Model Theory Nomenclature

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Abstract

This is a tourist's guide to model theory, including the most basic and important concepts of the theory, along with some elementary facts about them. To keep the presentation concise, proofs are omitted but may be found in the references cited.

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1 Signatures, Languages, Terms, Formulas, Sentences

1.1 Signatures

A **signature** $S = (\mathrm{C}, \mathrm{F}, \mathrm{R}, \rho)$ consists of three (possibly empty) sets C, F, and R of *constant*, *function*, and *relation* symbols (resp.), along with a function $\rho : \mathrm{C} + \mathrm{F} + \mathrm{R} \to \mathbb{N}$ that assigns an *arity* to each symbol.

1.2 Languages

The **language** $L = L(S)$ of signature S is a certain collection of strings of symbols from the following **alphabet** of symbols:

1. **logical symbols**:
   a. *logical connectives*: $\neg$, $\wedge$, $\vee$, $\top$, $\bot$ (for "negation," "conjunction," "disjunction," "true," "false," resp.),
   b. *existential quantifier*: $\exists$
   c. *equality*: $=$
2. **variable symbols** (countably many)
3. **non-logical symbols** from S (the constant, function, and relation symbols)
4. **parentheses**: (,).

We often denote the (countable) set of variable symbols by X. To specify which strings of symbols belong to the language L we must define the *terms* and *formulas* of S (called S-terms and S-formulas, respectively). Eventually, we will *define* L as follows:

The language L is the set of all S-formulas.

1.3 Terms

The set $\mathrm{Term}_S(X)$ of **terms** in the signature S generated by the set X of variable symbols is defined recursively as follows:[1]

1. All variables symbols (in X) are S-terms.
2. All constant symbols (in C) are S-terms.
3. If $f \in \mathrm{F}$ and t is a sequence of S-terms of length $\rho\, f$, then $f\ t = f(t_0, t_1, \dots)$ is an S-term.

[1] The set $\mathrm{Term}_S(X)$ may be denoted by $\mathrm{Term}(X)$ when the signature is clear from the context, and the members of this set go by the following synonymous names: S-**terms**, $L(S)$-terms, or L-terms.

4. Any string of symbols that can be obtained in finitely many steps from some combination of 1, 2, and 3 above is an S-term.

1.4 Formulas

The **formulas** of S (also known as S-formulas, or $L(S)$-formulas, or L-formulas) are defined recursively as follows:

1. $\top$ and $\perp$ are S-formulas (the "trivially true" and "trivially false" formulas).
2. If s and t are S-terms, then $s = t$ is an S-formula.
3. If $r \in \mathrm{R}$ and t is a sequence of S-terms of length $\rho\ r$, then $r\ t = r\ (t_0,\ t_1,\ \ldots\ )$ is an S-formula.
4. If φ and ψ are S-formulas and $x \in X$, then $\neg\ \varphi$, $\varphi \wedge \psi$, and $\exists\ x\ \varphi$ are S-formulas.
5. Any string of symbols that can be obtained in finitely many steps from some combination of 1–4 above is an S-formula.

The **language of the signature** S, denoted $L(S)$ (or just L), is the set of all S-formulas.

1.5 Sentences

A variable is **bound** if it is captured by (falls within the scope of) a quantifier or a λ. For example, x and y are bound and z is free in the following formulas: $(\forall\ \mathrm{x})(\exists\ \mathrm{y})(\mathrm{x} = \mathrm{z} + \mathrm{y})$, and $\lambda\ \mathrm{x}\ \mathrm{y}\ (\mathrm{x} + \mathrm{y} + \mathrm{z}) = \lambda\ \mathrm{y}\ (\mathrm{z} + \lambda\ \mathrm{x}\ (\mathrm{x} + \mathrm{y}))$.

