Functions and Spaces

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1 Introduction

Vector spaces and function spaces are essential to all of mathematical physics. Fortunately, most of undergrad work does not require rigorous

This is by no means a rigorous guide. There is a slight but important difference between *mathematical physics* and *physical mathematics*[1].

2 Functions

Note 2.1 Best approximations

The reason one uses an orthonormal basis is because adding an extra term, i.e. g_{N+1}, to the set of expansion coefficients, is independent of any previous terms. In a **non-orthogonal** basis, having g_{N+1} requires tweaking all N terms (again). However, both are able to give a *best* approximation.

Def 2.1 L^2 convergence

A sequence $\{f_n\} \in L^2$ of periodic functions is said to *converge in L^2* **to a function** if the sequence $\int_0^1 \mathrm{d}x \, |f_n(x) - f(x)|^2$ converges to 0.

Def 2.2 Parceval's theorem

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Def 2.3 Complete orthonormal set

- If $\{U_i\}$ spans a dense subset of the entire Hilbert space V, then it is a **complete orthonormal set**.
- This gives an **arbitrarily close** approximation to $f(x) \in V$.
- Leads to

  $$\|f\|^2 = \sum_n^\infty |a_n|^2 \iff L^2 \quad \text{convergence}$$

  and 2.

[1]Better told in Spanish, *matemáticas físicas* vs. *física matemática*

2.1 Dirac delta function

The the *Dirac delta function* is not a function, and it was introduced (although not in its current form) by Fourier and later by Cauchy (as an infinitesimal form of a Cauchy distribution). However, it was Dirac who gave it its analogous meaning to *Kronecker's delta*, and its proper introduction to quantum mechanics. Field's medalist Laurent Schwartz gave the δ-distribution its meaning and rigour.

Consider the approximation to δ, which we name δ_ϵ.

This issue of convergence leads to an issue with the δ_ϵ function. Since the L^2 norm $\|\delta_\epsilon\| \to \infty$ as $\epsilon \to 0$, it cannot be in L^2.

> **Note 2.2**
> Since elements of L^2 are in reality *equivalence classes* of functions, $f(0)$ is undefined, and thus the paring $(\delta, f) = f(0)$ is doomed to fail for an arbitrary $f \in L^2$.