $\overline{A} \models \psi[a^{i\ j}_{u\ w}]$. It follows that $a^i_u \neq a^{i\ j}_{u\ w}$, and the only way this can happen is that $a_j \neq w$. We now consider two cases.

Case 1. $a_i \neq w$. Then $\overline{A} \models \neg(v_i = v_j)[a^j_w]$, $\overline{A} \models \varphi[a^{i\ j}_{u\ w}]$, hence $\overline{A} \models \exists v_i\varphi[a^j_w]$, so $\overline{A} \models \exists v_j[\exists v_i\varphi \wedge \neg(v_i = v_j)][a]$, as desired.

Case 2. $a_i = w$. Then $\overline{A} \models \neg(v_i = v_j)[a]$, $\overline{A} \models \varphi[a^i_u]$, hence $\overline{A} \models \exists v_i\varphi[a]$, so $\overline{A} \models \exists v_j[\exists v_i\varphi \wedge \neg(v_i = v_j)][a]$, as desired.

$\boxed{4.5}$ *Show that*

$$\models \exists x\varphi \wedge \exists y\psi \wedge \exists z\chi \rightarrow \exists x\exists y\exists z[\exists x(\exists y\chi \wedge \exists z\psi) \wedge \exists y(\exists z\varphi \wedge \exists x\chi) \wedge \exists z(\exists x\psi \wedge \exists y\varphi)].$$

Again, let x, y, z be v_i, v_j, v_k respectively, with i, j, k different. Suppose that $\overline{A}$ is an $\mathscr{L}$-structure and $a \in {}^\omega A$. Suppose that $\overline{A} \models (\exists v_i\varphi \wedge \exists v_j\psi \wedge \exists v_k\chi)[a]$. Accordingly, choose $r, s, t \in A$ such that

$$\overline{A} \models \varphi[a^i_r] \tag{1}$$
$$\overline{A} \models \psi[a^j_s] \tag{2}$$
$$\overline{A} \models \chi[a^k_t] \tag{3}$$

We claim that

$$\overline{A} \models [\exists v_i(\exists v_j\chi \wedge \exists v_k\psi) \wedge \exists v_j(\exists v_k\varphi \wedge \exists v_i\chi) \wedge \exists v_k(\exists v_i\psi \wedge \exists v_j\varphi)][a^{i\ j\ k}_{r\ s\ t}]. \tag{4}$$

Clearly this will prove the desired conclusion. By (3) we have $\overline{A} \models \exists v_j\chi[a^{j\ k}_{s\ t}]$, and by (2) we have $\overline{A} \models \exists v_k\psi[a^{j\ k}_{s\ t}]$. Hence

$$\overline{A} \models (\exists v_j\chi \wedge \exists v_k\psi)[a^{j\ k}_{s\ t}];$$

hence

$$\overline{A} \models \exists v_i(\exists v_j\chi \wedge \exists v_k\psi)[a^{i\ j\ k}_{r\ s\ t}]. \tag{5}$$

By (1) we have $\overline{A} \models \exists v_k\varphi[a^{i\ k}_{r\ t}]$, and by (3) we have $\overline{A} \models \exists v_i\chi[a^{i\ k}_{r\ t}]$. Hence

$$\overline{A} \models (\exists v_k\varphi \wedge \exists v_i\chi)[a^{i\ k}_{r\ t}];$$

hence

$$\overline{A} \models \exists v_j((\exists v_k\varphi \wedge \exists v_i\chi)[a^{i\ j\ k}_{r\ s\ t}]. \tag{6}$$

By (2) we have $\overline{A} \models \exists v_i\psi[a^{i\ j}_{r\ s}]$, and by (1) we have $\overline{A} \models \exists v_j\varphi[a^{i\ j}_{r\ s}]$. Hence

$$\overline{A} \models (\exists v_i\psi \wedge \exists v_j\varphi)[a^{i\ j}_{r\ s}];$$

hence

$$\overline{A} \models \exists v_k(\exists v_i \psi \wedge \exists v_j \varphi)[a^{i\ j\ k}_{r\ s\ t}]. \tag{7}$$

Clearly (5)–(7) give the desired result (4).

$\boxed{4.6}$ *In $(\omega, 0, \mathbb{S})$, show that every singleton $\{m\}$ for $m \in \omega$ is definable, i.e., there is a formula $\varphi_m(x)$ with only x free such that $\{m\} = \{a \in \omega : (\omega, 0, \mathbb{S}) \models \varphi[a]\}$. Here $\mathbb{S}$ is the successor function, which assigns $m+1$ to each natural number m.*

By recursion, let $\overline{0} = 0$ and $\overline{m+1} = S\overline{m}$. Then a formula φ_m as required is $v_0 = \overline{m}$.

$\boxed{4.7}$ *Let $\mathscr{L}$ be a relational language. A formula φ is standard if every nonequality atomic part of φ has the form $\mathbf{R}v_0 \ldots v_{m-1}$, where $\mathbf{R}$ is m-ary. (Normally any sequence of m variables is allowed.) Show that for every formula φ there is a standard formula ψ such that $\models \varphi \leftrightarrow \psi$.*

Clearly it suffices to do this only for atomic formulas $\mathbf{R}v_{i_0} \ldots v_{i_{m-1}}$. Let j be greater than both m and each i_k. Let φ be the formula

$$\exists v_j \ldots \exists v_{j+m-1}\left[\bigwedge_{k<m}(v_{j+k} = v_{i_k}) \wedge \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right]\right].$$

To show that this works, let $\overline{M}$ be any $\mathscr{L}$-structure and $a \in {}^{\omega}M$. Then

$\overline{M} \models \varphi[a]$ iff there are $b(0), \ldots, b(m-1) \in M$ such that

$$\overline{M} \models \bigwedge_{k<m}(v_{j+k} = v_{i_k}) \wedge \exists v_0 \ldots \exists v_{m-1}$$

$$\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right][a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and

$$\overline{M} \models \exists v_0 \ldots \exists v_{m-1}\left[\bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}\right][a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that

$$\overline{M} \models \bigwedge_{k<m}(v_k = v_{j+k}) \wedge \mathbf{R}v_0 \ldots v_{m-1}[(a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)})^{0}_{c(0)} \cdots {}^{m-1}_{c(m-1)}]$$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that $c(k) = b(k)$ for all $k < m$ and $\overline{M} \models \mathbf{R}v_0 \ldots v_{m-1}[(a^{j}_{b(0)} \cdots {}^{j+m-1}_{b(m-1)})^{0}_{c(0)} \cdots {}^{m-1}_{c(m-1)}]$

iff there are $b(0), \ldots, b(m-1) \in M$ such that $b(k) = a(i_k)$ for each $k < m$ and there are $c(0), \ldots, c(m-1) \in M$ such that $c(k) = b(k)$ for all $k < m$ and

$$\langle c(0),\ldots,c(m-1)\rangle \in \mathbf{R}^{\overline{M}}$$
$$\text{iff}\quad \langle a(i_0),\ldots,a_{i_{m-1}}\rangle \in \mathbf{R}^{\overline{M}}$$
$$\text{iff}\quad \overline{M} \models \mathbf{R}v_{i_0}\ldots v_{i_{m-1}}[a].$$

$\boxed{4.8}$ *In the language of rings, write down a single sentence whose models are exactly all rings.*

$$\forall x\forall y[x+y=y+x] \wedge \forall x\forall y\forall z[x+(y+z)=(x+y)+z] \wedge \forall x[x+0=x]$$
$$\wedge\, \forall x[x+(-x)=0] \wedge \forall x\forall y\forall z[x\cdot(y\cdot z)=(x\cdot y)\cdot z]$$
$$\wedge\, \forall x\forall y\forall z[x\cdot(y+z)=(x\cdot y)+(x\cdot z)] \wedge \forall x\forall y\forall z[(y+z)\cdot x=(y\cdot x)+(z\cdot x)]$$

$\boxed{4.9}$ *We describe an extension of first-order logic that can be used to make the set theoretical notation $\{a\in A:\varphi\}$ formal (rather than being treated as an abbreviation). Let $\mathscr{L}$ be a first order language, with an individual constant $\mathbf{Z}$ which will play a special role (in set theory, this can be the empty set as introduced in a definition). We define description terms and description formulas simultaneously:*

(a) Any variable or individual constant is a description term.