1. A term t is said to be **constant** (or **closed**) if it contains no variables.
2. A formula φ is called a **sentence** (or **closed formula**) if it contains no *free* variables; that is, all variables appearing in φ are bound.
3. $L_0 :=$ all sentences in the language L (aka "L-sentences");
4. An **atomic** L-formula has one of the following two forms:
   a. $s = t$, where s and t are L-terms;
   b. $r\ t$, where $r \in \mathrm{R}$ and t is a sequence of L-terms of length $\rho\ r$.
5. $\mathbf{at}_L$ (or just **at** when the context makes L clear) is the class of all atomic L-formulas.
6. An **atomic** L-sentence is either an equation of constant terms or a relational sentence, $R(t_0, \ldots, t_{n-1})$, where all t_i are closed terms;
7. A **literal** L-formula (or, L-**literal**) is an atomic or negated atomic L-formula;
8. $\mathbf{lt}_L :=$ the set of all L-**literals**; that is, $\mathbf{at}_L \cup \{\neg\varphi : \varphi \in \mathbf{at}_L\}$;
9. $\mathbf{cl}_L :=$ the set of all **closed** L-**literals** (literal L-sentences; that is, literals without free variables).

▶ Remarks.

1. Every constant symbol is a constant term.
2. An atomic sentence contains no variables at all.
3. *Languages without constant symbols have no atomic sentences.*
4. Every language comes equipped with a countable supply of variables, so the cardinality of L is $|L| = \max\ \{\aleph_0,\ |\ \mathrm{C} \cup \mathrm{F} \cup \mathrm{R}\ |\}$.

2 Boolean comibinations and quantifier-free formulas

Let $\mathcal{E}$ be a set of formulas.

1. A **boolean combination** of $\mathcal{E}$ is obtained by connecting formulas from $\mathcal{E}$ using only connectives from $\{\wedge,\ \vee,\ \neg\}$.

2. A **positive boolean combination** of $\mathcal{E}$ is obtained by connecting formulas from $\mathcal{E}$ using only connectives from $\{\wedge, \vee\}$.
3. The **boolean closure** of $\mathcal{E}$ is the set $\tilde{\mathcal{E}}$ of all boolean combinations of formulas from $\mathcal{E}$.
4. A **positive** formula is obtained from atomic formulas using only connectives or quantifiers from $\{\wedge, \vee, \exists, \forall\}$.
5. The class of all positive formulas (of all possible languages) is denoted by $+$.
6. A **negative** formula is a negated positive formula, and the class of such is denoted by $-$.
7. A **quantifier-free** formula is one with no quantifiers.
8. The class of all quantilier-free formulas (of arbitrary signature) is denoted by $\mathbf{qf}$.

▶ Remarks.

1. In the definition of boolean combinations, we could allow $\rightarrow$ and $\leftarrow$; we could do without $\vee$.
2. Formulas in $\mathbf{qf}$ have no occurrences of $\exists$ or $\forall$; $\top$ and $\bot$ are such formulas, so $\top, \bot \in \mathbf{qf}$.
3. $\mathbf{qf}$ is the class of all boolean combinations of atomic formulas.

3 Expansion by Constants, Validity, Truth

Fix a signature $S = (\mathrm{C}, \mathrm{F}, \mathrm{R}, \rho)$ and a let

- $L = L(S)$, the language of S;
- $\Delta =$ an arbitrary class of formulas (not necessarily from L);
- $\mathcal{M} = (M, \dots)$ and $\mathcal{N} = (N\ , \dots)$, S-structures (aka L-structures);
- $x =$ a tuple of variable symbols;
- $\varphi = \varphi(x)$, an L-formula.

3.1 Expansion by Constants

1. A **new constant symbol** for L is any symbol not occuring in the alphabet of L.
2. If c is a set of new constant symbols, then $L(c)$ is the **expansion of** L **by new constants** and is defined to be the (uniquely determined) language of the signature $(\mathrm{C} \cup c, \mathrm{F}, \mathrm{R}, \rho)$.
3. $\Delta(c)$ is the **expansion of** Δ **by new constants** (from c, where $c \cap \Delta = \emptyset$) and is defined to be the class of formulas obtained from formulas $\varphi \in \Delta$ upon substituting (at will) elements from c for variables in φ. ("At will" indicates that $\Delta \subseteq \Delta(c)$.)
4. $\mathbf{lt}_{L(M)} :=$ the set of all atomic and negated atomic $L(M)$-formulas.
5. $\mathbf{cl}_{L(M)} :=$ the set of all atomic and negated atomic $L(M)$-setnences.

