Fields

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1 Calculus of variations

1.1 Functional Derivative

Remark 1.1 Definitions

- **Field**: a map from a base manifold $\mathcal{M}$ into a target space T.
- **Functional**: assigns each φ a number $S[\varphi]$, i.e. it maps a field onto $\mathbb{R}$. Formally, it is defined through the map

$$J : C^\infty(\mathbb{R}) \to \mathbb{R}$$

$$\text{where} \quad J[y] = \int_{x_1}^{x_2} \mathrm{d}x \; f(x, y, y', y'', \cdots, y^{(n)})$$

Entity	Discrete	Continuous
Argument	vector $\mathbf{f}$	function f
Differential	$\mathrm{d}F_{\mathbf{f}}\,(\mathbf{g})$	$\delta F_f(g)$
Cartesian basis	$\mathbf{e}_n$	δ_x
Scalar product	$\sum_n f_n g_n$	$\int \mathrm{d}x\, f(x)g(x)$
Derivative	$\partial/\partial f_n\; F(\mathbf{f})$	$\delta F[f]/\delta f(x)$

A functional derivative (also called a *Frechet* derivative) is the generalization of a directional derivative $\partial J/\partial y_i$ for a continuum of coordinates $y_i \to y(x)$. Similar to multivariate calculus, finding the function that gives the extrema of $J[y]$ amounts to setting its functional derivative to zero.

$$\delta J = \sum_i \frac{\partial J}{\partial y_i} \delta y_i \to \int_{x_1}^{x_2} \mathrm{d}x \left(\frac{\delta J}{\delta y(x)} \right) \delta y(x) \tag{1}$$

For the general case where f depends on n derivatives of $y(x)$, one has

$$\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial f}{\partial y'''}\right) + \cdots = 0. \tag{2}$$

Relating the functional derivative to partial derivatives is essential for analyzing problems. In Lagrangian mechanics, the action functional and the functional derivative take the form of

Note 1.1 Lagrangian functional

$$S[q] = \int_{t_0}^{t_f} \mathrm{d}t\, L(t, q^i, q'^i), \qquad \frac{\delta L}{\delta q(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{3}$$

1.2 Continuous systems

Continuum systems and field theories require infinite degrees of freedom. For notational purposes, we define

$$\boxed{\varphi_\mu \equiv \frac{\partial \varphi}{\partial x^\mu}}. \tag{4}$$

The Lagrangians evaluated over a manifold $\mathcal{M}$ will usually take the form of

$$\boxed{S[\varphi] = \int_{\mathcal{M}} \mathrm{d}t\, L = \int_{\mathcal{M}} \mathrm{d}^m x\, \mathcal{L}(x^\mu, \varphi, \varphi_\mu)} \tag{5}$$

Note 1.2 Vanishing at boundary

We assume the field variation vanishes at the boundary, $\theta\big|_{\partial\mathcal{M}} = \delta\varphi\big|_{\partial\mathcal{M}} = 0$. This eliminates any boundary integrals when integrating by parts. However, this vanishing need not be always the case.

Remark 1.2 Divergence theorem

Using the established notation, the divergence theorem looks like this

$$\int_\Omega \mathrm{d}^m x \left(\frac{\partial A^\mu}{\partial x^\mu}\right) = \int_{\partial\Omega} \mathrm{d}S\, A^\mu n_\mu$$

Note 1.3 First Principles

In continuum problems, it is often easier to derive the equations of motion from *first principles*, rather than using the continuum Euler-Lagrange equations,

$$\frac{\delta S}{\delta \varphi^i(x)} = \frac{\partial \mathcal{L}}{\partial \varphi^i(x)} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi^i_\mu(x)}\right). \tag{6}$$

Variating by hand will usually require the following identity,

$$\boxed{\delta(\varphi_\mu(x)) = \frac{\partial}{\partial x^\mu}\delta\varphi(x)}.$$

1.3 Noether's Theorem

Remark 1.3 Continuum Noether's Theorem

A continuous symmetry entails a classically conserved current.

Note 1.4 Momenta

Imagine a wave on a string with fixed ends. The distinction between momentum and pseudo-momentum is appreciated by analyzing the associated symmetry,

- Pseudo momentum $\Leftrightarrow$ invariance under $y(x) \to y(x-a)$ (loc. of wave)
- Momentum $\Leftrightarrow$ invariance under translation of whole system

Tip 1.1 Curl

Curl curl identity

$$\text{curl}\left(\frac{1}{2}\right)$$

1.4 Electrodynamics

More fundamental than Maxwell's equations, for modern electrodynamics, is the concept of **symmetry**; an electromagnetic theory must be Lorenz invariant. We restrict our analysis to vacuum theory, where $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$.

The Lagrangian (or *action density*) that gives rise to Maxwell's equations is

$$\boxed{\mathcal{S} = (\varphi, \varphi_\mu) = \frac{1}{2}\left(\dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - |\text{curl } A|^2\right)}. \tag{7}$$

The terms follow a similar relation to $T - U$. Should there be an external current, the Lagrangian includes an additional source term

Note 1.5 Gauge fixing

It means removing the freedom one has in specifying the four-vector potential A_μ, i.e. the axial-gauge and Lorenz gauge, $n_\mu A^\mu = 0$ and $\partial_\mu A^\mu$, do not completely fix the gauge.

Remark 1.4 Contravariant and Covariant

Consider a change of basis from $\mathbf{e} \to \mathbf{e}'$, where $\mathbf{e}, \mathbf{e}' \in V$, given by

$$\boxed{\mathbf{e}_\nu = a^\mu_\nu \mathbf{e}'_\mu}.$$

	Covariant	**Contravariant**
Elements	covectors $\mathbf{f} \in V^*$	vectors $\mathbf{x} \in V$
Transformation	$f_\nu = a^\mu_\nu {f'}_\mu$	$x'^\mu = a^\mu_\nu x^\nu$ i.e. $x^\nu = (a^\mu_\nu)^{-1} x'^\mu$
Derivative	$\partial_\mu = (+\partial_t, \boldsymbol{\nabla})$	$\partial^\mu = (-\partial_t, \boldsymbol{\nabla})$

2 Quantizing fields