(b) If $\mathbf{O}$ is an operation symbol of positive rank m and $\tau_0,\ldots,\tau_{m-1}$ are description terms, then $\mathbf{O}\tau_0\ldots\tau_{m-1}$ is a description term.

(c) If $i<\omega$ and φ is a formula, then $Tv_i\varphi$ is a description term. (This is the description operator. $Tv_i\varphi$ should be read "the v_i such that φ, or $\mathbf{Z}$ if there is not a unique v_i such that φ".

(d) If σ and τ are description terms, then $\sigma=\tau$ is an atomic description formula.

(e) If $\mathbf{R}$ is an m-ary relation symbol and $\tau_0,\ldots,\tau_{m-1}$ are description terms, then $\mathbf{R}\tau_0\ldots\tau_{m-1}$ is an atomic description formula.

(f) If φ and ψ are description formulas and $i<\omega$, then the following are description formulas: $\neg\varphi$, $(\varphi\wedge\psi)$, and $\exists v_i\varphi$.

Next we define the value of description terms, and satisfaction of description formulas in an $\mathscr{L}$-structure simultaneously. Let $\overline{A}$ be an $\mathscr{L}$-structure, and let $a\in{}^{\omega}A$.

(a) $v_i^{\overline{A}}=a_i$.

(b) If τ is the term $\mathbf{F}\sigma_0\ldots\sigma_{m-1}$, then

$$\tau^{\overline{A}}(a)=\mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a),\ldots,\sigma_{m-1}^{\overline{A}}(a)).$$

(c) If τ is the term $Tv_i\varphi$, then

$$\tau^{\overline{A}}=\begin{cases}\text{the } x\in A \text{ such that } \overline{A}\models\varphi[a] \text{ and } a_i=x & \text{if there is a unique such } x,\\ \mathbf{Z}^{\overline{A}} & \text{otherwise.}\end{cases}$$

(d) If φ is $\sigma=\tau$, then $\overline{A}\models\varphi[a]$ iff $\sigma^{\overline{A}}(a)=\tau^{\overline{A}}(a)$.

(e) If φ is $\mathbf{R}\sigma_0\ldots\sigma_{m-1}$, then $\overline{A}\models\varphi[a]$ iff $(\sigma_0^{\overline{A}}(a),\ldots,\sigma_{m-1}^{\overline{A}}(a))\in r_{\mathbf{R}}$.

(f) $\overline{A}\models\neg\varphi[a]$ iff it is not the case that $\overline{A}\models\varphi[a]$.

(g) $\overline{A}\models(\varphi\wedge\psi)[a]$ iff $\overline{A}\models\varphi[a]$ and $\overline{A}\models\psi[a]$.

(h) $\overline{A} \models \exists v_i \varphi[a]$ *iff there is an* $x \in A$ *such that* $\overline{A} \models \varphi[a^i_x]$.

Show that for any description formula φ *there is an ordinary formula* ψ *with the same free variables such that* $\models \varphi \leftrightarrow \psi$.

We prove by simultaneous induction on terms and formulas that if φ is a description formula and τ is a description term, and if v_i does not occur in τ, then there are ordinary formulas ψ and χ such that $\models \varphi \leftrightarrow \psi$ and $\models (v_i = \tau) \leftrightarrow \chi$.

(1) Suppose that τ is v_j. Take χ to be $v_i = v_j$.

(2) Suppose that τ is an individual constant $\mathbf{c}$. Take χ to be $v_i = \mathbf{c}$.

(3) Suppose that $\mathbf{O}$ is an operation symbol of rank $m > 0$, and $\tau_0, \ldots, \tau_{m-1}$ are description terms about which we know our result. Choose n greater than i and all v_j occurring in any term τ_k. Choose ordinary formulas $\theta_0, \ldots, \theta_{m-1}$ such that $\models (v_{n+j} = \tau_j) \leftrightarrow \theta_j$. Then the following formula works for $\mathbf{O}\tau_0 \ldots \tau_{m-1}$:

$$\exists v_n \ldots v_{n+m-1} \left(\bigwedge_{j<m} \theta_j \wedge v_i = \mathbf{O} v_n \ldots v_{n+m-1} \right).$$

(4) Suppose that τ is $T v_j \psi$. Let σ be an ordinary formula such that $\models \psi \leftrightarrow \sigma$. Then the following formula works for τ, where v_k is a new variable:

$$[\sigma(v_i) \wedge \forall v_k(\sigma(v_k) \rightarrow v_i = v_k)] \vee [\neg \exists v_i[\sigma(v_i) \wedge \forall v_k(\sigma(v_k) \rightarrow v_i = v_k)] \wedge v_i = \mathbf{Z}].$$

(5) If φ is $\sigma = \rho$ for description terms σ, ρ, choose ordinary formulas η, ξ such that $\models (v_m = \sigma) \leftrightarrow \eta$ and $\models (v_n = \rho) \leftrightarrow \xi$, where v_m and v_n are new variables. Then we can take the following formula for ψ:

$$\exists v_m \exists v_n [\eta \wedge \xi \wedge v_m = v_n]$$

The other cases are straightforward.

$\boxed{4.10}$ *We modify the definition of first-order language by using parentheses. Thus we add two symbols (and) to our logical symbols.*

We retain in this context the same definition of terms as before. But we change the definition of formula as follows:

An atomic formula is a sequence of one of the following two sorts: $(\sigma = \tau)$, *with* σ *and* τ *terms; or* $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, *where* $\mathbf{R}$ *is a relation symbol of rank* m *and* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms. Then we define the collection of formulas to be the intersection of all sets of sequences of symbols such that every atomic formula is in* A, *and if* φ *and* ψ *are in* A *and* $i < \omega$, *then each of the following is in* A: $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\exists v_i \varphi)$.

Prove the analog of Proposition 4.1(i), adding an additional condition, that in a formula the number of left parentheses is equal to the number of right parentheses, while in any proper initial segment of a formula, either there are no parentheses, or there are more left parentheses than right ones.

We go by induction on the formula φ. It is clear for atomic formulas. Suppose that it is true for φ and ψ.

Then $(\neg\varphi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\neg\varphi)$ are of these kinds: $\emptyset$; (; $(\neg$; and $(\neg\sigma$ with σ an initial segment of φ, possibly equal to φ. Our condition is clear in each case.

Similarly, $(\varphi\wedge\psi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\varphi\wedge\psi)$ are of these kinds: $\emptyset$; (; $(\sigma$ with σ an initial segment of φ, possibly equal to φ; $(\varphi\wedge$; and $(\varphi\wedge\sigma$ with σ an initial segment of ψ, possibly equal to ψ. Our condition is clear in each case.

Similarly, $(\exists v_i\varphi)$ has one more of each kind of parenthesis, so it has an equal number of both. The proper initial segments of $(\exists v_i\varphi)$ are of these kinds: $\emptyset$; (; $(\exists$; $(\exists v_i$; and $(\exists v_i\sigma$ with σ an initial segment of φ, possibly equal to φ. Our condition is clear in each case.

$\boxed{4.11}$ *Show that* 0 *and* $\mathbb{S}$ *are definable in* $(\omega,<)$. *That is, there are formulas* $\varphi(x)$ *and* $\psi(x,y)$ *with only the indicated free variables such that for all* $a\in\omega$, $a=0$ *iff* $(\omega,<)\models\varphi[a]\}$, *and for all* $a,b\in\omega$, $\mathbb{S}a=b$ *iff* $(\omega,<)\models\psi[a,b]$. *Here we are working in the language of orderings.*

Let φ be the formula $\neg\exists y[y<x]$. Clearly $a=0$ iff $(\omega,<)\models\varphi[a]\}$. Let ψ be the formula

$$x<y\wedge\neg\exists z[x<z\wedge z<y].$$

Clearly $\mathbb{S}a=b$ iff $(\omega,<)\models\psi[a,b]$.