3.2 Validity and Truth

$\mathcal{M} \models \varphi$ means that φ is **valid** in $\mathcal{M}$ which means that, for every tuple a from M that is at least as long as x, the L-sentence $\varphi(a)$ is *true in* $\mathcal{M}$.

Question. What is meant by "$\varphi(a)$ is true in $\mathcal{M}$?"

Intuitively, for each i we subsitute the element $a\ i$ for the variable $x\ i$ in the formula $\varphi(x)$, which yields a sentence $\varphi(a)$ that is "decidable" in $\mathcal{M}$. That is, there is a finite procedure by which we can determine whether or not $\varphi(a)$ holds (is "true") in $\mathcal{M}$.

Formally, however, we may follow a more careful procedure for judging the truth of a given formula φ in a given structure $\mathcal{M}$. This is the "simple" matter of how syntactically to denote interpretation of variables. As Hodges puts it (in [1]), this is "one of the more irksome parts of model theory." The issue is explained clearly in [1, Section 1.4], so we defer to that treatment:

> "Instead of interpreting a variable symbol as a name of an element b, we can add a new constant for b to the signature. The price we pay is that the language changes every time another element is named. When constants are added to a signature, the new constants and the elements they name are called parameters.
>
> For example, suppose that $\mathcal{A}$ is an L-structure, a is a sequence of elements of $\mathcal{A}$, and we want to name the elements in a. Then we choose a sequence c of distinct new constant symbols of the same length as a, and we form the signature $L(c)$ by adding the constants c to L. Then $(\mathcal{A}\ ,\ a)$ is an $L(c)$-structure and, for each i, the element $a\ i$ is $(c\,i)^{(\mathcal{A},a)}$, which is the interpretation of $c\ i$ in the structure $(\mathcal{A}\ ,\ a)$.
>
> Likewise if $\mathcal{B}$ is another L-structure and b a sequence of elements of $\mathcal{B}$ of the same length as c, then there is an $L(c)$-structure $(\mathcal{B},\ b)$ in which these same constants in c name the elements of b. The next lemma is about this situation. It comes straight out of the definitions and it is often used silently."

▶ **Lemma 1** ([1, 1.4.1]). Let $\mathcal{A}$, $\mathcal{B}$ be L-structures and suppose $(\mathcal{A},\ a)$, $(\mathcal{B},\ b)$ are $L(c)$-structures. Then a homomorphism $h : (\mathcal{A},\ a) \to (\mathcal{B},\ b)$ is the same thing as a homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that $h \circ a = b$.

▶ Remarks.

1. $h \circ a$ denotes the function whose value at i is $h\ (a\ i)$;
2. $\varphi_x(a)$ denotes $[a/x]\varphi(x) = [a0/x0\ ,\ a1/x1, \ldots]\varphi(x)$, which is the sentence obtained from φ by substitution: for each i replace $x\ i$ with $a\ i$.
3. If $t(x)$ is an L-term, then $t^{\mathcal{A}}(a)$ and $t(c)^{(\mathcal{A},a)}$ are the same element.
4. If $\varphi(x)$ is an atomic formula then $\mathcal{A} \models \varphi_x(a) \ \Leftrightarrow\ (\mathcal{A}, c) \models \varphi_x(c)$.

4 Models, Theories, Diagrams

Let $\varphi \in L_0$, $\mathcal{E} \subseteq L_0$, and let $\mathcal{M} = (M, \ldots)$, $\mathcal{N} = (N, \ldots)$ be L-structures. Let Δ be an arbitrary class of formulas (not necessarily from L).

4.1 Models

1. If $M \neq \emptyset$ and $\mathcal{M} \models \mathcal{E}$, then $\mathcal{M}$ is a **model** of $\mathcal{E}$; we also say "$\mathcal{M}$ *models* $\mathcal{E}$."
2. $\mathrm{Mod}_L\,\mathcal{E}$:= the class of L-structures that model $\mathcal{E}$.
3. $\mathrm{Mod}_L\,\emptyset$:= the class of all nonempty L-structures.
4. $\mathcal{E}$ **entails** φ, denoted $\mathcal{E} \vdash \varphi$, just in case every model of $\mathcal{E}$ also models φ.
5. φ is a **logical consequence** of $\mathcal{E}$ just in case $\mathcal{E}$ entails φ.
6. The **deductive closure** of $\mathcal{E}$ is the set $\mathcal{E}^{\vdash} = \{\varphi \in L_0 : \mathcal{E} \vdash \varphi\}$ of logical consequences of $\mathcal{E}$.
7. $\mathcal{E}$ is **deductively closed** provided $\mathcal{E}^{\vdash} \subseteq \mathcal{E}$.
8. The sets $\mathcal{E}_0$, $\mathcal{E}_1 \subseteq L_0$ are $\mathcal{E}$-**equivalent** if $(\mathcal{E} \cup \mathcal{E}_0)^{\vdash} = (\mathcal{E} \cup \mathcal{E}_1)^{\vdash}$.
9. **logically equivalent** means $\emptyset$-equivalent.
10. A **contradiction** is an L-sentence of the form $\varphi \wedge \neg\, \varphi$.
11. $\mathcal{E}$ is **consistent** if $\mathcal{E}^{\vdash}$ contains no contradictions; otherwise, $\mathcal{E}$ is **inconsistent**.

▶ Remark. No model satisfies a contradiction, so the deductive closure of a contradiction is the set L_0 of all L-sentences, and L_0 is the only deductively closed inconsistent set of L-sentences.

4.2 Theories

1. An L-**theory** is a *consistent* and *deductively closed* set of L-sentences.
2. The **cardinality** or **power** of an L-theory T (denoted $|T|$) is the cardinality of L.
3. $T_\Delta = (T \cap \Delta)^\vdash$, the Δ-**part** of the L-theory T (here Δ is an arbibtrary class of formulas).
4. $\forall$ is the class of formulas in which $\exists$ does not appear; $T_\forall = (T \cap \forall)^\vdash$, the *universal part* of T.
5. $\mathrm{Th}_\Delta\, \mathcal{M} := \{\varphi \in L_0 : \varphi \in \Delta,\, \mathcal{M} \vdash \varphi\}$, the set of L-sentences in Δ that are true in $\mathcal{M}$.
6. $\mathrm{Th}\, \mathcal{M} := \mathrm{Th}_{L_0}\, \mathcal{M}$, the set of L-sentences true in $\mathcal{M}$.
7. An L-theory T is **complete** if for all $\varphi \in L_0$, either $\varphi \in T$ or $\neg\, \varphi \in T$.

▶ **Lemma 2** ([2, 3.5.1])**.** If T is an L-theory, the following are equivalent:
1. T is complete.
2. T is a maximal L-theory.
3. T is a maximal consistent set of L-sentences.
4. $T = \mathrm{Th}\ \mathcal{M}$ for all $\mathcal{M} \vdash T$.
5. $T = \mathrm{Th}\ \mathcal{M}$ for some $\mathcal{M} \vdash T$.

▷ Examples. T^∞ is the theory of the class of all infinite models of T, and $T_=$ is the **theory of pure identity**, which is the $L_=$-theory of all sets (regarded as $L_=$-structures).