Solutions of exercises in Chapter 5

$\boxed{\text{E5.1}}$ *Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Let F be a nonprincipal ultrafilter on a set I. For each $a \in A$ let $f(a) = [\langle a : i \in I\rangle]_F$. Show that f is an embedding of $\overline{A}$ into ${}^I\overline{A}/F$, and $\overline{A}$ is elementarily equivalent to ${}^I\overline{A}/F$.*

For brevity let $\overline{B} = {}^I\overline{A}$ and $\overline{C} = \overline{B}/F$. See the definition of $\overline{C}$ following Theorem 1.15.

Suppose that $f(a) = f(b)$. Then $[\langle a : i \in I\rangle]_F = [\langle b : i \in I\rangle]_F$, hence $\{i \in I : a = b\} \in F$. Since the empty set is not in F, it follows that $a = b$. So f is one-one.

If k is an individual constant, obviously $f(k^{\overline{A}}) = k^{\overline{B}}$.

Suppose that G is an m-ary function symbol. Then

$$\begin{aligned} f(G^{\overline{A}}(a^0, \ldots, a^{m-1})) &= [\langle G^{\overline{A}}(a^0, \ldots, a_{m-1})]_F \\ &= [G^{\overline{B}}(\langle a^0 : i \in I\rangle, \ldots, \langle a^{m-1} : i \in I\rangle)]_F \\ &= G^{\overline{C}}(f(a^0), \ldots, f(a^{m-1})). \end{aligned}$$

If R is an m-ary relation symbol, then

$$\begin{aligned} \langle f(a^0), \ldots, f(a^{m-1})\rangle \in R^{\overline{C}} \quad &\text{iff} \quad \{i \in I : \langle a^0, \ldots, a^{m-1}\rangle \in R^{\overline{A}}\} \in F \\ &\text{iff} \quad \langle a^0, \ldots, a^{m-1}\rangle \in R^{\overline{A}}. \end{aligned}$$

Hence f is an isomorphism of $\overline{A}$ into $\overline{C}$. The last statement of the exercise is true by Corollary 5.2.

$\boxed{5.2}$ *We work in the language for ordered fields; see Chapter 1. In general, an element $a \in M$ is definable iff there is a formula $\varphi(x)$ with one free variable x such that $\{b \in M : \overline{M} \models \varphi[b]\} = \{a\}$.*

(i) Show that 1 is definable in $\mathbb{R}$.

(ii) Show that every positive integer is definable in $\mathbb{R}$.

(iii) Show that every positive rational is definable in $\mathbb{R}$.

(iv) If $\overline{M}$ is an extension of $\mathbb{R}$, an element ε of M is infinitesimal iff $0 < \varepsilon < r$ for every positive rational r. Let $\overline{M}$ be an ultrapower of $\mathbb{R}$ using a nonprincipal ultrafilter on ω. Thus $\overline{M}$ is isomorphic to an extension of $\mathbb{R}$ by exercise 5.1. Show that $\overline{M}$ has an infinitesimal.

(v) Use the compactness theorem to show the existence of an ordered field $\overline{M}$ which has an infinitesimal, and is elementarily equivalent to $\mathbb{R}$.

(i): Let $\varphi(x)$ be the formula $\forall y[x \cdot y = y]$.

(ii): Let φ be as in (i). By induction we define a formula ψ_m which defines m, for each positive integer m. Let ψ_1 be φ. Having defined ψ_m, let ψ_{m+1} be the formula $\exists y \exists z[\psi_m(y) \wedge \varphi(z) \wedge x = y + z]$.

(iii) Let r be a positive rational. Say $r = m/n$ with m and n positive integers. Let χ_r be the formula $\exists y \exists z[\psi_m(y) \wedge \psi_n(z) \wedge y = x \cdot z]$.

(iv) Let F be a nonprincipal ultrafilter on ω. Define $e \in {}^\omega\mathbb{R}$ by setting $e(n) = 1/(n+1)$ for every $n \in \omega$. We claim that $[e]$ is an infinitesimal. To prove this, take any positive rational r. Choose $p \in \omega$ with $\frac{1}{p} < r$. Let $x(m) = r$ for all $m \in \omega$. Thus $[x]$ is the image of r under the isomorphism of exercise 5.1, so it suffices to show that $[0] < [e] < [r]$. We have

$$\{m \in \omega : 0 < e(m)\} = \omega \in F$$

and

$$\{m \in \omega : e(m) < r\} \supseteq \{m \in \omega : m \geq p\} \in F;$$

hence $[0] < [e] < [r]$.

(v) Adjoin a new individual constant $\mathbf{c}$ to our language, and consider the following set of sentences:

$$\begin{aligned}&\{\varphi : \varphi \text{ is a sentence and } \mathbb{R} \models \varphi\}\\ &\cup \{0 < \mathbf{c}\} \cup \{\forall x[\mathbf{c} < \chi_r(x)] : r \text{ a positive rational}\}.\end{aligned}$$

Clearly every finite subset of this set has a model; the compactness theorem gives a model of the whole set, and this give the desired conclusion. (The denotation of the constant $\mathbf{c}$ is ignored in order to make the final model an ordered field with no extra fundamental constant.)

$\boxed{5.3}$ *Consider the structure* $\overline{N} = (\omega, +, \cdot, 0, 1, <)$. *We look at models of* $\Gamma = \{\varphi : \varphi$ *is a sentence and* $\overline{N} \models \varphi\}$.

(i) For every $m \in \omega$ *there is a formula* φ_m *with one free variable* x *such that* $\overline{N} \models \varphi_m[m]$ *and* $\overline{N} \models \exists! x \varphi_m(x)$.

(ii) $\overline{N}$ *can be embedded in any model of* Γ.

(iii) Show that Γ *has a model with an infinite element in it, i.e., an element greater than each* $m \in \omega$.

(i): We define φ_m by recursion; clearly the ones defined work: φ_0 is $x = 0$. Having defined φ_m, φ_{m+1} is the formula $\exists y[\varphi_m(y) \wedge x = y + 1]$.

(ii) For each $m \in \omega$, let $f(m)$ be the unique $a \in M$ such that $\overline{M} \models \varphi_m[a]$. If $m \neq n$, then $\overline{N} \models \neg(\varphi_m(x) \wedge \varphi_n(x))$, so also $\overline{m} \models \neg(\varphi_m(x) \wedge \varphi_n(x))$, hence $f(m) \neq f(n)$.

Next, $\overline{N} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow \varphi_{m+n}(x+y)$, so also $\overline{M} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow \varphi_{m+n}(x+y)$. Now $\overline{M} \models \varphi_m[f(m)]$ and $\overline{M} \models \varphi_m[f(n)]$, so $\overline{M} \models \varphi_{m+n}[f(m) + f(n)]$. Hence $f(m + n) = f(m) + f(n)$.

Similarly for $\cdot$.

If $m < n$, then $\overline{N} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow x < y$, so $\overline{M} \models \forall x \forall y[\varphi_m(x) \wedge \varphi_n(y) \rightarrow x < y$, so $f(m) < f(n)$. It folows that $m < n$ iff $f(m) < f(n)$.

This finishes the proof of (ii).

For (iii), adjoin a new individual constant $\mathbf{c}$ and consider the set

$$\begin{aligned}\Gamma =&\{\varphi : \varphi \text{ is a sentence and } \overline{N} \models \varphi\}\\ &\cup \{\forall x[\varphi_m(x) \rightarrow x < \mathbf{c}] : m \in \omega\}.\end{aligned}$$

By the compactness theorem, Γ has a model, which gives the desired conclusion.

$\boxed{5.4}$ *(Continuing exercise 5.3.) An element p of a model $\overline{M}$ of Γ is a prime iff $p > 1$ and for all $a, b \in M$, if $p = a \cdot b$ then $a = 1$ or $a = p$.*

(i) Prove that if $\overline{M}$ is a model of Γ with an infinite element, then it has an infinite prime element.

(ii) Show that the following conditions are equivalent:

(a) There are infinitely many (ordinary) primes p such that $p + 2$ is also prime. (The famous twin prime conjecture, unresolved at present.)

(b) There is a model $\overline{M}$ of Γ having at least one infinite prime p such that $p+2$ is also a prime.

(c) For every model $\overline{M}$ of Γ having an infinite element, there is an infinite prime p such that $p + 2$ is also a prime.

(i): The sentence $\forall m \exists p[m < p \wedge$ "p is a prime"$]$ holds in $\overline{N}$ and hence in $\overline{M}$. Applying this with m an infinite element yields an infinite prime.

(ii): (a)⇒(c): Assume (a), and let $\overline{M}$ be any model of Γ having an infinite element e. Now $\forall m \exists p[p$ "is a prime, and also $p + 2$ is a prime$]$ holds in $\overline{N}$, and hence also in $\overline{M}$. Applying this with m an infinite element gives the desired conclusion.

(c)⇒(b): By exercise 5.3, there is a model $\overline{M}$ of Γ having an infinite element. Hence (c) gives the conclusion of (b).

(b)⇒(a): Suppose that (a) is false. Choose $m \in \omega$ such that if p and $p+2$ are primes, then $p < m$. Then the sentence

$$\forall x[\varphi_m(x) \rightarrow \forall p[p \text{ “is a prime, and also } p + 2 \text{ is a prime” } \rightarrow p < x]]$$

holds in $\overline{N}$, and hence also in $\overline{M}$. But then there cannot exist an infinite prime p of $\overline{M}$ such that $p + 2$ is also a prime.

$\boxed{5.5}$ *Let G be a group which has elements of arbitrarily large finite order. Show that there is a group H elementarily equivalent to G which has an element of infinite order.*

Add an individual constant **c** to our language. Then the set

$$\{\varphi : \varphi \text{ is a sentence and } G \models \varphi\} \cup \{\neg(\mathbf{c} = e\} \cup \{\neg(\mathbf{c}^m = e) : m \in \omega\backslash 1\}$$

has a model by the compactness theorem, giving the desired result.

$\boxed{5.6}$ *Suppose that Γ is a set of sentences, and φ is a sentence. Prove that if $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

We prove the contrapositive: Suppose that for every finite subset Δ of Γ, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg\varphi\}$ has a model, so $\Gamma \cup \{\neg\varphi\}$ has a model, proving that $\Gamma \not\models \varphi$.

Solutions of exercises in Chapter 6

$\boxed{\text{E6.1}}$ *A subset X of a structure $\overline{M}$ is definable iff there is a formula $\varphi(x)$ with only x free such that $X = \{a \in M : \overline{M} \models \varphi[a]\}$. Similarly, for any positive integer m, a subset X of ${}^mM$ is definable iff there is a formula $\varphi(\overline{x})$ with $\overline{x}$ a sequence of m distinct variables including all variables occurring free in φ, such that $X = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$.*

For the language with no nonlogical symbols and for any structure $\overline{M}$ in that language, determine all the definable subsets and m-ary relations over $\overline{M}$. Hint: use Theorem 6.1.

Let $\overline{M}$ be any $\mathscr{L}$-structure. Thus $\overline{M}$ is essentially just a set. Its definable subsets are $\emptyset$ and M, which are clearly defined by $x \neq x$ and $x = x$ respectively. Suppose that $A \subseteq M$ is definable by $\varphi(x)$, with $\emptyset \neq A \neq M$. Let $a \in A$ and $b \in M \backslash A$. Let f be the bijection of M which interchanges a and b. Then $\overline{M} \models \varphi[a]$ but $\overline{M} \not\models \varphi[f(a)]$, contradicting Theorem 6.1.

Now suppose that $m > 1$. For each equivalence relation $\equiv$ on m let

$$R_{\equiv} = \{a \in {}^mM : \forall i, j < m[a_i = a_j \text{ iff } i \equiv j\}.$$

We claim that the definable m-are relations over $\overline{M}$ are just $\emptyset$ and unions of these relations $R_{\equiv}$. To show that these are definable, for $\emptyset$ take $\bigwedge_{i<m} x_i \neq x_j$; and for a nonempty set E of equivalence relations, take the formula

$$\bigvee_{\equiv \in E} \left(\bigwedge_{i \equiv j} (x_i = x_j) \wedge \bigwedge_{i \not\equiv j} (x_i \neq x_j) \right),$$

which we denote by φ_E. Now suppose that K is a nonempty definable m-ary relation on M; say $K = \{a \in {}^mM : \overline{M} \models \psi[a]\}$. For each $a \in K$ let $\equiv_a = \{(i, j) \in m \times m : a_i = a_j\}$, and let $E = \{\equiv_a : a \in K\}$. If $a \in K$, then $\equiv_a \in E$, and hence $\overline{M} \models \varphi_E[a]$. So $K \subseteq \{a \in {}^mM : \overline{M} \models \varphi_E[a]\}$. Now suppose that $\overline{M} \models \varphi_E[a]$ but $a \notin K$. Choose $b \in K$ such that

$$\overline{M} \models \left(\bigwedge_{i \equiv_b j} (x_i = x_j) \wedge \bigwedge_{i \not\equiv_b j} (x_i \neq x_j) \right) [a].$$

Then there is a bijection f of M onto M such that $f(b_i) = a_i$ for all $i < m$. Now $\overline{M} \models \psi[b]$ but $\overline{M} \not\models \psi[a]$, contradiction.

$\boxed{\text{E6.2}}$ *Let Γ be the set of all sentences holding in the structure $(\omega, S, 0)$, where $S(n) = n+1$ for all $n \in \omega$. Prove an elimination of quantifiers theorem for Γ.*

By the general procedure at the beginning of this chapter it suffices to eliminate the quantifier in a formula of the form $\exists x\varphi$, where φ is a conjunction of atomic formulas and their negations, where x actually occurs in each conjunct. Moreover, for any natural numbers m, n we have $\Gamma \models S^m x = S^n x$ if $m = n$, and $\Gamma \models S^m x \neq S^n x$ if $m \neq n$. So we may assume that the atomic and negated atomic formulas have the form $S^m x = S^n u$ and $S^m x \neq S^n u$, where u is 0 or a variable different from x. Now $\Gamma \models S^m x = S^n u \leftrightarrow$

$S^{m+1}x = S^{n+1}u$, so we may assume that m does not depend on any particular conjunct. If a conjunct $S^m x = S^n u$ actually appears, then

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \wedge \psi,$$

where ψ is obtained from φ by replacing $S^m x$ by $S^n u$. Now if $m \leq n$, then $\Gamma \models S^m x = S^n u \leftrightarrow x = S^{n-m}u$, and so

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \leftrightarrow 0 = 0,$$

eliminating the quantifier. If $n < m$, then

$$\Gamma \models \exists x\varphi \leftrightarrow \exists x(S^m x = S^n u) \leftrightarrow u \neq 0 \wedge u \neq S0 \wedge \ldots \wedge u \neq S^{m-n}0,$$

again eliminating the quantifier.

Hence we may assume that no conjunct $S^m x = S^n u$ actually appears in φ. Thus φ has the form

$$S^m x \neq S^{n(0)}u_0 \wedge \ldots \wedge S^m x \neq S^{n(k)}u_k,$$

where each u_i is a variable not equal to x, or is the individual constant 0. Then we claim that $\Gamma \models \exists x\varphi \leftrightarrow 0 = 0$, i.e, $\Gamma \models \exists x\varphi$. For, suppose that $a \in {}^\omega\omega$. Say x is v_j. Choose $v \in \omega$ such that $S^m v \neq S^{n(0)}u_0(a), \ldots, S^m v \neq S^{n(k)}u_k(a)$. Thus $(\omega, S, 0) \models \varphi[a^i_v]$, as desired.