4.3 Diagrams

1. The **diagram** of $\mathcal{M}$ is the set $D(\mathcal{M}) := \mathbf{cl}_{L(M)}$ of all atomic and negated atomic $L(M)$-sentences;
2. $\mathcal{M} \Rrightarrow_\Delta \mathcal{N}$ means $\forall\ \varphi \in \Delta \cap L_0\ (\mathcal{M} \vdash \varphi \to \mathcal{N} \vdash \varphi)$.
3. $\mathcal{M} \Rrightarrow \mathcal{N}$ means $\mathcal{M} \Rrightarrow_L \mathcal{N}$.
4. $\mathcal{M} \equiv \mathcal{N}$ means $\mathcal{M} \Rrightarrow \mathcal{N}$ and $\mathcal{M} \Lleftarrow \mathcal{N}$ hold, and this is equivalent to $\mathrm{Th}\ \mathcal{M} = \mathrm{Th}\ \mathcal{N}$. We call $\mathcal{M}$ and $\mathcal{N}$ **elementarily equivalent** in this case.
5. $f : \mathcal{M} \hookrightarrow \mathcal{N}$ means f is an L-structure-monomorphism from $\mathcal{M}$ to $\mathcal{N}$.
6. $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ means all L-formulas in Δ are preserved by f; that is, $\mathcal{M} \vdash \varphi(a)$ implies $\mathcal{N} \vdash \varphi(f \circ a)$, for all $\varphi \in \Delta \cap L$ and all tuples a from M.
7. $f : \mathcal{M} \xrightarrow{\equiv} \mathcal{N}$ means $f : \mathcal{M} \xrightarrow{L} \mathcal{N}$.

4.4 Some Facts

Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures and let Δ be a set of L-formulas. If $f :\ M \to N$, then denote by $f[M]$ the **image** of M under f.

1. $f : \mathcal{M} \xrightarrow{\mathbf{at}} \mathcal{N}$ iff $f :\ \mathcal{M} \to \mathcal{N}$
2. $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ iff $(\mathcal{M}, M) \Rrightarrow_{\Delta(M)} (\mathcal{N}, f[M])$.
3. If $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ and Δ contains **at** and all negations of unnested relational atomic formulas, then f is a strong homomorphism. (The converse is not true.)
4. If $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ and Δ is closed under negation, then $\mathcal{M} \vdash \varphi(a)$ implies $\mathcal{N} \vdash \varphi(f \circ a)$, for all $\varphi \in \Delta$ and tuples a from M.
5. $f :\ M \to N$ is injective iff $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ for the set $\Delta = \{x \neq y\}$.
6. If $\Delta \subseteq L_0$, then $f : \mathcal{M} \xrightarrow{\Delta} \mathcal{N}$ iff $\mathcal{M} \Rrightarrow_\Delta \mathcal{N}$ and $f :\ M \to N$.
7. If $\Delta \subseteq L_0$ and Δ is closed under negation, then $\mathcal{M} \Rrightarrow_\Delta \mathcal{N}$ implies $\mathcal{M} \equiv_\Delta \mathcal{N}$.

4.5 The Lemma on Constants

Above we remarked that if $(\mathcal{A},\ a)$ is an $L(c)$-structure with $\mathcal{A}$ an L-structure, then for every atomic formula φ of L, we have $\mathcal{A} \vdash \varphi(a)$ if and only if $(\mathcal{A},\ a) \vdash \varphi(c)$.

▶ **Lemma 3** ([1, 2.3.2]). Let L be a language, T a theory in L, and $\varphi(x)$ a formula in L. Let c be a sequence of distinct constants that are not in L. Then $T \vdash \varphi(c)$ if and only if $T \vdash \forall\ x,\ \varphi$.

4.6 The Diagram Lemma

▶ **Lemma 4** ([2, 6.1.2]). Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures.

1. $f : \mathcal{M} \hookrightarrow \mathcal{N} \Leftrightarrow f : \mathcal{M} \xrightarrow{\mathbf{qf}} \mathcal{N} \Leftrightarrow (\mathcal{N}, f[M]) \vdash D(\mathcal{M})$. In particular $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N} \Rightarrow f : \mathcal{M} \hookrightarrow \mathcal{N}$.
2. $\mathcal{M} \hookrightarrow \mathcal{N} \Leftrightarrow \mathcal{N}$ has an $L(M)$-expansion that models $D(\mathcal{M})$.

4.7 The Diagram Lemma (ver. 2)

Let's consider an alternative version of the Diagram Lemma that makes the role of new constants more explicit. For this version, we will introduce some new notation.