$\boxed{\text{E6.3}}$ *Let T be the theory of an infinite equivalence relation each of whose equivalence classes has exactly two elements. Use an Ehrenfeucht game to show that T is complete.*

To be precise, let T consist of the following sentences:

$\forall x[xEx]$

E is symmetric and transitive

there are infinitely many elements in the model

$\forall v_0$ there are exactly two elements equivalent to v_0.

Now assume that $\overline{A}$ and $\overline{B}$ are models of Γ and m is a positive integer. The strategy of ISO is as follows. Suppose that we are at the i-th turn and NON-ISO chooses 0 and an element $a \in A$. The move of ISO depends on the following possibilities. If the turns so far have not produced a partial isomorphism, then ISO selects any element of B. Suppose that the turns so far have produced a partial isomorphism f.

Case 1. No element of A equivalent to a has been selected yet. Then ISO picks an element of B not equivalent to any element selected so far.

Case 2. There is an element $a' \in A$ which has already been selected which is equivalent to a, while a itself has not been previously selected. Then ISO picks an element of B equivalent to $f(a')$ which has not yet been selected.

Case 3. a has already been selected. Then ISO picks $f(a)$.

If NON-ISO chooses 1 and an element of B, ISO does a similar thing, interchanging the roles of A and B.

Clearly this produces a partial isomorphism. $\square$

$\boxed{\text{E6.4}}$ *Let T be any theory. Show that the class of all substructures of models of T is the class of all models of a set of universal sentences, i.e., sentences of the form $\forall\overline{x}\varphi$ with φ quantifier free and $\overline{x}$ a finite string of variables containing all variables free in φ.*

Let $\Gamma = \{\forall\overline{x}\varphi : \overline{x}$ is a finite string of variables containing all variables free in φ, and $T \models \forall\overline{x}\varphi\}$.

Suppose that $\overline{A} \leq \overline{B} \models T$. Clearly then $\overline{A} \models \Gamma$.

Conversely, suppose that $\overline{A} \models \Gamma$. Then in order to show that $\overline{A}$ can be embedded in a model of T it suffices to show that $T \cup \mathrm{Diag}(\overline{A})$ has a model. Suppose not. Then $T \cup \mathrm{Diag}(\overline{A}) \models \exists x(x \neq x)$. Hence by Lemma 6.29 there is an existential sentence ψ such that $T \models \psi \to \exists x(x \neq x)$; so $T \models \neg\psi$. Hence $\neg\psi \in \Gamma$. But also by Lemma 6.29, $\overline{A} \models \psi$, contradicting $\overline{A} \models \Gamma$.

$\boxed{\text{E6.5}}$ *Suppose that $\Gamma \cup \{\varphi\}$ is a set of sentences in a language $\mathscr{L}$. Suppose that Γ and φ have the same models. Prove that there is a finite subset Δ of Γ with the same models as Γ.*

Applying Lemma 6.28 with Γ, Δ, φ replaced by $\emptyset, \Gamma, \varphi$ respectively, we get a finite conjunction ψ of members of Γ such that $\models \psi \to \varphi$. On the other hand, obviously $\models \varphi \to \psi$. Thus the collection of conjuncts of ψ has the same models as Γ.

$\boxed{\text{E6.6}}$ *Suppose that T and T' are theories in a language $\mathscr{L}$. Show that the following conditions are equivalent:*

(i) Every model of T' can be embedded in a model of T.

(ii) Every universal sentence which holds in all models of T also holds in all models of T'.

(i)$\Rightarrow$(ii): Assume (i), suppose that φ is a universal sentence holding in all models of T, and suppose that $\overline{A}$ is a model of T'. By (i), choose $\overline{B} \models T$ such that $\overline{A} \leq \overline{B}$. Clearly $\overline{A} \models \varphi$, as desired.

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $\overline{C}$ is a model of T'. Let Γ be a set of universal sentences as given in exercise E6.4:

$$\{\overline{A} : \overline{A} \models \Gamma\} = \{\overline{A} : \exists\overline{B} \models T(\overline{A} \leq \overline{B})\}.$$

Clearly then $T \models \Gamma$, so $\overline{C} \models \Gamma$ by (ii), and hence (i) holds.

$\boxed{\text{E6.7}}$ *Let T be a theory in a language $\mathscr{L}$. Let $\mathbf{K}$ be the class of all models of T. Show that the following conditions are equivalent:*

(i) $\mathbf{SK} = \mathbf{K}$.

(ii) There is a collection Γ of universal sentences such that $\mathbf{K}$ is the class of all models of Γ.

This is immediate from exercise E6.4.

$\boxed{\text{E6.8}}$ *Suppose that $\overline{A} \leq \overline{B}$. Prove that $\overline{A} \preceq \overline{B}$ iff $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)_{a\in A}$.*

$\Rightarrow$: Suppose that $\overline{A} \preceq \overline{B}$ and $(\overline{A}, a)_{a\in A} \models \varphi$. Thus $\varphi \in \mathrm{Eldiag}(\overline{A})$, so by Theorem 6.15, $(\overline{B}, a)_{a\in A} \models \varphi$. The converse follows by applying this argument to $\neg\varphi$.

$\Leftarrow$. Assume that $(\overline{A}, a)_{a\in A} \equiv (\overline{B}, a)_{a\in A}$. In particular, $(\overline{B}, a)_{a\in A}$ is a model of $\mathrm{Eldiag}(\overline{A})$, so by Theorem 6.15, $\overline{A} \preceq \overline{B}$.

$\boxed{\text{E6.9}}$ *Suppose that m is a positive integer, $\varphi(\overline{x})$ is a formula with free variables $\overline{x}$ of length m, and $\overline{M}$ is a structure. Define $\varphi(\overline{M}) = \{a \in {}^mM : \overline{M} \models \varphi[a]\}$. Show that the following conditions are equivalent:*

(i) $\varphi(\overline{M})$ is finite.

(ii) $\varphi(\overline{M}) = \varphi(\overline{N})$ whenever $\overline{M} \preceq \overline{N}$.

Assume (i), and suppose that $\overline{M} \preceq \overline{N}$. Say $|\varphi(\overline{M})| = n$. Since $\overline{M} \preceq \overline{N}$, the m-tuples from M that satisfy φ in $\overline{M}$ also satisfy φ in $\overline{N}$. The statement (i) can be expressed by a sentence (with fixed m), and it holds in $\overline{M}$, hence in $\overline{N}$. so (ii) follows.

Now assume that (i) fails; we show that (ii) fails. To the language $\mathscr{L}_A$ adjoin new constants $\overline{d}$ for $i < m$, of length m. We consider the following set of sentences:

$$\mathrm{Eldiag}(\overline{M}) \cup \left\{ \bigvee_{i<m} c_{b(i)} \neq d_i : \overline{M} \models \varphi[b(0), \ldots, b(m-1)] \right\}.$$

By (i) failing and the compactness theorem this set has a model, and this gives an elementary extension $\overline{N}$ of $\overline{M}$ in which $\varphi(\overline{M}) \subset \varphi(\overline{N})$.

$\boxed{\text{E6.10}}$ *Prove that if K is a set of models of a complete theory T then there is a structure $\overline{M}$ such that every member of K can be elementarily embedded in $\overline{M}$.*

Let κ be an infinite cardinal greater than the size of all members of K, and let $\overline{A}$ be a κ^+-saturated model of T. Then $\overline{A}$ is as desired, by Theorem 6.24.

$\boxed{\text{E6.11}}$ *Suppose that $\overline{A}$ and $\overline{B}$ are elementarily equivalent, κ-saturated, and both of size κ. Show that they are isomorphic.*

Write $A = \{a_\alpha : a < \kappa\}$ and $B = \{b_\alpha : \alpha < \kappa\}$. We now define $\langle c_\alpha : \alpha < \kappa\rangle$ and $\langle d_\alpha : \alpha < \kappa\rangle$ by recursion. Suppose they have been defined for all $\beta < \alpha$ so that $(\overline{A}, c_\beta)_{\beta<\alpha} \equiv (\overline{B}, d_\beta)_{\beta<\alpha}$. We now define $c_\alpha, d_\alpha, d_{\alpha+1}, c_{\alpha+1}$. Let $c_\alpha = a_\gamma$, with γ minimum such that $a_\gamma \notin \{c_\beta < \beta < \alpha\}$. Let Γ be the set of all formulas $\varphi(x)$ in $\mathscr{L}_{\langle c_\beta:\beta<\alpha\rangle}$ such that $(\overline{A}, c_\beta)_{\beta<\alpha} \models \varphi[c_\alpha]$. Then $(\overline{A}, c_\beta)_{\beta<\alpha} \models \exists x\Delta$ for every conjunction D of finitely many elements of Γ, so also $(\overline{B}, d_\beta)_{\beta<\alpha} \models \exists x\Delta$. Hence since $\overline{B}$ is κ-saturated we get an element d_α of B such that $(\overline{B}, d_\beta)_{\beta<\alpha} \models \exists\varphi[d_\alpha]$ for every $\varphi \in \Gamma$. It follows that $(\overline{A}, c_\beta)_{\beta\leq\alpha} \equiv (\overline{B}, d_\beta)_{\beta\leq\alpha}$. We define $d_{\alpha+1}$ and $c_{\alpha+1}$ analogously.

After this construction we have $A = \{c_\alpha : \alpha < \kappa\}$, $B = \{d_\alpha : \alpha < \kappa\}$, and $(\overline{A}, c_\alpha)_{\alpha<\kappa} \equiv (\overline{B}, d_\alpha)_{\alpha<\kappa}$. Hence $\{(c_\alpha, d_\alpha) : \alpha < \kappa\}$ is the desired isomorphism.

$\boxed{\text{E6.12}}$ *For any natural number n and any structure $\overline{M}$, an n-type of $\overline{M}$ is a collection Γ of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, a sequence of distinct variables of length n, such that $\overline{M}_M \models \exists\overline{x}\varphi$ for every conjunction of finitely many members of Γ. Prove that if Γ is a collection of formulas in $\mathscr{L}_M$ with free variables among $\overline{x}$, then Γ is an n-type* *over $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $\overline{N} \models \varphi[\overline{a}]$ for every $\varphi \in \Gamma$.*

$\Rightarrow$: Expand $\mathscr{L}_M$ with new individual constants $\overline{d}$ of length n. Every finite subset of

$$\mathrm{Eldiag}(\overline{M}) \cup \{\varphi(\overline{d}) : \varphi \in \Gamma\}$$

has a model by hypothesis, so the compactness theorem yields the required $\overline{N}$.

$\Leftarrow$: $\overline{N} \models \exists \overline{x}\varphi$ for every finite subset of Γ, so $\overline{M}$ models this too, as desired.

$\boxed{\text{E6.13}}$ *If $\overline{M}$ is a structure, $A \subseteq M$, and $n \in \omega$, then an n-type over A of $\overline{M}$ is an n-type of $\overline{M}$ all of whose additional constants come from A. Given an n-tupe $\overline{a}$ of elements of M, the n-type over A of $\overline{a}$ in $\overline{M}$, denoted by $\mathrm{tp}^{\overline{M}}(\overline{a})/A)$, is the set $\{\varphi(\overline{x}) : \varphi$ is a formula with free variables among $\overline{x}$, $\overline{x}$ has length n, and $\overline{M}_A \models \varphi[\overline{a}]\}$. An n-type S over A is complete iff $\varphi \in S$ or $\neg\varphi \in S$ for every formula in the language $\mathscr{L}_A$ with free variables among $\overline{x}$.*

Prove that S is a complete n-type over A in $\overline{M}$ iff there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S = \mathrm{tp}^{\overline{N}}(\overline{a}/A)$.

$\Rightarrow$: Assume that S is complete. By Exercise E6.12, there is an elementary extension $\overline{N}$ of $\overline{M}$ which has a sequence $\overline{a}$ of elements such that $S \subseteq \mathrm{tp}^{\overline{N}}(\overline{a}/A)$. Clearly equality holds since S is complete.

$\Leftarrow$: clear.

$\boxed{\text{E6.14}}$ *Let t be an n-type over A of $\overline{M}$. We say that t is isolated iff there is a formula $\varphi(\overline{x})$ in $\mathscr{L}_A$ such that $\overline{M}_A \models \exists\overline{x}\varphi$ and $\overline{M}_A \models \forall\overline{x}(\varphi \to \psi)$ for every $\psi \in t$. We then say that φ isolates t.*

Prove that if φ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

Assume the hypothesis. Choose $\overline{b}$ such that $\overline{M}_A \models \varphi[\overline{b}]$. If $\psi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$, then $\overline{M}_A \models \varphi(\overline{b}) \to \psi(\overline{b})$. Hence $\mathrm{tp}^{\overline{M}}(\overline{a}/A) \subseteq \mathrm{tp}^{\overline{M}}(\overline{b}/A)$, so $\mathrm{tp}^{\overline{M}}(\overline{a}/A) = \mathrm{tp}^{\overline{M}}(\overline{b}/A)$. Hence $\varphi \in \mathrm{tp}^{\overline{M}}(\overline{a}/A)$.

$\boxed{\text{E6.15}}$ *Show that $\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$ is isolated iff both $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated.*

$\Rightarrow$: Suppose that $\mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$ is isolated. Let $\varphi(\overline{x},\overline{y})$ be such that $\overline{M}_A \models \exists\overline{x}\exists\overline{y}\varphi$ and $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$ for every $\psi \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. We claim that $\varphi(\overline{x},\overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$ and $\exists\overline{x}\varphi$ isolates $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$.

By Exercise E6.14 we have $\overline{M}_A \models \varphi[\overline{a},\overline{b}]$. Hence $\overline{M}_{A\cup\mathrm{rng}(\overline{b})} \models \exists\overline{x}\varphi(\overline{x},\overline{b})$. Now suppose that $\psi(\overline{x}) \in \mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$. Then we can write $\psi(\overline{x}) = \psi(\overline{x},\overline{b})$, and $\psi(\overline{x},\overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. Hence $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$. Hence $\overline{M}_{A\cup\mathrm{rng}(\overline{b})} \models \forall\overline{x}[\varphi(\overline{x},\overline{b}) \to \psi(\overline{x})]$. This proves that $\varphi(\overline{x},\overline{b})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}/A \cup \mathrm{rng}(\overline{b}))$.

For the second type, clearly $\overline{M}_A \models \exists\overline{y}\exists\overline{x}\varphi$. Now suppose that $\psi(\overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{b}/A)$. Then also $\psi(\overline{y}) \in \mathrm{tp}^{\overline{M}}(\overline{a}\frown\overline{b}/A)$. Hence $\overline{M}_A \models \forall\overline{x}\forall\overline{y}[\varphi \to \psi]$. So $\overline{M}_A \models \forall\overline{y}[\exists\overline{x}\varphi \to \psi]$. This proves that $\exists\overline{x}\varphi$ isolates $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$.

$\Leftarrow$: Assume that $\mathrm{tp}^{\overline{M}}(\overline{a}/A\cup\mathrm{rng}(\overline{b}))$ and $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$ are isolated. Let $\psi(\overline{x},\overline{b})$ isolate the type $\mathrm{tp}^{\overline{M}}(\overline{a}/A\cup\mathrm{rng}(\overline{b}))$ and $\varphi(\overline{y})$ isolate $\mathrm{tp}^{\overline{M}}(\overline{b}/A)$. We claim that $\theta \overset{\mathrm{def}}{=} \psi(\overline{x},\overline{y})\wedge\varphi(\overline{y})$ isolates $\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$.