- c is a tuple of distinct symbols not appearing in L;
- $a : \{0, 1, \ldots, n-1\} \to M$ and $b : \{0, 1, \ldots, n-1\} \to N$ are n-tuples in M and N respectively.
- $(\mathcal{M},\ c)$ and $(\mathcal{N},\ c)$ are $L(c)$-structures, where $(c\,i)^{\mathcal{M}} = a\,i$ and $(c\,i)^{\mathcal{N}} = b\,i$ are the interpretations in $\mathcal{M}$ and $\mathcal{N}$ of the new constant symbols;
- $\langle\ a\ \rangle$ is the substructure of $\mathcal{M}$ generated by the elements of the tuple a;

▶ **Lemma 5** ([1, 1.4.2]). The following are equivalent:

1. For every atomic sentence $\varphi(c)$ of $L(c)$, if $(\mathcal{M},\ c) \vdash \varphi(c)$ then $(\mathcal{N},\ c) \vdash \varphi(c)$.
2. There is a homomorphism $f : \langle\ a\ \rangle \to \mathcal{N}$ such that $f \circ a = b$.
3. The homomorphism is unique, if it exists, and it is an embedding if and only if for every atomic sentence φ of $L(c)$, we have $(\mathcal{M},\ c) \vdash \varphi \Leftrightarrow (\mathcal{N},\ c) \vdash \varphi$.

5 Elementary equivalence

5.1 Isomorphic structures are elementarily equivalent

▶ **Lemma 6** ([2, 6.1.3]). If $f : \mathcal{M} \cong \mathcal{N}$, then $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$, hence also $\mathcal{M} \equiv \mathcal{N}$.

The second implication is $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$. That's true because, if $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$, then we have not only $\varphi \in \mathrm{Th}\ \mathcal{M}$ implies $\varphi \in \mathrm{Th}\ \mathcal{N}$, but also $\neg\ \varphi \in \mathrm{Th}\ \mathcal{M}$ implies $\neg\ \varphi \in \mathrm{Th}\ \mathcal{N}$. From the former, $\mathcal{M} \Rrightarrow \mathcal{N}$. From the latter, $\mathcal{M} \Lleftarrow \mathcal{N}$. Thus, $\mathcal{M} \equiv \mathcal{N}$. (For proof of the first implication, see [2, page 70].) The converse of 6.1.3 holds if and only if the structures involved are finite, as the next proposition shows.

▶ **Lemma 7** ([2, 8.1.1]). Let $\mathcal{M}$ be an L-structure. The following are equivalent:

1. For every L-structure $\mathcal{N}$, $\mathcal{N} \equiv \mathcal{M}$ implies $\mathcal{N} \cong \mathcal{M}$.
2. $\mathcal{M}$ is finite.

5.2 All models of a complete theory are elementarily equivalent

▶ **Lemma 8** ([2, 8.1.2]). A theory is complete iff its models are elementarily equivalent.

▶ **Corollary 9** ([2, 8.1.3]). A complete theory has a finite model iff it has only one model up to isomorphism.

6 Elementary maps

Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures. A map $f : M \to N$ is called **elementary** if $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$. The structure $\mathcal{M}$ is **elementarily embeddable** in $\mathcal{N}$, in symbols $\mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$, if there is an elementary map $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$.

▶ Remarks.

1. If $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.
2. While elementary equivalence is weaker than isomorphism, every elementary map is automatically an isomorphic embedding (by [2, Lemma 6.1.2(1)]). Therefore, elementary maps are aka **elementary embeddings**, and the notation $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$ is justified.
3. The converse is not true in general, unless the embedding is surjective (i.e., an isomorphism), since every isomorphism $f : \mathcal{M} \cong \mathcal{N}$ is an elementary map (by [2, Prop 6.1.3]).

6.1 Elementary Diagram Lemma

The **elementary diagram** of an L-structure $\mathcal{M}$ is the complete $L(M)$-theory $\mathrm{Th}(\mathcal{M}, M)$.

▶ **Lemma 10** ([2, 8.2.1]). Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures and $f : M \to N$.

1. f is elementary $\Leftrightarrow (\mathcal{M}, M) \equiv (\mathcal{N}, f[M]) \Leftrightarrow (\mathcal{N}, f[M]) \vdash \mathrm{Th}(\mathcal{M}, M)$.
2. $\mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N} \Leftrightarrow \mathcal{N}$ has an expansion that is a model of $\mathrm{Th}(\mathcal{M}, M)$.

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