By exercise E6.14 we have $\varphi\in\mathrm{tp}^{\overline{M}}(\overline{b}/A)$, so $\overline{M}_A\models\varphi[\overline{b}]$. Now $\overline{M}_A\models\exists\overline{x}\psi(\overline{x},\overline{b})$, so $\overline{M}\models\exists\overline{x}\exists\overline{y}(\psi\wedge\varphi)$.

Now suppose that $\chi\in\mathrm{tp}^{\overline{M}}(\overline{a}^\frown\overline{b}/A)$; we want to show that $\overline{M}_A\models\forall\overline{x}\forall\overline{y}[\theta\to\chi]$. Now

$$\overline{M}_A\models\forall\overline{x}[\psi(\overline{x},\overline{b})\to\chi(\overline{x},\overline{b})].$$

Hence

$$\overline{M}_A\models\varphi(\overline{d})\to\forall\overline{x}[\psi(\overline{x},\overline{d})\to\chi(\overline{x},\overline{d})],$$

so

$$\overline{M}_A\models\varphi(\overline{d})\to[\psi(\overline{c},\overline{d})\to\chi(\overline{c},\overline{d})],$$

hence

$$\overline{M}_A\models\theta(\overline{c},\overline{d})\to\chi(\overline{c},\overline{d})],$$

as desired.

$\boxed{\text{E6.16}}$ *Let T be a complete theory with a model. A formula $\varphi(\overline{x})$ is complete in T iff $T\cup\{\exists\overline{x}\varphi\}$ has a model, and for every formula $\psi(\overline{x})$, either $T\models\varphi\to\psi$ or $T\models\varphi\to\neg\psi$. Here $\overline{x}$ is a sequence of variables containing all variables free in φ or ψ.*

A formula $\theta(\overline{x})$ is completable in T iff there is a complete formula $\varphi(\overline{x})$ such that $T\models\varphi\to\theta$.

A structure $\overline{M}$ is atomic iff every tuple $\overline{a}$ of elements of M satisfies a complete formula in the theory of $\overline{M}$.

A theory T is atomic iff for every formula $\theta(\overline{x})$ such that $T\cup\{\exists\overline{x}\theta(\overline{x})\}$ has a model, θ is completable in T.

Show that if T is a complete theory in a countable language, then T has a countable atomic model iff T is atomic. Hint: in the direction $\Leftarrow$, for each $n\in\omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0,\ldots,v_{n-1}$, and apply the omitting types theorem.

$\Rightarrow$: Assume that T has an atomic model $\overline{M}$, and suppose that $\theta(\overline{x})$ is a formula such that $T\cup\{\exists\overline{x}\theta(\overline{x})\}$ has a model. Since T is complete, we have $T\models\exists\overline{x}\theta(\overline{x})$, and so $\overline{M}\models\exists\overline{x}\theta(\overline{x})$. Choose $\overline{a}$ in M such that $\overline{M}\models\theta[\overline{a}]$. Since $\overline{M}$ is atomic, there is a complete formula $\varphi(\overline{x})$ such that $\overline{M}\models\varphi[\overline{a}]$. Since $\varphi(\overline{x})$ is complete, it follows that $T\models\varphi\to\theta$. Thus θ is competable. This proves that T is atomic.

$\Leftarrow$: Assume that T is atomic. For each $n\in\omega$ let t_n be the set of all negations of complete formulas with free variables among $v_0,\ldots,v_{n-1}$. Then t_n is not isolated. For suppose it is, and let $\varphi(\overline{v})$ be a formula such that $T\cup\{\exists\overline{v}\varphi(\overline{v})$ has a model, and $T\models\varphi\to\psi$ for every $\psi\in t_n$. Now T is atomic, so φ is completable. Let χ be a complete formula such that $T\models\chi\to\varphi$. But $\neg\chi\in t_n$, so $T\models\varphi\to\neg\chi$. Hence $T\models\chi\to\neg\chi$, contradicting the fact that $T\cup\{\exists\overline{v}\chi\}$ has a model.

Now by the omitting types theorem, let $\overline{M}$ be a countable model of T which omits each type t_n. Thus for each $\overline{a}$ in M, say of length m, there is a φ in t_m such that $\overline{M}\models\neg\varphi[\overline{a}]$. Since $\neg\varphi$ is complete, this shows that $\overline{M}$ is atomic.

Solutions of exercises in Chapter 7

$\boxed{\text{E7.1}}$ *Let $\overline{M}$ be a field, A a subfield, and $a \in M$. Suppose that a is algebraic over A in the usual sense of field theory. Show that a is algebraic over A in the model-theoretic sense.*

Let $f(x)$ be a polynomial with coefficients in A such that $f(a) = 0$. Let $\varphi(x, \overline{b})$ be the formula $f(x) = 0$, where $\overline{b}$ is the system of coefficients of $f(x)$.

$\boxed{\text{E7.2}}$ *Let $\overline{M} = (\omega, <)$. Show that every element of ω is algebraic over $\emptyset$.*

For any $m \in \omega$ let $\varphi(x)$ be the formula "There are exactly m elements less than x."

$\boxed{\text{E7.3}}$ *Let $\overline{A} = ([\omega]^2, R)$, where*

$$R = \{(a,b) : a, b \in [\omega]^2,\ a \neq b \text{ and } a \cap b \neq \emptyset\}.$$

(i) Show that $\{a \in [\omega]^2 : (a, \{0,1\}) \in R\}$ is neither finite nor cofinite.

(ii) Infer from (i) that $[\omega]^2$ is not minimal.

(iii) If f is a permutation of ω, define $f^+ : [\omega]^2 \to [\omega]^2$ by setting $f^+(a) = f[a]$ for any $a \in [\omega]^2$. Show that f^+ is an automorphism of $\overline{A}$.

(iv) Let $X = \{a \in [\omega]^2 : 0 \in a \text{ and } a \cap \{1,2\} = \emptyset\}$. Show that X is definable in $\overline{A}$ with parameters.

(v) Show that X is minimal.

(i): For each $m > 1$ we have $(\{0,m\}, \{0,1\}) \in R$, so the set is not finite. Also, for each $m > 1$ we have $(\{m, m+1\}, \{0,1\}) \notin R$, so the set is not cofinite.

(ii): The set is definable with parameters, since it is

$$\{a \in [\omega]^2 : \overline{A} \models \mathbf{R}a\{0,1\}\}.$$

(iii): f^+ is one-one, since $f[a] = f[b]$ implies that $a = f^{-1}[f[a]] = f^{-1}[f[b]] = b$. f^+ maps onto $[\omega]^2$ since for any $a \in [\omega]^2$ we have $f[f^{-1}[a]] = a$. Now suppose that $a, b \in [\omega]^2$. Suppose that aRb. Then $a \neq b$ and $a \cap b \neq \emptyset$. Hence $f[a] \neq f[b]$ and $f[a] \cap f[b] \neq \emptyset$, so $f^{(}a)Rf^+(b)$. The converse is similar.

(iv): $X = \{a \in [\omega]^2 : \overline{A} \models aR\{0,1\} \wedge \neg(aR\{1,2\}) \wedge \neg(a = \{1,2\})\}$.

(v): For any $m > 2$ we have $\{0,m\} \in X$, so X is infinite. Now suppose that Z is first-order definable with parameters, and $X \cap Z$ is infinite; we want to show that $X \backslash Z$ is finite. Say $Z = \{b \in [\omega]^2 : \overline{A} \models \varphi(\overline{c}, b)\}$, where $\overline{c}$ is a finite sequence of elements of $[\omega]^2$. We claim that

$$X \backslash Z \subseteq \{\{0,n\} : n \in \bigcup \mathrm{rng}(\overline{c})\}.$$

To prove this, take any $\{0,n\} \in X \backslash Z$. Thus $n \notin \{0,1,2\}$. Suppose that $n \notin \bigcup \mathrm{rng}(\overline{c})$. Since $X \cap Z$ is infinite, there is a p such that $p \notin \{0,1,2,n\} \cup \bigcup \mathrm{rng}(\overline{c})\}$ and $\{0,p\} \in X \cap Z$. Let f be the transposition (n,p), considered as a permutation of ω. Then f induces an automorphism f^+ of A, defined by $f^+(a) = f[a]$, for any $a \in A$, as above. Note that $f^+(u) = u$ for each $u \in \mathrm{rng}(\overline{c})$. Now $A \models \varphi(\overline{c}, \{0,p\})$, so it follows that $A \models \varphi(\overline{c}, \{0,n\})$. Hence $\{0,n\} \in Z$, contradiction.

$\boxed{\text{E7.4}}$ *Let V be an infinite vector space over a finite field F. We consider V as a structure $(V, +, f_a)_{a\in F}$, where $f_a(v) = av$ for any $v \in V$ and $a \in F$. Show that V is minimal.*

Suppose that X is definable with parameters; say

$$X = \{a \in A : A \models \varphi(\overline{c}, a)\}.$$

Let Y be the span of $\text{rng}(\overline{c})$. Thus Y is finite. We claim that $X \subseteq Y$ or $A\backslash X \subseteq Y$. Suppose that $X \not\subseteq Y$; choose $x \in X\backslash Y$. Take any $y \in A\backslash Y$. Let B be a basis for Y. Both x and y are not in Y, and thus there exist bases B', B'' of A such that $B \subseteq B', B''$, $x \in B'$, and $y \in B''$. Hence there is a one-one function f from B' onto B'' such that f is the identity on B and $f(x) = y$. Now f extends to an automorphism of A and it follows that $y \in X$. So we have shown that $A\backslash Y \subseteq X$. Hence $A\backslash X \subseteq Y$, as desired.

$\boxed{\text{E7.5}}$ *(continuing exc. 7.4) Prove that for any subset A of V, $\text{acl}(A) = \text{span}(A)$.*

First suppose that $b \in \text{span}(A)$. Then there exist $a_0, \ldots, a_{n-1} \in A$ and $\varepsilon \in {}^nF$ such that $b = \varepsilon_0 a_0 + \cdots + \varepsilon_{n-1}a_{n-1}$. The formula $x = \varepsilon_0 a_0 + \cdots + \varepsilon_{n-1}a_{n-1}$ shows that $b \in \text{acl}(A)$.

Now suppose that $b \in \text{acl}(A)\backslash\text{span}(A)$; we want to get a contradiction. Say $\overline{V} \models \varphi(b, \overline{a})$ and $\varphi(\overline{V}, \overline{a})$ is finite, where $\overline{a} \in A$. Thus $b \notin \text{span}(\overline{a})$. Choose c not in $\text{span}(a)$ and also $c \notin \varphi(\overline{V}, \overline{a})$. Then there is an automorphism f of $\overline{V}$ such that $f \circ \overline{a}$ is the identity and $f(b) = c$. Then $\overline{V} \models \varphi(c, \overline{a})$, contradiction.

$\boxed{\text{E7.6}}$ *(continuing excs. 7.4, 7.5) By exercise 7.4 and Lemma 7.2, the following holds in $\overline{V}$: if $a \in \text{span}(A \cup \{b\})\backslash\text{span}(A)$, then $b \in \text{span}(A \cup \{a\})$. Prove this statement using ordinary linear algebra.*

Assume the hypothesis. Then we can write $a = \varepsilon_0 c_0 + \cdots + \varepsilon_{n-1}c_{n-1} + \delta b$ with each $c_i \in A$. Since $a \notin \text{span}(A)$, we must have $\delta \neq 0$. Then

$$b = a + -\varepsilon_0 c_0 + \cdots + -\varepsilon_{n-1}c_{n-1},$$

which shows that $b \in \text{span}(A \cup \{a\})$.

$\boxed{\text{E7.7}}$ *Give an example of a set Γ of sentences and two sentences φ and ψ, such that, $\Gamma \models \varphi$ iff $\Gamma \models \psi$ but $\Gamma \not\models (\varphi \leftrightarrow \psi)$.*

Let $\mathscr{L}$ have just one non-logical symbol, a unary relation symbol $\mathbf{P}$. Let $\Gamma = \emptyset$, $\varphi = \exists x\mathbf{P}x$, $\psi = \exists x\neg\mathbf{P}x$. Then $\not\models \varphi$, using the structure $(\omega, \emptyset)$. Also $\not\models \psi$, using the structure $(\{0\}, \{0\})$. Hence $\Gamma \models \varphi$ iff $\Gamma \models \psi$. Finally $\not\models (\varphi \leftrightarrow \psi)$, using the structure $(\omega, \emptyset)$.

$\boxed{\text{E7.8}}$ *Show that for Γ a set of sentences and for sentences φ, ψ, if $\Gamma \models \varphi \leftrightarrow \psi$ then $\Gamma \models \varphi$ iff $\Gamma \models \psi$.*

Assume the hypothesis, and suppose that $\Gamma \models \varphi$. Suppose that $\overline{M}$ is a model of Γ. Since $\Gamma \models \varphi \leftrightarrow \psi$ and $\Gamma \models \varphi$, it follows that $\Gamma \models \psi$. Similarly in the other direction.

$\boxed{\text{E7.9}}$ *Prove that the following two conditions are equivalent:*

(i) $\overline{M} \models \varphi[a]$ *iff* $\overline{M} \models \psi[a]$.

(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)[a]$.

Obvious.

$\boxed{\text{E7.10}}$ *Prove that the following two conditions are equivalent, for any sentences φ, ψ:*
(i) $\overline{M} \models \varphi$ iff $\overline{M} \models \psi$.
(ii) $\overline{M} \models (\varphi \leftrightarrow \psi)$.

Obvious.

$\boxed{\text{E7.11}}$ *In the language with no non-logical symbols, show that ω is an indiscernible set in ω.*

Given $\varphi(\overline{x})$ and two sequences $\overline{a}$ and $\overline{b}$ of distinct elements of ω, there is a permutation f of ω such that $f \circ \overline{a} = \overline{b}$. Thus $\omega \models \varphi(\overline{a})$ iff $\omega \models \varphi(\overline{b})$. Now use exercise E7.9.

$\boxed{\text{E7.12}}$ *(Continuing exercises 7.4, 7.5, 7.6) Let $A = \{w_1, w_2\}$, two members of V, and let $b = w_1$. Thus $b \in \operatorname{span}(A)$. According to Lemma 7.8, $\operatorname{tp}^{\overline{V}}(b/A)$ is isolated. Give a formula $\varphi(v_0, \overline{a})$ with $\overline{a} \in A$ which isolates $\operatorname{tp}^{\overline{V}}(b/A)$.*

Let φ be $v_0 = w_1$. Thus $\varphi \in \operatorname{tp}^{\overline{V}}(b/A)$. Suppose that $\psi \in \operatorname{tp}^{\overline{V}}(b/A)$. Then $\overline{V} \models \psi(w_1)$, and hence $\overline{V} \models \forall v_0[\varphi \rightarrow \psi]$.

$\boxed{\text{E7.13}}$ *Suppose that $\overline{M}$ is an infinite structure, $\varphi(v_0)$ is a formula with at most v_0 free, and $\varphi(\overline{M})$ is infinite. Show that $\overline{M}$ has a proper elementary extension $\overline{N}$ such that $(\overline{M}, \overline{N})$ is not a Vaughtian pair.*

Let L be the language of $\overline{M}$, and expand the language L_M by adding another individual constant d. Then every finite subset of

$$\operatorname{Eldiag}(\overline{M}) \cup \{a \neq d : a \in \varphi(\overline{M})\} \cup \{\varphi(d)\}$$

has a model, so the result follows by the compactness theorem and Theorem 6.15.